INTRODUCTION TO PROBABILITY

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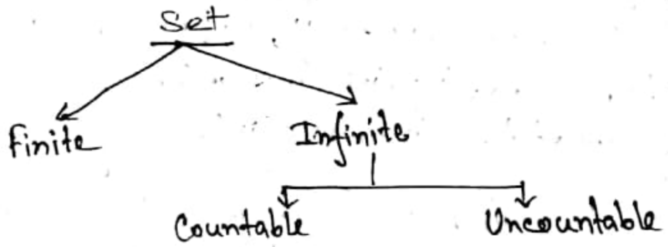
The concept of a SET :> A set is a collection of some elements which are its members. And the members are called the elements of the set. Synonyms for set are class, aggregate and collection. A set can be defined by actually listing its elements or, if this is not possible, by describing some property held by all members and by no nonmembers. The first is called the roster method and the second is called the property method.

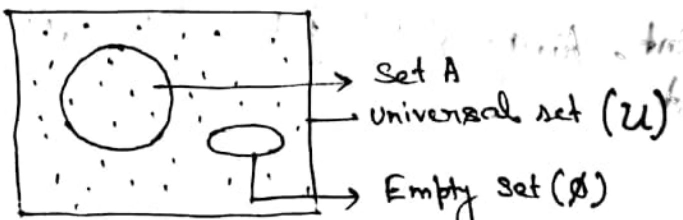
Ex. 1. The set of all vowels in the english alphabet can be defined by the roster method as {a, e, i, o, u} or by property method as {x | x is a vowel}, the vertical line | is read "such that" or "given that".

Ex. 2. The set {x | x is a triangular in a plane} is the set of all triangles in a plane. Note that the rooster method can't be used here.



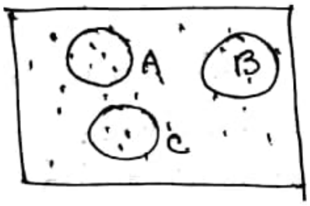
Depending on how many elements it has, a set may be finite or infinite. The set M = {1, 2, ..., 50} is finite and contains 50 elements. The set of all natural numbers N1 = {1, 2, ..., n, ...} is infinite. The set of all even numbers N2 = {2, 4, ..., 2n, ...} is also infinite. An infinite is said to be countable if all its elements can be enumerated. Both the set N1, N2 above are countable. The set C of all points with in or on a circle of radious r > 0, $C = \{(x, y) : x^2 + y^2 \leq r^2\}$ is infinite and uncountable. Its elements can't be enumerated.

Universal set and Null set :> Empty set is a set containing no elements, It's a member of all other sets.



The diagram representing a set is called a Venn diagram.

<u>Disjoint Sets</u> :-> Two sets A, B are disjoint if they have no common element, Similarly sets $A_1, A_2, \dots, A_m$ are mutually disjoint if no two them have any common element. Here the sets A, B, C are mutually exclusive.



The set containing all the possible points representing the elementary events of a random expt. i.e. the universal set is called the 'sample space'. It is represented as S. Thus in tossing a coin once, $S = \{H, T\}$. In tossing a coin twice, $S = \{HH, HT, TH, TT\}$. In throwing ~~two dice~~ two dice, $S = \{(1,1), (1,2), \dots\dots, (6,6)\}$.

<u>Ex. 1.</u> A coin is tossed until a head appears. Here the sample space consists of elementary events $H, TH, TTH, TTTH, \dots$, these points are countable and infinite in number. the sample space consists of countably infinite number of cases.

<u>SUBSETS</u> :-> If each element of a set A also belongs to a set B we call A, a subset of B, written $A \subset B$ or $B \supset A$ and read "A is contained in B" or "B contains A" respectively. If $A \subset B$ but $A \neq B$ we call A a proper subset of B.

<u>Ex. 1.</u> $\{a, i, u\}$ is a proper subset of $\{a, e, i, o, u\}$.

Some elements of set theory :->

(a) <u>Union</u> -> If the sets $A_1, \dots, A_m$ are mutually disjoint, then we shall sometime write $\bigcup_{1}^{m} A_i = A_1 + A_2 + \dots + A_m$. $\bigcup_{i=1}^{m} A_i$ represents the event that at least one of $A_1, \dots, A_m$ occurs. $\bigcup_{i=1} A_i$ is called a total event.

(b) <u>Intersection</u> -> The set of all elements which belong to both A and B is called the intersection. If the sets $A_1, \dots, A_m$ are mutually disjoint, $A_i \cap A_j = \emptyset \ \forall\ i \neq j = 1, \dots, m$. $\bigcap_{i=1}^{m} A_i$ is called a compound event.

(c) Complementation → The complement of set A, denoted as A^c or A' or $\bar{A}$ is the set of all elements not contained in A. Hence $A^c = \mathcal{U} - A$.
The set A^c represents the event then A does not occur.
Clearly, $\mathcal{U}^c = \phi$, $\phi^c = \mathcal{U}$, $(A^c)^c = A$, $P(A^c) = 1 - P(A)$.

(d) Difference :→ The difference $A - B$ is the set of all elements contained in A but not in B.

SOME THEOREMS INVOLVING SETS :→

1) Commutativity : $A \cup B = B \cup A$; $A \cap B = B \cap A$.

2) Associativity : $A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$;
$A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$.

3) Distributivity : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

4) Idempotency : $A \cup A = A$; $A \cap A = A$.

5) $A - B = A \cap B'$

6) If $A \subset B$, then $A' \supset B'$ or $B' \subset A'$.

7) $A \cup \phi = A$, $A \cap \phi = \phi$

8) $A \cup \mathcal{U} = \mathcal{U}$, $A \cap \mathcal{U} = A$.

9) For any sets A and B, $A = (A \cap B) \cup (A \cap B^c)$

10) De Morgan's 1st law : →

$$(A \cup B)^c = A^c \cap B^c$$

For m sets, A_i, $i = 1, \ldots, m$.

$$\left(\bigcup_{i=1}^{m} A_i\right)^c = \bigcap_{i=1}^{m} A_i^c.$$

11) De Morgan's 2nd law : →

$$(A \cap B)^c = A^c \cup B^c$$

For m sets, A_i, $i = 1, \ldots, m$.

$$\left(\bigcap_{i=1}^{m} A_i\right)^c = \bigcup_{i=1}^{m} A_i^c.$$

Examples :→

1) Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof :→ We've → $A \cap (B \cup C) = \{x \mid x \in A, x \in B \cup C\}$

$= \{x \mid x \in A, x \in B \text{ or } x \in C\}$

$= \{x \mid x \in A, x \in B \text{ or } x \in A, x \in C\}$

$= \{x \mid x \in A \cap B \text{ or } x \in A \cap C\}$

$= (A \cap B) \cup (A \cap C)$

2) $(A \cup B)' = A' \cap B'$.

Proof :→ We have, $(A \cup B)' = \{x \mid x \notin A \cup B\}$

$= \{x \mid x \notin A, x \notin B\}$

$= \{x \mid x \in A', x \in B'\}$

$= A' \cap B'$.

3) Prove that for any sets A and B we have $A = (A \cap B) \cup (A \cap B')$.

Proof :→ Method-1. $A = \{x \mid x \in A\}$

$= \{x \mid x \in A \cap B \text{ or } x \in A \cap B^c\}$

$= (A \cap B) \cup (A \cap B^c)$

Method-2. let $C = B'$.

from ①, we know, $A \cap (B \cup B') = (A \cap B) \cup (A \cap B')$

$A \cap U = (A \cap B) \cup (A \cap B')$

$\Rightarrow A = (A \cap B) \cup (A \cap B')$

4) Prove that if $A \subset B$ and $B \subset C$, then $A \subset C$.

Proof :→ Let x be ~~the~~ any element of A, i.e. $x \in A$. Then since $A \subset B$, i.e. every element of A is in B. we have $x \in B$. Also since $B \subset C$, we have $x \in C$. Thus every element of A is an element of C and so $A \subset C$.

5) Let A be the set of all real numbers whose squares are equal to 25. Show how to describe A by (a) the property method, (b) the roster method

Ans:→ (a) $A = \{x \mid x^2 = 25\}$
(b) $x^2 = 25$, $x = 5$ and $x = -5$. $\therefore A = \{5, -5\}$.

6) Let $A = \{x \mid x$ is an odd integer$\}$, $B = \{x \mid x^2 - 8x + 15 = 0\}$. Show that $B \subset A$.

Ans:→ Since, $x^2 - 8x + 15 = 0$; $x = 3, x = 5$, we have $B = \{3, 5\}$. Then since the elements 3 and 5 ~~both~~ are both odd integer, they belongs to A. so, $B \subset A$.

7) Is it true that $\{2\} = 2$?

Ans:→ No, 2 is a real number while $\{2\}$ is a set which consists of the real number 2. A set such as $\{2\}$ consisting of only one element is sometimes called a singleton set and must be distinguished from the element which it contains.

8) Prove that $(A-B) \cap C \subset A - (B \cap C)$ and construct an example to such that $(A-B) \cap C$ is a proper subset $A - (B \cap C)$.

Ans:→ Let $x \in (A-B) \cap C$ i.e., $x \in (A-B)$ and $C \Rightarrow x \in A - B \cap C$. Hence $(A-B) \cap C \subset A - (B \cap C)$.

Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{1, 6, 7, 8\}$.
$(A-B) \cap C = \{1\}$. $A - (B \cap C) = \{1, 2, 3, 4\}$.

9) Prove that — $(A-B) \cap (C-D) \subset (A \cap C) - (B \cup D)$

Ans:→ Let $x \in (A-B) \cap (C-D)$
i.e., $x \in (A-B)$ and $(C-D)$.
i.e. $x \in A$, $x \notin B$, $x \in C$, $x \notin D$
i.e. $x \in A$, $x \in C$, $x \notin B \cup D$.
i.e. $x \in A \cap C$, $x \notin B \cup D$,
i.e. $x \in A \cap C - B \cup D \Rightarrow x \in A \cap C - B \cup D$

10) Prove that — $(A-B) \cap (A-C) \subset A - (B \cap C)$.

Proof:→ Let, $x \in (A-B) \cap (A-C)$
i.e., $x \in A$, $x \notin B$, $x \notin C$,
i.e., $x \in A$, $x \notin B \cup C$.
i.e. $x \in A - (B \cup C) \Rightarrow x \in A - (B \cap C)$

Field of events :→ We now construct a field of events F which has the following properties:

i) F contains the set S as one of its elements.

ii) If A, B, subsets of S, are elements of F, so are $A \cup B$, $A \cap B$, $\bar{A}$ and $\bar{B}$. Thus the null set is contained in F (because $S \in F$). The union, intersection and complements of a finite number of events belong to F.

Example :→ A die is thrown. The sample space $S = \{E_1, \ldots, E_6\}$, E_i denoting the elementary event, roll of point i. The field F consists of $2^6 = 64$ sets, $F = \{(\phi); (E_1), \ldots, (E_6); (E_1, E_2), \ldots, (E_5, E_6); (E_1, E_2, E_3), \ldots, (E_4, E_5, E_6); (E_1, \ldots, E_4), (E_3, \ldots, E_6); (E_1, \ldots, E_5), \ldots, (E_2, \ldots, E_6); (E_1, \ldots, E_6)\}$

Problem on Classical definition of probability :→

1) **Ques :→** Two unbiased dice are thrown, what is the prob. that the sum of the top faces is 9.

Ans :→ The possible cases are (1, 1), (1, 2), ..., (6, 6), being 36 in number. All these are mutually exclusive and equally likely. The event A (sum of the face values in 9) occurs if any of (3, 6), (4, 5), (5, 4), (6, 3) occurs. Thus $n(A) = 4$. Hence $P(A) = \frac{4}{36} = \frac{1}{9}$.

2) **Ques :→** Find the prob. that among two digited numbers formed by 1, 2, ..., 5, there is no repetition.

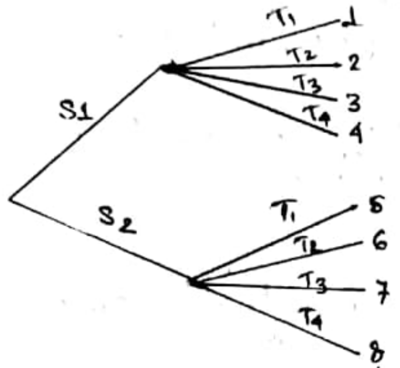
Ans :→ Let (x, y) stand for the number formed by the above digits. Total number of possible cases is 25 since each of x and y can be any of 1, ..., 5. All these cases are mutually exclusive and equally likely. To find n(A) we note that A occurs if any of the following occurs: (x, y), $x \neq y = 1, \ldots, 5$. Suppose x is chosen first and then y. x may be anything between 1 to 5 and thus x may be chosen in 5 ways. After x has been chosen, y may be chosen in any of the remaining 4 ways. Since with each choice of x there are 4 ways of choosing y, the total number of cases, (x, y), $x \neq y = 1, \ldots, 5$ is $5 \cdot 4 = 20$. Thus $n(A) = 20$ and hence $P(A) = \frac{20}{25} = \frac{4}{5}$.

Fundamental principle of counting . Tree diagrams

If one thing can be accomplished in n_1 different ways and after this a second thing can be accomplished in n_2 different ways, ..., and finally a kth thing can be accomplished in n_k different ways, then all k-things can be accomplished in the specified order in $n_1 n_2 \cdots n_k$ different ways.

Ex. If a man has 2 shirts and 4 ties then he has $2 \cdot 4 = 8$ ways of chosing a shirt and then a tie.

Ex. Letting the shirts be represented by S_1, S_2 and ties by T_1, T_2, T_3, T_4, the various ways of choosing a shirt and then a tie are indicated in the below tree diagram.



Permutations :- Suppose that we are given n distinct objects and wish to arrange r of these objects in a line. Since there are n ways of choosing the 1st object, and after this is done n-1 ways of choosing the 2nd object, ..., and finally $n-r+1$ ways of choosing the rth object, it follows by the fundamental principle of counting that the number of different arrangements or permutations as they are often called, is given by

$$^nP_r = n(n-1)(n-2)\cdots(n-r+1) \quad \text{——— (1)}$$

where it is noted that the product has r factors, we call ^nP_r the number of permutations of n objects taken r at a time.

where, $r = n$, then $^nP_n = n(n-1)\ldots 1 = n!$

$$^nP_r = \frac{n!}{(n-r)!} \quad \text{——— (2)}$$

If $r = n$ we see that (1) and (2) agree only if we have $0! = 1$ and we shall actually take this as the defn. of $0!$.

Examples:→

e.g. The number of different arrangements or permutations consisting of 3 letters each which can be formed from the 7 letters A, B, C, D, E, F, G is

$$^7P_3 = \frac{7!}{4!} = 7\cdot6\cdot5.$$

1.11 Suppose that a set consists of n objects of which n_1 are of one type (i.e. indistinguishable from each other), n_2 are of a second type, ..., n_k are of a kth type. Here, of course, $n = n_1 + n_2 + \dots + n_k$. Then the number of different permutation of the objects is

$$^nP_{n_1, n_2, \dots, n_k} = \frac{n!}{n_1!\, n_2! \dots n_k!}$$

e.g. The number of different permutations of the 11 letters of the word MISSISSIPPI, which consists of 1M, 4 I's, 4 S's and 2P's, is $\frac{11!}{1!\,4!\,4!\,2!} = 34,650.$

Problems:→

1) In how many ways can 5 different colored marbles be arrange in a row.

Ans:→ 5! ways.

2) In how many ways can 10 people be seated on a bench if only 4 seats are available?

Ans:→ $\frac{10!}{6!} = 10.9.8.7 = 5040$ ways.

3) It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible?

Ans:→ The men may be seated in $^5P_5$ ways and the women in $^4P_4$ ways. Each arrangements of the men may be associated with each arrangements of the women. Hence, Number of arrangements $= {}^5P_5 \cdot {}^4P_4 = 5!\,4! = 2880$ ways.

4) How many 4-digit numbers can be formed with the 10 digits 0, 1, 2, 3, ..., 9 if (a) repetitions are allowed, (b) repetitions are not allowed, (c) the last digit must be zero and repetitions are not allowed?

Ans:→ (a) the first digit can be any one of 9 (since 0 is not allowed). The second, third, fourth digits can be any of 10. Then 9.10.10.10 = 9000 numbers can be formed.

(b) The first digit can be any one of 9 (any one but 0).
The second " " " " " 9 (any but that used for the first digit)
The third " " " " " 8 (any but those used for the first two digits)
The fourth " " " " " 7 (any but those used for the first three digits).

Then 9.9.8.7 = 4536 numbers can be formed.

Alt. method:→ The first digit can be any one of 9 and the remaining three can be chosen in $^9P_3$ ways.
Then, $9 \times {}^9P_3 = 9.9.8.7 = 4536$ numbers can be formed.

(c) The first digit can be chosen in 9 ways.
The second " " " " " 8 ways.
The third " " " " " 7 ways.

Then 9.8.7 = 504 numbers can be formed.

Alt. method:→ the first digit can be chosen in 9 ways and the next two digits in $8P_2$ ways. Then $9.8P_2 = 9.8.7 = 504$ numbers can be formed.

5) Four different mathematics books, six different physics books and two different chemistry books are to be arranged on a shelf. How many different arrangements are possible if (a) the books in each particular subject must all stand together, (b) only the mathematics books must stand together?

Ans:→ (a) the mathematics books can be arranged among themselves in ${}^4P_4 = 4!$ ways, the physics books in ${}^6P_6 = 6!$ ways, the chemistry books in ${}^2P_2 = 2!$ ways, and the three groups in ${}^3P_3 = 3!$ ways, thus,—

Number of arrangements $= 4!\,6!\,2!\,3! = 207{,}360$.

(b) Consider the four mathematics books as one big book. Then we have 9 books which can be arranged in ${}^9P_9 = 9!$ ways. In all of these ways the mathematics books are together. But the mathematics books can be arranged among themselves in ${}^4P_4 = 4!$ ways. Hence

Number of arrangements $= 9!\,4! = 8{,}709{,}120$.

6) Five red marbles, two white marbles, and three blue marbles are arranged in a row. If all the marbles of the same color are not distinguishable from each other, how many different arrangements are possible?

Ans:→ Assume that there are N different arrangements. Multiplying N by the numbers of ways of arranging
(a) the five red marbles among themselves,
(b) the two white marbles among themselves,
(c) the three blue marbles among themselves,
(i.e. multiplying N by $5!\,2!\,3!$), we obtain the number of ways of arranging the 10 marbles if they were all distinguishable, i.e. $10!$.

Then $(5!\,2!\,3!)N = 10!$ and $N = 10!/(5!\,2!\,3!)$

In general, the number of different arrangements of n objects of which n_1 are alike, n_2 are alike, ..., n_k are alike is

$$\frac{n!}{n_1!\,n_2!\cdots n_k!} \quad \text{where } n_1+n_2+\cdots+n_k = n.$$

7) In how many ways can 7 people be seated at a round table if (a) they can sit anywhere, (b) 2 particular people must not sit next to each other?

Ans:→ (a) Let 1 of them be seated anywhere. Then the remaining 6 people can be seated in $6! = 720$ ways, which is the total number of ways of arranging the 7 people in a circle.

(b) Consider the 2 particular people as one person. Then there are 6 people altogether and they can be arranged in $5!$ ways. But the 2 people considered as 1 can be arranged among themselves in $2!$ ways. Thus the number of ways of arranging 7 people at a round table with 2 particular people sitting together $= 5!\,2! = 240$.

Then, using (a), the total number of ways in which 7 people can be seated at a round table so that the 2 particular people do not sit together $= 720 - 240 = 480$ ways.

COMBINATIONS :→ In a permutation we are interested in the order of arrangement of the objects. Thus abc is a different permutation from bca. In many problems, however, we are interested only in selecting or choosing objects without regard to order. Such selections are called combinations. For example abc and bca are the same combination.

The total number of combinations of r objects selected from n (also called the combinations of n things taken r at a time) is denoted by ${}^{n}C_{r}$ or $\binom{n}{r}$. We have —

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

It can also be written

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{{}^{n}P_{r}}{r!}$$

It is easy to show that $\binom{n}{r} = \binom{n}{n-r}$ or ${}^{n}C_{r} = {}^{n}C_{n-r}$.

Ex. The number of ways in which 3 cards can be chosen or selected from a total of 8 different cards is

$$^8C_3 = \binom{8}{3} = \frac{8.7.6}{3!} = 56$$

Problems:→

1) In how many ways can 10 objects be split into two groups containing 4 and 6 objects respectively?

Ans:→ This is the same as the number of arrangements of 10 objects of which 4 objects are alike and 6 other objects are alike. this is $\frac{10!}{4!\,6!} = 210$.

2) In how many ways can a committee of 5 people be chosen out of 9 people?

Ans:→ $\binom{9}{5} = {}^9C_5 = \frac{9!}{5!\,4!} = 126$.

3) Out of 5 mathematicians and 7 physicists, a committee consisting of 2 mathematicians and 3 physicists is to be formed. In how many ways can this be done if
(a) any mathematician and any physicist can be included,
(b) one particular physicist must be on the committee,
(c) two particular mathematicians cannot be on the committee?

Ans:→ (a) Total no. of possible selections $= {}^5C_2 \cdot {}^7C_3 = 10.35 = 350$

(b) " " " " " $= {}^5C_2 \cdot {}^6C_2 = 10.15 = 150$

(c) " " " " " $= {}^3C_2 \cdot {}^7C_3 = 3.35 = 105$

4) From 7 consonants and 5 vowels, how many words can be formed consisting of 4 different consonants and 3 different vowels? The words need not have meaning.

Ans:→ The 4 different consonants can be selected in $^7C_4$ ways, the 3 different vowels can be selected in $^5C_3$ ways, and the resulting 7 different letters (4 consonants, 3 vowels) can then be arrange among themselves in $^7P_7 = 7!$ ways.

Then — Number of words $= {}^7C_4 \cdot {}^5C_3 \cdot 7! = 1,764,000$

Elements of Combinatorial Analysis :→

Result 1. If there are two groups G_1 and G_2, G_1 consisting of m elements $a_1, \ldots, a_m$ i.e., if $G_1 = (a_1, \ldots, a_m)$ and similarly $G_2 = (b_1, \ldots, b_n)$ consisting of n elements, number of pairs (a_i, b_j) formed by taking an element 'a' from G_1 and an element 'b' from G_2 is mn.

Ex. A college has 27 students in the first year hons. class and 20 students in the second year hons. class. Number of pairs that can be formed by taking one student from each class is $= 27 \times 20 = 540$.

Result 2. If there are k groups $G_1, G_2, \ldots, G_k$ containing $m_1, m_2, \ldots, m_k$ elements respectively, i.e. if

$$G_1 = \{u_1, u_2, \ldots, u_{m_1}\}$$
$$G_2 = \{y_1, y_2, \ldots, y_{m_2}\}$$
$$\cdots\cdots\cdots$$
$$G_k = \{t_1, t_2, \ldots, t_{m_k}\}$$

the the number of multiplets $(u_i, y_j, \ldots, t_n)$ formed by taking one element from each group is $m_1 m_2 \ldots m_k$.

Ex. Number of 3 digited numbers that can be formed out of 7 digits $1, 2, \ldots, 7$ is 7^3. Here each of G_1, G_2, G_3 contains 7 elements.

Result 3. The number of ways in which r things can be chosen out of n different things and arranged among themselves is

$$n(n-1) \ldots (n-r+1) = {}^nP_r \quad (r \le n).$$

the permutation of n things taken r at a time.

We shall often denote ${}^nP_r$ as $(n)_r$

Ex. 1) An elevator contains 5 passengers and stops at 10 floors, what's the prob. that no two passenger get down at the same floor?

Soln :→ Number of possible cases is 10^5 since each of the 5 passengers has 10 choices to get down. All these cases are mutually exclusive and equally likely. Number of favourable cases for the event under consideration is $(10)_5$, since the first passenger can choose his floor in 10 ways, the second in 9 ways and so on. Hence probability is $(10)_5 / 10^5$.

Ex. 2. If n distinct balls are placed in n cells what is the probability that each cell will be occupied?

Soln: Total no. of possible arrangements is n^n because each ball can go to any cell and thus each ball has n choices. All these cases are mutually exclusive and equally likely.

The no. of ways each cell may be kept occupied is the number of ways in which n balls may be arranged among themselves and this number is $n!$ Hence probability is $n!/n^n$.

Ex. 3. The numbers $1, 2, \ldots, n$ are arranged in a random order. Find the prob. that digits (i) 1 and 2, (ii) 1, 2, 3 appears as neighbours in the order named.

Soln: Total no. of mutually exclusive and equally likely arrangements is $n!$

(i) Assume the digits 1 and 2 are tied together. Then the total no. of digits taking 1 and 2 as a single digit is $(n-1)$ and these can be arranged among themselves is $(n-1)!$ ways. Hence the prob. is $\frac{(n-1)!}{n!} = \frac{1}{n}$

(ii) Here the prob. is $\frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$.

Result 4. The number of ways in which a set of r objects can be formed out of n different objects is — $\frac{n!}{r!\,(n-r)!} = \binom{n}{r}$

This is the number of combinations of r things taken out of n different things. Two sets are different if and only if each has at least one uncommon element. How arrangements of elements within a set is not considered. Thus (a, b, c) and (a, b, d) are two different sets; but (a, b, c) and (b, a, c) are the same set. It follows that —

(i) $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$, (ii) $\sum_{i=0}^{n} \binom{n}{i} = 2^n$, (iii) $\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$,

(iv) $\sum_{i=0}^{n} i\binom{n}{i} = n2^{n-1}$, (v) $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$.

Ex. 4. What is the prob. that the 2 cards drawn at random from an ordinary deck will be all spades.

Soln. :-
Total number of possible cases is the total number of ways in which a set of 2 cards can be formed out of 52 cards, this is $\binom{52}{2}$. All these cases are mutually exclusive and equally likely.

Number of favourable cases is the number of ways in which 2 spades can be choosen out of 13 spades. This number is $\binom{13}{2}$.

Hence prob. is $= \dfrac{\binom{13}{2}}{\binom{52}{2}}$.

Ex. 5. What is the prob. that a bridge hand will be a complete suit?

Soln. :- Total no. of mutually exclusive and equally likely cases is $\binom{52}{13}$. Since there are only 4 suits number of favourable cases is 4. Hence prob. is $\dfrac{4}{\binom{52}{13}}$.

Ex. 6 The face cards are removed from a full pack. Out of the remaining cards, 4 are drawn. (i) What is the prob. that they belong to different suits? (ii) What is the prob. that the 4 cards belong to different suits and different denominations?

Soln. :-
Total no. of mutually exclusive and equally likely cases is $\binom{40}{4}$.
(i) If the cards are (a, b, c, d) where 'a' is spade, 'b' a hearts, 'c' a club, 'd' a diamond, then each of a, b, c, d has 10 choices. Thus total number of choices for (a, b, c, d) is 10^4. Hence the prob. is $\dfrac{10^4}{\binom{40}{4}}$.

(ii) Here the total no. of favourable cases is $(10)_4$. Hence the prob. is $(10)_4 / \binom{40}{4} = 0.05515$.

Result 5. Let a population consist of two types of elements of sizes N_1 and N_2 respectively. The no. of ways in which n_1 elements of the first type and n_2 elements of the second type may be chosen is $\binom{N_1}{n_1}\binom{N_2}{n_2}$.

Similarly if the population consists of k types of elements of size $N_1, N_2, \ldots, N_k$ respectively, the number of ways in which n_1 elements of the first type, ..., ..., n_k elements of the kth type may be chosen is

$$\binom{N_1}{n_1}\binom{N_2}{n_2}\cdots\cdots\binom{N_k}{n_k}.$$

Ex. 7. → What is the prob. that of 6 cards taken from a full ~~deck~~ pack, 3 will be black and 3 will be red?

Soln. :→ Number of possible mutually exclusive and equally likely cases is $\binom{52}{6}$. Number of ways in which 3 black and 3 red cards may be chosen is $\binom{26}{3}\binom{26}{3}$. Hence prob. $= \dfrac{\binom{26}{3}\binom{26}{3}}{\binom{52}{6}}$

Ex. 7. Find the prob. that a hand at bridge will consist of 3 spades, 5 diamonds, 3 hearts, 2 clubs.

Soln. :→ There will be 13 cards of each type in the pack.
Hence the prob. is $= \dfrac{\binom{13}{3}\binom{13}{5}\binom{13}{3}\binom{13}{2}}{\binom{52}{13}}$.

Result. 6. The number of ways in which N elements can be grouped into k specific groups $G_1, G_2, \ldots, G_k$ of sizes $N_1, N_2, \ldots, N_k$ respectively ($N_1 + N_2 + \cdots + N_k = N$) is

$$\frac{N!}{N_1!\, N_2! \cdots N_k!}$$

If the groups are of equal sizes, $N_i = M$, $i = 1, 2, \ldots, k$, the number of unordered groups into which N elements can be grouped is $\dfrac{N!}{k!\,(M!)^k}$

Ex. 8. What is the probability that in a throw of 12 dice each face occurs twice?

Soln. :→ Number of all possible, equally likely and mutually exclusive cases is 6^{12} since each die may result in one of the 6 faces.

Number of favourable cases is the number of ways in which 12 dice may be arranged in 6 groups of size 2 each, group 1 consisting of two dice each showing 1, group 2 of two dice each showing 2 etc. and this number is $\frac{12!}{(2!)^6}$.

Hence the prob. is $\dfrac{\frac{12!}{(2!)^6}}{6^{12}} = 0.00344$.

Ex. 9. Given 30 people, find the prob. that among the twelve months there are six containing two birthdays and six containing three.

Soln. :- The number of ways 30 people can be grouped into twelve groups according to the month of birth is $-\frac{30!}{(2!)^6(3!)^6}$.

Hence the prob. is $\dfrac{\frac{30!}{(2!)^6(3!)^6}}{12^{30}} \simeq 0.00035$.

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4 Probability using Combinatorial Analysis :-

1) A box contains 8 red, 3 white and 9 blue balls. If 3 balls are drawn at random without replacement, determine the prob. that (a) all 3 are red, (b) all 3 are white, (c) 2 are red and 1 is blue, (d) at least 1 is white, (e) 1 of each color is drawn, (f) the balls are drawn in the order red, white, blue.

Ans:-

(a) Required probability = $\dfrac{\text{number of selection of 3 out of 8 red balls}}{\text{number of selection of 3 out of 20 balls}}$

$$= \frac{\binom{8}{3}}{\binom{20}{3}} = \frac{14}{285}$$

(b) $P(\text{all 3 are white}) = \dfrac{\binom{3}{3}}{\binom{20}{3}} = \dfrac{1}{1140}$

(c) $P(\text{2 are red and 1 is white}) = \dfrac{\binom{8}{2}\binom{3}{1}}{\binom{20}{3}} = \dfrac{7}{95}$

(d) $P(\text{none is white}) = \dfrac{\binom{17}{3}}{\binom{20}{3}} = \dfrac{34}{57}$,

$\therefore P(\text{at least one is white}) = 1 - \dfrac{34}{57}$

$$= \frac{23}{57}$$

(e) $P(1 \text{ of each color is drawn}) = \dfrac{\binom{8}{1}\binom{3}{1}\binom{9}{1}}{\binom{20}{3}} = \dfrac{18}{95}$.

(f) $P(\text{balls drawn in order red, white, blue}) = \frac{1}{3!} P(1 \text{ of each color is drawn})$

$$= \frac{1}{3!} \times \left(\frac{18}{95}\right)$$

$$= \frac{3}{95}.$$

2) In the game of poker five cards are drawn from a deck of 52 well-shuffled cards. Find the prob. that — (a) 4 are aces, (b) 4 are aces and 1 is a king, (c) 3 are tens and 2 are jacks, (d) a nine, ten, jack, queen, king are obtained in any order, (e) 3 are of any one suit and 2 are of another, (f) at least 1 ace is obtained.

Ans:→

(a) $P(4 \text{ aces}) = \dfrac{\binom{4}{4}\binom{48}{1}}{\binom{52}{5}}$

(b) $P(4 \text{ aces and } 1 \text{ king}) = \dfrac{\binom{4}{4}\binom{4}{1}}{\binom{52}{5}}$

(c) $P(3 \text{ are tens and } 2 \text{ are jacks}) = \dfrac{\binom{4}{3}\binom{4}{2}}{\binom{52}{5}}$

(d) $P(\text{nine, ten, jack, queen, king any order}) = \dfrac{\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}}$.

(e) $P(3 \text{ of any one suit, } 2 \text{ of another}) = \dfrac{4 \cdot \binom{13}{3} \cdot 3 \cdot \binom{13}{2}}{\binom{52}{5}}$

since, there are 4 ways of choosing the first suit and 3 ways of choosing the second suit.

(f) $P(\text{no ace}) = \dfrac{\binom{48}{5}}{\binom{52}{5}}$

$\therefore P(\text{at least one ace}) = 1 - \dfrac{\binom{48}{5}}{\binom{52}{5}}$.

Models in Statistical Mechanics :→

So far we have assumed when n balls are distributed in m cells each of the m^n sample points has the prob. $1/m^n$. The balls here are assumed to be each distinct.

When the balls are indistinguishable, an outcome of an expt. is described by a sequence of m members, $(k_1, \ldots, k_m)$, $k_1 + k_2 + \cdots + k_m = n$, giving the number of balls in each cell. The prob. of such an outcome, using assumption of equal likelihood of each of m^n sample points (as in the case of distinguishable balls as above) is

$$P(k_1, \ldots, k_m) = \frac{n!}{k_1! \cdots k_m!} \cdot \frac{1}{m^n}$$

This is called the Maxwell-Boltzman statistic. (If the balls were distinct there were $\frac{n!}{k_1! \cdots k_m!}$ distinct arrangements for the event $(k_1, \ldots, k_m)$ all of which have now become indistinguishable)

- Under Bose-Einstein statistics only distinguishable arrangements of n indistinguishable balls in m cells are considered and each is given equal probability. There are $\binom{m+n-1}{n}$ such distinguishable arrangements and probability of each distinct arrangement is $\left[\binom{m+n-1}{n}\right]^{-1}$

Proof :- Suppose m cells are represented by a sequence of $(m+1)$ bars $(|)$, each bar representing an wall of a cell. With in these cells the balls (crosses) are placed. For example, $|\times|\times\times|||$ shows a configuration of 3 balls in 4 cells with $k_1 = 1, k_2 = 2, k_3 = k_4 = 0$. Any arrangement of n balls in m cells is thus represented by a sequence of $(m+1)$ bars and n crosses provided the sequence must start with a bar and end with a bar. Hence the number of distinguishable arrangements equal the number of choices of n positions for crosses out of $(m+n-1)$ available positions, this is $\binom{m+n-1}{n}$.

The Fermi-Dirac statistics is based on the following hypothesis: (a) it is impossible for two or more balls to be in the same cell; (b) all distinguishable arrangements satisfying (a) have equal probability. With F-D statistics there are thus $\binom{m}{n}$ arrangements each having prob. $= \frac{1}{\binom{m}{n}}$, $n \le m$.

The model of distributing balls over cells is an appropriate one for various problems in statistical mechanics. The n balls are physical particles and the m cells represent m equal parts into which a special region is divided. B-E statistics hold good for photons, nuclei and atoms containing an even number of elementary particles. F-D statistic apply to electrons, neutrons and photons. However, different situations fit different models and it is impossible to select or justify probability models by a priori arrangements.

Theorem on Probability of Union of Events :→

Theorem :→ If $A_1, \dots, A_n$ are all mutually exclusive then prob. of union of events $A_1 \dots A_n$ is,

$$P\left(\bigcup_{i=1}^{n}\right) = P(A_1) + P(A_2) + \dots + P(A_n) \qquad (*)$$

Proof :→ Let M be the total number of mutually exclusive and equally cases in the sample space of which m_i are favourable for the event A_i, $i = 1, \dots, n$, since $A_1, \dots, A_n$ are mutually exclusive events, the sets representing them are mutually disjoint. Hence, the total number of cases favourable for the event $\bigcup_{i=1}^{n} A_i \left(= \sum_{i=1}^{n} A_i\right)$ is $\sum_{i=1}^{n} m_i$. Thus

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \frac{m_1 + \dots + m_n}{M} = \frac{m_1}{M} + \dots + \frac{m_n}{M} = P(A_1) + \dots + P(A_n)$$

1. Corollary :→ If an event A consists of n mutually exclusive forms $A_1, A_2, \ldots, A_n$ so that A happens whenever any of these occurs and vice-versa, then $A = A_1 + \cdots + A_n$ and —

$$P(A) = P(A_1) + \cdots + P(A_n).$$

2. Corollary :→ Let the event B imply A. Then the set B is a subset of A. Now A can be decomposed into two disjoint subsets B and A − B. Hence $P(A) = P(B) + P(A-B)$

thus $P(A-B) = P(A) - P(B)$, $B \subset A$.

Ex. 13 cards are drawn randomly from a well-shuffled pack of 52 cards. What is the prob. all are black or red.

Ans :→ Let A_1, A_2 be the events, all the cards are black, red respectively the two events are mutually exclusive. Here $P(A_1) = P(A_2) = \dfrac{\binom{26}{13}}{\binom{52}{13}} = a$ (say)

Hence, $P(A_1 + A_2) = 2a$.

Poincare's Theorem :→

We now consider the case when $A_1, \ldots, A_n$ are not necessarily mutually exclusive events. For simplicity we introduce the following notation. Let

$$\sum_{i=1}^{n} P(A_i) = S_1, \quad \sum\sum_{i<j=1}^{n} P(A_i \cap A_j) = S_2$$

$$\sum\sum\sum_{i<j<k=1}^{n} P(A_i \cap A_j \cap A_k) = S_3$$

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$$P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n) = S_n$$

Thus S_1 is the sum of n terms each of probability of an event A_i ; S_2 is the sum of $\binom{n}{2}$ terms each of prob. of joint occurance of a pair A_i, A_j ; S_3 is the sum of $\binom{n}{3}$ terms each of prob. of joint occurance of a triplet of events A_i, A_j, A_k etc. Clearly there are $\binom{n}{r}$ terms in the expression for S_r, $1 \le r \le n$. The following useful result due to Poincare, and is also known as the principle of inclusion and exclusion.

Theorem :→ Let $A_1, A_2, \ldots, A_n$ be events not necessarily mutually exclusive, the prob. that at least one of $A_1, \ldots, A_n$ occurs is

$$P\left(\bigcup_{i=1}^{n} A_i\right) = S_1 - S_2 + S_3 - \cdots + (-1)^{n-1} S_n \quad \text{———} (1)$$

or,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j=1}^{n}\sum P(A_i \cap A_j) + \sum\sum_{i<j<k=1}^{n}\sum P(A_i \cap A_j \cap A_k) - \cdots$$
$$\cdots + (-1)^{n-1} P(A_1 \cap A_2 \cap \cdots \cap A_n).$$

Proof :→ (By method of induction). First consider the events A_1, A_2.

Now, $A_1 \cup A_2 = A_1 + (A_2 - A_1 \cap A_2)$. Hence by theorem (*)

$$P(A_1 \cup A_2) = P(A_1) + P(A_2 - A_1 \cap A_2)$$

Since $A_1 \cap A_2 \subset A_2$, by corollary (2)

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

Thus the theorem is true for $n=2$, Assume (1) holds for $n=t$ (≥ 2). (t is an arbitrary positive integer, ≥ 2);

Thus

$$P\left(\bigcup_{i=1}^{t} A_i\right) = \sum_{i=1}^{t} P(A_i) - \sum_{i<j=1}^{t}\sum P(A_i A_j) + \sum\sum_{i<j<k=1}^{t}\sum P(A_i A_j A_k) - \cdots$$
$$\cdots + (-1)^{t-1} P(A_1 A_2 \cdots A_t) \quad \text{———} (2)$$

Then for $n = t+1$, we have —

$$P\left(\bigcup_{i=1}^{t+1} A_i\right) = P\left(\bigcup_{i=1}^{t} A_i\right) + P(A_{t+1}) - P\left(\bigcup_{i=1}^{t}(A_i \cap A_{t+1})\right) \quad \text{—} (3)$$

Now,

$$P\left(\bigcup_{i=1}^{t}(A_i \cap A_{t+1})\right) = \sum_{i=1}^{t} P(A_i \cap A_{t+1}) - \sum_{i<j=1}^{t}\sum P(A_i \cap A_j \cap A_{t+1})$$
$$+ \cdots + (-1)^{t-1} P(A_1 \cap A_2 \cap \cdots \cap A_t \cap A_{t+1}) \quad \text{———} (4)$$

Combining (2), (3), (4), we get —

$$P\left(\bigcup_{i=1}^{t+1} A_i\right) = \sum_{i=1}^{t+1} P(A_i) - \sum_{i<j=1}^{t+1}\sum P(A_i A_j) + \sum\sum_{i<j<k=1}^{t+1}\sum P(A_i A_j A_k) - \cdots$$
$$\cdots + (-1)^{t} P(A_1 \cdots A_{t+1})$$

Thus the result is true for $n = m+1$. The proof is, therefore complete by the principle of mathematical induction. ///

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An alternative proof :→ To calculate $P\left(\bigcup_{i=1}^{n} A_i\right)$ one should add the probabilities of all elementary points contained in at least one $A_i\ (i=1,\dots,n)$, each point being taken ~~once~~ only once. Consider an arbitrary point E contained in exactly m of the sets A_i, say, $A_1, A_2, \dots, A_m$ and find its contribution in the RHS of (1). P(E) occurs m times in S_1, since S_1 is the sum of the probabilities $P(A_1), \dots, P(A_m)$. Similarly it occurs $\binom{m}{2}$ times out of $\binom{n}{2}$ times in S_2, $\binom{m}{3}$ times in S_3, ... $\binom{m}{j}$ times in S_j, ..., $\binom{m}{m}$ times in S_m. Its contribution to $S_{m+1}, \dots, S_n$ is zero since E occurs only in m of the sets $A_1, \dots, A_n$. Therefore co-efficient of P(E) in the expression in the RHS of (1) is

$$\binom{m}{1} - \binom{m}{2} + \dots \pm \binom{m}{m} = \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} = 1 - (1-1)^m = 1.$$

which equals 1, thus each P(E) occurs only once in RHS of (1).

Hence the proof.

Example :→ Find the probability that in a bridge game, at least one of the players will get a complete suits of cards.

Soln :→

Let A_i be the event that a player i gets a complete suit of cards.

Thus

$$P\left(\bigcup_{1}^{4} A_i\right) = \sum_{1}^{4} P(A_i) - \sum_{i<j=1}^{4}\sum P(A_iA_j) + \sum_{i<j<k=1}^{4}\sum\sum P(A_iA_jA_k) - P(A_1A_2A_3A_4)$$

Let, $P(A_i) = \dfrac{4}{\binom{52}{13}} = P_1$; $P(A_iA_j) = \dfrac{4.3}{\binom{52}{13}\binom{39}{13}} = P_2$

$$P(A_iA_jA_k) = \frac{4.3.2}{\binom{52}{13}\binom{39}{13}\binom{26}{13}} = P_3; \quad P(A_1A_2A_3A_4) = \frac{4!}{\binom{52}{13}\binom{39}{13}\binom{26}{13}} = P_4$$

Hence the required probability $= 4P_1 - 6P_2 + 4P_3 - P_4$.

Theorem :→ 1. (Boole's inequality). Given $n(>1)$ events $A_1, A_2, \ldots, A_n$
$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

Proof :→ Consider first A_1 and A_2. Now
$$A_1 \cup A_2 = A_1 + (A_2 - A_1),$$
the event $A_1, A_2 - A_1$ being disjoint. Hence
$$P(A_1 \cup A_2) = P(A_1) + P(A_2 - A_1)$$
$$\leq P(A_1) + P(A_2)$$
Since $(A_2 - A_1) \subset A_2$, thus the inequality is proved for $n=2$. Now
$$\bigcup_{i=1}^{n} A_i = \left(\bigcup_{i=1}^{n-1} A_i\right) \cup A_n$$
$$\Rightarrow P\left(\bigcup_{i=1}^{n} A_i\right) \leq P\left(\bigcup_{i=1}^{n-1} A_i\right) + P(A_n) \leq P\left(\bigcup_{i=1}^{n-2} A_i\right) + P(A_{n-1}) + P(A_n)$$

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$$\leq \sum_{i=1}^{n} P(A_i)$$

Corollary :→ Given n events $A_1, A_2, \ldots, A_n$
$$P\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} P(A_i^c) = \sum_{i=1}^{n} P(A_i) - (n-1)$$

Proof :→ Since $\bigcap_{i=1}^{n} A_i = \left(\bigcup_{i=1}^{n} A_i^c\right)^c$
$$P\left(\bigcap_{i=1}^{n} A_i\right) = 1 - P\left(\bigcup_{i=1}^{n} A_i^c\right) \geq 1 - \sum_{i=1}^{n} P(A_i^c)$$
by the above theorem, the second expression follows by putting $P(A_i^c) = 1 - P(A_i)$.

Theorem 1.2. (Bonferroni's inequality).

Given $n (\geq 2)$ events $A_1, A_2, \ldots, A_n$,

$$P\left(\bigcup_{i=1}^{n} A_i\right) \geq S_1 - S_2 \quad , \text{or,}$$

$$\sum_{i=1}^{n} P(A_i) - \sum_{i<j=1}^{n}\sum P(A_i A_j) \leq P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{m} P(A_i)$$

Proof:- (By mathematical induction)

for $n=2$.

$$P(A_1)+P(A_2)-P(A_1 \cap A_2) = P(A_1 \cup A_2)$$

and thus the inequality is true for $n=2$ (by equality).

For $n=3$

$$P\left(\bigcup_{i=1}^{3} A_i\right) = \sum_{i=1}^{3} P(A_i) - \sum_{i<j=1}^{3}\sum P(A_i \cap A_j) + P(A_1 A_2 A_3)$$

and the result holds. Assuming that it holds m events $(3 < m \leq n-1)$, we show that it holds for $m+1$.

$$P\left(\bigcup_{i=1}^{m+1} A_i\right) = P\left(\bigcup_{i=1}^{m} A_i \cup A_{m+1}\right)$$

$$= P\left(\bigcup_{i=1}^{m} A_i\right) + P(A_{m+1}) - P\left\{A_{m+1} \cap \left(\bigcup_{i=1}^{m} A_i\right)\right\}$$

$$\geq \left\{\sum_{i=1}^{m} P(A_i) - \sum_{i<j=1}^{m}\sum P(A_i A_j)\right\} + P(A_{m+1})$$

$$- P\left(\bigcup_{i=1}^{m} (A_i \cap A_{m+1})\right)$$

$$\geq \sum_{i=1}^{m+1} P(A_i) - \sum_{i<j=1}^{m}\sum P(A_i A_j) - \sum_{i=1}^{m} P(A_i A_{m+1})$$

[By Boole's Inequalities]

$$= \sum_{i=1}^{m+1} P(A_i) - \sum_{i<j=1}^{m+1}\sum P(A_i A_j)$$

* Note that Boole's inequality and Bonferroni's inequality imply that if $P(A_i \cap A_j) = 0$ for each pair (i,j) with $i \neq j$. then $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$;

Conditional probability :→

We know that –

$$P(A \cap B) = P(A)\,P(B|A)$$

$$P(A_1 A_2 A_3) = P(A_1)\,P(A_2|A_1)\,P(A_3|A_1A_2), \text{ provided } P(A_1A_2) > 0.$$

$$P(A_1A_2A_3A_4) = P(A_1)\,P(A_2|A_1)\,P(A_3|A_1A_2)\,P(A_4|A_1A_2A_3), \text{ provided } P(A_1A_2A_3) > 0.$$

$$P(A_1 \ldots A_m) = P(A_1)\,P(A_2|A_1) \ldots P(A_m|A_1 \ldots A_{m-1}), \text{ provided } P(A_1 \ldots A_{m-1}) > 0.$$

Example :→ 1) An urn contains 6 reds and 4 black balls. Two balls are drawn without replacement. What's the prob. that the second ball is red if it is known that the first is red.

Ans :→ Let A and B be the events that the first and the second ball are red respectively.

Hence $P(AB) = \dfrac{\binom{6}{2}}{\binom{10}{2}} = \dfrac{1}{3}$. $P(A) = \dfrac{6}{10}$. thus,

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\cancel{10}^{\,5}}{3 \times \cancel{6}_{\,3}} = \frac{5}{9}.$$

Ex. 2) Two unbiased dice are thrown. Find the conditional prob that two fives occur if it is known that the total is divisible by 5 ?

Ans :→ ~~Let A be the event the total is divisible by 5, B be the event two fives occurs; then AB is the event two~~

$P(A) = \dfrac{7}{36}$, $P(AB) = \dfrac{1}{36}$. Hence $P(B|A) = \dfrac{1}{7}$.

Here A be the event the total is divisible by 5, AB be the event two five occurs, Ex. 3) A die is loaded in such a way that the prob. of a given number turning up is proportional to that number (e.g. a 6 is twice as probable as 3). What is the prob. of a 3, given that an odd number is rolled?

Ans:→ Here, the prob. of getting number i is $\frac{i}{21}$, since $\sum_{1}^{6} j = 21$.
Let A be the event, an odd number turns up, B the event 3 is obtained, then $P(B|A) = P(AB)/P(A) = \frac{1}{3}$.

* **Theorem:→** For any fixed event A in $\mathcal{Q}$, such that $P(A) > 0$, $P(\cdot|A)$ defined by poincare's theorem behaves like an ordinary prob. function on $\mathcal{Q}$ / Show that conditional prob. satisfies Kolmogorov's axioms on the axiomatic defn. of probability.

Proof:→ We define prob. as a (finite) real-valued function P or P(·) on a σ-field of events, say $\mathcal{Q}$, such that —

(A) $P(A) \geq 0$ for any $A \in \mathcal{Q}$;

(B) $P(\Omega) = 1$;

and (C) if $\{A_i\}$ is a sequence of disjoint events, each belonging to $\mathcal{Q}$, then
$$P\left(\sum_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

— Kolmogorov's axiom on prob.

Now we have to show that $P(\cdot|A)$ satisfies axioms (A), (B) and (C).

(a) We have $P(A \cap B) \geq 0$. Hence on dividing both sides by $P(A)$, we get — $\frac{P(A\cap B)}{P(A)} \geq 0$, $\Rightarrow P(B|A) \geq 0$, for any B in $\mathcal{Q}$.

(b) Since $(\Omega \cap A) = A$,
$$P(\Omega|A) = \frac{P(\Omega \cap A)}{P(A)} = 1.$$

(c) If $\{B_i\}$ is a sequence of disjoint events in $\mathcal{Q}$, then $\{A \cap B_i\}$ is also a sequence of disjoint events in $\mathcal{Q}$. Also since —
$$A \cap \left(\sum_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} (A \cap B_i),$$
$$P\left(A \cap \left[\sum_{i=1}^{\infty} B_i\right]\right) = P\left(\sum_{i=1}^{\infty} [A \cap B_i]\right) = \sum_{i=1}^{\infty} P(A \cap B_i)$$
$$\therefore \frac{P\left(A \cap \left[\sum_{i=1}^{\infty} B_i\right]\right)}{P(A)} = \frac{\sum_{i=1}^{\infty} P(A \cap B_i)}{P(A)} \quad \text{[dividing both side by } P(A)]$$
$$\Rightarrow P\left(\sum_{i=1}^{\infty} B_i \middle| A\right) = \sum_{i=1}^{\infty} P(B_i|A). \quad ///.$$

Theorem:→ ~~Let $P(AB)>0$~~, Let $P(B\cap C)>0$. Then

$$P(A\cap B|C) = P(B|C)\,P(A|B\cap C)$$

Proof:→ Note that the conditional probabilities are all defined because $(B\cap C)\subset C \Rightarrow P(B\cap C)\le P(C) \Rightarrow P(C)>0$. Further,

$$P(A\cap B|C) = \frac{P([A\cap B]\cap C)}{P(C)}$$

$$= \frac{P(B\cap C)}{P(C)} \times \frac{P[A\cap [B\cap C]]}{P(B\cap C)}, \text{ since } P(B\cap C)>0$$

$$= P(B|C)\,P(A|B\cap C)$$

Theorem of Total probability :→

Statement:→ Suppose the event A can occur only along with the event B. Suppose also B can occur only in m mutually exclusive ways $B_1, B_2, \ldots, B_m$. Then

$$P(A) = \sum_{i=1}^{m} P(B_i)P(A|B_i), \text{ provided } P(B_i)>0, i=1,\ldots,m. \quad (*)$$

Proof:→

The events $AB_1, \ldots, AB_m$ are mutually exclusive. Thus

$$P(A) = P(AB)$$

$$= \sum_{i=1}^{m} P(AB_i)$$

$$= \sum_{i=1}^{m} P(B_i)\,P(A|B_i)$$

[Actually $(*)$ remains valid even if $P(B_i)=0$ for some i, when its contribution to the RHS of $(*)$ is zero.]

Corollary:→ For any event B which can occur along with (not necessarily always) A, $P(A) = P(AB) + P(AB^c)$.

Examples:→

1) An urn contains a white balls and b black balls; another contains c white balls and d black balls. One ball is transferred from the first into the second and then one ball is drawn from the later. What is the prob. that it is a white ball?

Ans:→ The ball transfered from the first urn may be a black (B) one or a white (W) one. Thus the possible events are BW, WW.

Now —

$$P(BW) = \frac{bc}{(a+b)(c+d+1)}, \quad P(WW) = \frac{a(c+1)}{(a+b)(c+d+1)}$$

Hence, the required prob. $= P(BW) + P(WW)$

$$= \frac{ac + bc + a}{(a+b)(c+d+1)}.$$

Ex.

2) Two players play a game as follows. Taking turns, they draw the balls out of an urn containing 'a' white and 'b' black balls, one ball at a time. He who extracts the first white ~~and~~ one wins the game. What's the prob. that the player who starts the game will win the game.

Ans:→ Let A, B be two players, A starting the game, the prob. of A to win $= P(W) + P(BBW) + P(BBBBW) + \cdots$

$$= \frac{a}{a+b}\left[1 + \frac{b(b-1)}{(a+b-1)(a+b-2)} + \frac{b(b-1)(b-2)(b-3)}{(a+b-1)(a+b-2)(a+b-3)(a+b-4)} + \cdots\right]$$

Ex. 3) Find the prob. that a single toss of a die results in a number less than 4 if (a) no other information is given, (b) it is given that the toss resulted in an odd number.

Soln:→ (a) Let B be the event {less than 4}. Since B is the union of the events 1, 2 or 3 turning up that —

$$P(B) = P(1) + P(2) + P(3) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}.$$ [assuming equal probabilities for the sample points]

(b) Letting A be the event {odd number} we see that $P(A) = \frac{3}{6} = \frac{1}{2}$. Also, $P(A \cap B) = \frac{2}{6} = \frac{1}{3}$. Then —

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{1/2} = \frac{2}{3}.$$

T.E

Statistical Independence of events

We have seen in the previous section that $P(B|A)$ does not generally equal $P(B)$. Thus the info. whether A has happened changes the prob. of occurance of B. If however, $P(B|A) = P(B)$, probabilities of B does not depend on whether A has or has not happened. In this case events A, B are said to be stochastically independent.

Independence of a pair of events: Two events A and B are said to be independent if the conditional prob. of occurance of B given the occurance of A is equal to the unconditional prob. of B i.e., if $P(B|A) = P(B)$, provided $P(B|A)$ is defined.

Thus for independent events A and B,

$$P(AB) = P(A)P(B|A) = P(A)P(B).$$

It follows that

$$P(B|A) = P(B) \Leftrightarrow P(A|B) = P(A),$$

because $P(A)P(B|A) = P(AB) = P(B)P(A|B)$.

Thus the independence of A and B implies that that the unconditional prob. of either event is equal to its conditional prob. given the other event.

Events A and B are stochastically independent iff —

$$P(AB) = P(A)P(B).$$

The defn of independence of events can easily be extended to any finite number of events. Given n events $A_1, \ldots, A_n$, $P(A_i | A_1 \ldots A_{i-1} A_{i+1} \ldots A_n)$ denotes the conditional prob. of A_i given that the events $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n$ have occured.

$$P(A_i | A_1 \ldots A_{i-1} A_{i+1} \ldots A_n) = \frac{P(A_1 \ldots A_n)}{P(A_1 \ldots A_{i-1} A_{i+1} \ldots A_n)}$$

if $P(A_1 \ldots A_{i-1} A_{i+1} \ldots A_n) > 0$.

* A collection $\mathcal{C}$ of events is called pairwise independent if whenever A, B are two distinct members of $\mathcal{C}$, A and B are independent.

4) Independence of a number of events :→ n events $A_1, \ldots, A_n$ are said to be independent if the conditional probabilities of any one of these events, say A_i, given one or more of the remaining events, is equal to the unconditional prob. of A_i.

Theorem :→ (Compound prob. for independent events) If $A_1, \ldots, A_n$ are independent events,

$$P(A_1 \cdots A_n) = P(A_1)P(A_2)\cdots\cdots P(A_n)$$

Proof :→ We have, for any set of events $A_1, \ldots, A_n$

$$P(A_1 \cdots A_n) = P(A_1)P(A_2|A_1)\cdots\cdots P(A_n|A_1\cdots A_{n-1})$$

Since $A_1, \ldots, A_n$ are independent we have

$$P(A_2|A_1) = P(A_2), \ldots\ldots, P(A_n|A_1\cdots A_{n-1}) = P(A_n).$$

The theorem follows.

Defn. :→ three events A, B, C are independent, by defn., if

$$P(A|BC) = P(A|B) = P(A|C) = P(A)$$
$$P(B|AC) = P(B|A) = P(B|C) = P(B)$$
$$P(C|AB) = P(C|A) = P(C|B) = P(C)$$

1) It is easy to veryfy three events A, B, C are stochastically independent iff they are pairwise independent i.e. if —

$$P(AB) = P(A)P(B),$$
$$P(AC) = P(A)P(C),$$
$$P(BC) = P(B)P(C)$$

and if also $P(ABC) = P(A)P(B)P(C)$. ///

Note that if A, B and C are independent, then $P(C|AB) = P(C)$

Defn. :→

2) The $m (\geq 2)$ events $A_1, A_2, \ldots, A_m$ are stochastically independent if —

$$\left.\begin{array}{ll} P(A_iA_j) = P(A_i)P(A_j) & i<j=1,\ldots,m \\ P(A_iA_jA_k) = P(A_i)P(A_j)P(A_k) & i<j<k=1,\ldots,m \\ \cdots\cdots\cdots & \\ P(A_1\cdots A_m) = P(A_1)P(A_2)\cdots P(A_m). & \end{array}\right\} \longrightarrow (*)$$

The number of conditions in $(*)$ is —

$$\binom{m}{2} + \binom{m}{3} + \cdots + \binom{m}{m} = 2^m - m - 1.$$

Theorem:- The events A, B are independent. Determine whether the events (a) A and B^c, (b) A^c and B^c are independent.

Proof:-

i) We have — $(AB^c) \cup (AB) = A$; also AB^c and AB are disjoint sets. Thus $P(A) = P(AB^c) + P(AB)$, $\Rightarrow P(A) - P(AB) = P(AB^c)$

Now, $P(A)P(B^c) = P(A)[1 - P(B)]$

$= P(A) - P(AB)$

$= P(AB^c)$.

Thus A, B^c are independent.

Or,

i) Let A and B be independent. Then

$P(A \cap B) = P(A)P(B)$.

Since $P(A \cap B) + P(A \cap B^c) = P(A)$,

this implies $P(A \cap B^c) = P(A) - P(A \cap B)$

$= P(A) - P(A)P(B)$

$= P(A)[1 - P(B)]$

$= P(A)P(B^c)$.

Hence, A and B^c are independent.

ii) We have $(AB^c) \cup (A^cB^c) = B^c$;

also $P(B^c) = P(AB^c) + P(A^cB^c)$.

Now, $P(A^c)P(B^c) = P(B^c)\{1 - P(A)\}$

$= P(B^c) - P(B^cA)$

$= P(A^cB^c)$

Hence A^c, B^c are independent.

[Similarly A^c and B are independent.]

Or,

ii) Let A and B be independent. Then

$P(A \cap B) = P(A)P(B)$.

Hence $P(A^c \cap B^c) = 1 - P(A \cup B)$

$= 1 - \{P(A) + P(B) - P(A \cap B)\}$

$= \{1 - P(A)\} - \{P(B) - P(A)P(B)\}$

$= \{1 - P(A)\}\{1 - P(B)\}$

$= P(A^c)P(B^c)$

Thus A^c and B^c are independent if A and B are.

Examples of ~~pairwise~~ pairwise and mutual independences :-

Example 1. Give an example that three events are pairwise independent but A, B, C are not independent (mutually).

Ans :- Let us suppose that for an experiment the sample space consists of four points only : $\omega_1, \omega_2, \omega_3, \omega_4$. Let

$$P(\{\omega_i\}) = p_i = \frac{1}{4}, \text{ for } i = 1, 2, 3, 4,$$

as would be the case in two throws of a perfect coin.

Consider three events A, B, C and assume that the outcomes are equally likely, and defined as follows :

$$A = \{\omega_1, \omega_2\}, \quad A_2 = \{\omega_1, \omega_3\} \text{ and } A_3 = \{\omega_1, \omega_4\}.$$

Suppose, $\Omega = \{1, 2, 3, 4\}$, $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{1, 4\}$.

Then, $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, $P(C) = \frac{1}{2}$ as they are equally likely cases.

$$\text{Now,} \quad P(A \cap B) = \frac{1}{4} = P(A)P(B)$$
$$P(A \cap C) = \frac{1}{4} = P(A)P(C)$$
$$P(B \cap C) = \frac{1}{4} = P(B)P(C),$$

So three events are pairwise independent but not mutually independent ~~exclusive~~ ~~independent~~ because ~~the~~ in case of mutually independence, it is necessary that $P(A \cap B \cap C) = P(A)P(B)P(C)$

However, in the present case,

$$P(A \cap B \cap C) = \frac{1}{4},$$

while, $P(A)P(B)P(C) = \frac{1}{8}$.

Thus this example is due to Bernstein.

Note that $P(C \mid A \cap B) \neq P(C)$ ///.

So, A, B, C are pairwise independent but A, B, C are not independent.

Example. 2. Consider families with 3 children.

$\Omega = \{(bbb), (bbg), (bgb), (bgg), (gbb), (gbg), (ggb), (ggg)\}$
where 'b' stands for boy and 'g' stands for girl. Assume the cases all are equally likely. Let A be the event that the family has children of both sexes and B be the event there ~~are~~ is at most one girl. Examine if A and B are stochastically independent.

Ans.: Here, $P(A) = \frac{6}{8} = \frac{3}{4}$, $A = \{bbg, bgb, bgg, gbb, gbg, ggb\}$

$P(B) = \frac{4}{8} = \frac{1}{2}$ $\quad B = \{bgb, gbb, bbg, bbb\}$

The event AB means the family has exactly one girl.

$P(AB) = \frac{3}{8}$, $\quad AB = \{bgb, gbb, bbg\}$

$$P(AB) = P(A)P(B) = \frac{3}{8}.$$

Thus, A, B are independent.

Ex. 3. Two dice are tossed. Let A denote the event of an odd total, B that of an ace on the first die, C that of having a total of seven. Are A, B, C mutually independent?

Ans.: Here $P(A) = \frac{18}{36} = \frac{1}{2}$,

where $A = \{(1,2), (1,4), (1,6), (2,1), (4,1), (6,1),$
$(2,3), (2,5), (3,2), (5,2),$
$(3,4), (3,6), (4,3), (6,3),$
$(4,5), (5,4),$
$(5,6), (6,5)\}$

$P(B) = \frac{1}{6}$, Where, $n = 6$, $n(A) = 1$.

$P(C) = \frac{6}{36} = \frac{1}{6}$, where, $C = \{(1,6), (6,1), (2,5), (5,2), (3,4), (4,3)\}$

$P(AB) = \frac{1}{12}$, $P(BC) = \frac{1}{36}$, $P(AC) = \frac{1}{6}$.

$P(ABC) = \frac{1}{36}$. Thus A, B, C are not mutually independent.

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Ex. 4. This is an example of pairwise independent events that are not independent. An urn contains four tickets bearing numbers 1234, 2341, 3412 and 4123 and one ticket is drawn. Let the events A, B, C be as follows:

A = 1st digit of the ticket drawn is 1 or 4.

B = 2nd digit of the ticket drawn is 2 or 4.

C = 3rd digit of the ticket drawn is 3 or 4.

Ans:→ $P(A) = \frac{2}{4} = \frac{1}{2}$, where $A = \{1234, 4123\}$

$P(B) = \frac{2}{4} = \frac{1}{2}$, where $B = \{1234, 3412\}$

$P(C) = \frac{2}{4} = \frac{1}{2}$, where $C = \{1234, 2341\}$

$P(AB) = \frac{1}{4}$,

$P(BC) = \frac{1}{4}$,

$P(AC) = \frac{1}{4}$.

Thus $P(ABC) = \frac{1}{4}$.

Obviously A, B, C are pairwise independent but $P(ABC) \neq P(A)P(B)P(C)$.

Hence A, B, C are not independent.

Ex. 5. Two cards are drawn one after another without replacement. Find the prob. that both are spades.

Ans:→ Let A_i be the event of drawing a spade in the i-th drawing, $i = 1, 2$. Then the required prob. $= P(A_1 \cap A_2) = P(A_1)P(A_2|A_1)$

$$= \frac{13}{52} \times \frac{12}{51}$$

$$= \frac{1}{17}.$$

Ex. 6. An urn contains 3 black and 4 white balls, another urn contains 4 black and 6 white balls. One ball is drawn from each urn. Find the prob. that both are white.

Ans:→ Let A_i be the event of drawing a white ball from the i-th urn, $i = 1, 2$. Note that A_1 and A_2 are independent. Hence the required prob. is $P(A_1 \cap A_2) = P(A_1)P(A_2) = \frac{4}{7} \times \frac{6}{10} = \frac{12}{35}$. ///

Ex. 7. Let $\Omega = \{1,2,3,4\}$ and $A_1 = \{1,2\}$, $A_2 = \{1\}$, $A_3 = \{1,2,3\}$, $A_4 = \{1,2,4\}$, $A_5 = \{1,3\}$ and $A_6 = \{4\}$.
Then $P(A_1)P(A_2) < P(A_1 \cap A_2)$, $P(A_3)P(A_4) > P(A_3 \cap A_4)$, $P(A_1)P(A_5) = P(A_1 \cap A_5)$. Note that A_1 and A_5 are independent but not mutually exclusive, while A_1 and A_6 are mutually exclusive but not independent.

Ex. 8. If two independent events A and B are such that $P(A \cap B^c) = \frac{3}{25}$ and $P(A^c \cap B) = \frac{8}{25}$ and $P(A) > \frac{1}{2}$. find $P(A)$ and $P(B)$.

Ans:

Let $P(A) = a$, $P(B) = b$. Then we are given that —

$$P(A \cap B^c) = \frac{3}{25} = P(A) - P(A \cap B) = a - ab, \quad \text{——— (1)}$$
$$P(A^c \cap B) = \frac{8}{25} = P(B) - P(A \cap B) = b - ab, \quad \text{——— (2)}$$

Hence solving (1) and (2),

$a = \frac{1}{5}$ and $b = \frac{2}{5}$, or $a = \frac{3}{5}$ and $b = \frac{4}{5}$.

As $a > 1/2$, we must have $P(A) = \frac{3}{5}$, $P(B) = \frac{4}{5}$. ///

~~Example 9~~

Independence of events :→

Defn 3. :→ A collection $\mathcal{C}$ of events is called independent if every finite subfamily of $\mathcal{C}$ is independent.

Ex. 9. Let $A_1, \dots, A_n$ ($n \geq 3$) be independent. Show that $A_1 \cup A_2, A_3, \dots, A_n$ are independent.

Soln :→ Let $B_1 = A_1 \cup A_2$, $B_i = A_{i+1}$ for $2 \leq i \leq n-1$.
Let $1 \leq i_1 < i_2 < \dots < i_k \leq n-1$, $k = 2, \dots, n-1$. Then —

$$P\left(\bigcap_{j=1}^{k} B_{i_j}\right) = \prod_{j=1}^{k} P(B_{i_j})$$

For, if $i_1 \neq 1$, then this is true since $\{B_{i_1}, \ldots, B_{i_k}\}$ being a subfamily of $\{A_3, \ldots, A_n\}$ are independent. If $i_1 = 1$ then –

$$P((A_1 \cup A_2) \cap B_{i_2} \cap \ldots \cap B_{i_k})$$

$$= P(A_1 \cap B_{i_2} \cap \ldots B_{i_k}) + P(A_2 \cap B_{i_2} \cap \ldots \cap B_{i_k}) - P(A_1 \cap A_2 \cap B_{i_2} \cap \ldots \cap B_{i_k})$$

$$= P(A_1) \prod_{j=2}^{k} P(B_{i_j}) + P(A_2) \prod_{j=2}^{k} P(B_{i_j}) - P(A_1)P(A_2) \prod_{j=2}^{k} P(B_{i_j})$$

[as $\{B_{i_2}, \ldots, B_{i_k}\}$ is a subfamily of $\{A_3, \ldots, A_n\}$]

$$= [P(A_1) + P(A_2) - P(A_1)P(A_2)] \prod_{j=2}^{k} P(B_{i_j})$$

$$= P(A_1 \cup A_2) \prod_{j=2}^{k} P(B_{i_j}) \,. \quad ///$$

Ex. 10. The events $A_1, \ldots, A_n$ $(n \geq 2)$ are independent iff $P(\bigcap_{i=1}^{n} B_i) = \prod_{i=1}^{n} P(B_i)$ for all choices of the B_i such that $B_i = A_i$ or A_i^c for $1 \leq i \leq n$.

Proof :–

Only if part :– The result is true for $n=2$. (before proved)

Suppose that the result is true for $n = m (\geq 2)$. Let $A_1, \ldots, A_m, A_{m+1}$ be independent.

Case I. $B_1 = A_1$ and $B_2 = A_2$

Let $E_1 = B_1 \cap B_2$. Hence the m events $E_1, A_3, A_4, \ldots, A_{m+1}$ are independent as can be easily checked using defn. By our inductive hypothesis

$$P(E_1 \cap B_3 \cap \ldots \cap B_{m+1}) = P(E_1)P(B_3) \ldots P(B_{m+1})$$

$$= P(B_1)P(B_2)P(B_3) \ldots P(B_{m+1})$$

$$= \prod_{i=1}^{m+1} P(B_i)$$

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Case II. $B_1 = A_1, B_2 = A_2^c$

$$P\left(\bigcap_{i=1}^{m+1} B_i\right) = P(A_1 \cap A_2^c \cap B_3 \cap \dots \cap B_{m+1})$$

$$= P(A_1 \cap B_3 \cap \dots \cap B_{m+1}) - P(A_1 \cap A_2 \cap B_3 \cap \dots \cap B_{m+1})$$

$$= P(A_1)\prod_{i=3}^{m+1} P(B_i) - P(A_1)P(A_2)\prod_{i=3}^{m+1} P(B_i)$$

$$= P(A_1)(1 - P(A_2))\prod_{i=3}^{m+1} P(B_i)$$

$$= P(A_1)P(A_2^c)\prod_{i=3}^{m+1} P(B_i)$$

$$= \prod_{i=1}^{m+1} P(B_i)$$

Case III. $B_1 = A_1^c$ and $B_2 = A_2^c$.

$$P\left(\bigcap_{i=1}^{m+1} B_i\right) = P((A_1 \cup A_2)^c \cap B_3 \cap \dots \cap B_{m+1})$$

$$= P[(A_1 \cup A_2)^c]\, P(B_3) \dots P(B_{m+1})$$

$$= (1 - P(A_1))(1 - P(A_2))\prod_{i=3}^{m+1} P(B_i)$$

$$= \prod_{i=1}^{m+1} P(B_i).$$

So, the result is true for $n = m+1$, and the proof is complete by the principle of mathematical induction.

If part :→ Let $P\left(\bigcap_{i=1}^{n} B_i\right) = \prod_{i=1}^{n} P(B_i)$ for all choices of B_i such that $B_i = A_i$ or A_i^c for $1 \le i \le n$. Then

$$P(A_1 \cap A_2 \cap \dots \cap A_{n-1})$$

$$= P(A_1 \cap A_2 \cap \dots \cap A_{n-1} \cap A_n) + P(A_1 \cap A_2 \cap \dots \cap A_{n-1} \cap A_n^c)$$

$$= \prod_{i=1}^{n} P(A_i) + \prod_{i=1}^{n-1} P(A_i)P(A_n^c)$$

$$= \prod_{i=1}^{n-2} P(A_i)$$

Similarly we can prove that $P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \dots P(A_{i_k})$ for $1 \le i_1 < i_2 < \dots < i_k \le n$ and $2 \le k \le n$. Hence $A_1, \dots, A_n$ are independent.

Ex. 11. The prob. that a teacher will give a surprise test during any class meeting is 3/5. If a student is absent on two days, what is the prob. that he will miss at least one test?

Ans:→ The required probability is $= 1 - P(\text{no test is missed})$

$= 1 - P(\text{no test on his first day of absence and no test on his second day of absence})$

$= 1 - P(\text{no test on his first day of absence}) \times P(\text{no test on his second day of absence})$

$= 1 - \left(\frac{2}{5}\right) \times \left(\frac{2}{5}\right)$

$= \frac{1}{5}$.

Ex. 12. If the prob. of n independent events are $p_1, \ldots, p_n$. find the prob. that (a) none of the events will occur, (b) at least one of the events will occur, (c) at most one of the events will occur.

Soln.:→ Let the events be $A_1, \ldots, A_n$ and $P(A_i) = p_i$ for $1 \le i \le n$.

(a) Clearly, $P(\text{none of the events will occur})$

$= P\left(\bigcap_{i=1}^{n} A_i^c\right) = \prod_{i=1}^{n} P(A_i^c) = \prod_{i=1}^{n} (1 - p_i)$

(b) $P(\text{at least one of the events will occur})$

$= 1 - P(\text{none of the events will occur})$

$= 1 - \prod_{i=1}^{n} (1 - p_i)$.

(c) $P(\text{exactly one of the events will occur})$

$= P(A_1 \cap A_2^c \cap A_3^c \cap \ldots \cap A_n^c) + P(A_1^c \cap A_2 \cap A_3^c \cap \ldots \cap A_n^c) + \ldots$

$+ P(A_1^c \cap A_2^c \cap \ldots \cap A_{n-1}^c \cap A_n)$

$= p_1(1-p_2)\ldots(1-p_n) + p_2(1-p_1)(1-p_3)\ldots(1-p_n) + \ldots$

$\ldots + (1-p_1)(1-p_2)\ldots(1-p_{n-1})p_n$ [By ex. 10]

$= \prod_{i=1}^{n} (1-p_i) \left(\sum_{i=1}^{n} p_i/(1-p_i) \right)$

Since, $P(\text{at most one of the events will occur})$

$= P(\text{exactly one of the events will occur}) + P(\text{none of the events will occur})$

$= \prod_{i=1}^{n} (1-p_i) + \prod_{i=1}^{n} (1-p_i) \left(\sum_{i=1}^{n} p_i/(1-p_i) \right) = \prod_{i=1}^{n} (1-p_i) \left[1 + \sum_{i=1}^{n} p_i/(1-p_i) \right]$

Ex. 13. A can solve 75% of problems of a Maths. book while B can solve 70% of problems of the book. What is the chance that a problem selected at random will be solved when both ~~both~~ A and B try?

Ans:→ $P(A_1) = 0.75$ and $P(A_2) = 0.70$ where A_1 and A_2 are, respectively, the event that A can solve the problem and B can solve the problem. Since A_1 and A_2 are independent, the required prob. is –

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$
$$= P(A_1) + P(A_2) - P(A_1)P(A_2)$$
$$= 0.925.$$

Ex. 14. A die is rolled twice. Let A, B and C denote the events that the sum of scores is 6, the sum of scores is 7 and the first score is 4. Are A, C independent? Are B, C independent?

Ans:→ $P(A) = \frac{5}{36}$, where $A = \{(1,5), (5,1), (2,4), (4,2), (3,3)\}$

$P(B) = \frac{6}{36} = \frac{1}{6}$, where $B = \{(1,6), (6,1), (2,5), (5,2), (4,3), (3,4)\}$

$P(C) = \frac{1}{6}$.

$P(A \cap C) = \frac{1}{36}$, $P(B \cap C) = \frac{1}{36}$.

$P(A)P(C) \neq P(A \cap C)$ and $P(B)P(C) = P(B \cap C)$

So that, A, C are not independent but B, C are independent.

Problems on Conditional Probability and Independent events

1) A fair die is tossed twice. Find the prob. of getting a 4, 5 or 6 on the first toss and a 1, 2, 3 or 4 on the second toss.

Ans: Let A_1 be the event "4, 5 or 6 on first toss", and A_2 be the event "1, 2, 3 or 4 on second toss". Then we are looking for $P(A_1 \cap A_2)$.

Method. 1. $P(A_1 \cap A_2) = P(A_1) P(A_2|A_1) = P(A_1)P(A_2) = \left(\frac{3}{6}\right)\left(\frac{4}{6}\right) = \frac{1}{3}$.

We have used here the result of the second toss is independent of the first so that $P(A_2|A_1) = P(A_2)$. Also we have used $P(A_1) = \frac{3}{6}$ (since 4, 5 or 6 are 3 out of 6 equally likely probabilities) and $P(A_2) = \frac{4}{6}$ (since 1, 2, 3 or 4 are 4 out of 6 equally likely possibilities).

Method. 2. Each of the 6 ways in which a die can fall on the first toss can be associated with each of the 6 ways in which it can fall on the second toss, a total of $6.6 = 36$ ways, all equally likely.

Each of the 3 ways in which A_1 can occur can be associated with each of the 4 ways in which A_2 occur to give $3.4 = 12$ ways in which both A_1 and A_2 can occur. then

$$P(A_1 \cap A_2) = \frac{12}{36} = \frac{1}{3}.$$

2) One bag contain 4 white balls and 2 black balls; another contains 3 white balls and 5 black balls. If one ball is drawn from each bag. Find the prob. that (a) both are white, (b) both are black, (c) one is white and one is black.

Ans: Let W_1 = event "white ball from first bag", W_2 = event "white ball from second bag."

(a) $P(W_1 \cap W_2) = P(W_1) P(W_2|W_1) = P(W_1) P(W_2) = \left(\frac{4}{4+2}\right)\left(\frac{3}{3+5}\right) = \frac{1}{4}$.

(b) $P(W_1' \cap W_2') = P(W_1') P(W_2'|W_1') = P(W_1') P(W_2') = \left(\frac{2}{4+2}\right)\left(\frac{5}{3+5}\right) = \frac{5}{24}$.

(c) The required prob. is —

$$1 - P(W_1 \cap W_2) - P(W_1' \cap W_2') = 1 - \frac{1}{4} - \frac{5}{24} = \frac{13}{24}.$$

3) Three balls are drawn successively from a box containing 6 red balls, 4 white balls and 5 blue balls. Find the prob. that they are drawn in the order red, white and blue if each ball is (a) replaced, (b) not replaced.

Ans:- Let R_1 = event "red on first draw", W_2 = event "white on second draw", B_3 = event "blue on third draw". We require $P(R_1 \cap W_2 \cap B_3)$.

(a) If each ball is placed, then the events are independent and

$$P(R_1 \cap W_2 \cap B_3) = P(R_1)P(W_2|R_1)P(B_3|R_1 \cap W_2)$$
$$= P(R_1)\,P(W_2)P(B_3)$$
$$= \left(\frac{6}{6+4+5}\right)\left(\frac{4}{6+4+5}\right)\left(\frac{5}{6+4+5}\right)$$
$$= \frac{8}{225},$$

(b) If each ball is not replaced, then the events are dependent and

$$P(R_1 \cap W_2 \cap B_3) = P(R_1)\,P(W_2|R_1)P(B_3|R_1 \cap W_2)$$
$$= \left(\frac{6}{6+4+5}\right)\left(\frac{4}{5+4+5}\right)\left(\frac{5}{5+3+5}\right)$$
$$= \frac{4}{91}.$$

4) Find the prob. of a 4 turning up at least once in two tosses of a fair dice.

Ans:- Let A_1 = event "4 on first toss" and A_2 = event "4 on second toss". Then $A_1 \cup A_2$ = event "4 on first toss or 4 on second toss or both".
= event "at least one 4 turns up."

and we require $P(A_1 \cup A_2)$

Method 1. Events A_1 and A_2 are not mutually exclusive, but they are independent.

We consider

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$
$$= P(A_1) + P(A_2) - P(A_1)P(A_2)$$
$$= \frac{1}{6} + \frac{1}{6} - \frac{1}{36}$$
$$= \frac{11}{36}$$

Method 2. $P(\text{at least one 4 comes up}) = 1 - P(\text{no 4 comes up})$

$$= 1 - P(\text{no 4 on 1st toss and no 4 on 2nd toss})$$

$$= 1 - P(A_1' \cap A_2')$$

$$= 1 - P(A_1')P(A_2')$$

$$= 1 - \left(\frac{5}{6}\right)\left(\frac{5}{6}\right) = \frac{11}{36}.$$

5) Two cards are drawn from a well-shuffled ordinary deck of 52 cards. Find the prob. that they are both aces if the first card is (a) replaced, (b) not replaced.

Ans:

Method 1. Let A_1 = event "ace on first draw" and A_2 = event "ace on second draw". Then we are looking for $P(A_1 \cap A_2) = P(A_1)P(A_2|A_1)$

(a) Since for the first drawing there are 4 aces in 52 cards, $P(A_1) = \frac{4}{52}$. Also, if the card is replaced for the second drawing then $P(A_2|A_1) = \frac{4}{52}$, since there are also 4 aces out of 52 cards for the second drawing. Then –

$$P(A_1 \cap A_2) = P(A_1)\, P(A_2|A_1) = \left(\frac{4}{52}\right)\left(\frac{4}{52}\right) = \frac{1}{169}.$$

(b) Here $P(A_1) = \frac{4}{52}$. However if an ace occurs on the first drawing there will be only 3 aces left in the remaining 51 cards, so that $P(A_2|A_1) = \frac{3}{51}$. Then –

$$P(A_1 \cap A_2) = P(A_1)\, P(A_2|A_1) = \left(\frac{4}{52}\right)\left(\frac{3}{51}\right) = \frac{1}{221}.$$

Method 2.

(a) The first card can be drawn in any one of 52 ways and since there is replacement, the second card can also be drawn in any one of 52 ways. Then both cards can be drawn in (52)(52) ways, all equally likely.

In such case there are 4 ways of choosing an ace on the first draw and 4 ways of choosing an ace on the second draw show the number of ways of choosing aces on the first and second draws is (4)(4), then the required probability is –

$$\frac{(4)(4)}{(52)(52)} = \frac{1}{169}.$$

(b) Here in this case, replacement is not allowed.

So the required probability is –

$$\frac{(4)(3)}{(52)(51)} = \frac{1}{221}.$$

Bayes Theorem :→

Suppose an event A can occur if and only if one of the hypothesis $B_1, \ldots, B_k$ is true; the prob. $P(B_i)$ of occurance of B_i is known for each i, $i = 1, \ldots, k$. Also known as the conditional probability $P(A|B_i)$ of occurance of A given that B_i has already occured, $i = 1, \ldots, k$. We want to find the conditional prob. $P(B_i|A)$ of occurance of B_i given that A has already occurred. This is given in Bayes theorem that follows.

Generally, $P(B_i)$ and $P(B_i|A)$ will not be the same. Thus the occurence of A generally changes one's assignment of probabilities to different hypotheses. The probabilities $P(B_i)$ which are assigned to B_i without any ~~preference~~ reference to A are called 'a priori' probabilities, $i = 1, \ldots, k$. The probabilities $P(B_i|A)$ which are calculated after A has been observed are called 'a posteriori' probabilities, $i = 1, \ldots, k$. Our main interest lies here in the hypothesis $B_1, \ldots, B_k$.

Theorem :→ Suppose the events $B_1, \ldots, B_k$ are mutually exclusive and exhaustive and none of them has zero probability. Further, let A be an event which, too, has non-zero probability, ~~then, for any event~~ Then, the posterior prob. $P(B_i|A)$ is given by

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_{j=1}^{k} P(B_j)P(A|B_j)}, \text{ for } i = 1, 2, \ldots, k.$$

Proof :→

Since $P(B_i) > 0$ for $i = 1, \ldots, k$ and $P(A) > 0$, we have, for each i,

$$P(B_i \cap A) = P(B_i)P(A|B_i) = P(A)P(B_i|A)$$

or, $$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{P(A)}$$

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But, as the events $B_1, \ldots, B_k$ are exhaustive and mutually exclusive,

$$P(A) = \sum_{j=1}^{k} P(B_j \cap A) = \sum_{j=1}^{k} P(B_j) . P(A \mid B_j)$$

So, $$P(B_i \mid A) = \frac{P(B_i)\, P(A \mid B_i)}{\sum_{j=1}^{k} P(B_j)\, P(A \mid B_j)}.$$

— It's known as Bayes' theorem.

Theorem:→ Let C be an event in Ω. Then under the conditions of Baye's theorem together with the condition $P(B_i \cap A) > 0$ for each i,

$$P(C \mid A) = \frac{\sum_{i=1}^{\infty} P(B_i)\, P(A \mid B_i)\, P(C \mid B_i \cap A)}{\sum_{i=1}^{\infty} P(B_i)\, P(A \mid B_i)}$$

Proof:→ Since $B_1, B_2, \ldots$ are exhaustive and mutually exclusive events,

$$P(C \mid A) = \sum_{i=1}^{\infty} P(B_i \cap C \mid A)$$

$$= \sum_{i=1}^{\infty} P(B_i \mid A)\, P(C \mid B \cap A_i)$$

$$= \frac{\sum_{i=1}^{\infty} P(B_i)\, P(A \mid B_i)\, P(C \mid B \cap A_i)}{\sum_{i=1}^{\infty} P(B_i)\, P(A \mid B_i)} \quad \text{[By Baye's theorem]}$$

Hence, the result is proved.

Examples on Baye's Theorem :→

Ex.1. The first of three urns contains 7 white and 10 black balls, the second contains 5 white and 12 black balls and the third contains 17 white balls (and no black ball). A person chooses an urn at random and draws a ball from it. The ball is white. Find the probabilities that the ball came from (i) the first, (ii) the second, (iii) the third urn.

Soln :→ Let H_i be the hypothesis that the ith urn was chosen and E be the event a white ball is drawn.

$$P(H_i) = \frac{1}{3}, \quad i = 1, 2, 3.$$

$$P(E|H_1) = \frac{7}{7+10} = \frac{7}{17},$$

$$P(E|H_2) = \frac{5}{5+12} = \frac{5}{17},$$

$$P(E|H_3) = \frac{17}{17} = 1.$$

From Baye's theorem, we know —

$$P(H_i|E) = \frac{P(H_i)\,P(E|H_i)}{\sum_{i=1}^{3} P(H_i)\,P(E|H_i)}$$

So,
$$P(H_1|E) = \frac{P(H_1)\,P(E|H_1)}{\sum_{i=1}^{3} P(H_i)\,P(E|H_i)}$$

$$= \frac{\frac{1}{3}\cdot\frac{7}{17}}{\left(\frac{1}{3}\cdot\frac{7}{17}\right)+\left(\frac{1}{3}\cdot\frac{5}{17}\right)+\left(\frac{1}{3}\cdot 1\right)} = \frac{7}{29},$$

$$P(H_2|E) = \frac{5}{29}, \quad P(H_3|E) = \frac{17}{29}.$$

Ex. 2.

The production lines manufacture the same type of items. In a given time line 1 turns out n_1 items of which n_1p_1 are defectives; in the same line, line 2 turns out n_2 items of which n_2p_2 are defectives. Suppose a unit is selected at random from the combined lot produced by the two lines. Let D be the event of a defective item, A the event the unit was produced by line 1 and B the event it was produced by line 2. Determine $P(A|D)$, $P(B|D)$.

Sol^n :→

$$P(A) = \frac{n_1}{N}, \quad P(B) = \frac{n_2}{N} \quad \text{where } N = n_1 + n_2$$

$$P(D|A) = p_1, \quad P(D|B) = p_2 .$$

Hence $$P(A|D) = \frac{P(A)\,P(D|A)}{P(A)P(D|A) + P(B)P(D|B)} = \frac{n_1p_1}{n_1p_1 + n_2p_2}$$

Similarly, $$P(B|D) = \frac{n_2p_2}{n_1p_1 + n_2p_2}$$

If $n_1p_1 > n_2p_2$; $P(A|D) > P(B|D)$.

Problems:-

1) It is known that the population of a certain city is 45% female and 55% male. Suppose that 70% of the males and 10% of the females smoke. Find the prob. that a smoker is male.

Ans:- Let M and F denote respectively the events that a person selected is a male and a female. Let S denotes the event the person selected smokes. We are given that –

$$P(S|M) = 0.70, \quad P(S|F) = 0.10, \quad P(M) = 0.55, \quad P(F) = 0.45.$$

Hence $$P(M|S) = \frac{P(S|M)\,P(M)}{P(S|M)P(M) + P(S|F)P(F)}$$

$$= \frac{0.70 \times 0.55}{(0.70 \times 0.55) + (0.10 \times 0.45)} = 0.895 .$$

2) There are 3 boxes each having two drawers. The first box has a gold coin in each drawer, the second box has a gold coin in one drawer and a silver coin in the other drawer, and the third box has a silver coin in each drawer. A box is chosen at random and a drawer opened. If the drawer contains a gold coin, what's the prob. that the other drawer also contains a gold coin?
(example of Bayes' theorem)

Soln :→ Let A_i be the event that the ith box is selected. Let B be the event that the coin observed was gold. Then $P(A_i) = \frac{1}{3}$ for $i = 1, 2, 3$; $P(B|A_1) = 1$, $P(B|A_2) = \frac{1}{2}$ and $P(B|A_3) = 0$. The required probability is —

$$P(\text{the second drawer has a gold coin}|B)$$

$$= P(A_1|B) = \frac{P(A_1)\ P(B|A_1)}{\sum_{i=1}^{3} P(A_i)\ P(B|A_i)}$$

$$= \frac{1/3 \times 1}{\left(\frac{1}{3} \times 1\right) + \left(\frac{1}{3} \times \frac{1}{2}\right) + \left(\frac{1}{3} \times 0\right)}$$

$$= \frac{2}{3} . ///$$

3) A box has 12 red balls and 6 black balls. A ball is selected from the box. If the ball is red, it is returned to the box. If the ball is black, it and 2 additional black balls are added to the box. Find the prob. that a second ball drawn from the box is (a) red ; (b) black.
(example of total probability theorem)

Soln :→ [redacted]

Let R_i and B_i be respectively the event that the i-th ball drawn is red and the event that the i-th ball drawn is black, $i = 1, 2$.

$$P(R_1) = \frac{12}{12+6} = \frac{2}{3} , \quad P(B_1) = \frac{6}{12+6} = \frac{1}{3} ,$$

[redacted] =

(a) The required prob. is

$$P(R_2) = P(R_1)P(R_2|R_1) + P(B_1)P(R_2|B_1)$$
$$= \frac{2}{3} \times \frac{2}{3} + \frac{1}{3} \times \frac{12}{20}$$
$$= \frac{29}{45}.$$

(b) the required prob. is

$$P(B_2) = P(R_1)P(B_2|R_1) + P(B_1)P(B_2|B_1)$$
$$= \frac{2}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{8}{20} = \frac{16}{45}.$$

4) In a certain factory, machines 1, 2 and 3 are all producing pens. Of their production, machines 1, 2 and 3 produces 2%, 1% and 3% defective pens respectively. Of the total production of pens, machine 1 produces 35%, machine 2 produces 25% and machine 3 produces 40%. If one pen is selected at random from the total pens produced, show that the prob. that it is defective is 0.0215. Moreover, if the pen selected is defective, show that the conditional probability that it was produced by machine 3 is $\frac{120}{215}$.

Soln :> Let A_i be the event that the pen was produced by the ith machine, $i = 1, 2, 3$. Let B the event that the pen is defective. Then

$$P(B) = \sum_{i=1}^{3} P(A_i)P(B|A_i)$$
$$= (0.035 \times 0.02) + (0.25 \times 0.01) + (0.40 \times 0.03)$$
$$= 0.0215.$$

Also, $P(A_3|B) = \frac{P(A_3)P(B|A_3)}{P(B)}$.

$$= \frac{0.012}{0.0215} = \frac{120}{215}. ///$$

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5) In answering a question on a multiple choice test, an examinee either knows the answer (with probability p) or he guesses (with probability $1-p$). Let the prob. of answering the question correctly be 1 for an examinee who knows the answer and $\frac{1}{n}$ for one who guesses, n being the number of multiple-choice alternatives. What is the prob. p_n that an examinee knows the answer, given that he answers the question correctly? Show that $p_n > p$ and that p_n is a strictly increasing sequence for fixed p $(0 < p < 1, n \geq 2)$. [Parzen]

Soln. :→ Let A be the event that the examinee knows the answer and B the event that he answers correctly. The required probability is

$$p_n = P(A|B) = \frac{P(A)\,P(B|A)}{P(A)\,P(B|A) + P(A^c)\,P(B|A^c)}$$

$$= \frac{p \times 1}{p \times 1 + (1-p) \times \frac{1}{n}}$$

$$= \frac{np}{1 + (n-1)p}.$$

That $p_n > p$ follows from $p + (1-p)/n < 1$. Finally,

$$p_{n+1} - p_n = p(1-p) / \{(1+np)(1+(n-1)p)\} > 0. ///$$

6) If in a bridge game North and South have a total of 8 spades among them, what is the prob. that East has 3 of the remaining 5 spades?

Soln. :→ We work with the reduced sample space. That is, given that North-South have a total of 8 spades among their 26 cards, there remains a total of 26 cards, exactly 5 of them being spades to be distributed among the East-West hands. Assuming each distribution is equally likely, the required probability is

$$\frac{\binom{5}{3}\binom{21}{10}}{\binom{26}{13}} = 0.339. ///$$

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7) There are a white balls and b black balls in a urn, n balls are drawn from the urn and it is found that all of them are black. What is the the prob. that another ball drawn from those remaining will also be black? ($n < b$).

Ans:→ We work with the reduced sample space. Given that the n balls drawn are black, there remains $(a+b-n)$ balls of which $(b-n)$ are black. Hence the required probability is

$$(b-n)/(a+b-n). ///$$

Sample Spaces having Equally Likely Outcomes

1) A company while purchasing goods tests 5 articles from each consignment of 50 articles and if all tested articles are nondefective, the consignment is accepted. What's the prob. that a consignment containing 10 defective articles will be accepted?

Soln.:→ Since, the specific consignment had been accepted after testing, 5 articles from the consignment were tested and were found non-defective. The $\binom{50}{5}$ combinations of drawing 5 articles from 50 articles constitute the sample space, and are assumed to be equally likely. Let A denote the event of finding all the tested articles nondefective. Then these five articles must have been chosen from the 40 nondefective articles of the consignment. Hence $n(A) = \binom{40}{5}$. Thus the required probability is

$$P(A) = \binom{40}{5} \Big/ \binom{50}{5} = 0.3106. ///$$

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2) What is the prob. that the 4 children in a family have different birthdays? What is the prob. that (a) at least two of them have the same birthday, (b) only the oldest and the youngest have the same birthday? (Assume one year = 365 days)

Soln :->
The four children of the family can have birthdays in $(365)^4$ ways, since each of them can be born in any one of 365 days. We assume that these ways are equally likely. In order that they have different birthdays, the birthday of the first child can be selected in 365 days, the birthday of the second child can be selected in 364 days, the birthday of the third child can be selected in 363 ways, the birthday of the fourth child can be selected in 362 ways. So the number of ways favourable to the event of having different birthdays for all the four children is $(365)_4$ and the required probability is

$$\frac{(365)_4}{(365)^4} = 0.9836.$$

(a) P(at least two have the same birthday)

$$= 1 - P(\text{all four have different birthday})$$

$$= 1 - \frac{(365)_3}{(365)^3} = 0.0164$$

(b) Since the oldest and the youngest will have the same birthday. The number of ways it can take place is 365×1. So the number of ways the birthdays of the four childrens can be selected subject to the restriction that only the oldest and the youngest have the same birthday is $365 \times 1 \times 364 \times 363$. Hence the required prob. is

$$(365 \times 1 \times 364 \times 363) / (365)^4 = \frac{364 \times 363}{(365)^3} = 0.0027.$$

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3) A lady had five dissimilar pairs of gloves. If she selects four gloves at random, what is the prob. that there will be no complete pair among them?

Soln :→ Since there are 10 gloves altogether and four of them are selected, the selection can be made in $10\times9\times8\times7$ ways which are assumed to be equally likely. In order that there is no complete pair, after the first is selected in 10 ways, the second is selected in 8 ways, the third is selected in 6 ways, and the fourth in 4 ways. Hence the number of favourable ways of the desired event is $10\times8\times6\times4$. So, the required prob. is

$$(10\times8\times6\times4)/10\times9\times8\times7 = \frac{8}{21}. ///$$

4) Five cards are drawn from a full pack. Find the prob. that there are 3 spades.

Soln :→ Five cards can be drawn in $\binom{52}{5}$ ways which assumed to be equally likely. In order that 3 of them are spade, 3 cards must be from 13 spade cards and the remaining two must be drawn from 39 non-spade cards. This drawing of 5 cards can thus be done in $\binom{13}{3}\binom{39}{2}$ ways. Hence the required prob. is

$$\binom{13}{3}\binom{39}{2} \Big/ \binom{52}{5} = 0.0815. ///$$

5) A committee of 5 persons is formed from 8 gentleman and 3 ladies. What is the prob. that the committee contains (a) exactly 2 ladies, (b) at least one lady?

Soln :→ Five persons can be selected from 11 persons in $\binom{11}{5}$ ways which are assumed to be equally likely. In order that exactly two of them are ladies, they must be selected from 3 ladies and the remaining 3 must be selected from 8 gentlemen; this selection of 5 persons can therefore be made in $\binom{3}{2}\binom{8}{3}$ ways. Hence the prob. in case (a) is $\binom{3}{2}\binom{8}{3}\Big/\binom{11}{5} = \frac{12}{33}$.

The prob. in case (b) is $= 1 - P(\text{committee include no lady})$

$$= 1 - \frac{\binom{8}{5}}{\binom{11}{5}} = \frac{29}{33} ///$$

6) Find the prob. that in a throw of 12 dice, each face occurs twice.

Soln. :→ As each die can show any one of the 6 faces, there are 6^{12} possible cases which are assumed to be equally likely. The number of favourable cases to the event that each face occurs twice is the number of ways in which 12 dice can be arranged in 6 groups of size 2 each, the first group consisting of two dice each showing the point 1, the second group of two dice each showing the point 2 and so on; this number is $\frac{12!}{(2!)^6}$.

Hence the prob. is —

$$\frac{\frac{12!}{(2!)^6}}{6^{12}} = 0.00344. ///$$

7) Five executives including the secretary and the president sit in a round table. Find the prob. that the secretary and the president sit side by side.

Soln. :→ Five executives can sit in a round table in $4! = 24$ ways which are assumed to be equally likely. Of course 24 cases, ~~[illegible]~~ Consider them as a single person, then another three can sit in $3!$ ways, then they can sit $2!$ ways with in themselves. So, the no. of favourable cases to the event is $= 3!\,2!$, of secretary and president sitting side by side. Hence the required prob. is $\frac{3!\,2!}{4!} = \frac{1}{2}. ///$

8) In a bridge game the North and the South get 9 spades between them. Find the prob. that East does not get any spade.

Soln. :→ 26 cards are distributed to north and south. So, the number of ways the East can get his share of 13 cards in $\binom{26}{13}$ which are assumed to be equally likely. In order that he does not get any spade, East and west get 26 cards in total, each 13 cards, all the 13 cards of East come from non-spades. There being $(26-4) = 22$ non-spades, his 13 cards are chosen from these 22 cards which can be done in $\binom{22}{13}$ ways. Reqd. prob. $= \binom{22}{13} / \binom{26}{13} = 0.0478. ///$

Q) (a) The digits 1, 2, 3, 4, 5, 6 and 7 are written down at random order to give a 7-digit number. What is the prob. that the number is divisible by 4?

Soln. :→ The digits can be arranged in a total of 7! ways, As these digits are written down in random order, the outcomes can be assumed to be equally likely. A 7-digit number may be looked upon as the sum of two parts:

(a) 100 times the number formed by the first 5 digit and

(b) the number formed by the last 2 digit.

The first part is always divisible by 4. Hence for the 7-digit number to be divisible by 4, it is only necessary that the last two digits form a multiple of 4. This will be the case if the last two digits are 12, 16, 24, 32, 36, 52, 56, 64, 72 or 76. In each case the first five places of the number can be filled in 5! ways, so that there is a total of 10×5! favourable elementary events. The required prob. is, therefore,

$$\frac{10 \times 5!}{7!} = \frac{5}{21}.$$

(b) The digits 1, 2, 3, 4 and 5 are written down at random order to give a 5-digit number. ~~what is the~~ what's the prob. that the number is divisible by 4.

Soln. :→ The sample space consists of 5! permutations of the given five digits. As these digits are written down in random order, the outcomes can be assumed to be equally likely. To get number of cases favourable to the event 'the number is divisible by 4', note that each such number can be expressed as $100\alpha + \beta$ where α is the number formed by the first three digits and β is the number formed by the last two digits, and so is divisible by 4 iff its last two digits are any one of 12, 24, 32, 52. Thus the number of favourable cases is $3! \times 4$. The required probability is therefore $= \frac{3! \times 4}{5!} = \frac{1}{5}$. ///

10) What is the probability of getting 25 points in 5 throws of a die?

Soln :→ There are six possible cases as to the number of points obtained in each throw. An elementary event may therefore be represented by a vector

$$(i_1, i_2, i_3, i_4, i_5),$$

where $i_1 = 1, 2, 3, 4, 5$ or 6 and represents the number of points obtained in the 1st throw, and similarly for the other components. The total no. of elementary events in the sample space is, then,

$$6 \times 6 \times 6 \times 6 \times 6 = 6^5.$$

Provided the die is perfect and provided each throw is made without giving any conscious advantage to any particular face to turn up, these 6^5 elementary events may be supposed to be equally likely.

As to the number of elementary events favourable to the occurance of 25 points, we see that it is the same as the co-efficient of x^{25} in the expansion of

$$(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^5 = x^5\left(\frac{1-x^6}{1-x}\right)^5.$$

But this again, is the same as the co-efficient of x^{20} in the expansion of

$$(1-x^6)^5(1-x)^{-5}. \quad \text{——— (1)}$$

Now,

$$(1-x^6)^5(1-x)^{-5}$$

$$= (1 - 5x^6 + 10x^{12} - 10x^{18} + \dots)\left(1 + 5x + \frac{5\times6}{2!}x^2 + \frac{5\times6\times7}{3!}x^3 + \dots\right)$$

Hence the coefficient of x^{20} in (1) is —

$$\frac{(24)_{20}}{20!} - 5\times\frac{(18)_{14}}{14!} + 10\times\frac{(12)_8}{8!} - 10\times\frac{(6)_2}{2!}$$

$$= 126.$$

As such, the required probability is —

$$\frac{126}{6^5} = \frac{7}{432}.$$

11) What is the probability of getting 26 points in a throw of 6 perfect dice?

Soln :> Let $A = \{1,2,3,4,5,6\}$; the sample space for this problem is $\Omega = \underbrace{A \times A \times A \times \cdots \times A}_{n\ \text{times}}$. Hence the total number of outcomes is 6^6. Assuming that the throw of 6 dice is done at random, all these outcomes can be assumed to be equally likely. If m is the number of favourable cases to the event of getting 26 points, then m is the number of 6-tuples $(i_1, i_2, \dots, i_6)$ such that $i_1 + \cdots + i_6 = 26$ and each $i_k \in A$. Thus m is the co-efficient of x^{26} in

$$(x^1+x^2+x^3+\cdots+x^6)^6 = x^6(1-x^6)^6(1-x)^{-6}.$$

Now,

$$(1-x^6)^6(1-x)^{-6} = (1-6x^6+15x^{12}-20x^{18}+\cdots)\sum_{n=0}^{\infty}\binom{n+5}{n}x^n \quad \text{------ (2)}$$

So that, m = the coefficient of x^{20} in the right side of (2). Hence

$$m = \binom{25}{20} - 6\times\binom{19}{14} + 15\times\binom{13}{8} - 20\binom{7}{2}$$

$$= 2247.$$

Hence, the required probability $= 2247 \times 6^{-6} = 0.0482.$ ///

12) Two fair dice are tossed ten times. Find the prob. that the first three tosses result in a sum of seven and the last seven result in a sum of eight.

Soln :> Here the sample space is $\Omega = \prod_{i=1}^{10} \Omega_i$ where $\Omega_1 = \Omega_2 = \cdots = \Omega_{10} = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$. Hence the total number of outcomes is 36^{10}, which are assumed to be equally likely. Let

$A = \{$Sum of first 3 tosses is 7 and the last 7 tosses is 8$\}$

$B_s = \{(i,j) \in \Omega_1 : i+j = s\}$, $s = 2, 3, \dots, 12$

Then $A = B_7 \times B_7 \times B_7 \times B_8 \times B_8 \times B_8 \times B_8 \times B_8 \times B_8 \times B_8$. By direct counting $n(B_7) = 6$, where $\Omega = \{(2,5),(5,2),(4,3),(3,4),(1,6),(6,1)\}$, and $n(B_8) = 5$, where $\Omega = \{(3,5),(5,3),(4,4),(6,2),(2,6)\}$.

So, $n(A) = 6\times6\times6\times5\times5\times5\times5\times5\times5\times5 = 6^3 \times 5^7$.

So, the required prob. is $= (6^3 \times 5^7)/36^{10}$.

13) There are n men ~~along~~ among whom are A and B.
(a) If they stand at random in a row, what's the prob. that A and B will be separated by exactly r men,
(b) If they stand in a ring, show that the prob. is $\frac{1}{(n-1)}$ (consider only the positive direction).

Soln. :→
(a) The n men can stand in a row in $n!$ ways which are assumed to be equally likely. Consider now an arrangement in which A stands to the left of B and there are exactly r men between A and B. A can stand in any ~~one~~ one of the positions 1st, 2nd, ..., $(n-r-1)$th, and for such a position of A, the remaining $(n-2)$ men can stand in $(n-2)$ positions in $(n-2)!$ ways, the position of B being fixed and such that there are exactly r men between A and B. Hence the number of ways in which A stands to the left of B and r men stand between A and B is $(n-r-1)(n-2)!$. But B can stand to the left of A as well. Hence the number of favourable cases is $2(n-r-1)(n-2)!$, the required prob. is $2(n-r-1)/\{n(n-1)\}$.

(b) We fix the position of A in the ring. Then the total number of ways n men can stand in a ring is $(n-1)!$. To count the number of favourable cases, note that the position of B is also fixed and $(n-2)$ men can stand in $(n-2)!$ ways. Thus the required probability is $\frac{(n-2)!}{(n-1)!} = \frac{1}{(n-1)}$. ///

14) If n balls are distributed into m boxes so that each ball is equally likely to fall in any box, show that the prob P_r that a specified box will contain, r balls is $\binom{n}{r}(m-1)^{n-r}/m^n$

If $m=n$, $P_r \to e^{-1}/r!$.

Soln. The ~~total~~ no. of ways of distributing n balls into m boxes is m^n which are assumed to be equally likely. To find the number of favourable cases, not that the r balls which should go to the specified box can be chosen in $\binom{n}{r}$ ways and for any such way the remaining $(n-r)$ balls can be distributed to the $(m-1)$ boxes in $(m-1)^{n-r}$ ways. Hence, we have the desired expression for P_r.

$$P_r = \frac{\binom{n}{r}(m-1)^{n-r}}{m^n}.$$

(*) If $m=n$, then

$$P_r = \frac{1}{r!}\cdot\frac{n(n-1)\cdots(n-r+1)}{(n-1)^r}\left(1-\frac{1}{n}\right)^n \to \frac{e^{-1}}{r!}. ///$$

15) If n balls are distributed at random into n boxes, find the prob. that exactly one box remains empty.

Soln. The n balls can be distributed into n boxes in n^n way which are assumed to be equally likely. In order that exactly one [illegible] box remains empty, there must be one box containing exactly ~~two~~ 2 balls ~~can be chosen in $n(n-1)$ ways~~ and $(n-2)$ boxes each containing exactly 1 ball. The empty box and the box containing 2 balls can be chosen in $n(n-1)$ ways, and for each such way the balls can be distributed into the boxes satisfying the above condition in $\frac{n!}{2!}$ ways (consider the box containing 2 balls as two boxes and ignore the order of balls in this box). Thus the prob. asked for is

$$\frac{(n)_2 \times n!}{2! \times n^n} = \frac{\binom{n}{2}\times n!}{n^n}. ///$$

16) Two cards are drawn at random without replacement from a standard deck. What is the prob. of drawing either an ace or a spade?

<u>Soln.</u> :> Let C be the set of 52 cards. Then the sample space for this problem is $\Omega = \{(a,b) : a \neq b, a \in C, b \in C\}$. Thus the total number of outcomes is 52×51. Let A_1 and A_2 stand respectively for the events 'an ace is drawn' and 'a spade is drawn'. Then the required probability is

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$$= \frac{\{n(A_1) + n(A_2) - n(A_1 \cap A_2)\}}{52 \times 51}.$$

To get $n(A_1)$, note that $A_1 = B_1 \cup B_2 \cup B_3$ where B_1, B_2 and B_3 stand respectively for the events 'the first card is an ace' and 'the second one is not an ace' and 'both the cards are ace'. Obviously $n(B_1) = n(B_2) = 4 \times 48$, $n(B_3) = 4 \times 3$. Since the events B_1, B_2, B_3 are pairwise disjoint, we must have $n(A_1) = \sum_{i=1}^{3} n(B_i)$ $= 2 \times \{4 \times 48\} + 4 \times 3 = 396$. A similar argument shows that $n(A_2) = 2 \times \{13 \times 39\} + 13 \times 12 = 1170$, also $A_1 \cap A_2 = D_1 \cup D_2 \cup D_3$ where D_1 and D_2 are respectively the events 'the first card is the ace of spades' and 'the second card is the ace of spades', and $D_3 = \{(a,b)$: either a is an ace and b is a spade but neither is the ace of spade or b is an ace and a is a spade but neither is the ace of spade$\}$. Thus $n(A_1 \cap A_2) = 51 + 51 + (3 \times 12 + 3 \times 12) = 174$. Hence the required prob. is – $\frac{396 + 1170 - 174}{2652} = \frac{116}{221}$.

17) Let A_1, A_2, A_3 be three given events. Find the prob. that – (a) at least one event (b) exactly one event (c) exactly two events (d) at least two events occur, in terms of $P(A_i)$, $P(A_i \cap A_j)$ and $P(A_1 \cap A_2 \cap A_3)$.

<u>Soln.</u> :> (a) P(at least one event event occurs)

$$= P(A_1 \cup A_2 \cup A_3)$$

$$= P(A_1 \cup A_2) + P(A_3) - P((A_1 \cup A_2) \cap A_3)$$

$$= P(A_1) + P(A_2) - P(A_1 \cap A_2) + P(A_3) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

$$= \sum_{i=1}^{3} P(A_i) - \sum_{1 \le i < j \le 3} P(A_i \cap A_j) + P(A_1 \cap A_2 \cap A_3)$$

(b) P(exactly one event occurs)

$$= P\left((A_1 \cap A_2^c \cap A_3^c) \cup (A_1^c \cap A_2 \cap A_3^c) \cup (A_1^c \cap A_2^c \cap A_3)\right)$$

$$= P(A_1 \cap A_2^c \cap A_3^c) + P(A_1^c \cap A_2 \cap A_3^c) + P(A_1^c \cap A_2^c \cap A_3)$$

Now, $P(A_1 \cap A_2^c \cap A_3^c) = P(A_1 \cap (A_2 \cup A_3)^c)$

$$= P(A_1) - P((A_2 \cup A_3) \cap A_1)$$

$$= P(A_1) - P(A_1 \cap A_2) - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

thus, $P(A_1^c \cap A_2 \cap A_3^c)$

$$= P(A_2) - P(A_1 \cap A_2) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

and $P(A_1^c \cap A_2^c \cap A_3) = P(A_3) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$.

Hence the required prob. is —

$$= \sum_{i=1}^{3} P(A_i) - 2 \sum_{1 \le i < j \le 3} P(A_i \cap A_j) + 3P(A_1 \cap A_2 \cap A_3)$$

(c) P(exactly two events occur)

$$= P\left((A_1 \cap A_2 \cap A_3^c) \cup (A_1 \cap A_3 \cap A_2^c) \cup (A_2 \cap A_3 \cap A_1^c)\right)$$

$$= P(A_1 \cap A_2 \cap A_3^c) + P(A_1 \cap A_2^c \cap A_3) + P(A_1^c \cap A_2 \cap A_3)$$

Now,

$$P(A_1 \cap A_2 \cap A_3^c) = P(A_1 \cap A_2) - P(A_1 \cap A_2 \cap A_3)$$

$$P(A_1 \cap A_2^c \cap A_3) = P(A_1 \cap A_3) - P(A_1 \cap A_2 \cap A_3)$$

$$P(A_1^c \cap A_2 \cap A_3) = P(A_2 \cap A_3) - P(A_1 \cap A_2 \cap A_3)$$

thus the required prob. is

$$\sum_{1 \le i < j \le 3} P(A_i \cap A_j) - 3P(A_1 \cap A_2 \cap A_3)$$

(d) P(at least two events occur)

= P(exactly two events occur) + P(exactly three events occur)

$$= \left\{ \sum_{1 \le i < j \le 3} P(A_i \cap A_j) - 3P(A_1 \cap A_2 \cap A_3) \right\} + P(A_1 \cap A_2 \cap A_3)$$

$$= \sum_{1 \le i < j \le 3} P(A_i \cap A_j) - 2P(A_1 \cap A_2 \cap A_3)$$

18) If n balls are distributed into 3 boxes, what is the prob. that at least one box is empty?

Soln. :→ Here the sample space is $\Omega = \{(x_1, \ldots, x_n) ;$ each x_i is 1, 2 or 3$\} = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_n$ where $\Omega_1 = \Omega_2 = \ldots = \Omega_n = \{1, 2, 3\}$; $(x_1, \ldots, x_n)$ stands for the distribution of n balls to 3 boxes where the i-th ball has fallen into the box numbered x_i. Thus the total number of outcomes is 3^n which are assumed to be equally likely. The desired prob. is $P(A_1 \cup A_2 \cup A_3)$ where A_i is the event that the ~~box~~ i-th box is empty.

$$P(A_1 \cup A_2 \cup A_3) = \sum P(A_i) - \sum P(A_i \cap A_j) + P(A_1 \cap A_2 \cap A_3)$$

$$= 3 \{ P(A_1) - P(A_1 \cap A_2) \}$$

for reasons of symmetry. Now, $P(A_1) = \frac{n(A_1)}{3^n} = \frac{2^n}{3^n}$ and $P(A_1 \cap A_2) = \frac{n(A_1 \cap A_2)}{3^n} = \frac{1}{3^n}$. Thus the required prob. is $\frac{3(2^n - 1)}{3^n}$.

Basic concepts of Probability theory

Set theory	Probability theory
Ω (the universal set)	Ω (the sample space)
The empty set	the impossible event
A subset of Ω	An event in Ω
ω (an element of Ω)	ω (an outcome of an experiment)
$\omega \in A$	The event A occurs
$A \subset B$	the occurance of A implies that of B
$A \cup B$	At least one of the events A and B
$A \cap B$	Both of the events A and B
A^c	the complementary event of A
$\{A_i\}_{i \in I}$ is pairwise disjoint	The event $\{A_i\}_{i \in I}$ are mutually exclusive
$\bigcup_{i \in I} A_i = \Omega$	The events $\{A_i\}_{i \in I}$ are exhaustive

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<u>Example</u>:→ 1. For the experiment of throwing a die, the sample space is $\Omega = \{1,2,3,4,5,6\}$. If A is the event of getting an even number and B the event of getting an odd number. C is the event of getting a multiple of 3, D the event of getting more than 5, E the event of getting more than 6 and F the event of getting at most 6, then $A = \{2,4,6\}$, $B = \{1,3,5\}$, $C = \{3,6\}$, $D = \{6\}$, $E = \phi$ and $F = \Omega$; the events A and B are mutually exclusive and exhaustive since $A \cap B = \phi$, $A \cup B = \Omega$. The events A and C are neither mutually exclusive (as $A \cap C \neq \phi$) nor exhaustive (as $A \cup C \neq \Omega$). The events $A_1 = \{1,2,3\}$, $A_2 = \{4\}$ and $A_3 = \{5,6\}$ are pairwise disjoint (or, mutually exclusive), since $A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3 = \phi$. The events $B_1 = \{1,2\}$, $B_2 = \{2,3\}$ and $B_3 = \{4,5,6\}$ are exhaustive since $B_1 \cup B_2 \cup B_3 = \Omega$. ///

<u>Ex</u>. 2. The sample space Ω of the experiment of throwing two dice consists of the following 36 points:

$$(1,1), (1,2), (1,3), (1,4), (1,5), (1,6).$$
$$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6).$$

Then $\Omega = \Omega_1 \times \Omega_2$, where $\Omega_1 = \Omega_2 = \{1,2,3,4,5,6\}$. Let A, B and C denote respectively the events of getting a sum of 7 points, a difference of 3 points and a product of 12 points. Then.

$$A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$
$$B = \{(1,4), (2,5), (3,6), (4,1), (5,2), (6,3)\}$$
$$C = \{(3,4), (4,3), (2,6), (6,2)\}$$

The events A and B are not mutually exclusive.
The events A, B and C, are not exhaustive. ///

Some theorem of Classical (or a priori) approach :→

Theorem. 1.→ If A is an impossible event, then $P(A)=0$, and if A is a sure event, then $P(A)=1$.

Proof :→ Let the total number of elementary events for our experiment be N and let these be equally likely.

Now in case A is an impossible event, there is no elementary event favourable to A such that

$$P(A)=\frac{0}{N}=0.$$

On the other hand, if A is a sure event, each of the N elementary events is favourable to A so that

$$P(A)=\frac{N}{N}=1.$$

Note: The result may be written in the forms $P(\phi)=0$, $P(\Omega)=1$.

Theorem. 2.→ If the occurance of A implies the occurance of B, then $P(A)\le P(B)$.

Proof :→ Since A implies B, every elementary event that is favourable to A is also favourable to B. Hence $N(A)\le N(B)$, implying that

$$\frac{N(A)}{N}\le\frac{N(B)}{N}$$

i.e. $P(A)\le P(B)$.

Corollary :→ Since for any event A, $\phi\subset A\subset\Omega$, we have $P(\phi)\le P(A)\le P(\Omega)$, i.e. $0\le P(A)\le 1$.

Proof :→

we know $n(\phi)=0$ and $n(\Omega)=N$. for an arbitrary event A, one has $0\le n(A)\le N$ and so $0\le P(A)\le 1$.

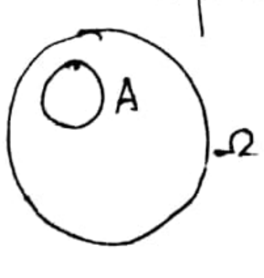
Limitations of Classical Definition :→

Though the classical definition of prob. is easy to understand and it suffers from some serious limitations as follows:

It is limited to situations in which there is only a finite number of possible outcomes. Consider an experiment in which trials are performed until a particular event occurs, say, tossing a coin until a head occurs. Here the sample space is $\Omega = \{H, TH, TTH, TTTH, \ldots\}$. So, the sample space contains a countably infinite number of points rendering N infinite. Thus the prob. of the event can't be defined by classical defn.. In practice such an experiment will terminate in a finite number of trials, but there is no 'a priori' assurance that this will happen.

It is desirable, both for theoretical and practical reasons, to extend the theory to situations where there is a continuum of possibilities. Suppose a physical variable like the height of an individual, the value of electric current in a wire etc. are observed. Here each of the continuum of possible values of these variables is to be regarded as a possible outcome. The number of such outcomes being infinite in number, prob. of an event, say the height of a man lies between 5'2" and 5'3", can't be defined in classical theory.

Consider, again, the following problem. A point object is thrown at random on the space Ω. We want to find the prob. that the object lies within the subspace A in Ω. Here the sample space Ω is a domain on a plane and the elementary events ω are points within the domain. The conditions of the experiment are such that all the points ω are equally likely. Hence both the total number of points in Ω and A are uncountably infinite and as such prob. can't be defined in classical theory. In this case geometric probability is applicable.



• The classical definition is based on the assumption that the cases are equally likely. This assumption may not be fulfilled in many cases. A simple example is the loaded die. For a die which is asymmetrical in mass or shape, it is not intuitively expected that each side has the same likelihood of turning up. The classical definition fails to answer the question like what is the prob. of obtaining a six.

In the classical defn. the value of prob. comes necessarily as a rational number.

II) Statistical or Empirical or Frequency or a posteriori Definition :>

The experiments conducted under identical conditions for a large number of times show a statistical regularity, namely, the relative frequency of an outcome in several sets of sequences of trials is more or less constant provided each set consists of a large number of trials. The rate of convergence of relative frequencies to this particular value increases rapidly as the number of trials increases. This constant value may be taken as the probability of the outcome. The basic requirement (assumption) for this defn. is that the experiment must be conducted under identical conditions and the number of trials must be large.

Definition :> Let $f_n(A)$ be the number of times in which an event A, the outcome of an experiment, occurs in a series of n repetitions of the trial conducted under identical conditions. The relative frequency of A is $r_n(A) = \frac{f_n(A)}{n}$. The prob. of the event A is defined as

$$P(A) = \lim_{n \to \infty} \frac{f_n(A)}{n},$$

provided the limit exists and is unique.

We note that even if the conditions of an experiment is such that its elementary events are not equally likely, probability can be defined in the statistical sense, though, however, it remains undefined in the classical sense. This is the improvement over the statistical definition.

AXIOMATIC APPROACH

— KOLMOGOROV'S APPROACH

Algebra of Sets :

[1] Notations :→

(a) $N = \{1,2,3,\ldots\}$ = the set of all natural numbers.
$Z = \{\ldots,-2,-1,0,1,2,\ldots\}$ = the set of all integers.
$Q = \{p/q : p, q \in Z, q \neq 0\}$ = the set of all rationals.
R = the set of all real numbers,
$C = \{x+iy : x, y \in R\}$ = the set of all complex numbers ($i = \sqrt{-1}$). ///

(b) If Ω is the set of real numbers, then ~~the~~ we define,—

the closed interval $[a,b]$ is

$$[a,b] = \{\omega \mid a \leq \omega \leq b\};$$

the open interval (a,b) is

$$(a,b) = \{\omega \mid a < \omega < b\};$$

the half open interval $(a,b]$ is

$$(a,b] = \{\omega \mid a < \omega \leq b\};$$ [left open, right closed interval]

the half open interval $[a,b)$ is

$$[a,b) = \{\omega \mid a \leq \omega < b\}.$$ [left-closed, right open interval]

Mutually exclusive or disjoint or non-overlapping Sets :→

Two sets A and B are said to be disjoint if they have no element in common, i.e. if $A \cap B = \phi$. In the same way, a number of sets are said to be disjoint if no two of them have any element in common.

Pairwise disjoint Sets :→ A family of sets is called pairwise disjoint if every pair of distinct members of the family are disjoint. An indexed family $\{A_i\}_{i \in I}$ is called pairwise disjoint if $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

Theorem :→ If $\{A_n\}$ is a sequence of sets, then there exists a sequence of disjoint sets $\{D_n\}$ such that

$$\bigcup_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} D_n .$$

[**Hints :→** Take, for example, $D_1 = A_1$ and, for all $n > 1$, $D_n = A_n - \bigcup_{i=1}^{n-1} A_i$]

Proof :→

Some elements of theory of Measure.

The union and intersection of a countable number of sets $A_1, A_2, \dots$ are written as $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1} A_i$ respectively.

Given n sets $A_1, \dots, A_n$ we can write

$$\bigcup_{i=1}^{n} A_i = A_1 + A_1' A_2 + A_1' A_2' A_3 + \dots + A_1' \dots A_{n-1}' A_n$$

as the sum of n disjoint sets $A_1, A_1'A_2, A_1'A_2'A_3 \dots$ etc. For a countably infinite number of sets $A_1, A_2, \dots$

$$\bigcup_{i=1}^{\infty} A_i = A_1 + A_1'A_2 + A_1'A_2'A_3 + \dots$$

Sequence and limits of sets :→ To every integer $n = 1, 2, \dots$ we assign a set. The ordered class of sets $A_1, A_2 \dots$ is a sequence $\{A_n\}$ of sets.

$\{A_n\}$ is monotonically increasing or expanding if

$$A_n \subseteq A_{n+1}, \text{ for } n \geq 1.$$

Since in this case $\bigcup_{k=1}^{n} A_k = A_n$; we define

$$\lim A_n = \bigcup_{k=1}^{\infty} A_k .$$

Symbolically $A_n \uparrow A$.

$\{A_n\}$ is monotonically decreasing or contracting if

$$A_n \supseteq A_{n+1}, \text{ for } n \geq 1 .$$

Since in this case $\bigcap_{k=1}^{n} A_k = A_n$; we define

$$\lim A_n = \bigcap_{k=1}^{\infty} A_k .$$

Symbolically $A_n \downarrow A$.

Limit of an arbitrary sequence of sets :→ For any sequence $\{A_n\}$ we define —

$$B_n = \inf_{k \geq n} A_k = \bigcap_{k=n}^{\infty} A_k$$

$= \{\omega : \omega$ belong to all A_k except perhaps for $A_1, \dots A_{n-1}\}$

$$C_n = \sup_{k \geq n} A_k = \bigcup_{k=n}^{\infty} A_k$$

$= \{\omega : \omega$ belong to at least one of $A_n, A_{n+1}, \dots\}$.

Thus B_n is a monotonically increasing sequence with limit $B = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \lim \text{inferior } A_k = \underline{\lim} A_n$. B is the set of all points which belong to almost all A_n (all but any finite number of sets).

Thus C_n is a monotonically decreasing sequence with limit $C = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \lim \text{superior } A_k = \overline{\lim} A_n$. C is the set of all those points which belongs to infinitely many A_n.

Since every point which belongs to almost all A_n belongs to infinitely many A_n, $\underline{\lim} A_n \subseteq \overline{\lim} A_n$. If $\underline{\lim} A_n = \overline{\lim} A_n = A$ (say), the limit of $\{A_n\}$ is said to exist and A is called the limit of $\{A_n\}$.

Class of sets :-> A class of sets is a collection of sets. The elements of this class are sets. An example is

$$\mathcal{A} = \{\{a\}, \{b\}, \{a, b\}\}$$

Power Set :-> The biggest class is, of course, the class of all sets of Ω (called the power set). In case Ω is finite (say with N elements), the number of members of this class will be 2^N.

Field or Algebra of Sets :->

Let Ω be a set. A family $\mathcal{A}$ of subsets of Ω is called a field (or algebra) on Ω if -

(i) $\emptyset \in \mathcal{A}$,

(ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$ (closure under complementation) and

(iii) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$ (closure under the operation union).

Sigma field of sets :-> Let Ω be a set. A family $\mathcal{A}$ of subsets of Ω is called a σ-field or σ-algebra if -

(i) $\phi \in \mathcal{A}$,

(ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$ (closure under complementation) and

(iii) $A_n \in \mathcal{A}$ for each $n \geq 1 \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ (closure under countable unions).

Several implications can be deduced : if $\mathcal{A}$ is a sigma-field, then $A_1, \ldots, A_n \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{A}$ and so $\mathcal{A}$ is a field. A field contain Ω and is closer under intersectione, difference, symmetric difference : if $A, B \in \mathcal{A}$, then $A \cap B$, $A \setminus B$, $A \Delta B$ are in $\mathcal{A}$. A sigma-field is closed under countable intersections. Further, if $\{A_n\}$ is a sequence of sets in a field $\mathcal{A}$ and we disjointify A_n to get B_n then $B_n \in \mathcal{A}$ for each $n \geq 1$.) Finally, if $\{A_n\}$ is a sequence of sets in a sigma-field $\mathcal{A}$, then limsup A_n and lim inf A_n belong to $\mathcal{A}$.

Example: $\rightarrow$ Let Ω be a set. then

$$\{\phi, \Omega\} \text{ and } \mathcal{P}(\Omega)$$

are sigma-fields. If $a \in \Omega$, then

$$\{\phi, \{a\}, \{a\}^c, \Omega\}$$

is a field (note that a finite field is a sigma-field).

e.g. Let Ω be a set. Call a subset A of Ω cofinite if A^c is finite. then the set of all subsets of Ω which are either finite or cofinite is a field on Ω. If Ω is infinite, then this field is not a sigma-field.

e.g. Let Ω be an uncountable set. Call a subset A of Ω cocountable if A^c is countable. Then the set of all subsets of Ω which are either countable or cocountable is a sigma-field on Ω.

Finitely Additive class of sets :$\rightarrow$ A clas $\mathcal{A}$ of sets is called finitely additive if it has the following properties : (i) $\phi \in \mathcal{A}$, (ii) if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$ (iii) if $A_i, i=1, \ldots, n \in \mathcal{A}$ then $\bigcup_{i=1}^{n} A_i \in \mathcal{A}$. It follows that $\bigcap_{i=1}^{n} A_i \in \mathcal{A}$. Clearly the universal set $\in \mathcal{A}$. Such a class is also called a Boolean field of sets (or a Boolean algebra).

Completely Additive class of sets :$\rightarrow$ A class $\mathcal{A}$ of sets is completely additive if it has the following properties: (i) $\phi \in \mathcal{A}$ (ii) if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$ (iii) if A_i, $i = 1, 2, \ldots \in \mathcal{A}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$. It follows that $\bigcap A_i \in \mathcal{A}$. Such a class is also called a Borel field on 'a' σ-field (or a σ-algebra of sets.

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1] Axiomatic Definition of Probability :- Let $\mathcal{A}$ be a sigma-field on Ω. Then a probability P on $(\Omega, \mathcal{A})$ is a function from $\mathcal{A}$ to $\mathbb{R}$ such that (i) $P(\Omega)=1$, (ii) $P(A) \geq 0$ for each $A \in \mathcal{A}$ and (iii) whenever $\{A_n\}$ is a sequence of pairwise disjoint elements of $\mathcal{A}$, the series $\sum_{i=1}^{\infty} P(A_i)$ converges to $P(\bigcup_{i=1}^{\infty} A_i)$.

Condition (iii) of this defn is called countable additivity of the set function P.

By Finite additivity of the set function P :-> Let $\mathcal{A}$ be a field on Ω. Then a finitely additive Probability P on $(\Omega, \mathcal{A})$ is a function from Ω to $\mathbb{R}$ such that (i) and (ii) conditions of the above defn hold and (iii) $A, B \in \mathcal{A}$ and $A \cap B = \phi \Rightarrow P(A \cup B) = P(A) + P(B)$.

Often a probability is defined on a field $\mathcal{A}$ of subsets of Ω, rather than on a sigma-field on Ω. By such a probability P, we mean a set-function from $\mathcal{A}$ to $\mathbb{R}$ such that conditions (i) and (ii) of the defn. of axiomatic defn of prob. hold and $P(\bigcup_{i=1}^{\infty} A_n) = \sum_{i=1}^{\infty} P(A_n)$ whenever $\{A_n\}$ is a sequence of pairwise disjoint elements of $\mathcal{A}$ such that $\bigcup_{i=1}^{\infty} A_n \in \mathcal{A}$. The last condition is also known as the countable additivity of P.

Probability Space :-> Let P be a probability on $(\Omega, \mathcal{A})$ where $\mathcal{A}$ is a sigma-field of subsets of Ω. Then the triplet $(\Omega, \mathcal{A}, P)$ is called a probability space.

4] Properties of Probability function :-> Let P be a finitely additive probability on $(\Omega, \mathcal{A})$. Let $A, A_1, \ldots, A_n$ be in $\mathcal{A}$. Then—

1) $P(A) \leq 1 \ \forall A \in \mathcal{A}$.

Proof :-> As $A \subset \Omega$ for each $A \in \mathcal{A}$, we know $P(A) \leq P(\Omega)$ and from the axiomatic defn of prob., we know $P(\Omega) = 1$.

So, we get — $P(A) \leq 1$.

2) $P(\bigcup_{i=1}^{n} A_i) = 0$ if $P(A_i) = 0$ for $i = 1, \ldots, n$.

3) $P(\bigcap_{i=1}^{n} A_i) = 1$ if $P(A_i) = 1$ for $i = 1, \ldots, n$.

4> $P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - n + 1$

Proof :→ We know – $P(A^c) = 1 - P(A)$ & $P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$ [Boole's inequality]

Using this we get –

$$P\left(\bigcap_{i=1}^{n} A_i\right) = 1 - P\left(\bigcup_{i=1}^{n} A_i^c\right)$$

$$\leq 1 - 1\cdot\sum_{i=1}^{n} P(A_i^c) = \sum_{i=1}^{n} P(A_i) - n + 1$$

Lemma :→ If $A_1, \ldots, A_n$ $(n \geq 2)$ are sets in $\mathcal{A}$ such that $P(A_i \cap A_j) = 0$ whenever $i \neq j$, then $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$

Proof :→ The result is true if $n = 2$. Assume that it is true for $n = m (\geq 2)$. Let $A_1, \ldots, A_m, A_{m+1}$ be in $\mathcal{A}$ such that $P(A_i \cap A_j) = 0$ whenever $1 \leq i \neq j \leq m+1$.

For $n = m$, we get – $P\left(\bigcup_{i=1}^{m} A_i\right) = \sum_{i=1}^{m} P(A_i)$ ——— (1)

For $n = m+1$, $P\left(\bigcup_{i=1}^{m+1} A_i\right) = P\left(\bigcup_{i=1}^{m} A_i\right) + P(A_{m+1}) - P\left[\bigcup_{i=1}^{m} A_i \cap A_{m+1}\right]$

[By $P(A \cup B) = P(A) + P(B) - P(A \cap B)$]

$$= P\left(\bigcup_{i=1}^{m} A_i\right) + P(A_{m+1}) - P\left[\bigcup_{i=1}^{m} (A_i \cap A_{m+1})\right]$$

Since $P(\underbrace{A_i \cap A_{m+1}}_{B}) = 0$ for $1 \leq i \leq m$,

Let $(A_i \cap A_{m+1}) = B_j$, $P(B_j) = 0$, for $j = 1, \ldots, m$, so that

we know, $P\left(\bigcup_{j=1}^{n} B_j\right) = 0$. [Using 2> (P.T.O)]

$$\therefore P\left(\bigcup_{i=1}^{m} (A_i \cap A_{m+1})\right) = 0.$$

$$\therefore P\left(\bigcup_{i=1}^{m+1} A_i\right) = P\left(\bigcup_{i=1}^{m} A_i\right) + P(A_{m+1})$$

$$= \sum_{i=1}^{m} P(A_i) + P(A_{m+1}) \quad \text{[By (1)]}$$

$$= \sum_{i=1}^{m+1} P(A_i)$$, the result is true for $n = m+1$. ///.

(*) We know, $P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$ & $P(A) \geq 0$.

So if $P(A_i) = 0$, for $i \geq 1$, we get $P\left(\bigcup_{i=1}^{n} A_i\right) = 0$ [2> proved]

(*) We know, $P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - n + 1$ & $P(A) \leq 1$.

So if $P(A_i) = 1$ for $i \geq 1$, we get $P\left(\bigcap_{i=1}^{n} A_i\right) = 1$. [3> proved]

7 Show that, the classical def^n. of prob. can be taken as a special case of the axiomatic def^n. of probability.

Ans:→ Let Ω be a finite set $\{\omega_1, \ldots, \omega_N\}$. Take $\mathcal{A}$ as the set of all subsets of Ω. Let P be a finitely additive probability on $(\Omega, \mathcal{A})$ such that $P(\{\omega_i\}) = \frac{1}{N}$ for $1 \le i \le N$. Then if $A = \{\omega_{i_1}, \ldots, \omega_{i_m}\} \subset \Omega$ where $1 \le m \le N$, we have

$$P(A) = \sum_{j=1}^{m} P(\{\omega_{i_j}\}) = \frac{m}{N} = \frac{n(A)}{N}.$$ Hence the proof.

8 Theorem of Total Probability :→

Statement:→ Let P be a prob. on $(\Omega, \mathcal{A})$ where $\mathcal{A}$ is a field of subsets of Ω. If $\{A_n\}$ is a sequence of disjoints sets in $\mathcal{A}$ such that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ and B is a set in $\mathcal{A}$ such that $B \subset \bigcup_{n=1}^{\infty} A_n$, then $P(B) = \sum_{n=1}^{\infty} P(B \cap A_n)$.

Proof:→ Note that B is the union of the disjoint sets $B \cap A_n$ in $\mathcal{A}$: $B = B \cap \left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} (B \cap A_n)$, and $(B \cap A_n)(B \cap A_m) = B \cap (A_n \cap A_m) = \phi$ if $n \ne m$. ///

— X —

MISCELLANEOUS PROBLEMS :

1) A and B play 12 games of chess of which 6 are won by A, 4 are won by B, and 2 end in a drawn. They agree to play a tournament consisting of 3 games. Find the prob. that – (a) A wins all three games, ~~(b) two games end in a draw~~ (c) A and B win alternatively (d) B wins at least one game.

Soln :> Let A_1, A_2, A_3 denote the events "A wins" in 1st, 2nd and 3rd games respectively,

B_1, B_2, B_3 denote the event "B wins" in 1st, 2nd and 3rd games respectively.

On the basis of their past performance (empirical probability) we shall assume that —

$P(\text{A wins any one game}) = \frac{6}{12} = \frac{1}{2}$, $P(\text{B wins any one game}) = \frac{4}{12} = \frac{1}{3}$.

(a) $P(\text{A win all 3 games}) = P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$

$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{8}$.

assuming that all the results of each game are independent of the results of any others.

(c) $P(\text{A and B win alternatively}) = P(\text{A wins then B wins then A wins or B wins then A wins then B wins})$

$= P(A_1 \cap B_2 \cap A_3) + P(B_1 \cap A_2 \cap B_3)$

$= P(A_1) P(B_2)P(A_3) + P(B_1) P(A_2)P(B_3)$

$= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)$

$= \frac{5}{36}$,

(d) $P(\text{B wins at least one game}) = 1 - P(\text{B wins no game})$

$= 1 - P(B_1' \cap B_2' \cap B_3')$

$= 1 - P(B_1') P(B_2') P(B_3')$

$= 1 - \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \frac{19}{27}$

2) Find the prob. that ~~the~~ n people ($n \leq 365$) selected at random will have n different birthdays.

Soln. :→ We assume that there are only 365 days in a year and that all birthdays are equally probable, assumptions which are not quite met in reality.

The first of the n people has of course ~~a~~ some birthday with probability $365/365 = 1$. then, if the second is to have a different birthday, it must occur on one of the other 364 ways. thus the prob. that the second person has a birthday different from the first is $364/365$. Similarly the prob. that the third person has a birthday different from first two is $363/365$. finally, the prob. that the nth person has a birthday different from the others is $(365-n+1)/365$. thus we have,

$$P(\text{all } n \text{ birthdays are different}) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365-n+1}{365}$$

$$= \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right) \cdots \left(1 - \frac{n-1}{365}\right)$$

3) The probabilities that a husband and wife will be alive 20 years from now are given by 0.8 and 0.9 respectively. Find the prob. that in 20 years (a) both, (b) neither, (c) at least one, will be alive.

Soln. :→ Let H, W be the events that the husband and wife, respectively, will be alive in 20 years. Then $P(H) = 0.8$, $P(W) = 0.9$. H and W are independent events, which we suppose that H and W are may or may not be reasonable.

(a) $P(\text{both will be alive}) = P(H \cap W) = P(H)P(W) = 0.72$

(b) $P(\text{neither will be alive}) = P(H' \cap W') = P(H')\,P(W') = 0.02$

(c) $P(\text{at least one will be alive}) = 1 - P(\text{neither will be alive})$

$= 1 - 0.02$

$= 0.98$.

4) A die is thrown 10 times. What is the prob. of getting six points in each of 4 throws?

Soln :→ Under the usual assumptions, the probability of getting a six in a single throw is $\frac{1}{6}$. Since the throws may be supposed to be independently of each other, the events $A_1, A_2, \ldots, A_{10}$, where A_i stands either for the appearance of a six or for the non-appearance of a six in the ith throw, are to be taken as statistically independent. Hence, by the theorem of compound prob., the prob. of getting a six in each of 4 particular throws, e.g. in each of the first 4 throws, (and a number other than six in each of the other 6 throws)

is $\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^6 = \frac{5^6}{6^{10}}$.

But this is also the prob. of getting 4 sixes in any other order. Since the total number of ways in which 4 sixes may appear is $\binom{10}{4}$,

the required probability is, by the theorem of probability for the union of mutually ~~exclu~~ exclusive events,

$$\binom{10}{4}\frac{5^6}{6^{10}} = 0.05427, \text{ approximately.}$$

5) Among the 270 tickets sold in a lottery, 100 are coloured red, 90 coloured blue and 80 coloured green. What's the prob. that (i) blue tickets will win both the first and second prizes? (ii) a blue ticket will win the first prize and a green ticket the second?

Soln :→ Let B_i denote that the ith prize goes to a blue ticket and G_i denote that it goes to a green ($i=1,2$).

(i) The probability is $P(B_1 \cap B_2)$. The first ticket drawn at random from the whole set of 270 that wins the first prize may be any one of the 270, and these are equally likely. Since the total number of blue tickets is 90,

$P(B_1) = \frac{90}{270}$, again, the second ticket drawn may be any one of the remaining 269, of which 89 are blue. Hence —

$$P(B_2|B_1) = \frac{89}{269}. \text{ Thus}$$

$$P(B_1 \cap B_2) = \frac{90 \times 89}{270 \times 269} = 0.110. ///$$

(ii) Here the required prob. is $P(B_1 \cap G_2)$. $P(B_1) = \frac{90}{270}$, while $P(G_2|B_1) = \frac{80}{269}$ so that

$$P(B_1 \cap G_2) = \frac{90 \times 80}{270 \times 269} = 0.099. ///$$

6) In an attempt to land an unmanned rocket on the moon, the probability of a successful landing is known to be 0.4 and the prob. of the monitoring system giving the correct information of landing is 0.9 in either case. Find the prob. of a successful landing, it being known that the monitoring system indicated it correctly.

<u>Solution:</u> Let A denote the event of a successful landing of the rocket, B_1 denote the event of the monitoring system indicating successful landing and B_2 denote the event of its indicating unsuccessful landing.

Then by the given conditions,

$$P(A) = 0.4, \; P(B_1|A) = 0.9, \; P(B_2|A^c) = 0.9.$$

The required prob $= P(A|B_1) = \frac{P(B_1|A)\,P(A)}{P(B_1)}$

$$\begin{aligned}
\text{Now,} \quad P(B_1) &= P(A \cap B_1) + P(A^c \cap B_1) \\
&= P(B_1|A)P(A) + P(B_1|A^c)P(A^c) \\
&= P(B_1|A)\,P(A) + [1 - P(B_1^c|A^c)]\,P(A^c) \\
&= P(B_1|A)\,P(A) + [1 - P(B_2|A^c)]\,P(A^c) \\
&= 0.9 \times 0.4 + (1 - 0.9) \times 0.6 \\
&= 0.42
\end{aligned}$$

Hence, the required probability $= \frac{0.9 \times 0.4}{0.42} = \frac{6}{7}$. ///

(*) 7) A lot of N objects contains Np objects of one kind (say, Np defectives) and Nq objects of another (say, Nq non-defectives.) Out of the lot, n objects are chosen at random. What is the prob. that there will be k defectives among the chosen objects?

Soln. :-

Case 1: Drawing with replacements

In this case, the first object drawn may be any one of the N in the lot; so may be the second; and so on. Hence the total number of elementary events, i.e. the total number of ways in which the n objects may be chosen is

$$N \times N \times \cdots \times N = N^n$$

Since the selection is made at random, these are to be regarded as equally likely.

Now, consider the number of ways in which k defectives and $n-k$ non-defectives may be chosen in a specified order, e.g. such that the first k drawings give defectives and the last $n-k$ non-defectives. The number is

$$\underbrace{Np \times Np \times \cdots \times Np}_{k \text{ factors}} \times \underbrace{Nq \times Nq \times \cdots \times Nq}_{n-k \text{ factors}} = N^n p^k q^{n-k}$$

Hence the prob. of having defective objects in the first k drawing and non-defective objects in the last $n-k$ is

$$N^n p^k q^{n-k} / N^n = p^k q^{n-k}.$$

But this obviously is also the prob. of having k defectives and $n-k$ non-defectives in any other particular order. For our problem, the order is immaterial, and hence the required probability of having k defectives and $n-k$ non defectives is,

$$c \times p^k q^{n-k},$$

where c is the total number of orders (permutations) in which k defectives and $n-k$ non-defectives may appear.

Since

$$c = \binom{n}{k},$$

the required prob. is

$$\binom{n}{k} p^k q^{n-k}.$$

Case 2: Drawing without replacements

Of the n objects selected, the first one may be any one of the N in the lot, the second one of the remaining $N-1$, and so on. Hence the total number of elementary events, i.e. the total number of ways in which the n objects may be chosen, regard being had to the order in which they appear, is

$$N(N-1)\cdots(N-n+1) = (N)_n .$$

Since the selection is made at random, these are to be regarded as equally likely.

Now consider the number of ways in which k defectives and $n-k$ non-defectives may be chosen in a specified order, e.g. such that the first k are defectives and the last $n-k$ non-defective. This number is

$$Np(Np-1)\cdots(Np-k+1)\,Nq(Nq-1)\cdots(Nq-n+k+1)$$

$$= (Np)_k (Nq)_{n-k}$$

Hence the prob. of having defective objects in the first k drawings and non-defective ones in the last $n-k$ is

$$(Np)_k (Nq)_{n-k} / (N)_n .$$

But this obviously is also the prob. of having k-defectives and $n-k$ non-defectives in any other particular order. For our problem, the order is immaterial, and hence the required prob. is,

$$\binom{n}{k} (Np)_k (Nq)_{n-k} / (N)_n$$

$$= \frac{(Np)_k}{k!} \times \frac{(Nq)_{n-k}}{(n-k)!} \Big/ \frac{(N)_n}{n!} = \binom{Np}{k}\binom{Nq}{n-k}\Big/\binom{N}{n}$$

Cor: → The conditions being the same, what's the prob. that the sample will contain at least one defective object?

Soln. We shall first obtain the prob. of the complementary event, i.e. there will be no defective object in the sample. The probability is $\binom{Nq}{n}\Big/\binom{N}{n}$

The prob. of getting at least one defective object is,

$$1 - \binom{Nq}{n}\Big/\binom{N}{n} = 1 - (Nq)_n / (N)_n .$$

(*) 8) There are N items in a lot consisting of Np items of the first grade and Nq items of the second $(q = 1-p)$. A quality control inspector checks a sample of n articles and finds that all of them are second-grade $(n < Nq)$. What's the prob. that another item selected at random from those remaining will also be second-grade?

Soln :→ Let us denote by A the event that the first n articles are second-grade and by B the event that the $(n+1)$-st article is second-grade. The probability we want to determine is the conditional prob. $P(B|A)$,

We know – $$P(B|A) = \frac{P(A \cap B)}{P(A)}, \text{ Now,}$$

$$P(A) = \frac{\binom{Nq}{n}}{\binom{N}{n}},$$

while $P(A \cap B)$, which is nothing but the prob. for the first $(n+1)$ articles to be second grade, is given by

$$P(A \cap B) = \frac{\binom{Nq}{n+1}}{\binom{N}{n+1}}.$$

Hence, $$P(B|A) = \frac{Nq - n}{N - n}.$$

Direct method :→ If the first n articles turn out to be of the second grade, it means that there ~~are~~ remain in the lot $(N-n)$ articles of which $(Nq - n)$ are second grade. Hence the probability for another article chosen to be second grade is

$$\frac{Nq - n}{N - n}.$$

9) In a randomized field experiment, a rectangular block of land is divided into k parallel strips of equal size, and k varieties of wheat, among which are A and B, are allotted to the strips at random. What's the prob. that there are r (naturally, $r = 0, 1, \ldots, k-2$) strips separating those occupied by A and B?

Soln :→ Here the sample space may be supposed to be composed of $k!$ elementary events corresponding to the $k!$ arrangements (permutations) that can be formed in allotting the varieties to the plots. That the arrangement is made at random means that these $k!$ elementary events are also equally probable.

If we consider a particular set of r varieties occupying the strips between A and B, we see that the number of such arrangements is

$$(k-r-1)!\,r!\,2!\,,$$

$(k-r-1)!$ being the number of ways in which (a) the other $(k-r-2)$ varieties and (b) the varieties A and B together with the intervening r varieties, taken as a whole, can be arranged among themselves.

Also these r varieties can be chosen in $\binom{k-2}{r}$ ways. Hence the total number of elementary events that are favourable to the event in question is

$$\binom{k-2}{r}(k-r-1)!\,r!\,2! = 2(k-2)!\,(k-r-1).$$

The required prob. is, therefore, $= \dfrac{2(k-r-1)(k-2)!}{k!} = \dfrac{2(k-r-1)}{k(k-1)}$.

10) Suppose r indistinguishable objects are allotted to n cells. We may require the prob. that a particular cell will contain k objects.

Soln :→ Here it is proper to take as the sample space the totality of distinguishable arrangements that we can have by the allotment. It will be assumed that the allocation is made at random, so that these arrangements are equally probable.

Now, the total number of such arrangements may be obtained if the n cells are likened to the n gaps between $(n+1)$ bars placed in a row and the r objects to r dots occupying these gaps. The number will be the same as that of the permutations of $(n-1)$ bars and r dots in a row (the two terminal bars being kept fixed), i.e. will equal,

$$\binom{r+n-1}{r}.$$

To get the number of arrangements favourable to the event in question, the two bars forming the given cell may be treated as one; in the other cells, may be arranged in $\binom{r-k+n-2}{r-k}$ ways. Hence the required prob. is

$$\binom{r-k+n-2}{r-k} \Big/ \binom{r+n-1}{r}.$$

Here one may also be interested in the prob. that exactly m of the cells will be empty. For $r < n-m$, obviously this probability will be zero. Let us consider the case for which $r \geq n-m$. The number of favourable cases may be obtained if, for each of a specified group of cells, the two bars forming the cell are treated as one, so that we have now $n+1-m$ bars; since the two terminal bars are kept fixed, we may say that we are to place $n-m-1$ bars in the $r-1$ gaps between the dots, which can be done in $\binom{r-1}{n-m-1}$ ways. Further, the m cells may be chosen in $\binom{n}{m}$ ways. The number of favourable arrangements is thus $\binom{n}{m} \times \binom{r-1}{n-m-1}$, and the required probability is

$$\binom{n}{m}\binom{r-1}{n-m-1} \Big/ \binom{r+n-1}{r}.$$

11) A group of $2N$ boys and $2N$ girls is divided into two equal groups. Find the prob. that each group will be equally divided into boys and girls.

Soln :→ Since there are $4N$ persons in all, the event of dividing them into two equal groups can take place in $\binom{4N}{2N}$ ways. We assume that these are equally likely. The number of ways favourable to the event that each group contains N boys and N girls is $\binom{2N}{N}\binom{2N}{N}$. Hence, the required probability is

$$\frac{\binom{2N}{N}\binom{2N}{N}}{\binom{4N}{2N}},$$

12) r distinguishable objects are distributed among n cells at random, each cell being free to receive any number of objects. What is the prob. that exactly m of the cells remain empty?

Soln :→ Let A_i denote, that the ith cell remains empty following the random distribution of the r objects among the n cells. Then the required probability is (by Jordan's theorem)

$$P_{[m]} = S_m - \binom{m+1}{m} S_{m+1} + \binom{m+2}{m} S_{m+2} \cdots\cdots + (-1)^{n-m}\binom{n}{m} S_n.$$

But each term of S_{m+i} equals $\frac{(n-m-i)^r}{n^r}$, while it includes in all $\binom{n}{m+i}$ terms, as such,

$$P_{[m]} = \binom{n}{m}\left(\frac{n-m}{n}\right)^r - \binom{n}{m+1}\binom{m+1}{m}\left(\frac{n-m-1}{n}\right)^r + \binom{n}{m+2}\binom{m+2}{m}\left(\frac{n-m-2}{n}\right)^r$$

$$- - - - - - - + (-1)^{n-m}\binom{n}{m}\binom{n}{m}\left(\frac{n-n}{n}\right)^r \quad (*)$$

But since $\binom{n}{m+i}\binom{m+i}{m} = \frac{n!}{(n-m-i)!\,m!\,i!} = \binom{n}{m}\binom{n-m}{i}$ — (1)

Using (1) in (*) we get —

$$P_{[m]} = \binom{n}{m}\left(\frac{n-m}{n}\right)^r\left[1 - \binom{n-m}{1}\left(\frac{n-m-1}{n-m}\right)^r + \binom{n-m}{2}\left(\frac{n-m-2}{n-m}\right)^r - \cdots\right]$$

12) From an urn containing n white balls and n black balls, select an random an even number of balls (all different ways of drawing an even number of balls are considered equally likely, irrespective of their number). Find the prob. that there will be the same number of black and white balls among them.

Soln:→ Since an even number of balls are drawn, this number may be 2, 4, 6, ..., 2n. So the total number of ways of drawing an even number of balls is

$$\sum_{k=1}^{n} \binom{2n}{2k} = 2^{2n-1} - 1.$$

It is given that these ways are equally likely. The number of ways in which 2k balls can be drawn such that among them there are k white balls and k black ones is $\binom{2n}{k}\binom{2n}{k}$. So, the number of ways favourable is

$$\sum_{k=1}^{n} \binom{2n}{k}^{2} = \binom{2n}{n} - 1.$$

Hence the required prob. is $\dfrac{\left\{\binom{2n}{n} - 1\right\}}{\left(2^{2n-1} - 1\right)}$.

13) Matching Problem: If n balls numbered 1 to n are placed at random in n urns numbered 1 to n, one ball in each urn, find the prob. that (a) ~~no ball goes~~ there is no matches, (b) there is at least one match, (c) there are exactly k matches.

Soln:→

(a) The desired prob. is $1 - P\left(\bigcup_{i=1}^{n} A_i\right)$, where A_i be the event that the ball numbered i goes to the urn numbered i, $1 \le i \le n$.

But

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{1 \le i \le n} P(A_i) - \sum_{1 \le i_1 \le i_2 < n} P(A_{i_1} \cap A_{i_2}) + \cdots\cdots + (-1)^{n-1} P\left(\bigcap_{i=1}^{n} A_i\right)$$

To calculate $P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k})$, note that the total number of ways in which n balls can be put in n urns, one ball in each urn, is $n!$ which are assumed to be equally likely; the number of ways in which the balls numbered $i_1, i_2, \dots, i_k$ go to their respective urns is $(n-k)!$; thus $P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = \frac{(n-k)!}{n!}$, which depends only on k and is free from $i_1, \dots, i_k$.

So, the required prob. is

$$1 - \binom{n}{1}\frac{(n-1)!}{n!} + \binom{n}{2}\frac{(n-2)!}{n!} - \dots + (-1)^n \binom{n}{n}\frac{1}{n!}$$

$$= 1 - \frac{1}{1!} + \frac{1}{2!} - \dots + (-1)^n \frac{1}{n!} = e^{-1} \text{ if } n \text{ is large}$$

[we remark that $e^{-1} = 0.36755$; for large n, the chance that no ball goes to its corresponding urn is approximately 0.37, although many people would have incorrectly thought that this prob. would go to 1 as $n \to \infty$.]

(b) $P(\text{at least one match}) = 1 - P(\text{no matches})$

$$P_1 \to\; \simeq 1 - e^{-1} = P_1$$

when n is large then $1 - e^{-1} = 0.63212$.

(c) The number of ways in which n balls can be put in n urns so that no ball goes to its corresponding urn is

$n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots + (-1)^n \frac{1}{n!}\right)$. We first fix attention on a particular set of r balls. The number of ways in which these and only these r balls go to their corresponding urn is equal to the number of ways in which the remaining $(n-r)$ balls go to $(n-r)$ urns in such a way that none of these goes to its corresponding urn, and hence is

$$(n-r)!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots + (-1)^{n-r}\frac{1}{(n-r)!}\right)$$

As there are $\binom{n}{r}$ possible selection of a group of r balls, it follows that there are

$$u_n := \binom{n}{r}(n-r)!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots + (-1)^{n-r}\frac{1}{(n-r)!}\right)$$

ways in which exactly r balls go to their corresponding urns. Hence the required prob. is

$$\frac{u_n}{n!} = \frac{1}{r!}\left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots + (-1)^{n-r}\frac{1}{(n-r)!}\right) \equiv e^{-1}/r! \text{ if } n \text{ is large.}$$

14) Each coefficient in the equation $ax^2+bx+c=0$ is determined by throwing a die. Find the prob. that the equation will have real roots.

Soln. :→ Clearly, a, b and c can assume values from 1, 2, 3, 4, 5, 6. So the total number of ways the coefficients can be chosen is $6^3 = 216$. The roots will be real iff $b^2 \geq 4ac$. Now, calculate the number of triplets of (a, b, c) satisfying this condition. Note that the maximum value of b^2 is 36 so that the maximum possible values of ac satisfying the above condition is 9.

Pair (a, c)	ac	4ac	b	No of b's	No. of cases
(1, 1)	1	4	2, 3, 4, 5, 6	5	$1\times5 = 5$
(1, 2), (2, 1)	2	8	3, 4, 5, 6	4	$2\times4 = 8$
(1, 3), (3, 1)	3	12	4, 5, 6	3	$2\times3 = 6$
(1, 4), (2, 2), (4, 1)	4	16	4, 5, 6	3	$3\times3 = 9$
(1, 5), (5, 1)	5	20	4, 6	2	$2\times2 = 4$
(1, 6), (2, 3), (3, 2), (6, 1)	6	24	5, 6	2	$4\times2 = 8$
(2, 4), (4, 2)	8	32	6	1	$2\times1 = 2$
(3, 3)	9	36	6	1	$1\times1 = 1$

So the total number of ways favourable to the desired event is 43, and the required prob. is $\frac{43}{216} = 0.1991$

15) If 2N married couples are seated at random at a round table, compute the prob. that no wife sits next to her husband.

Soln. :→ Let A_i be the event that the ith couple sit next to each other, $i = 1, 2, \ldots, N$. The required prob. is $1 - P(A_1 \cup \cdots \cup A_n)$. We now compute $P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j})$, $1 \leq i_1 \leq i_2 < \cdots < i_j \leq N$. There are $(2N-1)!$ ways of arranging 2N persons around a round table. The number of arrangements in which j specified group of men sit next to their wifes can be obtained by first treating each of the j married couples as being single entities; if this were the case, then we would need to arrange $(2N-j)$ entities around a round table, and there are $(2N-j-1)!$ such arrangements.

Finally, since each of the j married couples can be arranged among themselves in two possible ways, it follows that there are $2^j(2N-j-1)!$ arrangements such that a specified group of j men can sit next to their wifes. Thus

$$P(A_{i_1} \cap \dots \cap A_{i_j}) = \frac{2^j(2N-j-1)!}{(2N-1)!}$$

Therefore the required prob. is

$$1 + \sum_{j=1}^{N} (-1)^j \binom{N}{j} 2^j (2N-j-1)! \,/\, (2N-1)! \quad .///$$

Ques:→ Write down the limitations of classical definition of probability and relative frequency definition of probability?

Ans:→ i) Limitations of classical Def^n. :–

1) It is assumed here that all the cases are equally likely. This definition of probability is found useful when applied to the outcomes of the games of chance. If the outcomes of a random experiment are not equally likely then this def^n. is not applicable.

2) This definition breaks down if the no. of all possible cases is infinite.

3) In real life it is not easy to identify the outcomes as equally likely.

ii) Limitations of Statistical def^n. :–

1) If an experiment is repeated a number of times, the experimental conditions may not remain identical or homogeneous.

2) The $\lim_{n\to\infty} \frac{n(A)}{n}$ may not be unique.

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FIELD & σ-FIELD

▷ FIELD :

☐ **Theorem :→** Every field contains the empty set(ϕ) and the whole set (Ω).

Proof :→ If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$, by the defn. of field.

$\therefore A \cup A^c \in \mathcal{A}$

$\Rightarrow \Omega \in \mathcal{A}$

$\Rightarrow \Omega^c \in \mathcal{A}$

$\Rightarrow \phi \in \mathcal{A}$

☐ **Example of Field :→**

i) The class $\mathcal{A} = \{\phi, \Omega\}$ is a field. It is a minimal field of Ω.

The power set consisting of every subsets of Ω is also a field and it is the largest field of Ω.

If $A \subset \Omega$, then $\{\phi, A, A^c, \Omega\}$ is a field.

ii) Clearly $\{\phi, A, \Omega\}$ is not a field, where $A \subset \Omega$.

iii) If $A_1 = \{\phi, A, A^c, \Omega\}$ and $A_2 = \{\phi, B, B^c, \Omega\}$ are two fields, what can be said about $A_1 \cup A_2$ and $A_1 \cap A_2$?

Soln.→ $A_1 \cap A_2 = \{\phi, \Omega\}$ is a field.

$A_1 \cup A_2 = \{\phi, A, B, A^c, B^c, \Omega\}$ is not a field, in general since $A \cup B$ is not in $A_1 \cup A_2$. But if $A \subseteq B$, then $A \cup B = B$ or $A \in A_1 \cup A_2$ and it's a field.

▷ σ-FIELD :

☐ **Theorem :→** If $\mathcal{A}$ is a σ-field, then $A_1, A_2, \dots, A_n \in \mathcal{A}$

$\Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{A}$ and so $\mathcal{A}$ is a field.

Proof :→ Consider a sequence of sets $A_1, A_2, \dots, A_n$; $A_{n+1} = \phi$, $A_{n+2} = \phi, \dots\dots$; $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{A}$.

since $\mathcal{A}$ is a sigma field.

$\therefore \mathcal{A}$ is a field.

Remark :→ Since a σ-field is a field, it has all the properties of a field and it contains ϕ and Ω.

Example of σ-field :→ ~~If a set contains a finite number of~~ ~~[illegible]~~ Let Ω be a set, then $\{\phi, \Omega\}$ is a field. Clearly $\{\phi, \Omega\}$ is closed under complementation and any countable sequence of sets contains only the members ϕ or Ω and countable unions are ϕ or Ω. Hence $\{\phi, \Omega\}$ is closed under countable unions.

The σ-field containing $A \subset \Omega$ is:

(i) $\{\phi, A, A^c, \Omega\}$ which is the smallest σ-field containing A.

(ii) $\{\phi, A, A^c, B, B^c, A\cup B, A\cap B, A^c\cup B^c, A^c\cap B^c, A\cup B^c, B\cup A^c, A\cap B^c, B\cap A^c, (A\cap B^c)\cap(A^c\cup B), (A\cap B^c)\cup(B\cap A^c), \Omega\}$ is a σ-field. But $\{\phi, A, \Omega\}$ and $\{\phi, A, B, A^c, B^c, \Omega\}$ are not σ-field.

(iii) However, a field containing infinite number of sets may not be a σ-field.

⇨ Point Function & Set Function :—

A function whose domain is a collection of points is called the point function.

A function whose domain is a class of sets is called a set function.

If $\mathcal{A}$ is a class of sets and with each set $A \in \mathcal{A}$, we associate a value $f(A)$, then 'f' is a set function.

Ex. of Set Function :→

i) $n(A)$ = No. of members in the set A, $A \in \mathcal{A}$

$n(\cdot) : \mathcal{A} \to \{0, 1, 2, \ldots\}$

ii) area(A) = area of region A, $A \subseteq \mathbb{R}^2$.

⇨ Probability Space :—

The triplet $(\Omega, \mathcal{A}, P)$, where Ω is the sample space of a random experiment, $\mathcal{A}$ is a σ-field of events or of subsets of Ω and $P[\cdot]$ is the probability function defined on $\mathcal{A}$, is called the probability space.

Uncountable (or Continuous) Sample Space :-

If Ω is uncountable, then it is not possible to assign positive probability to each sample point $x \in \Omega$. If $\Omega = (a, b)$, then there are uncountable no. of sample points and if we assign positive probability to each sample point then $P[\Omega] = 1$ doesn't hold.

Borel Field :-> Now, $P[\cdot]$ defines on $\mathcal{A}$. As $P[(-\infty, x]] = F(x)$, say $\forall x \in R$. then we can find uniquely $P[A]$, for any $A \in \mathcal{A}$.

Example :-> Let $\Omega = \{\omega : 0 \le \omega \le a\}$. Now Ω is uncountable. Suppose the σ-field $\mathcal{A}$ is the Borel field restricted to the subsets $[0, a]$.

Here $F^* = \{(0, x] : 0 < x \le a\}$ and $\mathcal{A} = \sigma\{F^*\}$

Define $P[(0, x]] = \frac{x}{a}$, $0 < x \le a$.

Note that $P[\Omega] = P[(0, a]]$.

$$= \frac{a}{a} = 1.$$

Statistical Defn. Satisfies all the axioms of Axiomatic Defn. of Probability :-

$A \subseteq \Omega$, a probability function. In other words, show that the statistical definition of probability satisfies all the axioms of axiomatic defn of probability.

Soln -> Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n, \ldots\}$ be a countable sample. Then $\{A_n\}$, where $A_n = \{\omega_n\}$, forms a partition of Ω and $\mathcal{A}$ is the power set of Ω.

By statistical defn. of probability, for any $A \subseteq \Omega \Leftrightarrow A \subseteq \mathcal{A}$,

$$P[A] = \lim_{n \to \infty} \frac{n(A)}{n}$$

Clearly $P[\cdot]$ is real-valued set function of $\mathcal{A}$.

Axiom-I -> $P[A] = \lim_{n \to \infty} \frac{n(A)}{n} \geq 0$

Since $n(A) \geq 0$.

Axiom-II → $P[\Omega] = \lim\limits_{n\to\infty} \frac{n(\Omega)}{n}$

$= \lim\limits_{n\to\infty} \frac{n}{n}$

$= 1$ [normed]

Axiom-III → For any $A, B \in \mathcal{A}$ and $A \cap B = \phi$

$P[A \cup B] = \lim\limits_{n\to\infty} \frac{n(A \cup B)}{n}$

$= \lim\limits_{n\to\infty} \frac{n(A) + n(B)}{n}$

$= \lim\limits_{n\to\infty} \frac{n(A)}{n} + \lim\limits_{n\to\infty} \frac{n(B)}{n}$

$= P[A] + P[B]$ [finite additivity]

Hence, the set function $P[A] = \lim\limits_{n\to\infty} \frac{n(A)}{n}$, $A \in \mathcal{A}$, is a probability function.

Example :→ For any n events $A_1, A_2, \ldots, A_n \in \mathcal{A}$, express $\bigcup\limits_{i=1}^{n} A_i$ as a union disjoint events. Hence, obtain Boole's inequality.

Soln. → Note that – $A_1 \cup A_2 = A_1 + (A_2 - A_1)$

$= A_1 + A_2 A_1^c$

$\therefore A_1 \cup A_2 \cup A_3 = A_1 + A_2 \cap A_1^c + A_3 \cap A_1^c \cap A_2^c$

and lastly, $\bigcup\limits_{i=1}^{n} A_i = A_1 + A_2 A_1^c + A_3 A_1^c A_2^c + \cdots + A_n A_1^c A_2^c \ldots A_{n-1}^c$.

where A_1, $A_2 A_1^c$, $A_3 A_1^c A_2^c$, ..., $A_n A_1^c A_2^c \cdots A_n^c$ are mutually disjoint.

By finite additivity of $P[\cdot]$,

$P[\bigcup\limits_{i=1}^{n} A_i] = P[A_1 + A_2 A_1^c + A_3 A_1^c A_2^c + \cdots + A_n \cap A_1^c \cap A_2^c \cap \cdots \cap A_{n-1}^c]$

$= P[A_1] + P[A_2 A_1^c] + P[A_3 \cap A_1^c \cap A_2^c] + \cdots + P[A_n \cap A_1^c \cap A_2^c \cap \cdots \cap A_{n-1}^c]$

$\leq \sum\limits_{i=1}^{n} P[A_i]$

[By monotic property of $P[\cdot]$ as $A_i A_1^c A_2^c \ldots A_{i-1}^c \subseteq A_i$, $P[A_i A_1^c A_2^c \ldots A_{i-1}^c] \leq P[A_i]$, $\forall\ i = 2(1)n$.]

Probabilistic Independence of Events :→

Theorem :→ If two events $A, B \in \mathcal{A}$ are probabilistically independent then i) A and B^c
ii) A^c and B
iii) A^c and B^c
are also independent.

Proof :→ $P[A \cap B] = P[A] \cdot P[B]$

i) $P[A \cap B^c] = P[A] - P[A \cap B]$

$= P[A] - P[A] \cdot P[B]$

$= P[A]\,[1 - P[B]]$

$= P[A]\,P[B^c]$

ii) $P[A^c \cap B] = P[B] - P[A \cap B]$

$= P[B] - P[A]\,P[B]$

$= P[B]\,[1 - P[A]]$

$= P[A^c]\,P[B]$.

iii) $P[A^c \cap B^c] = P[(A \cup B)^c]$

$= 1 - P[A] - P[B] + P[A \cap B]$

$= 1 - P[A] - P[B] + P[A] \cdot P[B]$

$= \{1 - P[A]\}\{1 - P[B]\}$

$= P[A^c]\,P[B^c]$

Pairwise and Mutually independence of a set of events :→

a) A set of events $\{A_1, A_2, \ldots, A_n\}$ is said to be pairwise independent if

$$P[A_{i_1} \cap A_{i_2}] = P[A_{i_1}]\,P[A_{i_2}] \quad \forall\, i_1 < i_2 .$$

b) A set of events $\{A_1, \ldots, A_n\}$ is said to be mutually or totally independent if

$$P[A_{i_1} \cap A_{i_2}] = P[A_{i_1}]\,P[A_{i_2}]$$

$$P[A_{i_1} \cap A_{i_2} \cap A_{i_3}] = P[A_{i_1}]\,P[A_{i_2}]\,P[A_{i_3}] \quad \forall\, i_1 < i_2 < i_3 .$$

$$\vdots$$

$$P[A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_n}] = P[A_{i_1}]\,P[A_{i_2}] \ldots P[A_{i_n}]$$

$$P\left[\bigcap_{i=1}^{n} A_i\right] = \prod_{i=1}^{n} P[A_i]$$

Clearly, the mutually independence of a set of events implies the pairwise independence but the converse is not necessarily true.

Example 1. Consider a random experiment with four totally likely sample points. Construct three events which are pairwise independent but not mutually independent.

Soln.→ Consider a random experiment of tossing a fair coin twice.

Then, $\Omega = \{HH, HT, TH, TT\}$

and $P[\{\omega\}] = \frac{1}{4}$, $\omega \in \Omega$.

Define, $A_1 = \{HH, HT\}$

$A_2 = \{HH, TH\}$

$A_3 = \{HH, TT\}$

~~Then $A_i = \frac{2}{4}$~~ Then $P[A_i] = \frac{2}{4} = \frac{1}{2}$ $\forall\ i = 1, 2, 3$.

Now, $P[A_1 \cap A_2] = \frac{1}{4} = P[A_1].P[A_2]$

$\Rightarrow A_1, A_2, A_3$ are pairwise independent.

But $P[A_1 \cap A_2 \cap A_3] = P[\{HH\}] = \frac{1}{4} \neq \frac{1}{8} = P[A_1]\, P[A_2]\, P[A_3]$

i.e. A_1, A_2, A_3 are not mutually independent.

Example 2. Construct a random experiment with 8 ~~po~~ equally likely sample points and construct 3 events which are pairwise and not mutually exclusive.

Soln.→ Consider a random experiment of a coin is thrown thrice.

∴ Sample Space, $\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

and $P[\omega] = \frac{1}{8}$; $\omega \in \Omega$

Define, $A_1 = \{HHH, HHT, HTH, THH\}$

$A_2 = \{HHH, HHT, HTT, THT\}$

$A_3 = \{HHH, HHT, HTT, TTT\}$

$P[A_i] = \frac{4}{8} = \frac{1}{2}$ $\forall\ i = 1, 2, 3$.

Now, $P[A_1 \cap A_2] = P[\{HHH, HHT\}]$

$= \frac{2}{8} = \frac{1}{4}$

$= P[A_1]\, P[A_2]$

$P[A_1 \cap A_2 \cap A_3] = P[\{HHH, HHT\}]$

$= \frac{1}{4} \neq \frac{1}{8} = P[A_1]\, P[A_2]\, P[A_3]$.