MATHEMATICAL ANALYSIS

BY

TANUJIT CHAKRABORTY

Indian Statistical Institute

Mail : tanujitisi@gmail.com

Extrema of Functions of several variables :—

If $D \subseteq \mathbb{R}^n$ be the domain of a function 'f', we shall denote a point in D as $\underset{\sim}{x}$ and the value of the function at $\underset{\sim}{x}$ by $f(\underset{\sim}{x}) = f(x_1, x_2, \ldots, x_n)$.

Definition:→ A function 'f' has a local (or, relative) maximum or minimum at x_0 if there is some neighbourhood $N_\delta(x_0) = \{x : |x - x_0| < \delta\}$ such that, —

$$f(\underset{\sim}{x}) \leq f(\underset{\sim}{x}_0) \quad \text{or,} \quad f(\underset{\sim}{x}) \geq f(\underset{\sim}{x}_0) \quad \forall\ \underset{\sim}{x} \in N_\delta(\underset{\sim}{x}_0).$$

A necessary condition for an extemum of a differentiable function 'f' at x_0 is

$$\left(f_{x_1}(\underset{\sim}{x}_0), f_{x_2}(\underset{\sim}{x}_0), \ldots\ldots, f_{x_n}(\underset{\sim}{x}_0)\right) = \underset{\sim}{0}.$$

$\boxed{\text{OR}}$ $\nabla f(\underset{\sim}{x}_0) = \underset{\sim}{0}$ or, $f_{x_i}(\underset{\sim}{x}_0) = 0 \ \forall\ i = 1(1)n$,

[The partial derivative of 'f' w.r.t. x_i is $\dfrac{\partial f(\underset{\sim}{x})}{\partial x_i}$ or

$$f_{x_i}(\underset{\sim}{x}) = \lim_{h \to 0} \frac{\{f(x_1, \ldots, x_i + h, \ldots x_n) - f(x_1, \ldots, x_i, \ldots, x_n)\}}{h}$$

Notation: $f_i(\underset{\sim}{x})$, $\dfrac{\partial f}{\partial x_i}$, etc.]

$\boxed{A}$ Functions of two variables :—

Theorem: 1. If the 2nd order partial derivatives of 'f' are continuous at each point in D is an open origin in $\mathbb{R}^2$ and if $(x_0, y_0) \in D$ such that $\nabla f(x_0, y_0) = \underset{\sim}{0}$, i.e. $f_x(x_0, y_0) = f_y(x_0, y_0)$, then

i) $f(x_0, y_0)$ is a local maximum if

$$D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}_{(x,y) = (x_0, y_0)}$$

is negative definite.

i.e. $f_{xx}(x_0, y_0) < 0$ and $f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0) > 0$

ii> $f(x_0, y_0)$ is a local minimum if D is positive definite i.e. $f_{xx}(x_0, y_0) > 0$ and

$f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f^2_{xy}(x_0, y_0) > 0$

iii> $f(x_0, y_0)$ is neither a maximum nor a minimum if D is indefinite, i.e.

$f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f^2_{xy}(x_0, y_0) < 0$.

☆ **EXAMPLE** : → 1. Find all the local maximum and minimum of $f(x,y) = 2x^2 - xy + 2y^2 - 20x$.

Soln. → $f_x(x,y) = 4x - y - 20$

$f_y(x,y) = -x + 4y$

For points of extremum,

$$\left.\begin{array}{l} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} 4x - y = 20 \\ x = 4y \end{array} \Rightarrow \left\{\begin{array}{l} x = 16/3 \\ y = 4/3 \end{array}\right.$$

Now, $f_{xx}(x,y) = 4$

$f_{yy}(x,y) = 4$ and $f_{xy}(x,y) = -1$

Now,
$$D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}_{(x,y)=\left(\frac{16}{3}, \frac{4}{3}\right)} = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$$

Note that,

$f_{xx}\left(\frac{16}{3}, \frac{4}{3}\right) = 4 > 0$

and $|D| = f_{xx}\left(\frac{16}{3}, \frac{4}{3}\right) f_{yy}\left(\frac{16}{3}, \frac{4}{3}\right) - f^2_{xy}\left(\frac{16}{3}, \frac{4}{3}\right)$

$= 16 - 1$

$= 15 > 0$

Hence, D is positive definite.

Therefore, $f\left(\frac{16}{3}, \frac{4}{3}\right)$ is the minimum value.

Proof of the theorem 1. →

$$f(x_0+h, y_0+k) = f(x_0, y_0) + h \cdot f_x(x_0, y_0) + k \cdot f_y(x_0, y_0) + \frac{1}{\sqrt{2}}\{h^2 f_{xx}(x_0+\theta h, y_0+\theta k)$$

$$+k^2 f_{yy}(x_0+\theta h, y_0+k\theta) + 2hk f_{xy}(x_0+\theta h, y_0+\theta k)\}$$

$$= f(x_0, y_0) + \frac{1}{\underline{|2}}(h, k)\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\begin{pmatrix} h \\ k \end{pmatrix}.$$

as $f_x(x_0, y_0) = 0 = f_y(x_0, y_0)$

Now,

$$f(x_0+h, y_0+k) - f(x_0, y_0) = \frac{1}{2}(h, k)\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\begin{pmatrix} h \\ k \end{pmatrix}$$

If $D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}_{(x_0, y_0)}$ is n.d.,

then $f(x_0+h, y_0+k) \leq f(x_0, y_0)$

i.e. $f(x_0, y_0)$ is the maximum.

Remark: →

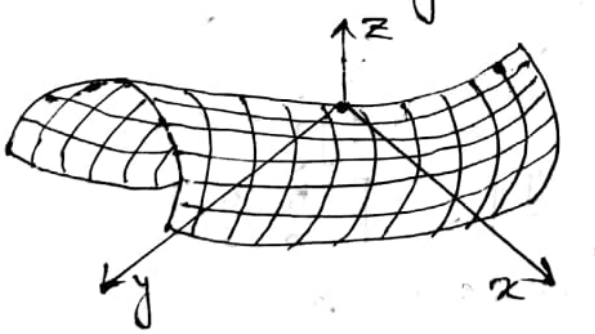
1) A function $f(x_0, y_0)$ has a saddle point if $f_x(x_0, y_0) = 0 = f_y(x_0, y_0)$ and

$$f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f^2_{xy}(x_0, y_0) < 0$$

i.e. it is neither a maximum, nor a minimum.

i.e. $f(x_0+h, y_0+k) - f(x_0, y_0)$ can take +ve or -ve values in every neighbourhood of (x_0, y_0).

The curve $z = -xy$ has a saddle point at $(0, 0)$



size z is -ve in 1st and 3rd quadrants and +ve in the 2nd and 4th quadrant.

2) If $f_x(x_0, y_0) = 0 = f_y(x_0, y_0)$ and $|D| = 0$

i.e. $f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - f^2_{xy}(x_0, y_0) = 0$.

Then it is a doubtful case and required further investigation.

Example 2. Give n points (x_i, y_i), where x_i's are distinct, find a and b such that the function $f(a,b) = \sum_{i=1}^{n} (ax_i + b - y_i)^2$ is minimum.

Soln. → Here,
$$f_a = \sum_{i=1}^{n} 2(ax_i + b - y_i)\, x_i$$
$$f_b = \sum_{i=1}^{n} 2(ax_i + b - y_i)\cdot 1$$

$$f_{aa} = 2\sum x_i^2$$
$$f_{bb} = 2 \quad \text{and} \quad f_{ab} = 2\sum x_i$$

For points of extreme $\left.\begin{array}{l} f_a = 0 \\ f_b = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} a = \dfrac{\sum(x_i-\bar{x})(y_i-\bar{y})}{\sum(x_i-\bar{x})^2} = \hat{a} \\ b = \bar{y} - \hat{a}\bar{x} = \hat{b}. \end{array}\right.$

Now,
$$D = \begin{bmatrix} f_{aa} & f_{ab} \\ f_{ab} & f_{bb} \end{bmatrix}_{(\hat{a},\hat{b})} = 2\begin{bmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & n \end{bmatrix}$$

Here, $f_{aa}(\hat{a},\hat{b}) = 2\sum x_i^2 > 0$

and $|D| = 4\left\{ n\sum x_i^2 - \left(\sum x_i\right)^2 \right\} = 4n\sum(x_i-\bar{x})^2 > 0$

Hence, $f(\hat{a},\hat{b})$ is the minimum value of $f(a,b)$.

Ex. 3. Show that $f(x,y) = y^2 + x^2y + x^4$ has a minimum at $(0,0)$.

Soln. → For points of extremum,
$$0 = f_x = 2xy + 4x^3$$
$$0 = f_y = 2y + x^2$$
$$\Rightarrow x = 0 = y$$

Again, $\left\{\begin{array}{l} f_{xx} = 2y + 12x^2 \\ f_{xy} = 2x \\ f_{yy} = 2. \end{array}\right.$

Note that,
$$D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}_{(x,y)=(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

$$\therefore |D| = f_{xx}(0,0)\, f_{yy}(0,0) - f_{xy}^2(0,0) = 0$$

Hence, it is a doubtful case and requires further investigation.

Note that, $f(x,y) = \left(y^2 + 2y\cdot\frac{x^2}{2} + \frac{x^4}{4}\right) + \frac{3x^4}{4}$

$$= \left(y + \frac{x^2}{2}\right)^2 + \frac{3x^4}{4} \geq 0$$

Now, $f(x,y) = 0$ iff $y + \frac{x^2}{2} = 0 = \frac{3x^4}{4}$ iff $x = 0 = y$,

Hence, $f(x,y)$ has a minimum value at $(0,0)$.

☆ **Example 4.** Show that the function:- $f(x,y) = 2x^4 - 3x^2y + y^2$ has neither a maximum or a minimum at $(0,0)$.

Soln. → $f_x = 8x^3 - 6xy$,

$f_y = -3x^2 + 2y$,

$f_{xx} = 24x^2 - 6y$, $f_{yy} = 2$, $f_{xy} = -6x$,

Now, $f_x(0,0) = 0 = f_y(0,0)$

Also, $f_{xx}(0,0)\, f_{yy}(0,0) - f^2_{xy}(0,0) = 0$

Hence, it is a doubtful case and requires further investigation

Now, $f(x,y) - f(0,0)$

$= 2x^4 - 3x^2y + y^2$

$= (y - x^2)(y - 2x^2)$

$$\begin{cases} < 0 & \text{as } x^2 < y < 2x^2 \\ > 0 & \text{as } y < x^2 \text{ or } y > 2x^2 \end{cases}$$

In every neighbourhood of $(0,0)$, $f(x,y)$ has neither a maximum nor a minimum, i.e. $f(x,y)$ has a saddle point at $(0,0)$.

☆ **Ex : 5)** Find all Local maxima and minima of $f(x,y) = xye^{-x}$.

Soln. → $f(x,y) = xye^{-x}$

$f_x = ye^{-x} - xye^{-x}$

$f_x = ye^{-x}(1-x)$

$f_y = xe^{-x}$

for points of extremum, $f_x = 0$, $f_y = 0$

$\Rightarrow ye^{-x}(1-x) = 0$ | $xe^{-x} = 0$

$\Rightarrow y = 0$ or $x = 1$, | $\Rightarrow x = 0$

$\Rightarrow (x,y) = (0,0)$

Now, $f_{xx} = -ye^{-x}(1-x) - ye^{-x}$

$= -ye^{-x}(2-x)$

$f_{yy} = 0$, $f_{xy} = e^{-x} - xe^{-x} = e^{-x}(1-x)$.

Now, at $(0,0)$, $f_{xx}f_{yy} - f_{xy}^2 = 0 - 1^2 = -1$

i.e. $D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}_{(0,0)}$ is indefinite.

Hence $f(0,0)$ is neither a maxima nor a minima.
and $f(x,y)$ has no maximum or no minimum at $(0,0)$.

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[B]. Functions of Several variables :—

Theorem :- If the function 'f' has 2nd order partial derivatives are continuous at every point in D, where $D \subseteq \mathbb{R}^n$ in an open region and $\underset{\sim}{x}_0 \in D$, such that $\nabla f(\underset{\sim}{x}) = \underset{\sim}{0}$.
i.e. $f_{x_i}(\underset{\sim}{0}) = 0$, $i = 1(1)n$, then

i) $f(\underset{\sim}{x}_0)$ is a local maximum if the matrix of 2nd order partial derivatives at x_0, i.e.

$$D = \left(\left(\frac{\partial^2 f(\underset{\sim}{x})}{\delta x_i \, \delta x_j}\right)\right)_{\underset{\sim}{x}_0}$$ is negative definite.

ii) $f(\underset{\sim}{x}_0)$ is local minimum if D is positive definite.

iii) $f(\underset{\sim}{x}_0)$ is neither a maximum or a minimum if D is indefinite.

Ex: 1) Find all the local maximum and minimum of

$$f(x,y,z) = x^2 + 2y^2 + z^2 - 2xy + yz.$$

Soln. → $f_x = 2x - 2y$

$f_y = 4y - 2x + z$

$f_z = 2z + y$

Now, for points of extremum,

$0 = f_x = 2x - 2y$

$0 = f_y = -2x + 4y + z$

$0 = f_z = y + 2z$

$$\Rightarrow \begin{pmatrix} 2 & -2 & 0 \\ -2 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{as, } \begin{vmatrix} 2 & -2 & 0 \\ -2 & 4 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 6 \neq 0$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

⇒ It is non-singular,
⇒ Inverse exists.

Now, $f_{xx} = 2$, $f_{yy} = 4$, $f_{xy} = -2$,
$f_{yz} = 1$, $f_{xz} = 0$, $f_{zz} = 2$.

Note that, $\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Here, $2>0$, $\begin{vmatrix} 2 & -2 \\ -2 & 4 \end{vmatrix} = 4 > 0$

$|D| = 6 > 0 \Rightarrow D$ is p.d. matrix.

$\Rightarrow f(0,0,0)$ is the minimum value.

☆ **Ex. 2.** Show that the function has a minimum at $(1,1,1)$,

$$f(x,y,z) = x^4 + y^4 + z^4 - 4xyz$$

Soln. Here, $f_x = 4x^3 - 4yz$,

$f_y = 4y^3 - 4xz$

$f_z = 4z^3 - 4xy$

For points of extremum,

$\left.\begin{array}{l} f_x = 0 \\ f_y = 0 \\ f_z = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x^3 = yz \\ y^3 = xz \\ z^3 = xy \end{array}$

$\Rightarrow x^3y^3z^3 = x^2y^2z^2$

$\Rightarrow (xyz)^2(xyz-1) = 0$

$\Rightarrow xyz = 0$, or 1.

Now, $x^4 = y^4 = z^4 = xyz = \begin{cases} 0 \\ 1 \end{cases}$

$\Rightarrow (x,y,z) = (0,0,0)$, $(1,1,1)$

Hence, $(1,1,1)$ is a point of extremum.

At the point $(1,1,1)$, $f_x = f_y = f_z = 0$.

Now, $f_{xx} = 12x^2$, $f_{xy} = -4z$, $f_{xz} = -4y$

$f_{yy} = 12y^2$, $f_{yz} = -4x$, $f_{zz} = 12z^2$

Now,

$$D = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}_{(1,1,1)} = \begin{bmatrix} 12 & -4 & -4 \\ -4 & 12 & -4 \\ -4 & -4 & 12 \end{bmatrix}.$$

Here, $12>0$, $\begin{vmatrix} 12 & -4 \\ -4 & 12 \end{vmatrix} > 0$, $|D|>0$,

i.e. D is p.d.

Hence, $f(1,1,1)$ is a minimum of $f(x,y,z)$.

Method of Lagrange's Multipliers :— $\boxed{\text{C.U.}}$

Suppose, we wish to minimize or maximize $z=f(x,y)$ where (x,y) is constrained to satisfy $g(x,y)=0$.

Assuming that these function have continuous derivative. We can visualise $g(x,y)=0$ as a curve along with the level curve of $f(x,y)=z$ (constant). Intuitively, if we move the level curve in the direction of increasing z, the largest or smallest z occurs at a point where a level curve touches $g(x,y)=0$.

The quadrants of 'f' and 'g' should be in the same or opposite direction. Then $\nabla f = -\lambda \nabla g$ for some λ (λ = constant) $\Rightarrow \nabla\{f+\lambda g\} = \underset{\sim}{0}$

[For $g(x,y)=0$, $\frac{dy}{dx} = -\frac{g_x}{g_y}$ and for $f(x,y)=c$, $\frac{dy}{dx} = \frac{-f_x}{f_y}$.

By problem, at point of tangency, $-\frac{f_x}{f_y} = \frac{dy}{dx} = -\frac{g_x}{g_y}$

$$\Rightarrow \frac{f_x}{g_x} = \frac{f_y}{g_y} = -\lambda \text{ (say)}$$

$\therefore (f_x, f_y) = -\lambda(g_x, g_y)$]

Hence, to find the maximum or minimum of $f(x,y)$ subject to $g(x,y)=0$, find all the solution of the equation,

$$\nabla\{f+\lambda g\} = \underset{\sim}{0} \quad \text{and} \quad g(x,y)=0$$

$$\Rightarrow \frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial y},\ g(x,y)=0 \text{ ; where } F(x,y) = f(x,y) + \lambda g(x,y)$$

Local maxima and minima will be among the solutions. If the curve $g(x,y)=0$ is closed and bounded, then the absolute maxima and minima of $f(x,y)$ exist and are among these solutions. (**Existance**)

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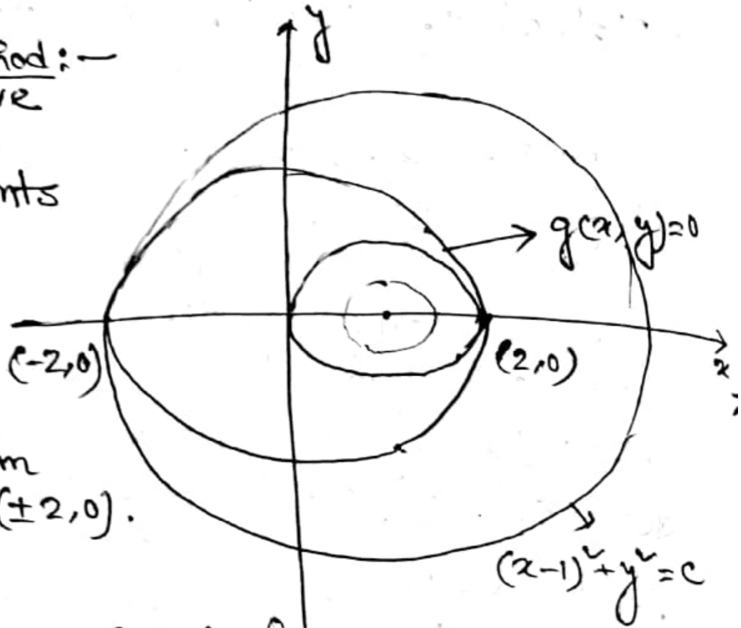
☆ **Ex: 1.** Find all the (local) maximum and minimum of $f(x,y) = 9-(x-1)^2-y^2$ on $x^2+y^2=4$.

Soln. → Here, $z = f(x,y)$ and $g(x,y) = x^2+y^2-4$

To maximize or minimize $z = f(x,y)$ subject to $g(x,y)=0$.

Graphical Method:-

[Clearly, the level curve touches the curve $g(x,y)=0$ at two points $(\pm 2, 0)$.



As the curve is closed and bounded, the absolute maximum and minimum are at $(\pm 2, 0)$.

Graph of $g(x,y)=0$ along with the level curves $f(x,y)=k$ $\Leftrightarrow (x-1)^2+y^2=c$.

Note that,

$f(2,0) = 9-(2-1)^2-0^2$
$\quad = 8$

and $f(-2,0) = 9-(-2-1)^2-0^2$
$\quad = 0$

Hence, $f(2,0)=8$ and $f(-2,0)=0$ are the maximum and minimum values of $f(x,y)$.]

• **Alt. Method:-**

Define, $F(x,y) = f(x,y) + \lambda g(x,y)$

$$= 9-(x-1)^2-y^2+\lambda(x^2+y^2-4)$$

For points of extremum,

$$\begin{cases} 0 = \dfrac{\partial F}{\partial x} = -2(x-1)+2\lambda x \\ 0 = \dfrac{\partial F}{\partial y} = -2x+2\lambda y \end{cases}$$

and $g(x,y)=0$

$$\Rightarrow \begin{cases} (1-\lambda)x = 1 & \text{——— (1)} \\ (1-\lambda)y = 0 & \text{——— (2)} \\ \text{and } x^2+y^2=4 \end{cases}$$

From ①, $\lambda \neq 1$, then ② $\Rightarrow y = 0$, from ① $\Rightarrow x = \frac{1}{1-\lambda}$.

$\therefore \left(\frac{1}{1-\lambda}\right)^2 + 0^2 = 4 \Rightarrow \lambda = 3/2, 1/2$.

$\therefore x = \pm 2$, $\therefore (x, y) = (\pm 2, 0)$

Since $g(x,y) = 0$, i.e. $x^2 + y^2 = 4$ is a closed bounded curve, the absolute maximum and minimum of $f(x,y)$ exist and are one among the solutions.

Note that, $f(2,0) = 8$ and $f(-2,0) = 0$

Clearly, $f(2,0) = 8$ and $f(-2,0) = 0$ are the absolute maximum and minimum.

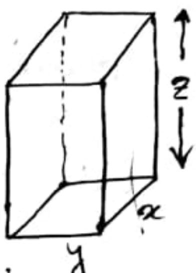
Ex. 2. Show that the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ is } \frac{abc}{3\sqrt{3}}.$$

Soln. →

The volume of the parallelopiped,

$$V = xyz$$



To maximize $V = xyz$ subject to

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. $\Rightarrow$ To max. $\ln V = \ln(x, y, z)$ sub. to $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Define, $F(x,y,z) = \ln(x.y.z) + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$

where λ is the Lagrange's multiplier.

For points of extremum,

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0 \quad \text{with} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

$$\Rightarrow \begin{cases} x^2 = -\frac{a^2}{2\lambda}, \\ y^2 = -b^2/2\lambda. \\ z^2 = -c^2/2\lambda. \end{cases}$$

and $1 = 3.\left(-\frac{1}{2\lambda}\right)$

$\Rightarrow \lambda = -\frac{3}{2}$. $\Rightarrow x^2 = \frac{a^2}{3}$, $y^2 = \frac{b^2}{3}$, $z^2 = \frac{c^2}{3}$

$\Rightarrow (x, y, z) = \left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$ as $(x, y, z > 0$

As $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is closed bounded curve, then absolute maximum or minimum exist and is at the solution.

At $(x,y,z)=\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)$, then $V=\frac{abc}{3\sqrt{3}}$.

At $(x,y,z)=\left(\frac{a}{2},\frac{b}{2},\frac{c}{\sqrt{2}}\right)$, $V=\frac{abc}{4\sqrt{2}}<\frac{abc}{3\sqrt{3}}$,

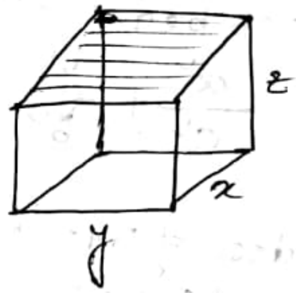
Hence the maximum volume is $\frac{abc}{3\sqrt{3}}$.

☆ **Ex. 3.** A rectangular box, open at the top, is to have a volume 32 cu. ft. What must be the dimensions so that the total surface area is minimum?

Ans: 48 (4, 4, 2)

Soln. → Volume $= V = xyz$,

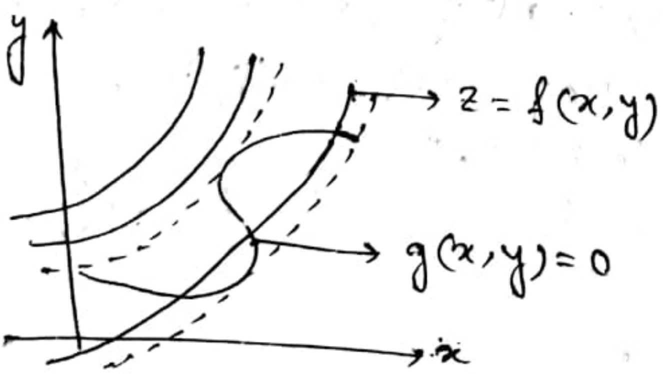
and total surface area
$$= xy + 2(yz+zx)$$



To maximize or minimize S subject to $V=32=xyz$

Let, $F(x,y,z)= xy+2(yz+zx)+\lambda\{xyz-32\}$

Geometry and definition of Lagrange's multiplier:—



Solve the following equations simultaneously
$$\nabla\left\{f+\sum_{i=1}^{k}\lambda_i g_i\right\}=\underset{\sim}{0} \text{ and } g_i(x_1,x_2,\ldots,x_n)=0 \;\forall\; i=1(1)k,$$
the numbers $\lambda_1,\lambda_2,\ldots,\lambda_k$ are called Lagrange's multipliers.

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☆ Ex. 4. The temperature T at any point is given by $T = 200(xyz^2)$. Find the highest and lowest temperature on the surface of sphere
$$x^2+y^2+z^2=1.$$

Soln. → To maximize or minimize $T = 200(xyz^2)$ subject to
$$x^2+y^2+z^2=1.$$

⇔ To maximize or minimize $(\ln T)$ subject to $x^2+y^2+z^2=1$.

Define, $F = \ln(200\,xyz^2) + \lambda(x^2+y^2+z^2-1)$.

For points of extremum,

$$0 = \frac{\partial F}{\partial x} = \frac{1}{x} + 2\lambda x \Rightarrow x^2 = -\frac{1}{2\lambda}.$$

$$0 = \frac{\partial F}{\partial y} = \frac{1}{y} + 2\lambda y \Rightarrow y^2 = -\frac{1}{2\lambda}.$$

$$0 = \frac{\partial F}{\partial z} = \frac{2}{z} + 2\lambda z \Rightarrow z^2 = -\frac{1}{\lambda}.$$

Now, $x^2+y^2+z^2=1$

$$\Rightarrow -\frac{1}{2\lambda} - \frac{1}{2\lambda} - \frac{1}{\lambda} = 1$$

$$\Rightarrow \lambda = -2.$$

$$\therefore (x^2, y^2, z^2) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$$

$$\Rightarrow (x, y, z) = \left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right)$$

The curve $x^2+y^2+z^2=1$ is closed and bounded, the absolute maximum and minimum of $f(x,y,z)$ exists and are among the solutions.

For points (x,y,z) for which $xy = \frac{1}{4}$,
$$f(x,y,z) = 200 \times \frac{1}{4} \times \frac{1}{2} = 25,$$

For points (x,y,z) for which $xy = -\frac{1}{4}$,
$$f(x,y,z) = 200 \times \left(-\frac{1}{4}\right) \times \frac{1}{2} = -25,$$

Hence the highest temperature is 25 and the lowest temperature is -25.

☆ **Ex. 5.** If $ax^2+by^2=ab$, S.T. the maximum and minimum value of (x^2+xy+y^2) will be the values of F given by the equation $4(\theta-a)(\theta-b)=ab$. $\boxed{\text{C.U.}}$ (5)

Soln. → Let, $F(x,y)=(x^2+xy+y^2)+\lambda(ax^2+by^2-ab)$

For points of extrema,

$$0=\frac{\partial F}{\partial x}=2x+y+2\lambda ax \quad \text{——— (1)}$$

$$0=\frac{\partial F}{\partial y}=x+2y+2\lambda by \quad \text{——— (2)}$$

and $ax^2+by^2=ab$.

[Solving (1) & (2) along with $ax^2+by^2=ab$, one will get the values of (x,y) for which $x^2+xy+y^2=0$ are the max. or min. values]

Now, (1) × x + (2) × y ⇒

$$2(x^2+xy+y^2)+2\lambda(ax^2+by^2)=0$$

$$\Rightarrow 2\theta+2\lambda ab=0$$

$$\Rightarrow \lambda=-\theta/ab.$$

Note that, $(x,y)=(0,0)$ is not a solution of $ax^2+by^2=ab$

Hence, $\left.\begin{aligned}2x(1+\lambda a)+y&=0\\ x+2y(1+\lambda b)&=0\end{aligned}\right\}$ has a non-trivial solution.

[$A\underset{\sim}{x}=\underset{\sim}{0}$, $\underset{\sim}{x}\neq\underset{\sim}{0}$
$r(A)<n$
$\Rightarrow |A|=0$]

$$\Rightarrow \begin{vmatrix} 2(\lambda a+1) & 1 \\ 1 & 2(\lambda b+1) \end{vmatrix}=0$$

$$\Rightarrow 4(1+\lambda a)(1+\lambda b)-1=0$$

$$\Rightarrow 4\left(1-\frac{\theta}{b}\right)\left(1-\frac{\theta}{a}\right)=1$$

$$\Rightarrow 4(b-\theta)(a-\theta)=ab$$

$$\Rightarrow 4(\theta-a)(\theta-b)=ab.$$

Remark:— In general, to maximize or minimize $z = f(x_1, x_2, \ldots, x_n)$ subject to the constraints

$g_i(x_1, x_2, \ldots, x_n) = 0$, $i = 1(1)k$,

solve the following equations simultaneously,

$$\nabla\left\{f + \sum_{i=1}^{k} \lambda_i g_i\right\} = \underset{\sim}{0} \text{ and}$$

$$g_i(x_1, x_2, \ldots, x_n) = 0, \quad i = 1(1)k.$$

The numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ are called the Lagrange's Multipliers. The method for finding extrema of a function subject to some constraints, is called the method of Lagrange's Multipliers.

Ex. 6. If $xyz = k^3$, show that $(x+a)(y+b)(z+c)$ is minimum when $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{k}{(abc)^{1/3}}$.

Soln. →

Let $F = \log\{(x+a)(y+b)(z+c)\} + \lambda(xyz - k^3)$

For points of extrema,

$$0 = \frac{\partial F}{\partial x} = \frac{1}{(x+a)} + \lambda yz = \frac{x}{x+a} + \lambda k^3 \Rightarrow \frac{x}{a} = \frac{-\lambda k^3}{1+\lambda k^3}.$$

$$0 = \frac{\partial F}{\partial y} = \frac{1}{(y+b)} + \lambda zx = \frac{y}{y+b} + \lambda k^3 \Rightarrow \frac{y}{b} = -\frac{\lambda k^3}{1+\lambda k^3}.$$

$$0 = \frac{\partial F}{\partial z} = \frac{1}{(z+c)} + \lambda xy = \frac{z}{z+c} + \lambda k^3 \Rightarrow \frac{z}{c} = -\frac{\lambda k^3}{1+\lambda k^3}.$$

$xyz = k^3$

$$\Rightarrow \frac{(abc).(-\lambda k^3)^3}{(1+\lambda k^3)^3} = k^3$$

$$\Rightarrow -\frac{\lambda k^3}{1+\lambda k^3} = \frac{k}{(abc)^{1/3}}.$$

$\therefore$ $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{k}{(abc)^{1/3}}$ is a point of maximum or minimum.

For $(x, y, z) = \left(au, b.\frac{u}{2}, c.2u\right)$, where $u = \frac{k}{(abc)^{1/3}}$

$$(x+a)(y+b)(z+c) = (abc)\cdot(u+1)\left(\frac{u}{2}+1\right)(2u+1)$$
$$= (abc)(u+1)\left(u^2+2u+1+\frac{u}{2}\right)$$
$$> (abc)\cdot(u+1)^3 = (x+a)(y+b)(z+c)$$

where $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{k}{(abc)^{1/3}} = k$

☆ **Ex. 7.** Given n positive numbers $c_1, c_2, \ldots, c_n$, find the maximum value of $\sum_{i=1}^{n} c_i x_i$, where the variables x_i's are restricted so that $\sum_{i=1}^{n} x_i^2 = 1$.

Soln → To maximize $f(\underset{\sim}{x}) = \sum_{i=1}^{n} c_i x_i$ subject to

$$g(\underset{\sim}{x}) = \sum_{i=1}^{n} x_i^2 - 1$$

Define, $F(\underset{\sim}{x}) = f(\underset{\sim}{x}) + \lambda g(\underset{\sim}{x})$

$$= \sum_{i=1}^{n} c_i x_i + \lambda\left(\sum_{i=1}^{n} x_i^2 - 1\right)$$

For points of extreema,

$$\frac{\partial F}{\partial x_i} = 0 \quad \forall\ i = 1(1)n, \text{ and } \sum_{i=1}^{n} x_i^2 = 1.$$

$$\Rightarrow \begin{cases} c_i + 2\lambda x_i = 0 & \forall\ i = 1(1)n \\ \text{and } \sum_{i=1}^{n} x_i^2 = 1. \end{cases}$$

$$\Rightarrow x_i = -\frac{c_i}{2\lambda}$$

$$\text{and } 1 = \sum_{i=1}^{n} x_i^2 = \frac{1}{4\lambda^2}\left(\sum_{i=1}^{n} c_i^2\right)$$

$$\Rightarrow 2\lambda = \pm\sqrt{\sum_{i=1}^{n} c_i^2}$$

$$\Rightarrow x_i = \pm\frac{c_i}{\sqrt{\sum_{i=1}^{n} c_i^2}}, \quad \forall\ i = 1(1)n$$

Since $\sum_{i=1}^{n} x_i^2 = 1$ is a closed and bounded curve, the absolute maximum and minimum are among the solutions.

For, $x_i = -\dfrac{c_i}{\sqrt{\sum_{i=1}^{n} c_i^2}}$, $\sum_{i=1}^{n} c_i x_i = -\sqrt{\sum_{i=1}^{n} c_i^2}$

For, $x_i = +\dfrac{c_i}{\sqrt{\sum_{i=1}^{n} c_i^2}}$, $\sum_{i=1}^{n} c_i x_i = +\sqrt{\sum_{i=1}^{n} c_i^2}$

Hence, $\sqrt{\sum_{i=1}^{n} c_i^2}$ is the maximum.

Remark:- By C-S inequality,

$$\left(\sum_{i=1}^{n} c_i x_i\right)^2 \le \left(\sum_{i=1}^{n} c_i^2\right)\left(\sum_{i=1}^{n} x_i^2\right) = \sum_{i=1}^{n} c_i^2$$

$$\Rightarrow -\sqrt{\sum_{i=1}^{n} c_i^2} \le \sum_{i=1}^{n} c_i x_i \le \sqrt{\sum_{i=1}^{n} c_i^2}$$

Ex. 8. Maximize $f(x_1, x_2) = x_1^2 x_2$ $\boxed{\text{C.U.}}$
subject to $x_1^2 + x_1 x_2 = 12$.

Soln. :> Let $F(x_1, x_2) = x_1^2 x_2 + \lambda(x_1^2 + x_1 x_2 - 12)$

$$0 = \frac{\partial F}{\partial x_1} = 2x_1 x_2 + \lambda(2x_1 + x_2) \quad \text{--- (1)}$$

$$0 = \frac{\partial F}{\partial x_2} = x_1^2 + \lambda x_1 \quad \text{--- (2)}$$

and $x_1^2 + x_1 x_2 = 12$.

(2) $\Rightarrow x_1(x_1 + \lambda) = 0$

$\Rightarrow x_1 = -\lambda$ as, $x_1 = 0$ is not a soln. of $x_1^2 + x_1 x_2 = 12$.

(1) $\Rightarrow -2\lambda x_1 - 2\lambda^2 + \lambda x_2 = 0$

$\Rightarrow -\lambda x_2 = 2\lambda^2$

$\Rightarrow x_2 = -2\lambda$.

Now, $x_1^2 + x_1 x_2 = 12$

$\Rightarrow \lambda^2 + 2\lambda^2 = 12$

$\Rightarrow \lambda^2 = 4$

$\Rightarrow \lambda = \pm 2$

$\therefore (x_1, x_2) = (-2, -4)$ or $(2, 4)$

Hence, the maximum value of $x_1^2 x_2$ is 16 and the minimum value is -16.

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☆Ex.9. Find the max. or min. values of $\underset{\sim}{x}' A \underset{\sim}{x}$ subject to $\underset{\sim}{x}'\underset{\sim}{x} = 1$.

Soln. → Let $F(\underset{\sim}{x}) = \underset{\sim}{x}' A \underset{\sim}{x} - \lambda(\underset{\sim}{x}'\underset{\sim}{x} - 1)$

For points of extrema,

$$\frac{\partial F}{\partial \underset{\sim}{x}} = \underset{\sim}{0} \quad \text{and} \quad \underset{\sim}{x}'\underset{\sim}{x} = 1$$

$$\Rightarrow \frac{\partial}{\partial x}\left\{\underset{\sim}{x}' A \underset{\sim}{x} - \lambda(\underset{\sim}{x}'\underset{\sim}{x} - 1)\right\} = 0 \quad \text{and} \quad \underset{\sim}{x}'\underset{\sim}{x} = 1$$

$$\Rightarrow 2A\underset{\sim}{x} - \lambda.2\underset{\sim}{x} = \underset{\sim}{0} \quad \text{and} \quad \underset{\sim}{x}'\underset{\sim}{x} = 1$$

$$\Rightarrow A\underset{\sim}{x} = \lambda\underset{\sim}{x} \quad \text{and} \quad \underset{\sim}{x}'\underset{\sim}{x} = 1$$

$$\Rightarrow \underset{\sim}{x}' A \underset{\sim}{x} = \lambda \underset{\sim}{x}'\underset{\sim}{x} = \lambda$$

$\therefore$ $\text{Max}\left\{\underset{\sim}{x}' A \underset{\sim}{x}\right\} = \max\{\lambda_i\} = \lambda_{(n)}$, the largest eigen value. $\underset{\sim}{x};\ \underset{\sim}{x}'\underset{\sim}{x} = 1$.

and

$$\underset{\underset{\sim}{x}\,:\,\underset{\sim}{x}'\underset{\sim}{x} = 1}{\text{Min}} \left\{\underset{\sim}{x}' A \underset{\sim}{x}\right\} = \lambda_{(1)}.$$

Remark:-

Note that, $$\max_{\underset{\sim}{x}} \left\{\frac{\underset{\sim}{x}' A \underset{\sim}{x}}{\underset{\sim}{x}'\underset{\sim}{x}}\right\} = \lambda_{(n)}$$

and

$$\min_{\underset{\sim}{x}} \left\{\frac{\underset{\sim}{x}' A \underset{\sim}{x}}{\underset{\sim}{x}'\underset{\sim}{x}}\right\} = \lambda_{(1)}$$

Ex. 10. Find the minimum value of $\sum_{i=1}^{n} a_i^2$ subject to $\sum_{i=1}^{n} a_i = 1$.

Soln. →

[By C-S inequality: $\left(\sum_{i=1}^{n} a_i \cdot 1\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} 1^2\right)$

$$\Rightarrow \sum_{i=1}^{n} a_i^2 \geq \frac{1}{n}$$

'=' holds iff $a_i \propto 1$, i.e. iff $a_i = k$, then,

$$\sum_{i=1}^{n} a_i = nk \Rightarrow a_i = \frac{1}{n}$$]

$$F(a_1, \ldots, a_n) = \sum_{i=1}^{n} a_i^2 + \lambda\left(\sum_{i=1}^{n} a_i - 1\right)$$

For points of extrema,

$\frac{\partial F}{\partial a_i} = 0 \quad \forall\ i = 1(1)n$ and $\sum_{i=1}^{n} a_i = 1$

$\Rightarrow 2a_i + \lambda = 0 \quad \forall\ i$ & $\sum_{i=1}^{n} a_i = 1$

$\Rightarrow a_i = -\frac{\lambda}{2}$, and $1 = n\left(-\frac{\lambda}{2}\right)$.

$\Rightarrow a_i = \frac{1}{n}$, $\forall\ i = 1(1)n$.

Note, $\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} \left(a_i - \frac{1}{n} + \frac{1}{n}\right)^2$

$$= \sum_{i=1}^{n}\left(a_i - \frac{1}{n}\right)^2 + 2\cdot\frac{1}{n}\sum_{i=1}^{n}\left(a_i - \frac{1}{n}\right) + n\cdot\frac{1}{n^2}$$

$$= \sum_{i=1}^{n}\left(a_i - \frac{1}{n}\right)^2 + \frac{2}{n}\left(\sum_{i=1}^{n} a_i - 1\right) + \frac{1}{n}$$

$$= \sum_{i=1}^{n}\left(a_i - \frac{1}{n}\right)^2 + \frac{1}{n}$$

$$\geq \frac{1}{n}, \quad \text{if } a_i = \frac{1}{n}$$

hence the minimum value is $\frac{1}{n}$ and it occurs when $a_i = \frac{1}{n}\ \forall\ i = 1(1)n$.

Lagrange Multiplier:—

1. **Problem** :— Minimize $x^2+y^2+z^2$ subject to condition $2x+3y+5z=30$.

Solution:—

Define,

$$F = x^2+y^2+z^2+\lambda\,(2x+3y+5z-30)$$

$$\frac{\partial F}{\partial x} = 2x+2\lambda = 0$$

$$\frac{\partial F}{\partial y} = 2y+3\lambda = 0$$

$$\frac{\partial F}{\partial z} = 2z+5\lambda = 0$$

$$\therefore x = -\lambda$$

$$\therefore y = -\frac{3\lambda}{2}$$

$$\therefore z = -\frac{5\lambda}{2}$$

$$\Rightarrow 2x+y+5z = 30$$

$$\lambda = -\frac{30}{19}.$$

$$\therefore x = \frac{30}{19},\quad y = \frac{45}{19},\quad z = \frac{75}{19}$$

so, therefore, the function has an extremum ~~of~~ at the point $\left(\frac{30}{19}, \frac{45}{19}, \frac{75}{19}\right)$

$$\therefore \frac{\partial^2 F}{\partial x^2} = 2$$

$$\therefore \frac{\partial^2 F}{\partial y^2} = 2$$

$$\therefore \frac{\partial^2 F}{\partial z^2} = 2$$

2. **Problem:**— Divide the number 27 into three parts x, y, z such that

$$2yz + 3zx + 4xy$$ is maximum.

Solution:—

Define,

$$F = 2yz + 3zx + 4xy + \lambda(x+y+z-27)$$

$$\frac{\partial F}{\partial x} = 3z + 4y + \lambda = 0 \longrightarrow (i)$$

$$\frac{\partial F}{\partial y} = 2z + 4x + \lambda = 0 \longrightarrow (ii)$$

$$\frac{\partial F}{\partial z} = 2y + 3x + \lambda = 0 \longrightarrow (iii)$$

3. **Problem:-** If $2x+3y+4z=a$, show that the maximum value of $x^2y^3z^4$ is $\left(\frac{a}{9}\right)^9$

Solution:- $F = x^2y^3z^4 + \lambda(2x+3y+4z-a)$

$$\frac{\partial F}{\partial x} = 2xy^3z^4 + 2\lambda = 0$$

$$\frac{\partial F}{\partial y} = 3x^2y^2z^4 + 3\lambda = 0$$

$$\frac{\partial F}{\partial z} = 4x^2y^3z^3 + 4\lambda = 0$$

$\therefore 2xy^3z^4 = -2\lambda$

$\therefore 3x^2y^2z^4 = -3\lambda$

$\therefore 4x^2y^3z^3 = -4\lambda$

$\therefore \frac{2xy^3z^4}{3x^2y^2z^4} = \frac{2}{3}$

$\Rightarrow x = y$

$\therefore \frac{3x^2y^2z^4}{4x^2y^3z^3} = \frac{3}{4}$

$\Rightarrow z = y$

$\therefore x = y = z$

$\therefore 3x + 2x + 4x = a$

$\Rightarrow x = \frac{a}{9} = y = z.$

$\therefore$ The function has an extremum in $\left(\frac{a}{9}, \frac{a}{9}, \frac{a}{9}\right)$.

We know AM $\geq$ GM.

The GM of x, y, z is $(x^2y^3z^4)^{1/9}$

$$AM = \frac{2x+3y+4z}{9} = \frac{a}{9}.$$

Now, AM $\geq$ GM.

$\therefore$ Maximum value $= \left(\frac{a}{9}\right)^9$.

4. **Problem:-** Find the maximum volume of a rectangular box open at the top and the given surface area $= a$.

Solution:- Let length $= x$, breadth $= y$, height $= z$

Total volume $= xyz$.

It is given that, $2(xy+yz+zx) - xy = a$

so, define,

$$F = xyz + \lambda(xy + 2yz + 2zx - a) = 0$$

$$\frac{\partial F}{\partial x} = yz + \lambda(y+2z) = 0$$

$$\frac{\partial F}{\partial y} = xz + \lambda(x+2z) = 0$$

$$\frac{\partial F}{\partial z} = xy + \lambda(2y+2x) = 0$$

$$\therefore z(y-x) + \lambda(y-x) = 0$$

$$\Rightarrow z = -\lambda$$

and $x = y = 2z$

$$\therefore \lambda = \pm\sqrt{\frac{a}{48}}$$

$$\therefore x = \sqrt{\frac{a}{3}} = y \quad \text{and} \quad z = 2\sqrt{\frac{a}{3}}.$$

$\therefore$ Extremum point is $\left(\sqrt{\frac{a}{3}}, \sqrt{\frac{a}{3}}, 2\sqrt{\frac{a}{3}}\right)$

$$\therefore \text{volume} = 2\left(\frac{a}{3}\right)^{3/2}$$

Now, we know, AM $\geq$ GM.

Let us consider three numbers $xy, 2yz, 2xz$

$$G.M = (xy \cdot 2yz \cdot 2zx)^{1/3}$$

$$= (2xyz)^{2/3}$$

Here, $AM = \frac{a}{3}$

$$\Rightarrow (2xyz)^{2/3} \leq \frac{a}{3} = \left(2 \cdot \frac{1}{2} \cdot \sqrt{\frac{a}{3}}\right)^{2/3}$$

$$= \frac{a}{3}.$$

A special Topic :- Let $f(\underset{\sim}{x})$ be a function defined on $\mathbb{R}^n$. Then,

$$\frac{\partial f}{\partial \underset{\sim}{x}} = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

Now, $\frac{\partial}{\partial x_i}(\underset{\sim}{a}'\underset{\sim}{x}) = \frac{\partial}{\partial x_i}\left(\sum_{i=1}^{n} a_i x_i\right) = a_i$

and

$$\frac{\partial}{\partial \underset{\sim}{x}}(\underset{\sim}{a}'\underset{\sim}{x}) = \begin{pmatrix} \frac{\partial(\underset{\sim}{a}'\underset{\sim}{x})}{\partial x_1} \\ \frac{\partial(\underset{\sim}{a}'\underset{\sim}{x})}{\partial x_2} \\ \vdots \\ \frac{\partial(\underset{\sim}{a}'\underset{\sim}{x})}{\partial x_n} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \underset{\sim}{a}.$$

Note that, $A^{m\times n}\underset{\sim}{x}^{n\times 1} = \begin{pmatrix} \underset{\sim}{a}_1' \\ \underset{\sim}{a}_2' \\ \vdots \\ \underset{\sim}{a}_m' \end{pmatrix}\underset{\sim}{x} = \begin{pmatrix} \underset{\sim}{a}_1'\underset{\sim}{x} \\ \underset{\sim}{a}_2'\underset{\sim}{x} \\ \vdots \\ \underset{\sim}{a}_m'\underset{\sim}{x} \end{pmatrix}$

$$\therefore \frac{\partial}{\partial x}(A\underset{\sim}{x}) = \begin{pmatrix} \frac{\partial}{\partial \underset{\sim}{x}}(\underset{\sim}{a}_1'\underset{\sim}{x}) \\ \frac{\partial}{\partial \underset{\sim}{x}}(\underset{\sim}{a}_2'\underset{\sim}{x}) \\ \vdots \\ \frac{\partial(\underset{\sim}{a}_m'\underset{\sim}{x})}{\partial \underset{\sim}{x}} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{a}_1' \\ \underset{\sim}{a}_2' \\ \vdots \\ \underset{\sim}{a}_m' \end{pmatrix} = A$$

Now,

$$\underset{\sim}{x}'A\underset{\sim}{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j$$

$$= \sum_{i=1}^{n} a_{ii}x_i^2 + \sum_{i=1}^{n}\sum_{j\neq i} a_{ij}x_i x_j$$

$$\therefore \frac{\partial(\underset{\sim}{x}'A\underset{\sim}{x})}{\partial x_i} = 2a_{ii}x_i + 2\sum_{j\neq i} a_{ij}x_j$$

$$= 2\sum_{j=1}^{n} a_{ij}x_j$$

$$= 2\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in}\end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= 2\underset{\sim}{a_i}'\underset{\sim}{x}$$

$$\therefore \frac{\partial}{\partial \underset{\sim}{x}}(\underset{\sim}{x}'A\underset{\sim}{x}) = \begin{pmatrix} \frac{\partial}{\partial x_1}(\underset{\sim}{x}'A\underset{\sim}{x}) \\ \vdots \\ \frac{\partial}{\partial x_n}(\underset{\sim}{x}'A\underset{\sim}{x}) \end{pmatrix}$$

$$= \begin{pmatrix} 2\underset{\sim}{a_1}'\underset{\sim}{x} \\ 2\underset{\sim}{a_2}'\underset{\sim}{x} \\ \vdots \\ 2\underset{\sim}{a_n}'\underset{\sim}{x} \end{pmatrix}$$

$$= 2\begin{pmatrix} \underset{\sim}{a_1}' \\ \underset{\sim}{a_2}' \\ \vdots \\ \underset{\sim}{a_n}' \end{pmatrix}\underset{\sim}{x}$$

$$= 2A\underset{\sim}{x}$$

<u>Alt:-</u> $\frac{\partial}{\partial \underset{\sim}{x}}(\underset{\sim}{x}'\underset{\sim}{a}) = \underset{\sim}{a}$ as $\frac{\partial}{\partial x}(xa) = a$

$\frac{\partial}{\partial \underset{\sim}{x}}(\underset{\sim}{x}'A\underset{\sim}{x}) = 2A\underset{\sim}{x}$ and $\frac{\partial}{\partial x}(xax) = 2ax$.

INTEGRALS :→

A. Double Integrals :— We assume that the real-valued function $f(x,y)$ is defined and bounded on the rectangle $R=[a,b]\times[c,d]$.

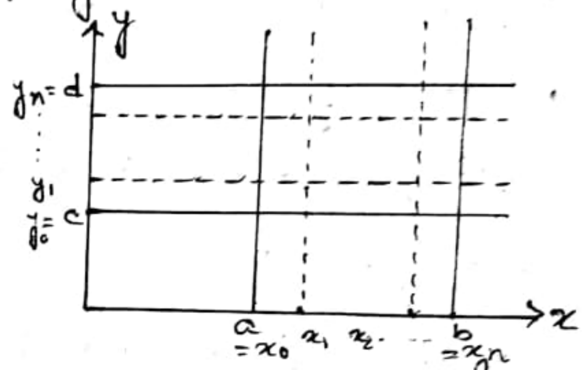
A partition of R is constructed by passing lines through parallel to y-axis : $x=x_i$ and parallel to x-axis : $y=y_j$ where,

$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ and

$c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d$.

We let, $M = \displaystyle\sup_{(x,y)\in R} f(x,y)$

and $m = \displaystyle\inf_{(x,y)\in R} f(x,y)$.



For the partition $P=\{R_{ij}\}$,

where,

$$R_{ij}=\{(x,y): x_{i-1}\le x\le x_i \text{ and } y_{i-1}\le y\le y_j\}$$

We define, $M_{ij} = \displaystyle\sup_{(x,y)\in R_{ij}} f(x,y)$

$$m_{ij} = \inf_{(x,y)\in R_{ij}} f(x,y)$$

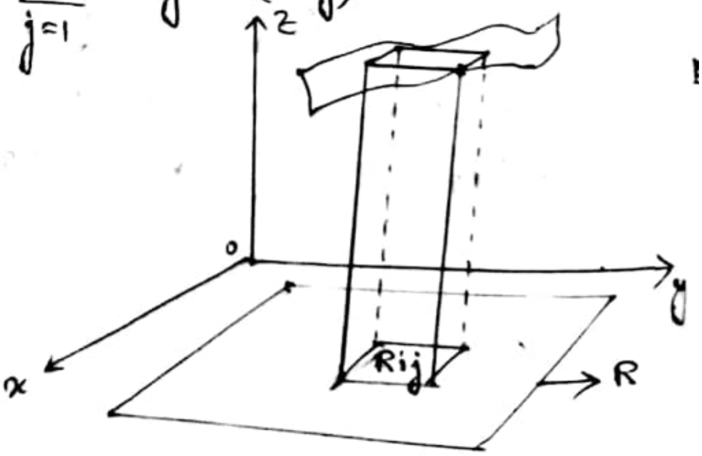
The upper sum and the lower sum are defined as below, ⟶

$$U(P,f) = \sum_{i=1}^{n}\sum_{j=1}^{m} M_{ij}\, A(R_{ij})$$

and, $$L(P,f) = \sum_{i=1}^{n}\sum_{j=1}^{m} m_{ij}\, A(R_{ij})$$

We define the upper double integral of f over R to be

$$\overline{\iint} f = \inf_P U(P,f)$$



Similarly, we define the lower double integral of 'f' over R by

$$\underline{\iint} f = \sup_P L(P,f).$$

[For any P, $L(P,f)\le$ actual volume $\le U(P,f)$]

• Definition:— A bounded function 'f' defined on a rectangle R is called integrable provided

$$\underline{\iint} f = \overline{\iint} f .$$

Ex: 1) Let $R = [0,1] \times [1,0]$

Define, $f(x,y) = \begin{cases} 1 & \text{, if } x \text{ and } y \text{ are rationals} \\ 0 & \text{, ow} \end{cases}$

on R. Does $\iint f$ exist?

Soln → Let $P = \{R_{ij}\}$ be any arbitrary partition of R. In each R_{ij}, necessarily contains points where f is 1 and also points where f is 0.

Therefore, $M_{ij} = 1$ and $m_{ij} = 0$, $\forall (i,j)$.

Thus $U(P,f) = \sum_i \sum_j M_{ij} A(R_{ij})$

$$= \sum_i \sum_j 1 . A(R_{ij})$$

$$= A(R)$$

$$= 1 .$$

and $L(P,f) = \sum_i \sum_j m_{ij} A(R_{ij})$

$$= \sum_i \sum_j 0 . A(R_{ij})$$

$$= 0$$

As P is arbitrary, it follows

$$\underline{\iint} f = \sup_P L(P,f) = 0 \text{ and } \overline{\iint} f = \inf_P U(P,f) = 1 .$$

Hence, $\iint_R f$ does not exist.

Theorem:— Suppose f is bounded on the rectangle R. Then 'f' is integrable on R iff given any $\epsilon > 0$ there is a partition P of R such that

$$\{ U(P,f) - L(P,f) \} < \epsilon .$$

1

☆ **Ex. 2.** Let $R = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$ and define f by,
$$f(x,y) = \begin{cases} 1 & , \text{if } x = 1 \\ 0 & , \text{if } 0 \le x < 1. \end{cases}$$

Does $\iint_R f$ exist?

Soln. → Let $\epsilon > 0$ be arbitrary and choose a partition P in such a way that $x_n - x_{n-1} < \epsilon$ and

$P = \{R_{ij}\}_{\substack{i=1(1)n \\ j=1(1)m}}$

Here, $0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1$
and, $0 = y_0 < y_1 < \dots < y_{m-1} < y_m = 1$

Now, $M_{nj} = 1, \quad j = 1(1)m$
$M_{ij} = 0, \quad i = 1(1)\overline{n-1}, \forall j$

Hence,
$$U(P,f) = \sum_{i=1}^{n}\sum_{j=1}^{m} M_{ij} A(R_{ij}) = \sum_{j=1}^{m} M_{nj} A(R_{nj})$$
$$= \sum_{j=1}^{m} A(R_{nj})$$
$$= \sum_{j=1}^{m} (x_n - x_{n-1})(y_j - y_{j-1})$$
$$< \epsilon \cdot \left\{\sum_{j=1}^{m} (y_j - y_{j-1})\right\}$$
$$= \epsilon(y_n - y_0) = \epsilon$$

Since $m_{ij} = 0 \ \forall (i,j)$,
$L(P,f) = \sum_i \sum_j m_{ij} A(R_{ij}) = 0$, for any P.

Now, $U(P,f) - L(P,f) < \epsilon$

$\Rightarrow \iint_R f$ exists and is equal to zero.

Evaluation of Double Integrals by Repeated or Iterated Integrals :

Evaluating double integral using defn., for most of the functions, is a difficult job if not impossible. Here, we shall develop a method called iterated or repeated integration, which will be quite useful in evaluating double integrals. We assume that the function 'f' is bounded on the rectangle $R = \{(x, y) : a \le x \le b, c \le y \le d\}$. If we fix $x \in [a, b]$, then $f(x, y)$ can be thought as a function of y, defined on $[c, d]$. If this function, for each $x \in [a, b]$ is integrable on $[c, d]$. Then this defines a function

$$g(x) = \int_c^d f(x, y)\, dy \text{ on } [a, b].$$

If $g(x)$ is integrable on $[a, b]$, then the integral,

$$\int_a^b g(x)dx = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx$$

is called an iterated integral or repeated integral of $f(x, y)$ over R.

Fubini's Theorem:— If a double integral $\iint_R f(x, y)\, dx\, dy$ exists on a rectangle $R = [a, b] \times [c, d]$ and if $\int_c^d f(x, y) dy$ exists for each $x \in [a, b]$, then the repeated integral $\int_a^b \left(\int_c^d f(x, y) dy \right) dx$ exists and is equal to the double integral $\iint_R f(x, y)\, dx\, dy$.

Similarly, for each $y \in [c, d]$, if $f(x, y)$ is integrable on $[a, b]$, then $h(y) = \int_a^b f(x, y)\, dx$ defines a function on $[c, d]$. If $h(y)$ is integrable on $[c, d]$, then $\int_c^d h(y) dy = \int_c^d \left(\int_a^b f(x, y) dx \right) dy$ is an iterated or repeated integral. By Fubini's theorem, if $\iint_R f(x, y)\, dx\, dy$ exists, then $\iint_R f(x, y) dx dy$ can be evaluated by

$$\int_c^d \left(\int_a^b f(x, y)\, dx \right) dy$$

Remark: — If a double integral exists, then the two repeated integrals can't exist without being equal. However, if the double integral does not exist, nothing can be said about the repeated integrals; they may or may not exist. However, if the two repeated integrals exist but are unequal, then the double integral can not exist.

☆ **Ex. 3.** Let $f(x,y) = \begin{cases} \frac{1}{y^2}, & 0 < x < y < 1 \\ -\frac{1}{x^2}, & 0 < y < x < 1 \end{cases}$

be defined on $R = [0,1] \times [0,1]$, evaluate two repeated integrals and hence comment on the existance of $\iint_R f$.

Soln. → For each $x \in [0,1]$.

$$g(x) = \int_0^1 f(x,y)\,dy = \int_0^x \left(-\frac{1}{x^2}\right) dy + \int_x^1 \frac{1}{y^2}\,dy$$

$$= -\frac{1}{x^2}[y]_0^x + \left[-\frac{1}{y}\right]_x^1$$

$$= -\frac{1}{x} + \frac{1}{x} - 1 = -1.$$

Now, $\int_0^1\left(\int_0^1 f(x,y)\,dy\right)dx = \int_0^1 g(x)\,dx = \int_0^1 (-1)\,dx = -1$

For each $y \in [0,1]$,

$$h(y) = \int_0^1 f(x,y)\,dx = \int_0^y \left(\frac{1}{y^2}\right)dx + \int_y^1 \left(-\frac{1}{x^2}\right)dx$$

$$= \frac{1}{y}[x]_0^y + \left[\frac{1}{x}\right]_y^1 = 1$$

Now, $\int_0^1\left(\int_0^1 f(x,y)\,dy\right)dx = \int_0^1 h(y)\,dy = \int_0^1 1.\,dy = 1.$

Two repeated integrals exist but are not equal. Hence the double integral $\iint_R f$ does not exist.

Ex.4. Let $f(x,y) = \begin{cases} 1, & x \text{ is rational} \\ 2y, & x \text{ is irrational} \end{cases}$

defined on $R = [0,1] \times [0,1]$. Show that one of the two repeated integral exist but that the other does not. What can be said about $\iint_R f$?

Soln. → For each $x \in [0,1]$,

$$g(x) = \int_0^1 f(x,y)\,dy = \begin{cases} \int_0^1 1.dy & , x \text{ is rational} \\ \int_0^1 2y\,dy, & x \text{ is irrational} \end{cases} = 1.$$

Now, $\int_0^1 \left(\int_0^1 f(x,y)\,dy \right) dx = \int_0^1 g(x)\,dx = \int_0^1 1.dx = 1.$

For each $y \in [0,1]$, but $y \neq \frac{1}{2}$,

$\int_0^1 f(x,y)\,dx$ does not exist, since $f(x,y)$ is discontinuous everywhere on $[0,1]$ as a function of x, except $y = \frac{1}{2}$.

[Let $x \in [0,1]$. If x approaches to x_0 through rational numbers, then $f(x,y) = 1$. If x approaches to x_0 through irrational numbers, then $f(x,y) = 2y$. Hence the $\lim_{x \to x_0} f(x,y)$ does not exist for $y \neq \frac{1}{2}$.

For $y = \frac{1}{2}$, $f(x,y) = 1 \;\forall\; x \in [0,1]$.

$\Rightarrow f(x,y)$ is continuous on $[0,1]$.]

Therefore $\int_0^1 \left(\int_0^1 f(x,y)\,dx \right) dy$ does not exist. We can't get any conclusion regarding the existance of $\iint_R f$ from the repeated integrals. Here, $f(x,y)$ is discontinuous everywhere on $R = [0,1] \times [0,1]$ except the line segment $y = \frac{1}{2}$, $0 \leq x \leq 1$.

Hence $\iint_R f$ does not exist.

Ex. 5. Compute the double integrals of the following functions over R.

i) $f(x,y) = x^2 y \cos(xy^2)$, $R = [0, \frac{\pi}{2}] \times [0, 2]$

ii) $f(x,y) = x \sin(x+y)$, $R = [0, \pi] \times [0, \pi/2]$

iii) $f(x,y) = x^2 y e^{xy}$, $R = [0, 1] \times [0, 2]$

Soln.

i) Clearly, $f(x,y)$ is continuous on R.

Hence, the double integral $\iint_R f(x,y)\,dx\,dy$ exists.

By Fubini's theorem, evaluating the double integral by repeated integral.

$$\iint_R f(x,y)\,dx\,dy = \int_0^{\pi/2} \left\{ \int_0^2 x^2 y \cos(xy^2)\,dy \right\} dx$$

$$= \int_0^{\pi/2} \frac{x}{2} \left(\int_0^2 2xy \cos(xy^2)\,dy \right) dx$$

$$= \int_0^{\pi/2} \frac{x}{2} \left(\int_0^{4x} \cos u\,du \right) dx, \quad \text{let, } u = xy^2.$$

$$= \int_0^{\pi/2} \frac{x}{2} \sin 4x \,.\, dx$$

$$= \left[\frac{x}{2} \left(-\frac{\cos 4x}{4} \right) \right]_0^{\pi/2} - \int_0^{\pi/2} \frac{1}{2} \left(-\frac{\cos 4x}{4} \right) dx$$

$$= -\frac{\pi}{16} + \frac{1}{8} \left[\frac{\sin 4x}{4} \right]_0^{\pi/2}$$

☆ Ex. 6. Evaluate: $\iint_R (x^2+y^2)\,dx\,dy$, where $R = [0,1]\times[1,2]$

Soln.

Note that, $f(x,y) = x^2+y^2$ is continuous on R.
Hence, the above double integral $\iint_R f(x,y)\,dx\,dy$ exists.
By Fubini's theorem, evaluating the double integral by repeated integrals, we get,

$$\iint_R f(x,y)\,dx\,dy = \int_0^1 \left\{ \int_1^2 (x^2+y^2)\,dy \right\} dx$$

$$= \int_0^1 \left[x^2y + \frac{y^3}{3} \right]_1^2 dx$$

$$= \int_0^1 \left[2x^2 + \frac{8}{3} - x^2 - \frac{1}{3} \right] dx$$

$$= \int_0^1 \left[x^2 + \frac{7}{3} \right] dx$$

$$= \left[\frac{x^3}{3} + \frac{7}{3}\cdot x \right]_0^1$$

$$= \frac{1}{3} + \frac{7}{3}$$

$$= 2\frac{2}{3} \text{ (Ans)}$$

☆ **Ex. 7.** Let $R = [0,1] \times [0,1]$.

Evaluate i) $\iint_R \frac{x^2}{1+y^2}\,dx\,dy$

ii) $\iint_R \frac{1}{(x+y+1)^2}\,dx\,dy$

Soln. →

i) Since $f(x,y)$ is bounded on R, $\iint_R f(x,y)\,dx\,dy$ exists, where $f(x,y) = \frac{x^2}{1+y^2}$.

Hence, evaluating the double integral by repeated integral, we get

$$I = \int_0^1 \left\{ \int_0^1 \frac{x^2}{1+y^2}\,dx \right\} dy$$

$$= \left(\int_0^1 \frac{1}{1+y^2}\,dy \right)\left(\int_0^1 x^2\,dx \right)$$

$$= \left[\tan^{-1} y\right]_0^1 \cdot \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{\pi}{12}.$$

ii) Since, $f(x,y)$ is bounded on R, $\iint_R f(x,y)\,dx\,dy$ exists, where $f(x,y) = \frac{1}{(x+y+1)^2}$.

Hence, evaluating the double integral by repeated integral, we get

$$\iint_R \frac{1}{(x+y+1)^2}\,dx\,dy = \int_0^1 \left\{ \int_0^1 \frac{1}{(x+y+1)^2}\,dy \right\} dx$$

$$= \int_0^1 \left[\frac{(x+y+1)^{-1}}{-1} \right]_0^1 dx = \int_0^1 \left[\frac{(x+2)^{-1} - (x+1)^{-1}}{-1} \right] dx$$

$$= \int_0^1 \left(\frac{1}{x+1} - \frac{1}{x+2} \right) dx = \left[\log_e\left(\frac{x+1}{x+2}\right) \right]_0^1$$

$$= \log_e\left(\frac{2}{3}\right) - \log\left(\frac{1}{2}\right)$$

$$= \log_e\left(\frac{4}{3}\right).$$

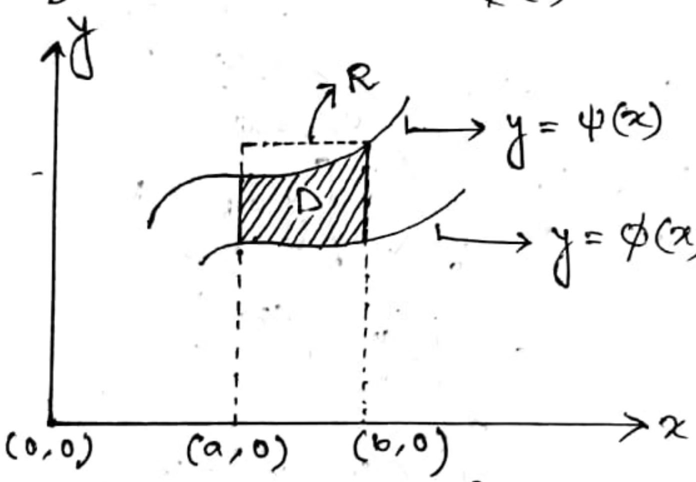
Calculation of Double integral over a closed region :

If $f(x,y)$ is integrable on a closed region $D = \{(x,y) : a \le x \le b, \ \phi(x) \le y \le \psi(x)\}$ and if $\int_{\phi(x)}^{\psi(x)} f(x,y)\,dy$ exists for each $x \in [a,b]$, then

$\int_a^b \left\{ \int_{\phi(x)}^{\psi(x)} f(x,y)\,dy \right\} dx$ exists and

$$\iint_D f(x,y)\,dx\,dy = \int_a^b \left\{ \int_{\phi(x)}^{\psi(x)} f(x,y)\,dy \right\} dx$$

[



Define, $f^*(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in D \\ 0, & \text{if } (x,y) \in R-D \end{cases}$

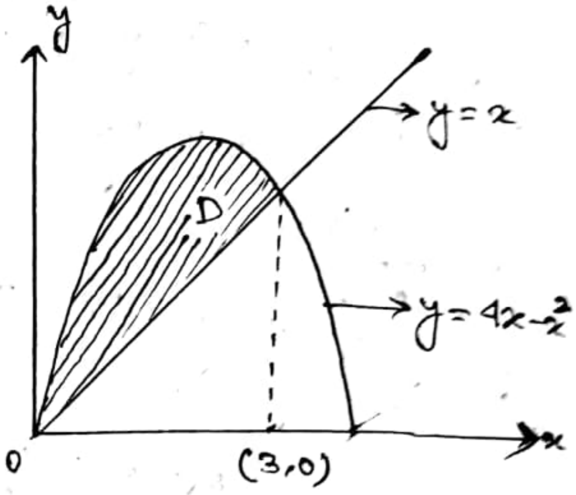
Now, $\iint_R f^*(x,y)\,dx\,dy$ can be evaluated by Riemann integral. Note that, $\iint_R f^*(x,y)\,dx\,dy$

$$= \iint_D f(x,y)\,dx\,dy + \iint_{R-D} 0\,dx\,dy$$

$$= \iint_D f(x,y)\,dx\,dy$$]

☆ **Ex. 1.** Evaluate: $\iint\limits_D y\,dx\,dy$, where D is the region bounded by $y = x$ and $y = 4x - x^2$.

Soln. →



For points of intersection:

$x = 4x - x^2$

$\Rightarrow x^2 - 3x = 0$

$\Rightarrow x(x-3) = 0$

$\Rightarrow x = 0, 3$.

The domain of the integration is,

$y = 4x - x^2$

$\Rightarrow y - 4 = -(x-2)^2 \leq 0$

$D = \{(x,y) : 0 \leq x \leq 3,\ x \leq y \leq 4x - x^2\}$

Hence, $\iint\limits_D y\,dx\,dy = \int_0^3 \left(\int_x^{4x-x^2} y\,dy \right) dx$

$= \int_0^3 \left[\frac{y^2}{2} \right]_x^{4x-x^2} dx$

$= \int_0^3 \frac{(4x-x^2)^2 - x^2}{2}\,dx$

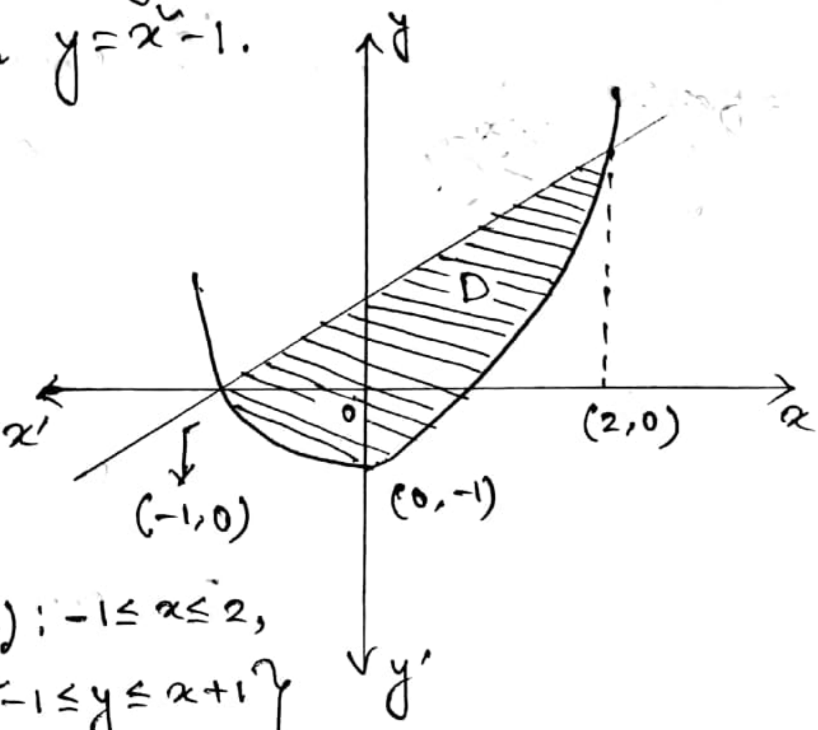
$= \int_0^3 \frac{x^4 - 8x^3 + 15x^2}{2}\,dx$

$= \frac{1}{2}\int_0^3 x^4\,dx - 4\int_0^3 x^3\,dx + \frac{15}{2}\int_0^3 x^2\,dx$

$= \frac{1}{2}\left[\frac{x^5}{5}\right]_0^3 - 4\left[\frac{x^4}{4}\right]_0^3 + \frac{15}{2}\left[\frac{x^3}{3}\right]_0^3$

$= \underline{10.8}$

Ex. 2. Evaluate $\iint_D (x^2+y^2)\,dx\,dy$ where D is the region bounded by $y=x+1$ and $y=x^2-1$.



Soln. $\rightarrow$ $y+1=x^2 \geq 0$

For points of intersection:

$$x+1=x^2-1$$

$$\Rightarrow x^2-x-2=0$$

$$\Rightarrow x=-1,\ 2.$$

The domain of the integral is $D=\{(x,y): -1\leq x\leq 2,\ x^2-1\leq y\leq x+1\}$

Hence, $\iint_D (x^2+y^2)\,dx\,dy$

$$=\int_{-1}^{2}\left\{\int_{x^2-1}^{x+1}(x^2+y^2)\,dy\right\}dx$$

$$=\int_{-1}^{2}\left[x^2y+\frac{y^3}{3}\right]_{x^2-1}^{x+1} dy$$

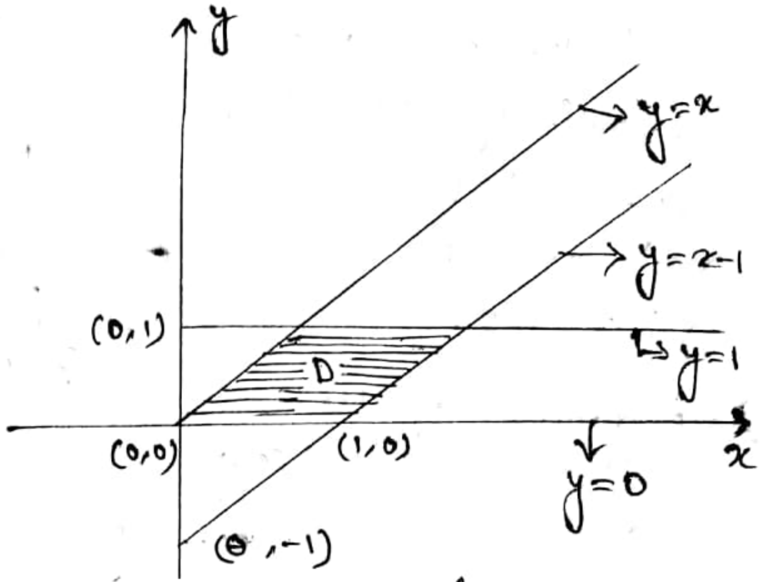
$$=\int_{-1}^{2}\left\{x^2(x-x^2+2)+\frac{(x+1)^3-(x^2-1)^3}{3}\right\}dx$$

$$=\frac{117}{14}.$$

☆ **Ex. 3.** Evaluate: $\iint_D x e^{x^2-y^2}\,dx\,dy$

where D is the region bounded by $y=x$, $y=x-1$, $y=0$ and $y=1$.

Soln.



Here, $D=\{(x,y): 0\le y\le 1,\ y\le x\le y+1\}$

Now, $\iint x e^{x^2-y^2}\,dx\,dy$

$$= \int_0^1 \left\{\int_y^{y+1} x e^{x^2}\,dx\right\} e^{-y^2}\,dy$$

$$= \int_0^1 \left[\frac{1}{2}e^{x^2}\right]_y^{y+1} . e^{-y^2}\,dy$$

$$= \int_0^1 \left(\frac{e^{(y+1)^2}-e^{y^2}}{2}\right) . e^{-y^2}\,dy$$

$$= \int_0^1 \frac{e^{2y+1}-1}{2}\,dy$$

$$= \frac{1}{2}\left[\left(\frac{e^{2y+1}}{2}-y\right)\right]_0^1$$

$$= \frac{1}{2}\left[\frac{e^3-e}{2}-1\right]$$

$$= \frac{e^3-e-2}{4}.$$

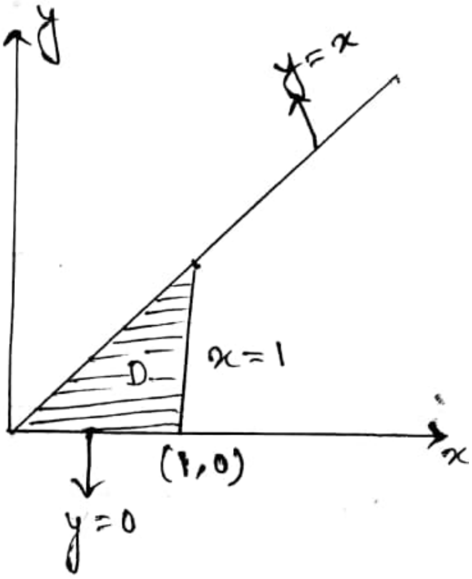
Ex. 4. Evaluate: $\iint_D e^{y/x}\,dx\,dy$, where D is the region bounded by $y=x$, $y=0$, $x=1$.

Soln. →

The domain of integration is

$\{(x,y): 0<x<1,\ 0<y<x\}$

Hence, evaluating the double integral by repeated integral,



$$\iint_D e^{y/x}\,dx\,dy = \int_0^1 \left(\int_0^x e^{y/x}\,dy \right) dx$$

$$= \int_0^1 \left[\frac{e^{y/x}}{1/x} \right]_0^x dx$$

$$= \int_0^1 x(e-1)\,dx$$

$$= (e-1)\left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2}(e-1).$$

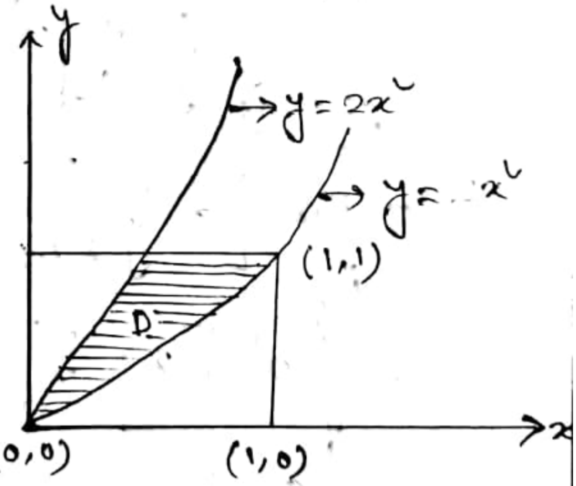
Ex. 5. Evaluate $\iint_R f(x,y)\,dx\,dy$, where $R=[0,1]\times[0,1]$

and $f(x,y) = \begin{cases} (x+y), & x^2<y<2x^2 \\ 0, & \text{ow} \end{cases}$

soln. →

Here,

$D = \{(x,y): 0<y<1,\ \sqrt{\frac{y}{2}} < x < \sqrt{y}\}$



Hence,

$$\iint_R f(x,y)\,dx\,dy = \int_0^1 \left\{ \int_{\sqrt{y/2}}^{\sqrt{y}} (x+y)\,dx \right\} dy$$

$$= \int_0^1 \left[\frac{x^2}{2} + yx\right]_{\sqrt{y/2}}^{\sqrt{y}} dy$$

$$= \int_0^1 \left\{ \frac{y}{4} + y\sqrt{y}\left(1 - \frac{1}{\sqrt{2}}\right) \right\} dy$$

$$= \left[\frac{y^2}{8} + \frac{2y^{5/2}}{5}\left(1 - \frac{1}{\sqrt{2}}\right) \right]_0^1$$

$$= \frac{1}{8} + \frac{2(\sqrt{2}-1)}{5\sqrt{2}}$$

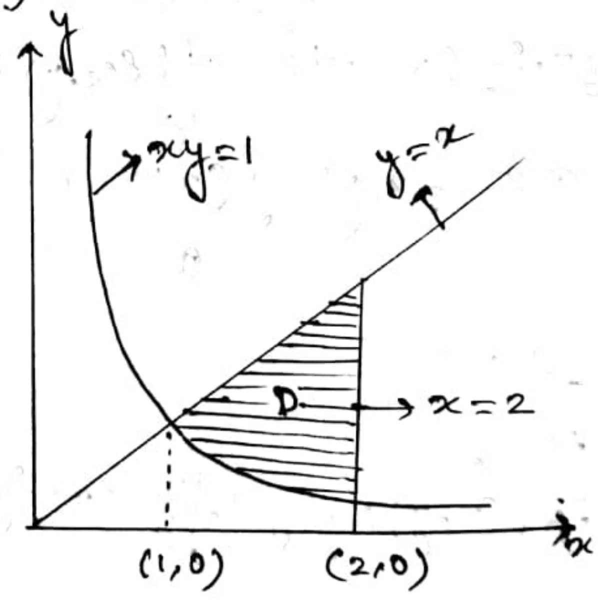
$$= \frac{21 - 8\sqrt{2}}{40}$$

☆ **Ex. 6.** Evaluate $\iint_D \frac{x^2}{y^2}\, dx dy$, where D is bounded by $x = 2$, $y = x$, $xy = 1$.

Soln. →

Here,

$D = \left\{(x,y) : 1 < x < 2,\ \frac{1}{x} < y < x\right\}$



Now, $\iint_D \frac{x^2}{y^2}\, dx dy$

$$= \int_1^2 \left[\int_{1/x}^{x} \frac{1}{y^2}\, dy \right] x^2\, dx$$

$$= \int_1^2 x^2 \left[-\frac{1}{y} \right]_{1/x}^{x} dx$$

$$= \int_1^2 x^2 \left[x - \frac{1}{x} \right] dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_1^2$$

$$= 4 - \frac{1}{4} - 2 + \frac{1}{2}$$

$$= \frac{9}{4}.$$

Ex. 7. Evaluate the following iterated integrals by changing the order of integrations:

i) $\int_0^1 \left\{ \int_{2y}^{2} e^{x^2} dx \right\} dy$, ii) $\int_0^1 \left\{ \int_x^1 y \sin(\pi y^3) dy \right\} dx$

Soln. →

i) The domain of integration is

$$D = \{(x,y) : 0<y<1,\ 2y<x<2\}$$
$$= \{(x,y) : 0<y<\tfrac{x}{2}<1\}$$
$$= \{(x,y) : 0<y<\tfrac{x}{2},\ 0<x<2\}$$

Note that, $f(x,y) = e^{x^2}$ is bounded on D. Hence $\iint_D f(x,y)\,dx\,dy$ exists and can be evaluated by any one of the two iterated integrals. Hence, by changing the order of integration,

$$\int_0^1 \left(\int_{2y}^{2} e^{x^2} dx \right) dy$$
$$= \int_0^2 \left(\int_0^{x/2} e^{x^2} dy \right) dx$$
$$= \int_0^2 e^{x^2} \left(\int_0^{x/2} dy \right) dx$$
$$= \int_0^2 e^{x^2} \cdot \frac{x}{2}\, dx$$
$$= \frac{1}{4} \left[e^{x^2} \right]_0^2$$
$$= \frac{e^4 - 1}{4}.$$

ii) The domain of integration is

$$D = \{(x,y) : 0<x<1,\ x<y<1\}$$
$$= \{(x,y) : 0<x<y<1\}$$
$$= \{(x,y) : 0<x<y,\ 0<y<1\}$$

Note that, $f(x,y) = y\sin(\pi y^3)$ is bounded on D.

Hence $\iint_D f(x,y)\,dx\,dy$ exists.

Hence, changing the order of the integration,

$$\int_0^1 \left\{ \int_x^1 y\sin(\pi y^3)\,dy \right\} dx$$

$$= \int_0^1 y\sin(\pi y^3) \left(\int_0^y dx \right) dy$$

$$= \int_0^1 y^2 \sin(\pi y^3)\,dy$$

$$= \frac{1}{3\pi} \int_0^1 3\pi y^2 (\sin \pi y^3)\,dy$$

$$= \frac{1}{3\pi} \left[-\cos(\pi y^3) \right]_0^1$$

$$= \frac{1}{3\pi} \left[-\cos\pi + \cos 0 \right]$$

$$= \frac{2}{3\pi}.$$

☆ **Ex. 8.** $\int_0^{\infty}\left(\int_x^{\infty} y^{-1}e^{-y}dy\right)dx$: evaluate.

Soln. →

The domain of integration is

$$D = \{(x,y) : 0 < x < \infty ,\ x < y < \infty\}$$

$$= \{(x,y) : 0 < x < y < \infty\}$$

$$= \{(x,y) : 0 < y < \infty,\ 0 < x < y\}$$

Interchanging the order of the integration is

$$\int_0^{\infty}\left(\int_x^{\infty} y^{-1}.e^{-y}dy\right)dx$$

$$= \int_0^{\infty}\left(\int_0^{y} y^{-1}.e^{-y}dx\right)dy$$

$$= \int_0^{\infty} e^{-y}.y^{-1}\left(\int_0^{y}dx\right)dy$$

$$= \int_0^{\infty} e^{-y}.y^{-1}.y\,dy$$

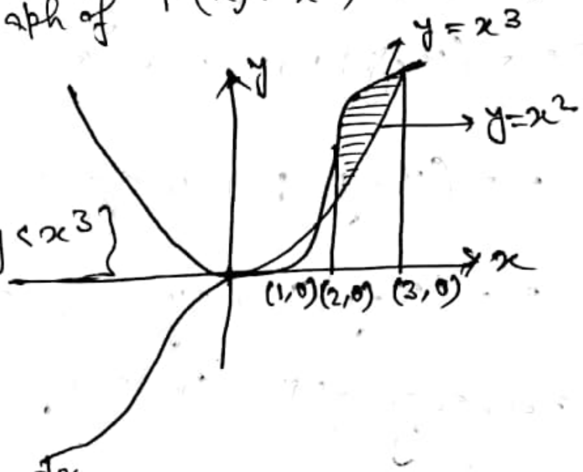
$$= \int_0^{\infty} e^{-y}dy$$

$$= \Gamma(1) = 1.$$

Ex. 9. i) $\iint_D \cos\left(\frac{\pi x^2}{2}\right) dx\,dy$, where

$D = \{(x, y) : 0 < x < y,\ 0 < y < 1\}$.

ii) Evaluate: $\iint_D (2x^2 + y)\,dx\,dy$, where D is the closed region bounded by the graph of $F(x) = x^2$, $g(x) = x^3$, $x = 2$ and $x = 3$.



Ans:-

$D = \{(x, y) : 2 < x < 3,\ x^2 < y < x^3\}$

$= \iint_D (2x^2 + y)\,dx\,dy$

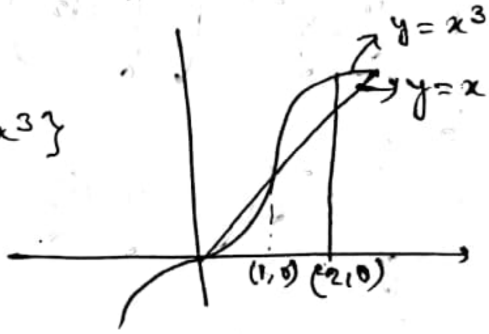
$= \int_2^3 \left\{ \int_{x^2}^{x^3} (2x^2 + y)\,dy \right\} dx$

$= \int_2^3 \left[2x^2 + y + \frac{y^2}{2} \right]_{x^2}^{x^3} dx$

iii) Evaluate: $\iint_D xy\,dx\,dy$, where D is bounded by graphs $y = x$, $y = x^3$, $x = 2$.

Ans:-

$D = \{(x, y) : 0 < x < 1,\ x^3 < y < x,$ and $1 < x < 2,\ x < y < x^3\}$



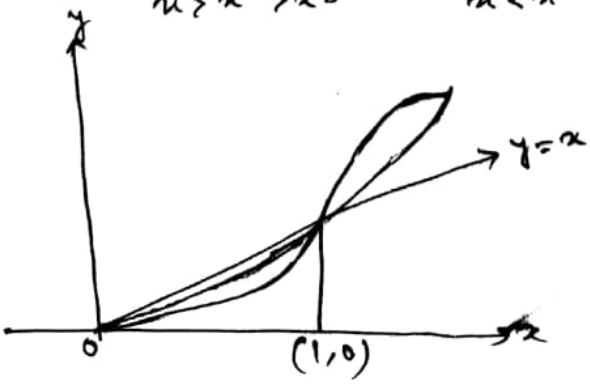
$\therefore \iint_D xy\,dx\,dy$

$= \int_0^1 \left\{ \int_{x^3}^{x} xy\,dy \right\} dx + \int_2^1 \left\{ \int_x^{x^3} xy\,dy \right\} dx$

Remark:-

For $0 < x < 1$: $x > x^2 > x^3$

For $x > 1$: $x < x^2 < x^3$



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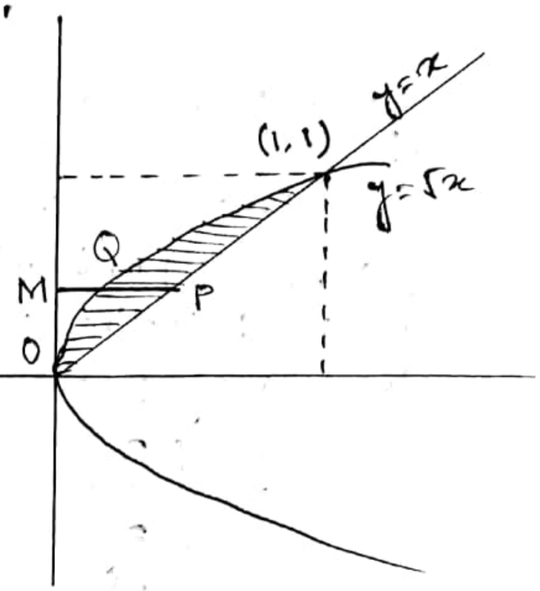
☆ **Ex. 10.** Change the order of integration in

$$I = \int_0^1 \left(\int_x^{\sqrt{x}} f(x,y)\,dy \right) dx .$$

Soln. →

The line $y = x$ and the parabola $y = \sqrt{x}$ cut at $(0,0)$ and $(1,1)$. The domain of integration is the shaded region. Any line parallel to x-axis cuts the region into two points such as Q and P.



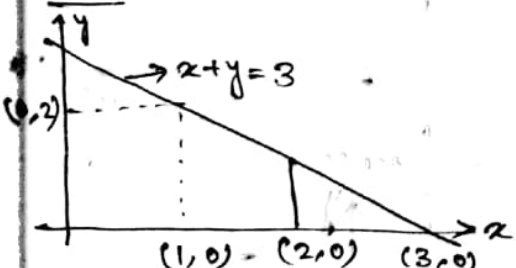
$$\therefore\ I = \int_0^1 dy \int_{y^2}^{y} f(x,y)\,dx$$

☆ **Ex. 11.** Let $F(x) = \begin{cases} x & \text{if } x \in [0,2] \\ 0 & \text{if } x \notin [0,2] \end{cases}$, and,

$$g(y) = \begin{cases} 1 & \text{if } y \in [0,2] \\ 0 & \text{if } y \notin [0,2] \end{cases}$$

Let $A = \{(x,y) : x + y \le 3\}$ then evaluate $\iint_A F(x)\,g(y)\,dx\,dy$

Soln. →



Now, $\iint_A f(x)\,g(y)\,dx\,dy$

$$= \iint_D f(x)\,g(x)\,dx\,dy + \iint_{A-D} f(x)\,g(x)\,dx\,dy$$

$$= \iint_D x\,dx\,dy + \iint_{A-D} 0.\,dx\,dy, \text{ where}$$

$D = \{(x,y) : 0 < x < 1,\ 0 < y < 2$ or, $1 < x < 2,\ 0 < y < 3 - x\}$

$$= \int_0^1 x \left(\int_0^2 dy \right) dx + \int_1^2 x \left(\int_0^{3-x} dy \right) dx$$

$$= \frac{19}{6}.$$

Change of variables in a Double Integral :

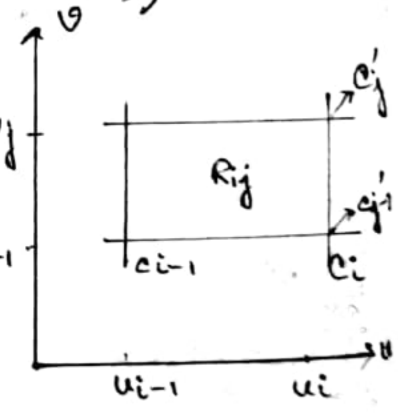
Jacobian : Consider the double integral $\iint_D f(x,y)\,dx\,dy$ over a bounded region D in XY - plane.

Consider the transformation : $u = \phi(x,y)$
$v = \psi(x,y)$

If the transformation represents a continuous one-to-one mapping of the closed region D in the xy-plane on a region D^* in the uv-plane and if $x = h_1(u,v)$ and $y = h_2(u,v)$ have the continuous partial derivative with jacobian (also called the local magnification factor).

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$$

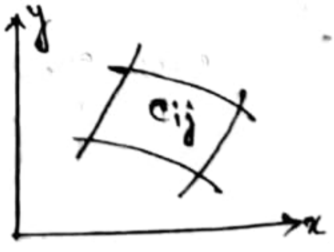
Then, $\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f\left(h_1(u,v), h_2(u,v)\right)|J|\,du\,dv$



[The tangent vectors are
$\left(\frac{\partial h_1}{\partial u}, \frac{\partial h_2}{\partial u}\right)\Delta u$ and $\left(\frac{\partial h_1}{\partial v}, \frac{\partial h_2}{\partial v}\right)\Delta v$.

Area $(c_{ij}) \simeq$ the area of the parallelogram formed by the tangent vectors

$$= \left|\left(\frac{\partial h_1}{\partial u}, \frac{\partial h_1}{\partial v}, 0\right) \times \left(\frac{\partial h_2}{\partial u}, \frac{\partial h_2}{\partial v}, 0\right)\right| \Delta u \Delta v$$



$$= \left\| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & 0 \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & 0 \end{matrix} \right\| \Delta u \Delta v$$

$$= \left| \hat{k} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right| \Delta u \Delta u = \text{mod.} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \Delta u \Delta v$$

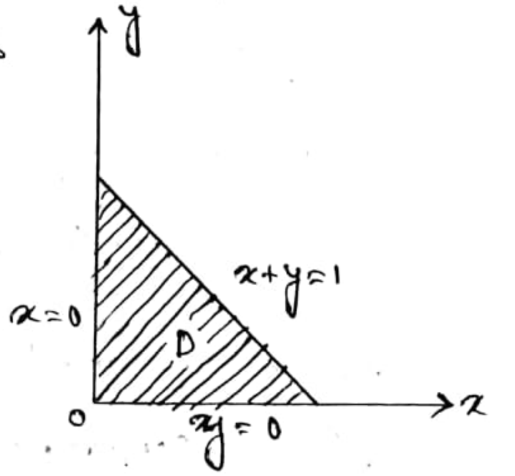
$= |J| \cdot \text{area}(R_{ij})$

This is the significance of the Jacobian.]

☆ **Example 1.** Evaluate: $\iint_D e^{\frac{y}{x+y}}\,dx\,dy$, where D is the region bounded by $x=0,\ y=0,\ x+y=1.$

Soln. → The domain of the integration is

$D=\{(x,y): 0\le x, y\le 1,\ x+y\le 1\}$



Let $u=\dfrac{y}{x+y}$, $v=x+y$

$\therefore\ y=uv,\ x=v(1-u)$

As, $(x,y)\in D$,

$0\le x\le 1,\ 0\le y\le 1$ and $x+y\le 1.$

$\Rightarrow 0\le uv\le 1,\quad 0\le v(1-u)\le 1,\ v\le 1$

$\Rightarrow 0\le u\le 1,\quad 0\le v\le 1.$

The new domain is

$$D^*=\{(u,v): 0\le u, v\le 1\}.$$

The Jacobian is $J=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix} -v & 1-u \\ v & u\end{vmatrix}$

$= -v.$

Hence, $\iint_D e^{\frac{y}{x+y}}\,dx\,dy$

$$=\iint_{D^*} e^{u}\cdot|-v|\,du\,dv$$

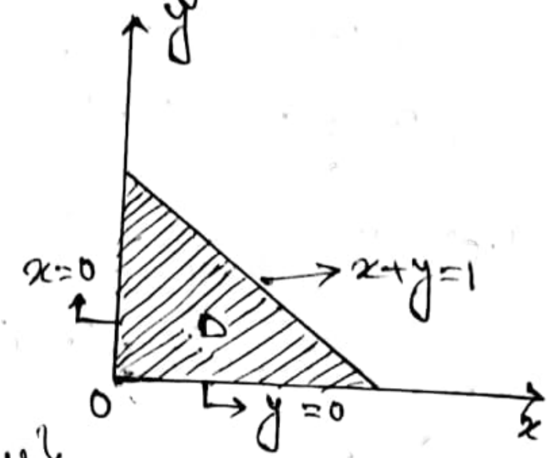
$$=\int_0^1\left\{\int_0^1 e^{u}\,v\,du\right\}dv$$

$$=\left(\int_0^1 e^{u}\,du\right)\left(\int_0^1 v\,dv\right)$$

$$=\frac{1}{2}(e-1).$$

☆ **Ex. 2.** Evaluate: $\iint_D \sin\left(\frac{x-y}{x+y}\right) dx\,dy$, where D is the region bounded by $x=0, y=0, x+y=1$.

Soln. →



Let, $u = x-y$, $v = x+y$

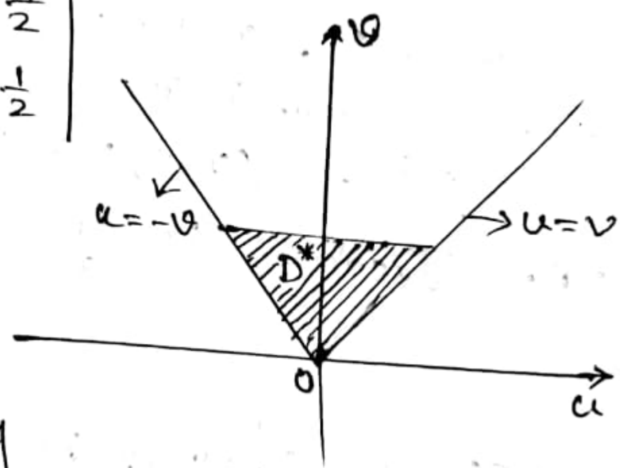
$\therefore x = \frac{u+v}{2}, \; y = \frac{v-u}{2}$.

The transformed domain is,

$D^* = \{(u,v) : 0 \le v \le 1, -v \le u \le v\}$

Jacobian is, $J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix}$

$= \frac{1}{2}$



Hence, $\iint_D \sin\left(\frac{x-y}{x+y}\right) dx\,dy$

$= \iint_{D^*} \sin\left(\frac{u}{v}\right) \frac{1}{2}\, du\,dv$

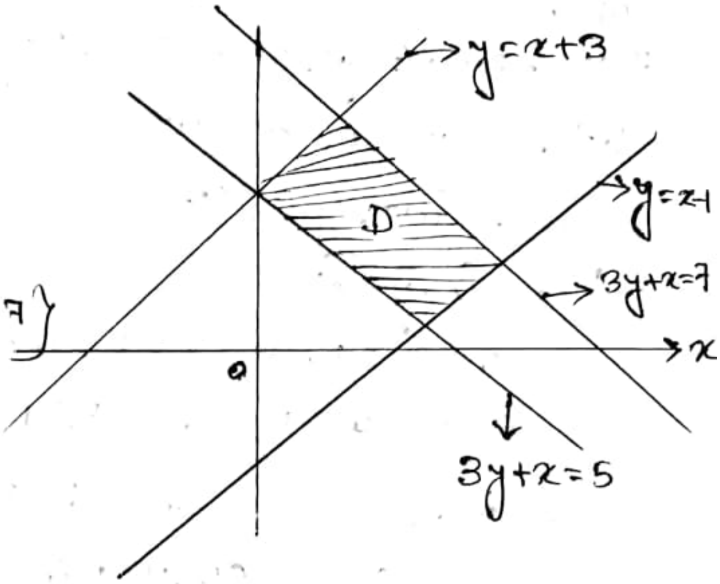
$= \frac{1}{2}\int_0^1 \left\{ \int_{-v}^{v} \sin\left(\frac{u}{v}\right) du \right\} dv$

$= \frac{1}{2}\int_0^1 \left[-v\cos\frac{u}{v} \right]_{-v}^{v} dv$

$= \frac{1}{2}\int_0^1 (-v)\left[\cos 1 - \cos(-1)\right] dv$

$= 0$.

☆ **Ex. 3.** Evaluate: $\iint_D (y-x)\,dx\,dy$ where D is the region bounded by the lines $y=x+3$, $y=x-1$, $3y+x=5$ and $3y+x=7$.

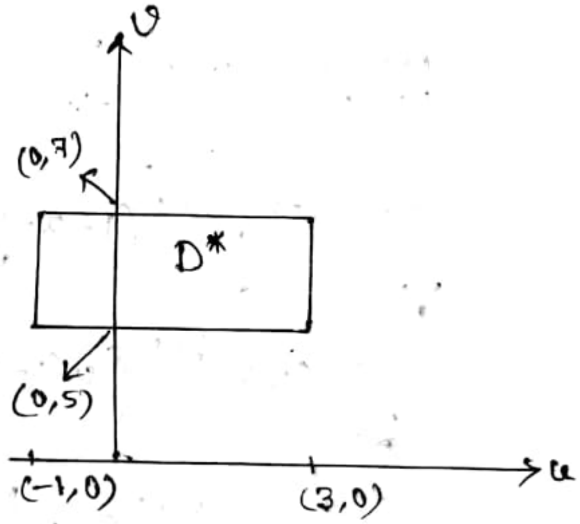


Soln. →

Let, $u=y-x$, $v=3y+x$

the new domain is

$D^* = \{(u,v) : -1 \le u \le 3,\ 5 \le v \le 7\}$



Here,

$$J = \begin{vmatrix} -3/4 & 1/4 \\ 1/4 & 1/4 \end{vmatrix}$$

$$= -\frac{3}{16} - \frac{1}{16}$$

$$= -\frac{1}{4}.$$

Hence, $\iint_D (y-x)\,dx\,dy$

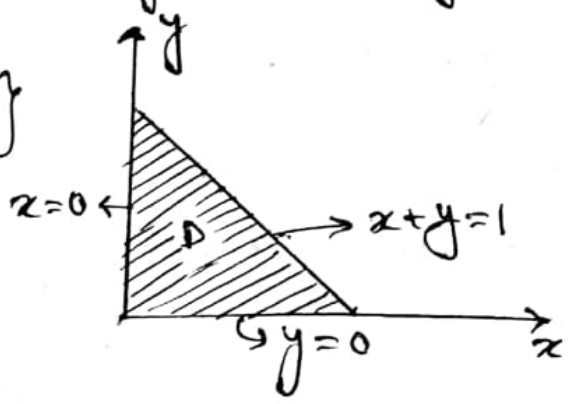
$$= \iint_{D^*} u \cdot \frac{1}{4}\,du\,dv$$

$$= \int_5^7 \int_{-1}^3 \frac{u}{4}\,du\,dv = \frac{1}{4}\left[\int_5^7 \left[\frac{u^2}{2}\right]_{-1}^3 dv\right.$$

$$= \frac{1}{4}\int_5^7 \left[\frac{9}{2} - \frac{1}{2}\right] dv = \frac{8}{2}\cdot\frac{1}{4}[7-5]$$

$$= 2.$$

☆ **Ex. 4.** Evaluate $\iint_D x^{l-1} y^{m-1} (1-x-y)^{p-1}\, dx\,dy$, where D is the region bounded by $x=0$, $y=0$ and $x+y=1$.



Soln. →

Here, $D = \{(x,y),\ 0 \le x, y \le 1,\ x+y \le 1\}$

Let, $x+y = u$, $x = uv$

$\therefore x = uv,\ y = u(1-v)$

As, $(x,y) \in D$

$0 \le x \le 1,\ 0 \le y \le 1,\ x+y \le 1.$

$\Rightarrow 0 \le u \le 1,\ 0 \le v \le 1$

The transformed domain is

$$D^* = \{(u,v) : 0 \le u, v \le 1\}$$

$$|J| = \text{mod.} \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = |-u| = u$$

$$\iint_D x^{l-1} y^{m-1} (1-x-y)^{p-1}\, dx\,dy$$

$$= \iint_{D^*} (uv)^{l-1} \{u(1-v)\}^{m-1} (1-u)^{p-1} \cdot u \cdot du\,dv$$

$$= \left\{\int_0^1 u^{l+m-1} (1-u)^{p-1}\, du\right\} \left\{\int_0^1 v^{l-1} (1-v)^{m-1}\, dv\right\}$$

$$= \beta(l+m, p) \cdot \beta(l, m)$$

$$= \frac{\Gamma(\cancel{l+m})\Gamma(p)}{\Gamma(l+m+p)} \cdot \frac{\Gamma(l)\Gamma(m)}{\Gamma(\cancel{l+m})}$$

$$= \frac{\Gamma(l)\Gamma(m)\Gamma(p)}{\Gamma(l+m+p)}$$ [This is a Dirichlet integral]

☆ Ex.5. Show that : $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, $m>0, n>0$.

Soln. ⇒ $\Gamma(m)\cdot\Gamma(n) = \left(\int_0^\infty e^{-x} x^{m-1} dx\right)\left(\int_0^\infty e^{-y} y^{n-1} dy\right)$

$$= \iint_D e^{-(x+y)} x^{m-1} y^{n-1} dx\,dy$$

, where, $D = \{(x,y) : 0 < x, y < \infty\}$

Let $u = x+y$, $v = \frac{x}{x+y}$,

$x = uv$, $y = u(1-v)$.

As $(x,y) \in D$,

$0 < x < \infty$, $0 < y < \infty$, $\left[\because 0 < x < x+y \right.$

$\Rightarrow 0 < uv < \infty$, $0 < u(1-v) < \infty$ $\quad \Rightarrow 0 < \frac{x}{x+y} < 1$

$\Rightarrow 0 < u < \infty$, $0 < v < 1$. $\quad \left. \Rightarrow 0 < v < 1 \right]$

The transformed domain is

$$D^* = \{(u,v) : 0 < u < \infty, 0 < v < 1\}$$

and $|J| = \text{mod} \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = |-u| = u$

Hence, $\Gamma(m)\Gamma(n) = \iint_{D^*} e^{-u} (uv)^{m-1} \{u(1-v)\}^{n-1} \cdot u\,du\,dv$

$$= \left(\int_0^\infty e^{-u} \cdot u^{m+n-1} du\right)\left(\int_0^1 v^{m-1}(1-v)^{n-1} dv\right)$$

$$= \Gamma(m+n) \cdot \beta(m,n)$$

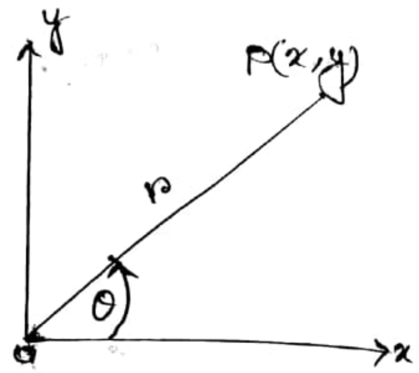
Hence, $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Polar transformation in $\mathbb{R}^2$:—

Let $x = r\cos\theta$, $y = r\sin\theta$

Here, $r = \sqrt{x^2+y^2}$ = the distance of $P(x,y)$ from the origin.

and, θ = the angle which OP makes with the direction of x-axis.



As $(x,y) \in \mathbb{R}^2$, $0 < r < \infty$, $0 < \theta < 2\pi$.

The transformed plane of $\mathbb{R}^2$ under polar transformation is $D^* = \{(r,\theta) : 0 < r < \infty, 0 < \theta < 2\pi\}$

Here, Jacobian is $J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

☆ **Ex.1** Evaluate: $\iint_D e^{x^2+y^2}\,dx\,dy$, where D is the region bounded by the semi-circle $y = \sqrt{1-x^2}$ and x-axis.

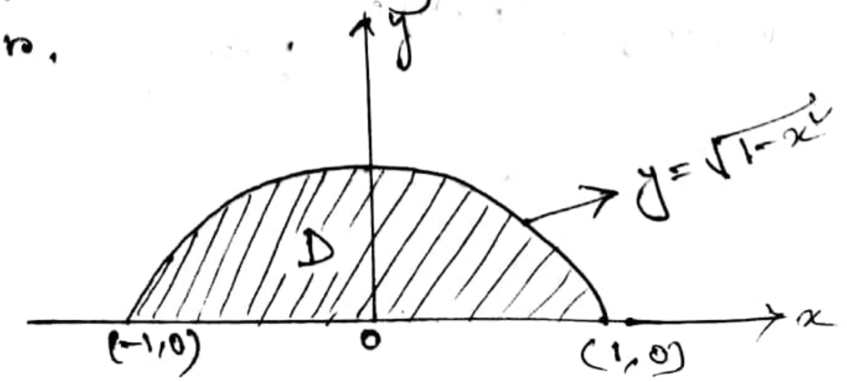
Soln. → Let $x = r\cos\theta$, $y = r\sin\theta$,

As $(x,y) \in D$, $0 < r < 1$ and $0 < \theta < \pi$.

Hence, the transformed domain is,

$D^* = \{(r,\theta) : 0 < r < 1, 0 < \theta < \pi\}$

and $|J| = r$.



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Hence, $\displaystyle\iint_D e^{x^2+y^2}\,dx\,dy$

$$= \iint_{D^*} e^{r^2}.\,r\,dr\,d\theta$$

$$= \int_0^1\int_0^{\pi} e^{r^2}.\,r\,dr\,d\theta$$

$$= \left(\int_0^1 e^{r^2}.\,r\,dr\right)\left(\int_0^{\pi} d\theta\right)$$

$$= \left[\frac{e^{r^2}}{2}\right]_0^1 \,[\theta]_0^{\pi}$$

$$= \frac{\pi(e-1)}{2}.$$

Ex. 2. Evaluate: $\displaystyle\iint_D \sqrt{x^2+y^2}\,dx\,dy$, where D is the region bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Soln. →

Here, $D = \{(x,y) : 1 \le x^2+y^2 \le 4\}$

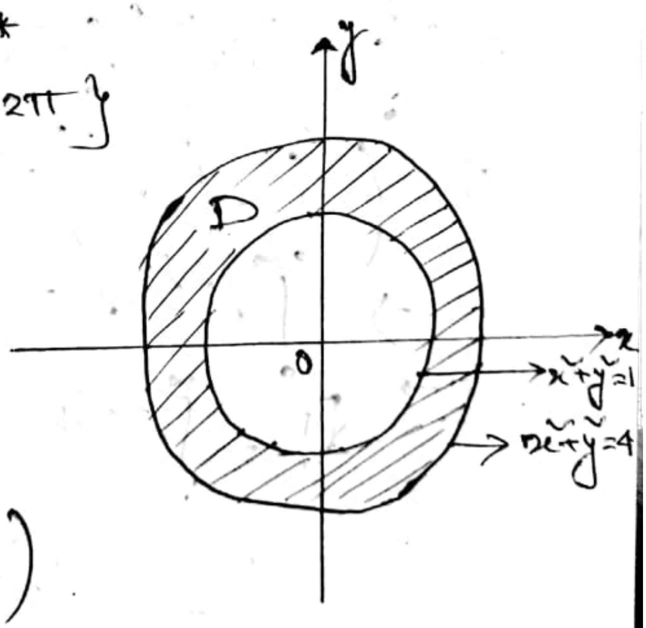
Let, $x = r\cos\theta,\ y = r\sin\theta$,

As, $(x,y) \in D$, $1 < r < 2$,
$0 < \theta < 2\pi$.

The transformed domain is, D^*

$= \{(r,\theta) : 1 < r < 2,\ 0 < \theta < 2\pi\}$

Here $|J| = r$.



Hence, $\displaystyle\iint_D \sqrt{x^2+y^2}\,dx\,dy$

$$= \iint_{D^*} r.\,r\,dr\,d\theta$$

$$= \left(\int_0^1 r^2\,dr\right)\left(\int_0^{2\pi} d\theta\right)$$

$$= \left[\frac{r^3}{3}\right]_1^2 .\,2\pi = \frac{7}{3}.\,2\pi = \frac{14\pi}{3}.$$

☆ **Ex. 3.** Evaluate: $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$, using Polar transformation. Hence, evaluate $\int_0^\infty e^{-x^2}dx$.

Soln. $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \iint_D e^{-(x^2+y^2)}\,dx\,dy$

where, $D = \{(x,y) : 0 < x, y < \infty\}$

Let, $x = r\cos\theta$, $y = r\sin\theta$.

As, $(x,y) \in D$, $0 < r < \infty$, $0 < \theta < \pi/2$

The transformed domain is

$$D^* = \left\{ (r,\theta) : 0 < r < \infty, 0 < \theta < \frac{\pi}{2} \right\}$$

and $|J| = r$.

Hence, $\iint_D e^{-(x^2+y^2)}dx\,dy = \iint_{D^*} e^{-r^2}\cdot r\,dr\,d\theta$

$$= \left(\int_0^\infty e^{-r^2}\cdot r\,dr\right)\left(\int_0^{\pi/2} d\theta\right)$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}.$$

Hence, $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \pi/4$.

$$\Rightarrow \left(\int_0^\infty e^{-x^2}dx\right)\left(\int_0^\infty e^{-y^2}dy\right) = \pi/4$$

$$\Rightarrow \left(\int_0^\infty e^{-x^2}dx\right)^2 = \pi/4$$

$$\Rightarrow \int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$$

Remark: →

$$\frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-x^2}dx \quad , \quad z = x^2 \Rightarrow 2x\,dx = dz$$

$$= \int_0^\infty e^{-z}\cdot\frac{dz}{2\sqrt{z}}$$

$$= \frac{1}{2}\int_0^\infty e^{-z} z^{1/2-1}\,dz = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$\Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

☆ **Ex.4.** Evaluate: $\int_0^\infty\int_0^\infty e^{-(ax^2+2hxy+by^2)}\,dx\,dy$, where $a>0$, $b>0$, $(ab-h^2)>0$.

Soln. → $ax^2+2hxy+by^2$

$= a\left\{x^2+2x\cdot\frac{h}{a}\cdot y\right\}+by^2$

$= a\left(x+\frac{h}{a}y\right)^2+\frac{ab-h^2}{a}\cdot y^2$

Let, $u=\sqrt{a}\left(x+\frac{h}{a}y\right)$ and $v=\sqrt{\frac{ab-h^2}{a}}\,y$

As, $0<x, y<\infty$

$0<u, v<\infty$.

Now, $J=\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}=\frac{1}{\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}}$

$=\frac{1}{\begin{vmatrix}\sqrt{a} & h/\sqrt{a}\\ 0 & \sqrt{\frac{ab-h^2}{a}}\end{vmatrix}}=\frac{1}{\sqrt{ab-h^2}}$.

Hence, $\int_0^\infty\int_0^\infty e^{-(ax^2+2hxy+by^2)}\,dx\,dy$

$=\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\cdot\frac{1}{\sqrt{ab-h^2}}\,du\,dv$

$=\frac{1}{\sqrt{ab-h^2}}\left(\int_0^\infty e^{-u^2}du\right)\left(\int_0^\infty e^{-v^2}dv\right)$

$=\frac{1}{\sqrt{ab-h^2}}\left(\int_0^\infty e^{-u^2}du\right)^2$

$=\frac{1}{\sqrt{ab-h^2}}\left(\int_0^\infty e^{-z}\frac{dz}{2\sqrt{z}}\right)^2$

$=\frac{\pi}{4\sqrt{ab-h^2}}$.

B. Multiple Integrals :

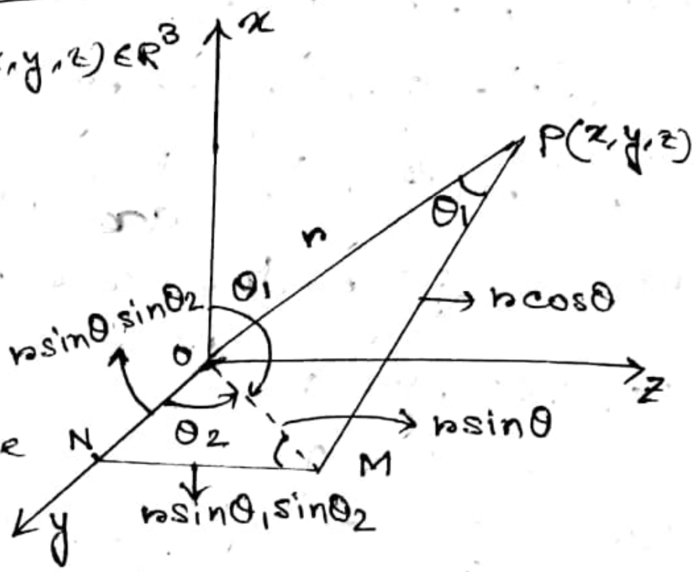
i Polar transformation in $\mathbb{R}^3$:

Polar trasformation of $(x,y,z)\in\mathbb{R}^3$ is

$x = r\cos\theta_1,$

$y = r\sin\theta_1\cos\theta_2$

$z = r\sin\theta_1\sin\theta_2$



[Note that, in yz-plane

$ON^2 + NM^2 = OM^2$

$\Rightarrow r^2\sin^2\theta_1\{\cos^2\theta_2 + \sin^2\theta_2\}$

$= r^2\sin^2\theta_1$

Again, $OM^2 + MP^2 = OP^2$

$\Rightarrow r^2\sin^2\theta_1 + r^2\cos^2\theta_1 = r^2$]

Clearly, $r = \sqrt{x^2+y^2+z^2}$ = the distance between O and P,

θ_1 = the angle which $\overrightarrow{OP}$ makes with the positive direction of x-axis. and

θ_2 = the angle which $\overrightarrow{OM}$ (the projection of OP on yz-plane) makes with the positive direction of y-axis.

From figure, $0<r<\infty$, $0<\theta_1<\pi$, $0<\theta_2<2\pi$.

The Jacobian of the tramformation is

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta_1,\theta_2)} = \begin{vmatrix} \cos\theta_1 & -r\sin\theta_1 & 0 \\ \sin\theta_1\cos\theta_2 & r\cos\theta_1\cos\theta_2 & -r\sin\theta_1\sin\theta_2 \\ \sin\theta_1\sin\theta_2 & r\cos\theta_1\sin\theta_2 & r\sin\theta_1\cos\theta_2 \end{vmatrix}$$

$$= r^2(\sin^2\theta_1\cos\theta_1\cos\theta_2\sin\theta_2) \times \begin{vmatrix} 1 & -\tan\theta_1 & 0 \\ 1 & \cot\theta_1 & -\tan\theta_2 \\ 1 & \cot\theta_1 & \cot\theta_2 \end{vmatrix}$$

$$= r^2 \left(\sin^2\theta_1 \cos\theta_1 \cos\theta_2 \sin\theta_2\right) \begin{vmatrix} 1 & -\tan\theta_1 & 0 \\ 0 & \frac{1}{\sin\theta_1\cos\theta_1} & -\tan\theta_2 \\ 0 & 0 & \frac{1}{\sin\theta_2\cos\theta_2} \end{vmatrix}$$

$$\left[R_3' = R_3 - R_2,\ R_2' = R_2 - R_1\right]$$

$$= r^2 \cdot \left(\sin^2\theta_1 \cdot \cos\theta_1 \cdot \cos\theta_2 \cdot \sin\theta_2\right) \cdot \frac{1}{\sin\theta_1 \cos\theta_1 \sin\theta_2 \cos\theta_2}$$

$$= r^2 \sin\theta_1 .$$

☆**Ex. 1.** Evaluate: $\iiint\limits_V dx\,dy\,dz$, where V is the sphere $x^2+y^2+z^2=1$.

Also evaluate $\iiint\limits_V x^2\,dx\,dy\,dz$.

Soln →

Let, $x = r\cos\theta_1$

$y = r\sin\theta_1 \cos\theta_2$

$z = r\sin\theta_1 \sin\theta_2$

As, $x^2+y^2+z^2 \le 1$.

$0 < r < 1$, $0 < \theta_1 < \pi$, $0 < \theta_2 \le 2\pi$

Here, $J = r^2 \sin\theta_1$

Hence, $$\iiint\limits_V dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^2 \sin\theta_1\, dr\, d\theta_1\, d\theta_2$$

$$= \left(\int_0^1 r^2 dr\right)\left(\int_0^{\pi} \sin\theta_1 d\theta_1\right)\left(\int_0^{2\pi} d\theta_2\right)$$

$$= \frac{1}{3}\left[-\cos\theta_1\right]_0^{\pi} \cdot 2\pi$$

$$= \frac{4\pi}{3}.$$

Now, $$\iiint\limits_V x^2\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^2\cos^2\theta_1 \cdot r^2 \sin\theta_1\, dr\, d\theta_1\, d\theta_2$$

$$= \left(\int_0^1 r^4 dr\right)\left(\int_0^{\pi} \cos^2\theta_1 \sin\theta_1 d\theta_1\right)\left(\int_0^{2\pi} d\theta_2\right)$$

$$= \frac{1}{5}\left\{-\int_1^{-1} z^2 dz\right\} \cdot 2\pi \qquad [\text{Let } z = \cos\theta_1]$$

$$= \frac{1}{5}\left[\frac{z^3}{3}\right]_{-1}^{1} \cdot 2\pi$$

$$= \frac{4\pi}{15}.$$

☆ **Ex. 2.** Evaluate: $\iiint\limits_D \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,dx\,dy\,dz$,

where D is the region bounded by $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$.

Soln. → Here, $D=\left\{(x,y,z): \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\right\}$

Let, $\frac{x}{a}=r\cos\theta_1$,

$\frac{y}{b}=r\sin\theta_1\cos\theta_2$

$\frac{z}{c}=r\sin\theta_1\sin\theta_2$

As, $(x,y,z)\in D$,

$0<r<1,\quad 0<\theta_1<\pi,\quad 0<\theta_2<2\pi$

Here the jacobian of the transformation

$(x,y,z)\longrightarrow(r,\theta_1,\theta_2)$ is

$$J=\frac{\partial(x,y,z)}{\partial(r,\theta_1,\theta_2)}$$

$$=abc.\,r^2\sin\theta_1$$

Hence, $\iiint\limits_D \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,dx\,dy\,dz$

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^{1}\sqrt{1-r^2}\,(abc\,r^2\sin\theta_1)\,dr\,d\theta_1\,d\theta_2$$

$$=(abc)\left(\int_0^1 r^2\sqrt{1-r^2}\,dr\right)\left(\int_0^{\pi}\sin\theta_1\,d\theta_1\right)\left(\int_0^{2\pi}d\theta_2\right)$$

$$=(abc).\left\{\int_0^1 (r^2)^{1/2}(1-r^2)^{1/2}\,\frac{dr^2}{2}\right\}.[\cos\theta_1]_0^{\pi}.2\pi$$

$$=2\pi(abc).\int_0^1 z^{3/2-1}(1-z)^{3/2-1}dz \quad , \text{ where } z=r^2.$$

$$=2\pi(abc).\,\beta(3/2,3/2)$$

$$=2\pi(abc).\frac{\Gamma(3/2)\,\Gamma(3/2)}{2\Gamma(3)}$$

$$=2\pi.(abc).\frac{\left(\frac{\sqrt{\pi}}{2}\right)^2}{2}$$

$$=\frac{\pi^2(abc)}{4}$$

☆**Ex. 3.** Evaluate $\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$, integral being extended to all +ve values of x, y, z for which the integrand is defined. [C.U.]

Soln.→ Domain $= \{(x,y,z) : x, y, z > 0 \text{ and } x^2+y^2+z^2 < 1\}$

Let, $x = r\cos\theta_1$

$y = r\sin\theta_1\cos\theta_2$

$z = r\sin\theta_1\sin\theta_2$

Transformed domain is $\{(r, \theta_1, \theta_2) : 0 < r < 1,\ 0 < \theta_1 < \pi,\ 0 < \theta_2 < 2\pi\}$

☆**Ex. 4.** Evaluate: $\iiint\limits_D x^{l-1} y^{m-1} z^{n-1} (1-x-y-z)^{p-1}\, dx\,dy\,dz$

where D is the region bounded by

$x=0,\ y=0,\ z=0$, and $x+y+z=1$.

Soln. →

$$\left.\begin{aligned} x+y+z &= u \\ x+y &= vu \\ z &= uvw. \end{aligned}\ \right|\ \left.\begin{aligned} \frac{x}{x+y} &= u \\ \frac{x+y}{x+y+z} &= v \\ x+y+z &= w \end{aligned}\right\} \Rightarrow \begin{aligned} x &= uvw \\ y &= vw(1-u) \\ z &= w(1-v) \end{aligned}$$

As $(x,y,z) \in D$, $0<u<1$, $0<v<1$ and $0<w<1$.

the transformed domain is $E = \{(u,v,w) : 0<u,v,w<1\}$

$$|J| = \begin{vmatrix} vw & uw & uv \\ -vw & w(1-u) & v(1-u) \\ 0 & -w & 1-v \end{vmatrix} = 1 \begin{vmatrix} -vw & w(1-u) \\ 0 & -w \end{vmatrix}$$

$$= vw^2.$$

Hence,

$$\iiint\limits_D x^{l-1} y^{m-1} z^{n-1} (1-x-y-z)^{p-1}\, dx\,dy\,dz$$

$$= \iiint\limits_E (uvw)^{l-1} \{vw(1-u)\}^{m-1} \{w(1-v)\}^{n-1} (1-w)^{p-1} |vw^2|\, du\,dv\,dw$$

$$= \left(\int_0^1 u^{l-1}(1-u)^{m-1}\,du\right)\left(\int_0^1 v^{l+m-1}(1-v)^{n-1}\,dv\right)\left(\int_0^1 w^{l+m+n-1}(1-w)^{p-1}\,dw\right)$$

$$= B(l,m)\, B(l+m,n)\, B(l+m+n,p)$$

$$= \frac{\Gamma l\, \Gamma m}{\cancel{\Gamma l+m}} \cdot \frac{\cancel{\Gamma l+m}\, \Gamma n}{\cancel{\Gamma l+m+n}} \cdot \frac{\cancel{\Gamma l+m+n}\, \Gamma p}{\Gamma l+m+n+p}$$

$$= \frac{\Gamma l\, \Gamma m\, \Gamma n\, \Gamma p}{\Gamma l+m+n+p}.$$

[ii] Polar transformation in $\mathbb{R}^n$:—

Let $\underset{\sim}{x} = (x_1, \ldots, x_n)$ be a point in $\mathbb{R}^n$.
Polar transformation of $(x_1, \ldots, x_n)$ to $(r, \theta_1, \ldots, \theta_{n-1})$ is

$$x_1 = r\cos\theta_1$$
$$x_2 = r\sin\theta_1 \cos\theta_2$$
$$x_3 = r\sin\theta_1 \sin\theta_2 \cos\theta_3$$
$$\vdots$$
$$x_{n-1} = r\sin\theta_1 \sin\theta_2 \cdots\cdots \sin\theta_{n-2} \cos\theta_{n-1}$$
$$x_n = r\sin\theta_1 \sin\theta_2 \cdots\cdots \sin\theta_{n-2} \sin\theta_{n-1}$$

where, $0 < r < \infty$,
$0 < \theta_i < \pi$, $i = 1(1)\overline{n-2}$, and $0 < \theta_{n-1} < 2\pi$.

It can be shown that, —

$$|J| = r^{n-1} \cdot \sin^{n-2}\theta_1 \cdot \sin^{n-3}\theta_2 \cdots\cdots \sin\theta_{n-2}.$$

☆ **Ex. 1.** Show that $\int\cdots\cdots\int_{\sum_{i=1}^{n} x_i^2 \le R^2} dx_1 dx_2 \cdots\cdots dx_n$

$$= \frac{R^n \pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

Also find,

i) $\int\cdots\cdots\int_{\sum_{i=1}^{n} x_i^2 \le R^2} x_i \, dx_1 dx_2 \cdots\cdots dx_n$

ii) $\int\cdots\cdots\int_{\sum_{i=1}^{n} x_i^2 \le R^2} x_i x_j \, dx_1 dx_2 \cdots dx_n \ , \ i \ne j$

iii) $\int\cdots\cdots\int_{\sum_{i=1}^{n} x_i^2 \le R^2} x_i^2 \, dx_1 dx_2 \cdots dx_n$

Soln. → Consider the transformation

$x_1 = r\cos\theta_1$,
$x_2 = r\sin\theta_1 \cos\theta_2$
$x_3 = r\sin\theta_1 \sin\theta_2 \cos\theta_3$

$\vdots$

$x_{n-1} = r\sin\theta_1 \sin\theta_2 \cdots\cdots \sin\theta_{n-2} \cdot \cos\theta_{n-1}$
$x_n = r\sin\theta_1 \sin\theta_2 \cdots\cdots \sin\theta_{n-2} \cdot \sin\theta_{n-1}$.

of $(x_1, x_2, \cdots, x_n)$ to $(r, \theta_1, \theta_2, \cdots\cdots, \theta_{n-1})$.

As, $\sum_{i=1}^{n} x_i^2 \le R^2$, $0 < r < R$, $0 < \theta_i < \pi$, $i = 1(1)n-2$.
$\therefore 0 < \theta_{n-1} < 2\pi$,

Here,
$|J| = r^{n-1} \cdot \sin^{n-2}\theta_1 \cdot \sin^{n-3}\theta_2 \cdots\cdots \cdot \sin\theta_{n-2}$

Then,

$$\int_{\sum_{i=1}^{n} x_i^2 \le R^2} \cdots\cdots \int dx_1 dx_2 \ldots dx_n$$

$$= \int_0^{2\pi}\int_0^{\pi} \cdots\cdots \int_0^{\pi}\int_0^{R} r^{n-1} \cdot \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \ldots \sin\theta_{n-2} \cdot dr\, d\theta_1 \ldots \ldots d\theta_{n-1}.$$

$$= \left(\int_0^R r^{n-1}\, dr\right) \cdot \left(\int_0^{\pi} \sin^{n-2}\theta_1 d\theta_1\right) \cdot \left(\int_0^{\pi} \sin^{n-3}\theta_2\, d\theta_2\right) \cdots \cdots \left(\int_0^{\pi} \sin\theta_{n-2}\, d\theta_{n-2}\right)\left(\int_0^{2\pi} d\theta_{n-1}\right).$$

$$= \frac{R^n}{n} \cdot B\left(\frac{n-1}{2}, \frac{1}{2}\right) \cdot B\left(\frac{n-2}{2}, \frac{1}{2}\right) \cdots\cdots B\left(1, \frac{1}{2}\right) \cdot (2\pi)$$

$$\left[\because \int_0^{\pi} \sin^p\theta\, d\theta = 2\int_0^{\pi/2} \sin^p\theta\, d\theta \quad, \text{as } \sin(\pi-\theta) = \sin\theta \right.$$

$$= \int_0^{\pi/2} (\sin^2\theta)^{\frac{p-1}{2}} (1-\sin^2\theta)^{-\frac{1}{2}} \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= \int_0^1 t^{\frac{p+1}{2}-1} (1-t)^{\frac{1}{2}-1} dt \,, \text{ where, } t = \sin^2\theta$$

$$\Rightarrow dt = 2\sin\theta\cos\theta\, d\theta$$

$$\left. = B\left(\frac{p+1}{2}, \frac{1}{2}\right) \right]$$

$$= \frac{R^n}{n} \cdot \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \cdot \frac{\Gamma\left(\frac{n-2}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \cdots\cdots \frac{\Gamma(1)\Gamma\left(\frac{1}{2}\right)}{\Gamma(3/2)} \cdot 2\pi$$

$$= 2\pi \cdot \frac{R^n}{n} \cdot \frac{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^{n-2}}{\Gamma\left(\frac{n}{2}\right)} = \frac{\pi^{n/2} \cdot R^n}{n/2 \cdot \Gamma\left(\frac{n}{2}\right)} = \frac{\pi^{n/2} \cdot R^n}{\Gamma\left(\frac{n}{2}+1\right)}.$$

i) $\displaystyle\int_{\sum_{i=1}^{n} x_i^2 \le R^2} \cdots \int x_i \, dx_1 \, dx_2 \cdots dx_n$

$\displaystyle = \int_{\sum_{i=1}^{n} x_i^2 \le R^2} \cdots \int x_1 \, dx_1 \, dx_2 \cdots dx_n$ since, the integral is symmetric w.r.t. the arguments $x_1, x_2, \ldots, x_n$.

$\displaystyle = \int_0^{2\pi}\int_0^{\pi} \cdots \int_0^{\pi}\int_0^{R} (r\cos\theta_1) \cdot |J| \cdot dr \cdot d\theta_1 \ldots d\theta_{n-2} \, d\theta_{n-1}$

$\displaystyle = \left(\int_0^R r^n \, dr\right) \cdot \left(\int_0^{\pi} \sin^{n-2}\theta_1 \cos\theta_1 \, d\theta_1\right)\left(\int_0^{\pi} \sin^{n-3}\theta_2 \, d\theta_2\right)$
$\displaystyle \cdots\cdots \left(\int_0^{\pi} \sin\theta_{n-2} \, d\theta_{n-2}\right) \cdot \left(\int_0^{2\pi} d\theta_{n-1}\right).$

$= 0$, since, $\displaystyle\int_0^{\pi} \sin^{n-2}\theta_1 \cos\theta_1 \, d\theta_1 = 0$ as

$\sin^{n-2}(\pi - \theta_1)\cos(\pi - \theta_1) = -\sin^{n-2}\theta_1 \cos\theta_1.$

TRIPLE INTEGRAL (Theorem) :—

If the transformation $u = \phi_1(x, y, z)$, $v = \phi_2(x, y, z)$ and $w = \phi_3(x, y, z)$ represents a continuous one-to-one transformation of the closed region D in xyz-plane on a region E in uvw-plane and if the functions ϕ_1, ϕ_2, ϕ_3 and if the functions ϕ_1, ϕ_2, ϕ_3 have continuous partial derivatives then the Jacobian is $J = \dfrac{\partial(x, y, z)}{\partial(u, v, w)}$

$$= \frac{\partial(\text{old variable})}{\partial(\text{new variable})} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

and $\displaystyle\iiint_D f(x, y, z)\,dx\,dy\,dz = \iiint_E f(\phi_1^*, \phi_2^*, \phi_3^*)\,|J|\,du\,dv\,dw,$

where
$$x = \phi_1^*(u, v, w)$$
$$y = \phi_2^*(u, v, w)$$
$$z = \phi_3^*(u, v, w).$$

☆ **Ex. 2.** (Continuation) Show that

$$\int \cdots \int_{\underset{\sim}{x}' A \underset{\sim}{x} \le R^2} dx_1 dx_2 \cdots dx_n = \frac{\pi^{n/2} R^n}{\sqrt{|A|} \cdot \Gamma\left(\frac{n}{2}+1\right)}$$, provided A is p.d.

Soln. → Since A is p.d., $\exists$ an n.s. B such that $A = B^T B$.

Then, $\underset{\sim}{x}' A \underset{\sim}{x} \le R^2$

$\Rightarrow \underset{\sim}{x}' B' B \underset{\sim}{x} \le R^2$

$\Rightarrow (B\underset{\sim}{x})'(B\underset{\sim}{x}) \le R^2$

$\Rightarrow \underset{\sim}{y}' \underset{\sim}{y} \le R^2$, where $\underset{\sim}{y} = B\underset{\sim}{x}$

Here $|J| = \text{mod}\left|\frac{\partial x}{\partial \underset{\sim}{y}}\right| = \frac{1}{\text{mod}\left|\frac{\partial \underset{\sim}{y}}{\partial \underset{\sim}{x}}\right|} = \frac{1}{\text{mod}\,|B|}$.

$$= \frac{1}{\sqrt{|B^T B|}}$$

$$= \frac{1}{\sqrt{|A|}}.$$

Hence, $\int \cdots \int_{\underset{\sim}{x}' A \underset{\sim}{x} \le R^2} dx_1 dx_2 \cdots dx_n$

$$= \int \cdots \int_{\underset{\sim}{y}' \underset{\sim}{y} \le R^2} |J|\, dy_1 dy_2 \cdots dy_n$$

$$= \int \cdots \int_{\sum_{i=1}^{n} y_i^2 \le R^2} \frac{1}{\sqrt{|A|}} \cdot dy_1 dy_2 \cdots dy_n$$

$$= \frac{1}{\sqrt{|A|}} \cdot \frac{R^n \cdot \pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}, \text{ (by Ex. 1.)}$$

☆ **Ex. 3.** [Dirichlet Integral]

Show that —

$$\int \cdots \int_D x_1^{p_1-1} x_2^{p_2-1} \cdots x_k^{p_k-1} (1-x_1-\cdots-x_k)^{p_{k+1}-1} \, dx_1 \cdots dx_k$$

$$= \frac{\Gamma(p_1) \cdots \Gamma(p_k)\Gamma(p_{k+1})}{\Gamma(p_1+\cdots+p_k+p_{k+1})}.$$

where $D = \left\{(x_1, \ldots, x_k) : x_i > 0, \forall i, \text{ and } \sum_{i=1}^{k} x_i < 1\right\}$

Soln. → Let, $x_1+x_2+\cdots+x_k = z_1$

$$x_2+\cdots+x_k = z_1 z_2$$

$$x_3+\cdots+x_k = z_1 z_2 z_3$$

$$\vdots$$

$$x_{k-1}+x_k = z_1 z_2 \cdots z_{k-1}$$

$$x_k = z_1 \cdots z_k.$$

$$\Rightarrow x_1 = z_1(1-z_2)$$

$$x_2 = z_1 z_2 (1-z_3)$$

$$\vdots$$

$$x_{k-1} = z_1 z_2 \cdots z_{k-1}(1-z_k)$$

$$x_k = z_1 z_2 \cdots z_{k-1} \cdot z_k.$$

Hence the jacobian is, —

$$J = \begin{vmatrix} 1-z_2 & -z_1 & 0 & 0 & \cdots & 0 & 0 \\ z_2(1-z_3) & z_1(1-z_3) & -z_1z_2 & 0 & & 0 & 0 \\ z_2z_3(1-z_4) & z_1z_3(1-z_4) & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & \\ z_2\cdots z_{k-1}(1-z_k) & z_1z_3\cdots z_{k-1}(1-z_k) & \vdots & \cdots & & z_1\cdots z_{k-2}(1-z_k) & -z_1\cdots z_{k-1} \\ z_2z_3\cdots z_k & z_1 z_3 \cdots z_k & \vdots & \cdots & & z_1\cdots z_{k-2} z_k & z_1\cdots z_{k-1} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ z_2(1-z_3) & z_1(1-z_3) & & & \vdots \\ z_2z_3(1-z_4) & z_1z_3(1-z_4) & \cdots & & \vdots \\ \vdots & \vdots & & & \vdots \\ z_2\cdots z_{k-1}(1-z_k) & \vdots & \cdots & & \vdots \\ z_2\cdots z_k & & & & \end{vmatrix}_{k\times k} \quad R_1' \to \sum_{i=1}^{n} R_i$$

$$= \begin{vmatrix} z_1(1-z_3) & -z_1z_2 & 0 & \cdots & 0 & 0 \\ z_1z_3(1-z_4) & z_1z_2(1-z_4) & -z_1z_2z_3 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & \\ z_1z_3\cdots z_{k-1}(1-z_k) & \vdots & & & & \\ z_1z_3\cdots z_k & & & & & \end{vmatrix}_{\overline{k-1}\times\overline{k-1}}$$

$$= z_1^{k-1} \begin{vmatrix} 1-z_3 & -z_2 & 0 & \cdots & 0 & 0 \\ z_3(1-z_4) & z_2(1-z_4) & -z_2z_3 & & & \\ \vdots & \vdots & \vdots & & & \\ z_3\cdots z_{k-1}(1-z_k) & & \cdots & & z_2\cdots z_{k-2}(1-z_k) & -z_2\cdots z_{k-1} \end{vmatrix}$$

$$= z_1^{k-1}\cdot z_2^{k-2}\cdots\cdots \tilde{z}_{k-2} \begin{vmatrix} 1-z_k & -z_{k-1} \\ z_k & z_{k-1} \end{vmatrix}$$

$$= z_1^{k-1} z_2^{k-2}\cdots\cdots \tilde{z}_{k-2}\, z_{k-1}.$$

As, $(x_1, x_2, \cdots, x_n) \in D$, $0 < x_i$ and

$$\sum_{i=1}^{k} x_i < 1$$

$\Rightarrow 0 < \sum_i x_i < 1$, $i = 1(1)k$.

Then;

$$I = \int \cdots \int \left\{z_1(1-z_2)\right\}^{p_1-1} \cdot \left\{z_1 z_2 (1-z_3)\right\}^{p_2-1} \cdots \cdots$$

$$\cdot \left\{z_1 \cdots z_k\right\}^{p_k-1} \cdot (1-z_1)^{p_{k+1}-1} \times z_1^{k-1} \cdot z_2^{k-2} \cdots z_{k-2} z_{k-1}$$

$$dz_1 dz_2 \cdots dz_k$$

$$= \left\{\int_0^1 z_1^{\sum_{i=1}^{k} p_i - 1} (1-z_1)^{p_{k+1}-1} dz_1\right\} \left\{\int_0^1 z_2^{\sum_{i=2}^{k} p_i - 1} (1-z_2)^{p_1-1} dz_2\right\}$$

$$\cdots \cdots \left\{\int_0^1 z_k^{p_k-1} (1-z_k)^{p_{k-1}-1} dz_k\right\}$$

$$= \beta\left(\sum_{i=1}^{k} p_i, p_{k+1}\right) \beta\left(\sum_{i=2}^{k} p_i, p_1\right) \cdots \beta\left(p_k, p_{k-1}\right)$$

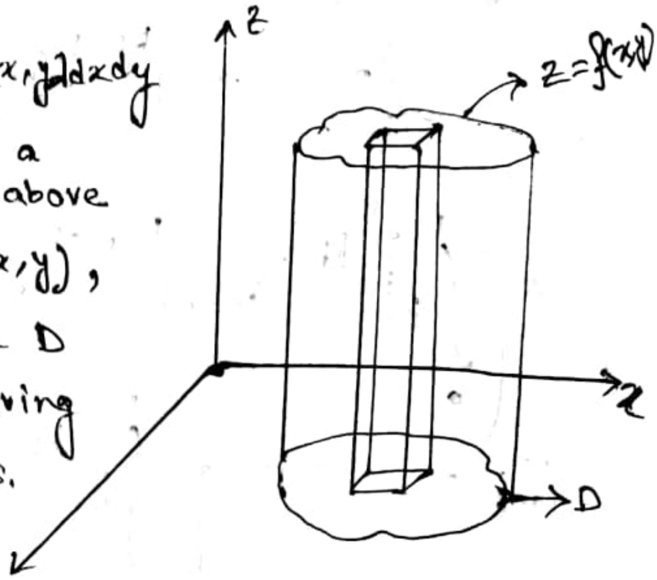
$$= \frac{\Gamma(p_1) \cdots \Gamma(p_k)\,\Gamma(p_{k+1})}{\Gamma\left(\sum_{i=1}^{k+1} p_i\right)}.$$

[Proved]

Geometric significance of Double and Triple Integrals:

[A]. Double Integral:—

(a) The double integral $\iint_D f(x,y)\,dx\,dy$ represents the volume of a cylindrical solid bounded above by the surface $z = f(x,y)$, below by the plane base D on the xy-plane and having sides parallel to z-axis.

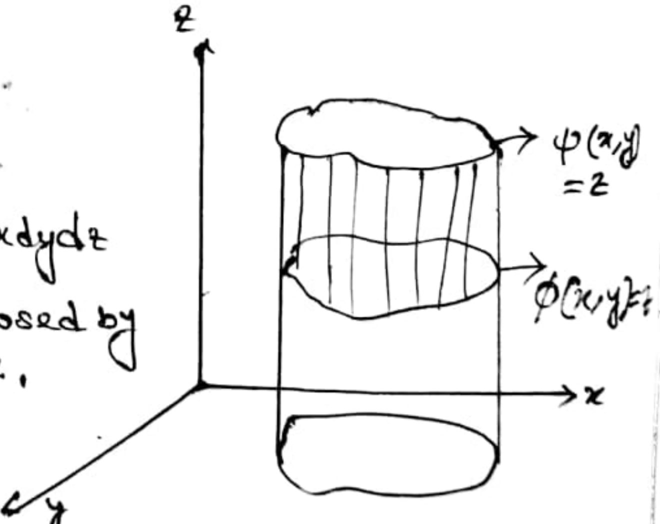


[The projection of $z = f(x,y)$]

(b) The double integral $\iint_D dx\,dy$ represents the area enclosed by the region D in xy-plane.

[B.] Triple Integral :—

(a) The triple integral $\iiint_E dx\,dy\,dz$ represents the volume enclosed by the solid E in xyz-plane.



(b) $\iiint_E dx\,dy\,dz$, where

$E = \{(x,y,z) : \phi(x,y) \le z \le \varphi(x,y),\ (x,y) \in D \subseteq \mathbb{R}^2\}$

represents the volume of the solid E.

☆ Exercises:—

1. Evaluate $\iiint\limits_{x^2+y^2+z^2\le 1} dx\,dy\,dz$, geometrically.

2. Compute the volume of the ellipsoid: → $\left[\iiint\limits_D dx\,dy\,dz,\right.$ where $D=\{(x,y,z):$ $\left.\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\}\right]$
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, using triple integral.

3. Show that the volume enclosed by the solid ;
$$\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}+\left(\frac{z}{c}\right)^{2/3}=1 \text{ is } \frac{4\pi abc}{35}.$$

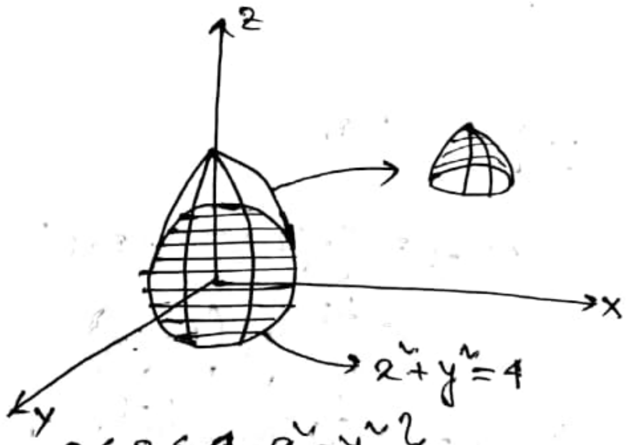
4. Find the volume of the parabolid $z=4-x^2-y^2$ and above the (x,y) plane.

5. Compute the volume of the solid bounded by the sphere $x^2+y^2+z^2=4$ and the surface of the paraboloid : $x^2+y^2=3z$.

Ans of Ques. 4.

Hints:-

Required volume:—



$=\iiint\limits_D dx\,dy\,dz$, where,

$D=\{(x,y,z): x^2+y^2\le 4,\ 0\le z\le 4-x^2-y^2\}$

$$=\iint\limits_{x^2+y^2\le 4}\left(\int_0^{4-x^2-y^2} dz\right)dx\,dy = \iint\limits_{x^2+y^2\le 4}(4-x^2-y^2)\,dx\,dy$$

$$=\int_0^{2\pi}\left(\int_0^2 (4-r^2)\cdot r\,dr\right)d\theta$$

| $x=r\cos\theta$, $y=r\sin\theta$, $0<r<2$, $0<\theta<2\pi$

$$=\left[2r^2-\frac{r^4}{4}\right]_0^2\left[\theta\right]_0^{2\pi}$$

$=(8-4)\cdot 2\pi$

$=8\pi$. (ANS)

SOME MULTIPLE INTEGRAL

1) $$\underset{\substack{x_i > 0 \, \forall i \\ \sum_i x_i \le 1}}{\int \cdots \int} x_1^{l_1 - 1} x_2^{l_2 - 1} \cdots x_n^{l_n - 1} \left(1 - \sum_{i=1}^{n} x_i\right)^{l_{n+1} - 1} dx_1 \cdots dx_n$$

$$= \frac{\prod_{i=1}^{n+1} \Gamma l_i}{\Gamma \sum_{i=1}^{n+1} l_i}$$

Soln → Consider the transformation

$(x_1, x_2, \dots, x_n) \longrightarrow (y_1, y_2, \dots, y_n)$ such that,

$$y_1 = x_1 + x_2 + \dots + x_n$$

$$y_1 y_2 = x_2 + \dots + x_n$$

$$y_1 y_2 y_3 = x_3 + \dots + x_n$$

$$\vdots$$

$$\prod_{i=1}^{n} y_i = x_n$$

$$\therefore x_{n-1} = y_1 y_2 \cdots y_{n-1} - y_1 y_2 \cdots y_n$$

$$= \prod_{i=1}^{n-1} y_i (1 - y_n)$$

$$\therefore x_{n-2} = \prod_{i=1}^{n-2} y_i - \prod_{i=1}^{n-1} y_i (1 - y_n) - \prod_{i=1}^{n} y_i$$

$$= \prod_{i=1}^{n-2} y_i (1 - y_n)(1 - y_{n-1}) - \prod_{i=1}^{n} y_i$$

$$= y_1 y_2 \cdots y_{n-2} (1 - y_n)(1 - y_{n-1}) - y_1 \cdots y_n$$

$$= y_1 \cdots y_{n-2} \left[(1 - y_n)(1 - y_{n-1}) - y_{n-1} y_n\right]$$

$$= \prod_{i=1}^{n-2} y_i \left\{(1 - y_n)(1 - y_{n-1}) - y_{n-1} y_n\right\}$$

$$= \prod_{i=1}^{n-2} y_i (1 - y_n - y_{n-1})$$

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$\vdots$

$x_2 = y_1 y_2 (1-y_3)$

$x_1 = y_1 (1-y_2)$

Jacobian of the transformation

$Z_1 = y_1 \quad \Rightarrow \quad x_1 = Z_1 - Z_2$

$Z_2 = y_1 y_2 \quad \Rightarrow \quad x_2 = Z_2 - Z_3$

$Z_3 = y_1 y_2 y_3$

$\vdots \qquad\qquad \vdots$

$Z_n = \prod_{i=1}^{n} y_i \qquad x_n = Z_n$

$$J\left(\frac{\underset{\sim}{x}}{\underset{\sim}{y}}\right) = J\left(\frac{\underset{\sim}{y}}{\underset{\sim}{Z}}\right) J\left(\frac{\underset{\sim}{Z}}{\underset{\sim}{x}}\right)$$

$$J\left(\frac{\underset{\sim}{Z}}{\underset{\sim}{x}}\right) = \begin{vmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & \\ & & & & \\ \cdots & & & & 1 \end{vmatrix} = 1$$

$$J\left(\frac{\underset{\sim}{y}}{\underset{\sim}{Z}}\right) = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ y_2 & y_1 & \cdots & 0 \\ y_2 y_3 & y_1 y_3 & \cdots & \\ \vdots & \vdots & & \\ y_2 \cdots y_n & y_1 y_3 \cdots y_n & \cdots & \end{vmatrix} = y_1^{n-1} y_2^{n-2} \cdots y_{n-1}$$

$$\therefore I = \int_0^1 \cdots\cdots \int_0^1 y_1^{l_1-1}(1-y_2)^{l_1-1} \cdots\cdots (\Pi y_i)^{l_n-1} (1-y_1)^{l_{n+1}-1} y_1^{n-1} y_2^{n-2} \cdots y_{n-1} \, dy_1 \, dy_2 \cdots dy_n$$

$$= \int_0^1 y_1^{\sum_{i=1}^{n} l_i - 1} (1-y_1)^{l_{n+1}-1} dy_1 \int_0^1 y_2^{\sum_{i=2}^{n} l_i - 1} (1-y_2)^{l_1-1} dy_2 \cdots\cdots \int_0^1 y_n^{l_n-1} (1-y_n)^{l_{n-1}-1} dy_n$$

$$= \frac{\Gamma\left(\sum_{i=1}^{n} l_i\right) \Gamma(l_{n+1})}{\Gamma\left(\sum_{i=1}^{n+1} l_i\right)} \cdot \frac{\Gamma\left(\sum_{i=2}^{n} l_i\right) \Gamma(l_1)}{\Gamma\left(\sum_{i=1}^{n} l_i\right)} \cdots\cdots \frac{\Gamma(l_n)\Gamma(l_{n-1})}{\Gamma(l_n + l_{n-1})}$$

$$= \frac{\prod_{i=1}^{n+1} \left\{ \Gamma(l_i) \right\}}{\Gamma\left(\sum_{i=1}^{n+1} l_i\right)},$$

2) $$\underset{\substack{x_i \geq 0\ \forall i \\ \sum_{i=1}^{n} x_i^2 \leq 1}}{\int \cdots\cdots \int} \frac{dx_1\, dx_2 \cdots dx_n}{\left(1 - \sum_{i=1}^{n} x_i^2\right)^{1/2}} = \frac{\pi^{\frac{n+1}{2}}}{2^n \Gamma\frac{n+1}{2}}$$

Soln. → Let us consider the tramformation,

$Y_i = X_i^2$, $i = 1, 2, \ldots, n$

$Y_1 = X_1^2$

$Y_2 = X_2^2$

$\vdots$

$Y_n = X_n^2$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots\cdots & \frac{\partial x_n}{\partial y_n} \\ \vdots & & & \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots\cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$

$$= \left[\begin{vmatrix} 2X_1 & 0 & \cdots\cdots & 0 \\ 0 & 2X_2 & \cdots & 0 \\ 0 & \cdots & \cdots & 2X_n \end{vmatrix} \right]^{-1}$$

$$= \frac{1}{2^n \prod_{i=1}^{n} X_i} = \frac{1}{2^n \prod_{i=1}^{n} \sqrt{Y_i}}$$

$$\underset{\substack{y_i > 0 \\ \sum y_i \leq 1}}{\int \cdots\cdots \int} \frac{dy_1\, dy_2 \cdots dy_n}{\left(1 - \sum y_i\right)^{1/2} 2^n \prod_{i=1}^{n} \sqrt{Y_i}}$$

$$= \frac{1}{2^n} \underset{\substack{y_i > 0 \\ \sum y_i \leq 1}}{\int \cdots\cdots \int} y_1^{\frac{1}{2}-1} y_2^{\frac{1}{2}-1} \cdots y_n^{\frac{1}{2}-1} \left(1 - \sum_{i=1}^{n} y_i\right)^{1/2-1}$$

$$= \frac{1}{2^n} \cdot \frac{\left\{\Gamma\frac{1}{2}\right\}^{n+1}}{\Gamma\frac{n+1}{2}}$$

3) $$\int \cdots \int x_1^{l_1-1} x_2^{l_2-1} \cdots x_n^{l_n-1}\, dx_1\, dx_2 \cdots dx_n$$

$x_i \geq 0\ \forall i$

$0 < \sum_{i=1}^{n} \frac{x_i}{a_i} < 1$

$$= \frac{\prod_{i=1}^{n} a_i^{l_i} \prod_{i=1}^{n} \Gamma l_i}{\Gamma \left(\sum_{i=1}^{n} l_i + 1 \right)}$$

Solution :-

Consider the following transformation

$(x_1, x_2, \ldots, x_n) \longrightarrow (y_1, y_2, \ldots, y_n)$

$$y_1 = \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n}$$

$$y_1 y_2 = \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n}$$

$$y_1 y_2 y_3 = \frac{x_3}{a_3} + \cdots + \frac{x_n}{a_n}$$

$\vdots$

$$y_1 y_2 \cdots y_n = \frac{x_n}{a_n}$$

$$x_n = a_n y_1 y_2 \cdots y_n$$

$$\frac{x_n}{a_n} + \frac{x_{n-1}}{a_{n-1}} = y_1 y_2 \cdots y_{n-1}$$

$$\Rightarrow \frac{x_{n-1}}{a_{n-1}} = y_1 y_2 \cdots y_{n-1} - y_1 y_2 \cdots y_n$$

$$= \prod_{i=1}^{n-1} y_i (1 - y_n)$$

$$\Rightarrow x_{n-1} = a_{n-1} \prod_{i=1}^{n-1} y_i (1 - y_n)$$

$\vdots$

$$x_2 = a_2 y_1 y_2 (1 - y_3)$$

$$x_1 = a_1 y_1 (1 - y_2)$$

$$Z_1 = y_1 \Rightarrow x_1 = a_1(Z_1 - Z_2)$$

$$Z_2 = y_1 y_2 \Rightarrow x_2 = a_2(z_2 - z_3)$$

$$Z_3 = y_1 y_2 y_3 \Rightarrow x_3 = a_3(Z_3 - Z_4)$$

$$\vdots$$

$$Z_n = y_1 y_2 \cdots y_n \Rightarrow x_n = a_n z_n$$

$$J\left(\frac{\underset{\sim}{x}}{\underset{\sim}{y}}\right) = y_1^{n-1} y_2^{n-2} \cdots y_{n-1} \prod_{i=1}^{n} a_i$$

$$\int_0^1 \cdots \cdots \int_0^1 a_1^{l_1-1} y_1^{l_1-1} (1-y_2)^{l_1-1} a_2^{l_2-2} y_2^{l_2-2} y_1^{l_2-2} (1-y_3)^{l_2-2} \cdots a_n^{l_n-1} (y_1 y_2 \cdots y_n)^{l_n-1} dy_1 dy_2 \cdots dy_n$$

$$= \prod_{i=1}^{n} a_i^{l_i} \int_0^1 y_1^{\sum_{i=1}^{n-1} l_i - 1} (1-y_1)^{1-1} \int_0^1 y_2^{\sum_{i=2}^{n-1} l_i - 1} (1-y_2)^{l_1-1} \cdots \int_0^1 y_n^{\Sigma} (1-y_n)^{l_{n-1}-1}$$

$$= \prod_{i=1}^{n} a_i^{l_i} \frac{\Gamma \sum_{i=1}^{n-1} l_i}{\Gamma \sum_{i=1}^{n-1} l_i + 1}$$

$$= \frac{\prod_{i=1}^{n} a_i^{l_i} \prod_{i=1}^{n} \Gamma l_i}{\Gamma \sum_{i=1}^{n} l_i + 1}$$

4) $$\underset{\sum x_i^2 \le 1}{\iint \cdots \int} x_1^2 \, dx_1 \cdots dx_n = \frac{\pi^{n/2}}{(n+2)\Gamma\left(\frac{n}{2}+1\right)}.$$

Solution:- Let us consider the following transformation

$$(x_1, x_2, \ldots, x_n) \longrightarrow (y_1, y_2, \ldots, y_n)$$

$$|J| = r^{n-1} \sin^{n-2}\theta_1 \cdots \sin\theta_{n-2}.$$

$$\therefore \int_0^1 \int_0^{\pi/2} \cdots \int_0^{\pi/2} r^{n+1} \cos^2\theta_1 \sin^{n-2}\theta_1 \cdots \sin\theta_{n-2}$$

$$= \frac{r^{n+2}}{n+2} \int_0^1 \int_0^{\pi/2} \sin\theta_1^{n-2} \cos^2\theta_1 \int_0^{\pi/2} \sin^{n-3}\theta_2 \cos^{1-1}\theta_2 \cdots \int_0^{2\pi} d\theta_{n-1}$$

$$= \frac{2\pi}{n+2} \cdot \frac{\Gamma\frac{n-1}{2} \cdot \frac{1}{2} \cdot \Gamma\frac{1}{2}}{\Gamma\left(\frac{n}{2} - \frac{1}{2} + \frac{3}{2}\right)}$$

$$= \frac{\pi^{n/2}}{(n+2)\Gamma\left(\frac{n}{2}+1\right)}$$

5) $$\underset{\sum_{i=1}^{n} x_i \le 1}{\int \cdots \int} dx_1 \cdots dx_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$$

Solution:-

Consider the transformation

$$(X_1, \ldots, X_n) \to (r, \theta_1, \ldots, \theta_{n-1})$$

$X_1 = r\cos\theta_1$

$X_2 = r\sin\theta_1\cos\theta_2$

$\vdots$

$X_{n-1} = r\sin\theta_1 \cdots \sin\theta_{n-2}\cos\theta_{n-1}$

$X_n = r\sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-1}.$

$r^2 = \sum x_i^2$

$\therefore \; 0 < r < 1$

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$$|J| = r^{n-1} \sin^{n-2}\theta_1 \cdots\cdots \sin\theta_{n-2}$$

$$I = \int_0^1 \int_0^\pi \cdots\cdots \int_0^{2\pi} r^{n-1} \sin^{n-2}\theta_1 \cdots\cdots \sin\theta_{n-2}$$

$$= \int_0^1 r^{n-1}\, dr \; 2\int_0^{\pi/2} \sin^{n-2}\theta_1 \, d\theta_1 \cdots\cdots 2\int_0^{\pi/2} \sin\theta_{n-2}\, d\theta_{n-2} \int_0^{2\pi} d\theta_{n-1}.$$

$$= \frac{r^n}{n}\Big|_0^1 \; B\left(\frac{n-1}{2}, \frac{1}{2}\right) B\left(\frac{n-2}{2}, \frac{1}{2}\right) \cdots\cdots B\left(\frac{1}{2}, \frac{1}{2}\right) 2\pi$$

$$= \frac{2\pi}{n} \cdot \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{1}{2}\right) \cdots\cdots \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{2}-\frac{1}{2}\right) \cdots\cdots \Gamma(1)}$$

$$= \frac{2\pi^{n/2}}{n\,\Gamma\left(\frac{n}{2}\right)}$$

$$= \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

Ques:-

1. Evaluate the double integral $\iint_R \sin(x+y)\,dx\,dy$ over $R = \{0 \le x \le \frac{\pi}{2},\ 0 \le y \le \frac{\pi}{2}\}$ [C.U.] (5)

ANS:- $\int_0^{\pi/2}\int_0^{\pi/2} \sin(x+y)\,dx\,dy$

$= \int_0^{\pi/2} dx \int_0^{\pi/2} \sin(x+y)\,dy$

$= -\int_0^{\pi/2} dx \left[\cos(x+y)\right]_0^{\pi/2}$

$= -\int_0^{\pi/2} \left\{\cos\left(\frac{\pi}{2}+x\right) - \cos x\right\} dx$

$= -\int_0^{\pi/2} (-\sin x - \cos x)\,dx$

$= \int_0^{\pi/2} (\sin x + \cos x)\,dx = \int_0^{\pi/2} \sin x\,dx + \int_0^{\pi/2} \cos x\,dx$

$= 1+1$
$= 2$

2. Evaluate:- $\int_0^{\pi/2}\int_0^{\pi/2} \cos(x+y)\,dx\,dy$ (5)

ANS:- $\int_0^{\pi/2} dx \int_0^{\pi/2} \cos(x+y)\,dy$

$= \int_0^{\pi/2} \left[\sin(x+y)\right]_0^{\pi/2} dx$

$= \int_0^{\pi/2} \left[\sin\left(\frac{\pi}{2}+x\right) - \sin x\right] dx$

$= \int_0^{\pi/2} [\cos x - \sin x]\,dx$

$= \left[\sin x + \cos x\right]_0^{\pi/2}$

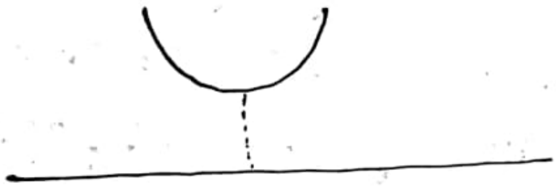
$= 1 - 1$

$= 0.$

MAXIMA & MINIMA

Theorem:— (1st Derivative Test):—

If $f'(x) < 0$ for $a-\delta < x < a$ and $f'(x) > 0$ for $a < x < a+\delta$ for some $\delta > 0$, then $f(a)$ is local minimum.



Proof:- Lagrange mean-value theorem states that — If $f(x)$ is continuous on $[a,b]$ and differentiable in (a,b), then $\dfrac{f(b)-f(a)}{b-a} = f'(\xi)$, where $\xi \in (a,b)$.

For $x \in (a-\delta, a)$

$$\frac{f(x)-f(a)}{x-a} = f'(\xi) < 0$$

$\Rightarrow f(x) - f(a) = (x-a) f'(\xi)$ as $x < a$ and $f'(\xi) < 0$.

For $x \in (a, a+\delta)$

$$\frac{f(x)-f(a)}{x-a} = f'(\xi) > 0$$

$\Rightarrow f(x) - f(a) = (x-a) f'(\xi)$

Hence for all $x \in (a-\delta, a+\delta)$, $f(x) \geq f(a)$

$\Rightarrow f(a)$ is a local minimum.

Theorem:- (2nd derivative test) :- If $f'(a)=0$ and $f''(a)<0$ then $f(a)$ is the local maximum of $f(x)$.

Proof:- By Taylor's theorem,

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(\xi) \; ; \xi \text{ lies between } a \,\&\, x.$$

Here $f'(a)=0$ and $f''(a)<0$.

Then $f(x)-f(a)=\frac{(x-a)^2}{2!}f''(\xi)<0$

assuming $f(x)$ is continuous at $x=a$.

For $x \in (a-\delta, a+\delta)$

$\Rightarrow f(x) \leq f(a) \;\forall\; x \in (a-\delta, a+\delta)$

$\Rightarrow f(a)$ is local maximum.

Ques:- Evaluate the double integral $\int_0^2 \int_0^2 [x+y]\,dx\,dy$

Solution:- $\int_0^2 \int_0^2 [x+y]\,dx\,dy$

$$= \underset{0<(x+y)<1}{\int_0^2 \int_0^2} [x+y]\,dx\,dy + \underset{1<(x+y)<2}{\int_0^2 \int_0^2} [x+y]\,dx\,dy + \underset{2<(x+y)<3}{\int_0^2 \int_0^2} [x+y]\,dx\,dy$$

$$+ \underset{3<(x+y)<4}{\int_0^2 \int_0^2} [x+y]\,dx\,dy$$

$$= 0 + \int_0^2 \int_0^2 dx\,dy + 2\int_0^2 \int_0^2 dx\,dy + 3\int_0^2 \int_0^2 dx\,dy$$

$$= 6\int_0^2 \int_0^2 dx\,dy$$

$$= 24.$$

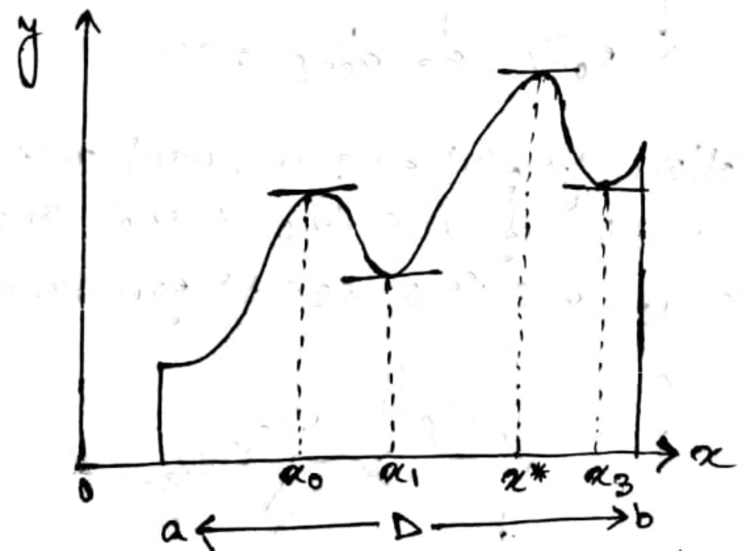
Extrema of Functions of single variable :—

Let $Y = f(x)$ be defined on D, domain.

Definition: a) A function $y = f(x)$ is maximum at 'a' $\in D$ if $f(x) \leq f(a)$, for all $x \in D$. A function $y = f(x)$ is minimum at 'a' $\in D$ if $f(x) \geq f(a)$ for all $x \in D$.
Here $f(a)$ is the global or absolute maximum or minimum at $y = f(x)$ on D.

Definition: (b) A function $y = f(x)$ has a local or relative maximum or minimum at x_0 if there exists a neighbourhood.
$N_\delta(x_0) = \{x : |x - x_0| < \delta\}$ of x_0 such that $f(x) \leq f(x_0)$
$\forall \; x \in N_\delta(x_0)$
$[\text{or}, \; f(x) \geq f(x_0) \;\; \forall \; x \in N_\delta(x_0)]$

Consider a graph of a function $y = f(x)$:

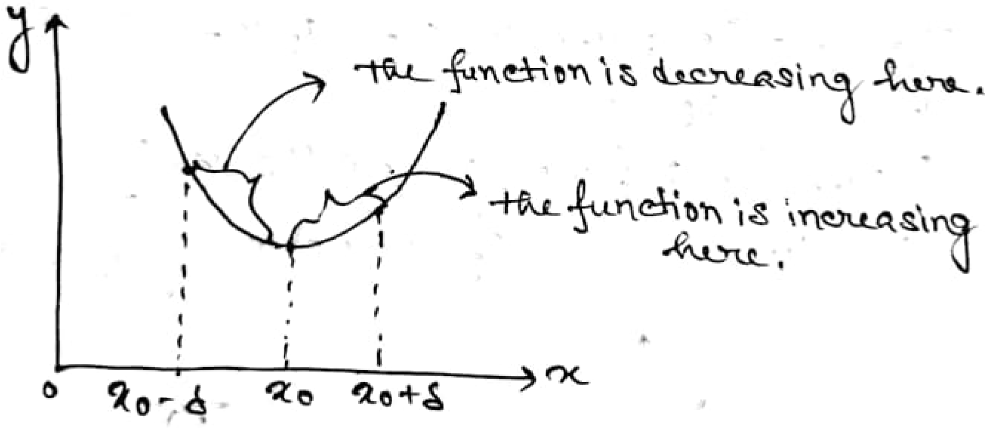


From graph $f(a)$ is the smallest / global minimum of $f(x)$ on D and $f(x^*)$ is the largest value of $f(x)$ on D.
Also, note that $f(x_0), f(x^*)$ are the local minimum and $f(x_1), f(x_3)$ are the local minimum of $f(x)$.
Here a local maximum $f(x_3)$ is larger than a local maximum $f(x_0)$.

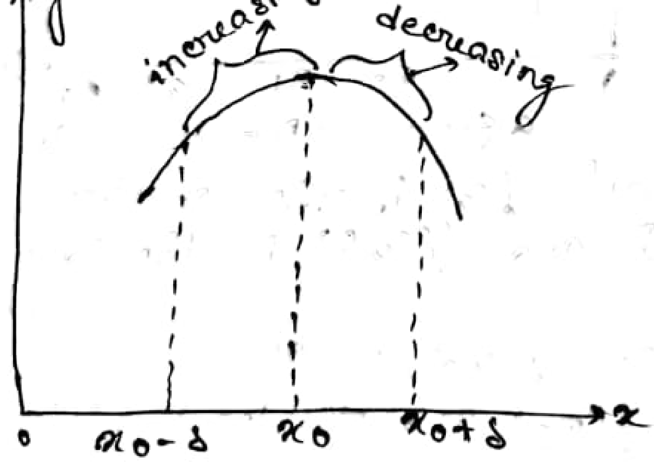
▨ A necessary condition for an extremum of a differentiable function $f(x)$ at x_0 is $f'(x_0)=0$. From graph, at the points of local maximum or minimum, the tangents of the $y=f(x)$ is parallel to x-axis. Then at the points of local extremum we must have $f'(x)=0$.

Tests for extremum :-

(i) First derivative test :- a) A function $y=f(x)$ has a local minimum at $x=x_0$ if $f'(x)<0$, if $x\in(x_0-\delta, x_0)$ and $f'(x)>0$, if $x\in(x_0, x_0+\delta)$ for some $\delta>0$.



b) A function $y=f(x)$ has a local maximum at $x=x_0$ if $f'(x)>0$ if $x\in(x_0-\delta, x_0)$ and $f'(x)<0$ if $x\in(x_0, x_0+\delta)$ for some $\delta>0$.



(ii) Second derivative test :-

(a) A function $y=f(x)$ has a local minimum at $x=x_0$ if $f'(x_0)=0$ and $f''(x_0)>0$.

(b) A function $y=f(x)$ has a local maximum at $x=x_0$ if $f'(x_0)=0$ and $f''(x_0)<0$.

Example:-

1. Find the max. or min. values of $y=f(x)$ over D :-
$y=|x|$, $x\in R=D$.

Soln. →
$$y=|x|=\begin{cases} x, & \text{if } x\geq 0 \\ -x, & \text{if } x<0 \end{cases}$$
$$\frac{dy}{dx}=\begin{cases} 1 & \text{if } x\geq 0 \\ -1 & \text{if } x<0 \end{cases}$$

N.T. when $x<0$, $\frac{dy}{dx}<0$ and when $x>0$, $\frac{dy}{dx}>0$.

Hence by first derivative test, $y=|x|$ has minimum at $x=0$.

2. Find the max. or min. value of $y=3-2(x-2)^{2/3}$, $x\in R=D$.

Soln :→
$$y=3-2(x-2)^{2/3}$$
$$\frac{dy}{dx}=-\frac{4}{3}\cdot\frac{1}{\sqrt[3]{x-2}}.$$

Note that, $\frac{dy}{dx}$ does not exist at $x=2$. Hence, $x=2$ may be a point of extremum, when $x\in(2-\delta,2)$, then
$$\frac{dy}{dx}>0$$
and when $x\in(2,2+\delta)$, $\frac{dy}{dx}<0$.

By 1st derivative test, y is maximum at $x=2$.

3. Find the extremum of the function
$$f(x) = 5 - 3(x-2)^{2/3}.$$

Solution:- $f(x) = 5 - 3(x-2)^{2/3}$

$$f'(x) = -3.\left(\frac{2}{3}\right)(x-2)^{-1/3}$$
$$= -\frac{2}{\sqrt[3]{x-2}}$$

$\therefore f'(x)$ does not exist.

$\Rightarrow x = 2$ is a stationary point.

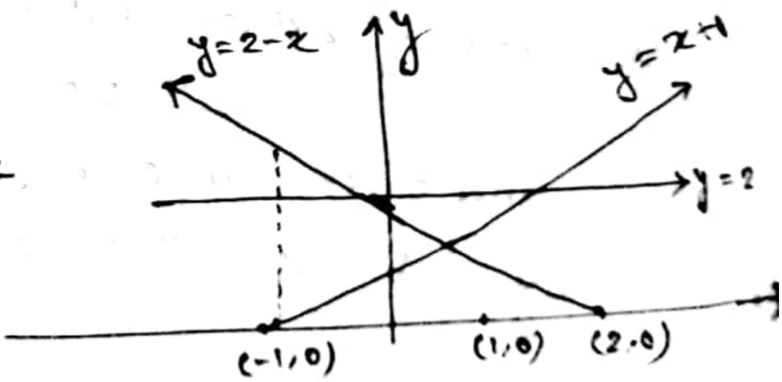
Now, $f'(2) \begin{cases} >0 & \text{if } x<2 \\ <0 & \text{if } x>2 \end{cases}$

$\Rightarrow f(x)$ has a unique local maximum at $x = 2$ and $f(2)$ is the absolute maximum or largest value.

A necessary Condition:- $f(x)$ has a local maximum or minimum at $x = a$, then $f'(a)$ does not exist or $f'(a) = 0$.

Ques:- $\int_{-1}^{1} \max\{2-x, 2, 1+x\}dx$ is (i) 4 (ii) 9/2 (iii) 2 (iv) None.

Solution:- (ii) :- Ans.



$$\int_{-1}^{1} \max\{2-x, 2, 1+x\}\, dx$$

$$= \int_{-1}^{0} (2-x)dx + \int_{0}^{2} 2.dx$$

$$= \int_{-1}^{0} 2dx + \int_{0}^{2} 2dx - \int_{-1}^{0} xdx$$

$=$

DEFINITE INTEGRALS

Ex. Evaluate the following definite integrals:—

(i) $\displaystyle\int_{-1}^{2} [x]\,dx$

Sol. $[x] = \begin{cases} -2 & , -2 \le x < -1 \\ -1 & , -1 \le x < 0 \\ 0 & , 0 \le x \le 1 \\ 1 & , 1 \le x < 2 \end{cases}$

$$I = \int_{-1}^{2} [x]\,dx$$

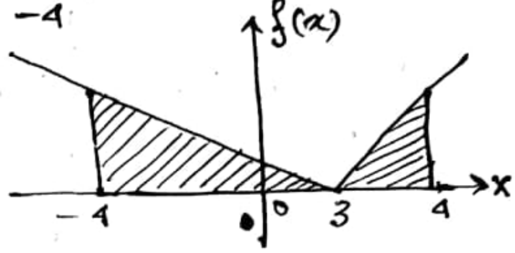
$$= \int_{-1}^{0} (-1)\,dx + \int_{0}^{1} (0)\,dx + \int_{1}^{2} (1)\,dx$$

$$= [-x]_{-1}^{0} + 0 + [x]_{1}^{2}$$

$$= 0.$$

(ii) $\displaystyle\int_{-4}^{4} |x-3|\,dx$

Sol.



Area of the shaded region $= \frac{1}{2}\times 7\times 7 + \frac{1}{2}\times 1\times 1$

$= 25$

Aliter:— $\displaystyle\int_{-4}^{4} |x-3|\,dx$

$$= -\int_{-4}^{3} (x-3)\,dx + \int_{3}^{4} (x-3)\,dx$$

$$= \frac{1}{2}\times 7\times 7 + \frac{1}{2}\times 1\times 1$$

$$= 25.$$

(iii) $\displaystyle\int_{0}^{100} e^{x-[x]}\,dx$

Sol. $\displaystyle\int_{0}^{100} e^{x-[x]}\,dx$

$$= \sum_{K=1}^{100} \int_{K-1}^{K} e^{x-[x]}\,dx$$

$$= \sum_{K=1}^{100} \left(\int_{K-1}^{K} e^{x-\overline{K-1}}\,dx \right)$$

$$= \sum_{K=1}^{100} \left(\int_{0}^{1} e^{y}\,dy \right), \text{ where } y = x-(k-1)$$

$$= 100\,(e-1)$$

(iv) $\int_0^{1.5} [x^2]dx$

Sol. $[x^2] = \begin{cases} 0 & , 0 \le x^2 < 1 \Leftrightarrow 0 \le x < 1 \\ 1 & , 1 \le x^2 < 2 \Leftrightarrow 1 \le x < \sqrt{2} \\ 2 & , 2 \le x^2 < 3 \Leftrightarrow \sqrt{2} \le x < \sqrt{3} \end{cases}$

$$\therefore \int_0^{1.5} [x^2]dx = \int_0^1 0.dx + \int_1^{\sqrt{2}} 1.dx + \int_{\sqrt{2}}^{1.5} 2.dx$$

$$= (\sqrt{2}-1) + 2(1.5-\sqrt{2})$$

$$= (2-\sqrt{2}).$$

(v) If $\int_0^1 \frac{e^t}{t+1} dt = a$, then $\int_{b-1}^{b} \frac{e^{-t}}{t-b-1} dt$ is equal to

(a) ae^{-b} (b) $-ae^{-b}$ (c) $-be^{-a}$. (d) ae^{b}.

Solution:- $a = \int_0^1 \frac{e^t}{t+1} dt$

$$\int_{b-1}^{b} \frac{e^{-t}}{t-b-1} dt = \int_1^0 \frac{e^{u-b}}{-(u+1)} (-du) \quad ; u = b-t$$

$$= -e^{-b} \int_0^1 \frac{e^u}{u+1} du$$

$$= -e^{-b}.a$$

Ans. (b) $-ae^{-b}$.

Ex.(1):- Find the difference between the greatest and least value of $f(x) = \int_0^x (t^2+t+1)\,dt$ on $[2,3]$.

Solution:- Let $x_1, x_2, \ldots, x_k$ be the stationary points and are solutions of $f'(x)=0$.

Then the absolute maximum value

$$= \max\{f(a), f(x_1), f(x_2) \ldots, f(x_k), f(b)\}$$

Here $f'(x) = x^2+x+1$

$= (x+\frac{1}{2})^2 + \frac{3}{4} > 0$ $\therefore f(x)$ is increasing.

$\therefore$ The required difference is $= f(3) - f(2)$

$$= \int_2^3 (t^2+t+1)\,dt$$

$$= \left[\frac{t^3}{3} + \frac{t^2}{2} + t\right]_2^3$$

$$= 8.\frac{2}{3} = \frac{16}{3}.$$