REAL ANALYSIS

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REAL/MATHEMATICAL ANALYSIS

- **SEQUENCES OF REAL NUMBERS:** The word "sequence" is used to convey the idea that the things are arranged in order.

Definition :→ A 'sequence' of real numbers is a function defined on the set N of natural numbers whose range is a subset of the set $\mathbb{R}$ of real numbers; i.e. if for every $n \in N$, $\exists$ a real number a_n, then the order set

$$a_1, a_2, \ldots, a_n, \ldots$$

is said to define a sequence of real nos.

Remark :→ 1) $f: A \to B$ is a mapping or function if for every $x \in A$, $\exists$ a unique value of $y \in B$.

Here, we write $y = f(x)$ where $x \in A$, $y \in B$.

2) $y = \pm x$ is not a function, it's a relation.

Notation :→ 1) If a_n is the n^{th} term of a sequence, then we write $a_1, a_2, \ldots, a_n, \ldots$, to describe the sequence.

2) $f: N \to \mathbb{R}$ is a sequence.

3) $\{a_n\}, \{b_n\}, \{x_n\}, \{y_n\}$.

The main question we are concerned with here is to decide whether or not the term a_n tends to a finite quantity when n increases indefinitely.

Definition :→ A sequence $\{a_n\}$ is said to have a limit $l \in \mathbb{R}$ if, for every $\epsilon > 0$, $\exists$ a natural number $N(\epsilon)$, $\ni |a_n - l| < \epsilon$, for all $n \geq N(\epsilon)$

Example.(1) Prove that $\lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$.

Soln.→ Let $\epsilon > 0$ be an arbitrary number, then

$$|a_n - l| < \epsilon$$
$$\Rightarrow \left|\frac{1}{n} - 0\right| < \epsilon$$
$$\Rightarrow \frac{1}{n} < \epsilon$$
$$\Rightarrow n > \frac{1}{\epsilon}$$

In particular if $\epsilon = 0.1$, then

$$\left|\frac{1}{n} - 0\right| < \epsilon = 0.1$$
$$\Rightarrow n > \frac{1}{\epsilon} = 10$$
$$\therefore \left|\frac{1}{n} - 0\right| < \epsilon = 0.1$$

whenever $n \geq 11 = N(\epsilon = 0.1)$

If $\epsilon = 0.01$, then

$$\left|\frac{1}{n} - 0\right| < \epsilon = 0.01$$
$$\Rightarrow n > \frac{1}{\epsilon} = 100$$
$$\Rightarrow n > 101$$

Hence take $N(\epsilon) = \left[\frac{1}{\epsilon}\right] + 1$

- Or, choose a natural no. $N(\epsilon)$ which is $> \frac{1}{\epsilon}$.

Then $\exists$ a natural no. $N(\epsilon) \ni \left|\frac{1}{n} - 0\right| < \epsilon \ \forall \ n \geq N(\epsilon)$

Hence, by definition, $\lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$

$$|a_n - l| < \epsilon \text{ for } n \geq N(\epsilon)$$
$$\Rightarrow l - \epsilon < a_n < l + \epsilon \text{ for } n \geq N(\epsilon)$$

For $0 < \epsilon' < \epsilon$, then in general,

$$N(\epsilon') > N(\epsilon)$$

If ϵ is small, then $N(\epsilon)$ will be sufficiently large to ensure $|a_n - l| < \epsilon$ for $n \geq N(\epsilon)$. And all the members a_n, $n \geq N(\epsilon)$ are in the small interval, $(l-\epsilon, l+\epsilon)$ i.e. then a_n is very close to l.

If a sequence $\{a_n\}$ has a finite limit 'l', then we say that the sequence $\{a_n\}$ converges to l or the sequence is convergent. If a sequence does not converge to a finite limit, then it is said to be divergent. If $\{a_n\}$ converges to 'l'; we write

$$\lim_{n\to\infty}(a_n) = l, \text{ or, } \lim(a_n) = l.$$

<u>Example. (2).</u> Prove that $\lim_{n\to\infty}\left(\frac{1}{n+1}\right) = 0$.

<u>Soln.</u> $\rightarrow$ Let $\epsilon > 0$ be an arbitrary number,

Then $|a_n - l| < \epsilon$

$$\Rightarrow \left|\frac{1}{n+1} - 0\right| < \epsilon$$
$$\Rightarrow \frac{1}{n+1} < \epsilon$$
$$\Rightarrow n < \frac{1}{\epsilon} - 1$$

Choose a natural number $N(\epsilon)$ which is $> \frac{1}{\epsilon} - 1$.

Then, $\exists$ a natural no. $N(\epsilon) \ni \left|\frac{1}{n+1} - 0\right| < \epsilon$ for all $n \geq N(\epsilon)$

By defn. $\lim_{n\to\infty}\left(\frac{1}{n+1}\right) = 0$

<u>Alt.</u> $\rightarrow$ $\left|\frac{1}{n+1} - 0\right| = \frac{1}{n+1} < \frac{1}{n} < \epsilon$

$$\Rightarrow n > \frac{1}{\epsilon}$$

Take $N(\epsilon) = \left[\frac{1}{\epsilon}\right] + 1$

Then $\exists$ a natural no. $N(\epsilon)$, $\ni \left|\frac{1}{n+1} - 0\right| < \epsilon \ \forall \ n \geq N(\epsilon)$

$\therefore$ By defn. $\lim_{n\to\infty}\left(\frac{1}{n+1}\right) = 0$.

Example (3). Prove that $\lim_{n\to\infty}\left(\frac{1}{n^2+1}\right)=0$

Soln. → Let $\epsilon>0$ be an arbitrary number.

$$|a_n-l|<\epsilon$$

$$\Rightarrow \left|\frac{1}{n^2+1}-0\right| = \frac{1}{n^2+1} < \frac{1}{n^2} < \epsilon$$

$$\Rightarrow n > \frac{1}{\sqrt{\epsilon}}$$

Take $N(\epsilon)=\left[\frac{1}{\sqrt{\epsilon}}\right]+1$

Then $\exists$ a natural no. $N(\epsilon)$, $\ni$ $\left|\frac{1}{n^2+1}-0\right|<\epsilon \;\; \forall\, n \geq N(\epsilon)$.

$\therefore$ By defn. $\lim_{n\to\infty}\left(\frac{1}{n^2+1}\right)=0$.

Example (4). Prove that $\lim_{n\to\infty}\left(\frac{2n^2+1}{n^2+n}\right)=2$.

Soln. → Let $\epsilon>0$ be an arbitrary number.

$$\therefore |a_n-l|<\epsilon$$

$$\Rightarrow \left|\frac{2n^2+1}{n^2+n}-2\right|<\epsilon$$

$$\Rightarrow \frac{\cancel{2n^2}+1-\cancel{2n^2}-2n}{n^2+n}<\epsilon$$

$$\Rightarrow \frac{1-2n}{n^2+n}<\epsilon$$

$$\Rightarrow \frac{2n-1}{n^2+n} < \frac{2n}{n^2} = \frac{2}{n} < \epsilon$$

$$\Rightarrow n > \frac{2}{\epsilon}.$$

Take $N(\epsilon)=\left[\frac{2}{\epsilon}\right]+1$

Then $\exists$ a natural no. $N(\epsilon)$, $\ni$ $\left|\frac{2n^2+1}{n^2+n}-2\right|<\epsilon \;\; \forall\, n\geq N(\epsilon)$

$\therefore$ By defn. $\lim_{n\to\infty}\left(\frac{2n^2+1}{n^2+n}\right)=2$.

Example (5). Prove that $\lim_{n\to\infty}\left(\frac{1}{n^p}\right)=0$, $p>0$.

Soln. → Let $\epsilon>0$ be an arbitrary number.

$$|a_n-l|<\epsilon$$

$$\Rightarrow \left|\frac{1}{n^p}-0\right|<\epsilon$$

$$\Rightarrow \frac{1}{n^p}<\epsilon$$

$$\Rightarrow n^p > \frac{1}{\epsilon}$$

$$\Rightarrow n > \left(\frac{1}{\epsilon}\right)^{1/p}$$

since $p>0$, Take $N(\epsilon)=\left[\frac{1}{\sqrt[p]{\epsilon}}\right]+1$.

Then $\exists$ a natural no. $N(\epsilon)$, $\ni$ $\left|\frac{1}{n^p}-0\right|<\epsilon \;\; \forall\, n\geq N(\epsilon)$

Example (7). ~~Proof~~ Prove that $\lim_{n\to\infty}(r^n)=0$ if $|r|<1$.

Soln.→ Let $\epsilon>0$ be an arbitrary number.

Then If $|r^n-0|<\epsilon$

$\Rightarrow |r^n|<\epsilon$

$\Rightarrow n\ln|r| < \ln\epsilon$

$\Rightarrow n > \dfrac{\ln\epsilon}{\ln|r|}$ [since $|r|<1$, $\ln|r|<0$]

Choose a natural no. $N(\epsilon)$ which is $\geqslant \dfrac{\ln\epsilon}{\ln|r|}$

Then $\exists$ a natural no. $N(\epsilon)$, $\ni |r^n-0|<\epsilon \;\forall\; n\geqslant N(\epsilon)$

$\therefore$ By defn. $\lim_{n\to\infty}(r^n)=0$ if $|r|<1$.

Example (8). Prove that $\lim_{n\to\infty}\left(2-\frac{1}{2^n}\right)=2$

Soln.→ Let $\epsilon>0$ be an arbitrary no.

$\therefore \left|2-\frac{1}{2^n}-2\right|<\epsilon$

$\Rightarrow \left|-\frac{1}{2^n}\right|<\epsilon$

$\Rightarrow \frac{1}{2^n}<\frac{1}{n}<\epsilon$

$\Rightarrow n>\frac{1}{\epsilon}$

Take $N(\epsilon)=\left[\frac{1}{\epsilon}\right]+1$

Then $\exists$ a natural no. $N(\epsilon)$, $\ni \left|2-\frac{1}{2^n}-2\right|<\epsilon \;\forall\; n\geqslant N(\epsilon)$

$\therefore$ By defn. $\lim_{n\to\infty}\left(2-\frac{1}{2^n}\right)=2$.

Example (9). Prove that $\lim_{n\to\infty}(2^{1/n})=1$.

Soln.→ Let $\epsilon>0$ be an arbitrary no.

$\therefore |2^{1/n}-1|<\epsilon$

$\Rightarrow 2^{1/n}<\epsilon+1$

$\Rightarrow \frac{1}{n}\ln 2<\ln(\epsilon+1)$

$\Rightarrow \frac{1}{n}<\frac{\ln(\epsilon+1)}{\ln 2}$.

$\Rightarrow n>\frac{\ln 2}{\ln(\epsilon+1)}$

Take $N(\epsilon)=\left[\frac{\ln 2}{\ln(\epsilon+1)}\right]$

$\therefore n>N(\epsilon)$. [P]

Example (10). Prove that $\lim_{n\to\infty}(3^{-n})=0$.

Soln. $\rightarrow$ Let $\epsilon>0$ be an arbitrary number.

$$\therefore |3^{-n}|<\epsilon$$

$$\Rightarrow \frac{1}{3^n}<\epsilon$$

$$\Rightarrow 3^n>\frac{1}{\epsilon}$$

$$\Rightarrow n\ln 3>\ln\frac{1}{\epsilon}$$

$$\Rightarrow n>\frac{\ln 1/\epsilon}{\ln 3}.$$

Take $N(\epsilon)=\left[\frac{\ln 1/\epsilon}{\ln 3}\right]$

$\therefore n>N(\epsilon)$ [P]

Example (11). Prove that $\lim_{n\to\infty}\left(\frac{10^6.n}{n^2+n}\right)=0$

Soln. $\rightarrow$ Let $\epsilon>0$ be an arbitrary number.

$$\therefore \left|\frac{10^6.n}{n^2+n}\right|=\frac{10^6}{n+1}<\epsilon$$

$$\Rightarrow n+1>\frac{10^6}{\epsilon}$$

$$\Rightarrow n>\frac{10^6}{\epsilon}-1$$

Take $N(\epsilon)=\left[\frac{10^6}{\epsilon}-1\right]+1$

Then $\exists$ a natural number $N(\epsilon)$ $\ni$ $\left|\frac{10^6.n}{n^2+1}\right|<\epsilon$ $\forall$ $n\geq N(\epsilon)$

$\therefore$ By defn. $\lim_{n\to\infty}\left(\frac{n.10^6}{n^2+n}\right)=0$ if $\left|\frac{n.10^6}{n^2+n}\right|<1$.

Example (12). Prove that $\lim_{n\to\infty}\left(\frac{\sqrt{n}}{n^2+n}\right)=0$

Soln. $\rightarrow$ Let $\epsilon>0$ be an arbitrary number.

$$\therefore \left|\frac{\sqrt{n}}{n^2+n}\right|<\frac{\sqrt{n}}{n}<\epsilon$$

$$\Rightarrow \frac{1}{\sqrt{n}}<\epsilon$$

$$\Rightarrow \frac{1}{n}<\epsilon^2$$

$$\Rightarrow n>\frac{1}{\epsilon^2}$$

Take $N(\epsilon)=\left[\frac{1}{\epsilon^2}\right]+1$.

Then $\exists$ a natural no. $N(\epsilon)$ $\ni$ $\left|\frac{\sqrt{n}}{n^2+n}\right|<\epsilon$ $\forall$ $n\geq N(\epsilon)$

$\therefore$ By defn. $\lim_{n\to\infty}\left(\frac{\sqrt{n}}{n^2+n}\right)=0$.

Example (13). Prove that $\lim_{n\to\infty}\left(\frac{n^2-1}{2n^2+3n}\right)=\frac{1}{2}$

Soln. → Let $\epsilon>0$ be an arbitrary number.

$$\therefore \left|\frac{n^2-1}{2n^2+3n}-\frac{1}{2}\right| = \left|\frac{\cancel{2n^2}-2-\cancel{2n^2}-3n}{2(2n^2+3n)}\right|$$

$$= \frac{2+3n}{2(2n^2+3n)} < \frac{3n}{4n^2} = \frac{3}{4n} < \epsilon$$

$$\Rightarrow n > \frac{3}{4\epsilon}$$

Take $N(\epsilon) = \left[\frac{3}{4\epsilon}\right]+1$

Then $\exists$ a natural no. $N(\epsilon)$ $\ni$ $\left|\frac{n^2-1}{2n^2+3n}-\frac{1}{2}\right| < \epsilon \ \forall\ n \geq N(\epsilon)$

$\therefore$ By defn., $\lim_{n\to\infty}\left(\frac{n^2-1}{2n^2+3n}\right)=\frac{1}{2}$

Example 14. Prove that $\lim_{n\to\infty}\left\{(-1)^n\frac{1}{n}\right\}=0$

Soln. → Let $\epsilon>0$ be an arbitrary no.

$$\therefore \left|(-1)^n\frac{1}{n}\right| = \frac{1}{n} < \epsilon$$

$$\Rightarrow n > \epsilon \quad [P]$$

Example 15. Show that → i) $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)=0$

ii) $\lim_{n\to\infty}\frac{\sin^2 \angle n}{n^2+1}=0.$

Soln. → i) Let $\epsilon>0$ be an arbitrary no.

$$\therefore \left|\sqrt{n+1}-\sqrt{n}\right|$$

$$= \left|\frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}}\right| = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \epsilon \Rightarrow \frac{1}{\sqrt{n+1}} < \epsilon$$

$$\Rightarrow \sqrt{n+1} > \epsilon$$

$$\Rightarrow n+1 > \epsilon^2$$

$$\Rightarrow n > \epsilon^2 - 1 \quad [P]$$

ii) Let $\epsilon>0$ be an arbitrary no.

$$\therefore \left|\frac{\sin^2\angle n}{n^2+1}-0\right| = \frac{\sin^2\angle n}{n^2+1} < \frac{1}{n^2} < \epsilon$$

$$\Rightarrow n > \frac{1}{\sqrt{\epsilon}}$$

Take $N(\epsilon) = \left[\frac{1}{\sqrt{\epsilon}}\right]+1$

Then $\exists$ a natural no. $N(\epsilon)$ $\ni$ $\left|\frac{\sin^2\angle n}{n^2+1}-0\right| < \epsilon \ \forall\ n > N(\epsilon)$

$\therefore \lim_{n\to\infty}\frac{\sin^2\angle n}{n^2+1}=0.$

REMARK :→

1) It is important to realize that the definition of the limit of a sequence is used to verify whether a propose value l is a limit of particular sequence or not. But in most of the sequences, we can make a guess about the limit by computing first few terms of the sequence.

2) It is also important to realize that the convergence and divergence of a sequence are the property of the tails of the sequence. In other words, the convergence and divergence depend on the behaviour of the ultimate terms.
Hence, the alternation of the finite number of term of a sequence doesnot effect the convergence or divergence of the sequence, i.e. a convergence sequence does not alter the limit. For e.g.
$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots, \frac{1}{n}, \dots$ & $1000, 100, 10, 1, \dots, \frac{1}{n}, \dots$ have the same limit.

Theorem :→ If a sequence is convergent then its limit is unique.

Proof :→ Let $\{a_n\}$ be a convergent sequence. If possible, let, l and l' be the two limits of $\{a_n\}$.
Then for a $\epsilon > 0$, $\exists$ a natural no. $N_1(\epsilon) \ni |a_n - l| < \epsilon \ \forall n \geq N_1(\epsilon)$.
Then also, $\exists$ a natural no. $N_2(\epsilon) \ni |a_n - l'| < \epsilon \ \forall n \geq N_2(\epsilon)$.

Take $N(\epsilon) = \max\{N_1(\epsilon), N_2(\epsilon)\}$

Then if $n \geq N(\epsilon)$, we have
$|a_n - l| < \epsilon$ and $|a_n - l'| < \epsilon$

Now, $0 \leq |l - l'| = |(a_n - l') - (a_n - l)| \leq |a_n - l| + |a_n - l'| < 2\epsilon \quad \forall n \geq N(\epsilon)$.

Hence, for any $\epsilon > 0$, we have $0 \leq |l - l'| < 2\epsilon$

Hence, $l - l' = 0 \Leftrightarrow l = l'$

[Lemma : If for every $\epsilon > 0$, $0 \leq a < \epsilon$ then $a = 0$.
Proof: If possible, let $a > 0$
Take $\epsilon = \frac{a}{2} > 0$, then $a = 2\epsilon$, i.e. $a \nless \epsilon$
which is the contradiction to the fact that for every $\epsilon > 0$, $0 \leq a < \epsilon$.

therefore, $a = 0$]

Bounded Sequences : A sequence $\{a_n\}$ is said to be bounded if $\exists$ two real number m and M $\ni$ $m \le a_n \le M$ $\forall n \in \mathbb{N}$.
If a sequence is not bounded then it is called unbounded sequence.

Example : The sequence $\{\frac{1}{n}\}$, $\{(-1)^n+1\}$ are bounded.
Since $0 < \frac{1}{n} \le 1$ and $(-1)^n+1 = \begin{cases} 0 & \text{if } n = \text{odd} \\ 2 & \text{if } n = \text{even}. \end{cases}$

- The sequence $\{n^2\}$, $\{n^2+n\}$ are unbounded.

Theorem : A convergent sequence is bounded.

Proof : Let $\{a_n\}$ converges to the limit 'l', then for a given $\epsilon > 0$, $\exists$ natural number $N(\epsilon)$, $\ni$ $|a_n - l| < \epsilon$ $\forall n \ge N(\epsilon)$

$$\Rightarrow |a_n| - |l| \le |a_n - l| < \epsilon \quad \forall n \ge N(\epsilon)$$

$$\Rightarrow |a_n| < |l| + \epsilon \quad \forall n \ge N(\epsilon)$$

Take $M = \max\{|l|+\epsilon, \ldots\ldots, |a_{N(\epsilon)-1}|\}$

Clearly, M is finite.

Then $|a_n| \le M$ $\forall n \in \mathbb{N}$

$$\Rightarrow -M \le a_n \le M \quad \forall n \in \mathbb{N}.$$

Corollary : Unbounded sequences are divergent.

~~Remark~~ Example : A bounded sequence may not be convergent.

- The sequence $\{(-1)^n+1\}$ is bounded but not convergent.

Soln: Clearly, the sequence is bounded.

$$(-1)^n+1 = \begin{cases} 0 & \text{if } n = \text{odd} \\ 2 & \text{if } n = \text{even}. \end{cases}$$

If $n \to \infty$ through odd integers then $a_n \to 0$, but if $n \to \infty$ through even integers then $a_n \to 2$.
Hence the sequence has no unique limit. But for a convergent sequence, limit is unique. $\therefore$ The sequence is not convergent.

Example : Note that the sequence $\{n^2\}$, $\{(-1)^n . n\}$ are unbounded, hence ~~they~~ they are divergent by the above corollary.

Divergent Sequence :→ The term "convergent sequence" is said to convey for the sequence whose limit is finite and unique. The sequence which is not convergent is called "divergent sequence". There are two possibility —

i) a sequence with no unique limit.

ii) a sequence with an infinite limit.

Definition :→

(a) We say that $\{x_n\}$ diverges to $+\infty$ and write $\lim(x_n) = +\infty$ if for every $\epsilon > 0$, $\exists$ a natural no. $N(\epsilon) \ni x_n > \epsilon \ \forall n \geq N(\epsilon)$.

(b) We say that $\{x_n\}$ diverges to $-\infty$ and write $\lim(x_n) = -\infty$ if for every $\epsilon < 0$, $\exists$ a natural no. $N(\epsilon), \ni x_n < -\epsilon \ \forall n \geq N(\epsilon)$.

Properly divergent :— If $\lim(x_n) = +\infty$ or $\lim(x_n) = -\infty$, then the sequence $\{x_n\}$ is called properly divergent.

Examples :→ show that →

(a) $\lim\limits_{n\to\infty}(n^2) = +\infty$

Soln.→ Let $\epsilon > 0$ be an arbitrary no.

Then $a_n > \epsilon \Rightarrow n^2 > \epsilon$

$\Rightarrow n > \sqrt{\epsilon}$

Take $N(\epsilon) = [\sqrt{\epsilon}] + 1$

Then $\exists$ a natural no. $N(\epsilon) \ni n^2 > \epsilon \ \forall n \geq N(\epsilon)$.

By defn. $\lim(n^2) = +\infty$.

(b) $\lim\left(\frac{n}{\sqrt{n-1}}\right) = +\infty$

Soln.→

Let $\epsilon > 0$ be an arbitrary no.

Then $\frac{n}{\sqrt{n-1}} > \frac{n}{\sqrt{n}} = \sqrt{n} > \epsilon$

$\Rightarrow n > \epsilon^2$

Take $N(\epsilon) = [\epsilon^2] + 1$

Then $\exists$ a natural no. $N(\epsilon) \ni$, $\frac{n}{\sqrt{n-1}} > \epsilon \ \forall n \geq N(\epsilon)$

Hence, $\lim \frac{n}{\sqrt{n-1}} = +\infty$.

(c) $\lim (r^n) = +\infty$ if $r > 1$.

Soln. → Let $\epsilon > 0$ be an arbitrary no.
Then $r^n > \epsilon$ if $r > 1$.

$$\Rightarrow n \ln r > \ln \epsilon$$
$$\Rightarrow n > \frac{\ln \epsilon}{\ln r}$$

choose a natural no. $N(\epsilon)$ which is $> \frac{\ln \epsilon}{\ln r}$.

Then $\exists$ a natural no. $N(\epsilon)$ $\ni$ $r^n > \epsilon$ $\forall$ $n \geq N(\epsilon)$

By defn, $\lim (r^n) = +\infty$ if $r > 1$.

Oscillatory Sequences :–

(a) A bounded sequence which is not convergent is said to be an oscillatory sequence with finite oscillation.

(b) An unbounded sequence which is not properly divergent is said to be an oscillatory sequence with infinite oscillation.

Example :→

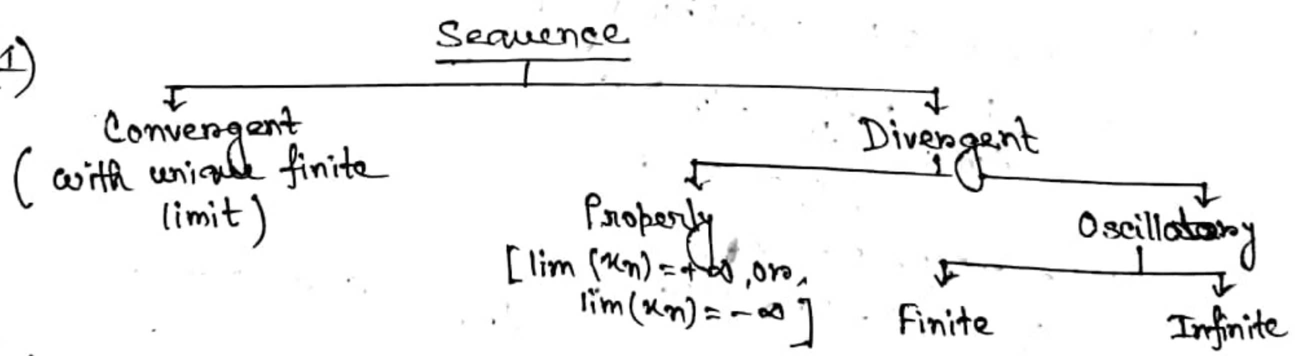
(a) The sequence $\{2^n\}$ is a properly divergent and diverges to $+\infty$.

(b) $\{-n^2\}$ is a properly divergent and diverges to $-\infty$.

(c) $\{(-1)^n + 1\}$ is a bounded sequence but not convergent. Hence, it is an oscillatory sequence with finite oscillation.

(d) $\{(-1)^n . n\}$ is unbounded and not properly divergent. It is an oscillatory sequence with infinite oscillation.

(1)



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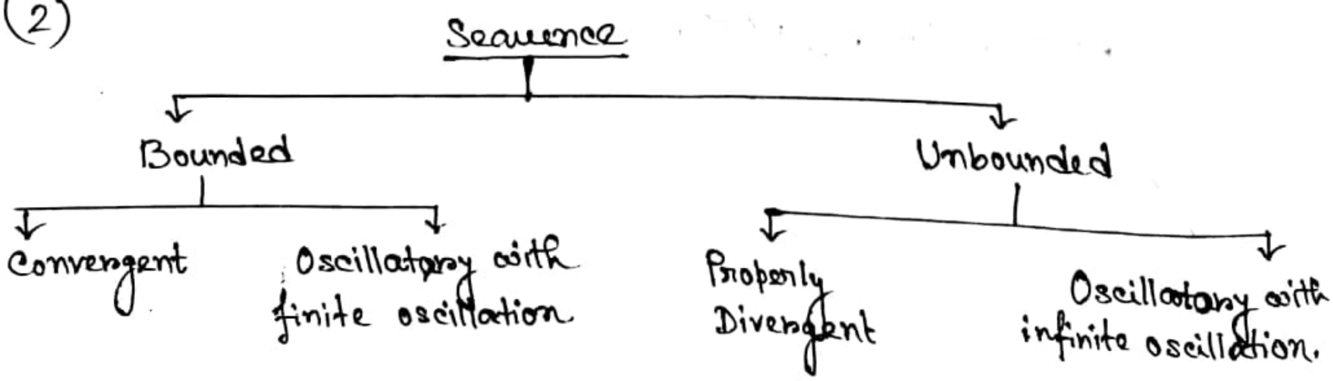

Example :→ Check the convergence or divergence of the sequence $\{r^n\}$ for different values of r.

Soln.→
If $|r| < 1$, $\lim (r^n) = 0$
If $r = 1$, then $\{r^n\}$ becomes an constant sequence:
$1, 1, 1, \ldots\ldots$ & $\lim (r^n) = 1$.
If $r = -1$, then $\{r^n\} = \{(-1)^n\}$ and
$\{r^n\}$ is an oscillatory sequence with finite oscillation.
If $r > 1$, $\lim (r^n) = +\infty$
If $r < -1$ then $r = -s$, $s > 1$.
and $\{r^n\} = \{(-1)^n . s^n\}$ is an oscillatory sequence with infinite oscillation.

Sandwich Theorem
(OR)
Squeeze Theorem :— If $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are three sequences of real numbers $\ni$ $a_n \leq b_n \leq c_n$ for $n \geq K$, and $\lim (a_n) = l = \lim (c_n)$, then $\lim (b_n) = l$.

Example :→ 1) Evaluate the following limits.

(a) $\lim\limits_{n \to \infty} (\sqrt{n+1} - \sqrt{n}) = 0$

Soln.→ Let, $b_n = \sqrt{n+1} - \sqrt{n}$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} = \frac{2}{\sqrt{n}}$$

Then $a_n = \frac{2}{\sqrt{n+1}} < b_n = \frac{1}{\sqrt{n+1} + \sqrt{n}} < c_n = \frac{2}{\sqrt{n}}$ $\forall$ $n \in N$

Note that $\lim (a_n) = 0 = \lim (c_n)$
$\therefore$ By squeeze theorem, $\lim (b_n) = 0$.

(b) $\lim\limits_{n \to \infty} \left\{ \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots\cdots + \frac{1}{(n+n)^2} \right\}$

Soln.→ Let, $b_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots\cdots + \frac{1}{(n+n)^2}$

Now, $n . \frac{1}{4n^2} < b_n < \frac{1}{n^2}$ $\forall$ $n \geq 1$

$\Rightarrow a_n = \frac{1}{4n} < b_n < \frac{1}{n^2} = c_n$ $\forall$ $n \geq 1$.

Note that, $\lim (a_n) = 0 = \lim (c_n)$

$\therefore$ By squeeze theorem, $\lim\limits_{n \to \infty} b_n = 0$.

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(c) $\lim_{n\to\infty} (2^n+3^n)^{1/n}$

Soln.→ Note that $(3^n)^{1/n} < b_n = (2^n+3^n)^{1/n} < (2.3^n)^{1/n}$

$\Rightarrow a_n = 3 < b_n < 3.2^{1/n} = c_n$

Clearly $\lim(a_n) = 3 = \lim(c_n)$

By squeeze theorem, $\lim(b_n) = 3$.

$\Rightarrow \lim_{n\to\infty} (2^n+3^n)^{1/n} = 3.$

Examples. 2> show that —

(a) $\lim_{n\to\infty} (a^{1/n}) = 1$, $a>0$.

Soln.→

(a) Case I :→ $a>1$.

Then $(a^{1/n}-1) > 0 \ \forall n \in \mathbb{N}$.

Define a new sequence $x_n = (a^{1/n}-1) > 0 \ \forall n \in \mathbb{N}$

Hence, $a = (1+x_n)^n \geq 1 + n.x_n$

[For $x>0$, $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}.x^2+\dots+x^n$

$\therefore (1+x)^n \geq 1+nx$, $n \geq 1$]

Now, $0 < x_n \leq \frac{a-1}{n}$, $n \geq 1$

Let ϵ be an arbitrary no.

then $|x_n - 0| = x_n \leq \frac{a-1}{n} < \epsilon$.

$\Rightarrow n > \frac{a-1}{\epsilon}$

Take $N(\epsilon) = \left[\frac{a-1}{\epsilon}\right]+1$.

Then $\exists$ a natural no. $N(\epsilon)$, $\ni$ $|x_n - 0| < \epsilon \ \forall n \geq N(\epsilon)$

Hence, $\lim(x_n) = 0$

$\Rightarrow \lim(a^{1/n}-1) = 0$

$\Rightarrow \lim(a^{1/n}) = 1.$

[Alt:→ We have $a_n = 0 < x_n \leq \frac{a-1}{n} = b_n \ \forall n \geq 1$

Now $\lim(a_n) = 0 = \lim(b_n)$

By squeeze theorem $\lim(x_n) = 0$]

Case II :→ $a=1$,

then the sequence $\{a^{1/n}\}$ becomes a constant sequence

$1, 1, 1, \dots\dots$

$\therefore \lim(a^{1/n}) = 1.$

Case III :→ $0 < a < 1$.

Let $b = \frac{1}{a} > 1$.

Then $\lim\limits_{n\to\infty}(a^{1/n}) = \lim\limits_{n\to\infty}\left(\frac{1}{b}\right)^{1/n} = \lim\limits_{n\to\infty}\left(\frac{1}{b^{1/n}}\right) = \frac{1}{\lim\limits_{n\to\infty}(b^{1/n})} = \frac{1}{1}$

[since $b > 1$, $\lim(b^{1/n}) = 1$ by case I]

Hence, $\lim(a^{1/n}) = 1$ for $a > 0$.

(b) $\lim\limits_{n\to\infty}(n^{1/n}) = 1$.

Soln. → Define a new sequence, —

$x_n = n^{1/n} - 1 \geq 0 \;\forall\; n \geq 1$

Now, $n = (1 + x_n)^n \geq \frac{n(n-1)}{2} x_n^2$, $n \geq 2$

[For $x > 0$, $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} \cdot x^2 + \cdots + x^n$

$\therefore (1+x)^n \geq \frac{n(n-1)}{2!} \cdot x^2 \;\forall\; n \geq 2$]

$\Rightarrow 0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$, $n \geq 2$

Let $\epsilon > 0$ be an arbitrary no.

Then $|x_n - 0| = x_n \leq \sqrt{\frac{2}{n-1}} < \epsilon$

$\Rightarrow n - 1 > \frac{2}{\epsilon^2}$

$\Rightarrow n > \frac{2}{\epsilon^2} + 1$

Take

$N(\epsilon) = \left[\frac{2}{\epsilon^2} + 1\right] + 1$

Then $\exists$ a natural no. $N(\epsilon) \ni |x_n - 0| < \epsilon \;\forall\; n \geq N(\epsilon)$

By defn., $\lim(x_n) = 0$

$\Rightarrow \lim(n^{1/n}) = 1$

Alt:-

[We have $a_n = 0 \leq x_n \leq \sqrt{\frac{2}{n-1}} = c_n \;\forall\; n \geq 2$

Note that $\lim(a_n) = 0 = \lim(c_n)$

By squeeze theorem, $\lim(x_n) = 0$]

(c) $\lim\left(\frac{\sin n}{n}\right) = 0$.

Sol. $-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \;\forall\; n \in \mathbb{N}$ as $-1 \leq \sin n \leq 1$

Hence we can apply squeeze theorem to infer that

$\lim(n^{-1} \sin n) = 0$.

Example 9) Show that —

(a) $\lim(n^{1/n^2}) = 1$

(b) $\lim(an)^{1/n} = 1$, $a>0$.

(c) $\lim(na^n) = 0$, $0<a<1$.

- Hints: →

(a) Define $x_n = n^{1/n^2} - 1 \geq 0 \quad \forall n \in \mathbb{N}$

$\therefore n = (1+x_n)^{n^2} \geq 1 + n^2 x_n$

$\Rightarrow 0 \leq x_n \leq \frac{n-1}{n^2} \leq \frac{1}{n} \quad \forall n \in \mathbb{N}$. [P]

(b) Case I → $a \geq 1$

Let $x_n = (an)^{1/n} - 1 \geq 0 \quad \forall n \in \mathbb{N}$

$an = (1+x_n)^n \geq \frac{n(n-1)}{2!} \cdot x_n^2$, $n \geq 2$

$\Rightarrow 0 \leq x_n \leq \sqrt{\frac{2a}{n-1}}$, $n \geq 2$.

Case II → $0<a<1$

Let $x_n = (an)^{1/n} - 1 < n^{1/n} - 1 \leq \sqrt{\frac{2}{n-1}}$, $n \geq 2$.

[Alt. → $\underset{n\to\infty}{\mathrm{Lt}} (an)^{1/n}$

$= \left(\lim_{n\to\infty} a^{1/n}\right) \cdot \left(\lim_{n\to\infty} n^{1/n}\right) = 1.1. = 1.$]

(c) $0<a<1$.

$\Rightarrow a = \frac{1}{1+b}$, $b>0$

$(na^n) = \frac{n}{(1+b)^n} \leq \frac{n}{\frac{n(n-1)}{2!} b^2}$

[For $b>0$, $(1+b)^n \geq \frac{n(n-1)}{2!} b^2$]

$\therefore 0 \leq na^n < \frac{2}{(n-1)b^2}$; $n \geq 2$

Let $\epsilon > 0$ be an arbitrary no.

Then $|na^n - 0| = na^n \leq \frac{2}{(n-1)b^2} < \epsilon$

$\Rightarrow n-1 > \frac{2}{b^2 \epsilon}$

$\Rightarrow n > \frac{2}{b^2 \epsilon} + 1$

Take $N(\epsilon) = \left[\frac{2}{b^2\epsilon} + 1\right] + 1$

$\therefore \lim_{n\to\infty} (na^n) = 0$

- **Limit Theorem :** $\longrightarrow$ If $\lim(a_n)=a$ and $\lim(b_n)=b$ then

i) $\lim(a_n \pm b_n) = \lim(a_n) \pm \lim(b_n) = a \pm b$

ii) $\lim(a_n b_n) = \lim(a_n) \cdot \lim(b_n) = a \cdot b$

iii) $\lim(k a_n) = k \cdot \lim(a_n) = k \cdot a$, $k \in \mathbb{R}$

iv) $\lim\left(\frac{a_n}{b_n}\right) = \frac{\lim(a_n)}{\lim(b_n)} = \frac{a}{b}$

provided $\lim(b_n) = b \neq 0$ and $b_n \neq 0$, $n \in \mathbb{N}$.

- **Proof of i)** Let $\epsilon > 0$ be an arbitrary number. then $\exists$ a natural number $N_1(\epsilon) \ni |a_n - a| < \epsilon$ $\forall$ $n \geq N_1(\epsilon)$ & $\exists$ a natural number $N_2(\epsilon) \ni |b_n - b| < \epsilon$ $\forall$ $n \geq N_2(\epsilon)$.

Let $N(\epsilon) = \max\{N_1(\epsilon), N_2(\epsilon)\}$

then if $n \geq N(\epsilon)$; then $n \geq N_1(\epsilon)$ and $n \geq N_2(\epsilon)$, and also

$$|(a_n + b_n) - (a+b)| \leq |a_n - a| + |b_n - b| < \epsilon + \epsilon = 2\epsilon$$

By definition, $\lim(a_n + b_n) = a + b$.

Remark : $\rightarrow$

1) If $\{a_n\} = \{(-1)^n\}$ and $\{b_n\} = \{(-1)^{n+1}\}$, then they are divergent but $\{a_n + b_n\} = \{0\}$ which converges to 0.

$$\left[\begin{array}{l} n = \text{odd, then } a_n = -1,\ b_n = +1 \\ n = \text{even, then } a_n = +1,\ b_n = -1 \end{array}\right] \forall\ n \in \mathbb{N}.$$

Hence $\lim(a_n + b_n)$ is a finite number but the sequence $\{a_n\}$, $\{b_n\}$ are not convergent.

2) Let $a_n = \frac{1}{n}$, $b_n = \frac{1}{n^2}$, then $\lim(a_n) = 0 = \lim(b_n)$. Using iv)

But $\left\{\frac{a_n}{b_n}\right\} = \{n\}$ diverges where as $\left\{\frac{b_n}{a_n}\right\} = \left\{\frac{1}{n}\right\}$ converges to zero.

Hence, when $\lim(a_n) = 0 = \lim(b_n)$, then nothing can be said about the sequence $\left\{\frac{a_n}{b_n}\right\}$, the sequence may converge or may diverge.

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Example 4)

(a) If $\lim(a_n) = 0$ then prove by using the definition of limit that $\lim(a_n^2) = 0$.

Soln. → Let $\epsilon > 0$ be an arbitrary quantity. Then $\exists$ a natural $N(\epsilon)$; $\ni$ $|a_n - 0| < \epsilon \ \forall \ n \geq N(\epsilon)$

$$\Rightarrow |a_n| < \epsilon \ \forall \ n \geq N(\epsilon)$$
$$\Rightarrow a_n^2 < \epsilon^2 \ \forall \ n \geq N(\epsilon)$$
$$\Rightarrow |a_n^2 - 0| < \epsilon^2 \ \forall \ n \geq N(\epsilon)$$

$\therefore$ By definition $\lim(a_n^2) = 0$.

(b) Prove that $\lim(a_n) = 0$ iff $\lim|a_n| = 0$, give an e.g. to establish that the convergence of $|a_n|$ may not imply the convergence of a_n.

Hint :→
$$\lim(a_n) = 0$$
$$\Rightarrow |a_n - 0| < \epsilon \quad \forall \ n \geq N(\epsilon)$$
$$\Rightarrow |a_n| < \epsilon \quad \forall \ n \geq N(\epsilon)$$
$$\Rightarrow ||a_n| - 0| < \epsilon \ \forall \ n \geq N(\epsilon)$$
$$\Rightarrow \lim|a_n| = 0$$

Ex.
Let $\{a_n\} = \{(-1)^n\}$
N.T. $\{a_n\}$ is oscillatory, i.e. it diverges.
but $|a_n| = \{1\}$ is constant sequence, which converges.

(c) If $\lim(a_n) = 0$ and $a_n \geq 0 \ \forall \ n \in \mathbb{N}$, then show that $\lim(\sqrt{a_n}) = 0$.

Example 5)

(a) Prove that if $\lim(a_n) = 0$ and sequence $\{b_n\}$ is bounded (but not necessarily convergent) then $\lim(a_n b_n) = 0$.

Soln. → Let $\epsilon > 0$ be an arbitrary quantity, then $\exists$ $N(\epsilon) \ni$
$$|a_n - 0| < \epsilon \ \forall \ n \geq N(\epsilon).$$

Again, $\{b_n\}$ is bounded,
$$\Rightarrow |b_n| < M \ \forall \ n \in \mathbb{N} \text{ for some } M$$

Now, $|a_n b_n - 0| = |a_n||b_n| < \epsilon \cdot M \ \forall \ n \geq N(\epsilon)$
$$\Rightarrow \lim(a_n b_n) = 0.$$

(b) Prove that if $\lim(a_n) = l$ then $\lim(a_n)^2 = l^2$.

Hint → Let $\epsilon > 0$

Then $|a_n - l| < \epsilon \quad \forall n \geq N(\epsilon)$

Now, $|a_n^2 - l^2| = |a_n - l||a_n + l| \leq |a_n - l| \{|a_n| + |l|\}$

Note that, since $\{a_n\}$ is convergent,

$\{a_n\}$ is bounded, i.e. $|a_n| < M \quad \forall n \in \mathbb{N}$ for some M.

$\therefore |a_n^2 - l^2| < \epsilon \cdot \{M + |l|\}$ for $n \geq N(\epsilon)$

By definition, $\lim(a_n^2) = l^2$.

Monotonic Sequences :–

(a) A sequence $\{a_n\}$ is said to be monotonic increasing if
$a_{n+1} \geq a_n \quad \forall n \in \mathbb{N}$.

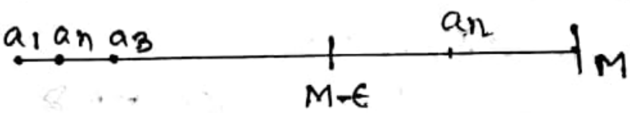
(b) A sequence $\{a_n\}$ is said to be monotic decreasing if
$a_{n+1} \leq a_n \quad \forall n \in \mathbb{N}$.

A sequence $\{a_n\}$ is called monotonic sequence if it is either monotonic increasing or decreasing.

• Theorem :→ A monotonic sequence converges iff it is bounded.

• Proof :→ Let $\epsilon > 0$,

Then, $M - \epsilon < a_n \leq M \quad \forall n \geq N(\epsilon)$



$\Rightarrow M - \epsilon < a_n \leq M < M + \epsilon \quad \forall n \geq N(\epsilon)$

$\Rightarrow |a_n - M| < \epsilon \quad \forall n \geq N(\epsilon)$

$\therefore \lim(a_n) = M$

Corollary :→ A monotone sequence is ~~not only~~ properly divergent if it is unbounded.

In fact, if $\{a_n\}$ is monotonic increasing and unbounded, then $\lim(a_n) = +\infty$.

If $\{a_n\}$ is monotonic decreasing and unbounded, then $\lim(a_n) = -\infty$.

- **Example 6>** For the following sequences examine whether or not they are bounded and whether or not they are monotonic, also establish the convergence or divergence of the sequences.

(a) $a_n = \frac{1}{n}$, $n \in \mathbb{N}$.

(b) $a_1 = 1$, $a_n = \sqrt{2a_{n-1}}$, $n = 2, 3, \ldots$

(c) $a_n = \left(1+\frac{1}{n}\right)^n$, $n \in \mathbb{N}$.

- **Soln>** (a) Note that →

$$a_{n+1} - a_n = \frac{1}{n+1} - \frac{1}{n} = \frac{1}{n(n+1)} < 0 \quad \forall\, n \in \mathbb{N}.$$

$\Rightarrow \{a_n\}$ is decreasing, clearly $0 < \frac{1}{n} \le 1 \ \forall\, n \in \mathbb{N}$.

i.e. $\{a_n\}$ is bounded,

Hence, $\{a_n\} = \{\frac{1}{n}\}$ converges.

(b) Here, $a_1 = 1$ and $a_2 = \sqrt{2a_1} = \sqrt{2}$

Note that, $a_2 > a_1$.

Let us suppose that $a_k > a_{k-1}$.

Now, $a_{k+1} = \sqrt{2a_k} > 2\sqrt{a_{k-1}} = a_k$.

Hence by mathematical induction, $a_{n+1} > a_n \ \forall\, n \in \mathbb{N}$.

So, $\{a_n\}$ is increasing.

Note that, $a_1 < 2$, $a_2 < 2$

Let us assume that $a_k < 2$.

Now, $a_{k+1} = \sqrt{2a_k} < \sqrt{2.2} = 2$

By mathematical induction, $a_n < 2 \ \forall\, n \in \mathbb{N}$.

Now, $1 = a_1 \le a_n < 2 \quad \forall\, n \in \mathbb{N}$

i.e. $\{a_n\}$ is bounded.

$\therefore$ $\{a_n\}$ converges.

(c) Here, $a_n = \left(1+\frac{1}{n}\right)^n$

$$= 1 + \frac{n}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3} + \cdots$$

$$\cdots + \frac{n(n-1)\ldots 3.2.1}{n!}\cdot\frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots$$

$$+ \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

$$< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{2^{n-1}}$$

$[\because n! \geqslant 2^{n-1} \ \forall\, n \in \mathbb{N}]$

$\therefore a_n < 1 + \dfrac{1\left(1-\frac{1}{2^n}\right)}{\left(1-\frac{1}{2}\right)} = 1 + 2\left(1-\frac{1}{2^n}\right) < 3 \;\forall\, n \in N.$

$\Rightarrow a_n < 3 \;,\; n \in N.$

Note that, $a_n = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots$
$\cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$

and $a_{n+1} = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \cdots$
$\cdots + \frac{1}{n!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right)$
$+ \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)$

Note that, $n+1 > n \Rightarrow \frac{1}{n+1} < \frac{1}{n}$.

$\Rightarrow 1 - \frac{1}{n+1} > 1 - \frac{1}{n} \;,\; 1 - \frac{2}{n+1} > 1 - \frac{2}{n}$, etc.

Hence, $a_{n+1} > a_n \;\forall\, n \in N$.

Now, $2 = a_1 \le a_n < 3 \;,\; \forall\, n \in N$.

$\therefore \{a_n\}$ is monotonic increasing and bounded, i.e. $\{a_n\}$ converges.

- **SERIES OF REAL NUMBERS** :— Consider a given sequence $\{a_n\}$, then the infinite sum $\sum_{n=1}^{\infty} a_n$ is called a series.

For a given sequence $\{a_n\}$, we define a second sequence $\{s_n\}$:

$s_1 = a_1$, $s_2 = a_1 + a_2$, $s_3 = a_1 + a_2 + a_3$, ...

$s_n = a_1 + a_2 + \cdots + a_n \left[= \sum_{i=1}^{n} a_i\right]$

If $\{a_n\}$ is a given sequence, then we define an infinite series or simply a series $\sum_{n=1}^{\infty} a_n$, by the sequence $\{s_n\}$ of partial sums, where $S_n = \sum_{k=1}^{n} a_k$.

Note that, $\lim_{n\to\infty}(s_n) = \lim_{n\to\infty}\left(\sum_{k=1}^{n} a_k\right) = \sum_{k=1}^{\infty} a_k$.

Hence, $\sum_{n=1}^{\infty} a_n$ converges ~~and~~ or diverges iff $\{s_n\}$ converges or diverges.

Definition :→ If the sequence $\{s_n\}$ of partial sums is convergent and has the limit 's'; the series $\sum_{n=1}^{\infty} a_n$ is called convergent and the number 's' is the value of ~~the~~ sum of the series.

If $\{s_n\}$ is divergent then the series $\sum_{n=1}^{\infty} a_n$ is said to diverge and has no sum or value.

Example 1) Check the convergence or divergence of the following series. Also find the sum of the following series if it converges:

i) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Soln→ The n^{th} partial sum is

$$s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{n+1}.$$

Let $\epsilon > 0$ be an arbitrary number.

Then $|s_n - 1| = \frac{1}{n+1} < \frac{1}{n} < \epsilon \Rightarrow n > \frac{1}{\epsilon}$

$$N(\epsilon) = \left[\frac{1}{\epsilon}\right] + 1$$

Then $\exists$ a natural number $N(\epsilon) \ni |s_n - 1| < \epsilon, \forall n \geq N(\epsilon)$

$\therefore \lim(s_n) = 1$

$\Rightarrow s_n$ converges to 1.

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and has the value 1.

ii) $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

Soln. → $S_n = \sum_{k=1}^{n} \frac{1}{4k^2-1}$

$= \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)}$

$= \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)$

$= \frac{1}{2}\left\{\left(1 - \frac{2}{2n+1}\right)\right\}$

$\therefore \lim(S_n) = \frac{1}{2}\left\{1 - \lim\left(\frac{1}{2n+1}\right)\right\} = \frac{1}{2}.$

Hence, the series converges and has the sum $1/2$.

iii) $\sum_{n=1}^{\infty} r^{n-1}$, $|r| < 1$.

Soln. → $S_n = \sum_{k=1}^{n} r^{k-1}$

$= \frac{(1-r^n)}{(1-r)}$

Now, $\lim_{n\to\infty}(S_n) = \frac{1 - \lim_{n\to\infty}(r^n)}{(1-r)}$.

$= \frac{1-0}{1-r} = \frac{1}{1-r}$ [since $\lim(r^n) = 0$, $|r| < 1$]

Hence the series $\sum_{n=1}^{\infty} r^{n-1}$ converges and has the sum $\frac{1}{1-r}$, if $|r| < 1$.

iv) $\sum_{n=1}^{\infty} n^2$

Soln. → $S_n = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

$\therefore S_n \to \infty$ as $n \to \infty$.

$\Rightarrow \{S_n\}$ diverges to $+\infty$

$\Rightarrow$ the series $\sum_{n=1}^{\infty} n^2$ diverges and has no value.

- **Theorem :→ (n^{th} term test)**

If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty}(a_n)=0$.

Proof :→

$$
\begin{aligned}
S_{n+1} &= a_1+a_2+\cdots\cdots+a_n+a_{n+1}\\
S_n &= a_1+a_2+\cdots\cdots+a_n\\
\hline
S_{n+1}-S_n &= a_{n+1}
\end{aligned}
$$

$\therefore a_n = S_n - S_{n-1}$.

$\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow$ S_n converges.

Let $\lim_{n\to\infty}(S_n)=S$

$$
\begin{aligned}
\therefore \lim(a_n) &= \lim(S_n - S_{n-1})\\
&= \lim_{n\to\infty}(S_n) - \lim_{n\to\infty}(S_{n-1})\\
&= S - S\\
&= 0.
\end{aligned}
$$

Corollary :→ If $\lim(a_n)\neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example→ Let a series be $\sum \cos\frac{\pi}{n}$,

we are going to check convergence or divergence of the following series by calculating its limiting value.

$$\therefore \lim_{n\to\infty}\left(\cos\frac{\pi}{n}\right) = \cos 0 = 1 \neq 0$$

By using the theorem, we can say that $\sum \cos\left(\frac{\pi}{n}\right)$ diverges.

- **Series of Non-negative items :→** If $a_n \geq 0$, $n\in\mathbb{N}$, then the series $\sum_{n=1}^{\infty} a_n$ is a series of non-negative terms.

- **Theorem :→** A series of non-negative terms $\sum_{n=1}^{\infty} a_n$ converges iff the sequence $\{S_n\}$ of partial sums is bounded above.

Proof :→ Let

$$
\begin{aligned}
S_n &= a_1+a_2+\cdots\cdots+a_{n-1}+a_n\\
S_{n-1} &= a_1+a_2+\cdots\cdots+a_{n-1}\\
\hline
S_n - S_{n-1} &= a_n \geq 0 \ \forall n
\end{aligned}
$$

$\Rightarrow \{S_n\}$ is increasing.

Hence, $\{S_n\}$ converges iff it is bounded above.

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges iff $\{S_n\}$ is bounded above.

NOTE: A series $\sum_{n=1}^{\infty} a_n$ of non-negative terms either converges or diverges to $(+\infty)$. If $\sum a_n$ is not bounded above then $\sum a_n$ is divergent, and diverges to ∞.

Example: Establish the convergence or divergence of the following series:

a) $\sum_{n=1}^{\infty} \frac{1}{n!}$

Soln. → Here, $s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \frac{1}{k!}$

$\therefore s_n - s_{n-1} = \frac{1}{n!} \geq 0 \quad \forall n \geq 2.$

$\Rightarrow \{s_n\}$ is increasing.

Now, $s_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} < 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$

$\left[\because n! = 1.2 \ldots n \geq 2^{n-1}, n \in N\right]$

$\Rightarrow s_n < \frac{1 - 1/2^n}{1 - 1/2} = 2\left(1 - \frac{1}{2^n}\right) < 2$

$\therefore 1 = s_1 \leq s_n < 2 \quad \forall n \in N.$

$\Rightarrow \{s_n\}$ is bounded above.

Hence $\{s_n\}$ converges.

$\boxed{C.U.} \Leftrightarrow \sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

** (b) For the Power Series, i.e. $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p \in R$.

[Power series : $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$ and diverges if $p \leq 1$]

Case I: $\boxed{p > 1}$

Note that, $s_n = \sum_{k=1}^{n} \frac{1}{k^p}$, $n \in N$.

$\therefore s_n - s_{n-1} = \frac{1}{n^p} > 0 \quad \forall n \in N.$

$\Rightarrow \{s_n\}$ is increasing.

Now,

$$s_{2^n - 1} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{(2^n - 1)^p}$$

$$= 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \left(\frac{1}{8^p} + \cdots + \frac{1}{15^p}\right) + \cdots + \left\{\left(\frac{1}{2^{n-1}}\right)^p + \left(\frac{1}{2^{n-1}+1}\right)^p + \cdots + \left(\frac{1}{2^n - 1}\right)^p\right\}$$

$$< 1 + 2 \cdot \frac{1}{2^p} + 2^2 \cdot \frac{1}{4^p} + 2^3 \cdot \frac{1}{8^p} + \cdots + 2^{n-1} \cdot \left(\frac{1}{2^{n-1}}\right)^p$$

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \cdots + \frac{1}{(2^{n-1})^{p-1}}$$

$$= 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \cdots + \left(\frac{1}{2^{p-1}}\right)^{n-1}$$

$$= 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \left(\frac{1}{2^{p-1}}\right)^3 + \cdots + \left(\frac{1}{2^{p-1}}\right)^{n-1}$$

$$= \frac{1\left\{1 - \left(\frac{1}{2^{p-1}}\right)^n\right\}}{\left(1 - \frac{1}{2^{p-1}}\right)} < \frac{2^{p-1}}{2^{p-1}-1} \quad \left[\begin{array}{l} \text{as } p>1 \\ \Rightarrow 2^{p-1} > 1 \\ \Rightarrow 2^{p-1} - 1 > 0 \end{array}\right]$$

$\therefore s_{2^{n-1}} < \dfrac{2^{p-1}}{2^{p-1}-1}$ if $p > 1$.

As $n \le 2^{n-1}$ and $\{s_n\}$ is increasing.

$$1 = s_1 \le s_n \le s_{2^{n-1}} < \frac{2^{p-1}}{2^{p-1}-1}$$

$\Rightarrow \{s_n\}$ is bounded if $p > 1$.

Hence, $\{s_n\}$ converges if $p > 1$,

$\Leftrightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$.

<u>Case II</u> $\boxed{p \le 1}$

Note that $n^p \le n$, $n \in \mathbb{N}$.

$\Rightarrow \frac{1}{n^p} \geq \frac{1}{n}$, $n \in \mathbb{N}$.

Now, $s_{2^n} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{(2^n)^p}$

$$= 1 + \frac{1}{2^p} + \left(\frac{1}{3^p} + \frac{1}{4^p}\right) + \left(\frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \frac{1}{8^p}\right) + \cdots + \left\{\left(\frac{1}{2^{n-1}+1}\right)^p + \left(\frac{1}{2^{n-1}+2}\right)^p + \cdots + \left(\frac{1}{2^n}\right)^p\right\}$$

$$\geq 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left\{\left(\frac{1}{2^{n-1}+1}\right) + \left(\frac{1}{2^{n-1}+2}\right) + \cdots + \left(\frac{1}{2^n}\right)\right\}$$

$$> 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + \cdots + 2^{n-1} \cdot \frac{1}{2^n}$$

$$s_{2^n} > 1 + \frac{n}{2}$$

For any $n \in \mathbb{N}$, $\exists$ a natural no. $m \ni 2^m \le n \le 2^{m+1}$

Hence, $s_n \geq s_{2^m} > 1 + \frac{m}{2}$

[since $\{s_n\}$ is increasing]

i.e. $\{s_n\}$ is unbounded.

As $\{s_n\}$ is ~~is~~ increasing and unbounded $\{s_n\}$ diverges to $+\infty$.

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, if $p \le 1$.

We consider the following tests for convergence or divergence of series of non-negative terms.

(I) Comparison Test :→ ~~[We consider the following tests for convergence or divergence of series of non-negative terms~~

● If $\sum a_n$ & $\sum b_n$ are two series of non-negative terms and $a_n \le K.b_n$, for $n \geq m$, then →

(a) $\sum a_n$ converges if $\sum b_n$ converges.

(b) $\sum b_n$ diverges if $\sum a_n$ diverges.

• Example :→ Test for converges or diverges of the following series:

i) $\sum\limits_{n=1}^{\infty} \frac{1}{n!}$, ii) $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$, iii) $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n+1000}}$, iv) $\sum\limits_{n=1}^{\infty} \frac{|\cos nx|}{n^2}$, v) $\sum\limits_{n=1}^{\infty} \frac{1}{n^2+n}$.

Proof :→ i) Note that, $0 < \frac{1}{n!} \le \frac{1}{2^{n-1}}$, $n \in \mathbb{N}$.

Note that $\sum\limits_{n=1}^{\infty} \frac{1}{n!}$ is a convergent. [Geometric series]

Since $\sum \frac{1}{2^{n-1}}$ converges, by comparison test, $\sum \frac{1}{n!}$ converges.

ii) Note that, $\frac{1}{2\sqrt{n+1}} < \frac{1}{\sqrt{n+1}+\sqrt{n}}$

Since, $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} = \sum\limits_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ is divergent.

Power series as $p = \frac{1}{2} < 1$, by comparison, $\sum \frac{1}{\sqrt{n+1}+\sqrt{n}}$ diverges.

iii) Note that, $\frac{1}{\sqrt{2n}} \le \frac{1}{\sqrt{n+1000}}$ $\forall\ n \geq 1000$

i.e. $\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{n}} \le \frac{1}{\sqrt{n+1000}}$ $\forall\ n \geq 1000$

Now, $\sum \frac{1}{\sqrt{n}}$ diverges, by comparison test, $\sum \frac{1}{\sqrt{n+1000}}$ diverges.

iv) Note that, $0 \le \frac{|\cos nx|}{n^2} \le \frac{1}{n^2}$ $\forall\ n \in \mathbb{N}$.

Note that $\sum \frac{1}{n^2}$ converges $\left[\because \sum\limits_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p > 1\right]$

By comparison test, $\sum \frac{|\cos nx|}{n^2}$ converges.

v) $0 \le \frac{1}{n^2+n} < \frac{1}{n^2}$ $\forall\ n \in \mathbb{N}$

As $\sum \frac{1}{n^2}$ converges, by comparison test,

$\sum \frac{1}{n^2+n}$ converges.

(II) Limit Comparison Test :→ If $\sum a_n$ and $\sum b_n$ are two series of positive terms and $\lim\left(\frac{a_n}{b_n}\right) = l \neq 0$, then both the series $\sum a_n$ and $\sum b_n$ converge / diverge simultaneously.

Example :→ Test the following series for convergence or divergence

i) $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$, ii) $\sum \frac{n^2+n}{n^4+\sqrt{n}}$, iii) $\sum_{n=1}^{\infty} \sin\frac{1}{n}$, iv) $\sum \frac{1}{n}\sin\frac{1}{n}$,

v) $\sum\left(a^{1/n}-1\right)$, vi) $\sum\left\{(n^3+1)^{1/3}-n\right\}$

Soln.→

i) Let $a_n = \frac{1}{n^2+n}$, $n \in \mathbb{N}$.

Take $b_n = \frac{1}{n^2}$, $n \in \mathbb{N}$. Now, $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim\left(\frac{n^2}{n^2+n}\right)$

$= \lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)$

$= 1 \neq 0$.

Since, $\sum b_n = \sum \frac{1}{n^2}$ converges by Limit Comparison test, $\sum a_n = \sum \frac{1}{n^2+n}$ converges.

ii) Let $a_n = \frac{n^2+n}{n^4+\sqrt{n}}$, $n \in \mathbb{N}$.

Take, $b_n = \frac{1}{n^2}$, $n \in \mathbb{N}$, Now, $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim\left(\frac{\frac{n^2+n}{n^4+\sqrt{n}}}{1/n^2}\right)$

$= \lim_{n\to\infty}\left(\frac{n^4+n^2}{n^4+\sqrt{n}}\right)$

$= 1 \neq 0$.

Since, $\sum b_n = \sum\frac{1}{n^2}$ converges by Limit comparison test.

$\sum a_n = \sum \frac{1}{n^2+n}$ converges.

iii) Note that, $\lim_{n\to\infty} \frac{\sin 1/n}{1/n} = \lim_{x=\frac{1}{n}\to 0} \frac{\sin x}{x} = 1 \neq 0$

Since, $\sum\frac{1}{n}$ diverges by Limit comparison test, $\sum \sin\frac{1}{n}$ diverges.

iv) Let $a_n = \frac{1}{n}\sin 1/n$, $n \in \mathbb{N}$

$b_n = \frac{1}{n^2}$, $n \in \mathbb{N}$

$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \frac{\sin 1/n}{1/n} = 1 \neq 0$

v) Note that, $\lim_{n\to\infty}\left(\frac{a^{1/n}-1}{1/n}\right) = \lim_{\substack{x\to 0 \\ x=\frac{1}{n}}} \left(\frac{a^x-1}{x}\right) = \log_e a \neq 0$

Since $\sum\frac{1}{n}$ diverges, $\sum\left(a^{1/n}-1\right)$ diverges by Limit comparison test.

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vi) $$a_n = \frac{(n^3+1)-n^3}{(n^3+1)^{2/3}+n(n^3+1)^{1/3}+n^2}$$

$$= \frac{1}{(n^3+1)^{2/3}+n(n^3+1)^{1/3}+n^2}$$

Take, $b_n = 1/n^2$

$\lim\left(\frac{a_n}{b_n}\right) = \frac{1}{3} \neq 0$

Remark: If $\lim_{n\to\infty} \frac{a_n}{b_n} \neq 0$, then we write $a_n \sim b_n$ for large n. Then $\sum a_n$ and $\sum b_n$ have similar nature of convergence or divergence.

Note that → i) $\frac{n^p}{(n+k)^q} \sim \frac{1}{n^{q-p}}$, ii) $\frac{n^p}{(an+b)^q} \sim \frac{1}{n^{q-p}}$,

iii) $\sin\frac{1}{n} \sim \frac{1}{n}$; iv) $a^{\frac{1}{n}}-1 \sim \frac{1}{n}$.

(III) D'alembert's Ratio Test :→ If $\sum a_n$ is a series of positive terms and $\lim\left(\frac{a_{n+1}}{a_n}\right) = l$, then $\sum a_n$ converges if $l < 1$ and diverges if $l > 1$.

Hint:→ Let $l < 1$, then $\exists$ an $\epsilon > 0 \ni l+\epsilon < 1$.

By definition of limit

$$\frac{a_{n+1}}{a_n} - l < \epsilon \quad \forall\ n \geq N(\epsilon)$$

$$\Rightarrow \frac{a_{n+1}}{a_n} < l+\epsilon = r\ (<1) \quad \forall\ n \geq N(\epsilon)$$

Now, $$\frac{a_k}{a_{N(\epsilon)}} = \frac{a_k}{a_{k-1}} \cdot \frac{a_{k-1}}{a_{k-2}} \cdots\cdots \frac{a_{N(\epsilon)+1}}{a_{N(\epsilon)}}$$

$$< r^{k-N(\epsilon)}$$

$$\Rightarrow a_k < \left(\frac{a_{N(\epsilon)}}{r^{N(\epsilon)}}\right) \cdot r^k.$$

Since, $\sum r^k$, $r < 1$ converges, $\sum a_n$ converges.

Note:- When $l=1$, the test fails to give a conclusion.

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(IV) **Cauchy's Root Test :→** If $\sum a_n$ is a series of non-negative terms and $\lim (a_n^{1/n}) = l$, then $\sum a_n$ converges if $l < 1$ and diverges if $l > 1$.

Hints : Let $l < 1$, then $\exists$ an $\epsilon > 0$, $\ni$ $l + \epsilon < 1$

By definition, $a_n^{1/n} < l + \epsilon = r \, (< 1) \quad \forall \, n \geq N(\epsilon)$

$\Rightarrow a_n = r^n \quad \forall \, n \geq N(\epsilon)$

Since $\sum r^n$, $r < 1$ converges,

$\sum a_n$ converges by comparison test.

Example :→ Test the following series for convergence or divergence.

(a) $\sum \frac{(n!)^2}{(2n)!}$, (b) $\sum (n^{1/n} - 1)^n$.

Soln. → (a) Let $a_n = \frac{(n!)^2}{\lfloor 2n}$, $n \in N$.

Now, $\lim_{n \to \infty} \left(\frac{a_{n+1}}{a_n} \right) = \lim \frac{(n+1)!}{(2n+2)!} \times \frac{(2n)!}{(n!)^2}$

$= \lim_{n \to \infty} \frac{(n+1)^2}{(2n+2)(2n+1)}$

$= \lim_{n \to \infty} \frac{(n+1)}{2(2n+1)}$

$= 1/4$ $\quad \therefore$ By ratio test, $\sum a_n = \sum \frac{(n!)^2}{(2n)!}$ converges

Here, $l = \lim \left(\frac{a_{n+1}}{a_n} \right) = \frac{1}{4} < 1$.

(b) Let $a_n = (n^{1/n} - 1)^n$, $n \in N$.

Here, $l = \lim (a_n^{1/n}) = \lim (n^{1/n} - 1)$

$= \lim (n^{1/n}) - 1$

$= 1 - 1 = 0 < 1$

$\left[\begin{array}{l} * \lim (n)^{1/n} = 1 \\ * \lim (a_n^{1/n}) = 1 \end{array} \right]$

By root test, $\sum a_n = \sum (n^{1/n} - 1)^n$ converges.

Remark :→

1) n^{th} term test : If $\sum a_n$ converges, then $\lim (a_n) = 0$.
But the converse is not true, (i.e. the converse theorem is not necessarily true).
Note that $\sum a_n = \sum \frac{1}{n}$ diverges and $\sum b_n = \sum \frac{1}{n^2}$ converges.
But $\lim (a_n) = 0 = \lim (b_n)$.

2) In the root test & ratio test if $l=1$ then we can't draw any conclusion. Let $\sum a_n = \sum \frac{1}{n}$, $\sum b_n = \sum \frac{1}{n^2}$

$\therefore \lim \left(\frac{a_{n+1}}{a_n}\right) = 1$ and $\lim \left(\frac{b_{n+1}}{b_n}\right) = 1$

In both cases $l=1$ but $\sum a_n = \sum \frac{1}{n}$ diverges whereas $\sum b_n = \sum \frac{1}{n^2}$ converges.

Again, $l = \lim (a_n^{1/n}) = \lim \left(\frac{1}{n}\right)^{1/n} = \frac{1}{\lim (n)^{1/n}} = 1$.

and, $l = \lim (b_n^{1/n}) = \frac{1}{\lim (n^2)^{1/n}} = \left\{\frac{1}{\lim (n^{1/n})}\right\}^2 = 1$.

- Series of arbitrary terms, not necessarily non-negative :-

⊛ Alternative Series :- If $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ with $a_n > 0$, is an alternating series where $\{a_n\}$ is decreasing and $\lim (a_n) = 0$, then the series converges. This is known as Leibnitz Test.

Example :-> Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ converges if $p>0$.

Soln.-> Here, $a_n = \frac{1}{n^p}$, $n \in \mathbb{N}$.

Note that, $a_{n+1} - a_n = \frac{1}{(n+1)^p} - \frac{1}{n^p} < 0 \quad \forall n \in \mathbb{N}$.

$\Rightarrow a_n$ is decreasing.

Note that, $\lim (a_n) = \lim \left(\frac{1}{n^p}\right) = 0$ if $p>0$

By Leibnitz test, the alternating series $\sum \frac{(-1)^{n-1}}{n^p}$ converges if $p>0$

⊛ Absolute Convergence :-> If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that the series $\sum a_n$ converges absolutely.

If a series converges absolutely (i.e. $\sum |a_n|$ if finite). Then it is (ordinary) convergent (i.e. $\sum a_n$ converges).

~~[Since $a_n \le |a_n|$. But absolute divergence, not necessarily implies the ordinary divergence]~~

Absolute Convergence $\Rightarrow$ Ordinary Convergence.

But absolute divergence $\not\Rightarrow$ Ordinary divergence.

Test for absolute Convergence :-

- **Ratio Test :→** If Σa_n is a series of arbitrary terms, with $a_n \neq 0$, and if $l = \lim \left|\frac{a_{n+1}}{a_n}\right|$, then
  i) Σa_n converges absolutely if $l < 1$.
  ii) Σa_n diverges if $l > 1$.

- **Root Test :→** If Σa_n is a series of arbitrary terms, with $a_n \neq 0$, and if $l = \lim\limits_{n\to\infty} |a_n|^{1/n}$, then
  i) Σa_n converges absolutely if $l < 1$.
  ii) Σa_n diverges if $l > 1$.

Example: Determine the convergence or divergence of the series:

i) $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \frac{2n+1}{n^3-4}$, ii) $\Sigma (-1)^{n-1} \frac{n^n}{n!}$

Soln.

i) Let $a_n = (-1)^{n-1} \frac{2n+1}{n^3-4}$

$b_n = \frac{1}{n^2}$, $n \in N$.

Now, $\lim\limits_{n\to\infty} \frac{|a_n|}{b_n} = \lim \left(\frac{2n+1}{n^3-4} \times n^2\right) = 2 \neq 0$

Hence, $\Sigma |a_n|$ & Σb_n both converge or diverge simultaneously.

As, $\Sigma b_n = \Sigma \frac{1}{n^2}$ converges, $\Sigma |a_n|$ also converges $\Leftrightarrow a_n$ converges absolutely $\Rightarrow \Sigma a_n = \Sigma (-1)^{n-1} \frac{2n+1}{n^3-4}$ converges (ordinarily).

ii) Here $a_n = (-1)^n \cdot \frac{n^n}{n!}$, $n \in N$.

Note that, $l = \lim \left|\frac{a_{n+1}}{a_n}\right|$

$= \lim\limits_{n\to\infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n} \right|$

$= \lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e > 1$

By ratio test (for absolutely convergence or divergence). The series Σa_n diverges.

Example: Show that $\sum \frac{\sin nx}{n^p}$ converge absolutely if $p>1$.

Soln.→ $0 \le \left|\frac{\sin nx}{n^p}\right| \le \frac{1}{n^p}$, $n \in \mathbb{N}$.

Now, $\frac{1}{n^p}$ converges if $p>1$.

By comparison test, $\sum \left|\frac{\sin nx}{n^p}\right|$ converges if $p>1$.

$\Rightarrow \sum \frac{\sin nx}{n^p}$ converges absolutely if $p>1$.

Example: Establish the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ for different values of x.

Soln.→ Note that, $l = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)^2} \Big/ \frac{x^n}{n^2}\right|$

$$= \lim_{n\to\infty} \left\{|x| \cdot \frac{n^2}{(n+1)^2}\right\}$$

$$= |x| \cdot \lim_{n\to\infty} \frac{n^2}{(n+1)^2}$$

$$= |x|.$$

By ratio test $|x|<1$, the series $\sum \frac{x^n}{n^2}$ converges (absolutely);
if $|x|>1$, the series $\sum \frac{x^n}{n^2}$ diverges.

The test fails to give any conclusion if $|x|=1$; when $|x|=1$, then
$\sum \left|\frac{x^n}{n^2}\right| = \sum \frac{1}{n^2}$ converges, i.e. $\sum \frac{x^n}{n^2}$ converges (absolutely).

Hence, $\sum \frac{x^n}{n^2}$ converges (absolutely) if $|x| \le 1$ and diverges if $|x|>1$

⊛ **Conditional Convergence:→** A series $\sum a_n$ is said to converge conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges.

$\sum a_n$ is a series of arbitrary terms.

$\sum a_n$

Convergent (Absolute convergence) | Divergent (absolute divergence)

$\sum a_n$ diverges | $\sum a_n$ converges [conditional convergence]

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Example :→ Test the following series for absolute and conditional convergence :

i) $\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{1}{n^p}$, $p>0$, ii) $\sum_{n=1}^{\infty}\frac{x^n}{n^p}$, $x\in\mathbb{R}$ and $p>0$.

Soln.→

i) By Leibnitz test, $\sum(-1)^{n-1}\cdot\frac{1}{n^p}$ converges if $p>0$

Note that, $\sum\left|(-1)^{n-1}\cdot\frac{1}{n^p}\right|=\sum\frac{1}{n^p}$ converges if $p>1$ and diverges if $p\le 1$.

Hence, the series $\sum(-1)^{n-1}\frac{1}{n^p}$ converges absolutely if $p>1$.

But, for $0<p\le 1$, $\sum\left|(-1)^{n-1}\cdot\frac{1}{n^p}\right|$ diverges but

$\sum(-1)^{n-1}\frac{1}{n^p}$ converges, i.e. $\sum(-1)^{n-1}\frac{1}{n^p}$ converges conditionally.

ii) Here, $a_n=\frac{x^n}{n^p}$, $n\in\mathbb{N}$, $x\in\mathbb{R}$

Note that,
$$\lim\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left\{\frac{|x|^{n+1}}{(n+1)^p}\times\frac{n^p}{|x|^n}\right\}$$
$$=|x|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^p$$
$$=|x|\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)^p$$
$$=|x|$$

If $|x|<1$, then $\sum a_n=\sum\frac{x^n}{n^p}$ converges absolutely.

If $|x|>1$, then $\sum a_n=\sum\frac{x^n}{n^p}$ diverges by ratio test.

If $|x|=1$, then the test fails to give conclusion.

If $x=1$, then $\sum\frac{x^n}{n^p}=\sum\frac{1}{n^p}$ converges (absolutely) if $p>1$, and diverges if $p\le 1$.

If $x=-1$, then $\sum\frac{x^n}{n^p}=\sum\frac{(-1)^n}{n^p}$ converges absolutely if $p>1$ and converges conditionally if $0<p\le 1$.

$$\sum_{n=1}^{\infty}\frac{x^n}{n^p}$$

- $|x|<1$: Absolute convergence
- $|x|=1$
  - $x=1$
    - $p>1$: Absolute convergence
    - $p\le 1$: Divergence
  - $x=-1$
    - $p>1$: Absolute Convergence
    - $0<p\le 1$: Conditional Convergence
- $|x|>1$: Divergence

Rearrangement of Series: It is a familiar fact that the sum of a finite numbers of terms is same, no matter how the terms are arranged. But it's not universally true for an infinite series, as an e.g., consider the series:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots$$
$$= s \text{ (say)}$$

If we arranged the terms of the series $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n}$ so that each positive terms are followed by two negative terms, then we have the series:

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots\cdots$$
$$= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \cdots\cdots$$
$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots\cdots$$
$$= \frac{1}{2}\left\{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\cdots\right\}$$
$$= \frac{s}{2}.$$

Hence, a rearrangement of the series $\sum \frac{(-1)^{n-1}}{n}$ alters the sum of the series. So, we naturally ask "Under what condition(s) may we arrange the terms of a series without altering its value"?

- **Theorem:** If a series $\sum a_n$ converges absolutely, its sum is unaltered by rearrangement.

Example: Consider the series $\sum \frac{(-1)^{n-1}}{n^2}$, what is the sum of the following series: $1 - \frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{3^2} - \frac{1}{6^2} - \frac{1}{8^2} + \cdots\cdots$

Soln. Since $\sum \frac{(-1)^{n-1}}{n^2}$ converges absolutely. Let $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = s$. Hence, any rearrangement of the series has the same value(s).

Riemann's Theorem: If a series converges conditionally, its sum can be made to have an arbitrary value by suitable rearrangement. It can also be made divergent or oscillatory.

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SEQUENCES & SERIES OF FUNCTIONS :

[A]. SEQUENCES OF FUNCTIONS :

It may happen that the terms of a sequence depend on some variable x in addition to the index $n \in N$ and this is indicated by $f_n(x)$. Suppose that for each $n \in N$, there is a function $f_n : A \to R$, then we say that $\{f_n(x)\}$ is a sequence of functions on A. For example, if $f_n(x) = x^n$, $x \in R$, $n \in N$. Then $\{x^n\}$ is a sequence of functions on R. For each $x \in A$, such a sequence becomes a sequence of real numbers. For some values of $x \in A$, the sequence may converge and for other values of n, the sequence may diverge. We assume that $\{f_n(x)\}$ is convergent for all $x \in A_0 \subseteq A$ and for each $x \in A_0$, we have $\lim_{n\to\infty} f_n(x)$ as a finite number.

Then the $\lim_{n\to\infty} f_n(x)$ defines a function of x, say $f(x)$ on A_0. The function $f(\cdot)$ defined on A_0 by the equation $f(x) = \lim_{n\to\infty} f_n(x)$ is called the limit function of the sequence $\{f_n(x)\}$ of functions.

Pointwise Convergence :

• Definition :

(I) The sequence $\{f_n(x)\}$ of functions converges pointwise to the function $f(x)$ on A_0 if for each $x \in A_0$, the sequence $\{f_n(x)\}$ of real numbers converges to the real number $f(x)$.

(II) The sequence $\{f_n(x)\}$ of functions converges pointwise to the function $f(x)$ on A_0 if for each $x \in A_0$ for any $\epsilon > 0$, there is a natural no. $N(\epsilon, x)$ may depend on ϵ & x $\ni$

$$|f_n(x) - f(x)| < \epsilon \quad \forall\ n \geq N(\epsilon, x).$$

NOTE:-

Uniform convergence $\Rightarrow$ pointwise convergence but not vice versa.

However, Non-pointwise convergence $\Rightarrow$ non-uniform convergence.

Example: –

1 (a) Show that the sequence $\{x^n\}$ of function converges pointwise to the function $f(x) = \begin{cases} 0 & , 0 < x < 1 \\ 1 & , x = 1 \end{cases}$ on the interval $[0,1]$

$f(x) = \lim x^n = 0$ for $0 \leq x < 1$.
$= 1$ for $x = 1$.

Soln. → Let $\epsilon > 0$ be an arbitrary quantity.

Then, for each $x \in (0,1)$, $|f_n(x) - f(x)| < \epsilon$

$$\Rightarrow |x^n - 0| < \epsilon$$

$$\Rightarrow x^n < \epsilon$$

$$\Rightarrow n \ln x < \ln \epsilon$$

$$\Rightarrow n > \frac{\ln \epsilon}{\ln x} \quad \left[\begin{array}{l} 0 < x < 1 \\ \Rightarrow \ln x < 0 \end{array}\right]$$

For $x = 0, 1$.

$$|f_n(x) - f(x)| = 0 < \epsilon \quad \forall n \geq 1.$$

Choose a natural number

$$N(\epsilon, x) = \left[\frac{\ln \epsilon}{\ln x}\right] + 1$$

Hence for each $x \in [0,1]$;

For any $\epsilon > 0$, there is a natural number $N(\epsilon, x)$, $\ni$

$|f_n(x) - f(x)| < \epsilon$ whenever $n \geq N(\epsilon, x)$.

By defn., $\{f_n(x)\} = \{x^n\}$ converges pointwise to $f(x) = \begin{cases} 0 & , 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$ on the interval $[0,1]$.

(b) Show that the sequence $\left\{\frac{\sin(nx+n)}{n}\right\}$ converges pointwise to $f(x) = 0$ on R.

Soln. → Let $\epsilon > 0$ be an arbitrary quantity.

Then for each $x \in R$.

$$|f_n(x) - f(x)| = \left|\frac{\sin(nx+n)}{n}\right| \leq \frac{1}{n} < \epsilon$$

$$\Rightarrow n > \frac{1}{\epsilon}$$

Take $N(\epsilon) = \left[\frac{1}{\epsilon}\right] + 1$

Hence for each $x \in R$, for any $\epsilon > 0$.

There is a natural no. $N(\epsilon)$ $\ni$ $\left|\frac{\sin(nx+n)}{n} - 0\right| < \epsilon$ $\forall n \geq N(\epsilon)$

Hence, $\left\{\frac{\sin(nx+n)}{n}\right\}$ converges pointwise to $f(x) = 0$ on R.

Remark:

1) In definition, "$|f_n(x) - f(x)| < \epsilon$, for $n \geq N(\epsilon, x)$", the natural number $N(\epsilon, x)$ is a measure of rate of convergence of the sequence at x. The value $N(\epsilon, x)$, in general, depends on both ϵ and x. In example 1.(a), $N(\epsilon, x)$ depends on ϵ & x both. For this the convergence of the sequence is significantly faster at some points than it is at other points. However, in example 1.(b), the value $N(\epsilon, x)$ does not depend on x; hence the rate of convergence is same for all x, i.e. the rate of convergence of the sequence is uniform on R.

2) The question arises: Is there a natural number $N(\epsilon)$ $\ni$ for a given $\epsilon > 0$,
$|f_n(x) - f(x)| < \epsilon \quad \forall\, n \geq N(\epsilon)$ and $\forall\, x \in A_0$?

Ans: If $N(\epsilon, x) \leq N(\epsilon)$ $\forall\, x \in A_0$, then the natural number $N(\epsilon)$ may be chosen instead of $N(\epsilon, x)$ and $n \geq N(\epsilon) \Rightarrow n \geq N(\epsilon, x)$. Since $N(\epsilon)$ does not depend on x, the same $N(\epsilon)$ serves equally well to satisfy,—

$$|f_n(x) - f(x)| < \epsilon \quad \forall \quad x \in A_0$$

$\therefore$ this question is the same question as following:
" For a given $\epsilon > 0$, is the function $N(\epsilon, x)$ bounded $\forall\, x \in A_0$?"

Uniform Convergence:

• **Definition:** A sequence $\{f_n(x)\}$ of functions converges uniformly to the function $f(x)$ on A_0 if, for every $\epsilon > 0$, there is a natural number $N(\epsilon)$, which does not depend on x $\ni$ $|f_n(x) - f(x)| < \epsilon$ $\forall\, x \in A_0$, whenever $n \geq N(\epsilon)$.

C.U

Example:

2) Show that the sequence $\{f_n(x)\} = \left\{\frac{1}{x+n}\right\}$ of functions is uniformly convergent on $[0, k]$, whatever k may be.

Soln. → For each $x \in [0, k]$,

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{1}{x+n} = 0.$$

$\therefore$ The limit function is $f(x) = 0$, $x \in [0, k]$.

Let $\epsilon > 0$ be an arbitrary quantity.

Then, for each $x \in [0, k]$

$$|f_n(x) - f(x)| < \epsilon$$

$$\Rightarrow \frac{1}{x+n} < \epsilon \Rightarrow n > \frac{1}{\epsilon} - x.$$

Choose a natural number $N(\epsilon, x) = \left[\frac{1}{\epsilon} - x\right] + 1$.
Note that $N(\epsilon, x)$ decreases as x increases.
Hence, $N(\epsilon, x) \leq \left[\frac{1}{\epsilon}\right] + 1 \;\forall\; x \in [0, K]$
Hence, it is possible to find a natural no. $N(\epsilon) = \left[\frac{1}{\epsilon}\right] + 1$, which does not depend on x, $\ni$ if $n \geq N(\epsilon)$,

$$|f_n(x) - f(x)| < \epsilon \;\forall\; n \in [0, K].$$

Hence, the sequence $\left\{\frac{1}{x+n}\right\}$ of functions converge uniformly [c.u] on $[0, K]$.

Example 3> Show that the sequence $\{x^n\}$ of functions is uniformly convergent on $[0, K]$, $K < 1$, but non-uniformly on $[0, 1]$.

Soln.-> i) For each $x \in [0,1]$, $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} (x^n) = 0$ for $0 \leq x < 1$, and for $x = 1$,

$\lim\limits_{n\to\infty} f_n(x) = 1$,

$\therefore$ The limit function is $f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1. \end{cases}$

Let $\epsilon > 0$ be an arbitrary quantity.
Then, for each $x \in (0, 1)$, $|f_n(x) - f(x)| < \epsilon$

$\Rightarrow x^n < \epsilon$

$\Rightarrow n > \frac{\ln \epsilon}{\ln x}$ $\left[\because 0 < x < 1, \ln x < 0\right]$

For $x = 0, 1$
$|f_n(x) - f(x)| = 0 < \epsilon \;\forall\; n \geq 1$.
Choose a natural number $N(\epsilon, x) = \left[\frac{\ln \epsilon}{\ln x}\right] + 1$.

Note that, $N(\epsilon, x)$ increases as x increases, $N(\epsilon, x) \to +\infty$ as $x \to 1^-$
i.e. $N(\epsilon, x)$ is not bounded in $[0, 1]$.
$\therefore$ It is not possible to find a natural number $N(\epsilon) \ni$, if $n \geq N(\epsilon)$, $|f_n(x) - f(x)| < \epsilon \;\forall\; x \in [0, 1]$
Hence, the convergence is non-uniform on $[0, 1]$, i.e. pointwise convergent.

ii> Now, consider the interval $[0, K]$, $K < 1$.
In the interval $[0, K]$, the greatest of $N(\epsilon, x)$ is

$$N(\epsilon, K) = \left[\frac{\ln \epsilon}{\ln K}\right] + 1 = N^*(\epsilon)$$

$[\because N(\epsilon, x)$ increases with $x \in [0, 1]]$

Hence it is possible to find a natural number $N^*(\epsilon) \ni$ if $n \geq N^*(\epsilon)$, $|f_n(x) - f(x)| < \epsilon \;\forall\; x \in [0, K]$, $K < 1$.
Hence, the convergence is uniform on $[0, K]$, $K < 1$.

NOTE:- Any unbounded sequence is divergent.

Remark: We see that $\{x^n\}$ is non-uniformly convergence on the interval $[0,1]$ and the non-uniform convergence is due to the influcation of point $x=1$. Hence $x=1$ can't be included in any uniform convergence of the given sequence of function; such a point is known as point of non-uniform convergence. Now if we eliminate the point of non-uniform convergence from the interval, the sequence of function is then uniformly convergence on the new interval. Therefore, $\{x^n\}$ is uniformly convergent on the interval $[0,K]$, $K<1$. C.U.

Example 4) Show that for the following sequences of function converges pointwise but not uniformly on $[0,1]$. Also determine an interval of uniform convergence of the following sequence $\to$ $f_n(x) = \dfrac{x^n}{1+x^n}$.

Soln. For $x \in [0,1]$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n} = \frac{0}{1+0} = 0 \text{ for } 0 \le x < 1.$$

For $x=1$, $\lim_{n\to\infty} f_n(x) = \frac{1}{2}$

Therefore the limit function is $f(x) = \begin{cases} 0, & 0 \le x < 1 \\ \frac{1}{2}, & x = 1. \end{cases}$

Let $\epsilon > 0$ be an arbitrary quantity. then, for each $x \in (0,1)$

$$|f_n(x) - f(x)| < \epsilon$$

$$\Rightarrow \frac{x^n}{1+x^n} < \epsilon \quad \Rightarrow \frac{1}{x^{-n}+1} < \epsilon$$

$$\Rightarrow x^{-n} > \frac{1}{\epsilon} - 1$$

$$\Rightarrow n > \frac{\ln\left(\frac{1}{\epsilon}-1\right)}{-\ln x} \quad \left[\because 0<x<1 \Rightarrow -\ln x > 0\right]$$

For $x = 0, 1$,

$$|f_n(x) - f(x)| = 0 < \epsilon \quad \forall\, n \ge 1.$$

Choose a natural no.

$$N(\epsilon, x) = \left[\frac{\ln\left(\frac{1}{\epsilon}-1\right)}{\ln x}\right] + 1 \text{ ; } N(\epsilon, x) \to +\infty \text{ as } x \to 1^-$$

i.e. $N(\epsilon, x)$ is not bounded on $[0,1]$.

Hence, it is not possible to find a natural no. $N(\epsilon)$ $\ni$ if $n \ge N(\epsilon)$, $|f_n(x) - f(x)| < \epsilon \quad \forall\, n \in [0,1]$

Hence, the convergence of $\{f_n(x)\}$ to $f(x)$ is pointwise but not uniformly on $[0,1]$.

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Here, the $x=1$ is the point of non-uniform convergence, i.e. non-uniform convergence is due to inclusion of the point $x=1$. Hence, $x=1$ can't be included any interval of uniform convergence of the given sequence of functions. Therefore, an interval $[0,k]$, $k<1$ is an interval of uniform convergence of $\{f_n(x)\}$ to $f(x)$.

C.U. **Example 5>** Construct an example of a sequence of functions $\{f_n(x)\}$ such that it is not uniformly convergent on $[0,1]$ but uniformly convergent on $[\frac{1}{4},1]$.

Hints:→ Consider the sequence $\{(1-x)^n\}$ of functions on $[0,1)$.

The limit function is

$$f(x) = \begin{cases} 1 & , x=0 \\ 0 & , 0<x\le 1 \end{cases}$$

Here $x=0$ is the point of non-uniform convergence.
Hence $\{(1-x)^n\}$ converges non-uniformly on $[0,1]$ but uniformly on $[1/4,1]$.

Example 6> Does $\{f_n(x)\}$ converges uniformly on A?

(a) $f_n(x) = \dfrac{x}{nx+1}$, $A=[0,1]$

C.U. (b) $f_n(x) = xe^{-nx}$, $A=[0,\infty)$

Soln.→ (a) For each $x\in[0,1]$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{x}{nx+1} = 0.$$

The limit function is $f(x)=0$; $x\in[0,1]$.
Let $\epsilon>0$ be an arbitrary quantity.
Then, for each $x\in(0,1]$,

$$|f_n(x)-f(x)| = \frac{x}{nx+1} < \frac{x}{nx} = \frac{1}{n} < \epsilon$$

$$\Rightarrow n > \frac{1}{\epsilon}$$

Take $N(\epsilon) = \left[\frac{1}{\epsilon}\right]+1$.

Hence, it is possible to find a natural no. $N(\epsilon)$ ∋ if $n\ge N(\epsilon)$, $|f_n(x)-f(x)|<\epsilon$ $\forall x\in A=[0,1]$

Therefore, $\left\{\dfrac{x}{nx+1}\right\}$ converges uniformly on $[0,1]$.

(b) For each $x \in [0, \infty)$,

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x e^{-nx} = \lim_{n\to\infty} \frac{x}{e^{nx}} = 0$.

Limit function is $f(x) = 0$, $x \in A$.

Let $\epsilon > 0$ be an arbitrary no.

Then, for each $x \in [0, \infty)$, $|f_n(x) - f(x)| = x . e^{-nx} \leq \frac{1}{ne} < \epsilon$.

Let $g(x) = x e^{-nx}$

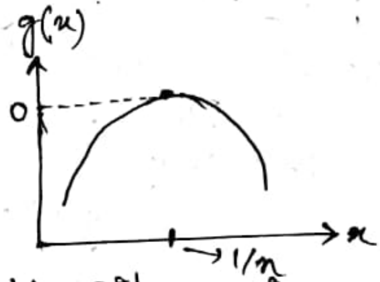
$\Rightarrow g'(x) = e^{-nx} + x . e^{-nx}(-n)$

$= e^{-nx} - nx e^{-nx}$

$= (1 - nx) e^{-nx}$

$= \left(\frac{1}{n} - x\right) . n e^{-nx}$

$= \begin{cases} > 0, & x < \frac{1}{n} \\ < 0 & x > \frac{1}{n} \end{cases}$



Note that $g(x)$ first increases, then achieves its maximum at $x = \frac{1}{n}$, finally decreases.

$\therefore \max_{x \in [0,\infty)} g(x) = \frac{1}{n} e^{-1} = \frac{1}{ne}$

$g(x)$ maximum when $g'(x) = 0 \Rightarrow x = \frac{1}{n} \Rightarrow g(x) = \frac{1}{ne}$.

$\therefore n > \frac{1}{e\epsilon}$

Take $N(\epsilon) = \left[\frac{1}{e\epsilon}\right] + 1$, which is independent of x.

Hence, it is possible to find a natural number $N(\epsilon)$ $\ni$ if $n \geq N(\epsilon)$, $|f_n(x) - f(x)| < \epsilon$ $\forall x \in [0, \infty)$

Hence, the convergence of $\{x e^{-nx}\}$ is uniform on $[0, \infty)$.

Remark: Let $\{f_n(x)\}$ be a sequence of function $\ni$ $\lim f_n(x) = f(x)$ on $x \in A$, and let $M_n = \max_{x \in [a,b]} |f_n(x) - f(x)|$ on any upper boundary of $|f_n(x) - f(x)|$.

then $\{f_n(x)\}$ converges uniformly to $f(x)$ on A iff $M_n \to 0$ as $n \to \infty$.

C.O. **Example 7** (a) S.T. $\left\{\frac{nx}{1+n^2x^2}\right\}$ is not uniformly convergent on any interval containing 0. Suggest an interval of uniform convergence.

C.O. (b) S.T. if $\{f_n(x)\} = x + \frac{1}{n}$, $n \in R$, then $\{f_n(x)\}$ converges uniformly on R but $\{f_n^2(x)\}$ does not converge uniformly on R.

Hints:→

(a) Limit function is $f(x)=0$, $x \in A$, where A is an interval containing 0.

$$|f_n(x)-f(x)| = \frac{n|x|}{1+n^2x^2} < \frac{n|x|}{n^2x^2} = \frac{1}{n|x|} < \epsilon$$

$$\Rightarrow n > \frac{1}{\epsilon|x|}$$

$$N(\epsilon,x) = \left[\frac{1}{\epsilon|x|}\right]+1.$$

Note that, $N(\epsilon,x) = \left[\frac{1}{\epsilon|x|}\right]+1 \to \infty$ as $x\to 0$,

i.e. $N(\epsilon,x)$ is not bounded on any interval containing 0.

Here $x=0$ is the point of non-uniform convergence. Hence, any integral of uniform convergence can't include the point $x=0$.

Therefore, i) $[k,\infty)$, $k>0$

ii) $(-\infty,-k]$, $k>0$

iii) $(-\infty,-k]\cup[k,\infty)$, $k>0$

are the intervals of uniform convergence.

(b) $f_n(x) = x+\frac{1}{n}$

$\therefore \lim_{n\to\infty} f_n(x) = x = f_1(x)$

$\therefore |f_n(x)-f(x)| = \frac{1}{n} < \epsilon$

$$\Rightarrow n > \frac{1}{\epsilon}$$

Choose a natural number $N(\epsilon) = \left[\frac{1}{\epsilon}\right]+1$

$\therefore$ $N(\epsilon)$ is independent on x.

$\therefore$ It is uniformly convergent.

$$\lim_{n\to\infty} f_n^2(x) = \lim_{n\to\infty}\left(x+\frac{1}{n}\right)^2 = x^2 \quad , x\in R$$
$$= f_1^2(x) = f_2(x)$$

Let $\epsilon>0$ be an arbitrary real no.

Then for each $x\in R$

$$|f_n^2(x)-f_1^2(x)| = |f_n^2(x)-f_2(x)| = \left|\frac{2x}{n}+\frac{1}{n^2}\right|$$
$$= \left|\frac{2nx+1}{n^2}\right| \le \frac{2n|x|+1}{n^2}$$
$$\le \frac{2n|x|+n}{n^2}$$
$$= \frac{2|x|+1}{n} < \epsilon$$

$$\Rightarrow n > \frac{2|x|+1}{\epsilon}$$

Take $N(\epsilon,x) = \left[\frac{2|x|+1}{\epsilon}\right]+1$

Note that, $N(\epsilon,x)\to+\infty$ as $x\to\pm\infty$, i.e. $N(\epsilon,x)$ is not bounded on R.

Hence, it is not possible to find a natural no. $N(\epsilon) \ni$ if $n \geq N(\epsilon)$, $|f_n^2(x) - f^2(x)| < \epsilon \ \forall x \in R$.

$\therefore \{f_n^2(x)\}$ does not converge uniformly on R.

B • **SERIES OF FUNCTIONS:** Consider a sequence $\{f_n(x)\}$ of functions defined on A, then the series $\sum_{n=1}^{\infty} f_n(x)$ is called a series of functions.

If the sequence $\{S_n(x)\}$ of partial sums, $S_n(x) = \sum_{K=1}^{n} f_K(x)$, converges pointwise to $S(x)$, we have

$$S(x) = \lim_{n\to\infty} S_n(x)$$
$$= \lim_{n\to\infty} \sum_{K=1}^{n} f_K(x)$$
$$= \sum_{K=1}^{\infty} f_K(x), \quad x \in A_0.$$

Definition:- The series $\sum_{n=1}^{\infty} f_n(x)$ of functions is said to converge uniformly to the function $S(x)$ on A_0 iff the sequence $\{S_n(x)\}$ of functions converges uniformly $S(x)$ on A_0.

Example 1) S.T. $\sum_{n=1}^{\infty} x^{n-1}$ converges uniformly to $S(x) = \frac{1}{1-x}$ on $[-a, a]$, $0 < a < 1$, but the convergence is not uniform on $(-1, 1)$.

Soln. → The n^{th} partial sum is

$$S_n(x) = \frac{1-x^n}{1-x}, \quad x \in (-1, 1)$$

$\left[\lim_{n\to\infty} S_n(x) = \frac{1 - \lim_{n\to\infty}(x^n)}{1-x} = \frac{1-0}{1-x} \right]$

Now, $\lim_{n\to\infty} S_n(x) = \frac{1}{1-x}$, $|x| < 1$

Let $\epsilon > 0$ be arbitrary.

Then for each $x \in (-1, 1)$

$$|S_n(x) - S(x)| = \frac{|x|^n}{|1-x|} \leq \frac{|x|^n}{1-|x|} < \epsilon$$

$$\Rightarrow n \ln|x| < \ln\{\epsilon(1-|x|)\}$$

$$\Rightarrow n > \frac{\ln\{\epsilon(1-|x|)\}}{\ln|x|}, \quad \left[x \neq 0 \Rightarrow 0 < |x| < 1 \Rightarrow \ln|x| < 0 \right]$$

Let $N(\epsilon, x) = \left[\frac{\ln\{\epsilon(1-|x|)\}}{\ln|x|} \right] + 1$

Note that $N(\epsilon, x) \longrightarrow +\infty$ as $x \longrightarrow +1^-$ or $x \longrightarrow -1^+$

i.e. $N(\epsilon, x)$ is not bounded on $(-1, 1)$.

Hence the convergence of $\{S_n(x)\}$ is not uniform on $(-1, 1)$.

Consider, the interval $(-a, a)$, $0 < a < 1$.

N.T. $N(\epsilon, x)$ is an increasing function in $|x|$, i.e.

$N(\epsilon, x) \leq N(\epsilon, a) = N^*(\epsilon)$, say.

$\therefore$ The convergence of $\{S_n(x)\}$ is uniform on $[-a, a]$

Example 2> Show that $\sum_{n=1}^{\infty} \frac{x}{(\overline{n-1}\,x+1)(nx+1)}$ is uniformly convergent on $[K,\infty)$, $K>0$ but is non-uniform on $[0,\infty)$.

Hints:→

$$S_n(x) = \sum_{k=1}^{n} \frac{x}{(\overline{k-1}\,x+1)(kx+1)}$$

$$= \sum_{k=1}^{n} \left\{ \frac{1}{\overline{k-1}\,x+1} - \frac{1}{kx+1} \right\}$$

$$= 1 - \frac{1}{nx+1} = \frac{nx}{nx+1}.$$

⇨ **Weirstrass M-test:** ↝ If $|f_n(x)| \le M_n$, $\forall x \in A_0$, for $n \geqslant m$, where M_n is a non-negative constant independent of x and if $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly as well as absolutely on A_0.

Hint: $\sum_{k=1}^{n} |f_k(x)| \le \sum_{k=1}^{n} M_k$,

By comparison test $\sum f_n(x)$ converges absolutely if $\sum M_n$ converges.

Example 3. $\boxed{\text{C.U.}}$

(a) S.T. $\sum_{n=1}^{\infty} \frac{1}{n^p} \sin nx$, $p>1$, is uniformly convergent on $\mathbb{R}$.

(b) Does the series $\sum_{n=1}^{\infty} (xe^{-x})^n$ converge uniformly on $[0,\infty)$?

Soln.→ (a) $|f_n(x)| = \left|\frac{1}{n^p} \sin nx\right| \le \frac{1}{n^p} = M_n$

Note that, $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$, $p>1$ converges.

By M. Test the series $\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{1}{n^p} \sin nx$ converges uniformly on $\mathbb{R}$.

(b) Let $g(x) = x.e^{-x}$

Now, $g'(x) = (1-x).e^{-x}$

$$= \begin{cases} >0 & \text{if } x<1 \\ <0 & \text{if } x>1 \end{cases}$$

Hence $g(x)$ first increases, then ~~achieves~~ achieves its maximum at $x=1$ and finally decreases.

$$\max g(x) = g(1) = e^{-1}$$

Hence, $|f_n(x)| \le (e^{-1})^n = M_n \ \forall n$

The series $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$ is a convergent geometric series with $r = \frac{1}{e} < 1$. By M. test, $\sum_{n=1}^{\infty} (xe^{-x})^n$ converges uniformly on $[0,\infty]$.

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- **POWER SERIES** :→ This is an important class of series of function and satisfies properties which are not valid in general.

C.U.
- **Definition** :→ A series of function of the form $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ where x_0 is a fixed real numbers and the coefficients a_n don't depend on x, is called a power series around x_0.

If we take $y = x - x_0$, then the Power series reduces to $\sum_{n=1}^{\infty} a_n y^n$, a power series around 0.

WLG, we shall consider only the power series of the form:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_n x^n + \cdots\cdots$$

Theorem :→ If for a power series $\sum_{n=0}^{\infty} a_n x^n$, $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}$ exists, then the power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely if $|x| < R$ and diverges if $|x| > R$. C.U.

Proof :→ $\lim_{n\to\infty} |a_n x^n|^{1/n} = |x| \cdot \lim_{n\to\infty} |a_n|^{1/n} = \frac{|x|}{R}$

By root-test the series converges absolutely if $|x| < R$ and diverges if $|x| > R$.

Thus any power series has an interval $(-R, R)$ within which it converges absoulutely and outside which convergence is not possible. The interval $(-R, R)$ is symmetrically located about '0' on the region of convergence or interval of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$. The numbers R is called the radious of convergence.

- **Definition** :→ 1) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. Then, we define the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ as:

$$R = \begin{cases} 0 & , \lim_{n\to\infty} |a_n|^{1/n} = \infty. \\ \dfrac{1}{\lim_{n\to\infty} |a_n|^{1/n}} & , \text{if } 0 < \lim_{n\to\infty} (a_n)^{1/n} < \infty. \\ \infty & , \text{if } \lim_{n\to\infty} |a_n|^{1/n} = 0. \end{cases}$$

provided the limit $|a_n|^{1/n}$ exists.

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Example 1. Find the radius of convergence of the following power series: (i) $\sum_{n=0}^{\infty} x^n$, (ii) $\sum_{n=0}^{\infty} \frac{x^n}{n}$, (iii) $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$.

Also check the series for convergence or divergence at the boundary points.

Soln. →

(i) Comparing the given power series with $\sum a_n x^n$, we have $a_n = 1$, $n = 0, 1, 2, \dots\dots$

$\therefore \lim\limits_{n\to\infty} |a_n|^{1/n} = 1.$

Hence the radius of convergence is $R = \frac{1}{\lim\limits_{n\to\infty} |a_n|^{1/n}} = 1.$

The boundary points are $|x| = 1$, i.e. $x = \pm 1$.
when $x = 1$, the series reduces to $\sum_{n=0}^{\infty} 1$, which diverges,
when $x = -1$, the series reduces to $\sum_{n=0}^{\infty} (-1)^n$ which diverges (oscillates). Hence the power series diverges at both the boundary points.

(ii) Comparing the given power series with $\sum a_n x^n$, we have $a_n = \frac{1}{n}$, $n = 1, 2, \dots\dots$ $\therefore \lim\limits_{n\to\infty} |a_n|^{1/n} = \lim\limits_{n\to\infty} \frac{1}{n^{1/n}} = \frac{1}{\lim\limits_{n\to\infty} (n)^{1/n}} = 1.$

Hence, the radius of convergence is $R = \frac{1}{\lim\limits_{n\to\infty} |a_n|^{1/n}} = 1.$

The boundary points are $|x| = 1$, i.e. $x = \pm 1$.
when $x = +1$, the series reduces to $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges.
when $x = -1$, the series reduces to $\sum_{n=0}^{\infty} (-1)^n / n$ which converges conditionally. Hence at $x = -1$, the power series $\sum \frac{x^n}{n}$ converges conditionally but at $x = +1$, the series diverges.

(iii) Comparing the given power series with $\sum a_n x^n$, we have, $a_n = \frac{1}{n^2}$, $n = 1, 2, \dots\dots$ $\therefore \lim\limits_{n\to\infty} |a_n|^{1/n} = \lim\limits_{n\to\infty} \left(\frac{1}{n^{1/n}}\right)^2 = \left\{\frac{1}{\lim\limits_{n\to\infty} (n^{1/n})}\right\}^2$

Hence, the radius of convergence is $R = \frac{1}{\lim\limits_{n\to\infty} |a_n|^{1/n}} = 1$

the boundary points are $|x| = 1$, i.e. $x = \pm 1$.
when $x = +1$, the series reduces to $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges absolutely.
when $x = -1$, the series reduces to $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ which converges absoultely.
Hence, the power series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges absolutely at $x = \pm 1$.

<u>Example 2</u>. Find the radius of convergence and the interval of convergence of the following series :

(i) $\sum \frac{(x+3)^n}{n}$, (ii) $\sum (2^n+3^n)x^n$, (iii) $\sum \frac{x^n}{n!}$, (iv) $\sum n!\, x^n$.

<u>Soln.</u> $\rightarrow$ (i) $y=(x+3)$.

Then the power series $\sum \frac{(x+3)^n}{n}$ reduces to $\sum \frac{y^n}{n}$.

Here, $a_n = \frac{1}{n}$, $n \in R$,

The radius of the convergence is $R = \frac{1}{\underset{n\to\infty}{Lt} |a_n|^{1/n}} = \underset{n\to\infty}{Lt} (n^{1/n}) = 1$.

Hence, the power series $\sum \frac{y^n}{n}$ converges absolutely if $|y|<1$,

i.e. $\sum \frac{(x+3)^n}{n}$ converges absolutely if $|x+3|<1$, i.e. if

$-1 < (x+3) < 1 \Rightarrow -4 < x < -2$ and diverges if $|x+3|>1$.

Hence the interval of convergence of $\sum \frac{(x+3)^n}{n}$ is $(-4,-2)$.

<u>(ii)</u> Here $a_n = (2^n+3^n)$, $n \in N$.

$\lim\limits_{n\to\infty} (a_n)^{1/n} = \lim\limits_{n\to\infty} (2^n+3^n)^{1/n} = 3$.

[$3 < (2^n+3^n)^{1/n} < (2.3^n)^{1/n} = 2^{1/n}.3$

$\therefore \lim\limits_{n\to\infty} (2^{1/n}.3) = 3$.

By Sauize theorem, $\lim\limits_{n\to\infty} (2^n+3^n)^{1/n} = 3$]

$\therefore R = \frac{1}{\lim\limits_{n\to\infty} (a_n)^{1/n}} = \frac{1}{3}$ is the radius of convergence.

Note that, the interval of convergence is $(-R,R) = \left(-\frac{1}{3}, \frac{1}{3}\right)$.

(*) (iii) Here, $a_n = \frac{1}{n!}$

The radius of the convergence is $R = \lim\limits_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim\limits_{n\to\infty} \left\{\frac{1}{n!} \times (n+1)!\right\}$

$= \lim\limits_{n\to\infty} (n+1) = \infty$.

Hence, the interval convergence is $(-\infty, \infty)$, i.e. the entire real line.

Therefore, the power series converges absolutely on the entire real line.

(*) <u>(iv)</u> $\sum\limits_{n=0}^{\infty} n!\, x^n$, Here $a_n = n!$

$R = \lim\limits_{n\to\infty} \left(\frac{a_n}{a_{n+1}}\right) = \lim\limits_{n\to\infty} \left(\frac{n!}{(n+1)!}\right) = \lim\limits_{n\to\infty} \left(\frac{1}{n+1}\right) = 0$ is the radius of convergence.

Note that, every power series has $\sum a_n x^n$ has the value a_0, when $n=0$, i.e. every power series is convergent at $x=0$.

Hence, the power series $\sum n!\, x^n$ has no interval of convergence but it converges only at $x=0$.

Theorem:- If, for a given power series $\sum a_n x^n$, $0 < \frac{1}{R} = \lim \left|\frac{a_{n+1}}{a_n}\right|$ exists, then $\sum a_n x^n$ converges absolutely if $|x| < R$ and ~~converges~~ diverges if $|x| > R$.

Definition 2) Let $\sum a_n x^n$ be a given power series then we define the radius of convergence of the power series as

$$R = \begin{cases} \dfrac{1}{\lim\left|\frac{a_{n+1}}{a_n}\right|} & , \text{if } 0 < \lim\left|\frac{a_{n+1}}{a_n}\right| < \infty \\ \infty & , \text{if } \lim\left|\frac{a_{n+1}}{a_n}\right| = 0 \\ 0 & , \text{if } \lim\left|\frac{a_{n+1}}{a_n}\right| = \infty . \end{cases}$$

$= \lim \left|\frac{a_n}{a_{n+1}}\right|$, provided the limit exists.

Example:-> Example 2)(iii) & (iv)* are the example of this type.

Example 3. Find the radius of convergence and the interval of convergence of the power series:

(i) $\sum \frac{n^n}{n!} \cdot x^n$, (ii) $\sum\limits_{n=0}^{\infty} \frac{x^{2n}}{2^{2n}}$.

Soln.->

(i) $a_n = \frac{n^n}{n!}$

The radius of convergence is

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left\{\frac{n^n}{n!} \times \frac{(n+1)!}{(n+1)^{n+1}}\right\} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \frac{1}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$$

The interval of convergence is $(-R, R) = \left(-\frac{1}{e}, \frac{1}{e}\right)$.

(ii) $\sum\limits_{n=0}^{\infty} \frac{x^{2n}}{2^{2n}} = \sum\limits_{n=0}^{\infty} \frac{(x^2)^n}{4^n} = \sum\limits_{n=0}^{\infty} \frac{y^n}{4^n}$, where $y = x^2$

Note that the radius of convergence of $\sum \frac{y^n}{4^n}$ is $R = \frac{1}{\lim\left|\frac{1}{4^n}\right|^{1/n}} = 4$

Hence, $\sum \frac{y^n}{4^n}$ converges absolutely if $|y| < 4$.

[Note that, for the series $\sum \frac{y^n}{4^n} = \sum u_n$, say, $\lim\limits_{n\to\infty} (u_n)^{1/n} = \lim\left(\frac{y^n}{4^n}\right)^{1/n} = \frac{y}{4}$

If $y < 4$, by root test the series $\sum \frac{y^n}{4^n}$ converges absolutely.]

Therefore, the series $\sum \frac{x^{2n}}{2^{2n}}$ converges absolutely if $x^2 < 4$, i.e. $-2 < x < 2$. and diverges if $|x| > 2$.

Hence, the radius of convergence of $\sum \frac{x^{2n}}{2^{2n}}$ is $R' = 2$.

Example 4) If the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is $R>0$. Find the radius of convergence of the following series:

i) $\sum_{n=0}^{\infty} a_n x^n/n$, (ii) $\sum a_n \lfloor n . x^n$

Hints: → i) Here, $R = \frac{1}{\log|a_n|^{1/n}}$ or $\lim\left|\frac{a_n}{a_{n+1}}\right|$, provided the limit exists.

Now, the radius of convergence of $\sum \frac{a_n}{n} . x^n$ is

$$R' = \frac{1}{\lim\left|\frac{a_n}{n}\right|^{1/n}} = \lim(n^{1/n}) . \frac{1}{\lim(a_n)^{1/n}} = 1.R = R.$$

ii) The radius of convergence of $\sum a_n \lfloor n . x^n$ is

$$R' = \lim\left|\frac{a_n \lfloor n}{a_{n+1}\lfloor n+1}\right| = \lim\left|\frac{a_n}{a_{n+1}}\right| . \lim\left(\frac{1}{n+1}\right) = R.0 = 0 .$$

Example 5)

(a) If $\lim a_n r^n = c$, then show that the radius of convergent of $\sum a_n x^n$ is $|r|$.

(b) If $0 < p \le |a_n| \le q$, $n \in N$. then show that the radius of convergent of $\sum_{n=0}^{\infty} a_n x^n$ is 1.

Soln. →

(a) Let $\epsilon > 0$ be an arbitrary quantity.

Then $|a_n . r^n - c| < \epsilon \ \forall \ n \geq N(\epsilon)$

$\Rightarrow c - \epsilon < a_n r^n < c + \epsilon \quad \forall \ n \geq N(\epsilon)$

$\Rightarrow |c-\epsilon|^{1/n} < |a_n r^n|^{1/n} < |c+\epsilon|^{1/n}$, assuming $|c-\epsilon| < |c+\epsilon|$.

Let, $|c-\epsilon| = l_1$, $|c+\epsilon| = l_2$.

$$\Rightarrow \frac{l_1^{1/n}}{|r|} < |a_n|^{1/n} < \frac{l_2^{1/n}}{|r|}$$

$$\therefore \frac{\lim_{n\to\infty}(l_1^{1/n})}{|r|} \le \lim|a_n|^{1/n} \le \frac{\lim(l_2^{1/n})}{|r|}$$

$$\Rightarrow \frac{1}{|r|} \le \lim|a_n|^{1/n} \le \frac{1}{|r|}$$

$$\Rightarrow \lim|a_n|^{1/n} = \frac{1}{|r|}$$

$$\Rightarrow R = \frac{1}{\lim|a_n|^{1/n}} = |r| .$$

(b) $p \le |a_n| \le q$

$$\Rightarrow p^{1/n} \le |a_n|^{1/n} \le q^{1/n}$$

$$\therefore \lim_{n\to\infty}(p^{1/n}) \le \lim_{n\to\infty}|a_n|^{1/n} \le \lim_{n\to\infty}(q^{1/n})$$

$$\Rightarrow 1 \le \lim_{n\to\infty}|a_n|^{1/n} \le 1$$

$$\Rightarrow \lim_{n\to\infty}|a_n|^{1/n} = 1$$

$$\Rightarrow R = 1 .$$

C.U.

Example 6) If a power series $\sum a_n x^n$ converges at $x = x_0$ then it is absolutely convergent on $(-|x_0|, |x_0|)$, i.e. the radius of convergence is $R \geq |x_0|$.

Soln. $\sum a_n x^n$ converges at $x = x_0$,

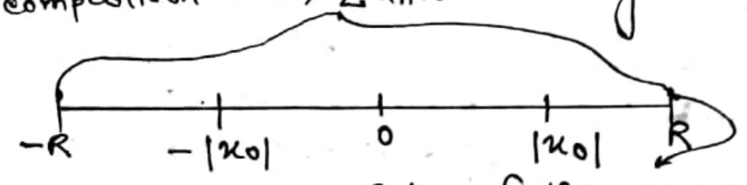
$\Rightarrow \lim (a_n x_0^n) = 0$

Let $\epsilon > 0$ then $|a_n x_0^n| < \epsilon$, $n \geq N(\epsilon)$

Now, $|a_n x^n| = |a_n \cdot x_0^n| \cdot \left|\frac{x}{x_0}\right|^n < \epsilon \cdot \left|\frac{x}{x_0}\right|^n$

As $\sum \left|\frac{x}{x_0}\right|^n$ converges if $|x| < |x_0|$,

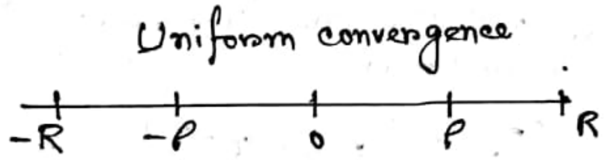
By comparison test, $\sum a_n x^n$ converges absolutely if $|x| < |x_0|$.



Uniform Convergence of Power Series :-

- **Theorem:-** If the radius of convergence of a power series $\sum a_n x^n$ is R, then $\sum a_n x^n$ converges uniformly on $[-\rho, \rho]$, where $0 < \rho < R$.

Proof:-



For $x \in [-\rho, \rho]$, i.e. $|x| \leq \rho$

Note that, $|f_n(x)| = |a_n x^n| = |a_n|\,|x^n|$

$\leq |a_n| \cdot \rho^n = M_n \;\forall n \in \mathbb{N}$.

As, $\sum M_n = \sum |a_n| \rho^n$ converges by M. test, the power series converges uniformly on $[-\rho, \rho]$.

Properties of Power Series :

1) A power series $\sum a_n x^n$ is a continuous function of x within its interval of convergence, i.e. $\lim_{n\to x_0}(\sum a_n x^n) = \sum a_n x_0^n$

$$= \sum\left(\lim_{n\to x_0} a_n x^n\right),$$

$x_0 \in (-R, R)$.

2) If $[c_1, c_2]$ is an interval within the interval of the convergence of $\sum a_n x^n$, then
$$\int_{c_1}^{c_2}(\sum a_n x^n)\,dx = \sum a_n \int_{c_1}^{c_2} x^n\,dx$$

The theorem may be stated as follows :
" A power series can be integrated term by term over any closed interval contained in the interval of convergence".

3) A power series can be differentiated by term by term within the interval of convergence.

i.e. $\frac{d}{dx}(\sum a_n x^n) = \sum \frac{d}{dx}(a_n x^n)$, $|x|<R$

$= \sum a_n . n x^{n-1}$, $|x|<R$.

Remark :→ It is important to note that the radius of convergence is not altered by differentiation or integration.

- **Uniqueness Property of Power Series :**

Theorem : If $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ converge on same interval $(-R, R)$, $R>0$, to the same limit function $f(x)$, then $a_n = b_n$, $n = 0, 1, 2, \ldots$, i.e. two power series are identical.

Proof :→ $f'(x) = \frac{d}{dx}(\sum a_n x^n) = \sum a_n n x^{n-1}$, $|x|<R$.

$$\therefore f^{(r)}(x) = \sum_{n=r}^{\infty} a_n\, n(n-1)\cdots(n-r+1).x^{n-r}$$

$$= \sum_{n=r}^{\infty} a_n . \frac{n!}{\lfloor n-r}.x^{n-r}$$

$$= \sum_{n=r+1}^{\infty} a_n . \frac{\lfloor n}{\lfloor n-r}.x^{n-r} + a_r . \lfloor r$$

when $x=0$,

$\therefore f^{(r)}(0) = a_r . \lfloor r$, $r \in N$.

Similarly, $f^{(r)}(0) = b_r \lfloor r$, $r \in N$

$a_r \lfloor r = b_r \lfloor r$, $\forall r \in N$.

$\Rightarrow a_r = b_r$, $r \in N$.

and $a_0 = b_0$.

Example: Find the radius of convergence and the interval of convergence of the following power series:

[C.U] i) $1+2x+x^2+2x^3+x^4+\cdots\cdots$

ii) $1+x+\left(\frac{x}{2}\right)^2+x^3+\left(\frac{x}{2}\right)^4+\cdots\cdots$

Soln. →

i) Here, $a_{2k}=1$, $k=0,1,2,\ldots$

$a_{2k+1}=2$, $k=0,1,2,\ldots$

Note that, $\frac{a_n}{a_{n+1}} = \begin{cases} \frac{1}{2} & , n=2k \\ 2 & , n=2k+1. \end{cases}$

Hence the limit $\lim\left|\frac{a_n}{a_{n+1}}\right|$ does not exist and we can't apply defn. 2) : $R=\lim\left|\frac{a_n}{a_{n+1}}\right|$.

But, $a_n^{1/n} = \begin{cases} 1^{1/n} & , n=2k \\ 2^{1/n} & , n=2k+1 \end{cases}$

$a_n^{1/n} \to 1$ as $n\to\infty$.

$\lim|a_n|^{1/n}=1$, Hence $R=1$.

The interval of convergence is $(-1,1)$.

ii) Here, $a_{2k}=\frac{1}{2^{2k}}$

$a_{2k+1}=1$, $k=0,1,2,\ldots$

Note that, $\frac{a_n}{a_{n+1}} = \begin{cases} \frac{1}{2^n} & , n=2k \\ 2^n & , n=2k+1 \end{cases}$

Here the limit $\lim\left|\frac{a_n}{a_{n+1}}\right|$ does not exist and we can't apply defn. 2)

$R=\lim\left|\frac{a_n}{a_{n+1}}\right|$.

But, $a_n^{1/n} = \begin{cases} 1/2 & n=2k \\ 2 & n=2k+1 \end{cases}$

$\Rightarrow \lim\left(a_n^{1/n}\right)$ does not exist.

Now, $1+x+\left(\frac{x}{2}\right)^2+x^3+\left(\frac{x}{2}\right)^4+\cdots\cdots$

$=\sum_{n=0}^{\infty}\frac{x^{2n}}{2^{2n}}+\sum_{n=0}^{\infty} x^{2n+1}$

Note that, $\sum\frac{x^{2n}}{2^{2n}}=\sum\left(\frac{x^2}{4}\right)^n$ converges absolutely if $\frac{x^2}{4}<1$, i.e. $|x|<2$ and diverges if $|x|>2$.

Again, $\sum_{n=0}^{\infty}x^{2n+1}=x\sum_{n=0}^{\infty}(x^2)^n$ converges absolutely if $x^2<1$. i.e. $|x|<1$ and diverges if $|x|>1$.

Therefore both $\sum\frac{x^{2n}}{2^{2n}}$, $\sum x^{2n+1}$ converge absolutely on $(-1,1)$ and the interval of convergence of $\sum\left(\frac{x^{2n}}{2^{2n}}+x^{2n+1}\right)$ is $(-1,1)$. Hence, the radius of convergence is $R=1$.

The Taylor's Polynomial & Taylor's Series :

[A]. TAYLOR'S POLYNOMIAL :→

There are several ways to approximate a given function by polynomials, depending on what use is to be made of the approximation.

Consider a function $f(x) = e^x$,

Note that, $f^{(r)}(x) = e^x$.

and $f(0) = 1 = f^{(r)}(0)$, $r \in N$.

The 1st degree polynomial $P_1(x) = 1 + x$ also has $P_1(0) = 1 = f(0)$

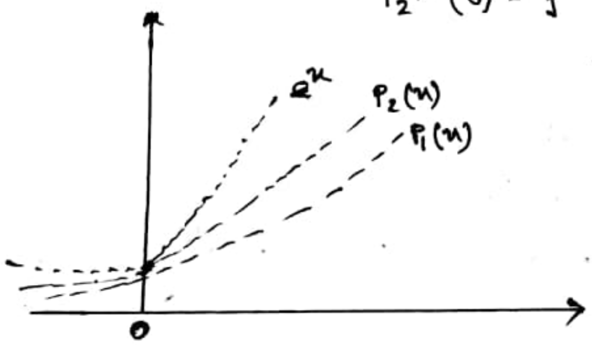
and $P_1'(0) = 1 = f'(0)$

The 2nd degree polynomial $P_2(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!}$ also has

$P_2(0) = f(0)$

$P_2'(0) = f'(0)$

$P_2''(0) = f''(0)$



→ Clearly, $P_2(x)$ is a better approximation than $P_1(x)$ around $x = 0$ for $f(x) = e^x$.

The Taylor's Polynomial generated by a function :

Suppose $f(x)$ has derivatives upto order n at $x = x_0$. Let us try to find out a polynomial $P_n(x)$ which agrees with $f(x)$ and its first n derivatives at $x = x_0$.

Note that there are $(n+1)$ conditions to be satisfied :

$P_n(x_0) = f(x_0)$

$P_n^{(r)}(x_0) = f^{(r)}(x_0)$, $r = 1(1)n$.

From the $(n+1)$ conditions, we can determine only $(n+1)$ unknown constants ; so we try with a polynomial of degree n, say, $P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n$.

When $x = x_0$, $P_n(x_0) = f(x_0) \Rightarrow a_0 = f(x_0)$

Note that $P_n^{(r)} x_0 = \underline{|r} \, . a_r$

and $P_n^{(r)}(x_0) = f^{(r)} x_0$

$\Rightarrow \underline{|r} . a_r = f^{(r)} x_0$

$\Rightarrow a_r = \frac{f^{(r)}(x_0)}{\underline{|r}}$

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Hence, we have $P_n(x) = \sum_{r=0}^{n} \frac{f^{(r)}(x_0)}{\underline{|r}} (x-x_0)^r$

And this is the one and only polynomial of exact degree $\leq n$, which agrees with $f(x)$ and its first n derivatives at $x=x_0$. This polynomial $P_n(x)$ is called the Taylor's Polynomial of degree n generated by $f(x)$ at $x=x_0$.

It is natural to expect that this polynomial will provide a reasonable approximation to $f(x)$ for points near x_0 and to measure the quality of approximation.

It is necessary to have information regarding the reminder $R_n(x) = f(x) - P_n(x)$.

<u>Taylor's Theorem :</u>- If $f(x)$ has continuous derivatives upto order $(n+1)$ is some neighbourhood of $x=x_0$, then

$$f(x) = \sum_{r=0}^{n} \frac{f^{(r)}(x_0)}{r!} (x-x_0)^r + R_n(x)$$

where $R_n(x) = \frac{f^{(n+1)}(c)}{\underline{|n+1}} (x-x_0)^{n+1}$,

for some c between x and x_0.

<u>Example 1)</u> Obtain the Taylor polynomial of degree '6' generated by $f(x) = \sin(x)$ at $x=0$. Also estimate the error in approximating the interval $[0, \frac{1}{2}]$.

<u>Soln.</u> → The Taylor's polynomial of degree 6 generated by $f(x)$ at $x=0$ is

$$P_6(x) = \sum_{r=0}^{6} \frac{f^{(r)}(0)}{\underline{|r}} \cdot x^r$$

Here, $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, ...
$f^{7}(x) = -\cos x$.

Hence, $P_6(x) = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + -\frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

By Taylors theorem :

$f(x) = P_6(x) + R_6(x)$ where $R_6(x) = \frac{f^{(7)}(c)}{7!} \cdot x^7$,

where c lies between 0 and x.

Hence, $R_6(x) = -\frac{\cos c}{7!} \cdot x^7$

and $|R_6(x)| = \frac{|-\cos c|}{7!} |x|^7 \leq \frac{|x|^7}{\underline{|7}}$

For $x \in [0, \frac{1}{2}]$, $|R_6(x)| \leq \frac{1}{2^7 \cdot 7!}$

Hence, the maximum error in approximating $f(x) = \sin x$ by $P_6(x)$ for $x \in [0, \frac{1}{2}]$ is $\frac{1}{2^7 \cdot 7!}$

Example 2) Obtain the Taylor polynomial $P_2(x)$ of degree 2 generated by $f(x) = \sqrt[3]{1+x}$ at $x=0$. Hence show that
$$\left|\sqrt[3]{1+x} - \left(1+\frac{x}{3}-\frac{x^2}{9}\right)\right| < \frac{5}{81}x^3 \text{ for } x>0.$$
Use this inequality to approximate $\sqrt[3]{1.6}$ with an estimate on the error in the approximation.

Soln. → Let $f(x) = \sqrt[3]{1+x}$

Then the 2nd degree Taylor Polynomial generated by $f(x)$ at $x=0$ is
$$P_2(x) = \sum_{r=0}^{2} \frac{f^{(r)}(0)}{\lfloor r}\cdot x^r$$
Now, $f(0)=1$, $f'(x) = \frac{1}{3}(1+x)^{-2/3}$
$$f''(x) = -\frac{2}{9}(1+x)^{-5/3},\ f^3(x) = \frac{10}{27}(1+x)^{-8/3}$$
therefore, $P_2(x) = 1 + \frac{1/3}{1!}x + \frac{-2/9}{2!}x^2$
$$= 1+\frac{x}{3}-\frac{x^2}{9}$$
By Taylor's theorem,
$$f(x) = P_2(x) + R_2(x)$$
where, $R_2(x) = \frac{x^3}{3!}f'''(c)$

where c lies between 0 and x.

Now, $\left|\sqrt[3]{1+x^2} - \left(1+\frac{x}{3}-\frac{x^2}{9}\right)\right|$
$$= \left|f(x)-P_2(x)\right| = \left|R_2(x)\right| = \frac{|x|^3}{6}\left|\frac{10}{27}(1+c)^{-8/3}\right|$$
For $x>0$, $0<c<x$ and $1+c>1$
$$\Rightarrow \frac{1}{1+c} < 1$$
$$\therefore \left|\sqrt[3]{1+x^2} - \left(1+\frac{x}{3}-\frac{x^2}{9}\right)\right| < \frac{x^3}{6}\cdot\frac{10}{27} = \frac{5}{81}x^3 \text{ for } x>0.$$
Note that $\sqrt[3]{1.06} = \sqrt[3]{1+0.6}$, and $x=0.6$

Hence we obtain the approximation of $\sqrt[3]{1.06}$ as
$$P_2(0.6) = 1+\frac{0.6}{3}-\frac{(0.6)^2}{9} \simeq 1.16$$
with a maximum error $\frac{5x^3}{81} = \frac{5(0.6)^3}{81} = \frac{4}{300}$.

Example 3) Obtain the Taylor's polynomial $P_n(x)$ generated by $f(x) = \ln(1+x)$ at $x=0$. Approximate $\ln(1.5)$ with an error less than 0.001.

Soln. → Let $f(x) = \ln(1+x)$

Hence, $f^{(r)}(x) = \frac{(-1)^{r-1}\lfloor r-1}{(1+x)^r}$; $r = 1,2,3,\ldots$

Hence, $P_n(x) = \sum_{r=0}^{n}\frac{f^{r}(0)}{\lfloor r}\cdot(x-0)^r = f(0) + \sum_{r=1}^{n}\frac{(-1)^{r-1}\lfloor r-1}{\lfloor r}\cdot x^r$
$$= 0 + \sum_{r=1}^{n}\frac{(-1)^{r-1}\cdot x^r}{r}$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n-1}\cdot x^n}{n}$$

By Taylor's theorem, we have, —

$$f(x) = P_n(x) + R_n(x)$$

$\Rightarrow$ the remainder term is $f(x) - P_n(x)$

$$= R_n(x) = \frac{f^{(n+1)}(c)}{\lfloor n+1} \cdot x^{n+1}$$

i.e. $$R_n(x) = \frac{(-1)^n \lfloor n}{(1+c)^{n+1}} \cdot \frac{x^{n+1}}{\lfloor n+1} = \frac{(-1)^n \cdot x^{n+1}}{(1+c)^{n+1}(1+n)}$$

For $x > 0$,

$$|R_n(x)| = \frac{x^{n+1}}{(n+1)(1+c)^{n+1}} < \frac{x^{n+1}}{n+1}$$

$$\left[\text{For } x > 0,\ 0 < c < x,\ (1+c) > 1\right]$$

$$\Rightarrow \frac{1}{1+c} < 1 \Rightarrow \frac{1}{(1+c)^{n+1}} < 1.$$

Here, $\ln(1.5) = \ln(1+0.5)$ is to be approximate by $P_n(1/2) \ni$ the maximum error $\frac{1}{2^{n+1}(n+1)} < 0.001$

$$\Rightarrow 2^{n+1}(n+1) > 1000$$

$$\Rightarrow 2^{n+1}(n+1) = \begin{cases} 896, & n = 6 \quad < 1000 \\ 2048, & n = 7 \quad > 1000 \end{cases}$$

and hence the smallest value of n is 7.

Thus, we obtain $P_7(0.5) = \sum_{r=0}^{7} (-1)^{r-1} \cdot \frac{(0.5)^r}{r}$

$= 0.405$ as an approximation to $\ln(1.5)$ with an error less than 10^{-3}.

Example 4) Obtain the Taylor's Polynomial $P_3(x)$ of degree 3 generated by $\ln x$ at $x = 1$. Also estimate the error in approximating $\ln x$ by $P_3(x)$ for $x \in \left[\frac{1}{2}, \frac{3}{2}\right]$

Soln. $\rightarrow$ $f(x) = \ln x$, $x_0 = 1$.

$$P_3(x) = \sum_{r=0}^{3} \frac{f^{(r)}(1)}{\lfloor r} \cdot (x - x_0)^r$$

$$f'(x) = \frac{1}{x},\ f''(x) = -\frac{1}{x^2},\ f'''(x) = \frac{2}{x^3},\ f^4(x) = -\frac{6}{x^4}$$

$$\therefore P_3(x) = \frac{1}{1!}(x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$$

By Taylors theorem,

$f(x) = P_3(x) + R_3(x)$, where

$$R_3(x) = \frac{f^{(4)}(c)}{4!} \cdot (x-1)^4 = \frac{-6/c^4}{4!}(x-1)^4 = \frac{-(x-1)^4}{4c^4}$$

where c lies between 1 and x.

$$|R_3(x)| = \left|\frac{-(x-1)^4}{4c^4}\right| \le \frac{(x-1)^4}{4c^4}$$

$$\le \frac{1}{2^4\, 4 \cdot 2^4} = \frac{1}{2^{10}}$$

$x \in \left[\frac{1}{2}, \frac{3}{2}\right]$

$\Rightarrow |x-1| \le \frac{1}{2}$

and $c \in \left[\frac{1}{2}, \frac{3}{2}\right]$

$\Rightarrow \frac{1}{c^4} < 2^4$

Hence the maximum error in approximating $\ln x$ by $P_3(x)$ for $x \in [\frac{1}{2}, \frac{3}{2}]$ is $\frac{1}{2^{10}}$.

[B]. TAYLOR'S SERIES :→ If a function $f(x)$ has derivatives of every order in an open interval about x_0, then we can compute the Taylor's coefficients $a_0 = f(x_0)$, $a_n = \frac{f^{(n)} x_0}{\lfloor n}$ and with these coefficients we can form the power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{\lfloor n} \cdot (x-x_0)^n$

The power series $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{\lfloor n} \cdot (x-x_0)^n$ converges to $f(x)$ for $|x-x_0| < R$ iff the sequence $\{R_n(x)\}$ of remainder terms converges to '0' for $|x-x_0|<R$.

Then we shall write $f(x) = \sum \frac{f^{(n)}(x_0)}{\lfloor n} (x-x_0)^n$ and we say that the power series $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{\lfloor n} (x-x_0)^n$ is the Taylor's series expansion of $f(x)$ at $x = x_0$.

C.U. **Example :→** Obtain the Taylor series expansions of the following functions at $x=0$.
(i) $\sin x$, (ii) e^x, (iii) $\log_e(1+x)$, where $-1 < x \le 1$.

Soln. →
(i) Let $f(x) = \sin x$,
Here, $f^{(2n-1)}(x) = (-1)^{n-1} \cdot \cos x$.
$f^{(2n)}(x) = (-1)^n \cdot \sin x$,

Taylor's coefficients are
$$a_{2n} = \frac{f^{(2n)}(0)}{\lfloor 2n} = 0$$
$$a_{2n-1} = \frac{f^{(2n-1)}(0)}{\lfloor 2n-1} = \frac{(-1)^{n-1}}{\lfloor 2n-1}, \; n \in \mathbb{N}.$$

Now, $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\lfloor n} \cdot x^n$ is the Taylor's series expansion of $f(x)$ at $x=0$ iff the sequence $\{R_n(x)\}$ of remainders terms converges to '0'.

Now, $R_n(x) = \frac{f^{(n+1)}(c)}{\lfloor n+1} \cdot x^{n+1}$

and $|R_n(x)| \le \frac{|x|^{n+1}}{\lfloor n+1}$ $\left[\because |f^{(n+1)}c| = \begin{cases} |\cos c|, n = \text{even} \\ |\sin c|, n = \text{odd}. \end{cases} \le 1. \right]$

Note that →

$$\lim \left\{ \frac{|x|^{n+2}}{\lfloor n+2} \Big/ \frac{|x|^{n+1}}{\lfloor n+1} \right\}$$

$$= |x| \cdot \lim \left(\frac{1}{n+2}\right) = 0 < 1$$

$$\Rightarrow \sum \frac{|x|^{n+1}}{\lfloor n+1} \text{ converges,}$$

$$\Rightarrow \lim_{n\to\infty} \frac{|x|^{n+1}}{\lfloor n+1} = 0$$ [By n^{th} term test: $\sum a_n$ converges $\Rightarrow \lim a_n = 0$]

Hence, $0 \le |R_n(x)| \le \frac{|x|^{n+1}}{\lfloor n+1} \longrightarrow 0$ as $n \to \infty$ $\forall\, x \in R$.

$$\Rightarrow \lim_{n\to\infty} R_n(x) = 0$$

Therefore, we have the Taylor's series expansion of $\sin x$ about $x = 0$ as:

$$\sin x = \sum_{n=0}^{\infty} a_n (x-0)^n$$

$$= \sum_{n=1}^{\infty} a_{2n-1} x^{2n-1} \text{ as } a_{2n} = 0$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot x^{2n-1}}{\lfloor 2n-1}, \quad x \in R$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^{n-1} \cdot x^{2n-1}}{\lfloor 2n-1} + \cdots$$

(ii) Let $f(x) = e^x$

Here, $f^{(n)}(x) = e^x$

and the Taylor's coefficient

$$a_n = \frac{f^{(n)}(0)}{\lfloor n} = \frac{1}{\lfloor n}, \quad n \in N$$

Now, $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\lfloor n} \cdot x^n$ is the Taylor's series expansion of $f(x)$ at $x = 0$ iff the sequence $\{R_n(x)\}$ of remainder terms converges to '0'.

Here, $$R_n(x) = \frac{f^{(n+1)}(c)}{\lfloor n+1} \cdot x^{n+1}$$

$$= \frac{e^c \cdot x^{n+1}}{\lfloor n+1}, \text{ where } c \text{ lies between } 0 \text{ and } x.$$

Now, $$|R_n(x)| \le \frac{e^{|x|} \cdot |x|^{n+1}}{\lfloor n+1}$$ as $|c| < |x|$ i.e. $e^{|c|} < e^{|x|}$

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Note that $\lim\limits_{n\to\infty} \left\{ \dfrac{e^{|x|} \cdot x^{n+2}}{\lfloor n+2} \bigg/ \dfrac{e^{|x|} \cdot |x|^{n+1}}{\lfloor n+1} \right\}$

$$= |x| \cdot \lim \left(\frac{1}{n+2}\right) = 0 < 1$$

$\Rightarrow \sum \dfrac{e^{|x|} \cdot |x|^{n+1}}{\lfloor n+1}$ converges

$\Rightarrow \lim \dfrac{e^{|x|} \cdot |x|^{n+1}}{\lfloor n+1} = 0$

$\therefore \lim\limits_{n\to\infty} R_n(x) \to 0$

Therefore, the Taylor's series expansion of e^x at $x = 0$ is

$$e^x = \sum_{n=0}^{\infty} a_n x^n \text{ or } \sum_{n=0}^{\infty} \frac{f^{(n)}(0) \cdot x^n}{\lfloor n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \cdot x^n \quad , x \in R.$$

(iii) Here $f(x) = \log_e(1+x)$, $-1 < x \leq 1$.

Now, $f^{(n)}(x) = \dfrac{(-1)^{n-1} \lfloor n-1}{(1+x)^n}$

$$a_n = \frac{f^{(n)}(0)}{\lfloor n} = \frac{(-1)^{n-1} \cdot \lfloor n-1}{\lfloor n} = \frac{(-1)^{n-1}}{n} \quad , n \in N.$$

Now, $\sum\limits_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{\lfloor n} \cdot x^n$ is the Taylor's series expansion of $f(x)$ at $x = 0$ iff the sequence $\{R_n(x)\}$ converges to zero.

Here, $R_n(x) = \dfrac{f^{(n+1)}(c)}{\lfloor n+1} \cdot x^{n+1} = \dfrac{(-1)^n}{(1+c)^{n+1}} \cdot \dfrac{x^{n+1}}{n+1}$

where c lies between 0 & x.

Case: $0 \leq x \leq 1$ →

Then $0 < c < x \leq 1$

$$|R_n(x)| = \frac{x^{n+1}}{n+1} \cdot \frac{1}{(1+c)^{n+1}} < \frac{x^{n+1}}{n+1}$$

$\Rightarrow |R_n(x)| < \dfrac{x^{n+1}}{n+1} \leq \dfrac{1}{n+1} \to 0$ as $n \to \infty$ for $0 \leq x \leq 1$.

$\Rightarrow \lim\limits_{n\to\infty} R_n(x) = 0 \quad , 0 \leq x \leq 1.$

Case-II :→ Cauchy's form of remainder

$$R_n = \frac{x^{n+1}(1-\theta)^n . f^{n+1}(\theta x)}{\lfloor n}$$

$$= \frac{x^{n+1}(1-\theta)^n}{\lfloor n} \cdot \frac{(-1)^n \lfloor n}{(1+\theta x)^{n+1}}$$

$$= (-1)^n . x^{n+1} \left(\frac{1-\theta}{1+\theta x}\right)^n \cdot \frac{1}{1+\theta x}$$

Now, $1-\theta < 1+\theta x$

$$\Rightarrow \frac{1-\theta}{1+\theta x} < 1$$

and $\left(\frac{1-\theta}{1+\theta x}\right)^n \to 0$ as $n \to \infty$

Also note that $x^{n+1} \to 0$ as $n \to \infty$.

and $\left[\frac{1}{1+\theta x} < \frac{1}{1-|x|}\right.$, moreover it is independent of n.$\left.\right]$

Hence, $R_n(x) \to 0$ as $n \to \infty$

therefore, the Taylor's series expansion of $\log_e(1+x)$ at $x=0$ is given by

$$\log_e(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} . x^n}{n}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\cdots, \quad -1 < x \leq 1.$$

—— x ——

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GAMMA AND BETA INTEGRALS
(DEFINITIONS & USES)

Gamma Function:- The Gamma function is defined by Euler to be the definite integral

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \quad (n>0) \text{ ------ } (1)$$

L.H.S will not exist for $n \le 0$.

The integral is known to exist for $n>0$ and the value of the integral is a positive number.

Now, by direct integration, we see that

$$\Gamma(1) = \int_0^\infty e^{-x} x^{1-1} dx = \int_0^\infty e^{-x} dx = \left[-e^{-x}\right]_0^\infty = [0-(-1)]$$
$$= 1.$$

Integrating (1) by parts, we get the following recurrence relation:

We have $\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx$

$$= \left[x^n \int e^{-x} dx\right]_0^\infty + n\int_0^\infty x^{n-1} e^{-x} dx$$

$$= \left[-x^n e^{-x}\right]_0^\infty + n\Gamma(n)$$

$$= [0-0] + n\Gamma(n)$$

$$= n\Gamma(n)$$

So, we have $\Gamma(n+1) = n\Gamma(n)$ or $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$

Relation between Factorial and Gamma Notation:-

If n is a positive integer, then $\Gamma(n+1) = n\Gamma(n)$

$$\therefore \Gamma(n+1) = n(n-1)\Gamma(n-2)$$
$$\vdots$$
$$= n(n-1)\cdots\cdots 1.\Gamma(1)$$
$$= n(n-1)\cdots\cdots 1.$$
$$= n!$$

$$\Gamma(n) = (n-1)!$$

$$\Gamma(0) = \frac{\Gamma(1)}{0} = \text{undefined}.$$

Thus $\Gamma(n)$ is not defined if $n=0$ or a negative integers.

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Example:-

1) S.T. $\Gamma\left(\frac{2k+1}{2}\right) = \frac{1.3.5.\ldots(2k-1)}{2^k}\sqrt{\pi}$

Proof:→ $\Gamma\left(\frac{2k+1}{2}\right) = \Gamma\left(k+\frac{1}{2}\right)$

$= \left(k-\frac{1}{2}\right)\Gamma\left(k-\frac{1}{2}\right)$

$= (k-1/2)(k-3/2)\,\Gamma(k-3/2)$

$\vdots$

$= \left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)\ldots\ldots\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$= \frac{(2k-1)(2k-3)\ldots\ldots 1}{2^k}\cdot\Gamma\left(\frac{1}{2}\right)$

$= \frac{(2k-1)(2k-3)\ldots\ldots 1}{2^k}\sqrt{\pi}$ $\left[\because \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\right]$

(Proved)

2) Evaluate: $\int_0^\infty e^{-x^4}dx$

Soln.→

$I = \int_0^\infty e^{-x^4}dx$

Let, $y = x^4$

$\therefore x = y^{1/4}$

or, $dx = \frac{1}{4}y^{1/4-1}\,dy$

x	0	∞
y	0	∞

$I = \frac{1}{4}\int_0^\infty e^{-y}\,y^{1/4-1}\,dy$

$= \frac{1}{4}\Gamma\left(\frac{1}{4}\right) = \Gamma\left(\frac{1}{4}+1\right) = \Gamma\left(\frac{5}{4}\right)$ (ANS)

Beta Integral :- Beta integral with parameters m and n is denoted by B(m,n) and defined as

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \; ; \; m,n>0 \quad \cdots\cdots (1)$$

Result:- S.T. $B(m,n) = B(n,m)$

Proof:→ We define $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

Let $y = 1-x$

$\therefore x = 1-y$

$\therefore dx = -dy$

x	0	1
y	1	0

So, $B(m,n) = \int_1^0 (1-y)^{m-1}\, y^{n-1}(-dy)$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}dy$$

$$= B(n,m).$$

Result:- $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$

Problem:-

1) Evaluate: $\int_0^1 \dfrac{x^n}{\sqrt{(1-x^2)}}$

ANS:-

$$I = \int_0^1 \frac{x^n}{\sqrt{(1-x^2)}}\,dx$$

Put, $x^2 = y \quad \therefore x = y^{1/2}$

or, $dx = \frac{1}{2}y^{-1/2}dy$

x	0	1
y	0	1

and $I = \int_0^1 \dfrac{y^{n/2}}{(1-y)^{1/2}} \cdot \frac{1}{2} y^{1/2-1}\,dy$

$$= \frac{1}{2}\int_0^1 y^{\frac{n+1}{2}-1}(1-y)^{\frac{1}{2}-1}dy$$

$$= B\left(\frac{n+1}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{2}\,\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$$

$$= \frac{1}{2}\sqrt{\pi}\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \quad \left[\text{Since, } \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}\right.$$

2) Evaluate: $\int_0^\infty \frac{dx}{(1+x^2)^n}$

ANS:- Let $x^2 = y$ $\therefore x = y^{1/2}$

$\therefore dx = \frac{1}{2} y^{\frac{1}{2}-1} dy$

x	0	∞
y	0	∞

$$I = \int_0^\infty \frac{\frac{1}{2} y^{1/2-1} dy}{(1+y)^n}$$

$$= \frac{1}{2} \int_0^\infty \frac{y^{1/2-1}}{(1+y)^{(n-\frac{1}{2})+\frac{1}{2}}} dy$$ [Beta integral of 2nd kind]

$$= \frac{1}{2} B\left(\frac{1}{2}, n-\frac{1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma(\frac{1}{2})\Gamma(n-\frac{1}{2})}{\Gamma(n)}$$

$$= \frac{1}{2}\sqrt{\pi} \cdot \frac{\Gamma(n-\frac{1}{2})}{\Gamma(n)}.$$

Show that $\rightarrow \Gamma(\frac{1}{2}) = \sqrt{\pi}$

Soln. We have $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

$$B\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \int_0^1 x^{1/2-1}(1-x)^{\frac{1}{2}-1} dx$$

$$\text{or, } \frac{\{\Gamma(\frac{1}{2})\}^2}{\Gamma(1)} = \int_0^1 \frac{dx}{\sqrt{\{x(1-x)\}}}$$

$$\text{or, } \{\Gamma(\tfrac{1}{2})\}^2 = \int_0^1 \frac{dx}{\sqrt{\{x(1-x)\}}} \quad [\because \Gamma(1) = 1].$$

Put, $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

x	0	1
θ	0	$\pi/2$

$$\text{or, } \{\Gamma(\tfrac{1}{2})\}^2 = \int_0^{\pi/2} \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} d\theta$$

$$= 2\theta\Big]_0^{\pi/2}$$

$$= \pi$$

$\therefore \Gamma(\frac{1}{2}) = \sqrt{\pi}$ [Negative sign ignored since it represents an area]

Result:→ Legendre's Duplication formula for Gamma Integral.

Statement:- If m is half an integer, then

$$\Gamma(2m) = 2^{2m-1} \cdot \frac{\Gamma(m)\,\Gamma(m+\frac{1}{2})}{\sqrt{\pi}}$$

Proof:→ $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

In particular, when $m=n$, we get

$$B(m,m) = \frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = \frac{\{\Gamma(m)\}^2}{\Gamma(2m)}$$

$$\therefore \int_0^1 x^{m-1}(1-x)^{m-1}dx = \frac{\{\Gamma(m)\}^2}{\Gamma(2m)}$$

or, $$\int_0^1 (x-x^2)^{m-1}dx = \frac{\{\Gamma(m)\}^2}{\Gamma(2m)}$$

or, $$\frac{1}{4^{m-1}}\int_0^1 (4x-4x^2)^{m-1}dx = \frac{\{\Gamma(m)\}^2}{\Gamma(2m)}$$

or, $$\frac{1}{4^{m-1}}\int_0^1 \{1-(2x-1)^2\}^{m-1}dx = \frac{\{\Gamma(m)\}^2}{\Gamma(2m)}$$

Let us put, $2x-1=y$

$dx = \dfrac{dy}{2}$

x	0	1
y	-1	1

$$\therefore \frac{1}{4^{m-1}}\int_{-1}^1 \{1-y^2\}^{m-1}\frac{dy}{2} = \frac{\{\Gamma(m)\}^2}{\Gamma(2m)}$$

or, $$\frac{1}{4^{m-1}}\int_0^1 (1-y^2)^{m-1}dy = \frac{\{\Gamma(m)\}^2}{\Gamma(2m)}$$ [$\because$ the integrand is even]

Now substitute, $y^2=t$, $\therefore y = t^{1/2}$

& $dy = \frac{1}{2}t^{1/2-1}dt$

y	0	1
t	0	1

$$\therefore \frac{1}{4^{m-1}}\int_0^1 (1-t)^{m-1}\cdot\frac{1}{2}t^{1/2-1}dt = \frac{\{\Gamma(m)\}^2}{\Gamma(2m)}$$

or, $$\frac{1}{2^{2m-1}}\int_0^1 t^{1/2-1}(1-t)^{m-1}dt = \frac{\{\Gamma(m)\}^2}{\Gamma(2m)}$$

or, $$\frac{1}{2^{2m-1}}\cdot B(m,\tfrac{1}{2}) = \frac{\{\Gamma(m)\}^2}{\Gamma(2m)}$$

or, $$\frac{1}{2^{2m-1}}\cdot\frac{\Gamma(m)\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})} = \frac{\{\Gamma(m)\}^2}{\Gamma(2m)}$$

$$\Rightarrow \Gamma(2m) = \frac{\Gamma(m)\Gamma(m+\frac{1}{2})\cdot 2^{2m-1}}{\Gamma(\frac{1}{2})} = 2^{2m-1}\frac{\Gamma(m)\Gamma(m+\frac{1}{2})}{\sqrt{\pi}}$$

(Proved)

◈ Alternative Approach:-

$$\Gamma(2m) = (2m-1)! = (2m-1)(2m-2)(2m-3)\cdots\cdots 1$$

$$\Gamma(2m) = 2^{2m-1}\left(m-\tfrac{1}{2}\right)(m-1)\left(m-\tfrac{3}{2}\right)\cdots\cdots\tfrac{1}{2}$$

$$\Gamma(2m) = 2^{2m-1}\left\{\left(m-\tfrac{1}{2}\right)\cdots\cdots\tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right)\right\}\left\{(m-1)(m-2)\cdots\cdots 1\right\}\cdot\frac{1}{\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{2^{2m-1}}{\sqrt{\pi}}\Gamma\left(m+\tfrac{1}{2}\right)(m-1)!$$

$$= \frac{2^{2m-1}}{\sqrt{\pi}}\Gamma(m)\Gamma(m+1/2) \quad \text{[Proved]}$$

Ques: Prove that if m, n are integers, $\frac{1}{B(m,n)} = m\binom{m+n-1}{n-1}$

$$= n\binom{m+n-1}{m-1}$$

Proof:→ $$\frac{1}{B(m,n)} = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} = \frac{(m+n-1)!}{(m-1)!\,(n-1)!}$$

$$= m.\frac{(m+n-1)!}{m!\,(n-1)!}$$

$$= m\binom{m+n-1}{n-1}$$

Again, $$\frac{1}{B(m,n)} = n.\frac{(m+n-1)!}{(m-1)!\,n!}$$

$$= n\binom{m+n-1}{m-1}$$

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DISCUSS THE CONVERGENCE OF THE INTEGRALS

1) $\int_0^1 \frac{dx}{x^{2/3}} = \lim_{\epsilon\to 0+} \int_\epsilon^1 \frac{dx}{x^{2/3}} = \lim_{\epsilon\to 0+}\left[\frac{x^{1/3}}{1/3}\right]_\epsilon^1$

$= \lim_{\epsilon\to 0} 3(1-\epsilon^{1/3})$

$= 3$, it converges.

2) $\int_1^\infty \frac{1}{x^2}dx$

$= \lim_{t\to\infty}\int_1^t \frac{dx}{x^2} = \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left(1-\frac{1}{t}\right)$

$= 1$ $\left[\text{As} \lim_{t\to\infty}\frac{1}{t}=0\right]$

$\therefore$ it converges.

3) $\int_1^\infty \frac{\sin^2 x}{x^2}dx$.

<u>ANS</u> since $0 \le \sin^2 x \le 1$

we can write, $\int_1^\infty \frac{\sin^2 x}{x^2} \le \int_1^\infty \frac{1}{x^2}$

Now, we shall examine the convergence of the improper integral $\int_1^\infty \frac{dx}{x^2}$ and if it is convergent then the improper integral $\int_1^\infty \frac{\sin^2 x}{x^2}dx$ is also convergent.

Now, $\int_1^\infty \frac{dx}{x^2} = \lim_{t\to\infty}\int_1^t \frac{dx}{x^2} = \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t$

$= \lim_{t\to\infty}\left(1-\frac{1}{t}\right) = 1$.

Hence, $\int_1^\infty \frac{dx}{x^2}$ converges, so $\int_1^\infty \frac{\sin^2 x}{x^2}dx$ converges

Improper Integral : We first explain the concept of Riemann Integral.

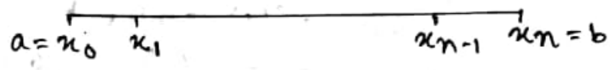
Riemann Integral : Let $f(x)$ be a bounded function in the closed interval $[a,b]$,

Define, $M = \sup\limits_{x\in[a,b]} f(x)$

$m = \inf\limits_{x\in[a,b]} f(x)$

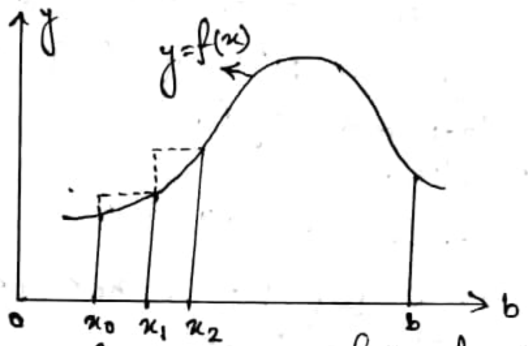
A partition of $[a,b]$ is a set of ~~fixed~~ points, called the points of subdivision, and denoted by $P : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$

For the partition P, we get



$M_i = \sup\limits_{x\in[x_{i-1},x_i]} f(x)$

$m_i = \inf\limits_{x\in[x_{i-1},x_i]} f(x)$



Clearly, $m \le m_i \le M_i \le M$, $\forall i$.

For the partition P, we define the upper sum of the function $f(x)$ as $U(P,f) = \sum_{i=1}^{n} M_i \cdot \Delta x_i$, where $\Delta x_i = x_i - x_{i-1}$.

Similarly, we define the lower sum as $L(P,f) = \sum_{i=1}^{n} m_i \cdot \Delta x_i$,

[From Graph, $U(P,f)$ overestimates the actual area and $L(P,f)$ underestimates the actual area; i.e. why they are known as upper sum and lower sum]

Note that, $m \le m_i \le M_i \le M$

$\Rightarrow m(b-a) \le \sum m_i \Delta x_i \le \sum M_i \Delta x_i \le M(b-a)$

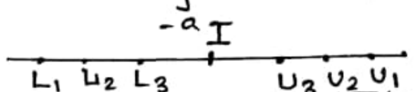
$\Rightarrow m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$

The upper integral of $f(x)$ on $[a,b]$ is defined by

$$\overline{\int_a^b} f(x)\,dx = \inf_P U(P,f)$$

and the lower integral of $f(x)$ on $[a,b]$ is defined by

$$\underline{\int_a^b} f(x)\,dx = \sup_P L(P,f)$$

[

L_i, U_i are the lower and upper sums for a partition P_i. 'I' may be the actual value of the integral. Then $\sup L(P_i,f)$ and $\inf U(P_i,f)$ will be very closely to I. If they are equal, then I exists, otherwise I does not exist]

Here $\int_{a}^{-b} f(x)dx$ and $\int_{-a}^{b} f(x)dx$ are finite.

- **Definition:** A function $f(x)$ is said to be Riemann integrable on $[a,b]$ if $\int_{a}^{-b} f(x)dx = \int_{-a}^{b} f(x)dx$.

Supremum: Supremum of a set is the least upper bound the set.

i) $A = \{0 \leq x \leq 1\}$, note that, $u = 1, 1.1, 1.2, \ldots\ldots$, i.e. any number $u \geq 1$ is an upper bound. The least upper bound is $U = 1$, that is supremum over x, $A = 1$.
Clearly the maximum over A is 1.

ii) $A = \{0 < x < 1\}$, Any number $u (\geq 1)$ is an upper bound, any number $u (< 1)$ can't be a upper bound, $\sup A = 1$,
Hence, the set A has no maximum.

Infremum: The greatest lower bound of a set A is called the infremum of A.

Example: 1) Show that every constant function $f(x) = k$ is ~~Riem~~ Riemann integrable on any interval $[a,b]$.

Soln.→ Let $P: a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$ be a ~~partiti~~ partition of $[a,b]$

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i = k \sum_{i=1}^{n} \Delta x_i = k(b-a)$$

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i = k(b-a)$$

Hence,

$$\int_{-a}^{b} f(x)dx = \sup_{P} L(P,f) = k(b-a)$$

and

$$\int_{a}^{-b} f(x)dx = \inf_{P} U(P,f) = k(b-a)$$

Hence,

$$\int_{a}^{-b} f(x)dx = \int_{-a}^{b} f(x)dx$$

i.e. $f(x)$ is integrable on $[a,b]$.

Example : 2) Let $f(x) = \begin{cases} 1 & , \text{if } x \in \mathbb{Q} \\ 0 & , \text{if } x \notin \mathbb{Q} \end{cases}$

where $\mathbb{Q}$ is the set of rational numbers. Show that $f(x)$ is not integral on $[a, b]$.

Soln. → Let $P : a = x_0 < x_1 < \cdots\cdots < x_{n-1} < x_n = b$ be a partition of $[a, b]$

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$= \sum 1 . \Delta x_i = (b - a)$$

$$L(P, f) = \sum m_i . \Delta x_i = \sum 0 . \Delta x_i = 0$$

Hence, $\underline{\int_a^b} f(x)dx = \sup_P L(P, f) = 0$

and $\overline{\int_a^b} f(x)dx = \inf_P U(P, f) = (b-a)$

** Hence, $f(x)$ is not integrable on $[a, b]$

Improper Integral : The concept of Riemann integrability of a function on an interval is subject to the following limitations :

i) the function is bounded,

ii) the interval is finite.

Here we extent the concept of Riemann integrability ; i.e. the integral $\int_a^b f(x)dx$ to integrate over unbounded intervals and / or integrands which are unbounded at some point.

Different types of Improper Integrals :

FIRST KIND : The integral of the form $\int_a^\infty f(x)\,dx$ or $\int_{-\infty}^b f(x)\,dx$, where $f(x)$ is bounded on the interval of integration, is an improper integral of the first kind.

For e.g. i) $\int_0^\infty \frac{1}{x+1}\,dx$, ii) $\int_{-\infty}^\infty \frac{1}{x^2+1}\,dx$ are improper integrals of first kind.

SECOND KIND : When 'a' and 'b' are real number (i.e. finite quantities) but $f(x)$ is unbounded at some point on the interval $[a, b]$, the integral $\int_a^b f(x)dx$ is an improper integral of second kind.

For e.g. i) $\int_0^1 \frac{1}{x}dx$, ii) $\int_{-1}^1 \frac{1}{x^2}dx$ are improper integral of second kind.

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THIRD KIND :- When combination of these two difficulties occur, i.e. the interval of integration is infinite as well as the integrand is unbounded at some point in the interval, then the integral is an improper integral of third kind.

For e.g. i) $\int_0^\infty \frac{1}{x} dx$, ii) $\int_0^\infty \frac{1}{x-1} dx$ are improper integrals of third kind.

FIRST KIND :-

- Definition :→ Let $f(x)$ be Riemann integrable on $[a, R]$, for every $R > a$, then

$$\int_a^\infty f(x)dx = \lim_{R\to\infty} \int_a^R f(x)dx .$$

similarly, if $f(x)$ is Riemann integrable on $[S, b]$ $\forall$ $S < b$, then

$$\int_{-\infty}^b f(x)dx = \lim_{S\to-\infty} \int_S^b f(x)dx .$$

If the ~~approximate~~ appropriate limit is finite, the integral is said to converge; otherwise it diverges.

Example 1) Does $\int_1^\infty \frac{1}{x^p} dx$ converge ?

Soln.→ Here the interval of integration is infinite but the integrand is bounded on the interval. Hence $\int_1^\infty \frac{1}{x^p} dx$ is of the first kind improper integral.

Now defn. gives :

$$\int_1^\infty \frac{1}{x^p} dx = \lim_{R\to\infty} \int_1^R \frac{1}{x^p} dx = \lim_{R\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^R \quad \text{if } p \neq 1.$$

$$= \lim_{R\to\infty} \frac{R^{1-p}}{1-p} + \frac{1}{p-1} = \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ +\infty & \text{if } p<1 \end{cases}$$

$$\left[\because \lim_{R\to\infty} R^{1-p} = \begin{cases} 0 & \text{if } p>1 \\ \infty & \text{if } p<1 \end{cases} \right]$$

If $p = 1$, then $\lim_{R\to\infty} \int_1^R \frac{1}{x} dx = \lim_{R\to\infty} [\ln x]_1^R = \lim_{R\to\infty} \ln R = +\infty$

Hence, $\int_1^\infty \frac{1}{x^p} dx$ converges if $p>1$ and diverges if $p \le 1$.

Remark: N.T. $\int_a^\infty \frac{1}{x^p} dx$ converges if $p>1$ and diverges if $p \le 1$, where $a>0$.

Example 2) Does $\int_0^\infty e^{-ax}dx$ converge?

Soln. → This is an improper integral of the first kind.

By defn., $\int_0^\infty e^{-ax}dx = \lim_{R\to\infty}\int_0^R e^{-ax}dx$

If $a \neq 0$, $\lim_{R\to\infty}\int_0^R e^{-ax}dx = \lim_{R\to\infty}\left[\frac{e^{-ax}}{-a}\right]_0^R$

$$= \lim_{R\to\infty}\left\{\frac{1}{a} - \frac{e^{-aR}}{a}\right\}$$

$$= \lim_{R\to\infty}\frac{1}{a}\left(1 - e^{-aR}\right)$$

$$= \frac{1}{a}\left\{1 - \lim_{R\to\infty} e^{-aR}\right\} = \begin{cases}\frac{1}{a} & \text{if } a>0 \\ \infty & \text{if } a<0\end{cases}$$

$$\left[\because \lim_{R\to\infty} e^{-aR} = \begin{cases} 0 , & a>0 \\ \infty , & a<0\end{cases}\right]$$

If $a=0$, then $\lim_{R\to\infty}\int_0^R e^{-aR}dx$

$$= \lim_{R\to\infty}\int_0^R 1.dx = \lim_{R\to\infty}(R) = +\infty .$$

Hence, $\int_0^\infty e^{-ax}dx$ converges if $a>0$ and diverges if $a\leq 0$.

SECOND KIND :-

• Definition :→ If $f(x)$ is Riemann integrable on $[a,R]$, for $R\in(a,b)$ and $f(x)$ is unbounded at $x=b$, then $\int_a^b f(x)dx = \lim_{R\to b^-}\int_a^R f(x)dx$

If $f(x)$ is Riemann integrable on $[s,b]$, for $s\in(a,b)$ and $f(x)$ is unbounded at $x=a$, then $\int_a^b f(x)dx = \lim_{s\to a^+}\int_s^b f(x)dx$.

If the appropriate limit is finite, the improper integral is said to converge ; otherwise it diverges.

Again, if $f(x)$ is unbounded at $x=x_0$, $a<x_0<b$ then we say that $\int_a^b f(x)dx$ converges iff $\int_a^{x_0} f(x)dx$ & $\int_{x_0}^b f(x)dx$ both converge.

C.U.

Example 1) Does $\int_0^1 \frac{1}{x^p}\,dx$ converge?

Soln.→ Here $f(x)$ is unbounded at $x=0$, if $p>0$. but the interval of integration is finite, This is an improper integral of 2nd kind.

Now, $\int_0^1 \frac{1}{x^p}\,dx = \lim_{s\to 0+} \int_s^1 \frac{1}{x^p}\,dx$

If $p \neq 1$,

$$\lim_{s\to 0+} \int_s^1 \frac{1}{x^p}\,dx = \lim_{s\to 0+} \left[\frac{x^{1-p}}{1-p}\right]_s^1 = \frac{1-\lim_{s\to 0+} s^{1-p}}{1-p}$$

$$= \begin{cases} \frac{1}{1-p} & , p<1 \\ \infty & , p>1 \end{cases}$$

$$\left[\because \lim_{s\to 0+} s^{1-p} = \begin{cases} 0 & , p<1 \\ \infty & , p>1 \end{cases} \right]$$

If $p=1$, $\lim_{s\to 0+} \int_0^1 \frac{1}{x}\,dx = \lim_{s\to 0+} [\ln x]_s^1$

$$= \lim_{s\to 0+} (-\ln s) = +\infty .$$

C.U. Hence, $\int_0^1 \frac{1}{x^p}\,dx$ converges if $p<1$ and diverges if $p\geq 1$.

Example 2) Show that $\int_a^b \frac{dx}{(b-x)^n}$ converges iff $n<1$.

Soln.→ The integrand $f(x) = \frac{1}{(b-x)^n}$ is unbounded at $x=b$, if $n>0$.

This is an improper of 2nd kind.

Hence, $\int_a^b \frac{dx}{(b-x)^n} = \lim_{R\to b^-} \int_a^R \frac{dx}{(b-x)^n}$

$$= \lim_{R\to b^-} \left[\frac{(b-x)^{-n+1}}{-n+1}\cdot(-1)\right]_a^R \quad \text{if } n\neq 1.$$

$$= \lim_{R\to b^-} \frac{(b-R)^{-n+1}}{n-1} - \frac{(b-a)^{-n+1}}{n-1}$$

$$= \begin{cases} \frac{(b-a)^{1-n}}{1-n} & \text{if } n<1 \\ +\infty & \text{if } n>1 \end{cases}$$

If $n=1$, $\lim_{R\to b^-} \int_a^R \frac{dx}{b-x} = \lim_{R\to b^-} \left[-\ln(b-x)\right]_a^R = \ln(b-a) - \lim_{R\to b^-} \ln(b-R) = +\infty$

Here, $\int_a^b \frac{dx}{(b-x)^n}$ converges iff $n<1$.

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Example 3) Examine the convergence of the following integrals:

i) $\int_{-1}^{1} \frac{1}{x} dx$, ii) $\int_{0}^{2} \frac{dx}{2x-x^2}$

Soln. → i) N.T. the integrand $f(x) = \frac{1}{x}$ is unbounded at $x = 0 \in (-1, 1)$. This is an improper integral of 2nd kind.

Write, $\int_{-1}^{1} \frac{1}{x} dx = \int_{-1}^{0} \frac{1}{x} dx + \int_{0}^{1} \frac{1}{x} dx$.

$\therefore$ $\int_{-1}^{1} \frac{1}{x} dx$ converges iff $\int_{-1}^{0} \frac{1}{x} dx$ and $\int_{0}^{1} \frac{1}{x} dx$ both converges.

Now, $\int_{-1}^{0} \frac{1}{x} dx = \lim_{R \to 0^-} \int_{-1}^{R} \frac{1}{x} dx = \lim_{R \to 0^-} \left[\ln|x| \right]_{-1}^{R}$

$= \lim_{R \to 0^-} \ln|R| = -\infty$

and, $\int_{0}^{1} \frac{1}{x} dx = \lim_{s \to 0+} \int_{s}^{1} \frac{1}{x} dx = \lim_{s \to 0+} \left[\ln|x| \right]_{s}^{1}$

$= \lim_{s \to 0^+} (-\ln s) = +\infty$

As $\int_{-1}^{0} \frac{1}{x} dx$ and $\int_{0}^{1} \frac{1}{x} dx$ both diverge, so $\int_{-1}^{1} \frac{1}{x} dx$ diverges.

ii) $f(x) = \frac{1}{2x-x^2} = \frac{1}{x(2-x)}$ is unbounded at $x = 0$ and $x = 2$ on $[0, 2]$

Hence, $\int_{0}^{2} \frac{dx}{2x-x^2} = \int_{0}^{1} \frac{dx}{x(2-x)} + \int_{1}^{2} \frac{dx}{x(2-x)}$

$= \lim_{s \to 0^+} \int_{s}^{1} \frac{dx}{x(2-x)} + \lim_{R \to 2^-} \int_{1}^{R} \frac{dx}{x(2-x)}$

$= \frac{1}{2} \lim_{s \to 0^+} \left[\log \frac{x}{2-x} \right]_{s}^{1} + \frac{1}{2} \lim_{R \to 2^-} \left[\log \frac{x}{2-x} \right]_{1}^{R}$

$= -\frac{1}{2} \lim_{s \to 0^+} \log \left(\frac{s}{2-s} \right) + \frac{1}{2} \lim_{R \to 2^-} \log \left(\frac{R}{2-R} \right)$

$= -(-\infty) + (+\infty) = \infty$

Comparison Test :- If $0 \le f(x) \le g(x)$ $\forall$ $x \in [a, R]$, $a < R < \infty$ and $f(x)$ and $g(x)$ are Riemann integrable on $[a, R]$, $\forall R > a$ then

i) $\int_a^\infty g(x)\,dx$ converges $\Rightarrow \int_a^\infty f(x)\,dx$ converges.

ii) $\int_a^\infty f(x)\,dx$ diverges $\Rightarrow \int_a^\infty g(x)\,dx$ diverges.

Example: 1) Does $\int_1^\infty \frac{dx}{x^2+\sqrt{x}}$ converge?

Soln.→ N.T. $0 < \frac{1}{x^2+\sqrt{x}} < \frac{1}{x^2}$ $\forall$ $x \geq 1$

Since $\int_1^\infty \frac{1}{x^2}\,dx$ converges,

Hence $\int_1^\infty \frac{dx}{x^2+\sqrt{x}}$ converges, by Comparison test.

Example 2) Does $\int_0^1 \frac{dx}{x^2+\sqrt{x}}$ converge?

Soln.→ Note that, $0 < \frac{1}{x^2+\sqrt{x}} < \frac{1}{\sqrt{x}}$ $\forall$ $x \geq 1$

Now, $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges, by p-integral.

By comparison Test, $\int_0^1 \frac{dx}{x^2+\sqrt{x}}$ converges.

Example 3) Does $\int_1^\infty \frac{x}{(1+x^2)^2}\,dx$ converge?

Soln.→ $0 < \frac{x}{(1+x^2)^2} < \frac{1}{x^3}$ $\forall$ $x > 1$

Now, $\int_1^\infty \frac{1}{x^3}$ converges by p-th integral.

By comparison test $\int_1^\infty \frac{x\,dx}{(1+x^2)^2}$ also converges.

Limit Comparison Test :— If $f(x)$ and $g(x)$ are two positive valued functions for $x \in [a, b]$ [or, for $x \in [a, b)$] and $\lim_{x \to a+} \frac{f(x)}{g(x)} = l$ $\left(\text{or, } \lim_{x \to b-} \frac{f(x)}{g(x)} = l\right)$

then the integral $\int_a^b f(x)dx$ and $\int_a^b g(x)\,dx$ both converge or diverge simultaneously.

C.U.

Example: Does $\int_0^\infty \frac{x}{(1+x^2)^2} dx$ converge?

Soln.→ Note that, $\int_0^\infty \frac{x}{(1+x^2)^2} dx$

$$= \int_0^1 \frac{x}{(1+x^2)^2} dx + \int_1^\infty \frac{x}{(1+x^2)^2} dx$$

Here, $\int_0^1 \frac{x}{(1+x^2)^2} dx$ is a proper integral or Riemann integrab and hence it is finite.

To examine the convergence of $\int_1^\infty \frac{x\,dx}{(1+x^2)^2}$

let $f(x) = \frac{x}{(1+x^2)^2}$ and $g(x) = \frac{1}{x^3}$ $\forall\ x \geq 1$

Note that, $\lim_{x \to +\infty} \frac{f(x)}{g(x)} = 1 \neq 0$

As, $\int_1^\infty g(x)\,dx = \int_1^\infty \frac{dx}{x^3}$ is a convergent p-integral, by limit comparison test $\int_1^\infty f(x)dx = \int_1^\infty \frac{x}{(1+x^2)^2} dx$ converges.

Hence, $\int_0^\infty \frac{x}{(1+x^2)^2} dx$ converges.

THIRD KIND :— Third kind improper integral is a combination of the difficulties of the first kind and second kind improper integrals.

The improper integrals of third kind are first written as a sum of the two integrals, one of the first kind and other of the second kind. If both, the first & second kind improper integrals converge then given the third kind improper integral converges.

C.U.

Example 1) Show that $\int_0^\infty x^p dx$ diverges for any value of p.

Soln → The interval of integration is infinite and the integrand $f(x) = x^p$ is unbounded at $x=0$ if $p<0$. This is an improper integral of third kind.

Now,
$$\int_0^\infty x^p dx = \int_0^1 x^p dx + \int_1^\infty x^p dx$$
$$= \lim_{s\to 0+}\int_s^1 x^p dx + \lim_{R\to\infty}\int_1^R x^p dx$$
$$= \lim_{s\to 0+}\left[\frac{x^{p+1}}{p+1}\right]_s^1 + \lim_{R\to\infty}\left[\frac{x^{p+1}}{p+1}\right]_1^R, \text{ if } p\neq -1$$
$$= \lim_{s\to 0+}\frac{1-s^{p+1}}{p+1} + \lim_{R\to\infty}\frac{R^{p+1}-1}{p+1}$$
$$= \frac{\lim_{R\to\infty} R^{p+1} - \lim_{s\to 0+} s^{p+1}}{p+1}$$
$$= +\infty, \text{ if } p\neq -1$$
$$= \begin{cases} \infty & \text{if } p+1<0 \\ \infty & \text{if } p+1>0 \end{cases}$$

$\left[\begin{array}{l} \text{For } p+1<0,\ \lim_{R\to\infty} R^{p+1} = 0 \text{ but } \lim_{s\to 0+} s^{p+1} = \infty. \\ \text{For } p+1>0,\ \lim_{R\to\infty} R^{p+1} = \infty \text{ but } \lim_{s\to 0+} s^{p+1} = 0 \end{array}\right]$

Hence for $p\neq -1$, $\int_0^\infty x^p dx$ diverges. For $p=-1$,
$$\int_0^\infty x^{-1}dx = \lim_{s\to 0+}\int_s^1 \frac{1}{x}dx + \lim_{R\to\infty}\int_1^R \frac{1}{x}dx = \lim_{s\to 0+}(-\ln s) + \lim_{R\to\infty}(\ln R)$$
$$= +\infty + (+\infty) = \infty$$

Hence $\int_0^\infty x^p dx$ diverges for all p.

Example 2) Examine the convergence of $\int_0^\infty \frac{1}{x^2+\sqrt{x}}dx$.

Soln → The interval of integration is infinite and the integrand $f(x) = \frac{1}{x^2+\sqrt{x}}$ is unbounded at $x=0$. This is an improper integral of third kind.

Now,
$$\int_0^\infty \frac{1}{x^2+\sqrt{x}}dx = \int_0^1 \frac{1}{x^2+\sqrt{x}}dx + \int_1^\infty \frac{1}{x^2+\sqrt{x}}dx$$

Note that, $0 < \frac{1}{x^2+\sqrt{x}} < \frac{1}{\sqrt{x}}\ \forall x \in (0,1]$

Here, $\int_0^1 \frac{1}{\sqrt{x}}dx$ converges; by comparison test, $\int_0^1 \frac{1}{x^2+\sqrt{x}}dx$ converges.

Again, $0 < \frac{1}{x^2+\sqrt{x}} < \frac{1}{x^2}\ \forall x>1$

As $\int_1^\infty \frac{dx}{x^2}$ converges; by comparison test $\int_1^\infty \frac{dx}{x^2+\sqrt{x}}$ converges.

C.U.

Gamma Integral :-

Result: Show that the Gamma integral $\int_0^\infty e^{-x}x^{n-1}dx$ converges iff $n>0$.

Proof: The interval of integration is infinite and the integrand $e^{-x}.x^{n-1}$ is ~~~~ unbounded at $x=0$ if $n<1$. this is an improper integral of third kind.

Write, $$\int_0^\infty e^{-x}x^{n-1}dx = \int_0^1 e^{-x}.x^{n-1}dx + \int_1^\infty e^{-x}.x^{n-1}dx$$

Convergence at $x=0$ for $n<1$.

Let $f(x) = e^{-x}.x^{n-1} = \dfrac{e^{-x}}{x^{1-n}}$

$g(x) = \dfrac{1}{x^{1-n}}$, $x \in (0,1]$

N.T. $\lim\limits_{x\to 0+} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to 0+} e^{-x} = 1$

As, $\int_0^1 g(x)dx = \int_0^1 \dfrac{1}{x^{1-n}}dx$ converges iff $1-n<1$, i.e. $n>0$,

by limit comparison test $\int_0^1 f(x)dx = \int_0^1 e^{-x}.x^{n-1}dx$ converges iff $n>0$.

Convergence at $x=\infty$.

N.T. $\lim\limits_{n\to\infty} \dfrac{x^{n+1}}{e^x} = 0 \quad \forall n$

For large x, e^x increases more rapidly than x^{n+1}.

Then $\exists$ 'b' $\ni$ $\dfrac{x^{n+1}}{e^x} < \epsilon = 1$ for $x>b$.

$\Rightarrow e^{-x}.x^{n-1} < \dfrac{1}{x^2} \quad \forall x>b$.

Since $\int_b^\infty \dfrac{1}{x^2}dx$ converges, by comparision test $\int_b^\infty e^{-x}.x^{n-1}dx$ converges $\forall n$.

Hence, $$\int_1^\infty e^{-x}.x^{n-1}dx = \underbrace{\int_1^b e^{-x}.x^{n-1}dx}_{\text{[proper Integral]}} + \int_b^\infty e^{-x}.x^{n-1}dx$$

converges for all n.

Here, $\int_0^1 e^{-x}.x^{n-1}dx$ converges iff $n>0$ and $\int_1^\infty e^{-x}.x^{n-1}dx$ converges for all n.

$\Rightarrow \int_0^\infty e^{-x}.x^{n-1}dx$ converges if $n>0$.

C.U.

Beta Integral :→

Result :→ Show that Beta integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ converges iff $m>0,\ n>0$

Proof :→ If $m>1,\ n>1$, then the integral is a proper integral (i.e. finite). The integrand $x^{m-1}(1-x)^{n-1}$ is unbounded at $x=0$ if $m<1$ and is unbounded at $x=1$ if $n<1$.

Let $m<1$ and $n<1$.

Write $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$$= \int_0^{1/2} x^{m-1}(1-x)^{n-1}\,dx + \int_{1/2}^1 x^{m-1}\cdot(1-x)^{n-1}\,dx$$

Now, we check the convergence of $\int_0^{1/2} x^{m-1}(1-x)^{n-1}\,dx$ at $x=0$ and the convergence of $\int_{1/2}^1 x^{m-1}(1-x)^{n-1}\,dx$ at $x=1$.

Convergence at $x=0$.

Let $f(x) = \dfrac{(1-x)^{n-1}}{x^{1-m}}$ and $g(x) = \dfrac{1}{x^{1-m}}$.

$$x \in (0, \tfrac{1}{2}).$$

N.T. $\lim\limits_{x\to 0+} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to 0+} (1-x)^{n-1} = 1$.

Since, $\int_0^{1/2} g(x)\,dx = \int_0^{1/2} \dfrac{1}{x^{1-m}}\,dx$ converges iff $1-m<1$, i.e. $m>0$.

By comparison test, $\int_0^{1/2} f(x)\,dx = \int_0^{1/2} x^{m-1}(1-x)^{n-1}\,dx$ converges iff $m>0$.

Convergence at $x=1$.

Here $f(x) = \dfrac{x^{m-1}}{(1-x)^{1-n}}$ and $g(x) = \dfrac{1}{(1-x)^{1-n}}$ $\forall\ x \in (\tfrac{1}{2}, 1)$.

$\lim\limits_{x\to 1-} \dfrac{f(x)}{g(x)} = 1 \neq 0$.

Since, $\int_{1/2}^1 g(x)\,dx = \int_{1/2}^1 \dfrac{1}{(1-x)^{1-n}}$ converges iff $1-n<1$, i.e. $n>0$,

By limit comparison test, $\int_{1/2}^1 f(x)\,dx = \int_{1/2}^1 x^{m-1}(1-x)^{n-1}\,dx$ converges iff $n>0$.

Hence, $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ converges iff $m>0$ and $n>0$.

Remark :→ $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \beta(m,n)$ if $m>0$, $n>0$

It can be shown that $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

⇨ FURTHER EXAMPLES :→

1) Does $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x^p}\,dx$ converge?

2) Show that $\displaystyle\int_0^{\pi/2} \frac{\sin^m x}{x^n}\,dx$ converges iff $n<m+1$.

Soln. of 1) Let $f(x) = \dfrac{\sin x}{x^p}$, $g(x) = \dfrac{1}{x^{p-1}}$, $x \in (0,\pi/2]$

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1.$$

$\displaystyle\int_0^{\pi/2} \frac{1}{x^{p-1}}\,dx$ converges iff $p-1<1$, i.e. $p<2$.

Hence, $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x^p}\,dx$ converges if $p<2$.

Absolute Convergence :~

• Definition :~ The improper integral $\int^{\infty} f(x)\,dx$ (1st kind) or $\int_a^b f(x)\,dx$ (2nd kind) is said to be a absolutely convergent if $\displaystyle\int_a^{\infty} |f(x)|\,dx$ or $\displaystyle\int_a^b |f(x)|\,dx$ converges.

Example :→ (a) [redacted] Show that $\displaystyle\int_1^{\infty} \frac{\sin x}{x^p}\,dx$ converges absolutely if $p>1$.

Soln. → $\left|\dfrac{\sin x}{x^p}\right| \le \dfrac{1}{x^p}$, $x \ge 1$

As $\displaystyle\int^{\infty} \frac{1}{x^p}\,dx$ converges if $p>1$, then

By comparison test, $\displaystyle\int_1^{\infty} \frac{|\sin x|}{x^p}$ converges if $p>1$

$\Rightarrow \displaystyle\int_1^{\infty} \frac{\sin x}{x^p}\,dx$ converges for $p>1$.

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Duplication Formula :

Result :→ $$\Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}} \cdot \Gamma(n)\,\Gamma\left(n+\tfrac{1}{2}\right)$$

C.U.

Proof :→

$$\beta(n,n) = \int_0^1 x^{n-1}(1-x)^{n-1}\,dx$$

$$= \int_0^1 \{x(1-x)\}^{n-1}\,dx$$

$$= 2\int_0^{1/2} \{x(1-x)\}^{n-1}\,dx \quad \text{———— } (*)$$

$\left[\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx, \text{ if } f(2a-x) = f(x), \text{ Here, } f(1-x) = f(x) \right]$

$\Big[$ Let $z = 4(x - x^2)$ $\quad \therefore x = \dfrac{4 \pm \sqrt{16 - 4.4.z}}{2.4}$

$\Rightarrow dz = 4(1-2x)\,dx$ $\quad \therefore x = \dfrac{1 - \sqrt{1-z}}{2}$, as $0 < x < \frac{1}{2}$

$\Rightarrow dx = \dfrac{dz}{4\sqrt{1-z}}$ $\Big]$

$(*)$ reduces to

$$= 2\int_0^1 \left(\frac{z}{4}\right)^{n-1} \frac{(1-z)^{1/2-1}}{4}\,dz$$

$$= \frac{1}{2^{2n-1}} \int_0^1 z^{n-1}(1-z)^{1/2-1}\,dz$$

$$= \frac{1}{2^{2n-1}}\,\beta\left(n, \tfrac{1}{2}\right)$$

$$\therefore \frac{\Gamma(n)\,\Gamma(n)}{\Gamma(2n)} = \frac{1}{2^{2n-1}} \cdot \frac{\Gamma(n)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)}$$

$$\Rightarrow \Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}} \cdot \Gamma(n)\,\Gamma\left(n+\tfrac{1}{2}\right)$$

[Proved]

~o~o~

//***

REAL ANALYSIS

1) (i) Give the classical definitions of uniform and pointwise convergences of a sequence of functions defined on a subset A of $(-\infty, \infty)$. (4)

(ii) Distinguish giving suitable example, between pointwise convergence and uniform convergence of a sequence of functions defined on a common set. (5)

(iii) Verify whether the sequence of functions $\{f_n(x), n \geq 1\}$, where $f_n(x) = \frac{\cos nx}{n}$, $x \geq 0$, $n \geq 1$ is uniformly convergent on $[0, \infty)$.

(iv) Let $f_n(x) = x + \frac{1}{n}$, $x \in R$, the set of all real numbers, $n = 1, 2, 3, \ldots$

(a) Show that $\{f_n(x)\}$ converges pointwise in R.

(b) Show that $\{g_n(x)\}$ does not converge uniformly in R where $g_n(x) = [f_n(x)]^2 \ \forall x \in R$ and $\forall n = 1, 2, 3, \ldots$

(5×4)

Pointwise Convergence of a sequence of functions:—

The sequence $\{f_n(x)\}$ of functions converges pointwise to the function $f(x)$ on A_0 if for each $x \in A_0$ for any $\epsilon > 0$, there is a natural number $N(\epsilon, x)$ may depend on ϵ and x such that

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N(\epsilon, x).$$

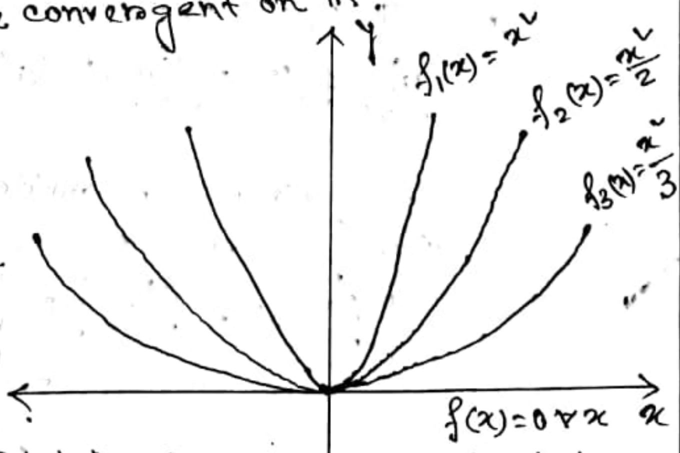
□ Here $f_n(x)$ is a sequence of function and $f(x)$ is a function, both of them are defined over the common domain A_0. Here, the natural number $N(\epsilon, x)$ is a measure of rate of convergence of the sequence at x. The value $N(\epsilon, x)$, in general, depends on both ϵ and x.

Example:- the sequence $\{f_n(x)\}_n$, $x \in \mathbb{R}$ defined by $f_n(x) = \frac{x^2}{n}$ is pointwise convergent on $\mathbb{R}$.

As for each $x \in R$, we have $\lim\limits_{n \to \infty} f_n(x) = \lim\limits_{n \to \infty} \frac{x^2}{n} = 0$

$\therefore f_n(x)$ converges pointwise to $f(x) = 0 \ \forall x \in \mathbb{R}$

i.e. $\{f_n(x)\}$ is pointwise convergent on $\mathbb{R}$.



* Uniform Convergence $\Rightarrow$ Pointwise convergence. but not vice versa.

Non-pointwise convergence $\Rightarrow$ Non-uniform convergence.

Uniform convergence of a sequence of functions:-

A sequence $\{f_n(x)\}$ of functions converges uniformly to the function $f(x)$ on A_0 if, for every $\epsilon > 0$, there is a natural number $N(\epsilon)$ which does not depend on x $\ni$ $|f_n(x) - f(x)| < \epsilon \;\forall\; x \in A_0$, whenever $n \geq N(\epsilon)$

□ Here $\{f_n(x)\}$ and $f(x)$ are both defined over the common domain A_0.

Example:- Let $f_n(x) = \dfrac{x}{nx+1}$, $A = [0,1]$, converges uniformly on A.

For each $x \in [0,1]$, $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \dfrac{x}{nx+1}$

$= 0$

$\epsilon (>0)$ be an arbitrary quantity, then

$$|f_n(x) - f(x)| = \frac{x}{nx+1} < \frac{x}{nx} = \frac{1}{n} < \epsilon$$

$$\Rightarrow n > \frac{1}{\epsilon}$$

Take $N(\epsilon) = \left[\frac{1}{\epsilon}\right] + 1$.

Hence, it is possible to find a natural number $N(\epsilon)$ $\ni$ if $n \geq N(\epsilon)$, $|f_n(x) - f(x)| < \epsilon \;\forall\; x \in A = [0,1]$

$\therefore \{f_n(x)\}$ converges uniformly on $[0,1]$.

(iv) (a) $f_n(x) = x + \frac{1}{n}$

$\lim\limits_{n\to\infty} f_n(x) = x = f_1(x)$

$\therefore |f_n(x) - f(x)| = \frac{1}{n} < \epsilon$

$\Rightarrow n > \frac{1}{\epsilon}$

Choose a natural number $N(\epsilon) = \left[\frac{1}{\epsilon}\right] + 1$

$\because N(\epsilon)$ is independent on x.

$\therefore$ It is uniformly convergent. so, $\{f_n(x)\}$ converges pointwise on R.

(b) $g_n(x) = \{f_n^2(x)\}$

$\lim\limits_{n\to\infty} f_n^2(x) = \lim\limits_{n\to\infty} (x+1/n)^2 = x^2$, $x \in R$

$= f_1^2(x) = f_2(x)$

Let $\epsilon > 0$ be an arbitrary real number.

Then for each $x \in R$,

$$|f_n^2(x) - f_1^2(x)| = |f_n^2(x) - f_2^2(x)| = \left|\frac{2x}{n} + \frac{1}{n^2}\right|$$

$$= \left|\frac{2nx+1}{n^2}\right| \leq \frac{2n|x|+1}{n^2}$$

$$\leq \frac{2n|x|+n}{n^2} = \frac{2|x|+1}{n} < \epsilon$$

$$\Rightarrow n > \frac{2|x|+1}{\epsilon}$$

Take $N(\epsilon, x) = \left[\frac{2|x|+1}{\epsilon}\right] + 1$

Note that $N(\epsilon, x) \to +\infty$ as $x \to \pm\infty$

i.e. $N(\epsilon, x)$ is not bounded on R.

Hence it is not possible to find a natural no. $N(\epsilon) \ni$ if $n \geq N(\epsilon)$, $|f_n^{\nu}(x) - f^{\nu}(x)| < \epsilon \;\; \forall x \in R$.

$\therefore \{f_n^{\nu}(x)\}$ does not converge uniformly on R.

— x —

END

(CU) Explain the convergence of $\int_5^7 (7-x)^p dx$ for different values of p.

<u>Solution:-</u> The integrand $f(x) = (7-x)^p$ is bounded at $x=7$, if $p<0$.

This is an ~~[illegible]~~ improper integral of 2nd kind.

Hence,
$$\int_5^7 (7-x)^p dx = \lim_{R\to 7^-} \int_5^R (7-x)^p$$

$$= \lim_{R\to 7^-} \left[\frac{(7-x)^{p+1}}{p+1} \cdot (-1)\right]_5^R \quad \text{if } p \neq -1.$$

$$= \lim_{R\to 7^-} \frac{(7-R)^{p+1}}{(p+1)} - \frac{(7-5)^{p+1}}{(p+1)}$$

$$= \begin{cases} \frac{(-2)^{p+1}}{p+1}, & \text{if } p > -1 \\ \infty, & \text{if } p < -1 \end{cases}$$

If $p = -1$,
$$\lim_{R\to 7^-} \int_5^R (7-x)^{-1} dx$$

$$= \lim_{R\to 7^-} \left[-\ln(7-x)\right]_5^R$$

$$= \ln(7-5) - \lim_{R\to 7^-} \ln(7-R)$$

$$= +\infty$$

Here, $\int_5^7 (7-x)^p$ converges iff $p > -1$.

SEQUENCE OF FUNCTIONS

Suppose we have a collection of countable number of functions defined over a common domain D. If these functions are arranged in ascending order of positive integers 1, 2, 3, ... Then the collection will be called a "sequence of functions" defined over the common domain D.

Example:— Let $f_1(x) = x, \quad x \in [0,1]$

$$f_2(x) = 2x^2, \quad x \in [0,1]$$

$$f_3(x) = 3x^3, \quad x \in [0,1]$$

$\vdots$

and so on.

Then the collection $\{ f_n(x) \mid f_n(x) = nx^n, x \in [0,1] \}$ will denote a "sequence of functions".

Sandwich Theorem / Squeeze Theorem:—

Let $A \subseteq \mathbb{R}$; $f, g, h : A \to \mathbb{R}$; and $c \in \mathbb{R}$ be a cluster point (or condensation or limit point) of A. If

$$f(x) \le g(x) \le h(x) \quad \forall x \in A, x \ne c.$$

and if $\lim\limits_{x \to c} f(x) = L = \lim\limits_{x \to c} h(x)$, then

$$\lim_{x \to c} g(x) = L.$$

Remark: Uniform convergence implies pointwise convergence but the converse is not necessarily true. However, pointwise limit function = Uniform limit function in case of uniform convergence.

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Ex. Show that the sequence $\{f_n(x) \mid f_n(x) = x^n, x \in \mathbb{R}, n = 1,2,\dots\}$ is pointwise convergent on $(-1, 1]$. Find the limit function of the sequence.

ANS:-

For $x = 1$, $f_n(1) = 1 \ \forall n \in \mathbb{N}$

Hence $f_n(1) \to 1$ for $x \in (-1, 1]$,

$f_n(x) = x^n \to 0$ as $n \to \infty$ and

hence $f_n(x) \to 0 \ \forall x \in (-1, 1)$, i.e. $\forall |x| < 1$.

But for $x = -1$, $f_n(x) = (-1)^n = \begin{cases} 1 & \text{if } n = \text{even} \\ -1 & \text{if } n = \text{odd} \end{cases}$

$\therefore f_n(-1)$ does not converge for $x = -1$.

For $|x| > 1$, $|x|^n$, for large n will be unbounded and hence $f_n(x)$ is not convergent for $|x| > 1$.

Let us define, $f(x) = \begin{cases} 0 & \text{whenever } |x| < 1 \\ 1 & \forall x = 1. \end{cases}$

$\therefore \{f_n(x)\}_n$ is pointwise convergent to $f(x) = \begin{cases} 0 & \text{for } |x| < 1 \\ 1 & \text{for } x = 1 \end{cases}$ on $(-1, 1]$

Ex. S.T. $\{f_n(x)\}$, where $f_n(x) = x^n$ is pointwise convergent on $[0, 1)$.

ANS:- $f_n(x) = x^n$, $x \in [0, 1)$

$\therefore \lim_{n \to \infty} f_n(x) = 0$

Let us define, $f(x) = 0$ if $x \in [0, 1)$

Then for $x \in [0, 1)$, $|f_n(x) - f(x)| < \epsilon$

$\Rightarrow |x^n - 0| < \epsilon$

$\Rightarrow n > \dfrac{\ln \epsilon}{\ln x}$ $\quad [\because \ln x < 0 \text{ as } 0 \le x < 1]$

i.e. $n \geq \left[\dfrac{\ln \epsilon}{\ln x}\right] + 1$

i.e. $n \geq \max\left\{1, \left[\dfrac{\ln \epsilon}{\ln x}\right] + 1\right\}$

$= N(\epsilon, x)$, say

Therefore ~~from~~ we can write each $\epsilon > 0$ and for every $x \in [0, 1)$, $\exists$ a positive integer $N(\epsilon, x) \ni$

$$|f_n(x) - f(x)| < \epsilon \ \forall n \geq N(\epsilon, x).$$

$\therefore f_n(x)$ is pointwise convergent to $f(x) = 0$ on $[0, 1)$.

Ex. S.T. $\{f_n\}_n$ defined by $f_n(x) = \frac{x}{x+n}$, $x \in [0, \infty)$ is pointwise convergent, but is not uniform convergent.

ANS:- For $x = 0$, $f_n(0) = 0 \quad \forall n = 1, 2, \ldots$

$$\Rightarrow \lim_{n\to\infty} f_n(0) = 0$$

For $0 < x < \infty$, $f_n(x) = \frac{x/n}{1 + x/n} \to 0$ as $n \to \infty$

Now, $\lim_{n\to\infty} f_n(x) = 0$, $x \in [0, \infty)$.

$\therefore f(x)$ is pointwise limit function of the sequence $\{f_n\}_n$.

Again, for any $\epsilon > 0$, we have

$$|f_n(x) - f(x)| = \left|\frac{x}{x+n} - 0\right| = \frac{x}{x+n} < \epsilon$$

$$\Rightarrow n > x\left(\frac{1}{\epsilon} - 1\right)$$

Let $N(\epsilon, x) = \left[x\left(\frac{1}{\epsilon} - 1\right)\right] + 1$.

$\therefore |f_n(x) - f(x)| < \epsilon \quad \forall n \geq N(\epsilon, x)$.

$\therefore$ there does not exist a natural number $N(\epsilon) \ni$

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N(\epsilon) \text{ and } \forall x \in [0, \infty)$$

So, the sequence $\{f_n\}_n$ is not uniform convergent on $[0, \infty)$.

The above sequence $\{f_n\}_n$ is uniformly convergent as well as pointwise convergent on $[0, a]$, a is a finite number.

Ex. S.T. the sequence $\{f_n\}_n$ defined by $f_n(x) = xe^{-nx}$, $x \in [0, \infty)$, $n = 1, 2, \ldots$ is uniformly convergent on $[0, \infty)$.

ANS:→ Since for every $x \in [0, \infty)$, we have

$$0 \leq xe^{-nx} < \frac{x}{nx}$$

$\therefore$ By Sandwich theorem, $\lim_{n\to\infty} f_n(x) = 0 \quad \forall x \in [0, \infty)$

$\therefore \quad |f_n(x) - f(x)| < \epsilon$ if $n \geq N(\epsilon, x)$ & for all $x \in [0, \infty)$ where $f(x) = 0$, $x \in [0, \infty)$ is the pointwise limit function.

Again, for any $\epsilon (> 0)$ and $\forall x \in [0, \infty)$

$$|f_n(x) - f(x)| < |xe^{-nx} - 0| = xe^{-nx} < \frac{1}{n} < \epsilon$$

$$\forall n \geq \left[\frac{1}{\epsilon}\right] + 1.$$

Hence, $\{f_n\}_n$ converges uniformly to $f(x)$ on $[0, \infty)$.

M_n-Test for Uniform Convergence: —

Statement: — Let $\{f_n(x)\}_n$ be a sequence of functions defined over a domain D. Also let $f(x)$ be another function defined on the same domain D.

Suppose, $|f_n(x) - f(x)| \le M_n \;\forall\, n = 1, 2, \ldots$

where $\{M_n\}$ is a sequence of non-negative real numbers. then $f_n(x) \to f(x)$ uniformly in D iff $M_n \to 0$ as $n \to \infty$.

Ex. $f_n(x) = \frac{x}{1+n^2x}$; $x \in [0,1]$ and $n = 1, 2, \ldots$

Examine whether $f_n(x)$ converges uniformly or not.

ANS: $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{1+n^2x} = 0$

Let us choose a function $f(x) = 0$ for $x \in [0,1]$

consequently, we have $|f_n(x) - f(x)| = \left|\frac{x}{1+n^2x} - 0\right|$

$$= \frac{x}{1+n^2x} < \frac{1}{n^2} = M_n \text{ (say)}$$

and $\lim_{n\to\infty} M_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$

$\therefore$ By M_n test for uniform convergence of

$\{f_n(x) \mid f_n(x) = \frac{x}{1+n^2x},\ x \in [0,1],\ n = 1, 2, \ldots\}$.

So, $\{f_n(x)\}$ converges uniformly on $f(x)$ on $D = [0,1]$

Ex. $f_n(x) = nxe^{-nx^2}$; $x \in [0,1]$ and $n = 1, 2, \ldots$

Examine whether $f_n(x)$ converges uniformly or not.

ANS: — $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nxe^{-nx^2}$

$$= \lim_{n\to\infty} \frac{nx}{e^{nx^2}}$$

$$= \lim_{n\to\infty} \frac{n}{ne^{nx^2}.2x}$$

$$= \lim_{n\to\infty} \frac{1}{e^{nx^2}.2x}$$

$$= 0 \quad [\text{By L' Hospital Rule}]$$

Let us choose a function $f(x) = 0$ for $x \in [0,1]$

consequently, we have $|f_n(x) - f(x)| = \left|\frac{nx}{e^{nx^2}} - 0\right|$

$$= \frac{nx}{e^{nx^2}} < ne^n = M_n$$

$$[\because -nx^2 < -n < n \;\forall\, x \in [0,1]]$$

& $\lim_{n\to\infty} M_n = \lim_{n\to\infty} ne^n = \infty$

$\therefore$ By M_n test for uniform convergence, $\{f_n(x) \mid f_n(x) = nxe^{-nx^2};\ x \in [0,1],\ n = 1, 2, \ldots\}$ on D, where $D = [0,1]$. It can be said that $f_n(x)$ does not converge uniformly to $f(x)$ on D.

Ex. Let $f_n(x) = nx(1-x^2)^n$; $x \in [0,1]$ and $n = 1, 2, \ldots$. Examine whether $f_n(x)$ converges uniformly.

Ans:- $0 \le f_n(x) = nx(1-x^2)^n \le (1-x^2)^n \to 0$ as $n \to \infty$ since $x \in [0,1]$

Since, by Sandwich theorem,

$$\lim_{n\to\infty} f_n(x) = 0 \quad \forall x \in [0,1]$$

$\therefore$ Let us therefore, choose a function $f(x) = 0$ for $x \in [0,1]$.

Consequently, we have $|f_n(x) - f(x)|$

$$= |nx(1-x^2)^n - 0|$$

$$\le n = M_n, \text{ say}$$

& $\lim_{n\to\infty} M_n = \lim_{n\to\infty} n = \infty$

$\therefore$ By M_n test for uniform convergence $\{f_n(x) \mid f_n(x) = nx(1-x^2)^n;\ x \in [0,1], n = 1, 2, \ldots\}$ on $D = [0,1]$, it can be said that $f_n(x)$ does not converge uniformly on D.

Result:- Let $\{f_n\}_n$ and $\{g_n\}_n$ both converge uniformly in D. Then $\{f_n + g_n\}$ converges uniformly on D.

Proof:→ Let us suppose $f_n \to f$ uniformly in D and $g_n \to g$ uniformly on D. [$\forall \to$ for every]

$\therefore \exists\ N(\epsilon_1) \ni \forall\ \epsilon_1 (>0)$,

$|f_n(x) - f(x)| < \epsilon_1\ \forall\ n \geq N(\epsilon_1)$ and $\forall\ x \in D$ ...... (1)

Similarly for $g_n(x)$, $\exists\ N(\epsilon_2) \ni \forall\ \epsilon_2 (>0)$

$|g_n(x) - g(x)| < \epsilon_2\ \forall\ n \geq N(\epsilon_2)$ and $\forall\ x \in D$ ...... (2)

Since, $\lim_{n\to\infty} f_n(x) = f(x)$ & $\lim_{n\to\infty} g_n(x) = g(x)$.

$$\therefore \lim_{n\to\infty} \{f_n(x) + g_n(x)\} = \lim_{n\to\infty} f_n(x) + \lim_{n\to\infty} g_n(x)$$

$$= f(x) + g(x)$$

$$\therefore |\{f_n(x) + g_n(x)\} - \{f(x) + g(x)\}|$$

$$= |\{f_n(x) - f(x)\} + \{g_n(x) - g(x)\}|$$

$$\leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$$

$< \epsilon_1 + \epsilon_2 = \epsilon(>0)$, say $\forall\ n \geq \min\{N(\epsilon_1), N(\epsilon_2)\}$ $= N(\epsilon)$, say and $\forall\ x \in D$

i.e. $\forall\ x \in D$ and $n \geq N(\epsilon) = \max\{N(\epsilon_1), N(\epsilon_2)\}$

$$\Rightarrow |\{f_n(x) + g_n(x)\} - \{f(x) + g(x)\}| < \epsilon$$

provided $f_n \to f$ and $g_n \to g$ uniformly on D.

$\therefore f_n + g_n \to f + g$ uniformly on D (provided).