SEQUENCE & SERIES OF REAL NUMBERS

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• USEFUL FORMULAE ON REAL NUMBERS •

Absolute Value of a Real Numbers :- $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Thus, we already have $|x| \geq 0$, also by definition $|x| = |-x|$.

Note:-
1. $|x| = \max(x, -x)$
2. $|-x| = \max(-x, -(-x)) = \max(-x, x) = |x|$
3. $-|x| = \min(x, -x)$
4. $|x|^2 = |-x|^2 = x^2$
5. $|xy| = |x| \cdot |y|$
6. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, provided $y \neq 0$.

Remark:- If a, b are real numbers, then show that
$\max(a,b) = \frac{a+b+|a-b|}{2}$ and $\min(a,b) = \frac{a+b-|a-b|}{2}$.

Triangle Inequalities :- For all real numbers x, y show that
(i) $|x+y| \leq |x| + |y|$, and
(ii) $|x-y| \geq ||x| - |y||$.

Proof:- (i) $|x+y|^2 = (x+y)^2 = x^2 + y^2 + 2xy$
$\leq |x|^2 + |y|^2 + 2|x| \cdot |y|$ $[\because xy \leq |xy| = |x| \cdot |y|]$
$= (|x| + |y|)^2$
since the quantities are non-negative, so taking the +ve sq. root,
$|x+y| \leq |x| + |y|$

(ii). similar method.

Ex.1. For real nos. $x, a, \epsilon > 0$, show that
(i) $|x| < \epsilon \Leftrightarrow -\epsilon < x < \epsilon$,
(ii) $|x-a| < \epsilon \Leftrightarrow a - \epsilon < x < a + \epsilon$.

Ex.2. If $a, b \in \mathbb{R}$ be $\ni$ $a < b + \epsilon$ for each $\epsilon > 0$, then $a \leq b$.

Ex.3. If $a, b \in \mathbb{R}$, show that if $a \leq b + \frac{1}{n}$, $\forall$ $n \in \mathbb{N}$, then $a \leq b$.

Ex.4. If for any $\epsilon > 0$, $|b-a| < \epsilon$, then $b = a$.

Ex.5. If $a, b \in \mathbb{R}$ and $a < c$ for each $c > b$, then $a \leq b$.

Sol. Do yourself.

Ex.6 $[a+b] \geq [a] + [b]$ for all real numbers a, b.

Ex.7. $[a] + [-a] = \begin{cases} 0, & a \text{ is an integer} \\ -1, & \text{ow} \end{cases}$

• SEQUENCE OF REAL NUMBERS •

The word "Sequence" is used to convey the idea that the things are arranged in order. Before introductiong the concept of sequence in $\mathbb{R}$, we define function or mapping or transformation between two sets A and B.

Let $f: A \to B$ is a mapping or function if for every $x \in A$, there exists ($\exists$) a unique value of $y \in B$. Then the rule f is called a mapping or a function of A into B.
Here we write $y = f(x)$, where $x \in A$ and $y \in B$.

Note that, $y = \pm x$ is not a function, it's a relation.

• DEFINITION:— A sequence of real numbers (or, a sequence in $\mathbb{R}$) is a function defined on the set $\mathbb{N}$ of natural numbers whose range is a subset of the set $\mathbb{R}$ of real numbers; i.e. if for every $n \in \mathbb{N}$ (a set of natural numbers), $\exists$ a real number a_n, then the order set

$$a_1, a_2, \ldots ; a_n, \ldots$$

is said to define a set of real numbers.

Notation:—

(1) If a_n is the n^{th} term of a sequence, then we write $a_1, a_2, \ldots, a_n, \ldots$ to describe the sequence.

(2) $f: \mathbb{N} \to \mathbb{R}$ is a sequence.

(3) A sequence f is generally denoted by the symbol $\{f(n)\}$.
e.g. $\{a_n\}$, $\{b_n\}$, $\{x_n\}$, $\{y_n\}$.

Examples:—

(a) If $b \in \mathbb{R}$, the sequence $B = \{b, b, \ldots\}$, all of whose terms are equal b, is called the constant sequence $\{b\}$.

(b) Let $f: \mathbb{N} \to \mathbb{R}$ is defined by $f(n) = \frac{n}{n+1}$, $n \in \mathbb{N}$. The sequence is $\left\{\frac{n}{n+1}\right\}$. It is also denoted by $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$

(c) Let $f: \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = (-1)^n$, $n \in \mathbb{N}$. The sequence is $\{(-1)^n\}$. It is also denoted by $\{-1, 1, -1, 1, \ldots\}$. The range of the sequence is $\{-1, 1\}$.

(d) The celebrated Fibonacci sequence $F = (f_n)$ is given by the inductive definition

$$f_1 = 1,\ f_2 = 1,\ f_{n+1} = f_{n-1} + f_n \quad (n \geq 2)$$

Thus each term past the second is the sum of its two immediate predecessors. The sequence is $\{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$.

(e) Null Sequence: A null sequence is one whose terms approach zero, i.e. $\lim a_n = 0 \Rightarrow \{a_n\}$ be a null sequence.
Example:- $\left\{\frac{1}{n}\right\}$ is a null sequence.
Note that, If $\{a_n\}$ be a null sequence then $\{|a_n|\}$ is a null sequence and conversely.

The Limit of a Sequence:—

- **DEFINITION:**— A sequence $\{a_n\}$ is said to have a limit $l \in \mathbb{R}$ if for every $\epsilon > 0$, $\exists$ a natural number $N(\epsilon)$ $\ni$ for all $n \geq N(\epsilon)$, the terms $\{a_n\}$ satisfy $|a_n - l| < \epsilon$.

Ex.(1):— Show that the sequence $\left\{\frac{1}{n}\right\}$ has the limit 0.

Solution:— We show that $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$.

Now, $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \epsilon$, whenever $n > \frac{1}{\epsilon}$

We choose $n_0 = \left[\frac{1}{\epsilon}\right] + 1$ $\therefore$ $\left|\frac{1}{n} - 0\right| < \epsilon$ holds when $n \geq n_0$.

By the above definition, $\lim_{n\to\infty} \frac{1}{n} = 0$. [Here $\left[\frac{1}{\epsilon}\right]$ denotes the integral part of $\frac{1}{\epsilon}$]

In other words, $\left\{\frac{1}{n}\right\}_n$ converges to 0.

Note:— Let $\{f(n)\}$ be a real sequence. A real number l is said to be a limit of the sequence $\{f(n)\}$ if corresponding to a pre-assigned positive quantity ϵ $\exists$ a natural number K (depending on ϵ) such that

$$\boxed{|f(n) - l| < \epsilon \quad \forall\ n \geq K}$$

i.e. $l - \epsilon < f(n) < l + \epsilon \ \forall\ n \geq K$.

To be explicit, a real number l is said to be a limit of the sequence $\{f(n)\}$, if for a pre-assigned positive ϵ $\exists$ a natural number K $\ni$ all elements of the sequence, excepting the first $K-1$ at most, lie in the ϵ-neighbourhood of l.

Theorem:— A sequence can have at most one limit.

Proof:— If possible, let a sequence $\{f(n)\}$ have two distinct limits l_1 and l_2, where $l_1 < l_2$.

Let $\epsilon = \frac{1}{2}(l_2 - l_1)$ Then $\epsilon > 0$ and $l_1 + \epsilon = l_2 - \epsilon$. Therefore the neighbourhoods $(l_1 - \epsilon, l_1 + \epsilon)$ and $(l_2 - \epsilon, l_2 + \epsilon)$ are disjoint.

But, as l_1 is a limit of a sequence, then

$$l_1 - \epsilon < f(n) < l_1 + \epsilon \quad \forall\ n \geq K_1$$

Similarly,

$$l_2 - \epsilon < f(n) < l_2 + \epsilon \quad \forall\ n \geq K_2.$$

Let, $K = \max\{K_1, K_2\}$

Then, $l_1 - \epsilon < f(n) < l_1 + \epsilon$ and $l_2 - \epsilon < f(n) < l_2 + \epsilon$ for all $n \geq K$.

This can't happen since the neighbourhoods $N(l_1, \epsilon)$ and $N(l_2, \epsilon)$ are disjoint. Therefore our assumption $l_1 \neq l_2$ is wrong.

Hence, $l_1 = l_2$ and this proves the theorem.

The Range:— The Range or the Range Set is the set consisting of all distinct elements of a ~~set of~~ sequence, without repetition and without regard to the position of a term. Thus, the range may be a finite or an infinite set, without ever being the null set.

Bounds of a Sequence:-

- **DEFINITION:-** A sequence $\{a_n\}$ is said to be bounded if $\exists$ two real numbers m and $M \ni m \le a_n \le M \;\forall\; n \in \mathbb{N}$.
Here, M is called the upper bound of the sequence and m is called the lower bound of the sequence.

If a sequence is not bounded then it is called unbounded sequence.

Ex:- (1) The sequence $\{\frac{1}{n}\}$ is a bounded sequence; as $0 < \frac{1}{n} \le 1 \;\forall n$; 0 and 1 are the lower and upper bound respectively.

(2) $\{(-1)^n+1\}$ is a bounded sequence as

$$(-1)^n+1 = \begin{cases} 0 & \text{if } n \text{ is odd.} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

(3) $\{(-1)^n+n\}$ is not a bounded sequence as $\nexists$ an $M \in \mathbb{R}$ such that $a_n \le M$.

(4) The sequence $\{n^2\}$, $\{n^2+n\}$ are unbounded.

(5) Let $f(n) = (-1)^n n$, $n \in \mathbb{N}$, the sequence $\{f(n)\}$ is unbounded above and unbounded below. The sequence is $\{-1, 2, -3, 4, \ldots\}$
Here $\sup\{f(n)\} = \infty$ and $\inf\{f(n)\} = -\infty$.

Note:- Therefore a real sequence is bounded if and only if it is bounded above as well as bounded below. In this case the range of the sequence is a bounded set.

The least upper bound of a real sequence $\{f(n)\}$ is a real number M (denoted by $\sup\{f(n)\}$) satisfy the following conditions:
(i) $f(n) \le M$ for all $n \in \mathbb{N}$,
(ii) for each pre-assigned positive ϵ, there exists a natural number k such that $f(k) > M - \epsilon$.

The greatest lower bound of a real sequence $\{f(n)\}$ is a real number m (denoted by $\inf\{f(n)\}$) satisfy the following conditions:
(i) $f(n) \ge m$ for all $n \in \mathbb{N}$,
(ii) for each pre-assigned positive ϵ, $\exists$ a natural number k such that $f(k) < m + \epsilon$.

For a real sequence $\{f(n)\}$ is said to be unbounded above if $\sup\{f(n)\} = \infty$.
For a real sequence $\{f(n)\}$ is said to be unbounded below if $\inf\{f(n)\} = -\infty$.

Convergent Sequence:-

- **DEFINITION:-** (1) A real sequence $\{a_n\}$ is said to be convergent and converge to a unique limit l (a real number) if for any given positive number ϵ $\exists$ a positive integer m depending on ϵ such that $|a_n - l| < \epsilon \;\forall\; n \geq m$ ....(1)

Inequality (1) may also be written as

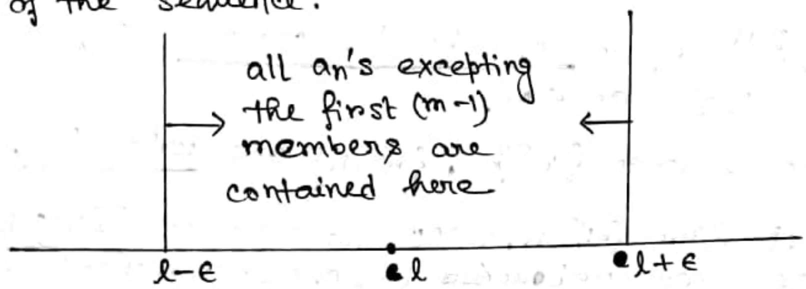
$$l - \epsilon < a_n < l + \epsilon \quad \text{when } n \geq m \text{ ....(2)}$$

This is also expressed in shorter form

(i) $\lim\limits_{n \to \infty} a_n = l$

(OR), (ii) $a_n \to l$ as $n \to \infty$

The significance of (1) or (2) is that given $\epsilon > 0$ all terms of the sequence lie between $l-\epsilon$ and $l+\epsilon$ excluding the first $(m-1)$ terms of the sequence.



In other words, the above definition of convergence of a sequence suggests a picture of the a_n's moving closer and closer to a unique limit point l on the real axis, so that any neighbourhood of l contains all the members with almost a finite number of exceptions.

(2) If a sequence $\{a_n\}$ has a finite limit 'l', we say that the sequence $\{a_n\}$ converges to l. Then $\{a_n\}$ is called a convergent sequence.

We write, $\lim\limits_{n\to\infty}(a_n) = l$ or, $\lim(a_n) = l$.

Divergent Sequence:- (1) If a sequence does not converge to a finite limit, then it is said to be divergent sequence.

(2) A sequence is said to be divergent sequence if it is not convergent.

Alternative Definition:-

(a) We say that $\{x_n\}$ diverges to $+\infty$ and write $\lim(x_n) = +\infty$ if for every $\epsilon > 0$, $\exists$ a natural number $N(\epsilon)$ $\ni$ $x_n > \epsilon \;\forall\; n \geq N(\epsilon)$.

(b) We say that, $\{x_n\}$ diverges to $-\infty$ and write $\lim(x_n) = -\infty$ if for every $\epsilon < 0$, $\exists$ a natural number $N(\epsilon) \ni x_n < -\epsilon \;\forall\; n \geq N(\epsilon)$.

Properly Divergent:- If $\lim(x_n) = +\infty$ or $\lim(x_n) = -\infty$, then the sequence $\{x_n\}$ is called properly divergent. **Example:-**

(a) $\{2^n\}$ is a properly divergent sequence and diverges to $+\infty$.

(b) $\{-n^2\}$ is a properly divergent sequence and diverges to $-\infty$.

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Remarks:- (1) The term "Convergent Sequence" is said to convey for the sequence whose limit is finite and unique.

The term "divergent Sequence" is said to convey for the sequence whose limit is infinite and not unique.

(2) It is important to note that the convergence and ~~and~~ divergence depend on the behaviour of the ultimate terms. Hence, the alternation of the finite number of term of a sequence does not effect the convergence and divergence of the sequence, i.e. a convergent sequence does not alter the limit. For example,

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{n}$ & $1000, 100, 10, 1, \ldots, \frac{1}{n}$ have the same limit.

Theorem 1:- Every convergent sequence is bounded.

Proof:- Let a sequence $\{a_n\}$ converges to a limit l.
Let $\epsilon > 0$, be a given number, so that $\exists$ a positive integer m such that

$$|a_n - l| < \epsilon \quad \forall\ n \geq m$$

$$\Leftrightarrow l - \epsilon < a_n < l + \epsilon \ \forall\ n \geq m$$

Let, $B = \max\{l+\epsilon, a_1, a_2, \ldots, a_{m-1}\}$
$b = \min\{l-\epsilon, a_1, a_2, \ldots, a_{m-1}\}$

Thus we have $b \leq a_n \leq B \ \forall\ n \in \mathbb{N}$

This proves that the sequence $\{a_n\}$ is a bounded sequence.

Remark:- The converse of this theorem is not always true, e.g.

A Bounded Sequence may not be convergent.

i) The sequence $\{(-1)^n + 1\}$ is bounded but not convergent.

Hints:- $(-1)^n + 1 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$

So, if $n \to \infty$ through odd integers then $a_n \to 0$
but if $n \to \infty$ through even integers then $a_n \to 2$
Hence the sequence is bounded.
But as it has no unique limit, so the sequence is divergent.

ii) The sequence $\{(-1)^n\}$, $n \in \mathbb{N}$ is bounded but not convergent.

Hints:- The given sequence is bounded.

If possible, let, $\lim_{n \to \infty} a_n = l$, then for $\epsilon = 1$, $\exists\ m \in \mathbb{N} \ni$

$$|a_n - l| < 1 \ \forall\ n \geq m$$

i.e. $|(-1)^{2m} - l| < 1$ and $|(-1)^{2m+1} - l| < 1$

or, $|1 - l| < 1$ and $|1 + l| < 1$

$\Rightarrow 2 = |1 - l + 1 + l| < 1 + 1 = 2$

which is absurd. So, the sequence is not convergent.

Corollary:- An unbounded sequence is not convergent.

e.g. The sequences $\{n^2\}$, $\{(-1)^n . n\}$ are unbounded and they are divergent.

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Theorem 2:- A convergent sequence has a unique limit.

Proof:- If possible, let $\{a_n\}$ converges to two real numbers l and l' $(l \neq l')$.

We choose, $\epsilon = \frac{1}{3}|l - l'| > 0$

Since the sequence $\{a_n\}$ converges to l and l'; therefore, $\exists$ positive integers m_1 and m_2 $\ni$

$$|a_n - l| < \epsilon \quad \forall\, n \geq m_1 \quad \cdots\cdots (1)$$

and $$|a_n - l'| < \epsilon \quad \forall\, n \geq m_2 \quad \cdots\cdots (2)$$

Now from (1) & (2), for $n \geq \max(m_1, m_2)$

$$|l - l'| = |l - a_n + a_n - l'| \leq |l - a_n| + |a_n - l'| \quad \text{[By Triangle inequality]}$$
$$< 2\epsilon$$

i.e. $|l - l'| < \frac{2}{3}|l - l'|$, which is not possible.

Hence, the sequence can't converge to two limits.

Theorem 3:- Every convergent sequence is bounded and has a unique limit.

Remark:- 1) It may be seen from the definition that the number to which a sequence converges is a limit point of the sequence. Consequently, the unique limit to which the sequence converges is called the limit point or the limit of the sequence.

2) The definition of the limit of a sequence of real numbers is used to verify that a proposed value 'l' is indeed the limit. It does not provide a means for initially determining what that value of 'l' might be.

3) A constant sequence is always a convergent sequence.

Eg. Let $a_n = 2 \ \forall\, n \in \mathbb{N}$. Then the sequence is $\{2, 2, 2, \ldots\}$.

Let us choose a positive ϵ.

Now $|a_n - 2| < \epsilon$ holds for all $n \geq 1$.

Therefore, $\lim a_n = 2$.

4) Every bounded sequence has a limit point.

e.g. Let us consider the sequence $\{a_n\}$, where $a_n = \frac{(-1)^{n-1}}{n!}, n \in \mathbb{N}$

i.e. $\{1, \frac{-1}{2!}, \frac{-1}{3!}, \ldots\}$ is bounded, for $-\frac{1}{2} \leq a_n \leq 1, \forall n$.

Here 0 being the only limit point, the upper and lower limit points coincide with 0, and so $\lim_{n\to\infty} a_n = 0$.

5) **Result:-** Let $\{a_n\}$ be a sequence of real numbers and let $l \in \mathbb{R}$. If $\{a_n\}$ is a sequence of all positive real numbers with $\lim(b_n) = 0$ and if for some constant $c > 0$ and some $m \in \mathbb{N}$, then we have $|a_n - l| < c b_n \ \forall\, n \geq m$, then it follows that $\lim(a_n) = l$. (Important Theorem)

WORKED EXAMPLES :- (Based on Convergent Sequences)

1) The sequence $\left\{\frac{n^2+1}{n^2}\right\}$ converges to 1. → Prove.

Solution:- $\epsilon > 0$

Now, $\left|\frac{n^2+1}{n^2} - 1\right| < \epsilon$ holds if $\frac{1}{n^2} < \epsilon$, i.e., if $n > \frac{1}{\sqrt{\epsilon}}$.

Let $k = \left[\frac{1}{\sqrt{\epsilon}}\right] + 1$, then $\left|\frac{n^2+1}{n^2} - 1\right| < \epsilon \ \forall \ n \geq k$

This proves $\lim \frac{n^2+1}{n^2} = 1$.

2) Prove that $\lim_{n\to\infty} \left(\frac{1}{n+1}\right) = 0$.

Solution:-

$$\left|\frac{1}{n+1} - 0\right| = \frac{1}{n+1} < \frac{1}{n} < \epsilon$$

$$\Rightarrow n > \frac{1}{\epsilon}.$$

Let $k = \left[\frac{1}{\epsilon}\right] + 1$, then $\left|\frac{1}{n+1} - 0\right| < \epsilon \ \forall \ n \geq k$

∴ By definition, $\lim_{n\to\infty} \left(\frac{1}{n+1}\right) = 0$.

3) Prove that $\lim_{n\to\infty} \left(\frac{1}{n^2+1}\right) = 0$.

Solution:-

$$\left|\frac{1}{n^2+1} - 0\right| = \frac{1}{n^2+1} < \frac{1}{n^2} < \epsilon$$

$$\Rightarrow n > \frac{1}{\sqrt{\epsilon}}$$

Take $k = \left[\frac{1}{\sqrt{\epsilon}}\right] + 1$

Then $\exists$ a natural no. $k \ni \left|\frac{1}{n^2+1} - 0\right| < \epsilon \ \forall \ n \geq k$

This proves $\lim_{n\to\infty} \left(\frac{1}{n^2+1}\right) = 0$.

4) Prove that $\lim_{n\to\infty} \frac{3n}{n+5n^{1/2}} = 3$.

Solution:-

$$\left|\frac{3n}{n+5n^{1/2}} - 3\right| < \epsilon$$

$$\Rightarrow \left|\frac{3n - 3n - 15n^{1/2}}{n+5n^{1/2}}\right| < \epsilon$$

$$\Rightarrow \frac{+15n^{1/2}}{n+5n^{1/2}} < \frac{15n^{1/2}}{n} < \epsilon$$

$$\Rightarrow \frac{15}{n^{1/2}} < \epsilon$$

$$\Rightarrow n > \frac{225}{\epsilon^2}.$$

$$\Rightarrow k = \left[\frac{225}{\epsilon^2}\right] + 1$$

Then $\exists$ a natural no. $k \ni \left|\frac{3n}{n+5n^{1/2}} - 3\right| < \epsilon \ \forall \ n \geq k$

$$\therefore \lim_{n\to\infty} \frac{3n}{n+5n^{1/2}} = 3.$$

5) Show that $\lim \left(\frac{3n+2}{n+1}\right) = 3$.

Solution:- For given $\epsilon > 0$, we want to obtain the inequality

$$\left|\frac{3n+2}{n+1} - 3\right| < \epsilon$$

when n is sufficiently large. Here we first simplify the expression on the left:

$$\left|\frac{3n+2}{n+1} - 3\right| = \left|\frac{3n+2-3n-3}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n} < \epsilon$$

$$\Rightarrow n > \frac{1}{\epsilon}$$

take $k = \left[\frac{1}{\epsilon}\right] + 1$

Then $\left|\frac{3n+2}{n+1} - 3\right| < \epsilon$ is true $\forall\ n \geq k$

$\therefore \lim \left[\frac{3n+2}{n+1}\right] = 3$.

6) $\left\{\frac{2n+1}{3n}\right\}_n$ converges to $\frac{2}{3}$.

Solution:- Now $\left|\frac{2n+1}{3n} - \frac{2}{3}\right| = \left|\frac{1}{3n}\right| < \epsilon$ if $n > \left[\frac{1}{3\epsilon}\right]$

$\therefore$ Choosing $k = \left[\frac{1}{3\epsilon}\right] + 1$, we see that

$$\left|\frac{2n+1}{3n} - \frac{2}{3}\right| < \epsilon \text{ whenever } n \geq k.$$

7) Show that $\lim\limits_{n\to\infty} \left(\frac{2n^2+1}{n^2+n}\right) = 2$.

Solution:-

$$\left|\frac{2n^2+1}{n^2+n} - 2\right| < \epsilon$$

$$\Rightarrow \left|\frac{1-2n}{n^2+n}\right| < \epsilon$$

$$\Rightarrow \frac{2n-1}{n^2+n} < \frac{2n}{n^2} = \frac{2}{n} < \epsilon$$

$$\Rightarrow n > \frac{2}{\epsilon}.$$

take, $k = \left[\frac{2}{\epsilon}\right] + 1$,

Then $\exists$ a natural no. $k \ni \left|\frac{2n^2+1}{n^2+n} - 2\right| < \epsilon\ \forall\ n \geq k$.

$\therefore \lim\limits_{n\to\infty} \left(\frac{2n^2+1}{n^2+n}\right) = 2$.

8. Prove that $\lim\limits_{n\to\infty} \left\{(-1)^n \cdot \frac{1}{n}\right\} = 0$

Solution:- $\epsilon > 0$ be an arbitrary no.

$$\left|(-1)^n \cdot \frac{1}{n} - 0\right| = \frac{1}{n} < \epsilon$$

$$\Rightarrow n > \frac{1}{\epsilon} \qquad \therefore k = \left[\frac{1}{\epsilon}\right] + 1$$

$\therefore \left|(-1)^n \cdot \frac{1}{n}\right| < \epsilon\ \forall\ n \geq k$

$\therefore \lim\limits_{n\to\infty} \left\{(-1)^n \cdot \frac{1}{n}\right\} = 0$.

9. Prove that $\lim_{n\to\infty}\left(\frac{1}{n^p}\right)=0$, where $p>0$.

Solution:-

$$\left|\frac{1}{n^p}-0\right|<\epsilon$$
$$\Rightarrow n^p>\frac{1}{\epsilon}$$
$$\Rightarrow n>\left(\frac{1}{\epsilon}\right)^{\frac{1}{p}}$$

We choose $k=\left[\frac{1}{\epsilon}\right]^{1/p}+1$

, Since $p>0$. .

Then $\exists$ a natural no. $k \ni \left|\frac{1}{n^p}-0\right|<\epsilon \;\forall\, n\geq k$.

10. Prove that $\lim_{n\to\infty}(r^n)=0$, if $|r|<1$. (A very useful result)

Solution:- Let $\epsilon>0$ be an arbitrary no.

$$|r^n-0|<\epsilon$$
$$\Rightarrow |r^n|<\epsilon$$
$$\Rightarrow n\ln|r|<\ln\epsilon$$
$$\Rightarrow n>\frac{\ln\epsilon}{\ln|r|}, \text{ since } |r|<1\Rightarrow \ln|r|<0.$$

Choose a natural no. k which is $>\frac{\ln\epsilon}{\ln|r|}$.

Then $\exists$ a natural no. $k \ni |r^n-0|<\epsilon \;\forall\, n\geq k$

$\therefore$ By defn. $\lim_{n\to\infty}(r^n)=0$ if $|r|<1$.

11. Show that $\{n^{1/n}\}_n$ converges to 1.

Solution:- We see that $n^{1/n}>1$ when $n>1$, and therefore, put $n^{1/n}=1+x_n\;(x_n>0)$

We proceed to prove that $x_n\to 0$ as $n\to\infty$.

Now, $n=(1+x_n)^n=1+nx_n+\frac{n(n-1)}{2!}x_n^2+\cdots+x_n^n$

$$\therefore n>\frac{n(n-1)}{2}x_n^2 \Rightarrow x_n<\sqrt{\frac{2}{n-1}}<\epsilon$$

if, $n-1>\frac{2}{\epsilon^2}$ or, $n>\frac{2}{\epsilon^2}+1$

We choose $k=\left[\frac{2}{\epsilon^2}+1\right]+1$

$\therefore |x_n|<\epsilon$ when $n\geq k$, proving that $x_n\to 0$ as $n\to\infty$.

$\therefore n^{1/n}\to 1$ as $n\to\infty$, or $\{n^{1/n}\}_n$ converges to 1.

12. Prove that $\lim_{n\to\infty}\left(2-\frac{1}{2^n}\right)=2$.

Solution:-

$$\left|2-\frac{1}{2^n}-2\right|<\epsilon$$

$$\Rightarrow \frac{1}{2^n}<\epsilon$$

$$\Rightarrow \frac{1}{2^n}<\frac{1}{n}<\epsilon$$

$$\Rightarrow n>\frac{1}{\epsilon}$$

Taking $k=\left[\frac{1}{\epsilon}\right]+1$

then $\exists$ a natural no. $k \ni \left|2-\frac{1}{2^n}-2\right|<\epsilon \ \forall\ n\geq k$

$$\therefore \lim_{n\to\infty}\left(2-\frac{1}{2^n}\right)=2.$$

13. Prove that $\lim_{n\to\infty}\left(2^{1/n}\right)=1$.

Solution:-

$$\left|2^{1/n}-1\right|<\epsilon$$

$$\Rightarrow \frac{1}{n}\ln 2<\ln(\epsilon+1)$$

$$\Rightarrow n>\frac{\ln 2}{\ln(\epsilon+1)}$$

$$\therefore k\geq\left[\frac{\ln 2}{\ln(\epsilon+1)}\right] \quad \text{or, } k=\left[\frac{\ln 2}{\ln(\epsilon+1)}\right]$$

$$\therefore n\geq k$$

14. Prove that $\lim_{n\to\infty}\left(3^{-n}\right)=0$.

Solution:-

$$|3^{-n}|<\epsilon$$

$$\Rightarrow 3^n>\frac{1}{\epsilon}$$

$$\Rightarrow n>\frac{\ln 1/\epsilon}{\ln 3}$$

$$\therefore k>\frac{\ln 1/\epsilon}{\ln 3}. \quad \text{or, } k=\left[\frac{\ln 2}{\ln(\epsilon+1)}\right]$$

$$\therefore n\geq k$$

15. Show that $\lim_{n\to\infty}\frac{3+2\sqrt{n}}{\sqrt{n}}=2$.

Solution:-

$$\left|\frac{3+2\sqrt{n}}{\sqrt{n}}-2\right|<\epsilon, \text{ if } \left|\frac{3}{\sqrt{n}}\right|<\epsilon \text{ or } n>\frac{9}{\epsilon^2}.$$

Let m be a positive integer greater than $9/\epsilon^2$.

Then to $\epsilon>0$, $\exists$ a positive m, such that

$$\left|\frac{3+2\sqrt{n}}{\sqrt{n}}-2\right|<\epsilon \quad \forall\ n\geq m$$

$$\therefore \lim_{n\to\infty}\frac{3+2\sqrt{n}}{\sqrt{n}}=2.$$

16. Prove that $\lim_{n\to\infty}\left(\frac{10^6 \cdot n}{n^2+n}\right) = 0.$

<u>Solution:-</u>

$$\left|\frac{10^6 \cdot n}{n^2+n}\right| < \epsilon$$

$$\Rightarrow n+1 > \frac{10^6}{\epsilon}$$

$$\Rightarrow n > \frac{10^6}{\epsilon} - 1$$

Take $K = \left[\frac{10^6}{\epsilon} - 1\right] + 1$

Then $\exists$ a natural no. $K \ni \left|\frac{10^6 \cdot n}{n^2+n}\right| < \epsilon \ \forall\ n \geq K$

$\therefore \lim\left(\frac{10^6 \cdot n}{n^2+n}\right) = 0.$

17. Prove that $\lim_{n\to\infty}\left(\frac{\sqrt{n}}{n^2+n}\right) = 0$

<u>Solution:-</u> Let $\epsilon > 0$ be an arbitrary no.

$$\left|\frac{\sqrt{n}}{n^2+n}\right| < \frac{\sqrt{n}}{n} < \epsilon$$

$$\Rightarrow \frac{1}{\sqrt{n}} < \epsilon$$

$$\Rightarrow \frac{1}{n} < \epsilon^2$$

$$\Rightarrow n > \frac{1}{\epsilon^2}.$$

$$\Rightarrow K = \left[\frac{1}{\epsilon^2}\right] + 1 \text{ (Taking)}$$

Then $\exists$ a natural no. $K \ni \left|\frac{\sqrt{n}}{n^2+n}\right| < \epsilon \ \forall\ n \geq K$

$\therefore$ By definition, $\lim_{n\to\infty}\left(\frac{\sqrt{n}}{n^2+n}\right) = 0.$

18. Prove that $\lim_{n\to\infty}\left(\frac{n^2-1}{2n^2+3n}\right) = \frac{1}{2}.$

<u>Solution:-</u>

$$\left|\frac{n^2-1}{2n^2+3n} - \frac{1}{2}\right| = \frac{2+3n}{2(2n^2+3n)} < \frac{3n}{4n^2} = \frac{3}{4n} < \epsilon$$

$$\Rightarrow n > \frac{3}{4\epsilon}$$

Taking $K = \left[\frac{3}{4\epsilon}\right] + 1.$

Then $\exists$ a natural no. $K \ni \left|\frac{n^2-1}{2n^2+3n} - \frac{1}{2}\right| < \epsilon \ \forall\ n \geq K$

$\therefore$ By definition, $\lim_{n\to\infty}\left(\frac{n^2-1}{2n^2+3n}\right) = \frac{1}{2}.$

19. Show that (i) $\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) = 0$

(ii) $\lim_{n\to\infty} \left(\frac{\sin^2 \angle n}{n^2+1}\right) = 0$

Solution:-

i) Let $\epsilon > 0$ be an arbitrary number.

$$\left|\sqrt{n+1} - \sqrt{n}\right| = \frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}} < \epsilon$$

$$\Rightarrow \frac{1}{\sqrt{n+1}} < \epsilon$$

$$\Rightarrow n+1 > \epsilon^2$$

$$\Rightarrow n > \epsilon^2 - 1.$$

ii) $$\left|\frac{\sin^2 \angle n}{n^2+1} - 0\right| < \frac{1}{n^2} < \epsilon$$

$$\Rightarrow n > \frac{1}{\sqrt{\epsilon}}$$

Take $K = \left[\frac{1}{\sqrt{\epsilon}}\right] + 1$

then $\exists$ a natural no. $K \ni \left|\frac{\sin^2 \angle n}{n^2+1} - 0\right| < \epsilon \ \forall \ n > K$

$$\therefore \lim_{n\to\infty} \frac{\sin^2 \angle n}{n^2+1} = 0.$$

20. Given that the sequence $\{x_n\}_n$ where, $x_n = 3 + \left(\frac{4}{5}\right)^n$ converges to 3, if $\epsilon = 10^{-5}$, find K.

Solution:-

$$\left|3 + \left(\frac{4}{5}\right)^n - 3\right| < 10^{-5}$$

$$\Rightarrow \left(\frac{4}{5}\right)^n < 10^{-5}$$

$$\Rightarrow \left(\frac{5}{4}\right)^n > 10^5$$

$$\Rightarrow n \log\frac{5}{4} > 5$$

$$\Rightarrow n > \frac{5}{\log\frac{5}{4}}$$

$$\therefore K = \left[\frac{5}{\log\frac{5}{4}}\right]$$

Further Examples: In establishing that a number l is the limit of a sequence (a_n), we often try to simplify the difference $|a_n - l|$ before considering an $\epsilon > 0$ and finding $N(\epsilon)$ as required by the definition of limit. This was done in some of the earlier examples. The next result is a more formal statement of this idea, and the examples that follow make use of this approach.

Theorem:- Let (x_n) be a sequence of real numbers and let $l \in \mathbb{R}$. If (a_n) is a sequence of positive real numbers with $\lim(a_n) = 0$ and if for some constant $c > 0$ and some $m \in \mathbb{N}$ we have

$$|x_n - l| \le c \cdot a_n \quad \forall \ n \ge m,$$

then it follows that $\lim(x_n) = l$.

Proof:- If $\epsilon > 0$ is given, then since $\lim(a_n) = 0$, we know there exists $K = N(\epsilon/c)$ such that $n \ge K$ implies

$$a_n = |a_n - 0| < \epsilon/c$$

Therefore it follows that if both $n \ge K$ and $n \ge m$, then

$$|x_n - l| \le c \cdot a_n < c\left(\epsilon/c\right) = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we conclude that $l = \lim(x_n)$. (QED)

Examples:-

(a) If $a > 0$, then $\lim\left(\frac{1}{1+na}\right) = 0$.

Since, $a > 0$, then $0 < na < 1 + na$,

$$\therefore \ 0 < \frac{1}{1+na} < \frac{1}{na}.$$

Thus, we have $\left|\frac{1}{1+na} - 0\right| \le \left(\frac{1}{a}\right) \cdot \frac{1}{n} \quad \forall \ n \in \mathbb{N}$

Since, $\lim\left(\frac{1}{n}\right) = 0$.

We may invoke the above theorem with $c = 1/a$ and $m = 1$ to infer that $\lim\left(\frac{1}{1+na}\right) = 0$.

(b) If $0 < b < 1$, then $\lim(b^n) = 0$.

Here we will give a 2nd proof that illustrate the use of Bernoulli's inequality:- If $x > -1$, then $(1+x)^n \ge 1 + nx \ \forall \ n \in \mathbb{N}$

(Proof can easily be done by using Mathematical Induction)

Since, here $0 < b < 1$, we can write $b = \frac{1}{(1+a)}$, where $a = \frac{1}{b} - 1$ so that $a > 0$, then by Bernoulli's inequality, we have $(1+a)^n \ge 1 + na$.

Hence, $0 < b^n = \frac{1}{(1+a)^n} \le \frac{1}{1+na} < \frac{1}{na}$.

Thus from the above theorem, we conclude that $\lim(b^n) = 0$.

WORKED EXAMPLES (Based on Divergent Sequence)

1. Show that $\lim_{n\to\infty} (n^2) = +\infty$.

Solution:- Let $\epsilon > 0$ be an arbitrary no.

Then $a_n > \epsilon \Rightarrow n^2 > \epsilon$
$\Rightarrow n > \sqrt{\epsilon}$

Take, $K = [\sqrt{\epsilon}] + 1$

Then $\exists$ a natural no. $K \ni n^2 > \epsilon, \forall n \geq K$

By definition, $\lim_{n\to\infty} (n^2) = +\infty$.

2. Show that $\lim \left(\frac{n}{\sqrt{n-1}}\right) = +\infty$.

Solution:- Let $\epsilon > 0$ be an arbitrary no.

Then $\frac{n}{\sqrt{n-1}} > \frac{n}{\sqrt{n}} = \sqrt{n} > \epsilon$

$\Rightarrow n > \epsilon^2$

Take $K = [\epsilon^2] + 1$

Then $\exists$ a natural no. $K \ni \frac{n}{\sqrt{n-1}} > \epsilon \ \forall n \geq K$

Hence, $\lim \frac{n}{\sqrt{n-1}} = +\infty$.

3. Show that $\lim (r^n) = +\infty$ if $r > 1$.

Solution:- Let $\epsilon > 0$ be an arbitrary no.

Then $r^n > \epsilon$ if $r > 1$

$\Rightarrow n > \frac{\ln \epsilon}{\ln r}$

Choose a natural number K which is greater than $\frac{\ln \epsilon}{\ln r}$.

Then $\exists$ a natural no. $K \ni r^n > \epsilon \ \forall n \geq K$.

By definition, $\lim (r^n) = +\infty$, if $r > 1$.

4. Show that $\{-2n\}_n$ diverges to $-\infty$.

Solution:- Now, $-2n < -K$ $(K > 0)$

if $n > \frac{K}{2} = m$.

$\therefore \lim_{n\to\infty} (-2n) = -\infty$.

Remark:- A sequence $\{x_n\}_n$ which is not convergent is not necessarily divergent. There are sequences which neither converge or diverge. Such sequences are called oscillatory.

Sequence

- Convergent (with unique finite limit)
- Divergent
  - Properly [$\lim(x_n) = \infty$ or $\lim(x_n) = -\infty$]
  - Oscillatory
    - Finite
    - Infinite

Sequence

- Bounded
  - Convergent
  - Oscillatory with finite oscillation
- Unbounded
  - Properly Divergent
  - Oscillatory with infinite oscillation

<u>Oscillatory sequence</u> :– An oscillatory sequence $\{x_n\}$ is said to oscillate finitely if there exists a constant $c \ni |x_n| < c \ \forall \ n \in \mathbb{N}$. Otherwise it is said to oscillate infinitely.

<u>DEFINITION</u> :– (a) A <u>bounded sequence</u> which <u>is not convergent</u> is said to be an oscillatory sequence with <u>finite oscillation</u>.

(b) An <u>unbounded sequence</u> which is <u>not properly divergent</u> is said to be an oscillatory sequence with <u>infinite oscillation</u>.

<u>Example</u> :– (a) $\{(-1)^n + 1\}$ is a bounded sequence but not convergent. Hence it is an oscillatory sequence with <u>finite oscillation</u>.

(b) $\{(-1)^n . n\}$ is unbounded and not properly divergent. Hence it is an oscillatory sequence with <u>infinite oscillation</u>.

(c) $\left\{\frac{(-1)^n}{n}\right\}$ <u>oscillates finitely</u> for $\left|\frac{(-1)^n}{n}\right| = \left|\frac{1}{n}\right| \leq 1 \ \forall \ n$.

<u>Ex.</u> Check the convergence or divergence of the sequence $\{r^n\}$ for different values of r.

<u>Solution</u> :– If $|r| < 1$, $\lim(r^n) = 0$.

If $r = 1$, then $\{r^n\}$ becomes an constant sequence. and $\lim(r^n) = 1$.

If $r = -1$, then $\{r^n\} = \{(-1)^n\}$ and $\{r^n\}$ is an oscillatory sequence with finite oscillation.

If $r > 1$, $\lim\limits_{n \to \infty}(r^n) = +\infty$.

If $r < -1$, then $r = -s$, $s > 1$ and $\{r^n\} = \{(-1)^n . s^n\}$ is an oscillatory sequence with infinite oscillation.

WORKED EXAMPLES:-

1. Show that the sequence $\{a_n\}$ where $a_n = \frac{1}{n}\cos\frac{n\pi}{2}$ is convergent.

Solution:- Here $|a_n - 0| = \left|\frac{1}{n}\cos\frac{n\pi}{2}\right| \le \frac{1}{n} < \epsilon$

if $n > \frac{1}{\epsilon}$ we choose $m = \left[\frac{1}{\epsilon}\right] + 1$.

$\therefore |a_n - 0| < \epsilon$ for $n > m$. $\therefore \{a_n\}$ converges to 0.

2. Given that $\{a_n\}$ as a sequence such that $a_2 \le a_4 \le a_6 \le \cdots \le a_5 \le a_3 \le a_1$ and a sequence $\{b_n\}$ where $b_n = a_{2n-1} - a_{2n}$ converges to 0, then show that the sequence $\{a_n\}$ is convergent.

Solution:- $\{a_n\}$ consists of two subsequences $\{a_{2n}\}$ and $\{a_{2n-1}\}$ of which $\{a_{2n}\}$ is monotone increasing and $\{a_{2n-1}\}$ is monotone decreasing. The sequence $\{a_{2n}\}$ is bounded above by a_1 and the sequence $\{a_{2n-1}\}$ is bounded below by a_2. Hence both the sequences are convergent.

Let $\{a_{2n}\}$ converges to l and $\{a_{2n-1}\}$ converges to l'.

Now, $\{b_n\}$ converges to 0.

$\therefore \lim_{n\to\infty} b_n = 0$, or, $\lim_{n\to\infty}(a_{2n-1} - a_{2n}) = 0$

or, $l' - l = 0$

or, $l = l'$.

$\therefore$ The sequence $\{a_n\}$ converges to l.

3. Let a sequence $\{S_n\}$ be defined as $S_{n+1} = \dfrac{4+3S_n}{3+2S_n}$, $n \geq 1$, $S_1 = 1$. Show that $\{S_n\}_n$ converges to $\sqrt{2}$.

Solution:- Here $S_{n+2} - S_{n+1} = \dfrac{(S_{n+1} - S_n)}{(3+2S_{n+1})(3+2S_n)}$

$\therefore S_{n+2} > S_{n+1}$ if $S_{n+1} > S_n$, i.e., according as $S_2 > S_1$.

Now, $S_2 = \frac{7}{5} > S_1$

$\therefore \{S_n\}$ is a monotone increasing sequence.

Now, $S_{n+1} - 1 = \dfrac{1+S_n}{3+2S_n} < 1$

$\therefore 0 < S_n < 2 \ \forall n$.

$\therefore \{S_n\}$ is a convergent sequence.

Let $\lim_{n\to\infty} S_n = l$.

$$\therefore \lim_{n\to\infty} S_{n+1} = \frac{4 + 3\lim_{n\to\infty} S_n}{3 + 2\lim_{n\to\infty} S_n}$$

$$\therefore l = \frac{4+3l}{3+2l}$$

$\therefore l^2 = 2$

$\therefore l = \sqrt{2}$.

Since the terms are all positive, l can not be negative.

Limit Theorems :—

Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences that converge to a and b respectively, i.e.

$\lim(a_n) = a$ and $\lim(b_n) = b$.

Then (i) $\lim(a_n \pm b_n) = \lim(a_n) \pm \lim(b_n) = a \pm b$

(ii) $\lim(a_n b_n) = \lim(a_n) \cdot \lim(b_n) = a \cdot b$

(iii) $\lim(c a_n) = c \lim(a_n) = ca$, $K \in \mathbb{R}$

(iv) $\lim\left(\frac{a_n}{b_n}\right) = \frac{\lim(a_n)}{\lim(b_n)} = \frac{a}{b}$, provided, $\lim(b_n) = b \neq 0$ and $\{b_n\} \neq 0 \ \forall \ n \in \mathbb{N}$

Proof:—

(i) Let $\epsilon > 0$ be a pre-assigned quantity, then $\exists$ a natural number $K_1 \ni |a_n - a| < \epsilon \ \forall \ n \geq K_1$;

Similarly, $|b_n - b| < \epsilon \ \forall \ n \geq K_2$;

Now, let $K = \max\{K_1, K_2\}$

then if $n \geq K$, then $n \geq K_1$ and $n \geq K_2$ and also using triangle inequality, we have

$|(a_n + b_n) - (a+b)| \leq |a_n - a| + |b_n - b| < \epsilon + \epsilon = 2\epsilon =$ an arbitrary.

$\therefore$ By definition $\lim(a_n + b_n) = a + b$.

(iii) Let us assume $k \neq 0$. When $k = 0$, the theorem is obvious. for a pre-assigned positive ϵ $\exists$ a natural number K such that

$|ca_n - ca| < \epsilon \ \forall \ n \geq K$

We have $|ca_n - ca| = |c|\,|a_n - a|$

Let $\epsilon > 0$, since $\lim a_n = a$, $\exists$ a natural number $K \ni$

$|a_n - a| < \frac{\epsilon}{|c|} \ \forall \ n \geq K$.

$\therefore$ It follows that $|ca_n - ca| < \epsilon \ \forall \ n \geq K$

Since ϵ is arbitrary, $\lim(ca_n) = ca$.

Note:—

1) If $K \in \mathbb{N}$ and $\lim(a_n) = (a_n)$ is a convergent sequence, then $\lim(a_n^K) = (\lim(a_n))^K$.

2) If $\{a_n\}, \{b_n\}, \{c_n\}$ be three convergent sequence of real numbers that converge to a and b and c respectively, then

(i) $\lim(a_n + b_n + c_n) = a + b + c$

and (ii) $\lim(a_n b_n c_n) = abc$.

The theorem can be generalised to the sum and the product of a finite number of convergent sequences.

Remark:- It may be noted that while the sum, difference, product, quotient under certain conditions, of two convergent sequences is convergent, the converse may not be true, i.e. if the sequence $\{a_n \pm b_n\}$, $\{a_n b_n\}$ and $\{\frac{a_n}{b_n}\}$ is convergent, then the component sequences $\{a_n\}$ and $\{b_n\}$ may not be convergent, however, it is not difficult to see that both shall behave alike.

For example:- Consider the sequences $\{a_n\}$ and $\{b_n\}$.

(1) $\{a_n\} = \{(-1)^n\}$ and $\{b_n\} = \{(-1)^{n+1}\}$

The sequence $\{a_n+b_n\}$ converges to zero, $\{a_n b_n\}$ converges to -1 and $\{\frac{a_n}{b_n}\}$ converges to -1, but $\{a_n\}$, $\{b_n\}$ are not convergent.

(2) $\{a_n\} = n^2$ and $\{b_n\} = -n^2$

The sequence $\{a_n+b_n\}$ converges to zero and the sequence $\{\frac{a_n}{b_n}\}$ converges to -1, but the two sequences $\{a_n\}$ and $\{b_n\}$ are divergent.

(3) $\{a_n\} = \{b_n\} = (-1)^n$

The sequence $\{a_n - b_n\}$ converges to zero, $\{\frac{a_n}{b_n}\}$ converges to 1 and $\{a_n b_n\}$ converges to 1, while both the sequences $\{a_n\}$ and $\{b_n\}$ oscillate finitely.

(4) Let $\{a_n\} = \frac{1}{n}$ and $\{b_n\} = \frac{1}{n^2}$

Then $\lim(a_n) = 0 = \lim(b_n)$; but $\{\frac{a_n}{b_n}\} = \{n\}$ diverges where as $\{\frac{b_n}{a_n}\} = \{\frac{1}{n}\}$ converges to zero.

Hence when $\lim(a_n) = \lim(b_n) = 0$, then nothing can be said about the sequence $\{\frac{a_n}{b_n}\}$, the sequence may converge or may diverge.

Theorem:- Let $\{a_n\}$ be a convergent sequence of real numbers converging to a. Then the sequence $\{|a_n|\}$ converges to $|a|$.

Proof:- We have $||a_n| - |a|| \le |a_n - a|$

Let $\epsilon > 0$. Since $\lim a_n = a$, $\exists$ a natural number k $\ni$
$|a_n - a| < \epsilon \ \forall\ n \ge k$.

It follows that $||a_n| - |a|| < \epsilon \ \forall\ n \ge k$

Since ϵ is arbitrary, $\lim |a_n| = |a|$.

Note:-

1. The converse of the theorem is not true. i.e. if $\{|a_n|\}$ is a convergent sequence then it does not necessarily imply that $\{a_n\}$ is a convergent sequence.

e.g. let $a_n = (-1)^n$, then the sequence $\{|a_n|\}$ converges to 1 but the sequence $\{a_n\}$ is a divergent sequence.

2. The theorem states that $\lim |a_n| = |\lim a_n|$, provided the limit in RHS exists.

Theorem:- Let $\{a_n\}$ be a convergent sequence of real numbers and $\exists$ a natural number $m \ni a_n > 0 \ \forall \ n \geq m$. Then $\lim a_n \geq 0$.

Proof:- Let $\lim a_n = a$ and if possible let $a < 0$

Let us choose a positive $\epsilon \ni a + \epsilon < 0$

Since $\lim a_n = a$, then $\exists$ a natural number $k_1 \ni$

$$a - \epsilon < a_n < a + \epsilon \ \forall \ n \geq k_1$$

Let $k = \max\{k_1, m\}$

Then by hypothesis, $a_n > 0$, for all $n \geq k$ and we have from above

$$a_n < a + \epsilon < 0 \ \forall \ n \geq k.$$

This is a contradiction. Therefore $\lim a_n \geq 0$.

Note:- 1. The theorem also says that a convergent sequence of positive numbers may converge to 0. For e.g., for the sequence $\{a_n\}$ where $a_n = \frac{1}{n}$, $a_n > 0 \ \forall \ n \in \mathbb{N}$ but $\lim a_n = 0$.

2. If $\{a_n\}$ be a convergent sequence and $a_n \geq 0$ for all $n \geq m$ (m being a natural number) then $\lim a_n \geq 0$.

Theorem:- Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences and $\exists$ a natural number $m \ni a_n > b_n \ \forall \ n \geq m$.

Then $\lim a_n \geq \lim b_n$.

Proof:- Let $\lim a_n = a$, $\lim b_n = b$; $c_n = a_n - b_n$.

Then $\{c_n\}$ is a convergent sequence $\ni c_n > 0 \ \forall \ n \geq m$ and $\lim c_n = a - b$.

By the previous theorem, $a - b \geq 0$

Consequently, we have $\lim a_n \geq \lim b_n$.

Note:- If $\{a_n\}$ is a convergent sequence of points in $[a,b]$, then $c \in [a,b]$ if $\lim a_n = c$.

Example:- (a) If $\lim(a_n) = 0$ then $\lim(a_n^2) = 0$.

(b) If $\lim(a_n) = 0$ and $\{b_n\}$ is bounded (not necessarily convergent) then $\lim(a_n b_n) = 0$.

(c) If $\lim(a_n) = l$ then $\lim(a_n^2) = l^2$.

Solution:- (a) Let $\epsilon > 0$ be an arbitrary quantity and $\exists$ a natural no. $k \ni$

$$|a_n - 0| < \epsilon \ \forall \ n \geq k$$

$$\Rightarrow |a_n| < \epsilon \ \forall \ n \geq k$$

$$\Rightarrow a_n^2 < \epsilon^2 \ \forall \ n \geq k$$

$$\Rightarrow |a_n^2 - 0| < \epsilon^2 \ \forall \ n \geq k$$

$\therefore$ By definition, $\lim(a_n^2) = 0$.

(b) Let $\epsilon > 0$ be an arbitrary quantity, then $\exists$ a natural no. $K \ni$

$$|a_n - 0| < \epsilon \quad \forall \; n \geq K$$

Again, $\{b_n\}$ is bounded, $\Rightarrow |b_n| < M \;\; \forall \; n \in \mathbb{N}$ for some M.

Now, $|a_n b_n - 0| = |a_n||b_n| < \epsilon M \quad \forall \; n \geq K$

$$\Rightarrow \lim (a_n b_n) = 0.$$

(c) Let $\epsilon > 0$. then $|a_n - l| < \epsilon \;\; \forall \; n \geq K$

Now, $|a_n^2 - l^2| = |a_n - l|\,|a_n + l| \leq |a_n - l|\{|a_n| + |l|\}$

Since, $\{a_n\}$ is convergent, so $\{a_n\}$ is bounded, i.e. $|a_n| < M$ $\forall \; n \in \mathbb{N}$

$\therefore |a_n^2 - l^2| < \epsilon . \{M + |l|\} \quad \forall \; n \geq K$

$\therefore$ By definition, $\lim a_n^2 = l^2$.

<u>Theorem:-</u> Let $\lim(x_n) = l$ and suppose that $x_n \geq 0$. Then the sequence $(\sqrt{x_n})$ of positive square roots converges and $\lim(\sqrt{x_n}) = \sqrt{l}$.

<u>Proof:-</u> <u>Case (i)</u>: $l = 0$, let $\epsilon > 0$ be given, since $x_n \to 0$ $\exists$ a natural number $K \ni$ if $n \geq K$ then,

$$0 \leq x_n = x_n - 0 < \epsilon^2$$

$$\therefore 0 \leq \sqrt{x_n} < \epsilon \quad \forall \; n \geq K$$

Since $\epsilon > 0$ is arbitrary, this implies that $\sqrt{x_n} \to 0$.

<u>Case (ii)</u>: if $l > 0$, then $\sqrt{l} > 0$ and we note that

$$\sqrt{x_n} - \sqrt{l} = \frac{(\sqrt{x_n} - \sqrt{l})(\sqrt{x_n} + \sqrt{l})}{\sqrt{x_n} + \sqrt{l}} = \frac{x_n - l}{\sqrt{x_n} + \sqrt{l}}.$$

Since, $\sqrt{x_n} + \sqrt{l} \geq \sqrt{l} > 0$, it follows that

$$|\sqrt{x_n} - \sqrt{l}| \leq \left(\frac{1}{\sqrt{l}}\right)|x_n - l|$$

The convergence of $\sqrt{x_n} \to \sqrt{l}$ follows from the fact that $x_n \to l$.

(*) <u>Theorem:-</u> If $\{x_n\}$ be a sequence of positive real numbers $\ni$ $\lim\left(\frac{x_{n+1}}{x_n}\right) = l$ exists $(0 \leq l < 1)$, then $\{x_n\}$ converges to zero.

<u>Proof:-</u>

$$\left|\frac{x_{n+1}}{x_n} - l\right| < \epsilon \quad \forall \; n \geq K$$

then, $\dfrac{x_{n+1}}{x_n} < l + \epsilon = l + (r - l) = r.$; (Let, $\epsilon = r - l > 0$)

Therefore, we obtain, for $n \geq K$

$$0 < x_{n+1} < x_n r < x_{n-1} r^2 < \dots < x_K r^{n-K+1}$$

Since $0 < r < 1$, $\therefore \lim r^n = 0$

$$\Rightarrow \lim x_n = 0.$$

* <u>Corollary:-</u> 1. Here if $l < 1$, where $l = \lim\left(\frac{x_{n+1}}{x_n}\right)$, then $\lim x_n = \infty$, i.e. $\{x_n\}$ diverges.

2. If $l = 1$, then no definite conclusion can be made about the nature of the sequence. e.g. $\{x_n\} = \left\{\frac{n+1}{n}\right\}$, then $\lim \frac{x_{n+1}}{x_n} = 1$ and $\lim(x_n) = 1$.

Application of this theorem:– (*)

Examples :- 1. Find $\lim(x_n)$, when $x_n = \frac{n}{2^n}$.

Sol.:- $\frac{x_{n+1}}{x_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2}\left(1+\frac{1}{n}\right)$

$\therefore \lim\left(\frac{x_{n+1}}{x_n}\right) = \frac{1}{2}$.

Since $\frac{1}{2} < 1$, so from the above theorem, $\lim\left(\frac{n}{2^n}\right) = 0$.

2. $\lim\limits_{n\to\infty} \frac{x^n}{n!} = 0$ for all real x → Prove it.

Sol. :- Here $x_n = \frac{x^n}{n!}$

$\frac{x_{n+1}}{x_n} = \frac{1}{n+1} \cdot x$

$\therefore \lim\left(\frac{x_{n+1}}{x_n}\right) = 0 = l$

$\therefore$ $0 \le l < 1$, so $\lim(x_n) = 0$.

3. For any positive real no. p and a, prove that —

$\lim\limits_{n\to\infty} \frac{n^p}{(1+a)^n} = 0$. [TIFR]

Sol. :- $x_n = \frac{n^p}{(1+a)^n}$ $\therefore x_{n+1} = \frac{(n+1)^p}{(1+a)^{n+1}}$

Now, $\frac{x_{n+1}}{x_n} = \left(1+\frac{1}{n}\right)^p \frac{1}{1+a}$

$\left(1+\frac{1}{n}\right) \to 1$ as $n\to\infty$

$\therefore$ Hence $\lim\limits_{n\to\infty}\left(\frac{x_{n+1}}{x_n}\right) = \frac{1}{1+a} < 1$.

$\therefore \lim\limits_{n\to\infty} \frac{n^p}{(1+a)^n} = 0$.

4. Show that → $\lim \frac{m(m-1)(m-2)\cdots\cdots(m-n+1)}{n!} x^n = 0$, $|x| < 1$.

Sol. :- Let $a_n = \frac{m(m-1)(m-2)\cdots\cdots(m-n+1)}{n!} \cdot x^n$

$\therefore \lim\left(\frac{a_{n+1}}{a_n}\right) = \lim\left(\frac{m-n}{n+1}\right)x = \lim \frac{\frac{m}{n}-1}{1+\frac{1}{n}} \cdot x = -x$

But $|-x| = |x| < 1$,

So, $\lim a_n = 0$.

5. Show that $\lim(nr^n) = 0$ if $|r| < 1$.

Sol. $a_n = nr^n$

$\lim\left(\frac{a_{n+1}}{a_n}\right) = \lim\left(\frac{n+1}{n} \cdot r\right) = \lim\left\{r\left(1+\frac{1}{n}\right)\right\} = \lim(r)$ $[\because |r| < 1]$

< 1.

$\therefore \lim(nr^n) = 0$.

WORKED EXAMPLES (Based on Limit theorems)

1. Prove that $\lim_{n\to\infty} \frac{3n^2+2n+1}{n^2+1} = 3.$

Solution:- $\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{\lim (u_n)}{\lim (v_n)} = \frac{3}{1} = 3.$

where $u_n = 3+\frac{2}{n}+\frac{1}{n^2}$ and $v_n = 1+\frac{1}{n^2}$.

2. Prove that $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0$.

Solution:- $\lim_{n\to\infty} (\sqrt{n+1}-\sqrt{n}) = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$

$= \lim_{n\to\infty} u_n v_n$; where $u_n = \frac{1}{\sqrt{n}}$, $v_n = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$

$= 0$, since $\lim u_n = 0$ and $\lim v_n = \frac{1}{2}$.

3. Prove that $\lim \left(\frac{2n+1}{n}\right) = 2.$

Solution:- $\lim \left(\frac{2n+1}{n}\right) = \lim \left(2+\frac{1}{n}\right) = \lim 2 + \lim \frac{1}{n} = 2.$

4. Prove that $\lim \left(\frac{2n+1}{n+5}\right) = 2.$

solution:-

$$\frac{2n+1}{n+5} = \frac{2+1/n}{1+5/n} = \frac{u_n}{v_n}$$

$\lim u_n = 2$, $\lim v_n = 1$

$\therefore \lim \left(\frac{u_n}{v_n}\right) = 2.$

5. $\lim \left(\frac{2n}{n^2+1}\right) = 0 \rightarrow$ Prove it.

Solution:-

$$\frac{2n}{n^2+1} = \frac{2/n}{1+1/n^2} = \frac{u_n}{v_n}$$

$\lim u_n = 0$

$\lim v_n = 1$

Therefore $\lim \left(\frac{u_n}{v_n}\right) = 0.$

6. Show that $\lim \frac{(3n+1)(n-2)}{n(n+3)} = 3.$

Solution:- Now, $\lim \frac{(3n+1)(n-2)}{n(n+3)}$

$$= \lim \frac{\left(3+\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\left(1+\frac{3}{n}\right)}$$

Now, we know that $\lim \frac{1}{n} = 0.$

$$\text{Ans.} = \frac{\lim \left(3+\frac{1}{n}\right) \lim \left(1-\frac{2}{n}\right)}{\lim \left(1+\frac{3}{n}\right)} = 3.$$

Sandwich Theorem / Squeeze Theorem :—

Statement:— If $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are three sequences of real numbers such that $a_n \le b_n \le c_n$ for $n \geq k$ and $\lim(a_n) = l = \lim(c_n)$, then $\{b_n\}$ is convergent and $\lim(b_n) = l$.

Proof:— Let $\epsilon > 0$ be given.
Now, since $\{a_n\}$ and $\{c_n\}$ both converge to l, therefore $\exists$ positive integers k_1, k_2 $\ni$

$$|a_n - l| < \epsilon \ \forall \ n \geq k_1$$

and

$$|c_n - l| < \epsilon \ \forall \ n \geq k_2$$

Let $k = \max\{k_1, k_2\}$

Then for $n \geq k$, we get $l - \epsilon < a \le b_n \le c_2 < l + \epsilon$ as $a_n \le b_n \le c_n$.

$$\Rightarrow l - \epsilon < b_n < l + \epsilon \quad \forall \ n \geq k$$

$$\Rightarrow |b_n - l| < \epsilon \quad \forall \ n \geq k$$

Hence, $\lim(b_n) = l$.

WORKED EXAMPLES :— (Based on Sandwich theorem)

1) Prove that $\lim\limits_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0$.

Solution:— Let, $b_n = \sqrt{n+1} - \sqrt{n} = \dfrac{1}{\sqrt{n+1} + \sqrt{n}} < \dfrac{1}{\sqrt{n}} + \dfrac{1}{\sqrt{n}}$

Then $a_n = \dfrac{2}{\sqrt{n+1}} < b_n = \dfrac{1}{\sqrt{n+1} + \sqrt{n}} < c_n = \dfrac{2}{\sqrt{n}} \quad \forall n \in \mathbb{N}$

Note that, $\lim(a_n) = 0 = \lim(c_n)$

$\therefore$ By squeeze theorem, $\lim(b_n) = 0$.

2) Prove that $\lim\limits_{n\to\infty} \left\{ \dfrac{1}{(n+1)^2} + \dfrac{1}{(n+2)^2} + \cdots + \dfrac{1}{(n+n)^2} \right\} = 0$. [TIFR]

Solution:— Let, $b_n = \dfrac{1}{(n+1)^2} + \dfrac{1}{(n+2)^2} + \cdots + \dfrac{1}{(n+n)^2} = \sum\limits_{r=1}^{n} \dfrac{1}{(n+r)^2}$

Now, $n \cdot \dfrac{1}{4n^2} < b_n < \dfrac{1}{n^2} \quad \forall \ n \geq 1$

$$\Rightarrow a_n = \frac{1}{4n} < b_n < \frac{1}{n^2} = c_n \quad \forall \ n \geq 1$$

Note that, $\lim(a_n) = 0 = \lim(c_n)$

$\therefore$ By squeeze theorem, $\lim(b_n) = 0$.

3) Prove that $\lim \left(\dfrac{\sin n}{n}\right) = 0$.

Solution:— Let, $b_n = \dfrac{1}{n} \sin n$

Now, we know $-1 \le \sin n \le 1 \quad \forall \ n \in \mathbb{N}$

$$\therefore -\frac{1}{n} \le \frac{1}{n} \sin n \le \frac{1}{n}$$

$$\Rightarrow a_n = -\frac{1}{n} \le b_n \le c_n = \frac{1}{n}$$

Note that, $\lim(a_n) = 0 = \lim(c_n)$

$\therefore$ By squeeze theorem, $\lim(b_n) = 0$.

4) Evaluate : $\lim_{n\to\infty} (2^n+3^n)^{1/n} = ?$

Solution:- Note that $(3^n)^{1/n} < b_n = (2^n+3^n)^{1/n} < (2.3^n)^{1/n}$

$\Rightarrow a_n = 3 < b_n < 3.2^{1/n} = c_n$

Clearly, $\lim(a_n) = \lim(c_n) = 3$.

$\therefore$ By Squeeze theorem, $\lim(b_n) = 3$.

$\Rightarrow \lim_{n\to\infty} (2^n+3^n)^{1/n} = 3$.

5) Show that the sequence $\{x_n\}$, where $x_n = \left\{\frac{1}{\sqrt{n^2+1}} + \cdots\cdots + \frac{1}{\sqrt{n^2+n}}\right\}$, converges to 1.

Solution:- $n^2+r \le n^2+n \;\; \forall\, r = 1(1)n \;\; \forall$ large n.

Aso, $n^2+1 < n^2+r \;\; \forall\, r = 2(1)n$.

$\therefore \frac{n}{\sqrt{n^2+n}} \le x_n \le \frac{n}{\sqrt{n^2+1}}$

Now, $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} = 1$.

Also, $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1$.

$\therefore \lim_{n\to\infty} x_n = 1$, by squeeze theorem.

6) Show that $\lim_{n\to\infty} (a^{1/n}) = 1$.

Solution:-

Case-I :- $a > 1$

Then $(a^{1/n} - 1) > 0 \;\; \forall\, n \in \mathbb{N}$

Define a new sequence $x_n = (a^{1/n} - 1) > 0 \;\; \forall\, n \in \mathbb{N} \;\therefore\; a = (1+x_n)^n$

From Bernoulli inequality : $(1+x_n)^n \ge 1+nx_n$

[for $x > 0$, $(1+x)^n \ge 1+nx \;\; \forall\, n \ge 1$]

Now, $0 < x_n \le \frac{a-1}{n}$

$\Rightarrow a_n = 0 < x_n \le \frac{a-1}{n} = b_n \;\; \forall\, n \ge 1$

Now, $\lim(a_n) = \lim(b_n) = 0$

$\therefore$ By squeeze theorem, $\lim_{n\to\infty} (x_n) = 0$

Case-II :- $a = 1$

Then the sequence $\{a^{1/n}\}$ becomes a constant sequence :

$\{1, 1, 1, \ldots\ldots\}$

$\therefore \lim_{n\to\infty} (a^{1/n}) = 1$.

Case-III :- $0 < a < 1$,

Let $b = \frac{1}{a} > 1$

Then $\lim_{n\to\infty} (a^{1/n}) = \lim_{n\to\infty} \left(\frac{1}{b}\right)^{1/n} = \lim_{n\to\infty} \left(\frac{1}{b^{1/n}}\right) = \frac{1}{\lim_{n\to\infty}(b^{1/n})} = \frac{1}{1}$

[Since $b > 1$, $\therefore \lim(b^{1/n}) = 1$ by Case-I]

Hence, $\lim(a^{1/n}) = 1 \;\; \forall\, a > 0$.

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7) Show that $\lim_{n\to\infty} (n^{1/n}) = 1$.

Solution:— Define, a new sequence : $x_n = n^{1/n} - 1 \geq 0 \;\forall n \geq 1$.

Now, $n = (1+x_n)^n \geq \frac{n(n-1)}{2!} x_n^2 \;\forall n \geq 2$

$\Rightarrow 0 \leq x_n \leq \sqrt{\frac{2}{n-1}} \;\forall n \geq 2$

We have, $a_n = 0 \leq x_n \leq \sqrt{\frac{2}{n-1}} = c_n \;\forall n \geq 2$

Note that, $\lim(a_n) = \lim(c_n) = 0$

$\therefore$ By squeeze theorem, $\lim(x_n) = 0$.

$\therefore \lim_{n\to\infty} (n^{1/n}) = 1$

8) Show that $\lim_{n\to\infty} (n^{1/n^2}) = 1$.

Solution:— Define, $x_n = n^{1/n^2} - 1 \geq 0 \;\forall n \in \mathbb{N}$

$\therefore n = (1+x_n)^{n^2} \geq 1 + n^2 . x_n$

$\Rightarrow 0 \leq x_n \leq \frac{n-1}{n^2} \leq \frac{1}{n} \;\forall n \in \mathbb{N}$

By Squeeze theorem:— $\lim(x_n) = 0 \Rightarrow \lim_{n\to\infty} (n^{1/n^2}) = 1$.

9) Show that $\lim (an)^{1/n} = 1$; $a > 0$

Solution:— Case-I:— $a \geq 1$.

Let $x_n = (an)^{1/n} - 1 \geq 0 \;\forall n \in \mathbb{N}$

$an = (1+x_n)^n \geq \frac{n(n-1)}{2!} x_n^2 \;\forall n \geq 2$

$\Rightarrow 0 \leq x_n \leq \sqrt{\frac{2n}{n-1}}$

$\therefore$ $\text{lt}(x_n) = 0$, by squeeze theorem.

Case-II:— $0 < a < 1$.

Let $x_n = (an)^{1/n} - 1 < n^{1/n} - 1 \leq \sqrt{\frac{2}{n-1}}$, $n \geq 2$

Then by squeeze theorem, $\lim(x_n) = 0$.

[Alternative:—

$\lim_{n\to\infty} (an)^{1/n} = \left(\lim_{n\to\infty} a^{1/n}\right)\left(\lim_{n\to\infty} n^{1/n}\right) = 1 . 1 = 1.$]

10) Show that $\lim(na^n) = 0$, $0 < a < 1$.

Solution:— $0 < a < 1$

$\Rightarrow a = \frac{1}{1+b}$, $b > 0$

$(na^n) = \frac{n}{(1+b)^n} \leq \frac{n}{\frac{n(n-1)b^2}{2!}} \;\forall b > 0$

$\therefore 0 \leq na^n < \frac{2}{(n-1)b^2} \;\forall n \geq 2$

$\therefore$ By Squeeze theorem, $\lim_{n\to\infty} (na^n) = 0$

[Alternative:— Let $\epsilon > 0$ be a pre-assigned number.

$|na^n - 0| = na^n \leq \frac{2}{(n-1)b^2} < \epsilon$

$\Rightarrow n-1 > \frac{2}{b^2\epsilon}$

$\Rightarrow n > \frac{2}{b^2\epsilon} + 1$

$\Rightarrow K = \left[\frac{2}{b^2\epsilon} + 1\right] + 1$

$\therefore \lim_{n\to\infty} (na^n) = 0$]

FURTHER EXERCISES:-

1. If $\lim x_n = 0$ and $a > 0$, then $\lim a^{x_n} = 1$.

Solution:- We have $\lim a^{1/n} = 1$ and $\lim a^{-1/n} = 1$.

Let us choose $\epsilon > 0$, $\exists$ natural numbers k_1, k_2 $\ni$

$1-\epsilon < a^{1/n} < 1+\epsilon \quad \forall n \geq k_1$

and $1-\epsilon < a^{-1/n} < 1+\epsilon \quad \forall n \geq k_2$

Let $k = \max\{k_1, k_2\}$

Then $1-\epsilon < a^{1/k} < 1+\epsilon$ and $1-\epsilon < a^{-1/k} < 1+\epsilon$

Since $\lim x_n = 0$, $\exists$ a natural number p $\ni$

$-\frac{1}{k} < x_n < \frac{1}{k}$ for all $n \geq p$.

Let $a > 1$, Then $a^{-1/k} < a^{x_n} < a^{1/k} \quad \forall n \geq p$

or, $1-\epsilon < a^{-1/k} < a^{x_n} < a^{1/k} < 1+\epsilon \quad \forall n \geq p$

Let $0 < a < 1$, Then $a^{+1/k} < a^{x_n} < a^{-1/k} \quad \forall n \geq p$

or, $1-\epsilon < a^{1/k} < a^{x_n} < a^{-1/k} < 1+\epsilon \quad \forall n \geq p$

Therefore if $a > 0$, $1-\epsilon < a^{x_n} < 1+\epsilon \quad \forall n \geq p$.

This implies $\lim a^{x_n} = 1$.

Cor. 1. If $\lim x_n = l$ and $a > 0$, then $\lim a^{x_n} = a^l$.

Cor. 2. If $\lim x_n = l$, then $\lim e^{x_n} = e^l$.

2. If $\lim x_n = 0$, then show that $\lim \log(1+x_n) = 0$.

Solution:- Let $\epsilon > 0$, $-\epsilon < \log(1+x_n) < \epsilon$ will hold if $e^{-\epsilon} - 1 < x_n < e^{\epsilon} - 1$.

Since $\epsilon > 0$, $e^{+\epsilon} - 1 > 0$ and $e^{-\epsilon} - 1 < 0$.

Consequently, $e^{-\epsilon} - 1 < x_n < e^{\epsilon} - 1 \quad \forall n \geq k$

or, $-\epsilon < \log(1+x_n) < \epsilon \quad \forall n \geq k$

This shows $\lim \log(1+x_n) = 0$.

3. If $u_n > 0$ and $\lim u_n = u > 0 \ \forall n \in \mathbb{N}$ and $\lim v_n = v$, then $\lim (u_n)^{v_n} = u^v$.

Solution:- By definition, $(u_n)^{v_n} = e^{v_n \log u_n}$

As $\log u_n = u$, $\lim \log u_n = \log u$. So, $\lim (v_n \log u_n) = v \log u$.

By cor. 2. of (1), $\lim e^{v_n \log u_n} = e^{v \log u} = u^v$.

or, $\lim (u_n)^{v_n} = u^v$.

Theorem:- Let $\{u_n\}$ be a sequence of positive real numbers $\ni$ $\lim \sqrt[n]{u_n} = l$.

(i) If $0 \leq l < 1$ then $\lim u_n = 0$.

(ii) If $l > 1$ then $\lim u_n = \infty$.

Solution:- (i) Let us choose a positive ϵ such that $l+\epsilon < 1$

Given $\lim \sqrt[n]{u_n} = l$ $\therefore$ $l-\epsilon < \sqrt[n]{u_n} < l+\epsilon \quad \forall n \geq k$

Let $l+\epsilon = r$ then $0 < r < 1$ and $\sqrt[n]{u_n} < r \quad \forall n \geq k$.

So, we have $0 < u_n < r^n \quad \forall n \geq k$.

Since $\lim r^n = 0$; $\therefore$ By Sandwich theorem, $\lim u_n = 0$.

(ii) Let $l-\epsilon = r'$ then $r' > 1$, and $\sqrt[n]{u_n} > r' \quad \forall n \geq m$

$\therefore$ $u_n > r'^n \quad \forall n \geq m$

Since $r' > 1$, $\lim (r')^n = \infty$, therefore $\lim u_n = \infty$.

Note:- If $\lim \sqrt[n]{u_n} = 1$, no definite conclusion can be made about the nature of the sequence $\{u_n\}$.

WORKED EXAMPLES:-

1. A sequence $\{u_n\}$ is defined by $u_{n+2} = \frac{1}{2}(u_{n+1}+u_n)$ for $n \geq 1$ and $0 < u_1 < u_2$. Prove that the sequence $\{u_n\}$ converges to $\frac{u_1+2u_2}{3}$.

Solution:- $u_2 - u_1 > 0$

$$u_3 - u_2 = \frac{1}{2}(u_2+u_1) - u_2 = -\frac{1}{2}(u_2-u_1)$$
$$u_4 - u_3 = \frac{1}{2}(u_3+u_2) - u_3 = \frac{1}{2}(u_2-u_3) = \left(-\frac{1}{2}\right)^2(u_2-u_1)$$
$$\vdots$$
$$u_n - u_{n-1} = \left(-\frac{1}{2}\right)^{n-2}(u_2-u_1)$$

Therefore, $u_n - u_1 = (u_2-u_1)\left[1+\left(-\frac{1}{2}\right)+\left(-\frac{1}{2}\right)^2+\cdots+\left(-\frac{1}{2}\right)^{n-2}\right]$

$$= \frac{2(u_2-u_1)}{3}\left[1-\left(-\frac{1}{2}\right)^{n-1}\right]$$

Now, $\lim(u_n-u_1) = \frac{2}{3}(u_2-u_1)$; since $\lim\left(-\frac{1}{2}\right)^{n-1} = 0$.

Therefore, $\lim u_n = u_1 + \frac{2}{3}(u_2-u_1) = \frac{u_1+2u_2}{3}$.

2. If $x_n = (a^n+b^n)^{1/n}$ for all $n \in \mathbb{N}$ and $0 < a < b$, show that $\lim x_n = b$.

Solution:- $x_n = b\left[\left(\frac{a}{b}\right)^n+1\right]^{1/n} > b \;\; \forall\, n \in \mathbb{N}$, since $\left(\frac{a}{b}\right)^n+1 > 1 \;\forall\, n \in \mathbb{N}$

Again, $0 < a < b \Rightarrow a^n < b^n \;\; \forall\, n \in \mathbb{N}$.

Therefore, $a^n + b^n < 2b^n$

or, $x_n < 2^{1/n}.b \;\; \forall\, n \in \mathbb{N}$.

Let $u_n = b \;\forall\, n \in \mathbb{N}$, $v_n = 2^{1/n}b \;\; \forall\, n \in \mathbb{N}$

Then $\lim u_n = b$ and $\lim v_n = b$, since $\lim 2^{1/n} = 1$.

Now, $u_n < x_n < v_n \;\; \forall\, n \in \mathbb{N}$.

Since $\lim u_n = \lim v_n = b$, $\lim x_n = b$, by Sandwich theorem.

3. A sequence $\{u_n\}$ is defined by $u_1 = \sqrt{2}$ and $u_{n+1} = \sqrt{2u_n}$ for $n \geq 1$. Prove that $\lim u_n = 2$.

Solution:- $u_1 = 2^{1/2}$, $u_2 = \sqrt{2\sqrt{2}} = 2^{1/2+1/2^2} = 2^{1-\frac{1}{2^2}}$,

$$u_3 = 2^{1/2+1/2^2+1/2^3} = 2^{1-1/2^3},$$
$$\vdots$$
$$u_n = 2^{1/2+1/2^2+\cdots+1/2^n} = 2^{1-1/2^n},$$

$\therefore \lim u_n = \lim 2^{1-1/2^n} \Rightarrow \lim u_n = \lim 2^{x_n}$ where $x_n = 1-\frac{1}{2^n}$

As, $\lim x_n = 1$, we have $\lim u_n = \lim 2^{x_n} = 2$, since, $\lim x_n = l$ and $a > 0 \Rightarrow \lim a^{x_n} = a^l$.

4. If $u_n > 0 \ \forall n$ and $\lim \sqrt[n]{u_n} = \mu > 0$, then prove that
$$\lim \sqrt[n]{(n+1)u_{n+1}} = \mu.$$

Solution:- $\lim \sqrt[n]{n+1} = \lim \left\{ (n+1)^{\frac{1}{n+1}} \right\}^{\frac{n+1}{n}}$

Since $\lim n^{1/n} = 1$, it follows that $\lim (n+1)^{\frac{1}{n+1}} = 1$.

Since $\lim \frac{n+1}{n} = 1$ and $\lim (n+1)^{\frac{1}{n+1}} = 1$, we have $\lim \sqrt[n]{n+1} = 1$.

$\lim \sqrt[n]{u_{n+1}} = \lim \left\{ (u_{n+1})^{\frac{1}{n+1}} \right\}^{\frac{n+1}{n}}$

Since $\lim \sqrt[n]{u_n} = \mu$, it follows that $\lim (u_{n+1})^{\frac{1}{n+1}} = \mu$.

Since $\lim \frac{n+1}{n} = 1$ and $\lim (u_{n+1})^{\frac{1}{n+1}} = \mu > 0$, we have $\lim \sqrt[n]{u_{n+1}} = \mu$.

Therefore, $\lim \sqrt[n]{(n+1)u_{n+1}} = \lim \left(\sqrt[n]{n+1} \cdot \sqrt[n]{u_{n+1}} \right) = \mu$.

Cantor's theorem on nested intervals:—

Theorem:— Let $\{[a_n, b_n]\}$ be a sequence of closed and bounded intervals such that

(i) $[a_{n+1}, b_{n+1}] \subset [a_n, b_n] \ \forall n \in \mathbb{N}$ and

(ii) $\lim \delta_n = 0$ where, $\delta_n = b_n - a_n$ = length of $[a_n, b_n]$.

Then $\bigcap_{n=1}^{\infty} [a_n, b_n]$ contains precisely one point.

Note:- The theorem says that a nested sequence of closed and bounded intervals has a non-empty intersection.

A nested sequence of open and bounded intervals $\{I_n\}$ may not have a non-empty intersection.

For example, let $I_n = \{x \in \mathbb{R} : 0 < x < \frac{1}{n}\}$ for all $n \in \mathbb{N}$. Then $\{I_n\}$ is a nested sequence of open bounded intervals since $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$. Here $\bigcap_{n=1}^{\infty} I_n = \phi$.

A nested sequence of closed and unbounded intervals $\{I_n\}$ may not have a non-empty intersection.

For example, let $I_n = \{x \in \mathbb{R} : x \geq n\}$ for all $n \in \mathbb{N}$. Then $\{I_n\}$ is a nested sequence of closed and unbounded intervals since $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$. Here $\bigcap_{n=1}^{\infty} I_n = \phi$.

Monotonic Sequence :-

Definition:-

(a) A real sequence $\{f(n)\}$ is said to be a monotone increasing sequence if $f(n+1) \geq f(n)$ for all $n \in \mathbb{N}$.

(b) A real sequence $\{f(n)\}$ is said to be a monotone decreasing sequence if $f(n+1) \leq f(n)$ for all $n \in \mathbb{N}$.

A sequence $\{f(n)\}$ is said to a monotone sequence if it is either a monotone increasing sequence or a monotone decreasing sequence.

Note:- If $f(n+1) > f(n)\ \forall n \in \mathbb{N}$, the sequence $\{f(n)\}$ is said to be a strictly monotone increasing sequence.

If $f(n+1) < f(n)\ \forall n \in \mathbb{N}$, the sequence $\{f(n)\}$ is said to be a strictly monotone decreasing sequence.

Theorem:-1. A monotone sequence converges iff it is bounded.

Proof:- Let $\epsilon > 0$

Then $M-\epsilon < f(n) \leq M\ \ \forall n \in \mathbb{N}$

Then $\{f(n)\}$ is a monotone increasing sequence,

$$M-\epsilon < f(k) \leq f(k+1) \leq f(k+2) \leq \cdots \leq M.$$

That is, $M-\epsilon < f(n) < M+\epsilon\ \ \forall n \geq k$

$\therefore \lim\{f(n)\} = M.$

This shows that $\{f(n)\}$ is convergent.

Similar proof can be provided for monotone decreasing sequence.

Cor.1. A monotone increasing sequence, if bounded above, is convergent and it converges to the least upper bound.

Cor.2. A monotone decreasing sequence, if bounded below, is convergent and it converges to the greatest lower bound.

Theorem 2. A monotone increasing sequence that is unbounded above diverges to ∞.

Proof:- For a pre-assigned positive number G, $\exists$ a natural no. $k \ni$
$f(k) > G$.

Since $\{f(n)\}$ is monotone increasing,

$G < f(k) \leq f(k+1) \leq f(k+2) \leq \cdots$

i.e. $f(n) > G\ \forall n \geq k$.

This shows $\lim(f(n)) = \infty$ and $\{f(n)\}$ is divergent.

Cor.1. If a monotone sequence is unbounded then it is properly divergent.

Note:- A monotone sequence has a definite behaviour. It is either convergent, or properly divergent. The limit of the sequence, however, can be determined if the l.u.b. of the increasing sequence (or the g.l.b. of the decreasing sequence) can be evaluated.

WORKED EXAMPLES (Based on Monotone Sequences)

1. Give an example of a sequence which is not monotone.

Sol. The sequence $\{(-2)^n\}$ is neither a monotone increasing sequence, nor a monotone decreasing sequence. Therefore it is not a monotone sequence.

2. Give an example of monotone increasing sequence.

Sol. Let $f(n) = 2^n, n \geq 1$

Then $f(n+1) = 2^{n+1} = 2 \cdot 2^n$

Thus $f(n+1) > f(n) \ \forall n \in \mathbb{N}$

So, $\{f(n)\}$ is strictly increasing monotone sequence.

OR

$f(n) = n^2, n \geq 1$

$f(n+1) > f(n)$

$\therefore$ the sequence is strictly increasing.

3. Give an example of monotone decreasing sequence.

Sol. Let $f(n) = \frac{1}{\sqrt{n}}, n \geq 1$

Then $f(n+1) < f(n) \ \forall n \in \mathbb{N}$

$\therefore \{f(n)\}$ is monotone decreasing sequence and it converges to zero.

i.e. $\lim (1/\sqrt{n}) = 0$.

4. For the following sequences examine whether or not these are bounded and whether or not they are monotone sequences and establish the convergence or divergence of the sequences.

(a) $f(n) = \frac{1}{n}, n \in \mathbb{N}$

(b) $f(n) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}; n \geq 1$

(c) $f(1) = 1, f(n) = \sqrt{2f(n-1)}; n \geq 2$

(d) $f(n) = \left(1 + \frac{1}{n}\right)^n; n \in \mathbb{N}$

(e) $f(n) = \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}, \forall n \in \mathbb{N}$

(f) $f(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n}, \forall n \in \mathbb{N}$

(g) $f(1) = \sqrt{2}, f(n+1) = \sqrt{2f(n)} \ \forall n \geq 1$

(h) $f(1) = 1, f(n+1) = \sqrt{3f(n)}$

Solution:-

(a) $f(n+1) - f(n) = \frac{1}{n+1} - \frac{1}{n} < 0 \ \forall n \in \mathbb{N}$

$\therefore \{f(n)\}$ is strictly monotone decreasing sequence.

Clearly, $0 < \frac{1}{n} \leq 1 \ \forall n \in \mathbb{N}$

i.e. $\{f(n)\}$ is bounded.

Hence, $\{f(n)\} = \{\frac{1}{n}\}$ converges.

(b) $f(n+1) - f(n) = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{2(n+1)(2n+1)} > 0 \ \forall n$

$\therefore \{f(n)\}$ is a monotone increasing sequence.

Again $f(n) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$

$< \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} = 1$.

$\therefore 0 < f(n) < 1$.

$\therefore$ the sequence is bounded as well as convergent.

(c) Here $f(1)=1$, $f(2)=\sqrt{2a_1}=\sqrt{2}$

Note that $f(2)>f(1)$

By induction $f(n+1)>f(n)$

so, $\{f(n)\}$ is increasing.

Note that, $f(1)<2$, $f(2)<2$;

$\therefore$ Let us assume that $f(k)<2$.

Now $f(k+1)=\sqrt{2f(k)}<\sqrt{2.2}=2$

$\therefore\ 1\leq f(n)<2\ \forall n\in\mathbb{N}$

$\therefore\ \{f(n)\}$ is bounded and $\{f(n)\}$ converges.

(d)

$$f(n+1)=\left(1+\frac{1}{n+1}\right)^{n+1}$$

$$f(n)=\left(1+\frac{1}{n}\right)^{n}$$

Let us consider $(n+1)$ positive numbers $1+\frac{1}{n}, 1+\frac{1}{n}, \ldots\ldots, 1+\frac{1}{n}$ (n-times) and 1.

Applying AM > GM, we have $\dfrac{n\left(1+\frac{1}{n}\right)+1}{n+1}>\left(1+\frac{1}{n}\right)^{\frac{n}{n+1}}$

or, $\left(1+\frac{1}{n+1}\right)^{n+1}>\left(1+\frac{1}{n}\right)^{n}$

i.e. $f(n+1)>f(n)\ \forall n\in\mathbb{N}$

This shows that $\{f(n)\}$ is monotone increasing sequence.

$$\text{Now, } f(n)=1+1+\frac{n(n-1)}{2!}\cdot\frac{1}{n^2}+\cdots+\frac{1}{n^n}$$

$$=1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

$$2\leq f(n)<1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$$

$$<1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}\quad(\because n!>2^{n-1})$$

$$<3$$

$\therefore\ 2\leq f(n)<3\ \forall n\in\mathbb{N}$

$\therefore\ \{f(n)\}$ is bounded and hence convergent.

Note that, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$, where $2<e<3$.

(e) Now, $f(n+1)-f(n)=\frac{1}{(n+1)!}>0,\forall n$.

$\therefore$ The sequence $\{f(n)\}$ is monotone increasing.

Again, $f(n)=\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}$

$$<1+\frac{1}{2}+\cdots+\frac{1}{2^{n-1}}=\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}=2-\frac{1}{2^{n-1}}<2$$

$\therefore 0<f(n)<2$

$\Rightarrow$ The sequence is bounded.

$\therefore\ \{f(n)\}$ is convergent.

(f) $f(n+1)-f(n)=\frac{1}{n+1}>0$

$\Rightarrow \{f(n)\}$ is monotone increasing sequence.

Since $\{f(n)\}$ is increasing, for $n>2^m$ we get,

$f(n)>f(2^m)=1+\frac{1}{2}+\cdots+\frac{1}{2^m}$

$\therefore f(n)>1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\cdots+\left(\frac{1}{2^{m-1}+1}+\cdots+\frac{1}{2^m}\right)$

$>1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\cdots+\left(\frac{1}{2^m}+\cdots+\frac{1}{2^m}\right)$

$=1+\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}=1+\frac{m}{2}$.

So, $\{f(n)\}$ is unbounded above.

So, $\{f(n)\}$ is divergent.

(g) The sequence is $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \cdots\}$

$\{f(n+1)\}^2-\{f(n)\}^2=2(f(n)-f(n-1))$

$\Rightarrow \{f(n+1)+f(n)\}\{f(n+1)-f(n)\}=2\{f(n)-f(n-1)\}$

Since $f(n)>0\ \forall n$

$\therefore f(n+1)>$ or $<f(n)$ according as $f(n)>$ or $<f(n-1)$.

But $f(2)>f(1)$ $\therefore$ Consequently, $f(3)>f(2)$, $f(4)>f(3)$ and so on.

Therefore $\{f(n)\}$ is a monotonic increasing sequence.

Again $2f(n)=\{f(n+1)\}^2>\{f(n)\}^2\ \forall n\in\mathbb{N}$

i.e. $(f(n))^2-2f(n)<0\ \forall n\in\mathbb{N}$

or, $f(n)\{f(n)-2\}<0\ \forall n\in\mathbb{N}$

But $f(n)>0$, therefore $f(n)<2\ \forall n\in\mathbb{N}$

This shows $\{f(n)\}$ is bounded above and therefore it is convergent

Let $\lim f(n)=l$,

By definition, $\{f(n+1)\}^2=2f(n)\ \forall n\in\mathbb{N}$

Taking lim as $n\to\infty$, we have $l^2=2l\Rightarrow l=0$ or 2,

As the seq. is monotone increasing so l can't be negative or 0.

Therefore the sequence converges to 2.

(h) The terms of the sequence $\{f(n)\}$ are $1, \sqrt{3}, \sqrt{3\sqrt{3}}, \sqrt{3\sqrt{3\sqrt{3}}}, \cdots$

Clearly, $f(2)>f(1)$

$f(3)>f(2)$

Also, $f(m+1)>f(m)\Rightarrow\sqrt{3f(m+1)}>\sqrt{3f(m)}\Rightarrow f(m+2)>f(m+1)$.

By induction, $\{f(n)\}$ is monotonic increasing.

Again, $f(1)<3$, $f(2)<3$, $f(3)=\sqrt{3f(2)}<3, \cdots, f(m)<3, f(m+1)<3$

By Mathematical Induction, $0<f(n)<3, \forall n$.

Let $\lim f(n)=l$ Since $\lim f(n+1)=\lim\sqrt{3f(n)}\Rightarrow l=\sqrt{3}\Rightarrow l=0$ or 3

But $l\neq 0$, since $f(n)\geq 1\ \forall n$. Hence $\lim f(n)=3$

5. Prove that the sequence $\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}, \ldots\ldots$ converges to the positive root of the equation $x^2-x-7=0$.

Solution:- Let $\{u_n\}_n$ be the sequence when for all n,

$$u_n = \sqrt{7+u_{n-1}} \text{ and } u_1 = \sqrt{7}.$$

Now, $u_{n+1}^2 = 7+u_n$ and $u_n^2 = 7+u_{n-1}$

$$\therefore u_{n+1}^2 - u_n^2 = u_n - u_{n-1}$$

Each u_n being positive $u_{n+1} \gtreqless u_n$ according as $u_n \gtreqless u_{n-1}$.

But $u_2 = \sqrt{7+u_1}$ $u_2 > u_1$, $u_3 > u_2$, $u_4 > u_3, \ldots\ldots$, so on.

$\therefore \{u_n\}_n$ is monotone increasing.

Again, $u_n^2 < u_{n+1}^2 = 7+u_n$

$$\therefore u_n^2 - u_n - 7 < 0 \quad \ldots\ldots\ldots (1)$$

Let us consider the equation $x^2-x-7=0$.
Product of two roots α and β of the equation is negative. Therefore, one root is negative. Let α be the negative root and the other root is positive.

Now, $\alpha+\beta = 1$, $\alpha\beta = -7$.

From (1), we can write $(u_n-\alpha)(u_n-\beta) < 0$

$$\therefore \alpha < u_n < \beta \;\forall\, n \in \mathbb{N}.$$

$\therefore \{u_n\}_n$ is bounded.

This proves that the sequence $\{u_n\}$ is bounded above and therefore the sequence $\{u_n\}$ is convergent.

Let $\lim u_n = l$.

By definition, $u_{n+1}^2 = 7+u_n \;\forall\, n \in \mathbb{N}$

Taking limit as $n \to \infty$, we have $l^2 = 7+l$

Therefore, $(l-\alpha)(l-\beta) = 0$

But $l \neq \alpha$, since each element of the sequence is positive and $\alpha < 0$.
Therefore $l = \beta$. That is, the sequence converges to the positive root of the equation $x^2-x-7=0$.

$$\therefore x = \frac{1+\sqrt{1+4.7}}{2} = \frac{1+\sqrt{29}}{2}.$$

Note:- The sequence $\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}, \ldots\ldots$ converges to

$$l = \frac{1+\sqrt{29}}{2}. \quad [\text{TIFR'2010}]$$

Generalised Form:- If k is positive and $\alpha, -\beta$ are the positive and negative roots of $x^2-x-k=0$, then $u_n = \sqrt{k+u_{n-1}}$, where $u_1 > 0$, $u_n \to \alpha$.

Alternatively, if $a_{n+1} = \sqrt{k+a_n}$, then $\{a_n\}$ converges to the positive root of the equation $x^2-x-k=0$.

6. Let $u_n = 1+\frac{1}{2}+\cdots+\frac{1}{n} - \log n$; $v_n = 1+\frac{1}{2}+\cdots+\frac{1}{n-1} - \log n$, $n \geq 2$. Show that $\{u_n\}$ is a monotone decreasing sequence and $\{v_n\}_{n=2}^{\infty}$ is a monotone increasing one and they converge to the same limit.

<u>Solution:-</u> $u_{n+1} - u_n = \frac{1}{n+1} - \log(n+1) + \log n = \frac{1}{n+1} - \log\left(1+\frac{1}{n}\right)$

$v_{n+1} - v_n = \frac{1}{n} - \log(n+1) + \log n = \frac{1}{n} - \log\left(1+\frac{1}{n}\right)$.

As the sequence $\left\{\left(1+\frac{1}{n}\right)^{n+1}\right\}$ is a strictly monotone decreasing sequence converging to e, $\left(1+\frac{1}{n}\right)^{n+1} > e \;\forall n \in \mathbb{N}$.

Therefore, $\log\left(1+\frac{1}{n}\right) > \frac{1}{n+1} \;\forall n \in \mathbb{N}$.

or, $u_{n+1} < u_n \;\forall n \in \mathbb{N}$.

This shows that the sequence $\{u_n\}$ is a strictly monotone decreasing sequence.

As the sequence $\left\{\left(1+\frac{1}{n}\right)^{n}\right\}$ is a strictly monotone increasing sequence converging to e, $\left(1+\frac{1}{n}\right)^{n} < e \;\forall n \in \mathbb{N}$.

Therefore, $\log\left(1+\frac{1}{n}\right) < \frac{1}{n} \;\forall n \in \mathbb{N}$

or, $v_{n+1} > v_n \;\forall n \geq 2$.

Therefore the sequence $\{v_n\}$ is a strictly monotone increasing sequence.

Again $\frac{1}{n} > \log\frac{n+1}{n} = \log(n+1) - \log n$.

Therefore, $1 > \log 2 - \log 1$, $\frac{1}{2} > \log 3 - \log 2$, $\ldots$, $\frac{1}{n} > \log(n+1) - \log n$.

So, we have $1+\frac{1}{2}+\cdots+\frac{1}{n} > \log(n+1) > \log n$,

Hence $u_n > 0 \;\forall n \in \mathbb{N}$,

Therefore $\{u_n\}$ is a monotone decreasing sequence bounded below.

Hence the sequence $\{u_n\}$ is convergent,

Let, $\lim u_n = \gamma$.

Now, $u_n - v_n = \frac{1}{n} \;\forall n \geq 2$. Therefore, $\lim v_n = \gamma$.

Thus the sequences $\{u_n\}$ and $\{v_n\}$ converge to the same limit γ. (γ is called Euler's constant)

7. Let $\{u_n\}$, $\{v_n\}$ be two real sequences with $\lim u_n = l$, $\lim v_n = m$. If $x_n = \max\{u_n, v_n\}$, $y_n = \min\{u_n, v_n\}$. Prove that the sequence $\{x_n\}$ converges to $\max\{l, m\}$ and the sequence $\{y_n\}$ converges to $\min\{l, m\}$.

<u>Hints:-</u> $\max\{a,b\} = \frac{1}{2}\{a+b+|a-b|\}$,

$\min\{a,b\} = \frac{1}{2}\{a+b-|a-b|\}$, for all $a, b \in \mathbb{R}$.

8. Show that the sequence $\{a_n\}$ defined by
$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{9}{a_n}\right), n \geq 1 \text{ and } a_1 > 0 \text{ converges to } 3.$$

Solution:- Now, $a_2 - a_1 = \frac{1}{2}\left(a_1 + \frac{9}{a_1}\right) - a_1$
$$= \frac{9 - a_1^2}{2a_1} \geq 0, \text{ if } a_1 \leq 3.$$
$$\Rightarrow a_2 \geq a_1, \text{ if } a_1 \leq 3.$$
Also $a_{n+1} - a_n = \frac{1}{2}\left(a_n + \frac{9}{a_n}\right) - a_n = \frac{9 - a_n^2}{2a_n} \geq 0$, if $a_n \leq 3$.

Thus the sequence $\{a_n\}$ is monotonically increasing if $a_n \leq 3\ \forall n$ and decreasing if $a_n \geq 3$, $\forall n$. In either case the sequence is monotonic and bounded and therefore it is convergent.

let, $\lim_{n \to \infty} a_n = l$.

Now, $\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \frac{1}{2}\left(a_n + \frac{9}{a_n}\right) = \frac{1}{2}\left(l + \frac{9}{l}\right)$

or, $l^2 = 9$

or, $l = \pm 3$

But $l \neq -3$ as $a_n > 0, \forall n$.

Hence, $l = 3$.

9. If x_1, y_1 are positive real numbers, $x_{n+1} = \frac{1}{2}(x_n + y_n)$ and $\frac{2}{y_{n+1}} = \frac{1}{x_n} + \frac{1}{y_n}$ for $n \geq 1$, then S.T. the sequences $\{x_n\}_n$ and $\{y_n\}_n$ are monotonic and approach to a common limit l, where $l^2 = x_1 y_1$.

Solution:- Let without loss of generality, $x_1 < y_1$

By the given condition,

Also, $x_2 = \frac{1}{2}(x_1 + y_1) \quad \therefore x_1 < x_2 < y_1$

$\frac{2}{y_2} = \frac{1}{x_1} + \frac{1}{x_1} \quad \therefore \frac{1}{y_1} < \frac{1}{y_2} < \frac{1}{x_1} \Rightarrow x_1 < y_2 < y_1$

Similarly, $x_2 < x_3 < y_2$ and $x_2 < y_3 < y_2$

Combing all these, finally we get, $x_1 < x_2 < x_3 < x_4 < y_3 < y_2 < y_1$, so

Proceeding in this way we get $\{x_n\}$ is monotone increasing on.

and $\{y_n\}$ is monotone decreasing.

Now, $\{x_n\}$ is bounded above by y_1 and $\{y_n\}$ is bounded below by x_1.

$\therefore \{x_n\}$ and $\{y_n\}$ are convergent.

Let, $\lim_{n \to \infty} x_n = l$ and $\lim_{n \to \infty} y_n = m$.

By the given condition taking the limit on
$$2x_{n+1} = x_n + y_n.$$
we get
$$2l = l + m$$
$$\therefore l = m.$$
$\therefore \{x_n\}_n$ and $\{y_n\}_n$ have a same limit.

Now, $2x_{n+1} = x_n + y_n$ and $\frac{2}{y_{n+1}} = \frac{x_n + y_n}{x_n y_n} = \frac{2x_{n+1}}{x_n y_n}$

$\therefore x_{n+1} y_{n+1} = x_n y_n = x_{n-1} y_{n-1} = \cdots = x_1 y_1$

In the limit $n \to \infty$,
$$l^2 = x_1 y_1$$

10. If x_1, x_2 are positive and $x_{n+1} = \frac{1}{2}(x_n + x_{n-1})$, then the sequences $x_1, x_3, x_5, \ldots$ and $x_2, x_4, x_6, \ldots$ are ~~are~~ decreasing and increasing sequence, respectively and they converge to the limit $\frac{1}{3}(x_1 + 2x_2)$.

<u>Solution:-</u> Let $x_1 > x_2$, then $x_1 + x_2 > 2x_2$

Also, $2x_1 > x_1 + x_2 \Rightarrow x_2 < x_3 < x_1$

Proceeding in this way, we find $x_4 < x_5 < x_3$, $x_4 < x_6 < x_5$ and so on.

Thus we get $x_2 < x_4 < x_6 < x_8 < \cdots < x_5 < x_3 < x_1$.

Thus the sequence $\{x_{2n}\}_n$ is monotone increasing and the sequence $\{x_{2n+1}\}_n$ is monotone decreasing; both of them being bounded are convergent.

Now we see that

$$x_3 - x_2 = \frac{1}{2}(x_1 - x_2)$$

$$x_4 - x_2 = x_4 - x_3 + x_3 - x_2 = -\frac{1}{4}(x_1 - x_2) + \frac{1}{2}(x_1 - x_2)$$

$$= (x_1 - x_2)\left(-\frac{1}{4} + \frac{1}{2}\right)$$

$$x_5 - x_2 = (x_1 - x_2)\left(\frac{1}{8} - \frac{1}{4} + \frac{1}{2}\right)$$

Proceeding in this way, we get, in general

$$x_n - x_2 = (x_1 - x_2)\left\{\underbrace{\frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \cdots\cdots}_{(n-2)\text{ term}}\right\}$$

$$= (x_1 - x_2)\cdot\frac{1}{2}\cdot\frac{\left\{1 - \left(-\frac{1}{2}\right)^{n-2}\right\}}{1 + \frac{1}{2}}$$

$$= \frac{1}{3}(x_1 - x_2)\left\{1 - \left(-\frac{1}{2}\right)^{n-2}\right\}$$

$$\longrightarrow \frac{1}{3}(x_1 - x_2)$$

$$\therefore \lim x_n = x_2 + \frac{1}{3}(x_1 - x_2)$$

$$= \frac{1}{3}(x_1 + 2x_2)$$

11. Let a sequence $\{x_n\}$ be defined by the recurrence relation $x_{n+1} = x_n(2 - x_n)$, where $x_n > 0\ \forall\ n$ and $0 < x_1 < 1$. S.T. $\{x_n\}$ is convergent.

<u>Solution:-</u> We see that $(1 - x_1)^2 > 0$

or, $1 + x_1^2 - 2x_1 > 0$.

or, $1 > 2x_1 - x_1^2 = x_2$, which again implies that $x_3 < 1$.

Proceeding in this way, we get that $0 < x_n < 1\ \forall n$.

Now, $x_{n+1} - x_n = x_n - x_n^2 = x_n(1 - x_n) > 0$

Hence, $x_{n+1} > x_n\ \forall n$,

$\therefore \{x_n\}$ is monotone increasing and bounded above.

$\therefore \{x_n\}_n$ is convergent.

12. Prove that the sequence $\{x_n\}$ defined by $x_{n+1} = \sqrt{Kx_n}$ for $n>1$ ~~convergent~~ and $x_1 = \sqrt{K}$ $(k>1)$ converges to K.

Solution:-

Here, $x_{n+1}^2 = kx_n$.

$x_{n+1}^2 - x_n^2 = k(x_n - x_{n-1})$

or, $(x_{n+1}+x_n)(x_{n+1}-x_n) = k(x_n - x_{n-1})$

$\therefore$ $x_{n+1} \gtrless x_n$ according as $x_n \gtrless x_{n-1}$

But $x_2 > x_1$ $\therefore$ $x_3 > x_2$ and $x_4 > x_3$ and so on.

$\therefore$ $\{x_n\}$ forms a monotone increasing sequence.

Again, $kx_n = x_{n+1}^2 > x_n^2$

$\therefore x_n(x_n - k) < 0 \Rightarrow x_n < k \;\forall n$.

$\therefore$ $\{x_n\}$ is a monotone increasing sequence and bounded above,

$\therefore$ $\{x_n\}$ is convergent,

Let, $\lim x_n = l$, $\quad\therefore \lim x_{n+1}^2 = k \lim x_n \Rightarrow l^2 = kl$

or, $l(l-k)=0$ $\quad\therefore l = k$ $(\because l \neq 0)$

$\therefore$ The sequence converges to K.

13. Let $u>v>0$ and $u_1 = \frac{1}{2}(u+v)$ and $v_1 = \sqrt{uv}$. $u_{n+1} = \frac{1}{2}(u_n + v_n)$ and $v_{n+1} = \sqrt{u_n v_n}$ for $n>2$.

Show that $\{u_n\}$ and $\{v_n\}$ both converge to a common limit.

Solution:- AM > GM $\Rightarrow u_1 > v_1$,

$u_2 = \frac{1}{2}(u_1+v_1) \Rightarrow v_1 < u_2 < u_1$ and $v_1 < v_2 < u_1$,

again by AM > GM, $u_2 > v_2$.

Finally, we get $v_1 < v_2 < v_3 < v_4 < \dots < u_3 < u_2 < u_1$

proving that $\{v_n\}$ is monotone increasing and $\{u_n\}$ is monotone decreasing and both the sequences are bounded.

So, they are convergent.

Let $\lim u_n = l$ and $\lim v_n = m$

$\lim u_{n+1} = \frac{1}{2}\lim(u_n+v_n)$

$\Rightarrow l = \frac{1}{2}(l+m)$

$\Rightarrow l = m$

$\therefore$ The sequences converge to the same limit.

Note:- In ex. (10), it is clear that if $\{s_n\}$ be a sequence of positive numbers such that

$$s_n = \frac{1}{2}(s_{n-1} + s_{n-2}) \;\forall n \geq 2$$

then $\lim_{n\to\infty} s_n = \frac{1}{3}(s_1 + 2s_2)$

as, $s_k + \frac{1}{2}s_{k-1} = \frac{1}{2}(s_1 + 2s_2)$

$\Rightarrow l + \frac{1}{2}l = \frac{1}{2}(s_1+2s_2) \Rightarrow l = \frac{1}{3}(s_1+2s_2)$.

14. If $x_1, x_2 > 0$ and $x_{n+2} = \sqrt{x_{n+1} \cdot x_n}$, prove that $\{x_n\}$ is composed of two sequences of which one is increasing and the other decreasing but both of them have the common limit $(x_1 \cdot x_2^2)^{1/3}$.

Solution:- Let $x_1 > x_2$, $x_3 = \sqrt{x_1 x_2} > x_2$ and $x_1 > \sqrt{x_1 x_2} = x_3$

$\therefore x_1 > x_3 > x_2$.

Also $x_4 = \sqrt{x_2 x_3} > x_2$ and $x_3 > x_4$.

$\therefore x_3 > x_4 > x_2$, which implies $x_5 > x_6 > x_4$.

Proceeding in this way, we find

$x_2 < x_4 < x_6 < \cdots\cdots < x_5 < x_3 < x_1$

$\therefore$ $\{x_{2n-1}\}$ is monotone decreasing and $\{x_{2n}\}$ is monotone increasing and the sequences are bounded.

Therefore they are convergent.

Let, $\lim x_{2n} = l$ and $\lim x_{2n-1} = m$

Now, $x_{n+2} = \sqrt{x_{n+1} \cdot x_n}$.

$\therefore l = \sqrt{lm}$ and $m = \sqrt{lm}$

$\therefore l = m$.

Now, $\dfrac{x_3}{x_2} = \dfrac{1}{x_2}\sqrt{x_1 x_2} = \sqrt{\dfrac{x_1}{x_2}}$

$\dfrac{x_4}{x_2} = \sqrt{\dfrac{x_1}{x_3}}$

Proceeding in this way, we finally get,

$$\frac{x_n}{x_2} = \sqrt{\frac{x_1}{x_{n-1}}}$$

or, $x_n\sqrt{x_{n-1}} = \sqrt{x_1 x_2^2}$

$\lim x_n\sqrt{x_{n-1}} = \lim (x_1 x_2^2)^{1/2}$

$\therefore \lim l^{3/2} = \lim (x_1 x_2^2)^{1/2}$

$\therefore l = (x_1 x_2^2)^{1/3}$

15. If a sequence $\{S_n\}$ is defined by $S_n = \frac{s}{1+S_{n-1}}$ where $s>0$ and $S_1>0$, then show that the sequence converges to the positive root of the equation $x^2+x-s=0$.

Solution:- Here, $S_n - S_{n-2} = \frac{-s(S_{n-1}-S_{n-3})}{(1+S_{n-1})(1+S_{n-3})} > 0$ ---------- (1)

$\Rightarrow S_{n-1} - S_{n-3} < 0$

Again from (1),

$$S_n - S_{n-2} = \frac{+s^2(S_{n-2}-S_{n-4})}{(1+S_{n-1})(1+S_{n-2})(1+S_{n-3})(1+S_{n-4})} \text{ -------- (2)}$$

$\therefore S_n - S_{n-2} > 0 \Rightarrow S_{n-2} - S_{n-4} > 0$.

(1) and (2) show that the sequence $\{S_{2n}\}$ is increasing and $\{S_{2n-1}\}$ is decreasing.

By definition, $S_1>0 \Rightarrow S_2>0$ so $S_3>0$, so on.

$\therefore 0 < S_n < s \ \forall n$ which shows that $\{S_n\}$ is a bounded sequence.

Here, let, $\lim\{S_{2n}\} = m$ and $\lim\{S_{2n-1}\} = l$.

If n is even, in the limit of $S_n = \frac{s}{1+S_{n-1}}$.

$\therefore m = \frac{s}{1+l}$

or, $m + ml = s$.

If n is odd, in the limit of $l = \frac{s}{1+m}$

or, $l + lm = s$.

$\therefore$ finally we have $l = m$.

$\therefore \{S_n\}$ converges to l.

Again, $l = \frac{s}{1+l} \Rightarrow l^2+l-s=0$.

l is a root of the equation $x^2+x-s=0$, which has two roots, one positive and the other is negative, since $s>0$. l is positive since the sequence of positive terms can never converge to a negative number.

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- Subsequence:

Definition: Let $\{k_n\}$ be an increasing sequence of natural numbers. For a sequence $\{x_n\}$, the collection $\{x_{k_n}\}$ is called a subsequence of $\{x_n\}$.

More precisely, if $f: \mathbb{N} \to \mathbb{R}$ be a real sequence and $g: \mathbb{N} \to \mathbb{N}$ be a strictly increasing function, then $f \circ g: \mathbb{N} \to \mathbb{R}$ is called a subsequence of $f: \mathbb{N} \to \mathbb{R}$.

Examples:

1. Let $x_n = \frac{1}{n}$ and $k_n = 2n \ \forall n \in \mathbb{N}$.
   Then $\{x_{k_n}\} = \{x_2, x_4, x_6, \ldots\}$
   $= \{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots\}$ is a subsequence of $\{\frac{1}{n}\}$.

2. Let $x_n = \frac{1}{n}$ and $k_n = 2n-1 \ \forall n \in \mathbb{N}$.
   Then $\{x_{k_n}\} = \{x_1, x_3, x_5, \ldots\}$
   $= \{1, \frac{1}{3}, \frac{1}{5}, \ldots\}$ is a subsequence of $\{\frac{1}{n}\}$.

3. Let $x_n = (-1)^n$ and $k_n = 2n \ \forall n \in \mathbb{N}$.
   Then $\{x_{k_n}\} = \{1, 1, \ldots, 1, \ldots\}$ is a subsequence of $\{(-1)^n\}$.

4. Let $x_n = 1 + \frac{1}{n}$ and $k_n = n^2 \ \forall n \in \mathbb{N}$.
   Then $\{x_{k_n}\} = \{1+1, 1+\frac{1}{2^2}, 1+\frac{1}{3^2}, \ldots\}$ is a subsequence of $\{1+\frac{1}{n}\}$.

Theorem: If a sequence $\{x_n\}$ converges to l then every subsequence $\{x_{k_n}\}$ of the sequence $\{x_n\}$ also converges to l.

Hints:

$k_1 < k_2 < \cdots < k_n < \cdots$

$\therefore \epsilon > 0$, since $\lim x_n = l$, $\exists$ a natural no. $N(\epsilon) \ni$
$l - \epsilon < x_n < l + \epsilon \ \forall n \geq N(\epsilon)$

Now, $k_n > N(\epsilon) \ \forall n \geq N(\epsilon_0)$

$\therefore l - \epsilon < x_{k_n} < l + \epsilon \ \forall n \geq N(\epsilon_0)$

$\therefore \lim_{n \to \infty} x_{k_n} = l$.

Note: If there exists two different subsequences $\{x_{k_n}\}$ and $\{x_{m_n}\}$ of a sequence $\{x_n\}$ such that $\{x_{k_n}\}$ and $\{x_{m_n}\}$ converges to two different limits, then the sequence $\{x_n\}$ is not convergent.

If a sequence $\{x_n\}$ has a divergent subsequence then $\{x_n\}$ is divergent.

WORKED EXAMPLES:-

1. Prove that $\lim \left(1+\frac{1}{2n}\right)^n = \sqrt{e}$.

Solution:- Let $u_n = \left(1+\frac{1}{n}\right)^n$, $v_n = \left(1+\frac{1}{2n}\right)^{2n}$ $\forall n \in \mathbb{N}$.

$\{u_n\}$ is a convergent sequence and $\lim \left(1+\frac{1}{n}\right)^n = e$.

Since $v_n = u_{2n}$ $\forall n \in \mathbb{N}$. $\{v_n\}$ is a sequence of $\{u_n\}$ and therefore, $\lim v_n = e$.

Now, $w_n = \sqrt{v_n}$ $\forall n \in \mathbb{N}$. Therefore $\lim w_n = \lim \sqrt{v_n} = \sqrt{e}$.

2. Prove that the sequence $\{(-1)^n\}$ is divergent.

Solution:- Let $u_n = (-1)^n$, $v_n = u_{2n}$, $w_n = u_{2n-1}$

$\therefore \lim v_n = 1$, $\lim w_n = -1$.

Since two different subsequences converge to two different limits, the sequence $\{u_n\}$ is divergent.

Theorem:- If the subsequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ of a sequence $\{u_n\}$ converge to a limit l then the sequence $\{x_n\}$ is convergent and $\lim x_n = l$.

Proof:- Let us choose $\epsilon > 0$. Since $\lim x_{2n} = l$, $\exists$ a natural no. $k_1 \ni$

$$|x_{2n} - l| < \epsilon \quad \forall n \geq k_1.$$

Since $\lim x_{2n-1} = l$, $\exists$ a natural no. $k_2 \ni$

$$|x_{2n-1} - l| < \epsilon \quad \forall n \geq k_2.$$

Let $k = \max\{k_1, k_2\}$. Then k is a natural no. and for all $n \geq k$,

$$l - \epsilon < x_{2n} < l + \epsilon \text{ and } l - \epsilon < x_{2n-1} < l + \epsilon.$$

That is $l - \epsilon < x_n < l + \epsilon$ and for all $n \geq 2k-1$,

As $2k-1$ is a natural number, it follows that $\lim x_n = l$.

Note:- If two subsequences of a sequence converges to the same limit l, the sequence $\{x_n\}$ may not be convergent.

Example:- let $x_n = \sin\frac{n\pi}{4}$

Then the subsequences $\{x_{8n-7}\}$ is $\left\{\sin\frac{\pi}{4}, \sin\frac{9\pi}{4}, \sin\frac{17\pi}{4}, \ldots\right\}$ converges to $\frac{1}{\sqrt{2}}$.

The subsequence $\{x_{8n-5}\}$ is $\left\{\sin\frac{3\pi}{4}, \sin\frac{11\pi}{4}, \sin\frac{19\pi}{4}, \ldots\right\}$ converges to $\frac{1}{\sqrt{2}}$.

But the sequence $\{x_n\}$ is not convergent.

WORKED EXAMPLES:-

1. Prove that the sequence $\{u_n\}$ defined by $0 < u_1 < u_2$ and $u_{n+2} = \frac{1}{2}(u_n + u_{n+1})$, is convergent.

Sol:- $u_3 - u_1 = \frac{u_1+u_2}{2} - u_1 = \frac{(u_2-u_1)}{2} > 0 \Rightarrow u_1 < u_3$

$u_3 - u_2 = \frac{u_1+u_2}{2} - u_2 = \frac{(u_1-u_2)}{2} < 0 \Rightarrow u_2 > u_3$

so, $u_1 < u_3 < u_2$.

This inequality gives $u_1 < u_3 < u_5 < \cdots < u_6 < u_4 < u_2$.

i.e. the sequence $\{u_{2n-1}\}$ is a monotone increasing sequence, bounded above and the sequence $\{u_{2n}\}$ is a monotone decreasing sequence, bounded below.

So, both the sequence $\{u_{2n}\}$ and $\{u_{2n-1}\}$ are convergent.

Let $\lim u_{2n} = l$ and $\lim u_{2n-1} = m$

Now, $2u_{2n+2} = u_{2n} + u_{2n+1}\ \forall n \in \mathbb{N}$.

$\therefore 2l = l + m$

$\Rightarrow l = m$.

Thus the sequences $\{u_{2n}\}$ and $\{u_{2n-1}\}$ converge to same limit l and therefore the sequence $\{u_n\}$ is convergent.

2. S.T $\{(-1)^n + \frac{1}{n}\}$ is not convergent.

Solution:- We consider the subsequences,

$u_{2n} = \left\{(-1)^{2n} + \frac{1}{2n}\right\} = \left\{1+\frac{1}{2}, 1+\frac{1}{4}, 1+\frac{1}{6}, \ldots\right\}$ which converges to 1.

$u_{2n+1} = \left\{(-1)^{2n+1} + \frac{1}{2n+1}\right\} = \left\{-1+\frac{1}{3}, -1+\frac{1}{5}, -1+\frac{1}{7}, \cdots\right\}$ which converges to -1.

So, two subsequences converge to different limits.

$\therefore \left\{(-1)^n + \frac{1}{n}\right\}$ is not convergent.

3. Show that the sequence $\{a_n\}_n$ defined by $a_n = \left(1 - \frac{1}{n}\right)\sin n.\frac{\pi}{2}$ $(n = 1, 2, 3, \ldots)$ has a convergent subsequence but the sequence is not convergent.

Solution:- For $n = 1, 5, 9, \ldots$; $\sin n\frac{\pi}{2} = 1$.

Then the sequence converges to 1.

For $n = 2, 4, 6, \ldots$; $\sin n\frac{\pi}{2} = 0$.

Then the sequence converges to 0.

$\therefore \{a_n\}$ is not convergent.

Theorem:– 1. Every subsequence of a monotone increasing (decreasing) sequence of real numbers is monotone increasing (decreasing).
2. Every sequence of real nos. has a monotone subsequence.

Example:– Let $n^{(-1)^n}$. The sequence is $\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \frac{1}{7}, \ldots\}$

Here the sequence $\{u_n\}$ has no peak.

u_1 is not a peak.

Thus $\{u_1, u_2, u_4, u_6, u_8, \ldots\}$ is a monotone increasing subsequence of the sequence $\{u_n\}$.

Subsequential Limit:– Let $\{u_n\}$ be a real sequence. A real number l is said to be a subsequential limit of the sequence $\{u_n\}$ if $\exists$ a subsequence of $\{u_n\}$ that converges to l.

Theorem:– A real number l is a subsequential limit of a sequence $\{u_n\}$ if and only if every neighbourhood of l contains infinitely many elements of the sequence $\{u_n\}$.

Note:– The limit of a sequence, it it exists, is also a subsequential limit of the sequence.

Theorem:– (Bolzano – Weierstrass Theorem)

Every bounded sequence of real nos. has a convergent subsequence.

Alter:– Every bounded sequence of real nos. has a subsequential limit.

Examples:–

1. The sequence $\{u_n\}$, where $u_n = \sin\frac{n\pi}{2}, n \geq 1$ is a bounded sequence since $|u_n| \leq 1 \;\forall\; n \geq 1$.

(i) The subsequence $\{u_1, u_5, u_9, \ldots\}$, i.e. $\{u_{4n-3}\}$ is a convergent subsequence that converges to 1.

(ii) The subsequence $\{u_2, u_4, u_6, \ldots\}$, i.e. $\{u_{2n}\}$ is a convergent subsequence that converges to 0.

(iii) The subsequence $\{u_1, u_3, u_5, \ldots\}$, i.e. $\{u_{2n-1}\}$ is a divergent subsequence. [This shows that a bounded sequence may have a divergent subsequence.]

2. The sequence $\{u_n\}$ where $u_n = n^{(-1)^n}$ is an unbounded sequence. The sequence is $\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \ldots\}$.

(i) The sequence $\{u_2, u_4, u_6, \ldots\}$ i.e. $\{u_{2n}\}$ is a properly divergent subsequence.

(ii) The sequence $\{u_1, u_3, u_5, \ldots\}$, i.e. $\{u_{2n-1}\}$ is a convergent subsequence. [This shows that an unbounded sequence may have a convergent subsequence.]

Note:– Limit points of a sequence should not be confused with limit of a convergent sequence. A convergent sequence will have one and only one limit point. The limit ξ_p of a sequence $\{x_{n_k}\}$ may or may not coincide with the limit of a sequence.

Characterisation of a Compact set :-

Theorem:- Let K be a non-empty ~~set~~ subset of $\mathbb{R}$. Then K is compact if and only if every sequence in K has a subsequence convergent to a point in K.

The upper limit and the lower limit :- Let $\{u_n\}$ be a bounded sequence of real numbers. Then by Bolzano-Weierstrass theorem there is a convergent subsequence of $\{u_n\}$.

Definition:- Let $\{u_n\}$ be a bounded sequence of real nos.

The greatest subsequential limit of $\{u_n\}$ is said to be the upper limit or the limit superior of $\{u_n\}$ and this is denoted by $\overline{\lim}\, u_n$ or ~~sup~~ $\lim\sup u_n$.

The least subsequential limit of $\{u_n\}$ is said to be the lower limit or the limit inferior of $\{u_n\}$ and this is denoted by $\underline{\lim}\, u_n$ or $\lim\inf u_n$.

If $\{u_n\}$ is unbounded above then we define $\overline{\lim}\, u_n = \infty$

If $\{u_n\}$ is unbounded below then we define $\underline{\lim}\, u_n = -\infty$.

Example:-

1. Let $u_n = (-1)^n\left(1+\frac{1}{n}\right)$, $n \geq 1$. Then the sequence $\{u_n\}$ is a bounded sequence. $\overline{\lim}\, u_n = 1$, $\underline{\lim}\, u_n = -1$.

2. Let $u_n = \frac{1}{n}$, $n \geq 1$. Then the sequence $\{u_n\}$ is a bounded sequence. $\overline{\lim}\, u_n = \underline{\lim}\, u_n = 0$.

Theorem:- A bounded sequence $\{u_n\}$ is convergent if and only if $\overline{\lim}\, u_n = \underline{\lim}\, u_n$.

Note:- The theorem can be restated as —

A bounded sequence is convergent if and only if it has only one sub-sequential limit.

Theorem:- If $\{u_n\}$ and $\{v_n\}$ be bounded sequence, then

(i) $\overline{\lim}\, u_n + \overline{\lim}\, v_n \geq \overline{\lim}\,(u_n+v_n)$

(ii) $\underline{\lim}\, u_n + \underline{\lim}\, v_n \leq \underline{\lim}\,(u_n+v_n)$.

Remark:- Consider the sequence $\{x_n\}_n$ where $x_{n+1} = \sqrt{k+x_n}$; x_1 and k being positive, is monotone increasing or decreasing according as x_1 is less or greater than the positive root of $x^2 - x - k = 0$, and that in either case the sequence converges to the positive root of the equation $x^2 - x - k = 0$.

Cauchy Sequence: Cauchy's Principle of Convergence

Cauchy's method of establishing convergence of a sequence does not required any knowledge of its limit, nor does it require the sequence to be monotone.

Theorem:- A necessary and sufficient condition for the convergence of a sequence $\{x_n\}$ is that for a pre-assigned positive $\epsilon\ (>0)$ there exists a natural number m such that

$$|x_{n+p} - x_n| < \epsilon \quad \forall\ n \geq m \text{ and for } p = 1, 2, 3, \ldots$$

Proof:- Let $\{x_n\}$ be convergent and $\lim x_n = l$. Then for a pre-assigned positive $\epsilon\ (>0)$ $\exists$ a natural number m $\ni$

$$|x_n - l| < \frac{\epsilon}{2} \quad \forall\ n \geq m$$

$$\therefore\ |x_{n+p} - l| < \frac{\epsilon}{2}\ \forall\ n \geq m \text{ and } p = 1, 2, 3, \ldots$$

Now, $|x_{n+p} - x_n| \leq |x_{n+p} - l| + |x_n - l|$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \forall\ n \geq m \text{ and } p = 1, 2, 3, \ldots$$

$$= \epsilon$$

This proves that the condition is necessary.

We now prove that the sequence $\{x_n\}$ is convergent under the stated condition.

First, we are to prove that the sequence $\{x_n\}$ is bounded.

Let $\epsilon = 1$. Then $\exists$ a natural no. k $\ni$

$$|x_{n+p} - x_n| < 1 \quad \forall\ n \geq k \text{ and } p = 1, 2, 3, \ldots$$

$$\therefore\ |x_{k+p} - x_k| < 1\ \forall\ p = 1, 2, 3, \ldots$$

or, $x_k - 1 < x_{k+p} < x_k + 1\ \forall\ p = 1, 2, 3, \ldots$

Let $B = \max\{x_1, x_2, \ldots, x_k, x_k + 1\}$, $b = \min\{x_1, x_2, \ldots, x_k - 1\}$

Then $b \leq x_n \leq B\ \forall\ n \in \mathbb{N}$.

This shows that $\{x_n\}$ is a bounded sequence.

Now, Bolzalo-Weierstrass theorem, $\{x_n\}$ has a convergent subsequence.

Let l be the limit of that subsequence, i.e. l is a subsequential limit of $\{x_n\}$.

Let $\epsilon > 0$. Then by the given condition, $\exists$ a natural number m $\ni$

$$|x_{n+p} - x_n| < \frac{\epsilon}{3} \quad \forall\ n \geq m \text{ and } p = 1, 2, 3, \ldots$$

Taking $m = n$, it follows that

$$|x_{m+p} - x_m| < \frac{\epsilon}{3} \quad \forall\ p = 1, 2, 3, \ldots \qquad \text{(i)}$$

Since l is a subsequential limit of $\{x_n\}$, each ϵ neighbourhood of l contains infinite number of elements of $\{x_n\}$. Therefore $\exists$ a natural number $q > m$ $\ni$ $|x_q - l| < \epsilon/3$.

As $q > m$, it follows from (i) that $|x_q - x_m| < \epsilon/3$.

Now, $|x_{m+p} - l| \leq |x_{m+p} - x_m| + |x_m - x_q| + |x_q - l|$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad \forall\ p = 1, 2, 3, \ldots$$

Therefore, $|x_n - l| < \epsilon \quad \forall\ n \geq m+1$.

Since ϵ is arbitrary, the sequence $\{x_n\}$ converges to l.

Hence the proof is complete.

<u>Note:-</u> The condition stated in the theorem is called the "Cauchy condition" for convergence of a sequence.

Therefore a sequence $\{x_n\}$ is convergent if and only if the cauchy condition is satisfied.

<u>WORKED EXAMPLES :—</u> Check the convergence of the following sequences using Cauchy's general principle of convergence:

(i) $\{u_n\} = \left\{\frac{n}{n+1}\right\}$

(ii) $\{u_n\} = \left\{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right\}$ (iv) $\left\{\frac{1}{n+1}\right\}$

(iii) $\{u_n\} = \left\{1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right\}$

<u>Solution:—</u>

(i) $u_n = \frac{n}{n+1}$

Let p be a natural number.

The $u_{n+p} = \frac{n+p}{n+p+1}$.

$$|u_{n+p} - u_n| = \left|\frac{n+p}{n+p+1} - \frac{n}{n+1}\right|$$

$$= \frac{p}{(n+p+1)(n+1)}$$

$$< \frac{1}{(n+1)} < \frac{1}{n} \ \forall p \quad \text{since } \frac{p}{n+p+1} < 1 \ \forall p.$$

Let $\epsilon > 0$. Then $\frac{1}{n} < \epsilon$ holds for $n > \frac{1}{\epsilon}$.

Let $m = \left[\frac{1}{\epsilon}\right] + 1$

Then m is a natural number and $|u_{n+p} - u_n| < \epsilon \ \forall\ n \geq m \ \forall\ p = 1, 2, 3, 4, \ldots$

This shows $\left\{\frac{n}{n+1}\right\}_n$ is a cauchy sequence and $\{u_n\}$ is convergent.

(ii) Let p be a natural number.

$$|u_{n+p} - u_n| = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p}$$

Let us choose $n = m$ and $p = m$.

Then $$|u_{2m} - u_m| = \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m}$$
$$> \frac{1}{2m} + \frac{1}{2m} + \cdots + \frac{1}{2m}$$
$$= \frac{1}{2}.$$

If we choose $\epsilon = \frac{1}{2}$ then no natural number k can be found such that $|u_{n+p} - u_n| < \epsilon$ will hold $\forall\, n \geq k$ and for every natural number k. So, $\{u_n\}$ is not convergent.

(iii) We know that $n! = 1.2.3.\ldots n > 2.2.\ldots 2 = 2^{n-1}$

$$\therefore \frac{1}{n!} < \frac{1}{2^{n-1}}.$$

Now for $m > n$

$$|u_m - u_n| = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{m!}$$
$$< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}}$$
$$= \frac{1}{2^n}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{m-n-1}}\right)$$
$$= \frac{1}{2^n} \cdot \frac{1 - \left(\frac{1}{2}\right)^{m-n}}{1 - \frac{1}{2}} < \frac{2}{2^n} = \frac{1}{2^{n-1}}.$$

$\therefore |u_m - u_n| < \epsilon$ if $\frac{1}{2^{n-1}} < \epsilon$

i.e. if $n > 1 + \frac{\log \frac{1}{\epsilon}}{\log 2}$

Let us choose $N(\epsilon) = \left[1 + \frac{\log \frac{1}{\epsilon}}{\log 2}\right] + 1$.

Then $N(\epsilon)$ is a natural no. and $|u_m - u_n| < \epsilon \;\forall\, n \geq N(\epsilon)$ and $m > N(\epsilon)$

$\therefore$ By Cauchy's condition $\{u_n\}$ is convergent.

(iv) Let $m > n$ where m and n are positive integers.

Then $\left|\frac{1}{m+1} - \frac{1}{n+1}\right| = \frac{1}{n+1}\left(1 - \frac{n+1}{m+1}\right) < \frac{1}{n+1} < \frac{1}{n} < \epsilon$ if $n > \frac{1}{\epsilon}$.

We choose $N(\epsilon) = \left[\frac{1}{\epsilon}\right] + 1$.

Then $\left|\frac{1}{m+1} - \frac{1}{n+1}\right| < \epsilon$ if $n > N(\epsilon)$ and $m > n$.

Thus $\left\{\frac{1}{n+1}\right\}_n$ is a cauchy sequence and is convergent.

Cauchy Sequence:

Definition:— A sequence $\{u_n\}$ is said to be a Cauchy Sequence if for a pre-assigned positive quantity ϵ $\exists$ a natural number k such that

$$|u_m - u_n| < \epsilon \text{ for all } m, n \geq k.$$

Replacing m by $n+p$ where $p = 1, 2, 3, \ldots$ The above condition can be equivalently stated as

$$|u_{n+p} - u_n| < \epsilon \text{ for all } n \geq k \text{ and } p = 1, 2, 3, \ldots$$

Theorem 1:— A cauchy sequence of real numbers is convergent. / Every cauchy sequence is convergent.

Theorem 2:— Every convergent sequence is a Cauchy sequence.

Proof:— Let $\{u_n\}$ be a convergent sequence and $\lim u_n = l$.
For a pre-assigned positive ϵ $\exists$ a natural number k $\ni$

$$|u_n - l| < \frac{\epsilon}{2} \ \forall\ n \geq k.$$

If m, n be natural numbers $\geq k$, then

$$|u_m - l| < \frac{\epsilon}{2} \text{ and } |u_n - l| < \frac{\epsilon}{2}.$$

Now, $|u_m - u_n| \leq |u_m - l| + |u_n - l|$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \ \forall\ m, n \geq k.$$

This proves that the sequence $\{u_n\}$ is a Cauchy Sequence.

Note:— Prove or disprove:— every bounded sequence is a cauchy sequence.

Solution:— The statement is not true.

Counter example:— Let us consider a sequence $\{(-1)^n\}_n$.

$-1 \leq \{(-1)^n\}_n \leq 1 \ \forall\ n \in \mathbb{N}$. So the sequence is bounded

Let $u_n = (-1)^n = -1$ if n is odd
$= 1$ if n is even.

$|u_m - u_n| = |(-1)^m - (-1)^n|$
$= 0$ if m and n are both odd or both even.
$= 2$ if one of m, n is odd and the other is even.

Let us choose $\epsilon = \frac{1}{2}$. then it is not possible to find a natural number k $\ni$ $|u_m - u_n| < \epsilon \ \forall\ m, n \geq k$.

WORKED EXAMPLES :-

1. Prove that the sequence $\{\frac{1}{n}\}$ is a cauchy sequence.

Solution:- Let $u_n = \frac{1}{n}$. Let us choose a positive $\epsilon (>0)$. There is a natural no. $K \ni \frac{2}{K} < \epsilon$.

Then $|u_m - u_n| = |\frac{1}{m} - \frac{1}{n}| \leq \frac{1}{m} + \frac{1}{n}$
$< \epsilon$ if $m, n \geq K$.

This proves that the sequence $\{u_n\}$ is a Cauchy sequence.

2. Show that $\{2^n\}_n$ is not a cauchy sequence.

Solution:-
Let, $u_n = 2^n$ and we choose $\epsilon = 1$. For $m > n$,

$|u_m - u_n| = |2^m - 2^n| = |2^n(2^{m-n} - 1)| > 2$ and can never be made less than arbitrary positive ϵ, in whatever way we choose $m > n > N(\epsilon)$, where $N(\epsilon)$ is a positive integer.

$\therefore \{2^n\}_n$ is not a cauchy sequence.

3. Prove that the sequence $\{u_n\}$ where $u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1} \cdot \frac{1}{n}$ is a convergent sequence.

Solution:- $|u_m - u_n| = \left| (-1)^n \frac{1}{n+1} + (-1)^{n+1} \frac{1}{n+2} + \cdots + (-1)^{m-1} \frac{1}{m} \right|$

$= \frac{1}{n+1} - \left(\frac{1}{n+2} - \frac{1}{n+3}\right) - \left(\frac{1}{n+4} - \frac{1}{n+5}\right) - \cdots$

$< \frac{1}{n+1}$ (each term within the bracket is positive)

$< \epsilon$

$\therefore |u_m - u_n| < \epsilon$ if $n > N(\epsilon)$, where $N(\epsilon)$ is integral part of $\left(\frac{1}{\epsilon} - 1\right)$.

We see that $|u_m - u_n| < \epsilon$ if $m, n > N(\epsilon)$.

$\therefore \{u_n\}$ is a cauchy sequence and hence is convergent.

4. Show by Cauchy's general principle of convergence that the sequence $\left\{\frac{n-1}{n+1}\right\}_n$ is convergent.

Solution:- $u_n = \frac{n-1}{n+1}$.

$|u_m - u_n| = \left| \frac{m-1}{m+1} - \frac{n-1}{n+1} \right| = \frac{2(m-n)}{(m+1)(n+1)}$

$< \frac{2}{n} \cdot \frac{(1 - \frac{n}{m})}{(n+1)^2} < \frac{2}{n^3} < \epsilon$

if $n^3 > \frac{2}{\epsilon}$ or $n > \left(\frac{2}{\epsilon}\right)^{1/3}$.

We choose $N(\epsilon) = \left[\frac{2}{\epsilon}\right]^{1/3} + 1$. Then $|u_m - u_n| < \epsilon \; \forall \; n > N(\epsilon)$.

Hence $\{x_n\}_n$ is convergent.

5. Prove that the sequence $\{u_n\}$ where $u_1 = 0$ and $u_2 = 1$ and $u_{n+2} = \frac{1}{2}(u_{n+1} + u_n)$ for all $n \geq 1$ is a cauchy sequence.

<u>Solution:-</u> $u_{n+2} - u_{n+1} = \frac{1}{2}(u_{n+1} + u_n) - u_{n+1}$

$$= \frac{1}{2}(u_n - u_{n+1}) = -\frac{1}{2}(u_{n+1} - u_n)$$

or, $|u_{n+2} - u_{n+1}| = \frac{1}{2}|u_{n+1} - u_n| \ \forall\ n \in \mathbb{N}$.

Therefore, $|u_{n+2} - u_{n+1}| = \frac{1}{2}|u_{n+1} - u_n| = \frac{1}{2^2}|u_n - u_{n-1}|$

$$= \cdots\cdots = \frac{1}{2^n}|u_2 - u_1| = \frac{1}{2^n}.$$

Let $m > n$ Then $|u_m - u_n|$

$$\leq |u_m - u_{m-1}| + |u_{m-1} - u_{m-2}| + \cdots + |u_{n+1} - u_n|$$

$$= \left(\frac{1}{2}\right)^{m-2} + \left(\frac{1}{2}\right)^{m-3} + \cdots + \left(\frac{1}{2}\right)^{n-1}$$

$$= \frac{4}{2^n}\left[1 - \left(\frac{1}{2}\right)^{m-n}\right] < \frac{4}{2^n}.$$

Let $\epsilon > 0$. Then $\exists$ a natural number $k \ni \frac{4}{2^n} < \epsilon$ for all $n \geq k$.

Hence $|u_m - u_n| < \epsilon \ \forall\ m, n \geq k$.

This proves that the sequence $\{u_n\}$ is a cauchy sequence.

6. Prove that the sequence $\{u_n\}$ satisfying the condition $|u_{n+2} - u_{n+1}| \leq c|u_{n+1} - u_n|$ for all $n \in \mathbb{N}$, where $0 < c < 1$, is a Cauchy sequence.

<u>Solution:-</u> $|u_{n+2} - u_{n+1}| \leq c|u_{n+1} - u_n|$

$$\leq c^2|u_n - u_{n-1}|$$

$$\vdots$$

$$\leq c^n|u_2 - u_1|$$

Let $m > n$.

Then $|u_m - u_n| \leq |u_m - u_{m-1}| + \cdots + |u_{n+1} - u_n|$

$$\leq |u_2 - u_1|\{c^{m-2} + c^{m-3} + \cdots + c^{n-1}\}$$

$$= |u_2 - u_1| \cdot c^{n-1} \cdot \frac{1 - c^{m-n}}{1 - c}$$

$$< \frac{c^{n-1}}{1-c} \cdot |u_2 - u_1|$$

Let $\epsilon > 0$. Since $0 < c < 1$, the sequence $\{c^{n-1}\}$ is a convergent sequence. Therefore $\exists$ a natural number $k \ni$

$$\frac{c^{n-1}}{1-c}|u_2 - u_1| < \epsilon \ \forall\ n \geq k.$$

It follows that $|u_m - u_n| < \epsilon \ \forall\ m, n \geq k$ and this proves that the sequence $\{u_n\}$ is a cauchy sequence.

Cauchy's theorems on limits :-

Theorem:-
(Cauchy's First Limit Theorem)

If $\lim u_n = l$ then $\lim \frac{u_1+u_2+\cdots+u_n}{n} = l$.

Note:- The converse of the theorem is not true. Let us consider the sequence $\{u_n\}$ where $u_n = (-1)^n$. Then $\lim \frac{u_1+u_2+\cdots+u_n}{n} = 0$ but the sequence $\{u_n\}$ is not convergent.

Corollary:- If $\lim u_n = l$ where $u_n > 0$ $\forall n$ and $l \neq 0$, then
$\boxed{\lim \sqrt[n]{u_1 u_2 \cdots u_n} = l.}$

Since each u_n is positive and $\lim u_n = l > 0$, the sequence $\{\log u_n\}$ converges to $\log l$.

Therefore $\lim \frac{\log u_1 + \log u_2 + \cdots + \log u_n}{n} = \log l$,

or, $\lim \log \sqrt[n]{u_1 \cdot u_2 \cdots u_n} = \log l$.

It follows that, $\lim \sqrt[n]{u_1 \cdots u_n} = l$, i.e. $(u_1 \cdots u_n)^{1/n} \to l$ as $n \to \infty$.

e.g. show that $\sqrt[n]{n} = 1$.

Sol. $\frac{n}{n-1} \to 1$ as $n \to \infty$

Let $u_1 = 1$, $u_2 = \frac{2}{1}, \ldots, u_n = \frac{n}{n-1}$.

$\therefore$ By the above example,

$(u_1 u_2 \cdots u_n)^{1/n} \to 1$ or $\left(1 \cdot \frac{2}{1} \cdot \frac{3}{2} \cdots \frac{n}{n-1}\right)^{1/n} \to 1$,

or, $n^{1/n} \to 1$.

WORKED EXAMPLES:-

1. Prove that $\lim \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n} = 0$.

Solution:- Let $u_n = \frac{1}{n}$. Then $\lim u_n = 0$.

so, By Cauchy's first limit theorem $\lim \frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{n} = 0$.

2. Prove that $\lim \frac{1+\sqrt{2}+\sqrt[3]{3}+\cdots+\sqrt[n]{n}}{n} = 1$.

Solution:- Let $u_n = \sqrt[n]{n}$. Then $\lim u_n = 1$.

By Cauchy's theorem, $\lim \frac{1+\sqrt{2}+\sqrt[3]{3}+\cdots+\sqrt[n]{n}}{n} = 1$.

3. Prove that $\lim \frac{1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}}{n} = 0$.

Solution:- Let $u_n = \frac{1}{n^2}$, then $\lim u_n = \lim \frac{1}{n^2} = 0$.

$\therefore$ By Cauchy's theorem, $\lim \frac{1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}}{n} = 0$.

Theorem:-

(Cauchy's second limit theorem)

If $\{x_n\}$ is a sequence of positive terms and $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = l$, then $\lim\limits_{n\to\infty} \sqrt[n]{x_n} = l$. [Proof:- Apply corollary of Cauchy's 1st limit theorem]

Note:- The converse of the theorem is not true.

For example, let us consider the sequence $\{u_n\}$ where $u_n = \frac{3+(-1)^n}{2}$

The sequence is $\{1, 2, 1, 2, 1, 2, \ldots\}$

Here $\lim \sqrt[n]{u_n} = 1$, since $\lim \sqrt[2n]{u_{2n}} = \lim 2^{1/2n} = 1$ and $\lim (u_{2n-1})^{\frac{1}{2(n-1)}} = 1$.

But $\lim \frac{u_{n+1}}{u_n}$ does not exist.

Note:- If $u_n > 0 \ \forall\, n \in \mathbb{N}$ and $\lim \frac{u_{n+1}}{u_n}$ exists, then it follows from the theorem that $\lim \sqrt[n]{u_n}$ also exists.

WORKED EXAMPLES:-

1. Prove that $\lim \frac{(n!)^{1/n}}{n} = \frac{1}{e}$.

Solution:- Putting $u_n = \frac{n!}{n^n}$, then $u_n > 0 \ \forall\, n \in \mathbb{N}$

and $\lim \frac{u_{n+1}}{u_n} = \frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!}$

$$= \left(\frac{n}{n+1}\right)^n \longrightarrow \frac{1}{e} \text{ as } n \to \infty.$$

$$> 0.$$

It follows from the theorem that $\lim \sqrt[n]{u_n} = \frac{1}{e}$, i.e., $\lim \frac{(n!)^{1/n}}{n} = \frac{1}{e}$.

2. Prove that $\lim \frac{\{(n+1)(n+2)\cdots(2n)\}^{1/n}}{n} = \frac{4}{e}$.

Solution:- Let $u_n = \frac{(n+1)(n+2)\cdots(2n)}{n^n}$, then $u_n > 0 \ \forall\, n \in \mathbb{N}$.

$$\lim \frac{u_{n+1}}{u_n} = \lim \frac{2(2n+1)}{n+1} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{4}{e} > 0.$$

It follows from the theorem that $\lim \sqrt[n]{u_n} = \frac{4}{e}$.

3. Prove that $\lim\limits_{n\to\infty} \left\{\left(\frac{2}{1}\right)^1 \left(\frac{3}{2}\right)^2 \cdots \left(\frac{n+1}{n}\right)^n\right\}^{1/n} = e$.

Solution:- Let $u_n = \left(\frac{2}{1}\right)^1 \left(\frac{3}{2}\right)^2 \cdots \left(\frac{n+1}{n}\right)^n$

Then $\frac{u_{n+1}}{u_n} = \left(\frac{n+2}{n+1}\right)^{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$

$$\lim \frac{u_{n+1}}{u_n} = \lim \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$= e.$$

$\therefore \lim\limits_{n\to\infty} \sqrt[n]{u_n} = e$.

SOME MORE EXAMPLES :—

1. A sequence $\{x_n\}$ is defined as follows:

$$x_{n+1} = \sqrt{\frac{ab^2 + x_n^2}{a+1}} \;\forall\, n \geq 1 \text{ and } x_1 = a > 0.$$

Prove that (i) $\{x_n\}$ is monotone decreasing and bounded if $x_1 > b$

(ii) $\{x_n\}$ is monotone increasing and bounded if $x_1 < b$.

(iii) in either case $\{x_n\}$ converges to b.

Solution:—

$$x_{n+1}^2 - x_n^2 = \frac{x_n^2 - x_{n-1}^2}{a+1} = \cdots = \frac{x_2^2 - x_1^2}{(a+1)^{n-1}} = \frac{a(b^2 - x_1^2)}{(a+1)^n}.$$

$\therefore x_{n+1} \gtrless x_n$ according as $b \gtrless x_1$.

$\therefore \{x_n\}$ is monotone increasing or decreasing according as $x_1 < b$ or $x_1 > b$.

Now $$x_{n+1}^2 - x_n^2 = \frac{x_n^2 - b^2}{a+1} = \frac{x_{n-1}^2 - b^2}{(a+1)^2} = \cdots = \frac{x_1^2 - b^2}{(a+1)^n}.$$

if $x_1 < b$, $0 < x_n < b$, then $\{x_n\}$ is monotone increasing and bounded above.

if $x_1 > b$, $x_n > b$, then $\{x_n\}$ is monotone decreasing and bounded below.

In either case $\{x_n\}$ is convergent.

Let $\lim_{n\to\infty} x_n = l$. $\therefore$ Taking limit we have,

$$x_{n+1} \longrightarrow \sqrt{\frac{ab^2 + x_n^2}{a+1}} \text{ as } n \to \infty$$

$$\Rightarrow l^2 = \frac{ab^2 + l^2}{a+1}$$

$$\Rightarrow l = b.$$

$\therefore \lim x_n = l.$

2. If the sequence $\{a_n\}$ and $\{b_n\}$ converges to A and B respectively, then $\lim \frac{1}{n}(a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1) = AB$.

Solution:– Let $a_n = A + x_n$ and $|x_n| = X_n$. Then $x_n \to 0$, and consequently $X_n \to 0$ also. Hence

$$\lim \frac{1}{n}(X_1 + X_2 + \cdots + X_n) = 0.$$

Now, on substituting for $a_1, a_2, \ldots, a_n$, we have

$$\frac{1}{n}(a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1)$$
$$= \frac{A}{n}(b_1 + b_2 + \cdots + b_n) + \frac{1}{n}(x_1 b_n + x_2 b_{n-1} + \cdots + x_n b_1).$$

Now, since $\{b_n\}$ is convergent, it is bounded. Hence $|b_n| \leq K \;\forall\, n$.

$$\left|\frac{1}{n}(x_1 b_n + x_2 b_{n-1} + \cdots + x_n b_1)\right| \leq \frac{K}{n}(|x_1| + |x_2| + \cdots + |x_n|)$$
$$= \frac{K}{n}(X_1 + X_2 + \cdots + X_n)$$
$$\to 0 \text{ since } K \text{ is fixed } \forall\, n.$$

Again since $\lim b_n = B$, we have

$$\lim \frac{1}{n}(b_1 + b_2 + \cdots + b_n) = B.$$

Hence, finally we have $\lim \frac{1}{n}(a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1) = AB$.

3. Show that the sequence $\left\{ \frac{\sin \frac{n\pi}{2}}{n} \right\}$ is a convergent sequence.

Solution:—

$$0 \leq \sin \frac{n\pi}{2} \leq 1$$

$$\Rightarrow 0 \leq \frac{\sin \frac{n\pi}{2}}{n} \leq \frac{1}{n}.$$

$\because \lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$ and $\left\{ \sin \frac{n\pi}{2} \right\}$ is a bounded sequence.

let $a_n = \frac{1}{n} \to 0$, $b_n = \sin \frac{n\pi}{2}$ (bounded)

so $\lim (a_n b_n) = 0$.

i.e. $\lim_{n\to\infty} \left(\frac{1}{n} \sin \frac{n\pi}{2} \right) = 0.$

4. Let a sequence $\{x_n\}$ be defined as follows:

$$x_n = \begin{cases} 0, & n = \text{even} \\ 1, & n = \text{odd} \end{cases}$$

Then find the limit of the sequence.

Solution:—

$$\lim_{n\to\infty} x_n = x$$

$\therefore |x_n - x| < \epsilon \ \forall\, n > n_0$. Taking $\epsilon = \frac{1}{2}$

if $n = \text{even}$ $\quad |0 - x| < \frac{1}{2} \quad \forall\, n > n_0$

$$\Rightarrow |x| < \frac{1}{2} \quad \forall\, n > n_0$$

if $n = \text{odd}$ $\quad |1 - x| < \frac{1}{2} \quad \forall\, n > n_0$

Now, $1 = |1 - x + x| \leq |1 - x| + |x|$

$$< \frac{1}{2} + \frac{1}{2}$$

$$< 1, \quad \forall\, n > n_0$$

which is impossible, $\therefore \lim_{n\to\infty} x_n$ does not exist in this case

5. Give an example of a sequence $\{x_n\}$ which is bounded but $\lim_{n\to\infty} \frac{x_n}{n} = 0$.

Solution:-

$x_n = \log n \ \forall\, n \in \mathbb{N}$.

then the sequence $\{x_n\}$ is unbounded.

and $\lim_{n\to\infty} \frac{x_n}{n} = \lim_{n\to\infty} \frac{\log n}{n}$

$$= 0.$$

6. Prove that a sequence $\{X_n\}$ bounded if $\lim_{n\to\infty} \frac{X_n}{n} = a \neq 0$.

Solution:- Since $\lim_{n\to\infty} \frac{x_n}{n} = a$ $[a \neq 0]$

Let for $0 < \epsilon < a$ $\exists$ $n_0 \in \mathbb{N}$.

$\Rightarrow -\epsilon < \frac{x_n}{n} - a < \epsilon \ \forall n > n_0$

$\Rightarrow a - \epsilon < \frac{x_n}{n} < a + \epsilon \ \forall n > n_0$

$\Rightarrow n(a-\epsilon) < x_n < n(a+\epsilon) \ \forall n > n_0$

$\Rightarrow \{X_n\}$ is a bounded sequence.

7. The sequence $\{1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \ldots, 2 - \frac{1}{2^{n-1}}, \ldots\}$ is —

(a) Bounded and decreasing
(b) Not bounded & not-decreasing.
(c) Not bounded and decreasing
(d) Bounded and not-decreasing.

Solution:- (D) Lower bound $= 1$, Upper bound $= 2$

since $s_{n+1} > s_n$

Hence given sequence is bounded is bounded and no-decreasing.

8. The sequence $\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots\}$ is

(a) Monotonic but not bounded
(b) Not monotonic & bounded.
(c) Monotonic & bounded
(d) Not-monotonic & not-bounded

Solution:- (B) Lower bound $= -\frac{1}{2}$,

Upper bound $= 1$.

But the sequence is neither decreasing non increasing.

9. Find the upper bound of the sequence $\left\{\frac{4n-1}{5n+2}\right\}_n$

Solution:-

$$\text{Upper bound} = \lim_{n\to\infty} \frac{4n-1}{5n+2}$$

$$= \lim_{n\to\infty} \frac{4 - 1/n}{5 + 2/n} = \frac{4}{5}.$$

10. What is the type of the following sequence $\left(\frac{n}{e^n}\right)_{n=1}^{\infty}$

Solution:-

$$u_n = \frac{n}{e^n}$$

$$u_{n+1} = \frac{(n+1)}{e^{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n}{e^n} \times \frac{e^{n+1}}{n+1} = e > 1.$$

$\therefore$ The given series is convergent.

11. Prove that $a_n = 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{(n-1)!}$ is monotonic and bounded.

Solution:-

$a_{n+1}-a_n = \frac{1}{n!} > 0 \ \forall n \in \mathbb{N}$

$\therefore \{a_n\}$ is monotonically increasing.

$0 \le a_n = 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{(n-1)!}$

$\le 1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-2}}$ $\quad [\because n! \geq 2^{n-1}]$

$= 1+\dfrac{1-\frac{1}{2^{n-1}}}{1-\frac{1}{2}}$

$\le 1+2 = 3.$

$\therefore \{a_n\}$ is bounded.

12. Let $a_1 = 1$ and $a_n = n(a_{n-1}+1) \ \forall n \geq 2$. Let $P_n = \left(1+\frac{1}{a_1}\right)\left(1+\frac{1}{a_2}\right)\cdots\left(1+\frac{1}{a_n}\right)$. Find $\lim_{n\to\infty} P_n$.

Solution:-

$a_n = n(a_{n-1}+1)$

$\therefore \frac{a_{n-1}+1}{a_n} = \frac{1}{n}$

$\therefore P_n = \frac{1}{a_1}\cdot\frac{a_1+1}{a_2}\cdot\frac{a_2+1}{a_3}\cdots\cdots\frac{a_{n-1}+1}{a_n}.(a_n+1)$

$\therefore P_n = \frac{1}{a_1}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdots\cdots\frac{1}{n-1}.(a_n+1)$

$\therefore \frac{a_n}{n!} = \frac{a_{n-1}}{(n-1)!}+\frac{1}{(n-1)!}$

$\frac{a_n}{n!} - \frac{a_{n-1}}{(n-1)!} = \frac{1}{(n-1)!}$

$\frac{a_{n-1}}{(n-1)!} - \frac{a_{n-2}}{(n-2)!} = \frac{1}{(n-2)!}$

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$\frac{a_2}{2!} - \frac{a_1}{1!} = \frac{1}{1!}$

Adding all these we have

$\Rightarrow \frac{a_n}{n!} - a_1 = \left\{\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{(n-1)!}\right\}$

$\Rightarrow \frac{a_n}{n!} = 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{(n-1)!}$

$\therefore P_n = \left(\frac{a_n}{n!}+\frac{1}{n!}\right) = 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!} \to e$ as $n\to\infty$.

• SERIES OF REAL NUMBERS •

Infinite Series :—

Definition:— If $\{u_n\}$ be a real sequence, then an expression of the form $u_1+u_2+\cdots+u_n+\cdots\cdots$ written as $\sum_{n=1}^{\infty} u_n$ or simply $\sum u_n$ is called an infinite series of real numbers.

Sequence of Partial sums:—

Let S_n denote the sum of first n terms of the series i.e., $S_n = u_1+u_2+\cdots+u_n$, Then $\{S_n\}$ is called the sequence of partial sums of the series $\sum u_n$.

Convergence of an Infinite series :—

An infinite series $\sum u_n$ is said to converge, diverge or oscillate according as $\{S_n\}$ converges, diverges or oscillates.

In case of convergence, if $\lim S_n = S$ then S is said to be the sum of the series $\sum u_n$.

If, however, $\lim S_n = \infty$ (or $-\infty$), the series $\sum u_n$ is said to diverge to ∞ (or $-\infty$).

Examples :— 1. The series $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots\cdots$ converges to $\frac{1}{1}$.

→ Let the series be $\sum_{n=1}^{\infty} u_n$, then $u_n = \frac{1}{n(n+1)}$.

Let $S_n = u_1+\cdots+u_n$.

Then
$$S_n = \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}$$
$$=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$
$$=1-\frac{1}{n+1}.$$

and $\lim S_n = 1$.

Hence the series $\sum u_n$ is convergent and the sum of the series is 1.

2. The series $1+2+3+\cdots\cdots$ diverges.

→ Let the series be $\sum_{n=1}^{\infty} u_n$, then $S_n = 1+2+\cdots+n = \frac{n(n+1)}{2}$.

$\lim S_n = \infty$.

Hence the series is divergent.

3. The series $1+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots$ converges to 2.

→ Let $S_n = 1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}$.
$$= 2\left(1-\frac{1}{2^n}\right)$$
$$= 2-\frac{1}{2^{n-1}}$$

and $\lim S_n = 2$

since $\lim \left(\frac{1}{2}\right)^{n-1} = 0$.

Therefore the series is convergent and the sum of the series is 2.

4. The series $1-1+1-1+\cdots\cdots$ is divergent.

→ Let $S_n = 1-1+1-1+\cdots\cdots+(-1)^{n-1}$

$$S_n = \begin{cases} 0 & \text{if } n \text{ be even} \\ 1 & \text{if } n \text{ be odd} \end{cases}$$

$\therefore$ The sequence $\{S_n\}$ is divergent. So, the series is divergent.

5. The series $1+a+a^2+\cdots$ is convergent when $|a|<1$.

→ Let $S_n = 1+a+a^2+\cdots+a^{n-1}$

$$= \frac{1-a^n}{1-a}$$

$$= \frac{1}{1-a} - \frac{a^n}{1-a}$$

$\lim S_n = \frac{1}{1-a}$, since $\lim a^n = 0$.

therefore the series is convergent and the sum of the series is

$$\frac{1}{1-a}.$$

<u>Note:-</u> The series $1+a+a^2+\cdots$ is divergent if $|a| \geq 1$.

6. The Harmonic series $1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots$ is divergent.

→ Let $\sum_{n=1}^{\infty} u_n$ be the series, then $u_n = \frac{1}{n}$.

Let, $S_n = u_1+u_2+\cdots+u_n$

then $S_2 = 1+\frac{1}{2}$

$S_4 = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} > 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right) = 1+2.\frac{1}{2}$

$S_8 = 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)$

$> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right) = 1+3.\frac{1}{2}$

$S_{16} > 1+4.\frac{1}{2}$

$\cdots\cdots\cdots$

$\cdots\cdots\cdots$

$S_{2^n} > 1+n.\frac{1}{2}$.

$\therefore \lim_{n\to\infty} S_{2^n} = \infty$

The sequence $\{S_n\}$ is a monotone increasing sequence since $S_{n+1}-S_n = a_{n+1} > 0 \ \forall n \in \mathbb{N}$. Since the subsequence $\{S_{2^n}\}$ diverges to ∞, the sequence $\{S_n\}$ is unbounded above and therefore the series $\sum_{n=1}^{\infty} u_n$ is divergent.

7. Check the convergence or divergence of the following series. Also find the sum of the following series if it converges:

i) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Sol. The n^{th} partial sum is $s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right)$

$= 1 - \frac{1}{n+1}$.

Let $\epsilon > 0$ be an arbitrary number.

Then $|s_n - 1| = \frac{1}{n+1} < \frac{1}{n} < \epsilon$

$\Rightarrow n > \frac{1}{\epsilon}$.

$N(\epsilon) = \left[\frac{1}{\epsilon}\right] + 1$.

Then $\exists$ a natural number $N(\epsilon)$ $\ni$ $|s_n - 1| < \epsilon \ \forall\ n \geq N(\epsilon)$.

$\therefore \lim(s_n) = 1$.

$\Rightarrow s_n$ converges to 1.

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and has the value 1.

ii) $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

Sol. $s_n = \sum_{k=1}^{n} \frac{1}{4k^2-1}$

$= \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)}$

$= \frac{1}{2} \sum_{k=1}^{n} \left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)$

$= \frac{1}{2}\left\{1 - \frac{2}{2n+1}\right\}$

$\therefore \lim_{n\to\infty}(s_n) = \frac{1}{2}\left\{1 - \lim_{n\to\infty} \frac{2}{2n+1}\right\} = \frac{1}{2}$.

Hence the series converges and has the sum $\frac{1}{2}$.

iii) $\sum_{n=1}^{\infty} r^{n-1}$; $|r| < 1$.

Sol. $s_n = \sum_{k=1}^{n} r^{k-1} = \frac{(1-r^n)}{(1-r)}$

$\therefore \lim s_n = \frac{1 - \lim_{n\to\infty} r^n}{1-r} = \frac{1}{1-r}$ $\left[\text{since } \lim_{n\to\infty} r^n = 0 \text{ when } |r| < 1\right]$

Hence the series $\sum_{n=1}^{\infty} r^{n-1}$ converges and has the sum $\frac{1}{1-r}$.

iv) $\sum_{n=1}^{\infty} n^2$

Sol.

$$s_n = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\lim_{n\to\infty} s_n = \infty.$$

$\therefore \{s_n\}$ diverges to $+\infty$.

$\Rightarrow$ The series $\sum_{n=1}^{\infty} n^2$ diverges and has no value.

8. The series $\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \cdots\cdots = \frac{1}{4}$.

Sol.

$$S_n = \frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \cdots\cdots + \frac{1}{n(n+1)(n+2)}$$

$$= \frac{1}{2}\left[\left(\frac{1}{1\cdot2} - \frac{1}{2\cdot3}\right) + \left(\frac{1}{2\cdot3} - \frac{1}{3\cdot4}\right) + \cdots + \left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{1\cdot2} - \frac{1}{(n+1)(n+2)}\right] \to \frac{1}{4} \text{ as } n\to\infty.$$

$\therefore \{S_n\}$ converges to $\frac{1}{4}$. Hence $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$.

9. The series $\log 2 + \log\frac{3}{2} + \log\frac{4}{3} + \cdots\cdots$ is divergent.

Sol.

$$S_n = \log 2 + \log\frac{3}{2} + \cdots\cdots + \log\frac{n+1}{n}$$

$$= \log\left(2 \times \frac{3}{2} \times \cdots\cdots \times \frac{n+1}{n}\right)$$

$$= \log(n+1) \longrightarrow \infty \text{ as } n\to\infty.$$

$\therefore \{S_n\}$ diverges. Hence the series diverges to $+\infty$.

10. The series $\sum_{n=1}^{\infty} (-1)^{n-1}$ oscillates finitely.

Sol. Here $S_n = 1 - 1 + 1 - 1 + \cdots\cdots$ to n terms

$S_n = 0$ or 1 according as n is even or odd.

$\therefore S_{2n} \to 0$ and $S_{2n+1} \to 1$ imply that $\{S_n\}$ is not convergent but oscillatory.

Moreover $|S_n| \le 1$.

$\therefore \{S_n\}$ oscillates finitely.

$\therefore \sum u_n$, where $u_n = (-1)^{n-1}$ oscillates finitely.

Theorem:- Let m be a natural number. Then the two series $u_1+u_2+u_3+\cdots$ and $u_{m+1}+u_{m+2}+\cdots\cdots$ converge or diverge together.

Proof:- Let $s_n = u_1+u_2+\cdots+u_n$;

$t_n = u_{m+1}+u_{m+2}+\cdots+u_{m+n}$,

Then $t_n = s_{m+n}-s_m$, where s_m is a fixed number.

If $\{s_n\}$ converges then $\{t_n\}$ converges and conversely.

If $\{s_n\}$ diverges then $\{t_n\}$ diverges and conversely.

Note:- The theorem states that we can remove from the beginning a finite number of terms from a given series, or add to the beginning a finite no. of terms to a given series without changing its behaviour regarding convergence or divergence.

Theorem:- If $\sum u_n$ and $\sum v_n$ be two convergent series having the sums s and t respectively then

(i) the series $\sum(u_n+v_n)$ converges to the sum $s+t$;

(ii) the series $\sum k u_n$, where k is a real number, converges to the sum ks.

Proof:- The proof is immediate.

Theorem (Cauchy's principle of convergence):-

A necessary and sufficient condition for the convergence of a series $\sum u_n$ is that corresponding to a pre-assigned positive ϵ $\exists$ a natural number $m \ni$

$|u_{n+1}+u_{n+2}+\cdots\cdots+u_{n+p}| < \epsilon \;\forall\; n \geq m$ and for every natural p.

Proof:- $\sum u_n$ converges iff $\{s_n\}$, where $s_n = \sum_{p=1}^{n} u_p$ converges.

Now, from the Cauchy's condition for convergence of a sequence, $\{s_n\}$ converges iff for any given $\epsilon(>0)$ $\exists$ a natural no. $m \ni$

$|s_{n+p}-s_n| < \epsilon$ holds for $n \geq m$ and $p = 1, 2, 3, \ldots\ldots$

i.e. $|u_{n+1}+u_{n+2}+\cdots\cdots+u_{n+p}| < \epsilon$ is true.

The proof is complete.

Definition:- The sequence $\{R_n\}$, where $R_n = u_{n+1}+u_{n+2}+\cdots\cdots$ is called the sequence of remainder after n terms of the series $\sum u_n$.

As an immediate result of the above theorem, we have:

Corollary:- The series $\sum u_n$ converges if $R_n \to 0$.

Proof:- $\sum u_n \to S$ implies that $s_n \to S$.

$\therefore$ From $S = s_n + R_n$, it follows that $R_n \to 0$.

$\therefore \sum u_n$ converges.

Theorem:- (n^{th} term test)

A necessary condition for the convergence of a series $\sum u_n$ is $\lim u_n = 0$.

i.e. If $\sum_{n=1}^{\infty} u_n$ converges then $\lim_{n\to\infty} u_n = 0$.

Proof:- Let $s_n = u_1 + u_2 + \cdots + u_n$ so that $\{s_n\}$ is the sequence of partial sums.

$$s_{n+1} = a_1 + a_2 + \cdots + a_n + a_{n+1}$$
$$s_n = a_1 + a_2 + \cdots + a_n$$

$$s_{n+1} - s_n = a_{n+1}$$

$$\therefore a_n = s_n - s_{n-1}$$

Let, $\lim_{n\to\infty}(s_n) = s$; $\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}(s_n) - \lim_{n\to\infty}(s_{n-1})$
$= s - s$
$= 0$.

Note:- The converse of the theorem is not true.

Corollary:- if $\lim(u_n) \neq 0$, then $\sum_{n=1}^{\infty} u_n$ diverges.

Example:- (i) Let a series be $\sum \cos\left(\frac{\pi}{n}\right)$,

we are going to check the convergence or divergence of the following series by calculating its limiting value.

$$\therefore \lim_{n\to\infty}\left(\cos\frac{\pi}{n}\right) = \cos 0 = 1 \neq 0$$

By using the theorem, we can say that $\sum \cos\left(\frac{\pi}{n}\right)$ diverges

(ii) It must be clearly understood that $\lim u_n = 0$ does not prove that a series is convergent.

e.g. We consider the series $\sum \frac{1}{n} = \sum u_n$, where $u_n = \frac{1}{n} \to 0$ as $n \to \infty$.

But $\sum \frac{1}{n}$ is divergent. [Ex. 6]

(iii) If $\lim_{n\to\infty} u_n \neq 0$, then we conclude that $\sum u_n$ is not convergent.

e.g. Consider the series $\sum \frac{n^n}{n!}$,

Here $u_n = \frac{n^n}{n!} = \frac{n.n.\cdots.n}{1.2.\cdots.n} = \left(\frac{n}{1}\right)\left(\frac{n}{2}\right)\cdots\left(\frac{n}{n}\right)$

whenever $n > 1$, $u_n > 1$, and therefore, cannot tend to zero.

$\therefore \sum u_n$ is not convergent.

WORKED EXAMPLES ON Cauchy's Principle of convergence :—

1) Prove that the series $\sum \frac{1}{n}$ does not converge.

Sol. $\sum u_n = \sum \frac{1}{n}$

$\therefore \quad |S_{n+p} - S_n| = \left| \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p} \right| < \epsilon \quad \forall\, n \geq m,\ p \geq 1$

If $n = m$ and $p = m$, then we get,

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p} = \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m}$$

$$> m \cdot \frac{1}{2m} = \frac{1}{2} > \epsilon$$

Thus, there is a contradiction, the series does not converge.

2) Prove that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots$ is convergent.

Sol. $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n}$.

Let $S_n = u_1 + u_2 + \cdots\cdots + u_n$.

$\therefore \quad |S_{n+p} - S_n| = \left| \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} \cdots\cdots + (-1)^{p-1} \cdot \frac{1}{n+p} \right|$

$$= \left| \frac{1}{n+1} - \left(\frac{1}{n+2} - \frac{1}{n+3} \right) \cdots\cdots \right|$$

$$< \frac{1}{n+1} \quad \text{(each term in the bracket is positive)}$$

Let $\epsilon > 0$, then $|S_{n+p} - S_n| < \epsilon$ holds if $n > \frac{1}{\epsilon} - 1$.

$\therefore \; m = \left[\frac{1}{\epsilon} - 1 \right] + 2$. Then m is a natural no. and

$|S_{n+p} - S_n| < \epsilon \;\; \forall\, n \geq m$ and $\forall\; p = 1, 2, 3, \ldots\ldots$

this proves that the sequence $\{S_n\}$ is convergent and consequently the series $\sum u_n$ is convergent.

3) Prove that the series $\sum_{n=1}^{\infty} u_n$, where $u_n = \frac{n}{n+1}$, is divergent.

Sol. Here $\lim u_n = 1$, Since $\lim u_n$ is not zero, $\sum u_n$ is divergent.

4) Prove that the series $1 + \frac{1}{3} + \frac{1}{5} + \cdots\cdots + \frac{1}{2n-1} + \cdots\cdots$ diverges.

Sol. Taking $\epsilon = \frac{1}{3}$,

$$\left| \frac{1}{2n+1} + \frac{1}{2n+3} + \cdots\cdots + \frac{1}{2n+2p-1} \right| < \epsilon = \frac{1}{3}$$

putting $p = n = m$.

$$\therefore \left| \frac{1}{2n+1} + \frac{1}{2n+3} + \cdots\cdots + \frac{1}{2n+2p-1} \right| = \frac{1}{2m+1} + \frac{1}{2m+3} + \cdots + \frac{1}{4m-1}$$

$$> \frac{m}{4m-1}$$

$$\therefore \frac{m}{4m-1} < \left| \frac{1}{2m+1} + \frac{1}{2m+3} + \cdots\cdots + \frac{1}{4m-1} \right| < \frac{1}{3}.$$

$$\Rightarrow m < -\frac{1}{4} \quad (\text{contradicting that } m \in \mathbb{N}.)$$

$\therefore$ the series is not convergent.

Series of Non-negative items:— If $a_n \geq 0$, $n \in \mathbb{N}$, then the series $\sum_{n=1}^{\infty} a_n$ is a series of non-negative terms.

Theorem:— A series of non-negative terms $\sum_{n=1}^{\infty} a_n$ converges iff the sequence $\{s_n\}$ of partial sums is bounded above.

Proof:— Let $s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$

$$s_{n-1} = a_1 + a_2 + \cdots + a_{n-1}$$

$$s_n - s_{n-1} = a_n \geq 0 \ \forall n$$

$\Rightarrow \{s_n\}$ is increasing.

Hence, $\{s_n\}$ converges iff it is bounded above.

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges iff $\{s_n\}$ is bounded above.

Note:— A series $\sum_{n=1}^{\infty} a_n$ of non-negative terms either converge or diverge to $+\infty$. If $\sum a_n$ is not bounded above then it ~~can~~ diverges to $+\infty$.

Example 1:— Establish the convergence or divergence of the following series:

(a) $\sum_{n=1}^{\infty} \frac{1}{n!}$

Solution:— Here, $s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \frac{1}{k!}$

$\therefore s_n - s_{n-1} = \frac{1}{n!} \geq 0 \ \forall n \geq 2.$

$\Rightarrow \{s_n\}$ is increasing.

Now, $s_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$ $\quad [\because n! = 1.2.3\ldots n \geq 2^{n-1} \ \forall n \in \mathbb{N}$

$$< 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$$

$$= \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 2\left(1 - \frac{1}{2^n}\right) < 2$$

$\therefore 1 = s_1 \leq s_n \leq 2 \ \forall n \in \mathbb{N}.$

$\Rightarrow \{s_n\}$ is bounded above.

Hence, $\{s_n\}$ is convergent.

~~Therefore~~ $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

(b) Powers series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p \in \mathbb{R}$. .

[$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$ and diverges iff $p \leq 1$.]

Solution:-

Case I:- $\boxed{p > 1}$

Note that, $s_n = \sum_{k=1}^{n} \frac{1}{k^p}$, $n \in \mathbb{N}$.

$\therefore s_n - s_{n-1} = \frac{1}{n^p} > 0 \quad \forall n \in \mathbb{N}$.

$\Rightarrow \{s_n\}$ is increasing.

Now, $s_{2^n-1} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots\cdots + \frac{1}{(2^n-1)^p}$

$$\therefore S_m = 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \cdots\cdots$$

$$+ \left\{\frac{1}{(2^{n-1})^p} + \frac{1}{(2^{n-1}+1)^p} + \cdots\cdots + \frac{1}{(2^n-1)^p}\right\}$$

$$\leq 1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) + \cdots\cdots$$

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^{p-1})^2} + \cdots\cdots$$

$$= 1 + a + a^2 + \cdots\cdots \; ; \text{ where } a = \frac{1}{2^{p-1}} < 1 \text{ if } p > 1.$$

$\therefore 0 < a^n < 1$.

$$\therefore S_m \leq 1 + a + a^2 + \cdots\cdots + a^{n-1} + \cdots\cdots$$

$$= \frac{1-a^n}{1-a} < \frac{1}{1-a} = \text{constant}.$$

$\therefore \{S_m\}$ is bounded and obviously monotonically increasing $\forall p > 1$

$\Leftrightarrow \{s_n\}$ " " " " " " $\forall p > 1$.

$\Leftrightarrow \{s_n\}$ converges if $p > 1$.

$\Leftrightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$.

Case-II :- $\boxed{p \leq 1}$

Note that $n^p \leq n \quad \forall n \in \mathbb{N}$.

$\Rightarrow \frac{1}{n^p} \geq \frac{1}{n} \quad \forall n \in \mathbb{N}$.

Now, $s_{2^n} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots\cdots + \frac{1}{(2^n)^p}$

$$= 1 + \frac{1}{2^p} + \left(\frac{1}{3^p} + \frac{1}{4^p}\right) + \left(\frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \frac{1}{8^p}\right) + \cdots\cdots$$

$$+ \left\{\frac{1}{(2^{n-1}+1)^p} + \cdots\cdots + \frac{1}{(2^n)^p}\right\}$$

$$\geq 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \cdots\cdots + \frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \cdots\cdots + \frac{1}{2^n}$$

$$\geq 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + \cdots\cdots + 2^{n-1} \cdot \frac{1}{2^n}$$

$\therefore s_{2^n} \geq 1 + n/2$ $\therefore \{s_n\}$ is unbounded above and diverges to $+\infty$.

$\therefore \sum \frac{1}{n^p}$ diverges iff $p \leq 1$.

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(c) Prove that the series $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\cdots\cdots$ is convergent when $p>2$ and diverges when $p\le 2$.

Solution:- The series can be written as $\sum \frac{n+1}{n^p}$

$$= \sum \frac{1}{n^{p-1}} + \sum \frac{1}{n^p}.$$

From the previous problem, we know,

$\sum \frac{1}{n^p}$ converges if $p>1$.

& $\sum \frac{1}{n^{p-1}}$ converges if $p-1>1$ i.e. $p>2$.

Hence $\sum \frac{n+1}{n^p}$ converges if $p>2$ and diverges if $p\le 2$.

Example 2:- If $\sum u_n$ be a convergent series of positive real numbers prove that $\sum u_n^2$ is convergent.

Solution:- $\{s_n\}$ is bounded, where $s_n = u_1+\cdots+u_n$.

$\sum u_n$ is convergent implies $\lim_{n\to\infty} u_n = 0$.

For $\epsilon=1$, $\exists$ an integer $m\in\mathbb{N}$ $\ni$ $u_n<1$ $\forall$ $n\geq m$.

$\therefore u_n^2 < u_n$ $\forall$ $n>m$.

$\{s_n\}$ is bounded.

$\therefore s_n = u_1+u_2+\cdots\cdots+u_n \le K$ for some real no. K.

Now $s_n' = u_1^2+u_2^2+\cdots+u_m^2+u_{m+1}^2+\cdots+u_n^2$

$< u_1^2+\cdots+u_m^2+u_{m+1}+\cdots+u_n$

$< r+K-(u_1+\cdots+u_n)$

$=M$, say

$\therefore \{s_n'\}$ sequence of the partial sums of $\sum u_n^2$ is bounded above.

Hence $\sum u_n^2$ is convergent.

Example 3:- If $\sum a_n$ be a convergent series of positive real nos., will the series $\sum a_{2n}$ be convergent? Give reason.

Solution:- $\sum a_n$ is convergent $\Rightarrow$ $\{s_n\}$ is bounded above.

$\therefore$ There is a real number K $\ni$ $a_1+\cdots+a_{2n}\le K$

or, $a_2+a_4+\cdots+a_{2n-2}+a_{2n} \le K-(a_1+a_3+\cdots+a_{2n-1})$ $= r$ (say)

$= K-r$, say.

$\therefore \{s_n'\}$ be the sequence of partial sums of the series $\sum a_{2n}$ is bounded above.

$\therefore \sum a_{2n}$ is convergent.

A necessary condition for Convergence of Positive Term Series :—

Theorem (Abel - Pringsheim's Theorem) :—
If a series $\sum u_n$ of positive monotonic decreasing terms converges then not only $u_n \to 0$ but also $nu_n \to 0$ as $n \to \infty$.

Proof :— We know that for a convergent series, for any $\epsilon > 0$, a positive integer N exists $\ni$

$$\left| u_{m+1} + u_{m+2} + \cdots\cdots + u_{m+p} \right| < \frac{\epsilon}{2} \quad \forall\, m \geq N,\ p \geq 1$$

Let us choose

$$m + p = n > 2N$$

and

$$m = \left[\frac{n}{2}\right] \leq \frac{n}{2}.$$

$$\therefore\ u_{m+1} + u_{m+2} + \cdots\cdots + u_n < \frac{\epsilon}{2}$$

But $\sum u_n$ is positive monotonic decreasing,

$$(n-m)u_n < u_{m+1} + u_{m+2} + \cdots\cdots + u_n < \epsilon/2$$

$$\text{or, } \therefore\ \frac{1}{2} nu_n < \frac{1}{2}\epsilon$$

$$\therefore\ nu_n < \epsilon \quad \forall\, n \geq N.$$

$$\therefore\ \lim_{n\to\infty} nu_n = 0$$

Notes :— 1. The condition $u_n \to 0$ holds for all types of convergent series but a convergent series of positive monotonic decreasing terms satisfies the additional condition $nu_n \to 0$.

2. The condition $nu_n \to 0$ is only a necessary not a sufficient condition for the convergence of the present type of series. If nu_n does not tend to zero then the series $\sum u_n$ is certainly divergent, e.g., the harmonic series $\sum \frac{1}{n}$ must diverge because it has positive monotonic decreasing terms and $n \cdot \frac{1}{n}$ does not tend to zero. However, $nu_n \to 0$ does not imply anything as to the possible convergence of $\sum u_n$.

Ex. 1 :— $\sum \frac{1}{2n+5}$ is divergent, but $\sum \frac{1}{2n^2+5n}$ is convergent.

Solution :— $u_n = \frac{1}{2n+5}$ is monotone decreasing

But $\lim_{n\to\infty} nu_n = \frac{1}{2} \neq 0$. $\therefore \sum \frac{1}{2n+5}$ is divergent.

In $\sum \frac{1}{2n^2+5n}$, $\lim nu_n = \lim_{n\to\infty} \frac{1}{2n+5} = 0$. (Some other test is necessary)

Ex. 2 :— $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots\cdots$ is divergent.

Solution :— Here $u_n = \frac{1}{\sqrt{n}}$ $\therefore\ nu_n = \sqrt{n} \to \infty$ as $n \to \infty$.

$\therefore\ nu_n$ does not tend to zero as $n \to \infty$.

$\therefore\ \sum \frac{1}{\sqrt{n}}$ is not convergent.

TEST FOR CONVERGENCE OF A SERIES OF POSITIVE TERMS :-

(I) Comparison Test:-

If $\sum a_n$ and $\sum b_n$ are two series of ~~two~~ positive terms and $a_n \leq k.b_n$ for $n \geq m$, then

(a) $\sum a_n$ converges if $\sum b_n$ converges.

(b) $\sum b_n$ diverges if $\sum a_n$ diverges.

Examples:- Test the convergence or divergence of the following series:

(i) $\sum_{n=1}^{\infty} \frac{1}{n!}$

Sol. Note, $0 < \frac{1}{n!} \leq \frac{1}{2^{n-1}}$

since $\sum \frac{1}{2^{n-1}}$ convergent (geometric series) so by comparison test $\sum \frac{1}{n!}$ converges.

(ii) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^p}$, $p>0$.

Sol. $(\log n)^p < n \quad \forall\ n>1$

$\Rightarrow \frac{1}{(\log n)^p} > \frac{1}{n} \quad \forall\ n>1$

Let us compare the series with the divergent series

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\cdots$$

So, By comparison test the series $\sum_{n=2}^{\infty}\frac{1}{(\log n)^p}$ is divergent. $(p>0)$

(iii) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$

Sol. $$\frac{1}{2\sqrt{n+1}} < \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

Since $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}} = \sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}$ is divergent. (Power series with $p=\frac{1}{2}$)

$\therefore$ By comparison test $\sum \frac{1}{\sqrt{n+1}+\sqrt{n}}$ diverges.

(iv) $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ $(x>0)$

Sol. For any real x, $\sin nx \leq 1$

$$\therefore \frac{\sin nx}{n^2} \leq \frac{1}{n^2}$$

We know $\sum \frac{1}{n^2}$ is convergent.

so, by comparison test $\sum \frac{\sin nx}{n^2}$ is convergent.

(v) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1000}}$

Sol. $\dfrac{1}{\sqrt{2n}} \leq \dfrac{1}{\sqrt{n+1000}} \;\forall\; n \geqslant 1000$

i.e. $\dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{\sqrt{n}} \leq \dfrac{1}{\sqrt{n+1000}} \;\forall\; n \geqslant 1000$

Now, $\sum \frac{1}{\sqrt{n}}$ diverges, $\therefore$ by comparison test, $\sum \frac{1}{\sqrt{n+1000}}$ diverges

(vi) $\displaystyle\sum_{n=1}^{\infty} \frac{|\cos nx|}{n^2}$

Sol. Note, $|\cos nx| \leq 1$, for real x.

$$\Rightarrow \frac{|\cos nx|}{n^2} \leq \frac{1}{n^2} \;\forall\; n \in \mathbb{N}$$

Note that $\sum \frac{1}{n^2}$ converges, so by comparison test $\sum \frac{|\cos nx|}{n^2}$ converge

(vii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

Sol. $0 \leq \dfrac{1}{n^2+n} < \dfrac{1}{n^2} \;\forall\; n \in \mathbb{N}$

Since $\sum \frac{1}{n^2}$ converges, $\therefore$ by comparison test $\sum \frac{1}{n^2+n}$ converges.

(II) Limit Comparison Test :—

If $\sum a_n$ and $\sum b_n$ are two series of positive terms and $\lim\left(\frac{a_n}{b_n}\right) = l \neq 0$, then both the series $\sum a_n$ and $\sum b_n$ converge or diverge simultaneously.

Note:— 1. If $\lim_{n\to\infty} \frac{a_n}{b_n} \neq 0$, then we say $a_n \sim b_n$ for large n.

Then $\sum a_n$ and $\sum b_n$ have similar nature of convergence or divergence.

Note that, (i) $\frac{n^p}{(n+k)^q} \sim \frac{1}{n^{q-p}}$ (ii) $\frac{n^p}{(an+b)^q} \sim \frac{1}{n^{q-p}}$,

(iii) $\sin\frac{1}{n} \sim \frac{1}{n}$ (iv) $a^{1/n} - 1 \sim \frac{1}{n}$

2. If $\lim \frac{u_n}{v_n} = 0$, then $\sum u_n$ is convergent if $\sum v_n$ is convergent.

3. If $\lim \frac{u_n}{v_n} = \infty$, then $\sum u_n$ is divergent if $\sum v_n$ is divergent.

Examples:— Test the convergence or divergence of the following series:

(i) $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

Sol. Let $a_n = \frac{1}{n^2+n}$, $n \in \mathbb{N}$.

Taking $b_n = \frac{1}{n^2}$, $n \in \mathbb{N}$.

$$\lim\left(\frac{a_n}{b_n}\right) = \lim\left(\frac{n^2}{n^2+n}\right) = \lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)$$
$$= 1 \neq 0.$$

Since $\sum b_n$ converges so does $\sum a_n$, by Limit comparison test.

$\therefore \sum \frac{1}{n^2+n}$ converges.

(ii) $\sum_{n=1}^{\infty} \frac{n^2+n}{n^4+\sqrt{n}}$

Sol. Let $a_n = \frac{n^2+n}{n^4+\sqrt{n}}$, $n \in \mathbb{N}$

Take $b_n = \frac{1}{n^2}$, $n \in \mathbb{N}$.

Now, $\lim\left(\frac{a_n}{b_n}\right) = \lim\left(\frac{n^2+n}{n^4+\sqrt{n}} \times n^2\right) = \lim_{n\to\infty}\left(\frac{n^4+n^2}{n^4+\sqrt{n}}\right) = \lim_{n\to\infty}\left(\frac{1+\frac{1}{n^2}}{1+\frac{1}{n^{7/2}}}\right)$

$$= 1 \neq 0.$$

Since $\sum b_n$ converges, so does $\sum a_n$, by limit comparison test.

$\therefore \sum_{n=1}^{\infty} \frac{n^2+n}{n^4+\sqrt{n}}$ converges.

(iii) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3+1}$

Sol. Let $a_n = \frac{\sqrt{n}}{n^3+1}$, $n \in \mathbb{N}$

$a_n = \frac{n^{1/2}}{n^3+1}$

Take $b_n = 1/n^{5/2}$, $n \in \mathbb{N}$

$\therefore \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \frac{n^3}{n^3+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^3}} = 1 \neq 0.$

As $\sum b_n$ converges, so does $\sum a_n$.

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(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+\sqrt{n}}$

Sol. Let $a_n = \dfrac{n}{n^2+\sqrt{n}} = \dfrac{\sqrt{n}}{n^{3/2}+1}$

Taking $b_n = \dfrac{1}{n}$

$\therefore \lim_{n\to\infty}\left(\dfrac{a_n}{b_n}\right) = \lim_{n\to\infty} \dfrac{n^{3/2}}{n^{3/2}+1} = \lim_{n\to\infty} \dfrac{1}{1+\frac{1}{n^{3/2}}} = 1 \neq 0.$

As $\sum b_n$ diverges, so does $\sum a_n$.

$\therefore \sum a_n$ is divergent.

(v) $\dfrac{1}{2\cdot3\cdot4} + \dfrac{3}{3\cdot4\cdot5} + \cdots\cdots$

Sol. Let $a_n = \dfrac{2n-1}{(n+1)(n+2)(n+3)}$

Taking $b_n = \dfrac{1}{n^2}$

$\therefore \lim \dfrac{a_n}{b_n} = \lim \dfrac{n^2(2n-1)}{(n+1)(n+2)(n+3)} = \lim \dfrac{n^2(2n-1)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)}$

$= \lim_{n\to\infty} \dfrac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)}$

$= \dfrac{2-0}{(1+0)(1+0)(1+0)}$

$= 2 \neq 0$

Since $\sum b_n$ converges, $\therefore$ by Limit comparison test $\sum a_n$ is convergent.

(vi) $\dfrac{1+2}{2^3} + \dfrac{1+2+3}{3^3} + \dfrac{1+2+3+4}{4^3} + \cdots\cdots$

Sol. Let $a_n = \dfrac{(n+1)(n+2)}{2(n+1)^3} = \dfrac{n+2}{2(n+1)^2}$

Taking $b_n = \dfrac{1}{n}$

$\therefore \lim\left(\dfrac{a_n}{b_n}\right) = \lim \dfrac{n(n+2)}{2(n+1)^2} = \dfrac{1}{2} \neq 0$

As $\sum b_n$ is divergent, $\sum a_n$ is divergent by Limit comparison test.

(vii) $\dfrac{1}{1\cdot2^2} + \dfrac{1}{2\cdot3^2} + \dfrac{1}{3\cdot4^2} + \cdots\cdots$

Sol. Let $a_n = \dfrac{1}{n(n+1)^2}$ $\quad\therefore \lim \dfrac{a_n}{b_n} = 1.$

Taking $b_n = \dfrac{1}{n^3}$. $\quad\therefore \sum a_n$ is convergent since $\sum b_n$ is convergent.

(viii) $\dfrac{1.2}{3^2.4^2} + \dfrac{3.4}{5^2.6^2} + \dfrac{5.6}{7^2.8^2} + \cdots\cdots$

Sol. Let $a_n = \dfrac{(2n-1)(2n)}{(2n+1)^2(2n+2)^2}$

Taking $b_n = \dfrac{1}{n^2}$ $\quad\therefore \lim \dfrac{a_n}{b_n} = \lim \dfrac{\left(2-\frac{1}{n}\right)(2)}{\left(2+\frac{1}{n}\right)^2\left(2+\frac{2}{n}\right)^2} = \dfrac{1}{4}$.

Since $\sum b_n$ converges so does $\sum a_n$.

(ix) $\sum_{n=1}^{\infty} \sin\frac{1}{n}$

Sol. Let $a_n = \sin\frac{1}{n}$, $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1 \neq 0.$$

Since $\sum b_n$ diverges, so does $\sum a_n$.

(x) $\sum_{n=1}^{\infty} \frac{1}{n}\sin\frac{1}{n}$

Sol. $a_n = \frac{1}{n}\sin\frac{1}{n}$

Choose $b_n = \frac{1}{n^2}$

$$\therefore \lim \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{1/n\to 0} \frac{\sin(1/n)}{(1/n)} = 1 \neq 0$$

Now $\sum \frac{1}{n^2}$ is convergent. $\therefore \sum a_n$ is convergent.

(xi) $\sum_{n=1}^{\infty} (a^{1/n} - 1)$

Sol. $a_n = a^{1/n} - 1$

Take, $b_n = 1/n$

$$\therefore \lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \left(\frac{a^{1/n}-1}{1/n}\right) = \lim_{x\to 0} \left(\frac{a^x - 1}{x}\right), \quad x = 1/n \to 0$$

$$= \log_e a \neq 0$$

Since $\sum \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} (a^{1/n}-1)$ diverges by Limit Comparison Test.

(xii) $\sum_{n=1}^{\infty} \{(n^3+1)^{1/3} - n\}$

Sol.
$$a_n = (n^3+1)^{1/3} - n$$
$$= n\left\{\left(1+\frac{1}{n^3}\right)^{1/3} - 1\right\}$$
$$= n\left\{\left(1 + \frac{1}{3n^3} + \cdots\cdots\right) - 1\right\}$$
$$= \frac{1}{3n^2}$$

Taking $b_n = \frac{1}{n^2}$

$$\therefore \lim\left(\frac{a_n}{b_n}\right) = \frac{1}{3} \neq 0$$

Since $\sum b_n$ converges, so does $\sum a_n$.

(xiii) $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

Sol. $a_n = \frac{1}{n^{1+1/n}}$, $\quad \lim \frac{a_n}{b_n} = \lim\left(\frac{1}{n^{1/n}}\right) = \frac{1}{n^0} = 1 \neq 0$

$b_n = \frac{1}{n}$ (Let),

Hence the two series $\sum a_n$ and $\sum b_n$ behave alike.

Since $\sum b_n$ is divergent, therefore, $\sum \frac{1}{n^{1+1/n}}$ is also divergent.

(xiv) $\sum_{n=1}^{\infty}\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$

<u>Sol.</u> $a_n=\sqrt{n^4+1}-\sqrt{n^4-1}=\dfrac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}=\dfrac{2}{n^2\left(\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}\right)}$

Consider $b_n=\dfrac{1}{n^2}$

$\therefore \lim \dfrac{a_n}{b_n}=\lim\limits_{n\to\infty}\dfrac{2}{\left(\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}\right)}$

$=1\neq 0$

Since $\sum\frac{1}{n^2}$ is convergent, $\sum a_n$ is so.

(xv) $\sum_{n=1}^{\infty}\left(\sqrt{n^3+1}-\sqrt{n^3}\right)$

<u>Sol.</u> $a_n=n^{3/2}\left(\sqrt{1+\frac{1}{n^3}}-1\right)$

$=n^{3/2}\left[\frac{1}{2n^3}-\cdots\cdots\right]$

$=\dfrac{1}{2n^{3/2}}$, as $n\to\infty$

Take $b_n=\dfrac{1}{n^{3/2}}$

$\therefore \lim\left(\frac{a_n}{b_n}\right)=\lim\left(\frac{1}{2}\right)=\frac{1}{2}\neq 0.$

But the series $\sum b_n$ is convergent. $\therefore \sum a_n$ is convergent.

(xvi) $\dfrac{5}{2\cdot 2\cdot 4}+\dfrac{7}{4\cdot 3\cdot 5}+\dfrac{9}{6\cdot 4\cdot 6}+\cdots\cdots$

<u>Sol.</u> Let $a_n=\dfrac{(2n+3)}{(2n)(n+1)(n+3)}$

Consider $b_n=\dfrac{1}{n^2}$

$\therefore \dfrac{a_n}{b_n}=\dfrac{n^2(2n+3)}{2n(n+1)(n+3)}=\dfrac{n^2\left(2+\frac{3}{n}\right)}{2n^2\left(1+\frac{1}{n}\right)\left(1+\frac{3}{n}\right)}=\dfrac{\left(2+\frac{3}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{n}\right)}$

$\therefore \lim\left(\frac{a_n}{b_n}\right)=1\neq 0$

Since $\sum\frac{1}{n^2}$ is convergent, so the given series is also convergent.

(xvii) $\sum_{n=1}^{\infty} \frac{n^p}{(n+1)^{p+\alpha}}$ $(\alpha > 1)$

Sol. let $a_n = \frac{n^p}{(n+1)^{p+\alpha}}$

Taking $b_n = 1/n^\alpha$.

$\therefore \lim \left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \frac{n^{p+\alpha}}{(n+1)^{p+\alpha}} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{p+\alpha}} = 1 \neq 0.$

As $\sum b_n$ converges if $\alpha > 1$, so $\sum a_n$ is convergent for $\alpha > 1$.

(xviii) If $\sum a_n$ is convergent where $a_n > 0$. Prove that $\sum \frac{a_n}{n}$ is convergent.

Solution:— We have seen that if $\sum a_n$ is convergent, then $\sum a_n^2$ is also convergent.

Again $\sum \frac{1}{n^2}$ is convergent.

$\therefore \sum \left(a_n^2 + \frac{1}{n^2}\right)$ is convergent.

We have by the property, AM $\geq$ GM.

$$\therefore \frac{a_n^2 + \frac{1}{n^2}}{2} > \frac{a_n}{n}$$

$\therefore$ Each term of the series $\sum \frac{a_n}{n}$ is less than the corresponding term of a convergent series.

$\therefore \sum \frac{a_n}{n}$ is convergent.

(xix) If $\sum u_n^2$ and $\sum v_n^2$ are both convergent, then the series $\sum u_n v_n$ is also convergent.

Sol. We have $\frac{u_n^2 + v_n^2}{2} > u_n v_n$

As $\sum \frac{u_n^2 + v_n^2}{2}$ is a convergent series, so by Comparison test, $\sum u_n v_n$ will also converge.

(xx) Is the series $\sum_{n=1}^{\infty} \frac{1}{n^2 - n}$ convergent ?

Sol. Here $u_n = \frac{1}{n^2 - n}$

and let $v_n = \frac{1}{n^2}$

$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{n}} = 1 \neq 0$

By Limit Comparison test, as $\sum v_n$ is convergent, so does $\sum u_n$.

$\therefore \sum u_n$ is also convergent.

(III) D'Alembert's Ratio Test :— If $\sum a_n$ is a positive term series, such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$, then the series:

(i) converges if $l < 1$.

(ii) diverges if $l > 1$.

(iii) the test fails if $l = 1$.

Proof:- Hints:— Let $l<1$, and $\epsilon > 0$ be an arbitrary quantity.

$$l+\epsilon < 1.$$

By defn. of limit, $\frac{a_{n+1}}{u_n} - l < \epsilon \quad \forall\ n \geq N(\epsilon)$

$$\Rightarrow \frac{a_{n+1}}{a_n} < l+\epsilon \quad \forall\ n \geq N(\epsilon)$$
$$= r\,(<1), \text{say}.$$

Now, $$\frac{a_K}{a_{N(\epsilon)}} = \frac{a_K}{a_{K-1}} \cdot \frac{a_{K-1}}{a_{K-2}} \cdots\cdots \frac{a_{N(\epsilon)+1}}{a_{N(\epsilon)}}$$

$$< r^{K-N(\epsilon)}$$

$$\Rightarrow a_K < r^K \cdot \left[\frac{a_{N(\epsilon)}}{r^{N(\epsilon)}}\right]$$

Since $\sum r^K$ converges when $r < 1$. So, $\sum a_n$ is also convergent when $l < 1$.

Let $l > 1$, $\epsilon > 0$

$$l - \epsilon > 1$$

Since $\lim \frac{a_{n+1}}{a_n} = l$, then $\exists$ K such that

$$p = l-\epsilon < \frac{a_{n+1}}{a_n} < l+\epsilon \quad \forall\ n \geq K.$$

We have $\frac{a_{K+1}}{a_K} > p$, $\frac{a_{K+2}}{a_{K+1}} > p$, ..., $\frac{a_n}{a_{n-1}} > p$, where $n > K$

Consequently $$\frac{a_n}{a_K} > p^{n-K} \quad \forall\ n > K$$

$\frac{a_K}{p^K}$ is a positive no. and $\sum p^n$ is a geometric series with $p>1$,

$\therefore \sum p^n$ is divergent, so $\sum a_n$ is also divergent by limit comparison test.

Ex. Test the convergence or divergence of the following series.

(i) $\sum \frac{(n!)^2}{(2n)!}$

Sol. $a_n = \frac{(n!)^2}{(2n)!}$, $n \in \mathbb{N}$.

Now, $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} \frac{\{(n+1)!\}^2}{(2n+2)!} \times \frac{(2n)!}{(n!)^2}$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty} \frac{(n+1)}{2(2n+1)}$$

$$= \frac{1}{4} < 1.$$

Here, $l < 1$, so $\sum a_n$ converges.

(ii) $2x + \frac{3x^2}{8} + \frac{4x^3}{27} + \cdots + \frac{(n+1)x^n}{n^3} + \cdots\cdots$

Sol. $a_n = \frac{(n+1)}{n^3}x^n$, $a_{n+1} = \frac{(n+2)}{(n+1)^3}.x^{n+1}$

Now, $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) \cancel{= \lim_{n\to\infty} \frac{(n+1)^4}{(n+2)n^3}\cdot\frac{1}{x}}$

$$= \lim_{n\to\infty} \frac{n^3(n+2)}{(n+1)^4}.x$$

$$= x.$$

Hence the series is convergent if $x < 1$ and divergent if $x > 1$.

If $x = 1$, then the test fails.

So, we have to apply limit comparison test.

Let $a_n = \frac{(n+1)}{n^3}$

Consider $b_n = \frac{1}{n^2}$

$\therefore \lim \frac{a_n}{b_n} = \lim \frac{(n+1)}{n}$

$= 1 \neq 0$

$\therefore$ As $\sum b_n$ is convergent, so does $\sum a_n$.

$\therefore$ thus the series $\sum a_n$ is convergent if $x \leq 1$ and divergent if $x > 1$.

(iii) $1 + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \cdots\cdots$

Sol. $\sum_{n=1}^{\infty} u_n$, where $u_n = \frac{2n-1}{n!}$ $\therefore \frac{u_{n+1}}{u_n} = \frac{2n+1}{(n+1)(2n-1)}$

$\therefore \lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = 0 < 1.$

$\therefore$ By D'Alembert's Ratio test $\sum u_n$ is convergent.

(iii) $x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \cdots\cdots$

Solution:— Let u_n denote the nth term of the series, then

$$u_n = \frac{1.3.5.\ldots(2n-3)}{2.4.6.\ldots(2n-2)}\cdot\frac{x^{2n-1}}{2n-1}, \; n>1.$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{(2n-1)^2}{2n(2n+1)}x^2$$

$$\therefore \lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\frac{(2-\frac{1}{n})^2}{2(2+\frac{1}{n})}x^2 = x^2.$$

$\therefore \sum u_n$ converges if $x<1$ and diverges if $x>1$.

Further investigation is required if $x=1$.

(iv) $\sum \frac{x^n}{n}$.

Sol. $\frac{u_{n+1}}{u_n} = \frac{nx}{n+1} = \frac{x}{1+\frac{1}{n}} \longrightarrow x$ as $n\to\infty$

$\sum u_n$ converges if $x<1$, diverges if $x>1$.

For $x=1$, the series is $\sum\frac{1}{n}$, we know this is a divergent series.

So $\sum u_n$ converges for $x<1$ and diverges if $x\geq 1$.

(vi) $\sum \frac{(n^2-1)}{n^2+1}\cdot x^n$; $x>0$.

Sol. Let $u_n = \frac{n^2-1}{n^2+1}\cdot x^n$.

$$\therefore \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{(n^2+1)\{(n+1)^2-1\}}{(n^2-1)\{(n+1)^2+1\}}\cdot\frac{x^{n+1}}{x^n} = x$$

Hence by D'Alembert's Ratio test the series converges if $x<1$ and diverges if $x>1$.

The test fails to give any information when $x=1$.

When $x=1$, $u_n = \frac{n^2-1}{n^2+1}$

$$\therefore \lim u_n = \lim\frac{n^2-1}{n^2+1} = 1 \neq 0$$

$\Rightarrow$ the series is divergent.

Hence the series converges if $x<1$ and diverges if $x\geq 1$.

(vii) $1 + \frac{1^2}{1!} + \frac{2^2}{2!} + \frac{3^2}{3!} + \cdots\cdots$

Sol. Ignoring the first term, let $\sum_{n=1}^{\infty} u_n$ be the series, where $u_n = \frac{n^n}{n!}$.

$$\therefore \frac{u_{n+1}}{u_n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n = e > 1.$$

$\sum u_n$ is divergent by D'Alembert's ratio test.

(IV) Cauchy's Root Test :—

If $\sum u_n$ is a positive term series, such that $\lim_{n\to\infty} (u_n)^{1/n} = l$, then the series
(i) converges if $l < 1$
(ii) diverges if $l > 1$
(iii) the test fails to give any definite information if $l = 1$.

Proof:— Case I :— $l < 1$.

Let us select a positive ϵ, $\ni$ $l+\epsilon < 1$.

Let $l+\epsilon = \alpha < 1$.

Since $\lim_{n\to\infty} (u_n)^{1/n} = l$, $\therefore$ $\exists$ a positive integer K $\ni$

$$|(u_n)^{1/n} - l| < \epsilon \;\; \forall\, n \geq K$$

$$\Rightarrow l-\epsilon < (u_n)^{1/n} < l+\epsilon \;\; \forall\, n \geq K$$

$$\Rightarrow (l-\epsilon)^n < u_n < (l+\epsilon)^n \;\; \forall\, n \geq K$$

$$\Rightarrow u_n < \alpha^n \;\; \forall\, n \geq K.$$

But since $\sum \alpha^n$ is a convergent geometric series (common ratio $\alpha < 1$), therefore, by comparison test, the series $\sum u_n$ converges.

Case II :— $l > 1$

Let us select a positive no. ϵ $\ni$ $l-\epsilon > 1$.

Let $l-\epsilon = \beta > 1$.

Since $\lim_{n\to\infty} (u_n)^{1/n} = l$, therefore $\exists$ a positive integer K $\ni$

$$l-\epsilon < (u_n)^{1/n} < l+\epsilon \;\; \forall\, n \geq K$$

$$\Rightarrow (l-\epsilon)^n < u_n < (l+\epsilon)^n \;\; \forall\, n \geq K$$

$$\Rightarrow u_n > (l-\epsilon)^n = \beta^n \;\; \forall\, n \geq K$$

But since $\sum \beta^n$ is a divergent series (geometric series with common ratio $\beta > 1$), therefore by comparison test the series $\sum u_n$ is divergent.

Remark:— For the root test & Ratio test:—

If $l = 1$, then we can't draw any conclusion.

Let $\sum a_n = \frac{1}{n}$, $\sum b_n = \frac{1}{n^2}$

$$\therefore \lim \left(\frac{a_{n+1}}{a_n}\right) = 1 \text{ and } \lim \left(\frac{b_{n+1}}{b_n}\right) = 1.$$

In both cases $l = 1$ but $\sum a_n$ diverges whereas $\sum b_n$ converges.

Again
$$l_1 = \lim (a_n)^{1/n} = \lim \left(\frac{1}{n}\right)^{1/n} = \frac{1}{\lim (n^{1/n})} = 1.$$

$$l_2 = \lim (b_n)^{1/n} = \lim \left(\frac{1}{n^2}\right)^{1/n} = \frac{1}{\lim (n^2)^{1/n}} = \left\{\frac{1}{\lim n^{1/n}}\right\}^2 = 1.$$

Ex. Examine the convergence or divergence of the following series:

(i) $\sum_{n=1}^{\infty}\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

Sol. Let $u_n = \frac{1}{(1+1/\sqrt{n})^{n^{3/2}}}$, then.

$$\lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \frac{1}{\sqrt{e}} < 1.$$

Hence the series converges.

(ii) $\frac{1}{2}+\frac{1}{3^2}+\frac{1}{4^3}+\cdots+\frac{1}{(n+1)^n}+\cdots$

Sol. $u_n = \frac{1}{(n+1)^n}$ $\therefore (u_n)^{1/n} = \frac{1}{n+1} \to 0$ as $n\to\infty$.

$\therefore \lim (u_n)^{1/n} = \lim \frac{1}{n+1} = 0 < 1,$

$\therefore \sum u_n$ is convergent.

(iii) $\sum_{n=1}^{\infty}\frac{n^{n^2}}{(1+n)^{n^2}}$.

Sol. $(u_n)^{1/n} = \left\{\frac{n^{n^2}}{(1+n)^{n^2}}\right\}^{1/n} = \frac{n^n}{(1+n)^n} = \left(\frac{n}{1+n}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e}$ as $n\to\infty$

$\therefore \lim (u_n)^{1/n} = \frac{1}{e} < 1.$

$\therefore \sum u_n$ is convergent.

(iv) $\sum_{n=1}^{\infty}\frac{n^n}{n!}$

Sol. $\sum u_n = \sum\frac{n^n}{n!}$, $u_n^{1/n} = \frac{n}{\sqrt[n]{n!}}$

Now, by Cauchy's second theorem, $\lim \frac{n}{\sqrt[n]{n!}} = e > 1.$

$\therefore$ the series diverges.

(v) $\sum_{n=1}^{\infty}\frac{(n+1)^n . x^n}{n^{n+1}}$.

Sol. $\sqrt[n]{u_n} = \frac{(n+1)x}{n^{\frac{n+1}{n}}} = \frac{(n+1)/n}{n^{1/n}}.x \to x$ $\quad [\sqrt[n]{n} \to 1$ as $n\to\infty]$

$\therefore \lim (u_n)^{1/n} = x.$

$\therefore \sum u_n$ is convergent if $x<1$ and divergent if $x>1$.

If $x=1$, $u_n = \left(\frac{n+1}{n}\right)^n . \frac{1}{n}$

Choose $v_n = \frac{1}{n}$, $\therefore \frac{u_n}{v_n} \to e$ as $n\to\infty$.

As $\sum v_n$ is divergent, so does $\sum u_n$.

$\therefore \sum u_n$ is convergent if $x<1$ and divergent if $x\geq 1$.

(vi) $\sum_{n=1}^{\infty} (n^{1/n}-1)^n$.

Sol. let $u_n = (n^{1/n}-1)^n$, $n \in \mathbb{N}$.

$\therefore l = \lim (u_n)^{1/n} = \lim (n^{1/n}-1)$

$= \lim n^{1/n} - 1$

$= 1 - 1 = 0 < 1$

$\therefore$ By Cauchy's Root test, $\sum u_n$ is convergent.

(vii) $1 + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \cdots\cdots$

Sol. $u_n = \dfrac{1}{2^{n+(-1)^n}}$.

$\therefore \sqrt[n]{u_n} = \dfrac{1}{2^{1+\frac{(-1)^n}{n}}}$.

$\therefore \lim_{n\to\infty} \sqrt[n]{u_n} = \frac{1}{2} < 1 \Rightarrow$ convergent of the series.

(viii) $a + b + a^2 + b^2 + a^3 + b^3 + \cdots\cdots$; where $0 < a < b < 1$.

Sol. $u_{2n} = b^n$, $u_{2n-1} = a^n$.

$\therefore \lim (u_{2n})^{1/2n} = b^{1/2}$; $\lim (u_{2n-1})^{1/2n-1} = \lim a^{n/2n-1} = \lim a^{1/2}$

$\therefore \overline{\lim} (u_n)^{1/n} \le b^{1/2} < 1$.

Hence by Cauchy's test the series is convergent.

(ix) $\frac{1}{2} + 2 + \frac{1}{2^2} + 2^2 + \frac{1}{2^3} + 2^3 + \cdots\cdots$

Sol. $u_{2n} = 2^n$, $u_{2n-1} = \frac{1}{2^n}$.

$\lim (u_{2n})^{1/2n} = 2^{1/2}$; $\lim (u_{2n-1})^{\frac{1}{2n-1}} = \frac{1}{\sqrt{2}}$.

$= \sqrt{2}$

$\therefore \overline{\lim}\, u_n = \sqrt{2} > 1$.

$\therefore \sum u_n$ is divergent.

(x) $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \cdots\cdots$

Sol. $u_n = \left\{ \left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n} \right\}^{-n}$

$\therefore (u_n)^{1/n} = \left\{ \left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right) \right\}^{-1} = \left(\frac{n+1}{n}\right)^{-1} \left\{ \left(\frac{n+1}{n}\right)^n - 1 \right\}^{-1}$

$= \left(1+\frac{1}{n}\right)^{-1} \left\{ \left(1+\frac{1}{n}\right)^n - 1 \right\}^{-1}$

$\longrightarrow (e-1)^{-1} < 1$.

$\therefore \lim (u_n)^{1/n} < 1$.

$\therefore$ The series converges.

(V) ~~[illegible]~~ Raabe's Test :-

If $\sum u_n$ is a positive term series, such that $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$, then the series

(i) converges if $l > 1$.
(ii) diverges if $l < 1$.
(iii) the test fails if $l = 1$.

Note:- Raabe's test is to be applied when D'Alembert's ratio test fails.

Examples:- Test the convergence or divergence of the following series:

i. $1 + \frac{1}{2}\cdot\frac{1}{3} + \frac{1.3}{2.4}\cdot\frac{1}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{7} + \cdots\cdots$

Sol. Let $\sum_{n=1}^{\infty} u_n$ be the given series.

Then $u_n = \frac{1.3.5.\cdots.(2n-3)}{2.4.6.\cdots.(2n-2)}\cdot\frac{1}{(2n-1)}$ $\forall\, n \geq 2$

$\therefore \frac{u_{n+1}}{u_n} = \frac{(2n-1)^2}{2n(2n+1)}$ and $\lim \frac{u_{n+1}}{u_n} = 1$.

$\therefore$ D'Alembert's Ratio test fails to give conclusion.

$\therefore$ ~~Lit~~ us apply Raabe's test.

$$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{2n(2n+1)}{(2n-1)^2} - 1\right) = \lim_{n\to\infty} \frac{6n^2-n}{(2n-1)^2} = \frac{3}{2} > 1.$$

Therefore $\sum u_n$ is convergent by Raabe's test.

ii. $\frac{a}{b} + \frac{1+a}{1+b} + \frac{(1+a)(2+a)}{(1+b)(2+b)} + \cdots\cdots$

Sol. Here $u_n = \frac{(1+a)(2+a)\cdots(n-1+a)}{(1+b)(2+b)\cdots(n-1+b)}$.

$$\therefore \lim \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n+b}{n+a} = 1$$

Hence the Ratio test fails, we proceed to Raabe's test.

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{n+b}{n+a} - 1\right) = \lim_{n\to\infty} \frac{b-a}{1+\frac{a}{n}} = b-a.$$

$\therefore$ By Raabe's test the series converges if $(b-a) > 1$ and diverges if $(b-a) < 1$.

For $(b-a) = 1$, then $u_n = \frac{1+a}{n+a}$, let $v_n = \frac{1}{n}$

$\sum u_n$ is divergent by comparison with $\sum \frac{1}{n}$.

$\therefore$ The given series is convergent if $b > 1+a$ and divergent if $b \leq (1+a)$.

(iii) $1+\frac{1}{2}\cdot\frac{x^2}{4}+\frac{1.3.5}{2.4.6}\cdot\frac{x^4}{8}+\frac{1.3.5.7.9}{2.4.6.8.10}\cdot\frac{x^6}{12}+\cdots\cdots$

Solution:- Here $u_n = \frac{1.3.5.\ldots(4n-7)}{2.4.6.\ldots(4n-6)}\cdot\frac{x^{2n-2}}{(4n-4)}$

and $u_{n+1} = \frac{1.3.5.\ldots(4n-7)(4n-5)(4n-3)}{2.4.6.\ldots(4n-6)(4n-4)(4n-2)}\cdot\frac{x^{2n}}{4n}$.

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(4n-4)(4n-2)}{(4n-5)(4n-3)}\cdot\frac{4n}{4n-4}\cdot\frac{1}{x^2}$$

$$= \frac{16n^2-8n}{16n^2-32n+15}\cdot\frac{1}{x^2}$$

$$\therefore \lim\frac{u_n}{u_{n+1}} = \frac{1}{x^2}.$$

By D'Alembert's ratio test the series converges if $x^2<1$ and diverges if $x^2>1$. In case $x^2=1$, the test fails.
Now, we apply Raabe's test.

$$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left(\frac{24n-15}{16n^2-32n+15}\right).$$

$$\therefore \lim\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\} = \frac{3}{2}>1.$$

$\therefore$ The series is convergent when $x^2\le 1$ and divergent if $x^2>1$.

(iv) The series $x^2+\frac{2^2}{3.4}x^4+\frac{2^2.4^2}{3.4.5.6}x^6+\frac{2^2.4^2.6^2}{3.4.5.6.7.8}\cdot x^8+\cdots\cdots$ is convergent if $x^2\le 1$ and divergent if $x^2>1$.

Solution:- Neglecting the first term, the series is $\sum u_n$, where

$$u_n = \frac{2^2.4^2.\ldots(2n)^2}{3.4.5.6.\ldots(2n+1)(2n+2)}x^{2n+2}$$

and $$u_{n+1} = \frac{2^2.4^2.\ldots(2n)^2(2n+2)^2}{3.4.5.6.\ldots(2n+1)(2n+2)(2n+3)(2n+4)}x^{2n+4}$$

$$\therefore \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{(2n+3)(2n+4)}{(2n+2)^2}\cdot\frac{1}{x^2}$$

$$=\frac{1}{x^2}$$

The series is convergent if $\frac{1}{x^2}>1$, i.e. $x^2<1$ and divergent if $x^2>1$.
The test fails if $x^2=1$.
We apply Raabe's test:-

$$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left[\frac{(2n+3)(2n+4)}{(2n+2)^2}-1\right]$$

$$=\frac{n(6n+8)}{(4n^2+8n+4)}$$

$$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty}\frac{n(6n+8)}{4n^2+8n+4}$$

$$=\frac{6}{4}=\frac{3}{2}>1.$$

$\therefore$ By Raabe's test the series is convergent at $x^2=1$.

(VI) LOGARITHMIC TEST:-

If $\sum u_n$ is a series of positive terms $\ni$,

$$\lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right)=l,$$

then the series converges for $l>1$ and diverges for $l<1$.

Ex. check the convergence or divergence of the following series;

(i) $1+\frac{x}{1!}+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\cdots\cdots,\ x>0.$

Sol. Ignoring the first term, let $\sum_{n=1}^{\infty} u_n$ be the given series.

Then $u_n=\frac{n^n x^n}{n!}$; $\frac{u_{n+1}}{u_n}=\left(1+\frac{1}{n}\right)^n.x$ and $\lim\frac{u_{n+1}}{u_n}=ex$.

By D'Alembert's ratio test,

$\sum u_n$ is convergent if $0<x<1/e$, $\sum u_n$ is divergent if $x>1/e$.

When $x=1/e$, let us apply logarithmic test.

$$\therefore \lim\left(n\log\frac{u_n}{u_{n+1}}\right)=\lim n\left[1+n\log\frac{n}{n+1}\right]=\lim\left[n+n^2\log\frac{n}{n+1}\right]=\frac{1}{2}$$

$$=\lim_{n\to\infty} n\left[1-n\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\cdots\right)\right]=\frac{1}{2}<1.$$

Therefore, by logarithmic test, the series diverges.

Hence the series converges for $x<1/e$, and diverges for $x\geq 1/e$.

(VII) **Gauss's Test:-** If $\sum u_n$ is a positive terms series $\ni$

$$\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\gamma_n}{n^p},$$

where $\alpha > 0$, $p > 1$, and $\{\gamma_n\}$ is a bounded sequence, then

(i) for $\alpha \neq 1$, $\sum u_n$ converges if $\alpha > 1$, and diverges if $\alpha < 1$

(ii) for $\alpha = 1$, $\sum u_n$ converges if $\beta > 1$, and diverges if $\beta \le 1$.

Ex. Test the convergent of the series:

(i) $\frac{2^2}{3^2} + \frac{2^2 \cdot 4^2}{3^2 \cdot 5^2} + \frac{2^2 \cdot 4^2 \cdot 6^2}{3^2 \cdot 5^2 \cdot 7^2} + \cdots\cdots$

Sol. ~~Neglecting the first term~~ let $\sum_{n=1}^{\infty} u_n$ be the given series.

Then $u_n = \frac{2^2 \cdot 4^2 \cdot 6^2 \cdots\cdots (2n-2)^2 (2n)^2}{3^2 \cdot 5^2 \cdot 7^2 \cdots\cdots (2n-1)^2 (2n+1)^2}$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(2n+3)^2}{(2n+2)^2} = \left(1 + \frac{3}{2n}\right)^2 \left(1 + \frac{1}{n}\right)^{-2}$$

$$= \left(1 + \frac{3}{n} + \frac{9}{4n^2}\right)\left(1 - \frac{2}{n} + \frac{3}{n^2} - \cdots\cdots\right)$$

$$= 1 + \frac{1}{n} - \frac{3}{4n^2} + \cdots \text{highest power of } \frac{1}{n}$$

so that $\alpha = 1$, $\beta = 1$.

Hence by Gauss's test, the series diverges.

(ii) $\left(\frac{1}{2}\right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 + \cdots\cdots\cdots$

Sol. $\sum_{n=1}^{\infty} u_n$ be the given series.

Then $u_n = \left\{\frac{1 \cdot 3 \cdot 5 \cdots\cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots\cdots (2n)}\right\}^2 \; \forall\, n \geq 1$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \left(1 + \frac{1}{n}\right)^2 \left(1 + \frac{1}{2n}\right)^{-2}$$

$$= \left(1 + \frac{2}{n} + \frac{1}{n^2}\right)\left(1 - \frac{1}{n} + \frac{3}{4n^2} + \cdots\cdots\right)$$

$$= \left(1 + \frac{1}{n} - \frac{1}{4n^2} + \cdots\cdots \text{highest power of } \frac{1}{n}\right)$$

So, by Gauss's Test, the series diverges.

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(VIII) Cauchy's Integral Test :—

If u is a non-negative monotonic decreasing integrable function such that $u(n) = u_n$, for all positive integral values of n, then the series $\sum_{n=1}^{\infty} u_n$ and $\int_1^{\infty} u(x)\,dx$ converges or diverge together.

Ex.(1):- Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$ and diverges if $p \le 1$.

Sol. $u(x) = \frac{1}{x^p}$, so that for $x \ge 1$, the function $u(x)$ is a non-negative monotonic decreasing integrable function $\ni$

$$u_n = u(n) = \frac{1}{n^p}, n \in \mathbb{N}.$$

By integral test, $\sum_{n=1}^{\infty} u_n$ and $\int_1^{\infty} u(x)\,dx$ converge or diverge together.

$$\int_1^{X} u(x)\,dx = \int_1^{X} \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{1-p}\left(X^{1-p}-1\right), & \text{if } p \ne 1 \\ \log X & \text{if } p = 1 \end{cases}$$

$$\therefore \int_1^{\infty} u(x)\,dx = \lim_{X\to\infty} \int_1^{X} u(x)\,dx = \begin{cases} \frac{1}{1-p} & \text{if } p>1 \\ \infty, & \text{if } 0<p\le 1. \end{cases}$$

Thus $\int_1^{\infty} u(x)\,dx$ converges if $p>1$, and diverges if $0<p\le 1$.

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$ and diverges if $0<p\le 1$.

Ex.(2):- The series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$, $p>0$ converges for $p>1$ and diverges for $p\le 1$.

Sol. By integral test, $\sum_{n=2}^{\infty} u_n$ and $\int_2^{\infty} u(x)\,dx$ converge or diverge together,

as $u(x) = \frac{1}{x(\log x)^p}$, $x \ge 2$, where u is a non-negative monotonic decreasing integrable function.

Let us now test the convergence of the indefinite integral.

$$\therefore \int_2^{X} u(x)\,dx = \int_2^{X} \frac{1}{x(\log x)^p}\,dx, \quad p>0$$

$$= \begin{cases} \frac{(\log X)^{1-p} - (\log 2)^{1-p}}{1-p}, & \text{if } p\ne 1 \\ \log\log X - \log\log 2, & \text{if } p=1. \end{cases}$$

$$\therefore \int_2^{\infty} u(x)\,dx = \lim_{X\to\infty} \int_2^{X} u(x)\,dx = \begin{cases} \frac{(\log 2)^{1-p}}{p-1}, & \text{if } p>1 \\ \infty, & \text{if } 0<p\le 1. \end{cases}$$

Thus $\int_2^{\infty} u(x)\,dx$ converges if $p>1$ and diverges if $0<p\le 1$.

Hence the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$, $p>0$, converges if $p>1$, and diverges if $p\le 1$.

Series of arbitrary terms, not necessarily non-negative :—

Alternating Series:— If $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ with $a_n>0$, is an alternating series where $\{a_n\}$ is decreasing sequence of positive real numbers and $\lim(a_n)=0$. Then the series is convergent.

This is known as Leibnitz Test.

Examples:— 1. Show that the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^p}$ converges if $p>1$.

Solution:— Here $a_n=\frac{1}{n^p}$, $n\in\mathbb{N}$.

Note that $a_{n+1}-a_n=\frac{1}{(n+1)^p}-\frac{1}{n^p}<0\ \forall n\in\mathbb{N}$

$\Rightarrow \{a_n\}$ is decreasing.

Note that $\lim(a_n)=\lim\left(\frac{1}{n^p}\right)=0$ if $p>1$.

By Leibnitz test, the alternating series $\sum\frac{(-1)^{n-1}}{n^p}$ converges if $p>1$.

Ex. 2. S.T. The series $2-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\cdots\cdots$ is not convergent.

Solution:— The given series is $\sum(-1)^{n+1}\frac{n+1}{n}=\sum(-1)^{n+1}U_n$

The terms are alternatively positive and negative.

Here, also the $\{U_n\}_n$ is monotonic decreasing.

But $\lim_{n\to\infty}U_n=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)=1\neq 0$.

Hence the given series is not convergent.

Ex. 3. S.T $\frac{1}{1+x^2}-\frac{1}{2+x^2}+\frac{1}{3+x^2}-\cdots\cdots$ is convergent.

Solution:— The terms are alternatively positive and negative.

Also, $x^2+n+1>x^2+n$.

$\therefore \frac{1}{n+x^2}>\frac{1}{n+1+x^2}\ \forall n$.

$\therefore$ The terms form a monotonic decreasing sequence, also $\lim_{n\to\infty}\frac{1}{x^2+n}=0$.

Ex. (4). S.T. $1-\frac{1}{2}+\frac{1.3}{2.4}-\frac{1.3.5}{2.4.6}+\cdots\cdots$ is convergent.

Solution:— Since $\lim\frac{1.3.\cdots(2n-1)}{2.4.6\cdots 2n}=0$

and $\frac{1.3.\cdots\cdots(2n-1)}{2.4.6\cdots\cdots 2n}>\frac{1.3.\cdots(2n+1)}{2.4.\cdots(2n+2)}$.

$\therefore \{a_n\}$ is decreasing sequence.

So, by Leibnitz test, the series is convergent.

Ex. (5). The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\cdots$ is convergent by Leibnitz test, but the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\cdots$ is divergent.

Absolute Convergence:— A series $\sum u_n$ is said to be absolutely convergent if the series $\sum |u_n|$ is convergent.

Theorem:— An absolutely convergent series is convergent.
i.e. if $\sum |u_n|$ converges, then $\sum u_n$ also converges.

Proof:— Let $\sum u_n$ be a series whose terms are positive and negative real number and $\sum |u_n|$ be convergent.

By Cauchy's principle of convergence, given $\epsilon > 0$, $\exists\ m \in \mathbb{N} \ni$ for all positive integers $n \geq m$ and for all natural numbers p,

$$\big|\, |u_{n+1}| + |u_{n+2}| + \cdots + |u_{n+p}| \,\big| < \epsilon \quad \forall\ n \geq m \text{ and } \forall\ p = 1, 2, 3, \ldots$$

$$|u_{n+1}| + |u_{n+2}| + \cdots + |u_{n+p}| < \epsilon \quad \forall\ n \geq m \text{ and } \forall\ p = 1, 2, 3, \ldots$$

i.e. $|u_{n+1}| + |u_{n+2}| + \cdots + |u_{n+p}| < \epsilon \ \forall\ n \geq m$

But $|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| \leq |u_{n+1}| + |u_{n+2}| + \cdots + |u_{n+p}|$

$\therefore \sum u_n$ is convergent by Cauchy's general principle.

Note:— The converse of the is not necessarily true.
i.e. Absolute convergence $\Rightarrow$ Ordinary Convergence.
But Absolutely divergence $\not\Rightarrow$ Ordinary divergence.

Examples. 1. The series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots\cdots$ is absolutely convergent.

Solution:— The series with absolute value of each term is

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\cdots$$

This series is convergent by comparison test,
Since, $\sum \frac{1}{n^p}$ is convergent when $p > 1$.
$\therefore$ the series is absolutely convergent.
$\Rightarrow$ the series is convergent.

2. Show that $\sum \frac{\sin nx}{n^p}$ converges absolutely if $p > 1$.

Solution:— $0 \leq \left| \frac{\sin nx}{n^p} \right| \leq \frac{1}{n^p}$, $n \in \mathbb{N}$.

Now, $\sum \frac{1}{n^p}$ converges if $p > 1$.
$\therefore$ By comparison test $\sum \left| \frac{\sin nx}{n^p} \right|$ converges if $p > 1$.
$\therefore \sum \frac{\sin nx}{n^p}$ converges absolutely if $p > 1$.

3. $\sum \frac{(-1)^n \sin nx}{n^2}$ is convergent for fixed $x \in \mathbb{R}$.

Solution:— $\left| \frac{(-1)^n \sin nx}{n^2} \right| \leq \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ is convergent.

$\therefore$ the series $\sum \frac{(-1)^n \sin nx}{n^2}$ is absolutely convergent and hence convergent.

Theorem:— If $\sum u_n$ is absolutely convergent and $\{v_n\}$ be a bounded sequence then the series $\sum u_n v_n$ is absolutely convergent.

Proof:— Let $|v_n| < K \ \forall n \in \mathbb{N}$.

Now, $|u_{n+1}v_{n+1}| + |u_{n+2}v_{n+2}| + \cdots + |u_{n+p}v_{n+p}|$

$< K\{|u_{n+1}| + |u_{n+2}| + \cdots + |u_{n+p}|\}$

since $\sum u_n$ is absolute convergent, $\sum |u_n|$ is convergent.

Therefore for a given positive ϵ $\exists$ a natural no. m $\ni$

$|u_{n+1}| + |u_{n+2}| + \cdots + |u_{n+p}| < \epsilon/K \ \forall \ n \geq m$ and $\forall p = 1, 2, 3, \ldots$

Therefore $|u_{n+1}v_{n+1}| + |u_{n+2}v_{n+2}| + \cdots + |u_{n+p}v_{n+p}| < \epsilon \ \forall \ n \geq m$

By Cauchy's principle of convergence, the series $\sum |u_n v_n|$ is convergent and consequently $\sum u_n v_n$ is absolutely convergent.

Ex.(1):— Test the series $\frac{2}{1^3} - \frac{3}{2^3} + \frac{4}{3^3} - \frac{5}{4^3} + \cdots\cdots$

Solution:— Let $\sum_{n=1}^{\infty} u_n$ be the given series.

Then $u_n = (-1)^{n+1} \cdot \frac{n+1}{n^3} = \frac{(-1)^{n+1}}{n^2}\left(1 + \frac{1}{n}\right) = a_n b_n$, where,

$a_n = \frac{(-1)^{n+1}}{n^2}$, $b_n = 1 + \frac{1}{n}$.

The series $\sum a_n$ is absolutely convergent and $\{b_n\}$ is a bounded sequence and therefore the series $\sum a_n b_n$ is absolutely convergent.

Theorem:— Every absolutely convergent series can be expressed as the difference of two convergent series of positive real nos.

As an illustration:— We see that the series

$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots\cdots$ is absolutely convergent

and the series can be expressed as

$\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots\right) - \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\cdots\right)$ where both the series $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$ and $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\cdots$ are convergent.

Ex.(2):— Test the series $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{2n+1}{n^3-4}$.

Solution:— $a_n = (-1)^{n-1} \cdot \frac{2n+1}{n^3-4}$; $b_n = \frac{1}{n^2}$, $n \in \mathbb{N}$.

Now, $\lim \frac{|a_n|}{b_n} = \lim \left(\frac{2n+1}{n^3-4} \cdot n^2\right) = 2 \neq 0$

Hence $\sum |a_n|$ and $\sum b_n$ both converge or diverge simultaneously.

As $\sum b_n$ converges, so $\sum |a_n|$ also converges $\Rightarrow \sum a_n$ converges.

Test for Absolute Convergence:—

Ratio Test:— If $\sum u_n$ be a series of arbitrary ~~non-negative~~ terms, with $u_n \neq 0$ and if $\lim \left|\frac{u_{n+1}}{u_n}\right| = l$.

Then (i) $\sum u_n$ is absolutely convergent if $l<1$.
(ii) $\sum u_n$ is divergent if $l>1$.

Root Test:— If $\sum u_n$ be a series of arbitrary terms and let $\lim |u_n|^{1/n} = l$.

Then (i) $\sum u_n$ is absolutely convergent if $l<1$.
(ii) $\sum u_n$ is divergent if $l>1$.

Ex. Test the convergence of the following series:— [By Ratio Test]

(i) $1 - \frac{2^2}{2!} + \frac{3^3}{3!} - \frac{4^4}{4!} + \cdots\cdots$

i.e. $\sum (-1)^{n+1} \cdot \frac{n^n}{n!}$

Solution:— Here $u_n = (-1)^{n+1} \cdot \frac{n^n}{n!}$

$\therefore \frac{|u_{n+1}|}{|u_n|} = \frac{(n+1)^{n+1}}{(n+1)} \cdot \frac{1}{n^n} = \left(1+\frac{1}{n}\right)^n$ and

$$\lim \left|\frac{u_{n+1}}{u_n}\right| = \lim \left(1+\frac{1}{n}\right)^n = e > 1.$$

$\therefore$ By Ratio test, the series is divergent.

(ii) $1 - \frac{(2!)^2}{4!} + \frac{(3!)^2}{6!} - \cdots\cdots\cdots$

i.e. $\sum (-1)^{n+1} \cdot \frac{(n!)^2}{(2n)!}$

Solution:— $u_n = (-1)^{n+1} \frac{(n!)^2}{(2n)!}$ for $n \geq 2$.

$$\therefore \lim \left|\frac{u_{n+1}}{u_n}\right| = \lim \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4} < 1.$$

By Ratio test, the series $\sum u_n$ is absolutely convergent.

(iii) $1 - \frac{2}{2!} + \frac{3^2}{3!} + \frac{4^3}{4!} - \cdots\cdots$

Solution:— $|u_n| = \frac{n^{n-1}}{n!}$, $|u_{n+1}| = \frac{(n+1)^n}{(n+1)!}$

$\therefore \left|\frac{u_{n+1}}{u_n}\right| = \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n}\right)} \to e$ as $n \to \infty$ $\therefore \lim \left|\frac{u_{n+1}}{u_n}\right| > 1$. The series is divergent.

(iv) $1 - \frac{1}{2} + \frac{2!}{3^2} - \frac{3!}{4^3} + \frac{4!}{5^4} - \cdots\cdots$

Sol. Here $|u_n| = \frac{(n-1)!}{n^{n-1}}$, $|u_{n+1}| = \frac{n!}{(n+1)^n}$.

$$\therefore \lim \left|\frac{u_{n+1}}{u_n}\right| = \lim \frac{n^n}{(n+1)^n} = \frac{1}{\lim\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1.$$

$\therefore$ By Ratio test the given series is convergent.

(v) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ for different values of x.

Solution:– Note that $l = \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right|$

$$= \lim_{n\to\infty} \left| \frac{x^{n+1} \cdot n^2}{x^n \cdot (n+1)^2} \right|$$

$$= \lim \left\{ |x| \cdot \frac{n^2}{(n+1)^2} \right\}$$

$$= |x|.$$

By Ratio test, $\sum \frac{x^n}{n^2}$ converges absolutely if $|x| < 1$. and diverges if $|x| > 1$.

For $|x| = 1$, the series is equal to $\sum \frac{1}{n^2}$, which also converges.

So, $\sum \frac{x^n}{n^2}$ converges absolutely if $|x| \le 1$ and diverges if $|x| > 1$.

(vi) $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$

Sol. Let $u_n = \frac{x^n}{n!}$

Now, $\lim \left| \frac{u_{n+1}}{u_n} \right| = \lim \left| \frac{x^{n+1} \cdot n!}{(n+1)! \cdot x^n} \right| = \lim \left| \frac{x}{(n+1)} \right|$

$$= \lim_{n\to\infty} \frac{1}{(n+1)} \cdot |x|$$

$= 0$ except when $x = 0$.

So, by Ratio test, the series converges absolutely $\forall x$ except $x = 0$.

But for $x = 0$, the series evidently converges absolutely.

Hence, the series converges absolutely for all values of x.

2. Show that $\lim_{n\to\infty} \frac{m(m-1)\cdots\cdots(m-n+1)}{(n-1)!} x^n = 0$, where $|x| < 1$ and m is any real number.

Sol.

$$\lim \left| \frac{u_{n+1}}{u_n} \right| = \lim \left| \frac{m-n}{n} \right| \cdot |x|$$

$$= \lim_{n\to\infty} \left| \frac{m}{n} - 1 \right| \cdot |x|$$

$$= |x|$$

Hence the series converges absolutely for $|x| < 1$

$\Rightarrow$ The series $\sum u_n$ converges for $|x| < 1$.

$\Rightarrow$ $\lim u_n = 0$, for $|x| < 1$

i.e. $\lim \frac{m(m-1)\cdots\cdots(m-n+1)}{(n-1)!} \cdot x^n = 0$, if $|x| < 1$.

Ex. Test the convergence of the following series :— [By Root Test]

(i) $(1+1)^{-1} - \left(1+\frac{1}{2}\right)^{-2^2} + \left(1+\frac{1}{3}\right)^{-3^2} + \cdots\cdots$

Sol. Here, $|u_n| = \left(1+\frac{1}{n}\right)^{-n^2}$, so that $|u_n|^{1/n} = \left(1+\frac{1}{n}\right)^{-n}$

$\therefore \lim |u_n|^{1/n} = \frac{1}{e} < 1.$

$\therefore \sum u_n$ is convergent.

(ii) $\frac{x}{1!} - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots\cdots$

Solution:— $\lim |u_n|^{1/n} = \lim \frac{x}{(n!)^{1/n}} = \lim \frac{n}{(n!)^{1/n}} \cdot \frac{x}{n}$

$= e . \lim \frac{x}{n}$ $\left[\text{since, } \lim \frac{n}{(n!)^{1/n}} = e\right]$

$= 0 < 1.$

$\therefore \sum |u_n|$ is convergent.

$\therefore \sum u_n$ is convergent.

$\therefore \lim \frac{x^n}{n!} = 0.$

(iii) $\frac{1^3}{3^2} - \frac{2^3}{3^2} + 1 - \frac{4^3}{3^4} + \cdots\cdots$

Solution:— $\lim |u_n|^{1/n} = \lim \frac{n^{3/n}}{3} = \frac{1}{3} < 1.$

$\therefore \sum u_n$ is convergent.

Test for Series of Arbitrary Terms:-

Theorem 1:- (Abel's Test)

If the sequence $\{b_n\}$ is a monotonic bounded sequence and $\sum a_n$ is a convergent series then the series $\sum a_n b_n$ is convergent.

Theorem 2:- (Dirichlet's Test)

If the sequence $\{b_n\}$ is a monotone sequence converging to 0 and the sequence of partial sums $\{s_n\}$ of the series $\sum a_n$ is bounded, then the series $\sum a_n b_n$ is convergent.

Note:- Leibnitz test is a particular case of Dirichlet's test. If $\{b_n\}$ is a monotone decreasing sequence converging to 0, then the series $\sum^{\infty} (-1)^{n+1} b_n$ is convergent by Dirichlet's test, since the sequence of partial sums $\{s_n\}$ of the series $\sum (-1)^{n+1}$ is bounded.

This is Leibnitz test for an alternating series.

Ex.(1):- [Using Abel's Test]

(i) Show that the series $\sum \frac{(-1)^{n+1} \cdot n^n}{(n+1)^{n+1}}$ is convergent.

Solution:- Here the series $\sum \frac{(-1)^{n+1}}{(n+1)}$ is convergent by Leibnitz's test and the sequence $\left\{\left(\frac{n}{n+1}\right)^n\right\}$ i.e. $\left\{\left(1+\frac{1}{n}\right)^{-n}\right\}$ is monotone decreasing and bounded below.

$\therefore$ By Abel's test the series is convergent.

(ii) S.T. the series $\sum_2^{\infty} \frac{(-1)^{n+1}}{n \log n}$ is convergent.

Solution:- $\sum \frac{(-1)^{n+1}}{n}$ is a convergent series and the sequence $\left\{\frac{1}{\log n}\right\}_2^{\infty}$ is a monotone decreasing sequence and bounded below.

$\therefore$ By Abel's test the series is convergent.

(iii) S.T. the series $0 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{3} + \frac{2}{3^2} - \frac{1}{4} + \frac{3}{4^2} - \cdots\cdots$ is convergent.

Solution:- Let us consider the series

$$1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \cdots\cdots \quad \text{——— (1)}$$

and the sequence,

$$0, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \frac{3}{4}, \frac{3}{4}, \cdots\cdots \quad \text{——— (2)}$$

The series (1) is convergent and the sequence (2) is monotone and bounded.

$\therefore$ By Abel's test the series converges.

(iv) If $\sum u_n$ be convergent series, S.T. the series $\sum \frac{u_n}{n}$ is convergent.

Solution:— $\sum u_n$ is a convergent series, and the sequence $\{\frac{1}{n}\}$ is monotone decreasing and bounded below.

$\therefore \sum \frac{u_n}{n}$ is convergent by Abel's test.

(v) Test the convergence of the series $\sum \frac{(n^3+1)^{1/3}-n}{\log n}$.

Solution:— Let $a_n = \{(n^3+1)^{1/3}-n\}$ and $b_n = \frac{1}{\log n}$

Since $\sum a_n$ converges and $\{b_n\}$ is a positive monotonic decreasing sequence, therefore, by Abel's test $\sum a_n b_n$ is convergent.

(vi) S.T. the series $1 - \frac{1}{3.2^2} + \frac{1}{5.3^2} - \frac{1}{7.4^2} + \ldots\ldots$ is convergent.

Solution:—

Let $\sum a_n = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots$ and $b_n = \frac{1}{2n-1}$

Then the given series can be written as $\sum a_n b_n$.

Since $\sum a_n$ converges and $\{b_n\}$ is positive and monotonic increasing sequence, therefore by Abel's test, the given series converges.

Ex. (2) :— [Using Dirichlet's Test]

(i) S.T. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ is convergent.

Sol. The sequence of partial sums $\{S_n\}$ of the series $\sum(-1)^{n+1}$ is bounded, and the sequence $\{\frac{1}{\sqrt{n}}\}$ is a monotone decreasing sequence converging to 0.

$\therefore$ By Dirichlet's test, the series is convergent.

(ii) S.T. the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\log(n+1)}$ is convergent.

Sol.

Since the sequence of partial sums $\{S_n\}$ of the series $\sum(-1)^{n+1}$ is bounded and the sequence $\left\{\frac{1}{\log(n+1)}\right\}$ is a monotone decreasing sequence converging to 0, i.e. $\lim_{n\to\infty} \frac{1}{\log(n+1)} = 0$.

$\therefore$ By Dirichlet's test the series is convergent.

Ques:— Let $\{a_n\}_{n=1}^{\infty}$ be a sequence converges to 0 and $\{b_n\}_{n=1}^{\infty}$ be a sequence that is bounded, then $\{a_n b_n\}_{n=1}^{\infty}$ is a convergent sequence and converges to 0.

ANS:—

$\lim_{n\to\infty} a_n = 0$

let $-K \le b_n \le K$.

$\therefore |b_n| \le K \ \forall n$.

$\therefore \lim \{a_n b_n\}_{n=1}^{\infty} = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

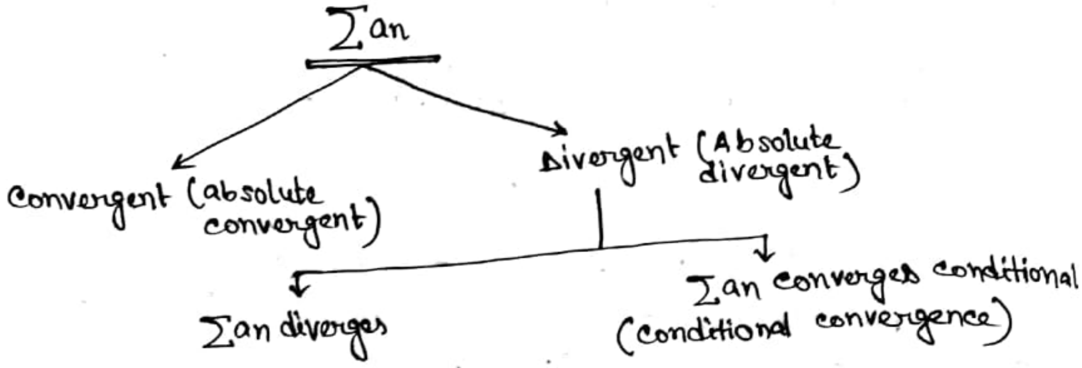
$\le 0 . \{K\} \ \forall n$

$\therefore$ Thus the sequence $\{a_n b_n\}_{n=1}^{\infty}$ converges to zero.

Conditional Convergence :— A series $\sum a_n$ is said to be conditionally convergent if $\sum a_n$ is convergent but $\sum |a_n|$ is not convergent.

A conditionally convergent series is also called a semi-convergent series or a non-absolutely convergent series.

Let $\sum a_n$ is a series of arbitrary terms.



Examples:— 1.

(i) The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots$ is conditionally convergent.

Solution:- Hence the series is $\sum (-1)^{n-1} \cdot u_n$, where $u_n = \frac{1}{n}$.

Now, $\{\frac{1}{n}\}$ is monotone decreasing and $\lim(\frac{1}{n}) = 0$.

Hence by Leibnitz's test the series is convergent. But the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots$ is divergent.

Hence the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots$ is divergent.

Hence the series is conditionally convergent.

(ii) The series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\cdots$ is conditionally convergent.

Solution:— The series $\sum (-1)^{n+1} u_n$, where $u_n = \frac{1}{2n-1}$ is an alternating series. Also $\left\{\frac{1}{2n-1}\right\}$ is monotone decreasing sequence of positive numbers and $\lim u_n = 0$.

$\therefore$ By Leibnitz test the given series is convergent.

But the series $1 + \frac{1}{3} + \frac{1}{5} + \cdots\cdots$ is a divergent series.

Hence, the series is conditionally convergent.

(iii) The series $\frac{1}{(1+x)^\alpha} - \frac{1}{(2+x)^\alpha} + \frac{1}{(3+x)^\alpha} - \cdots\cdots$ $(x > 0)$ is conditionally convergent when $0 < \alpha \leq 1$.

Solution:— The series is $\sum (-1)^{n+1} \frac{1}{(n+x)^\alpha} = \sum (-1)^{n+1} \cdot u_n$

$\therefore$ $\{u_n\}$ is a montone decreasing sequence and $\lim u_n = 0$

$\therefore$ The alternating series is convergent by Leibnitz's test.

But the series $\sum \frac{1}{(n+x)^\alpha}$ is divergent by comparing with the series $\sum \frac{1}{n^\alpha}$ which is divergent if $0 < \alpha \leq 1$.

$\therefore$ the given series is conditionally divergent.

Example 2. Test the following series for absolute and conditional convergence:

(i) $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n^p}$, $p>0$ (ii) $\sum_{n=1}^{\infty} \frac{x^n}{n^p}$, $x \in \mathbb{R}$ and $p>0$.

Solution:—

(i) By Leibnitz test, $\sum (-1)^{n-1} \cdot \frac{1}{n^p}$ converges if $p>0$.

Note that, $\sum \left| (-1)^{n-1} \cdot \frac{1}{n^p} \right| = \sum \frac{1}{n^p}$ converges if $p>1$ and diverges if $p \le 1$.

Hence the series $\sum (-1)^{n-1} \cdot \frac{1}{n^p}$ converges absolutely if $p>1$.

and the series $\sum \left|(-1)^{n-1} \cdot \frac{1}{n^p}\right|$ diverges for $0<p\le 1$, but

$$\sum (-1)^{n-1} \cdot \frac{1}{n^p} \text{ converges.}$$

i.e. $\sum (-1)^{n-1} \cdot \frac{1}{n^p}$ converges conditionally.

(ii) Here $a_n = \frac{x^n}{n^p}$, $n \in \mathbb{N}$, $x \in \mathbb{R}$

Note that, $\lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \left\{ \frac{|x|^{n+1}}{(n+1)^p} \cdot \frac{n^p}{|x|^n} \right\}$

$$= |x| \lim \left(\frac{n}{n+1}\right)^p$$

$$= |x| \cdot \frac{1}{\lim \left(1+\frac{1}{n}\right)^p}$$

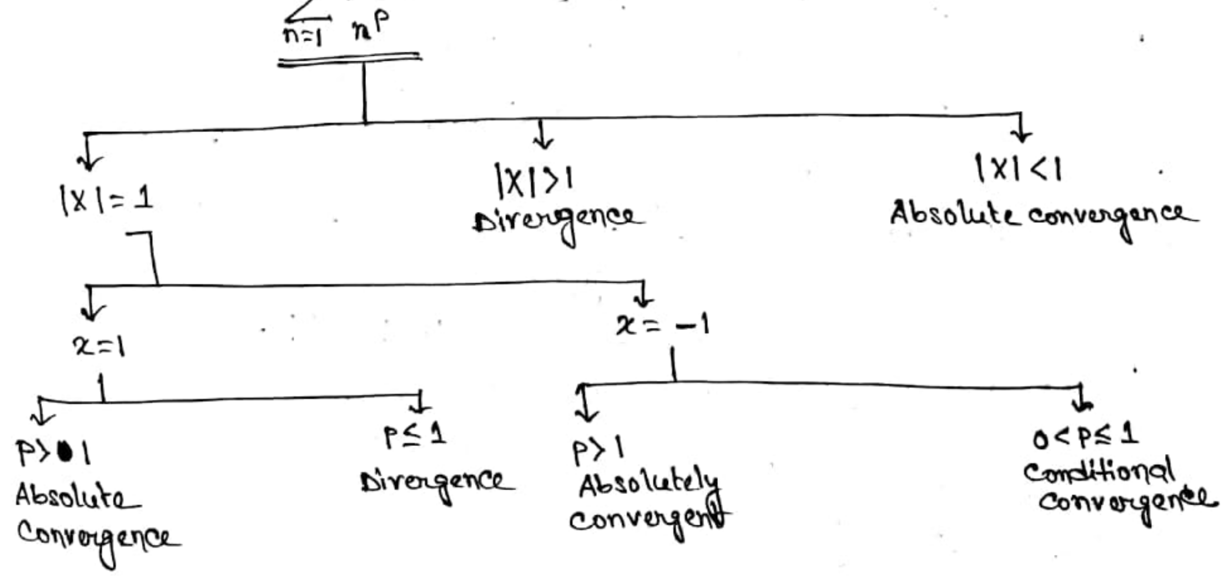
$$= |x|.$$

If $|x|<1$, then $\sum a_n = \sum \frac{x^n}{n^p}$ converges absolutely.

If $|x|>1$, then $\sum a_n = \sum \frac{x^n}{n^p}$ ~~converges~~ diverges.

If $x=1$, then $\sum a_n = \sum \frac{1}{n^p}$ converges if $p>1$, diverges if $p \le 1$.

If $x=-1$, then $\sum a_n = \sum \frac{(-1)^n}{n^p}$ converges absolutely if $p>1$ and converges conditionally if $0<p\le 1$.



Introduction and removal of brackets :—

Theorem:— Let $\sum u_n$ be a series of positive and negative real numbers and $\sum v_n$ is obtained from $\sum u_n$ by grouping its terms. Then (i) if $\sum u_n$ converges to the sum s, then $\sum v_n$ also converges to s,

(ii) if $\sum v_n$ converges, then $\sum u_n$ may not be convergent.

Proof:— (i) Let $v_1 = u_1 + u_2 + \cdots + u_{p_1}$, $v_2 = u_{p_1+1} + u_{p_2+2} + \cdots + u_{p_2}$, ..., $v_n = u_{p_{n-1}+1} + u_{p_{n-1}+2} + \cdots + u_{p_n}$, ...

Then $\{p_n\}$ is a strictly increasing sequence of natural numbers.

Let $s_n = u_1 + u_2 + \cdots + u_n$, $t_n = v_1 + v_2 + \cdots + v_n$.

Then $t_n = u_1 + u_2 + \cdots + u_{p_n} = s_{p_n}$.

Since $\sum u_n$ converges to the sum s, $\lim s_n = s$.

The sequence $\{t_n\}$ is a subsequence of the sequence $\{s_n\}$ and therefore the sequence $\{t_n\}$ also converge to the sum s.

In other words, the series $\sum v_n$ converges to the sum s.

(ii) That the converse is not true can be established by the following example.

Let $u_n = (-1)^{n+1}$. Then the series $\sum u_n$ is $1 - 1 + 1 - 1 + \cdots$

This is not a convergent series.

Let v_n be obtained from $\sum u_n$ by grouping the terms as $(1-1) + (1-1) + (1-1) + \cdots$

Then $\sum v_n$ is clearly a convergent series.

Examples:— 1. Prove that $\log 2 = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots$

Soln:— The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent, by Leibnitz's test.

We have $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ when $-1 < x \le 1$.

So, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2$.

Grouping these terms of the series as $\left(1-\frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots$ we have the series $\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots$.

∴ By above theorem, the sum of the series is $\log 2$.

Example. 2. Prove that $\frac{\pi}{8} = \frac{1}{1\cdot 3} + \frac{1}{5\cdot 7} + \frac{1}{9\cdot 11} + \cdots$

Soln:— The series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ is convergent by Leibnitz's test.

We have $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ when $-1 \le x \le 1$

So, $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$.

Grouping the terms of the series as $\left(1-\frac{1}{3}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots$

We have the series $\frac{2}{1\cdot 3} + \frac{2}{5\cdot 7} + \frac{2}{9\cdot 11} + \cdots$

By above theorem, the sum of the series is $\frac{\pi}{4}$.

Hence, $\frac{1}{1\cdot 3} + \frac{1}{5\cdot 7} + \frac{1}{9\cdot 11} + \cdots = \frac{\pi}{8}$.

Rearrangment of Series :–

It is a familiar fact that the sum of a finite number of terms is same, no matter how the terms are arranged. But it is not universally true for an infinite series, as an example, consider the series:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots$$

$$= s, \text{(say)}.$$

If we arrange the terms of the series $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ so that each positive terms are followed by two negative terms, then we have the series:

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{4} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots\cdots$$

$$= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{10} + \cdots\cdots$$

$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots\cdots$$

$$= \frac{1}{2}\left\{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\cdots\right\}$$

$$= \frac{s}{2}.$$

Hence, a rearrangement of the series $\sum \frac{(-1)^{n-1}}{n}$ alters the sum of the series. So, we naturally ask "Under what condition(s) may we arrange the terms of a series without altering its value?"

Theorem:– If a series $\sum a_n$ converges absolutely, then its sum is unaltered by rearrangement.

Example:– Find the sum of the following series:

$$1 - \frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{3^2} - \frac{1}{6^2} - \frac{1}{8^2} + \frac{1}{5^2} - \cdots\cdots$$

Solution:– consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$.

since $\sum \frac{(-1)^{n-1}}{n^2}$ converges absolutely. Let $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = s$.

Hence, any rearrangement of the series has the same value (s).

Riemann's Theorem:– If a series converges conditionally, its sum can be made to have an arbitrary value by suitable rearrangement. It can also be made divergent or oscillatory.

WORKED EXAMPLES :-

1. S.T. the series $\left(\frac{3}{2}-\frac{4}{3}\right)+\left(\frac{5}{4}-\frac{6}{5}\right)+\cdots\cdots$ is convergent but $\frac{3}{2}-\frac{4}{3}+\frac{5}{4}-\frac{6}{5}+\cdots\cdots$ is not.

Solution:- The series with the terms within the bracket is

$$\sum u_n = -\frac{1}{2\cdot 3}+\frac{1}{4\cdot 5}+\cdots\cdots\ ; \text{ where } u_n = \frac{1}{2n(2n+1)}$$

Let us consider $v_n = \frac{1}{n^2}$ & $\lim \frac{u_n}{v_n} = \frac{1}{4}$.

Now $\sum v_n$ is convergent, so by limit comparison test $\sum u_n$ is also convergent.

But when the brackets are removed, the series is $\sum (-1)^{n+1} u_n$, where $u_n = \frac{n+2}{n+1}$, $u_{n+1} = \frac{n+3}{n+2}$.

$$\therefore u_{n+1} - u_n = \frac{n+3}{n+2} - \frac{n+2}{n+1} < 0$$

$\therefore \{u_n\}$ is a monotonic decreasing sequence but $u_n \to 1$.

$\therefore$ Leibnitz test regarding convergence for alternating series fails.

Hence the series is not convergent.

2. S.T. the series $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot n^{-1/2}$ is convergent. Examine its absolute convergence.

Solution:- The series is an alternating one. Denoting the series by $\sum (-1)^{n-1} \cdot u_n$; we get $\{u_n\}$ is monotone decreasing since $\sqrt{n+1} > \sqrt{n}$ or $(n+1)^{-1/2} < n^{-1/2}\ \forall n$.

Again, $\lim u_n = 0$. $\therefore$ By Leibnitz's test the series is convergent.

Now, $|u_n| = \frac{1}{n^{1/2}}$. By comparison test $\sum \frac{1}{\sqrt{n}}$ is not convergent.

$\therefore$ The series is not absolutely convergent.

3. S.T. the series $\sum_{n=1}^{\infty} (-1)^n \sin\frac{1}{n}$ can't converge absolutely.

Solution:- For testing absolute convergence we are required to test the convergence of $\sum \sin\frac{1}{n}$.

Now, comparing the series with $\sum \frac{1}{n}$, we can find that

$$\lim_{n\to\infty} \frac{\sin 1/n}{1/n} = 1\ (\neq 0).$$

$\therefore$ The series $\sum \sin\frac{1}{n}$ is divergent since $\sum \frac{1}{n}$ is divergent.

$\therefore \sum (-1)^n \sin\frac{1}{n}$ is not absolutely convergent.

4. Discuss convergence of the series

$$1-\frac{1}{3^3}(1+2)+\frac{1}{4^3}(1+2+3)-\frac{1}{5^3}(1+2+3+4)+\cdots\cdots$$

Solution:- Let us denote $u_n = \frac{1}{(n+1)^3}(1+2+3+\cdots+n) = \frac{n}{2(n+1)^2}$,

The series is an alternating series.

Moreover $u_n > u_{n+1}\ \forall n$ and $\lim u_n = 0$.

$\therefore$ By Leibnitz's test the series is convergent.

5. If $u_1, u_2, \cdots\cdots$ is a decreasing sequence of positive terms and $\lim u_n = 0$, S.T. the series $u_1 - \frac{1}{2}(u_1+u_2)+\frac{1}{3}(u_1+u_2+u_3)-\cdots\cdots$ is convergent.

Solution:- Let us put $v_1 = u_1,\ v_2 = \frac{u_1+u_2}{2}$, etc.

then $v_n = \frac{1}{n}(u_1+u_2+\cdots+u_n)$.

Then the series is $v_1 - v_2 + v_3 - v_4 + \cdots\cdots$

Now, it is given that $\lim u_n = 0$.

$\therefore$ By Cauchy's first limit theorem, $\lim v_n = 0$.

Moreover, $u_1 > u_2 > u_3 \cdots\cdots$ $\therefore v_2 = \frac{u_1+u_2}{2} < u_1 = v_1$,

$$v_{n+1} = \frac{u_1+\cdots+u_n+u_{n+1}}{n+1} = \frac{nv_n+u_{n+1}}{n+1}$$

$$\therefore v_{n+1}-v_n = \frac{nv_n+u_{n+1}-(n+1)v_n}{n+1} = \frac{u_{n+1}-v_n}{n+1}$$

Now, $v_n = \frac{u_1+\cdots+u_n}{n} > u_1$

$\therefore u_{n+1} - v_n < 0$.

$\therefore \{v_n\}$ is a monotone decreasing sequence of positive terms.

$\therefore$ By Leibnitz's test the series is convergent.

6. Test the convergence of the series

$$1-\frac{1}{1\cdot2}+\frac{1}{2}-\frac{1}{2\cdot3}+\frac{1}{3}-\frac{1}{3\cdot4}+\cdots\cdots$$

Solution:- Here $S_{2n} = \left(1-\frac{1}{1\cdot2}\right)+\left(\frac{1}{2}-\frac{1}{2\cdot3}\right)+\cdots+\left\{\frac{1}{n}-\frac{1}{n(n+1)}\right\}$

$$= \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n+1}$$

Now, the series $1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots$ is divergent.

$\therefore \left(\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{n+1}\right) \longrightarrow \infty$. Hence $S_{2n} \to \infty$.

Again $S_{2n+1} = S_{2n} + \frac{1}{n+1} \longrightarrow \infty$.

$\therefore \{S_n\}$ is divergent and hence the series is divergent.

7. Prove that the series $1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots\cdots$ is divergent

Solution:- Let $S_n = 1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots\cdots$ to n terms.

$\therefore S'_{3n} = \left(1+\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{5}-\frac{1}{6}\right)+\cdots\cdots$ to n terms.

$> \left(\frac{1}{3}+\frac{1}{3}-\frac{1}{3}\right)+\left(\frac{1}{6}+\frac{1}{6}-\frac{1}{6}\right)+\cdots\cdots$ to n terms

$= \frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{n}\right)$

Now, the series $1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots$ is a well known divergent series.

$\therefore 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\to\infty$.

$\therefore S'_{3n}\to\infty$. Also $S'_{3n+1} = S'_{3n}+\frac{1}{3n+1}\to\infty$.

$S'_{3n+2} = S'_{3n+1}+\frac{1}{3n+2}\to\infty$.

$\therefore \{S_n\}$ is divergent and hence the series is divergent.

8. Show that the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\cdots$ converges to $\log 2$, but the series $1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\cdots\cdots$ converges to $\frac{1}{2}\log 2$.

Solution:- We know $\underset{n\to\infty}{Lt}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{n}-\log n\right)=\gamma$.

Let us put $\gamma_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{n}-\log n$, then $\underset{n\to\infty}{Lt}\gamma_n=\gamma$.

Now, $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\cdots+\frac{1}{2n-1}-\frac{1}{2n}$

$= \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\cdots+\frac{1}{2n-1}+\frac{1}{2n}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n}\right)$

$= \log 2n+\gamma_{2n}-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$

$= \log 2n+\gamma_{2n}-\log n-\gamma_n \longrightarrow \log 2 \quad \left(\because \gamma_{2n}-\gamma_n\to 0 \text{ as } n\to\infty\right)$

$\therefore \underset{n\to\infty}{Lt} S_{2n} = \log 2$

Also $S_{2n+1} = S_{2n}+\frac{1}{2n+1}$

$\therefore \lim S_{2n+1} = \lim S_{2n} = \log 2$.

$\therefore \lim S_n = \log 2$, and hence $\sum(-1)^{n-1}\cdot\frac{1}{n} = \log 2$. —— <i> no. proved.

Let $S'_n = 1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\cdots\cdots$ to n terms.

$\therefore S'_{3n} = \left(1-\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{6}-\frac{1}{8}\right)+\cdots\cdots+\left(\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n}\right)$

$= \left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right)-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{2n}\right)$

$= \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\cdots+\frac{1}{2n-1}+\frac{1}{2n}\right)-\left(\frac{1}{2}+\frac{1}{4}+\cdots\cdots+\frac{1}{2n}\right)$

$-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{2n}\right)$

$= \frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{2n}\right)-\frac{1}{2}\left(1+\frac{1}{2}+\cdots\cdots+\frac{1}{n}\right)$

$= \frac{1}{2}(\log 2n+\gamma_{2n})-\frac{1}{2}(\log n+\gamma_n)$

$\longrightarrow \frac{1}{2}\log 2 \quad [\because \gamma_{2n}-\gamma_n\to 0]$

So, $\lim S'_n = \frac{1}{2}\log 2$.

9. After re-arranging the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\cdots$, S.T. $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots\cdots$ converges to $\frac{3}{2}\log 2$.

Solution:– Let $S_n = 1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots\cdots$ to n terms.

$$\therefore S_{3n} = \left(1+\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{5}+\frac{1}{7}-\frac{1}{4}\right)+\cdots\cdots+\left(\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2n}\right)$$

$$= \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{4n-2}+\frac{1}{4n-1}+\frac{1}{4n}\right)$$

$$-\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{4n-2}+\frac{1}{4n}\right)-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n}\right)$$

$$= \log 4n + \gamma_{4n} - \frac{1}{2}\left(1+\frac{1}{2}+\cdots+\frac{1}{2n}\right)-\frac{1}{2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$$

$$= \log 4n + \gamma_{4n} - \frac{1}{2}\left(\log 2n+\gamma_{2n}\right)-\frac{1}{2}\left(\log n+\gamma_n\right)$$

$$= \log 4 + \log n - \frac{1}{2}\log 2 - \frac{1}{2}\log n - \frac{1}{2}\log n + \left(\gamma_{4n}-\frac{1}{2}\gamma_{2n}-\frac{1}{2}\gamma_n\right)$$

$$= \frac{3}{2}\log 2 + \left(\gamma_{4n}-\frac{1}{2}\gamma_{m}-\frac{1}{2}\gamma_n\right) \longrightarrow \frac{3}{2}\log 2$$

& $S_{3n+1} \to S_{3n}$ and $S_{3n+2} \to S_{3n}$

$\therefore \lim S_n = \frac{3}{2}\log 2$.

$\therefore$ The series converges to $\frac{3}{2}\log 2$.

10. Show that $1-1+\frac{1}{2}-\frac{1}{2}+\frac{1}{3}-\frac{1}{3}+\cdots\cdots$ is convergent. By rearranging the terms of the series prove that the series $1+\frac{1}{2}-1+\frac{1}{3}+\frac{1}{4}-\frac{1}{2}+\frac{1}{5}+\frac{1}{6}-\frac{1}{3}+\cdots\cdots$ is convergent.

Solution:– Here, if S_n denotes the sum of the first n terms of the series $S_{2n} = 0$,

$S_{2n+1} = S_{2n} + \frac{1}{n+1}$.

$\therefore \lim S_{2n+1} = 0$.

$\therefore \lim S_n = 0$.

Hence, the series converges to 0.

Now the second series is $\left(1+\frac{1}{2}-1\right)+\left(\frac{1}{3}+\frac{1}{4}-\frac{1}{2}\right)+\cdots\cdots$

$$= \left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\cdots$$

$$= \frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\cdots\cdots$$ is convergent by comparison test.

Hence we can remove the brackets and can write that the second series is $= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\cdots$

$= \log 2$.

11. The series $\sum \frac{1}{n+1}$ is ——

(A) Convergent (B) Divergent (C) Oscillate (D) None.

Sol. (B) $\sum u_n = \sum \frac{1}{n+1}$

$$\sum v_n = \frac{1}{n}$$

$$\therefore \sum \frac{u_n}{v_n} = \sum \frac{n}{n+1} = \sum_{n=1}^{\infty} \frac{1}{1+\frac{1}{n}} = 1.$$

$\because$ $\therefore$ $\sum v_n$ is divergent and $\sum u_n$ is also divergent.

12. The series $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$ is ——

(A) Convergent (B) Divergent (C) Unbounded (D) None.

Sol. (A) $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^2 = \frac{1}{3} < 1.$

$\Rightarrow$ the series is convergent.

13. The series $\sum \cos\left(\frac{1}{n}\right)$ is ——

(A) Convergent (B) Divergent (C) Oscillatory (D) None.

Sol. (B)

$$\sum u_n = \sum \cos \frac{1}{n}$$

$$\therefore \lim u_n = \lim_{n\to\infty} \cos \frac{1}{n}$$
$$= \cos 0$$
$$= 1 \neq 0$$

Hence the given series is divergent because the necessary condition to converge a series is $\lim_{n\to\infty} u_n = 0$.

~~14. S.T. the series~~ $\cancel{\sum_{n=1}^{\infty}\left(1+\frac{1}{\sqrt{n}}\right)^{-n}}$ ~~is convergent.~~

Sol.

~~We have~~ $\cancel{a_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n}}$

14. The series $\sum \sqrt{\frac{2^n-1}{3^n-1}}$ is convergent or not?

Sol. The given series is $\sum u_n = \sum \sqrt{\frac{2^n-1}{3^n-1}}$

$$\therefore \lim \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \sqrt{\frac{2^n-1}{3^n-1}} \times \sqrt{\frac{3^{n+1}-1}{2^{n+1}-1}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{1-\frac{1}{2^n}}{1-\frac{1}{3^n}}} \times \sqrt{\frac{3-\frac{1}{3^n}}{2-\frac{1}{2^n}}}$$

$$= \sqrt{\frac{1}{1} \times \frac{3}{2}} = \sqrt{1.5} > 1$$

$\therefore$ D'Alembert's Ratio test, the given series is convergent.

15. Is the series $\sum \frac{(n-2\log n)^n}{2^n n^n}$ convergent?

Solution:-

$$\lim (u_n)^{1/n} = \lim \frac{n-2\log n}{2n}$$
$$= \lim\left(\frac{1}{2} - \frac{\log n}{n}\right)$$
$$= \frac{1}{2} - 0 \quad \left[\because \lim_{n\to\infty} \frac{\log n}{n} = 0\right]$$
$$= \frac{1}{2} < 1.$$

$\therefore$ By Cauchy's root test, the given series is convergent.

16. Find the interval of convergence of $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots\cdots$

Solution:-

$$\lim \left|\frac{u_{n+1}}{u_n}\right| = \lim \left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}\right| = |x| \lim \frac{n}{n+1} = |x|$$

The series converges absolutely if $|x| < 1$ and diverges if $|x| > 1$.

For $|x| = 1$,

let $x = 1$, then, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots = \sum u_n$, then $\sum u_n$ is conditionally convergent.

For $x = -1$, then $-\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots\right)$ and its divergent.

Thus the given series convergent in the interval $-1 < x \le 1$.

17. Test the convergence of the series $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots\cdots + \frac{1}{n^n} + \cdots\cdots$

Solution:- For $n > 2$, $n^n > 2^n$

i.e. $\frac{1}{n^n} < \frac{1}{2^n}$

$\therefore \sum \frac{1}{n^n}$ is convergent by comparison test, since $\sum \frac{1}{2^n}$ is a convergent geometric series.

18. Test for convergent, the series

$$\frac{1}{3}x + \frac{1.2}{3.5}x^2 + \frac{1.2.3}{3.5.7}x^3 + \cdots\cdots + \frac{1.2.3.\cdots n}{3.5.7\cdots(2n+1)}x^n + \cdots\cdots$$

for all positive values of x.

Solution:-

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left(\frac{1.2.3\ldots n}{3.5.7\ldots(2n+1)} x^n\right)\left(\frac{3.5.\ldots[2(n+1)+1]}{1.2\ldots n(n+1)\, x^{n+1}}\right)$$
$$= \lim_{n\to\infty} \frac{2n+3}{n+1} \cdot \frac{1}{x}$$
$$= \frac{2}{x}.$$

$\therefore$ By ratio test, the given series is convergent if $2/x > 1$ and divergent if $2/x < 1$.

i.e. the series converges if $0 < x < 2$ and diverges if $x > 2$.

Now when $x = 2$,

$$\lim_{n\to\infty} n\left\{\frac{u_n}{u_{n+1}} - 1\right\} = \lim_{n\to\infty} n\left\{\frac{2n+3}{2n+2} - 1\right\} = \frac{1}{2} < 1.$$

$\therefore$ The series is diverges at $x = 2$ by Raabe's test.

19. Show that the series $\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} \cdots\cdots$ is convergent.

Solution:— The given series is an alternating series.

Here $u_n = (-1)^n \cdot \frac{\log(n+1)}{(n+1)^2}$

Since $\lim_{n\to\infty} \frac{\log n}{n} = 0$.

$\therefore \lim u_n = \lim \frac{\log(n+1)}{(n+1)^2} = 0$.

To satisfy the condition of convergence of alternating series, we need to show $u_{n+1} \le u_n$ for all n.

let $f(x) = \frac{\log x}{x^2}$

$\therefore f'(x) = \frac{1-2\log x}{x^3} < 0$ whenever $x > e^{1/2}$

Therefore $f(x)$ is monotonically decreasing for all $x > e^{1/2}$.

Thus $f(n+2) \le f(n+1) \ \forall n$

i.e. $u_{n+1} \le u_n \ \forall n$.

$\therefore$ The given series is convergent.

20. Check the convergence of the series

$\left(\frac{1}{2}\right)^p + \left(\frac{1.3}{2.4}\right)^p + \left(\frac{1.3.5}{2.4.6}\right)^p + \cdots\cdots$ for different values of p.

Solution:— Ignoring first term of the series,

$$u_n = \left(\frac{1.3.5.\cdots\cdots(2n-1)}{2.4.6.\cdots\cdots(2n)}\right)^p$$

then $$\frac{u_n}{u_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^p = \left[\frac{1+\frac{1}{n}}{1+\frac{1}{2n}}\right]^p$$

$\therefore \lim \frac{u_n}{u_{n+1}} = 1$,

and the test fails, applying logarithmic test.

Now, $$n\log\frac{u_n}{u_{n+1}} = n\left[p\log\left(1+\frac{1}{n}\right) - p\left(\log\left(1+\frac{1}{2n}\right)\right)\right]$$

$$= np\left[\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\cdots\right) - \left(\frac{1}{2n} - \frac{1}{8n^2} + \frac{1}{24n^3} - \cdots\right)\right]$$

$$= np\left[\frac{1}{2n} - \frac{3}{8n^2} + \frac{7}{24n^3} - \cdots\cdots\right]$$

$$= p\left[\frac{1}{2} - \frac{3}{8n} + \frac{7}{24n^2} - \cdots\right]$$

$\therefore \lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = \frac{p}{2}$.

$\therefore$ The series covergent if $p > 2$ and divergent if $p < 2$.

21. The series $x\log x + x^2\log 2x + \cdots + x^n\log nx + \cdots$ is convergent if (A) $x<1$ (B) $x\le 1$ (C) $x\ge 1$ (D) None.

Solution:— (A)

let $u_n = x^n \log nx$,

and $u_{n+1} = x^{n+1}\log(n+1)x$.

$$\therefore \frac{u_n}{u_{n+1}} = \left[\frac{\log nx}{\log(n+1)x}\right]\cdot\frac{1}{x}$$

$$= \left[\frac{\log(n+1)x}{\log nx}\right]^{-1}\cdot\frac{1}{x}$$

$$= \left[\frac{\log nx\left(1+\frac{1}{n}\right)}{\log nx}\right]^{-1}\cdot\frac{1}{x}$$

$$= \left[\frac{\log nx + \log\left(1+\frac{1}{n}\right)}{\log nx}\right]^{-1}\cdot\frac{1}{x}$$

$$= \left[1+\frac{\log\left(1+\frac{1}{n}\right)}{\log nx}\right]^{-1}\cdot\frac{1}{x}$$

$$= \left[1+\frac{\left(\frac{1}{n}\right)-\left(\frac{1}{2n^2}\right)+\cdots\cdots}{\log nx}\right]^{-1}\cdot\frac{1}{x}$$

$$= \left[1+\left(\frac{1}{n\log nx}-\frac{1}{2n^2\log nx}+\cdots\cdots\right)\right]^{-1}\cdot\frac{1}{x}$$

$$= \left[1-\frac{1}{n\log nx}+O\left(\frac{1}{n^2}\right)\right]\cdot\frac{1}{x}$$

$$\therefore \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

$\therefore$ From ratio test, the series is convergent, if $\frac{1}{x}>1$, i.e. $x<1$ and the series is divergent if $\frac{1}{x}<1$, i.e. $x>1$.

The ratio test fails if $x=1$.

$$\therefore n\left(\frac{u_n}{u_{n+1}}-1\right) = -\frac{1}{\log n}+O\left(\frac{1}{n}\right)$$

$$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = 0 < 1.$$

From Raabe's test, the series is divergent when $x=1$.

Hence the series is convergent if $x<1$ and divergent if $x\ge 1$.

22) The series $\sum n$ is
(A) convergent (B) Divergent (C) Oscillatory (D) None

Solution:- (B) $\sum n = 1+2+3+\cdots\cdots = \frac{n(n+1)}{2}$

$\therefore \lim_{n\to\infty} \frac{n(n+1)}{2} \to \infty$

$\therefore$ The series is divergent.

23) The series $\sum_{n=0}^{\infty} (2x)^n$ converges for x with
(A) $-1 \le x \le 1$ (B) $-\frac{1}{2} < x < \frac{1}{2}$ (C) $-2 < x < 2$ (D) $-\frac{1}{2} \le x \le \frac{1}{2}$.

Sol. (B) Let $\sum (2x)^n = 1 + 2x + (2x)^2 + \cdots\cdots$

This is a geometric series with common ratio $2x$.

$\therefore -1 < 2x < 1$.

$\Rightarrow -1/2 < x < \frac{1}{2}$.

24) If p and q are positive real nos, then the series
$\frac{2^p}{1^q} + \frac{3^p}{2^q} + \frac{4^p}{3^q} + \cdots\cdots$ ad inf. is convergent for
(A) $p < q-1$ (B) $p < q+1$ (C) $p \ge q-1$ (D) $p \ge q+1$.

Sol. (A) Here $u_n = \frac{(n+1)^p}{n^q}$

$\therefore v_n = \frac{n^p}{n^q} = \frac{1}{n^{q-p}}$.

$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{(n+1)^p}{n^p} = 1 \ne 0$.

$\therefore \sum v_n$ is convergent if $q-p > 1$, i.e. $p < q-1$.

$\therefore \sum u_n$ is convergent if $p < q-1$.

25) The series whose general term is $\sqrt{n+1} - \sqrt{n}$ is ——
(A) Convergent (B) Divergent (C) Oscillates finitely (D) Oscillates infinit

Sol. (B)

$$u_n = \sqrt{n+1} - \sqrt{n}$$
$$= \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})}$$
$$= \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Let $v_n = \frac{1}{\sqrt{n}}$

$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{2} \ne 0$

Since $\sum_{n=1}^{\infty} v_n$ is divergent, so the given series is divergent.

26) For the geometric series $1-k+k^2-k^3+\cdots\cdots$ which one of the following is FALSE ?

(A) The series is convergent if $k=\frac{-1}{3}$
(B) The series is divergent if $k=-3$
(C) the series is oscillatory if $k=-1$
(D) the series is divergent if $k=-2$.

Sol. (C) $1-k+k^2-k^3+\cdots\cdots = \frac{1}{1+k}$

∴ If $k=-1$, the series is $1+1+1+1+\cdots\cdots=\infty$.

The series is oscillatory if $k=-1$.

27) The series $1-\frac{1}{2}+1-\frac{3}{4}+1-\frac{7}{8}+\cdots\cdots$ is

(A) Convergent (B) Conditionally convergent (C) Absolutely convergent
(D) Oscillatory.

Sol. (D)

$$S_{2n}=\left(1-\frac{1}{2}\right)+\left(1-\frac{3}{4}\right)+\left(1-\frac{7}{8}\right)+\cdots\cdots+\left(1-\frac{2^{n-1}}{2^n}\right)$$

$$=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\cdots+\frac{1}{2^n}$$

$$=\frac{\frac{1}{2}\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}}$$

$$=\left[1-\left(\frac{1}{2}\right)^n\right]$$

$$\therefore \lim_{n\to\infty} S_{2n}=\lim_{n\to\infty}\left[1-\left(\frac{1}{2}\right)^n\right]$$

$$=1.$$

$$S_{2n+1}=\left(1-\frac{1}{2}\right)\left(1-\frac{3}{4}\right)+\cdots\cdots+\left(1-\frac{2^{n-1}}{2^n}\right)+1$$

$$=\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\cdots+\frac{1}{2^n}\right)+1$$

$$=\frac{\frac{1}{2}\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}}+1$$

$$=\left[1-\left(\frac{1}{2}\right)^n\right]+1$$

$$=2-\left(\frac{1}{2}\right)^n$$

$$\therefore \lim_{n\to\infty} S_{2n+1}=\lim_{n\to\infty}\left[2-\left(\frac{1}{2}\right)^n\right]=2.$$

∴ The series is oscillatory.

28) Which of the following is FALSE ?

(A) A series is said to be convergent if the sequence of partial sums converges.

(B) The nature of the series is not affected by adding or removing some terms.

(C) If $\lim_{n\to\infty} u_n = 0$, then the series $\sum u_n$ is convergent.

(D) If $u_n \geq v_n \geq 0$, then $\sum v_n$ is divergent if $\sum u_n$ is divergent.

Sol. (C) The necessary condition for convergence of a series $\sum u_n$ is

$$\lim_{n\to\infty} u_n = 0.$$

But if $\lim_{n\to\infty} u_n = 0$, then we can't say that the series is convergent. For example, $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\cdots$ is divergent but

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

29) The series $1-2x+3x^2-4x^3+\cdots\cdots$ where $0<x<1$, is ——

(A) Convergent (B) Divergent (C) Oscillatory (D) Semi-convergent

Sol. (A)

$$1-2x+3x^2-4x^3+\cdots\cdots = (1+x)^{-2}$$

For $-1<x<1$, the series is convergent.

Aliter:-

Here $|u_n| = nx^{n-1}$

$|u_{n+1}| = (n+1)x^n$

$\therefore$ Now, $\left|\frac{u_n}{u_{n+1}}\right| = \frac{n}{(n+1)}\cdot\frac{1}{x}$

$\therefore \lim \left|\frac{u_n}{u_{n+1}}\right| = \frac{1}{x}$

The series is convergent if $\frac{1}{x} > 1$, i.e. $x<1$.

" " " divergent if $\frac{1}{x} < 1$, i.e. $x > 1$.

30) The series $\sum \frac{x^n}{x^2+1}$ is divergent if

(A) $x<1$ (B) $x=1$ (C) $x>1$ (D) None.

Sol. (C) Here $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1}{x}\cdot\frac{(x+1)^2+1}{x^2}$

$$= \frac{1}{x}.$$

$\therefore$ the series is convergent if $\frac{1}{x} > 1$, i.e. $x<1$.

divergent if $\frac{1}{x} < 1$, i.e. $x > 1$.

31) Write down the conditions for which the alternating series $u_1 - u_2 + u_3 - u_4 + \cdots\cdots$ $(0<u_n<n)$ is convergent.

Sol. if $u_{n+1} < u_n$ and $u_n \to 0$ as $n\to\infty$.

32) The series

$x + x^{1+1/2} + x^{1+1/2+1/3} + \dots\dots$ is

(A) convergent if $x \geq \frac{1}{e}$ (B) Divergent if $x \geq \frac{1}{e}$ (C) convergent if $x > \frac{1}{e}$.

Solution:— (B)

Here $u_n = x^{1+1/2+\dots+1/n}$

$\therefore \frac{u_n}{u_{n+1}} = \frac{1}{x^{1/(n+1)}}$

$\therefore \lim \frac{u_n}{u_{n+1}} = 1.$

$\therefore$ D'Alembert's ratio test fails.

Now, $n\log\frac{u_n}{u_{n+1}} = n\log\frac{1}{x^{1/n+1}}$

$= n\log\left(\frac{1}{x}\right)^{1/(n+1)}$

$= \frac{n}{n+1}\log\frac{1}{x}$

$\therefore \lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{n}{n+1}\log\frac{1}{x}$

$= \log\frac{1}{x}$

$\therefore$ By Logarithmic test the series is convergent if $\log\frac{1}{x} > 1$

i.e. $\frac{1}{x} > e$

i.e. $x < \frac{1}{e}$.

The series is divergent if $x > \frac{1}{e}$.

The test fails at $x = 1/e$.

Now, $n\log\frac{u_n}{u_{n+1}} - 1 = \frac{n}{n+1}\log e - 1$

$= \frac{n}{n+1} - 1$

$= \frac{-1}{n+1}$

$\therefore \lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}} - 1\right)\cdot\log n = -\lim_{n\to\infty}\frac{1}{n+1}\cdot\log n$

$= -\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}\cdot\frac{\log n}{n}$

$= (-1)\times 0$

$= 0. < 1.$

$\therefore$ By Higher Logarithmic test the series is divergent at $x = 1/e$.

33) The absolute convergent series is

(A) $1-1+1-\cdots+(-1)^{n-1}+\cdots$ (B) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{(-1)^{n-1}}{n}+\cdots$

(C) $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots+\frac{(-1)^{n-1}}{\sqrt{n}}+\cdots$ (D) $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots+\frac{(-1)^{n-1}}{n^2}+\cdots$

Sol. Ans. (D)

(A) The series $1-1+1-1+\cdots+(-1)^{n-1}\cdots$ is oscillatory.

(B) The infinite series is $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{(-1)^{n-1}}{n}+\cdots$

By Leibnitz's test the series is convergent.

Now, $\sum|u_n| = 1+\frac{1}{2}+\frac{1}{3}+\cdots$

which is a p-series with $p=1$. Hence $\sum|u_n|$ is divergent.

i.e. the given series is conditionally convergent.

(C) same as (B), here $p=1/2$, so, the series is conditionally convergent series.

(D) The given series is $1-\frac{1}{2^2}+\frac{1}{3^2}\cdots$

By Leibnitz test the series is convergent.

Now, $\sum|u_n| = 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \sum\frac{1}{n^2}$

is convergent series.

So, the given series is absolutely convergent.

34) S.T. the series $1-2+3-4+5\cdots$ is oscillatory.

Sol. The series is an alternating series.

$$S_n = (1-2)+(3-4)+(5-6)+\cdots+\{(n-1)-n\}$$
$$= (-1)+(-1)+(-1)+\cdots+(-1)$$
$$= -\frac{n}{2}.$$

$\therefore \lim_{n\to\infty} S_n = -\infty.$

The series can be written as

$$S_n = 1-(2-3)-(4-5)-(6-7)-\cdots-\{(n-1)-n\}$$
$$= 1+1+1+\cdots+1$$
$$= \frac{n+1}{2}$$

$\therefore \lim_{n\to\infty} S_n = \infty.$

Since the limit does not exist because the sum of infinite terms of the series is not unique.

Therefore the series is oscillatory.

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35) Show that the series $\frac{1}{1+2^{-1}} + \frac{2}{1+2^{-2}} + \frac{3}{1+2^{-3}} + \cdots\cdots$ is divergent.

Sol. Here, $u_n = \frac{n}{1+2^{-n}}$

$$\therefore \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{1+2^{-n}}$$

$$= \lim_{n\to\infty} \frac{n}{1+\frac{1}{2^n}}$$

$$= \infty \neq 0.$$

Hence the series is divergent.

36) S.T. the series $\frac{2}{1} + \frac{3}{4} + \frac{4}{9} + \frac{5}{16} + \cdots + \frac{n+1}{n^2} + \cdots\cdots$ is divergent

Sol. $u_n = \frac{n+1}{n^2}$

$v_n = \frac{1}{n}$

$$\therefore \frac{u_n}{v_n} = \frac{n^2+n}{n^2} = 1+\frac{1}{n}$$

$$\therefore \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$

$$= 1 \neq 0$$

$\therefore$ As $\sum v_n$ is diverges so does $\sum u_n$.

37) S.T. The series $\frac{1}{2} + \frac{\sqrt{2}}{5} + \frac{\sqrt{3}}{10} + \cdots + \frac{\sqrt{n}}{n^2+1} + \cdots\cdots$ is convergent.

Sol. $u_n = \frac{\sqrt{n}}{n^2+1} = \frac{n^{1/2}}{n^2+1}$

$\therefore v_n = n^{3/2}$

$$\therefore \frac{u_n}{v_n} = \frac{n^2}{n^2+1}$$

$$\therefore \lim \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n^2}\right)}$$

$$= 1 \neq 0$$

As $\sum v_n$ is convergent, so does $\sum u_n$, by Comparison test.

38) Test the convergence of the following series

$$\frac{1}{\sqrt{2}-1} - \frac{1}{\sqrt{3}-1} + \frac{1}{\sqrt{4}-1} - \cdots\cdots\cdots$$

Sol. (i) The series is an alternating series

(ii) $\frac{1}{\sqrt{2}-1} > \frac{1}{\sqrt{3}-1} > \frac{1}{\sqrt{4}-1} > \cdots\cdots$

(iii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}-1} = 0.$

Hence all the conditions of Leibnitz's test are satisfied. So, The series is divergent.

39) Test the convergence of the following series

$$1+2-3+1+2-3+\cdots\cdots$$

<u>Sol.</u> The given series is $\sum u_n = 1+2-3+1+2-3+\cdots$

$s_{3n} = (1+2-3)+(1+2-3)+\cdots\cdots+(1+2-3) = 0$

$\therefore \lim s_{3n} = 0.$

$s_{3n+1} = (1+2-3)+\cdots\cdots+(1+2-3)+1$
$= 1.$

$\therefore \lim s_{3n+1} = 1.$

$s_{3n+2} = (1+2-3)+\cdots\cdots+(1+2-3)+1+2$
$= 3.$

$\therefore \lim s_{3n+2} = 3.$

$\therefore$ The limit does not exist. Hence the given series is oscillatory and so it is not convergent.

40) Test the convergence of the series

$$1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\cdots\cdots$$

<u>Sol.</u> The given series ~~can be~~ ~~considered as~~ $\sum u_n$, where

$$u_n = \frac{n^n}{(n+1)^{n+1}}$$

Taking $v_n = \frac{n^n}{n^{n+1}} = \frac{1}{n}$

$$\therefore \lim \frac{u_n}{v_n} = \frac{n^{n+1}}{(n+1)^{n+1}} = \lim \frac{1}{\left(1+\frac{1}{n}\right)^{n+1}}$$

$$= \lim \frac{1}{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)}$$

$$= \frac{1}{e}\times 1$$

$$= \frac{1}{e} \neq 0.$$

$\therefore$ By comparison test the two series $\sum u_n$ and $\sum v_n$ behave alike.

As $\sum v_n$ is divergent, so does $\sum u_n$.

$\therefore$ The series is not convergent.