PROBABILITY THEORY I

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BASIC PROBABILITY

THEORY OF PROBABILITY

Meanings of Probability :- It's a measure of chance of occurance of a phenomenon.

⟨1⟩ The word 'Probability' may be used to mean 'the degree of belief' of a person making a statement or proposition. It is used in the sense when we say that a certain football team will be the champion in a league or we say that the 'Mahabharat' is very probably the work of several authors.

⟨2⟩ On the other hand, the word has a different meaning, when we use it in the context of an experiment that can be repeated any no. of times under identical conditions. By the probability of any outcome of the experiment we shall now mean the long run relative frequency of any particular outcome of the experiment We use the probability in this sense when we say that the probability of getting a 'head' in tossing a coin is 3/4 or the probability that an article produced by a machine will defective is negligable. In statistics, we generally use the term in 2nd sense.

In probability and statistics, we concern ourselves to same special type of experiment.

(1) **Random Experiment :—**

A random experiment or statistical experiment is an experiment in which—

⟨i⟩ all possible outcomes of the experiment are known in advance.

⟨ii⟩ any performance of the experiment results in an outcome that is not known in advance.

⟨iii⟩ The experiment can be repeated under identical or similar condition.

Ex : Consider an experiment of 'tossing a coin'. If the coin does not stand on the side there are two possible outcomes : Head (H), Tail (T). On any performance of the experiment, one does not know what the result will be. coin can be tossed as many times as desired under identical or similar condition. Hence, tossing of one is a random experiment.

(2) **Sample Space** :— The collection or set of all possible outcomes of a random experiment is called the sample space of the random experiment. It's noted by Ω (or S). The elements of the sample space (Ω) are called the '<u>Sample Point</u>'.

Ex: (1) Consider a random experiment of 'tossing a coin' twice. Write down the sample space?

Sol. The sample space is — $\Omega = \{HH, HT, TH, TT\}$

The sample points are — HH, HT, TH, TT.

Ex: (2)

In each of the following experiment. What is the sample space?

i> a coin is tossed thrice.

ii> a die is rolled twice.

iii> a coin is tossed until a head appear.

Sol. i> $\Omega = \{HHH, HTH, THT, HHT, TTH, HTT, THH, TTT\}$

ii> $\Omega = \{(i,j) : i,j = 1(1)6\}$ [arithmatic progression [a(d)l]]

iii> $\Omega = \{H, TH, TTH, TTTH, ----\}$

Ex: (3) In each of the following experiments, what is the sample space?

i> In a survey of families with 3 children, the genders of the childrens are recorded in increasing of their age.

Sol.

$\Omega = \{BBB, BBG, BGG, GBG, GGB, GBB, BGB, GGG\}$

ii> The experiment consists of selecting four items from a manufacturers output and observing whether or not each item is defective.

Sol.

$\Omega = \{(a,b,c,d) : a, b, c, d$ is either defective or non-defective, consisting of 16 sample points$\}$

iii> Two cards are drawn from an ordinary deck of cards
(a) with replacement ; (b) without replacements

Sol. (a) $\Omega = \{(x,y) : x, y = 1(1)52\}$ [cosisting 52^2 sample points]

(b) $\Omega = \{(x,y) : x, y = 1(1)52$ but $x \neq y\}$ [consisting 52×51 sample points]

Ex. (4) In each of the following experiments what is the sample space?

(i) Noting the lifetime of an electronic bulb.

(ii) A point is selected from a rod of unit length.

Sol. (i) $\Omega = \{x : 0 < x < \infty\}$ [continuous sample space]

(ii) $\Omega = \{x : 0 \le x \le 1\}$ [Here x is the distance of the selected point from the origin]

(3) **Trial:-** A trial refers to a special type of experiment in which there are two possible outcomes — 'success' and 'failure' with varying probability of success.

(4) **Outcome:-** Result of an experiment.

(5) **Sample:-** It is a part of the population and is supposed to represent the characteristic of the population.

(6) **Event:-** An event is a subset of sample space

(i) **Elementary Event:-** If an event contains only one sample point, it's known as an elementary event.

(ii) **Composite Event:-** If an event contain more than one sample points, it's known as a composite event.

Ex. (1). Consider the random experiment of 'tossing a fair coin twice'. Identify elementary & composite events.

Sol. $\Omega = \{HH, HT, TH, TT\}$

The event (i) 'at least one head' is $A = \{HH, HT, TH\}$, is called a composite event.

(ii) 'no head' is $B = \{TT\}$, is called an elementary event.

Ex: 2: A club has 5 members A, B, C, D, E. It's required to select a chairman and a secretary. Assuming that 1 member can't occupy both positions. Write the sample space associated with this section. What's the event that member A is an officeholder.

Sol^n sample space is, $\Omega = \{(x, y) : x, y = A, B, C, D, E \text{ but } x \neq y\}$

Here x stands for chairman and y stands for secretary.

Event is, $P = \{AB, BA, AC, AD, AE, CA, DA, EA\}$

$= \{(x, y) : \text{If } x = A \text{ then } y = B, C, D, E. \text{ If } y = A \text{ then } x = B, C, D, E\}$.

Mutually Exclusive Events :→ Several events $A_1, A_2, \ldots, A_n$ in relation to a random experiment are said to be mutually exclusive (or disjoint) if any two of them can't occur simultaneously, everytime the experiment is performed i.e. $A_i \cap A_j = \phi, \forall (i \neq j) / i<j = 1(1)n$.

Exhaustive Events :→ Several events $A_1, A_2, \ldots, A_n$ in relation to a random experiment are said to be exhaustive events if any of them must necessarily occur, everytime the experiment is performed that is $\bigcup_{i=1}^{n} A_i = \Omega$.

Equally Likely Cases (or events) :→ Several cases $A_1, A_2, A_3, \ldots$ are said to be equally likely if, after taking into consideration all relevant evidance, there is no reason to believe that one is more likely than the other.

Ex :→ For a random experiment of 'tossing a coin twice', the sample space is — $\Omega = \{HH, HT, TH, TT\}$.

Let A be the event of getting at least one head and B be the event of getting at most one head.

Then — $A = \{HT, TH, HH\}$

$B = \{HT, TH, TT\}$

$A \cup B = \Omega$ and $A \cap B \neq \phi$.

Hence, the event A and B are exhaustive but not mutually exclusive.

Let C be the event of getting 'no head', then $C = \{TT\}$, $A \cup C = \Omega$, $A \cap C = \phi$,

Hence, the event A and C are exhaustive and mutually exclusive too.

The Classical Definition of Probability :→ If a random experiment can result in N (finite) mutually exclusive, exhaustive and equally likely cases and N(A) of them are favorable to the occurance of the event A, then the probability of occurance of A is —

$$P[A] = \frac{N(A)}{N}.$$

Remark: ① Since $0 \leq N(A) \leq N$

$0 \leq P(A) \leq 1$

② By classical definition of probability of an event is a rational number between 0 and 1. But in general probability is a real no. between 0 and 1.

③ $P[A^c] = \frac{N-N(A)}{N} = 1 - \frac{N(A)}{N} = 1 - P(A)$.

Ex:

① A fair coin is tossed 3 times, what's the prob. of getting 'exactly 2 heads'.

② What's the prob. of getting 'at least on tail'?

Solⁿ:

① $\Omega = \{HHH, HTH, THT, TTH, HTT, THH, TTT\}$

Since the coin is fair, $N=8$, elementary cases are equally likely. The events of getting two heads is $A = \{HHT, HTH, THH\}$. Hence the no. of favorable cases $N(A) = 3$.

By classical definition $P[A] = \frac{N(A)}{N} = \frac{3}{8}$.

② The event of getting 'at least one tail' is

$N(B) = \Omega - \{HHH\} = 8 - 1 = 7$.

∴ By classical definition, $P[B] = \frac{N(B)}{N} = \frac{7}{8}$.

Limitation of Classical Definition :→

1) It is assumed here that all the cases are equally likely. This defⁿ of probability is found useful when applied to the outcomes of the games of chance. If the outcomes of a random experiment are not equally likely then this defⁿ is not applicable.

2) This defⁿ breaks down if the no. of all possible cases is infinite.

3) In real life, it is not easy to identify the outcomes as equally likely.

Statistical or Empirical (Approach) Definition of Probability :→ Suppose A is an event of a random experiment. Suppose it is possible to repeat the experiment a large number of times under essentially similar condition.

Denote by $n(A)$, the number of occurance of A in 'n' repetition. $n(A)$ is called the frequency of A and $\frac{n(A)}{n}$, is the relative frequency. A kind of regularity is observed when a large number of repetition is considered. It is an observed fact that the relative frequencies stabilize to a certain value as 'n' become large. This tendency seems to be inherent in the nature of a random experiment and stability of relative frequencies for large values of n constitutes the basis of statistical theory or statistical definition of probability. This kind of regularity in a random experiment is known as statistical regularity. The limiting value of $\frac{n(A)}{n}$ as $n \to \infty$, is called the prob. of A, provided the limit exists.

Definition :→ If a random experiment is repeated under essentially similar conditions then the limiting value of the relative frequency of an event A, as the trials become indefinitely large, is called the probability of event A, provided the limit exists.

Consider the Question :→

1) If a coin is tossed, what is the probability that it will turn up head.

Ans :→ Examine the results of tosses given below:

No. of times the coin is tossed (n)	1	10	100	1000	2000	3000
No. of times the head turns up [N(A)]	0	6	61	605	1207	1718

Thus, we get the relative frequencies as :—

$\frac{6}{10}, \ldots, \frac{61}{100}, \ldots, \frac{605}{1000}, \ldots, \frac{1207}{2000}, \ldots, \frac{1718}{3000}, \ldots$

As the no. of tossing increases the relative frequency tents to stabilize at 0·6. Therefore the probability of getting a head in a tossing of a coin is 0·6.

Remark :→ If in a random experiment all possible cases are not equally likely, then we can't apply classical definition in this case, if the experiment can be repeated a large no. of times, then probability of an event A can be obtained by statistical definition, This is an improvement over the statistical def^n.

Limitations :→

1) If an experiment is repeated a number of times, the experimental conditions may not remain identical or homogeneous.

2) The $\lim_{n\to\infty} \frac{n(A)}{n}$ may not be unique.

Subjective Probability :→ In everyday's life we hear or make statements such as "probably I shall miss the train", "probably Mr. Raj will be at home now." Such statements can be made more precise by "the chance of missing the train is 60% "the chance that Mr. Raj will be at home now is 75%" etc. Here 60%, 75% etc. measures one's belief in the occurance of the event. This Subjective method is another method of consigning probabilities of various events based on the personal beliefs.

When the experiment is not repeatable, this method may be adopted for assigning probabilities to events. Since, different persons may assign different probabilities, one can't arrive at objective conclusion using probabilities assigned by subjective methods.

PROBABILITY & STATISTICS:- The problem in Probability is —
" Given a stochastic model what we can say about the outcome"

The problem in statistics is —
" Given a sample what we can say about the population".

Set Theory	Probability Theory
1. Point / Element	Elementary Event
2. Set	Event
3. Universal Set	Sample Space
4. Null set	Impossible event
5. A is a subset of B	A implies B
6. A is a superset of B	A is implied by B

Ex. 1. Let A, B, C are 3 events. Then the expression of following events in set notations:

(i) Only A occurs : $A \cap B^c \cap C^c$

(ii) A occurs : A

(iii) Both A and B, but not C occur : $A \cap B \cap C^c$

(iv) All 3 events occur: $A \cap B \cap C$

(v) At least one occur: $A \cup B \cup C$

(vi) At least two occur: $(A \cap B) \cup (B \cap C) \cup (A \cap C)$

(vii) One and no more occur: $(A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap A^c \cap B^c)$

(viii) Two and no more occur: $(A \cap B \cap C^c) \cup (B \cap C \cap A^c) \cup (A \cap C \cap B^c)$

(ix) None occurs: $A^c \cap B^c \cap C^c$

(x) If A occurs so does B: $A \subseteq B$.

Ex. 2. Eight students are arranged at random (a) in a row and (b) in a column. Find the probability that two given students will be next to each other.

Sol. (a) Req. prob. $= \dfrac{7!\,2!}{8!}$

(b) Req. prob. $= \dfrac{6!\,2!}{7!}$

Ex. 3. The nine digits 1, 2, 3, ..., 9 are arranged in random order to form a nine-digit number. Find the prob. that 1, 2 and 3 appears as neighbours in the order mentioned.

Sol. Req. prob $= \dfrac{7!}{9!} = \dfrac{1}{72}$.

Ex. 4. Find the prob. that seven people has birthdays on 7 different days of the week, assuming equal prob. for the seven days.

Sol. Req. prob. $= \dfrac{7!}{7^7}$.

No. of Distinguishable or distinct arrangement of r balls (objects) into n cells when —

(I) balls are distinguishable and exclusion principle followed.

(II) " " " but " " NOT " .

(III) " " indistinguishable and " " followed.

(IV) " " " but " " NOT " .

Exclusion Principle :- The principle of excluding a cell from taking more than one ball (object) while distributing r balls (objects) into n cells, i.e., to exclude or deban a ball (object) to be placed into a cell which is occupied.

CASE-I :- Let $u(r,n)$ denotes the no. of distinguishable distributions of r balls into n cells.

Hence, $u(r,n) = 0$ if $r > n$.

For $r \le n$, we have $u(r,n)$

$$= \begin{pmatrix} \text{no. of ways in which} \\ \text{1st ball can be placed} \\ \text{in any of the } n \\ \text{cells} \end{pmatrix} \times \begin{pmatrix} \text{no. of ways in} \\ \text{which 2nd ball} \\ \text{can be placed} \\ \text{in any of the} \\ (n-1) \text{ cells} \end{pmatrix} \times \cdots\cdots \times \begin{pmatrix} \text{no. of ways in} \\ \text{which } r\text{th ball} \\ \text{can be placed} \\ \text{in any of the} \\ (n-r+1) \text{ cells} \end{pmatrix}$$

$$= n(n-1)\cdots\cdots(n-r+1) = (n)_r .$$

CASE-II :- Maxwell-Boltzman Statistics

Here $u(r,n) = n.n.\cdots\cdots r\text{-times} = n^r$.

CASE-III :- Fermi-Dirac Statistics

Here $u(r,n) = 0$ for $r > n$

For $r \le n$, $u(r,n) = \dfrac{(n)_r}{r!} = \dbinom{n}{r}$.

CASE-IV :- Bose-Einstein Statistics

$u(r,n) =$ no. of distinguishable arrangements of r dots and $(n-1)$ bars

$$= \frac{(r+n-1)!}{r!\,(n-1)!}$$

$$= \binom{r+n-1}{r}.$$

SOLVED EXAMPLES

Ex.1. 2 cards are drawn from a well-shuffled cards. What's the probability that both extracted cards are aces.

Sol. Here, ~~total~~ no. of cases, = no. of ways in which 2 cards can be drawn from 52 cards WOR.

$= 52 \times 51$

No. of favourable cases = No. of ways of getting two aces from 4 aces WOR

$= 4 \times 3$

So, Required probability $= \dfrac{\text{No. of favourable cases}}{\text{Total no. of cases}} = \dfrac{4\times 3}{52 \times 51} = \dfrac{1}{221}$.

Ex.2. Two dice are thrown n times in succession. What's the prob. of obtaining double 6 at least one. Also determine the minimum no. of thrown required to accomplish the objective with a probability $> \frac{1}{2}$.

Sol. (i) No. of throws resulted in with required probability

$$= \frac{\text{a double six at least once}}{\text{total no. of all possible cases}}$$

$$= \frac{36^n - 35^n}{36^n} = 1 - \left(\frac{35}{36}\right)^n = p_n, \text{ say}$$

$$p_n > \frac{1}{2} \Rightarrow \left(\frac{35}{36}\right)^n < \frac{1}{n}$$

$$\Rightarrow n(\log 35 - \log 36) = -\log 2$$

$$\Rightarrow n < \frac{\log 2}{\log 36 - \log 35}.$$

$$\therefore n_{min} = \frac{\log 2}{\log 36 - \log 35} \cong 24.$$

Ex.3. A certain number h of distinguishable balls is distributed among N compartment. What is the prob. that a certain specified compartment will contain h balls?

Sol. Total no. of cases = No. of ways in which n distinguishable balls can be distributed among N compartment without following exclusion principle.

$= N^n$.

No. of favourable cases $= \begin{pmatrix} \text{No. of ways in which} \\ h \text{ balls can be chosen} \\ \text{from } n \text{ balls and} \\ \text{placed at the} \\ \text{specific compartment} \end{pmatrix} \times \begin{pmatrix} \text{No. of ways in} \\ \text{which the} \\ \text{remaining } (n-h) \\ \text{balls can be} \\ \text{distributed into} \\ (N-1) \text{ compartments} \end{pmatrix}$

$= \binom{n}{h} \times (N-1)^{n-h}$ $\therefore$ Req. prob. $= \dfrac{\binom{n}{h}(N-1)^{n-h}}{N^n}$.

Ex. 4. In an urn there are n groups of p objects in each. Objects in different groups are distinguished by some characteristic property. What's the prob. that among $(\alpha_1 + \cdots + \alpha_n)$ objects taken, $[0 \le \alpha_i \le p \ \forall \ i = 1(1)n]$, there are α_1 of one group, α_2 from another group ...... and so on.

Sol. The total no. of cases $= \binom{np}{\alpha_1 + \alpha_2 + \cdots + \alpha_n}$

Favourable cases # are $= \begin{pmatrix} \text{no. of distinguishable} \\ \text{arrangements of} \\ \alpha_1, \ldots, \alpha_n \end{pmatrix} \begin{pmatrix} \text{no. of ways in which} \\ \alpha_1 \text{ comes from one} \\ \text{group, } \alpha_2 \text{ from 2nd and} \\ \text{so on} \end{pmatrix}$

$$= \frac{n!}{1!\ 2! \cdots\cdots n!} \binom{p}{\alpha_1}\binom{p}{\alpha_2} \cdots \binom{p}{\alpha_n}$$

Ex. 5. There are N tickets numbered $1, 2, \ldots, N$ of which n are taken at random in an increasing order of their numbers $x_1 < x_2 < \cdots < x_n$. What's the prob. that $x_m = M$.

Sol. The n tickets can be taken in $\binom{N}{n}$ ways. We assume that these are equally likely.

In order that $x_m = M$, it is necessary & sufficient that $(m-1)$ tickets have numbered from 1 to $M-1$. Now $(n-m)$ tickets have numbers from $N-m$ to N and one ticket has the number M. Hence, the No. of favourable cases are $\binom{M-1}{m-1}\binom{N-M}{n-m}$.

$\therefore$ Req. prob. is $= \dfrac{\binom{M-1}{m-1}\binom{N-M}{n-m}}{\binom{N}{n}}$.

Ex. 6. An urn contains 'a' white and 'b' black balls. Balls are drawn one by one until only those of the same colour are left. What's the prob. that they are white.

Sol. Let E be the given experiment and A be the desired event. Let E' be the desired experiment of drawing all the balls one by one and A' the event that the last ball drawn is white. Then A happens in E iff A' happens in E'. Hence, $P(A) = P(A')$. Since the balls are drawn at random in E', $P(A')$ is also the prob. that the first ball drawn is white and hence is $\frac{a}{a+b}$.

Ex. 7. Three numbers are chosen from the first 30 natural numbers. What's the prob. that the chosen number will be in
(a) A.P. (b) G.P.

Solution:- (a) $N = \{1, 2, \ldots, 30\}$

Three numbers can be chosen from 30 natural numbers in $\binom{30}{3}$ ways which are assumed to be equally likely.
In order that, the three numbers will be of the form $m, m+K, m+2K$ $(K \geq 1)$, the k must satisfy $1 \leq K \leq 14$ and for any such value of K, m must satisfy $1 \leq m \leq 30-2K$.
Hence, the number of such A.P.s is $\sum_{K=1}^{14} (30-2K)$

$$= 30 \times 14 - 15 \times 14$$
$$= 14 \times 15.$$

So, the required probability is $\dfrac{14 \times 15}{\binom{30}{3}} = 0.0517.$

(b) We count the triplets (arranged in increasing order) whose terms form a G.P. by listing them as follows:

Common ratio	Triplet
2	$\{(i, 2i, 4i), 1 \leq i \leq 7\}$
3	$\{(i, 3i, 9i), 1 \leq i \leq 3\}$
4	$(1, 4, 16)$
5	$(1, 5, 25)$
$3/2$	$(4, 6, 9), (8, 12, 18), (12, 18, 27)$
$5/2$	$(4, 10, 15)$
$4/3$	$(9, 12, 16)$
$5/3$	$(9, 15, 25)$
$5/4$	$(16, 20, 25)$

$\therefore$ Hence the required prob. is $\dfrac{19}{\binom{30}{3}} = 0.0047.$

Elements of Combinatorial Analysis

Rule-I :→ If there are two groups G_1 & G_2 ; $G_1 = \{a_1, a_2, \ldots, a_n\}$ consisting of n elements and $G_2 = \{b_1, b_2, \ldots, b_m\}$ consisting of m elements then the no. of pairs (a_i, b_j) formed by taking one element a_i from G_1 and b_j from G_2 is $n \times m$.

If there are k groups $G_1, G_2, \ldots, G_K, \ldots$ such that

$$G_1 = \{a_1, a_2, \ldots, a_{n_1}\}$$
$$G_2 = \{b_1, b_2, \ldots, b_{n_2}\}$$
$$\vdots$$
$$G_K = \{t_1, t_2, \ldots, t_{n_K}\}$$

Then the number ordered k-tuples $(a_{i_1}, b_{i_2}, \ldots, t_{i_K})$ formed by taking one element from each group is $= n_1 \times n_2 \times \cdots \times n_K$

Example :→ 'Placing balls into the cells' amounts to choose one cell for each ball. Let there are r balls and n cells. For the 1st ball, we can choose any one of the n cells. Similarly, for each of the balls, we have n choices, assuming the capacity of each cell is infinite or we can place more than one ball in each cell. Hence the r balls can be placed in the n cells in n^r ways.

Applications :→

① A die is rolled r times. Find the prob. that —

i) No ace turns up. [ace-1] —

ii) No ace turns up.

Sol^n :→

i) The experiment of throwing a die r times has $6 \times 6 \times 6 \times \cdots r$ times $= 6^r$ possible outcomes. Assume that all possible cases are equally likely. The no. of cases favorable to the event (A), 'no ace turns up' is 5^r.

By Classical Def^n, $P[A] = \frac{N(A)}{N} = \frac{5^r}{6^r}$.

ii) P [an ace turns up] $= 1 - P$[no ace turns up]

$$= 1 - \frac{5^r}{6^r}.$$

Remark :→ The all possible outcomes of 'r' throw of a die correspond to the placing r balls into $n = 6$ cells.

Rule - II :-

Ordered Samples :→ Consider a population of n elements $a_1, a_2, \ldots, a_n$ any order arragement. $a_{j1}, a_{j2}, \ldots, a_{jr}$ of r elements is called an ordered sample of size r, drawn from the population. Two procedure are possible——

i) Sampling with replacement :→ Here an element is selected from the population and the selected element is returned to the population before the next selection is made. Each selection is made from the entire population, so that the same element can be drawn more than ones.

ii) Sampling without replacement :→ Here an element once choosen is removed from the population, so that the sample becomes an arrangement without repeatation.

☐ For a popl^n with n elements and a prescribed sample size r, there are n^r different ordered samples with replacement and $n(n-1)\ldots(n-r+1) = {}^nP_r$ or $(n)_r$ different ordered samples without replacement.

Remark :→

(1) ${}^nP_r = n(n-1)\ldots(n-r+1)$ is defined if $n \in N$ and r is a non-negative integers. But $(n)_r = n(n-1)\ldots(n-r+1)$ is defined if $n \in R$ and r is non-negative integer. In the some way if $n \in \mathbb{R}$ then ${}^nC_r = \binom{n}{r} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

Example:- 1. A random sample of size 'r' with replacement is taken from a population of n elements. Find the probability that in the sample no element appear twice.

Solution:- There are n^r sample in all. As the samples are drawn randomly, all samples are equally likely. The no. of the samples in which no element appears twice is the no. of samples drawn without replacement,

Favourable sample is $= n(n-1)\cdots\cdots(n-r+1) = (n)_r$

Hence, the probability is $= \dfrac{(n)_r}{n^r}$.

Example:- 2. If n balls are randomly placed into n cells, what is the probability that each cell will be occupied.

Solution:-

$$P(A) = \frac{n!}{n^n}.$$

- **SOLVED EXAMPLES:-**

Q.1. Find the probability that among five randomly selected digits, all digits are different.

Ans:- $P(A) = \dfrac{(10)_5}{10^5}$.

Q.2. In a city seven accidents occur each ~~week~~ weak in a particular week there occurs one accidents perday. Is it surprising?

Ans:- $P(A) = \dfrac{7!}{7^7}$.

Q.3. An elevator (lift) stands with 7 passengers and stops at 10^{th} floor. What's the prob. that no two passengers leave at the same floor?

Sol.
$$N = 10.10.\cdots\cdots10 \text{ (7 times)}$$
$$= 10^7.$$
$$N(A) = 10.9.8.7.6.5.4$$
$$= (10)_7$$
$$P(A) = \frac{(10)_7}{10^7}.$$

Q.4. What's the probability that n individuals have different birthdays? Also show that the prob. is approximately equal to $e^{-n(n-1)/730}$. How many people are required to make the prob. of distinct birthdays less than $\frac{1}{2}$?

Sol.
$$p = \frac{(365)_n}{365^n} = \frac{365 \cdot 364 \cdots\cdots (365-n+1)}{365 \cdot 365 \cdots\cdots 365}$$
$$= 1\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\cdots\cdots\left(1-\frac{n-1}{365}\right)$$
$$\therefore \ln p = \sum_{k=1}^{n-1} \ln\left(1-\frac{k}{365}\right)$$

For, $0 < x < 1$, $\ln(1-x) \simeq -x$.
$$\therefore \ln p \simeq \sum_{k=1}^{n-1}\left(-\frac{k}{365}\right) = -\frac{1}{365}\left(\sum_{k=1}^{n-1} k\right)$$
$$= -\frac{n(n-1)}{2(365)} = -\frac{n(n-1)}{730}$$
$$\therefore p = e^{-n(n-1)/730}$$

For $p = \frac{1}{2}$, $\ln p = -\ln 2 = -0.693$,
$$\therefore \frac{n(n-1)}{730} = 0.693$$
$$\therefore n^2 - n - 506 = 0$$
$$\Rightarrow (n-23)(n+22) = 0$$
$$\Rightarrow n = 23.$$

$\therefore$ more than 23 people are required.

Q.5. Six dice are thrown. What's the prob. that every possible number will appear.

Hints:-
$$p = \frac{6!}{6^6} = 0.0154$$

Q.6. There are four childrens in a family. Find the prob. that-
(a) at least two of them have the same birthday?
(b) only the oldest and the youngest have the same birthday?

Hints:- (a) $p_1 = 1 - \left\{\frac{(365)_4}{365^4}\right\} = 1 - P\{$them have different birthdays$\}$.

(b) $$p_2 = \frac{365 \times 364 \times 363}{365^4} = \frac{(365)_3}{365^4}.$$

Q.7. The number 1, 2, ..., n are arranging in a random order. Find the prob. that digits (a) 1, 2 (b) 1, 2, 3 appears as neighbours in the order named.

Hints:- Consider (1, 2) as a single digit then there are $(n-1)$ entities which can be arranged in $(n-1)!$ ways.

(a) Required prob. is $= \dfrac{(n-1)!}{n!} = \dfrac{1}{n}$.

(b) Required prob. is $= \dfrac{(n-2)!}{n!} = \dfrac{1}{n(n-1)}$.

Q.8 (i) In sampling with replacement find the prob. that a fixed element be included at least once.

(ii) In sampling without replacement find the prob. that a fixed element of a population of n elements to be included in a random sample of size r.

Hints:- (i) $P_1 = 1 - P$[the fixed element is not included in the sample WOR]

$$= 1 - \frac{(n-1)^r}{n^r}.$$

(ii) $P_2 = 1 - P$[a fixed element is not included in the sample WR]

$$= 1 - \frac{(n-1)_r}{(n)_r} = 1 - \frac{n-r}{n} = \frac{r}{n}.$$

Q.9. There is 3 volume dictionary among 30 books is arranged in a shelf in random way. Find the prob. ~~that~~ of 3 volume standing in an increasing order from left to right? (The vols. are not necessary side by side).

Sol. The order of the 3 vols. doesn't depend on the arrangement of the remaining books. Here 3 vols. can be arranged in 3! ways of which only one case V_1, V_2, V_3 is favourable. Hence prob. is $\dfrac{1}{3!}$.

Q.10. Two fair dice are thrown 10 times. Find the prob. that the first 3 throws result in a sum of 7 and the last 7 throws in a sum of 8.

Sol. $\Omega_K = \{(i,j) : i,j = 1(1)6\}$, $k = 1(1)10$, be the sample space of the k^{th} throw of a pair of dice, the sample space of the experiment is $\Omega = \Omega_1 \times \Omega_2 \times \Omega_3 \times \cdots \times \Omega_{10}$.

$$N = n(\Omega) = n(\Omega_1 \times \Omega_2 \times \Omega_3 \times \cdots \times \Omega_{10}) = 36^{10}.$$

Let, $A = \{(i,j) : i+j = 7,\ i,j = 1(1)6\}$, the event of getting a sum of 7 in a throw of a pair of dice.

and $B = \{(i,j) : i+j = 8,\ i,j = 1(1)6\}$, the event of getting a sum of 8 in a throw of a pair of dice.

Our event is = A X A X A X B X ... X B

favourable cases are = { (3,4), (2,5), (1,6), (2,6), (3,5), (4,4), ... }

$$N(A) = \{n(A)\}^3 \{n(B)\}^7 = 6^3 \times 5^7.$$

$\therefore$ Required Probability $= \dfrac{6^3 \times 5^7}{36^{10}}$

Q.11. (i) If n men, among whom A and B, stand in a row, What's the prob. that there will be exactly r men between A and B ?

(ii) If they stand in a ring instead of in a row, show that the prob. is independent of 'r'.

[In the circular arrangement, consider only that they are leading from A to B in the +ve direction]

Sol. (i) n persons can be arranged among themselves in $n!$ ways. Since, the persons are randomly, all possible cases are equally likely. For the favourable cases if A occupies a position to the left of B, then A may choose any of the positions : $1^{st}, 2^{nd}, \ldots\ldots (n-r-1)^{th}$ from the left, with r persons between A and B. The remaining $(n-2)$ persons can stand in $(n-2)$ places in $(n-2)!$ ways. Similar thing for B on the left of A. Hence, the no. of favourable cases, $N(A) = 2(n-r-1)(n-2)!$

$$\text{Reqd. prob.} = \frac{2(n-r-1)(n-2)!}{n!} = \frac{2(n-r-1)}{n(n-1)}.$$

(ii) If they form a ring, then the no. of possible arrangement is $(n-1)!$ which is obtained by keeping the place for any person fixed and arranging the remaining $(n-1)$ persons. For the favourable cases, we fixed the places for A and B, with r individuals between them and then remaining $(n-2)$ persons can be arranged in $(n-2)!$ ways.

$$\text{Reqd. prob.} = \frac{(n-2)!}{(n-1)!} = \frac{1}{n-1}, \text{ it is indep. of } r.$$

Rule - III :-

Subpopulations and Groups:- Consider a subpopulation of size 'r' from a given population of size 'n', let the no. of the groups of size r be x.

Now the r elements in a group can be arranged in $r!$ ways.
Hence $x \cdot r!$ ordered samples of size r.

$$\therefore \quad x \cdot r! = (n)_r$$

$$\text{So, } x = \binom{n}{r}$$

Application:-

1. Each of the 50 states has two senators. Find the prob. of the event that in a committee of 50 senators chosen randomly — (a) a given state is represented. (b) all states are represented.

Solution:- We can choose a group of 50 senators in $\binom{100}{50}$ ways & since 50 senators are chosen randomly so all possible outcomes are equally likely.

(a) There are 100 senators and 98 not from the given state

Required probability $= P[\text{the given state is not represented}]^c$

$$= 1 - \frac{\binom{98}{50}}{\binom{100}{50}}$$

(b) All states will be represented if one senators from each state is selected. A committee of 50 with one senetors from 50 states can be selected in $\underbrace{2 \times 2 \times \ldots \times 2}_{50\text{ times}}$ ways. Required prob. $= \dfrac{2^{50}}{\binom{100}{50}}$.

2. If r balls are placed at random in n cells. find the probability that exactly one cell remains empty.

Solution:-

$$N = n^r$$

Since k balls can be chosen in $\binom{r}{k}$ ways which are to be placed in the specified cells and the remaining $(r-k)$ balls can be placed in the remaining $(n-1)$ cells in $(n-1)^{r-k}$ ways.

$$\text{Required prob.} = \frac{\binom{r}{k}(n-1)^{r-k}}{(n)^r}$$

$$= \binom{r}{k}\left(\frac{1}{n}\right)^k\left(1-\frac{1}{n}\right)^{r-k}.$$

3. If n balls are placed at a random order in n cells, find the prob. that exactly one cell remains empty.

Sol. $N = n^n$

For the favourable cases, the empty cell can be chosen in n ways and the two balls to be kept in the same cell can be chosen in $\binom{n}{2}$ ways.

Consider the two balls as a single ball or entity, then $(n-1)$ entities can be arranged in $(n-1)$ cells in $(n-1)!$ ways.

So, required prob. $= \dfrac{n\binom{n}{2}(n-1)!}{n^n}$.

4. A closent contains n pairs of shoes. If $2r$ shoes chosen at random $(2r < n)$. What is the prob. that there will be —
(a) no complete pair
(b) exactly one complete pair
(c) exactly two complete pair among them.

Ans.

(a) Req. prob. $= \dfrac{\binom{n}{2r} 2^{2r}}{\binom{2n}{2r}}$

(b) Req. prob. $= \dfrac{\binom{n}{1}\binom{n-1}{2r-2} 2^{2r-2}}{\binom{2n}{2r}}$

(c) Req. prob. $= \dfrac{\binom{n}{2}\binom{n-2}{2r-4} \cdot 2^{2r-4}}{\binom{2n}{2r}}$.

5. A car is parked among N cars in a row, not at either end. On the return the car owner finds that exactly r of the N places are still occupied. What's the prob. that both neighbouring places are empty?

Ans. Req. prob. $= \dfrac{\binom{N-3}{r-1}}{\binom{N-1}{r-1}}$

Rule - IV The no. of ways in which a popl^n of n elements can be divided into K-ordered parts of which 1st contains n_1, 2nd contains n_2 elements and so on is

$$\frac{n!}{n_1!\, n_2! \cdots\cdots n_K!}, \text{ where } \sum_{i=1}^{K} n_i = n.$$

Application:-

1. In a bridge table, calculate the prob. that
   (a) each of the 4 players has an ace
   (b) one of the player receives all 13 spades.

Sol. (a) In a bridge table 52 cards are partitioned into four equals groups and the no. of different hands is

$$\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}$$

For the favourable cases, 4 aces can be arranged in 4! ways and each arrangement represents one possibility of given one ace to each player and the remaining 48 cards can be distributed equally among the 4 players in

$$\binom{48}{12}\binom{36}{12}\binom{24}{12}\binom{12}{12} \text{ ways}$$

$$\text{Req. prob.} = \frac{4!\binom{48}{12}\binom{36}{12}\binom{24}{12}\binom{12}{12}}{\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}} = \frac{4!\,\dfrac{48!}{(12!)^4}}{\dfrac{52!}{(13!)^4}}.$$

(b) $$\text{Req. prob} = \frac{4 \cdot \dfrac{39!}{(13!)^3}}{\dfrac{52!}{(13!)^4}}.$$

2. In a bridge hand of cards consists of 13 cards drawn at random WOR from a deck of 52 cards. Find the prob. that a hand of cards will contain
   (a) r_1 clubs, r_2 spades, r_3 diamonds
   (b) r aces
   (c) r_1 aces and r_2 kings.

Ans. (a) $$\text{Prob.} = \frac{\binom{13}{r_1}\binom{13}{r_2}\binom{13}{r_3}\binom{13}{13-r_1-r_2-r_3}}{\binom{52}{13}}$$

(b) $$\text{Prob.} = \frac{\binom{4}{r}\binom{48}{13-r}}{\binom{52}{13}}$$

(c) $$\text{Prob.} = \frac{\binom{4}{r_1}\binom{4}{r_2}\binom{44}{4-r_1-r_2}}{\binom{52}{3}}.$$

3. 4 cards are drawn at random from a full deck of 52 cards. What's the prob. that
(i) they are of different denominations?
(ii) they are of different suits?
(iii) Both?

Sol. (i) In a deck of cards there are 13 denominations and 4 suits.
For favourable cases select a group of 4 denominations from 13 and then choose one card from each of the 4 denomination
So, no of favourable cases $= \binom{13}{4}\binom{4}{1}^4$.

$$\text{Req. prob.} = \frac{\binom{13}{4}\binom{4}{1}^4}{\binom{52}{4}}$$

(ii). $$\text{Req. prob} = \frac{\binom{13}{1}^4}{\binom{52}{4}}$$

(iii) For favourable cases, selecting 4 denomination from 13 and then taking one card from the 1^{st} denomination in 4 ways from the 4 suits. Then taking 2^{nd} from the 2^{nd} denomination in 3 ways & so on.

$$\text{Req. prob.} = \frac{\binom{13}{4}\times 4!}{\binom{52}{4}}.$$

4. From a deck of 52 cards are drawn successively until an ace appears. What is the prob. that the 1^{st} ace will appears
(a) at the n^{th} draw,
(b) after the n^{th} draw.

Sol. (a) For the favourable cases, at the n^{th} draw an ace can occur in 4 ways and the first $(n-1)$ cards are to be taken from 48 non-ace cards which can be done in $(48)_{n-1}$ ways.

$$\therefore \text{Req. prob.} = \frac{4\times(48)_{n-1}}{(52)_n}$$

(b) For the favourable cases, 1^{st} n cards contain no ace.

$$\therefore \text{Req. prob} = \frac{(48)_n}{(52)_n}$$

5. (Spread of Rumours) In a town of $(n+1)$ inhabitants, a person tells a rumour to a second person, who in turn repeats it to a third person, etc. At each step the receipt of the rumour is chosen at random from n people available.

(i) Find the prob. that the rumour will be told r times without
(a) returning to the originator (b) being repeated to any person.

(ii) Do the same problem when at each step the rumour is told by one person to a gathering of N randomly chosen individuals.

Sol. (i) Since any person can tell the rumour to any one of the n available persons in n ways. Total possible cases $= n^r$.

(a) The originator can tell the rumour to anyone of the remaining n persons in n ways & each of the $(r-1)$ receipts of the rumour can tell to anyone of the remaining $(n-1)$ persons without returning to the originator in $(n-1)$ ways.

$$\text{Req. prob.} = \frac{n(n-1)^{r-1}}{n^r}.$$

(b) $$\text{Req. prob.} = \frac{(n)_r}{n^r}.$$

(ii)

(a) $$P_a = \frac{\binom{n}{N}\left\{\binom{n-1}{N}\right\}^{r-1}}{\left\{\binom{n}{N}\right\}^r}$$

$$= \left\{\frac{\binom{n-1}{N}}{\binom{n}{N}}\right\}^{r-1} = \left(1-\frac{N}{n}\right)^{r-1}$$

(b) $$P_b = \frac{\binom{n}{N}\binom{n-N}{N}\binom{n-2N}{N}\cdots\cdots\binom{n-\overline{r-1}N}{N}}{\left\{\binom{n}{N}\right\}^r}$$

$$= \frac{\dfrac{n!}{\{N!\}^r\,(n-rN)!}}{\left\{\binom{n}{N}\right\}^r}$$

$$= \frac{(n)_{rN}}{\{(n)_N\}^r}.$$

6. 5 cards are taken at random from a full deck. Find the probability that

(a) they are different denominations?

(b) 2 are of same denominations?

(c) one pair is of one denomination & other pair of a different denomination and one odd?

(d) there are of one denomination & two scattered?

(e) 2 are of one denomination and 3 of another?

(f) 4 are of one denomination and 1 of another?

Sol.

(a) $$P(a) = \frac{\binom{13}{5}\binom{4}{1}^5}{\binom{52}{5}}$$

(b) $$P(b) = \frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3}{\binom{52}{5}}$$

(c) $$P(c) = \frac{\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}}{\binom{52}{5}}$$

(d) $$P(d) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2}{\binom{52}{5}}$$

(e) $$P(e) = \frac{\binom{13}{2}\binom{4}{2}\binom{13}{3}\binom{4}{3}}{\binom{52}{5}}$$

(f) $$P(f) = \frac{\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{1}}{\binom{52}{5}}$$

Rule - V

- <u>Occupancy Problem</u>:- In many situations it is necessary to treat the balls indistinguishable. e.g., in statistical studies of the distribution of accidents among weekdays, here one is interested only in the number of occurances and not in the individuals involved.

Such an example is completely described by its occupancy numbers $r_1, r_2, \ldots, r_n$; where, r_k denotes the number of balls in the k^{th} cell.

~~Here we are interested in number of r identical balls in n cells.~~

Here we are interested in number of possible distribution, i.e., the number of different n-tuples $(r_1, r_2, \ldots, r_n)$ such that $r_1 + r_2 + \cdots + r_n = r \; (r_i \geq 0)$.

- <u>Theorem 1</u>:- The number of different distributions of 'r' indistinguishable balls in n cells, i.e., the number of different solution of the above fact is $\binom{n+r-1}{n-1}$.

- <u>Theorem 2</u>:- The number of different distribution of 'r' indistinguishable balls in the n cells in which no cell remains empty is $\binom{r-1}{n-1}$.

<u>Ex.</u> r indistinguishable balls are distributed into n cells and all possible distributions are equally likely. Find the prob. that exactly m cells remain empty.

<u>Sol.</u> The m cells which are to be kept empty can be chosen from n cells in $\binom{n}{m}$ ways and r indistinguishable balls can be distributed in the remaining $(n-m)$ cells so that no cell remain empty is in $\binom{r-1}{n-m-1}$ ways.

No. of favourable cases $= \binom{n}{m}\binom{r-1}{n-m-1}$

$$\therefore \text{ Required prob.} = \frac{\binom{n}{m}\binom{r-1}{n-m-1}}{\binom{n+r-1}{r}}.$$

- **Application:-**

1. Show that r indistinguishable balls can be distributed in n cells i.e., the no. of different solution $(r_1, r_2, \ldots, r_n)$ such that $\{r_1+r_2+\cdots+r_n=r\}$ is $\binom{n+r-1}{r}$, where $r_i \geq 0$.

Solution:- Denoting the choices of r_1, i.e., $0, 1, \ldots, r$ in the indices, we get the factor $(x^0+x^1+\cdots+x^r)$.
The no. of different solution $(r_1, r_2, \ldots, r_n)$ of $\sum_{i=1}^{n} r_i = r$ where $r_i \geq 0$ is

$$= \text{The coefficient of } x^r \text{ in } \underbrace{(x^0+x^1+\cdots+x^r)\cdots\cdots(x^0+x^1+\cdots+x^r)}_{n\text{ times}}$$

$$= \text{The coefficient of } x^r \text{ in } \left(\frac{1-x^{r+1}}{1-x}\right)^n$$

$$= \text{The coefficient of } x^r \text{ in the expression } (1-x^{r+1})^n(1-x)^{-n}$$

$$= \text{The coefficient of } x^r \text{ in}$$

$$\left\{1-nx^{r+1}+\binom{n}{2}x^{2r+2}+\cdots\right\}\left\{1+nx+\binom{n}{2}x^2+\cdots+\binom{n}{r}x^r+\cdots\right\}$$

$$= \binom{n+r-1}{r}.$$

2. Show that the no. of different distributions of r indistinguishable balls in n cells where no cell remains empty is $\binom{n-1}{r-1}$:

Hints:-

The co-efficient of x^r in $(x+x^2+\cdots+x^r)^n$

$$= \text{''} \quad \text{''} \quad \text{''} \; x^r \text{ ''} \; x^n\left(\frac{1-x^r}{1-x}\right)^n$$

$$= \text{''} \quad \text{''} \quad \text{''} \; x^{r-n} \text{ ''} \; (1-x^r)^n(1-x)^{-n}.$$

$$= \binom{n+r-n-1}{r-n} = \binom{r-1}{n-1}.$$

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ALGEBRA OF SETS

Class of sets (events):- A collection of some sets closed under one or more set operations.

Closed under Complementation:- A class C of sets is said to be closed under the operation of complementation if for every $A \in C$ implying $A^c \in C$.

Closed under finite (countable) unions (intersections):- A class C of sets is said to be closed under the operation of finite unions (intersections) if for a finite (countable) no. of sets $A_1, A_2, \ldots, A_K \in C \Rightarrow \bigcup_{i=1}^{K} A_i \in C$ $\left(\bigcap_{i=1}^{K} A_i \in C\right)$.

Result:- If a class C of sets is closed under complementation and finite (countable) unions (intersections), then it is necessarily closed under finite (countable) intersections (unions) i.e.,

(i) $A \in C \Rightarrow A^c \in C$

(ii) $A_1, \ldots, A_K \in C \Rightarrow \bigcup_{i=1}^{K} A_i \in C \Rightarrow \bigcap_{i=1}^{K} A_i \in C$

Proof:- Let $A_1, \ldots, A_K \in C$

$\Rightarrow A_1^c, \ldots, A_K^c \in C$

$\Rightarrow \bigcup_{i=1}^{K} A_i^c \in C$

$\Rightarrow \left(\bigcap_{i=1}^{K} A_i\right)^c \in C$ [By D'Morgans Law]

$\Rightarrow \left[\left(\bigcap_{i=1}^{K} A_i\right)^c\right]^c \in C$

$\Rightarrow \bigcap_{i=1}^{K} A_i \in C$.

Field of Sets:- A non-empty class C of sets is said to be a field of sets if (i) C is closed under complementation
(ii) C is closed under finite unions (intersections)

Example:- $C = \{\phi, \Omega\}$ is a trivial field.
$C = \{\phi, A, A^c, \Omega\}$ is a field.

Sigma field of Sets:- A non-empty class C of sets is called to be a sigma field of sets, if
(i) C is closed under complementation.
(ii) C is closed under countable unions (or, intersections).

So, C is a sigmafield (σ-field) if (i) $A \in C \Rightarrow A^c \in C$
(ii) $A_i \in C \Rightarrow \bigcup_i A_i \in C$.

σ-field generated by a class of sets ($\mathcal{A}$):- Let $\mathcal{A}$ be any arbitrary class of sets, then it is possible to extend $\mathcal{A}$ to a σ-field containing $\mathcal{A}$, written as $\sigma(\mathcal{A})$. The σ-field $\sigma(\mathcal{A})$ thus generated is called a σ-field generated by $\mathcal{A}$ (or, σ-field containing $\mathcal{A}$).

Minimal σ-field generated by a class of sets:- Let $\sigma_1(\mathcal{A}), \sigma_2(\mathcal{A}), \ldots$ be σ-fields containing given class of sets $\mathcal{A}$. Then the minimal σ-field generated by $\mathcal{A}$ will be given by $\sigma_0(\mathcal{A}) = \bigcap_{i=1}^{\infty} \sigma_i(\mathcal{A})$.

Symmetric difference:- $A \Delta B = (A \cap B^c) \cup (A^c \cap B)$

Sequence of Sets (events):- A sequence $\{A_n\}$ of sets (events) is said to be monotone increasing or expending if

$$A_n \subseteq A_{n+1} \quad \forall \; n = 1, 2, \ldots$$

and monotone decreasing or contracting if

$$A_{n+1} \subseteq A_n \quad \forall \; n = 1, 2, \ldots$$

A monotone increasing (decreasing) sequence of sets (events) is said to have the set A as the limit of $\{A_n\}$, where

$$A = \lim_{n \to \infty} A_n = \begin{cases} \bigcup_{n=1}^{\infty} A_n & \text{if } \{A_n\} \text{ is increasing.} \\ \bigcap_{n=1}^{\infty} A_n & \text{if } \{A_n\} \text{ is decreasing.} \end{cases}$$

Example:-

(i) $A_n = \left(-\infty, x + \frac{1}{n}\right)$ where $x \in R^1$ and $n = 1, 2, \ldots$

Here $\{A_n\}$ is a contracting sequence of intervals (sets).

$$\lim_{n \to \infty} A_n = \lim_{n \to \infty} \left(-\infty, x + \frac{1}{n}\right)$$

$$= \bigcap_{n=1}^{\infty} (-\infty, x + 0)$$

$$= (-\infty, x]$$

(ii) $B_n = \left(-\infty, x - \frac{1}{n}\right]$ where $x \in R^1$ and $n = 1, 2, \ldots$

Clearly, $\{B_n\}$ is an expending sequence of intervals (sets).

$$\lim_{n \to \infty} B_n = \lim_{n \to \infty} \left(-\infty, x - \frac{1}{n}\right)$$

$$= \bigcup_{n=1}^{\infty} \left(-\infty, x - \frac{1}{n}\right)$$

$$= (-\infty, x - 1].$$

Probability Measure:- A probability measure P on a σ-algebra $\mathcal{A}$ of subsets of Ω is a function $P: \mathcal{A} \to [0, 1]$ such that

(i) $P(\Omega) = 1$

(ii) $P(A) \geq 0 \quad \forall \; A \in \mathcal{A}$

(iii) If $\{A_n\}_{n=1}^{\infty}$ is a sequence of mutually disjoint set in $\mathcal{A}$, then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

Limit of an arbitrary sequence of sets :-

Let $\{A_n\}$ be any sequence of sets (not necessarily monotone increasing or decreasing) then for such an arbitrary sequence $\{A_n\}$ of sets, let us define,

$$B_n = \inf_{K \geq n} A_K = \inf\{A_n, A_{n+1}, \ldots\}$$

$$= \bigcap_{K=n}^{\infty} A_K$$

$$= \{\omega \mid \omega \in \text{all } A_K\text{'s except perhaps } A_1, A_2, \ldots, A_{n-1}\}$$

Clearly, $B_{n+1} = \bigcap_{K=n+1}^{\infty} A_K$

Therefore, $B_{n+1} \supseteq B_n \;\forall\; n = 1, 2, \ldots$

i.e. $\{B_n\}$ is an expending sequence of sets.

$$\therefore \lim_{n\to\infty} B_n = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} \bigcap_{K=n}^{\infty} A_K = \lim\inf A_n, \text{ say.}$$

(or, $\underline{\lim}\, A_n$)

Let us define, $C_n = \sup_{K \geq n} A_K$ (i.e., largest set that can be formed by $A_n, A_{n+1}, \ldots$)

$$= \bigcup_{K=n}^{\infty} A_K$$

$$= \{\omega \mid \omega \in \text{at least one of the sets } A_n, A_{n+1}, \ldots\}$$

Obviously, $C_{n+1} \subseteq C_n \;\forall\; n = 1, 2, \ldots$

i.e. $\{C_n\}$ is a contracting sequence of sets.

$$\therefore \lim_{n\to\infty} C_n = \bigcap_{n=1}^{\infty} C_n = \bigcap_{n=1}^{\infty} \bigcup_{K=n}^{\infty} A_K = \lim \sup A_n, \text{ say}$$

(or, $\overline{\lim}\, A_n$)

In general, $\underline{\lim}\, A_n \subseteq \overline{\lim}\, A_n$ and in particular if $\underline{\lim}\, A_n = \overline{\lim}\, A_n = A$, say, then the limit of the (arbitrary) sequence $\{A_n\}$ is said to ~~be~~ exist and we have $\lim_{n\to\infty} A_n = A$.

e.g. $A_n = \{(x, y) \mid 0 \leq x \leq n, 0 \leq y \leq \frac{1}{n} : x, y \in \mathbb{R}\}$.

Monotone class :- A class $\mathcal{C}$ of sets is said to be a monotone class if it is closed under monotone ~~class~~. operations :- i.e. for a monotone sequence $\{A_n\}$ of sets $\in \mathcal{C}$.

$$\Rightarrow \lim_{n\to\infty} A_n \in \mathcal{C}.$$

Remark :- σ-field is a monotone class since σ-field is closed under countable unions or intersections.

Borel σ-field/ Borel field:- The σ-field generated by a class $\mathcal{C}$ of sets (intervals) of the real line R^1, i.e., the minimal σ-field containing a class $\mathcal{C}$ of sets (intervals) of the real line R^1 is called a Borel field of sets of R^1 and is denoted by $\mathcal{B}$ (one-dimensional Borel field).

$\Omega = \{1, 2, 3, 4, 5\}$

$\mathcal{A} = \{\phi, \{1\}, \{2\}, \ldots \{5\}, \{1,2\}, \ldots, \{5,6\}, \ldots, \Omega\}$

Result:- Given a class $\{A_i, i = 1, 2, 3, \ldots, n\}$ of n sets, $\exists$ a class $\{B_i, i = 1, 2, \ldots, n\}$ of disjoint sets such that

$$\bigcup_{i=1}^{n} A_i = \sum_{i=1}^{n} B_i .$$

Proof:- This will be proved by induction. Evidently,

$$A_1 \cup A_2 = A_1 + A_1^c A_2 = B_1 + B_2, \text{ say}, \quad \text{(1)}$$

where B_1 and B_2 are disjoint.

This result is true for $n = 2$, suppose it is true for all $n \leq m \geq 2$. Then

$$\bigcup_{i=1}^{m+1} A_i = \left(\bigcup_{i=1}^{m} A_i\right) \cup A_{m+1},$$

$$= \left(\sum_{i=1}^{m} B_i\right) \cup A_{m+1}$$

$$= \sum_{i=1}^{m} B_i + \left(\sum_{i=1}^{m} B_i\right)^c A_{m+1}$$

$$= \sum_{i=1}^{m} B_i + B_{m+1}, \text{ say},$$

where, B_{m+1} and $\sum^{m} B_i$ are disjoint and hence B_{m+1} and B_i are distinct for $i = 1, 2, \ldots, m$. Hence, the result holds for $n = m+1$ and by induction it is proved.

Properties of Probability function:-

(i) $P(\phi) = 0$.

Proof:- We know that $\phi \in \mathcal{A}$, the σ-field of events of $\mathcal{E}$. Let us consider a sequence of events $\{A_i\}$ from $\mathcal{A} \ni$

$$A_i = \phi \text{ for all } i = 1, 2, \ldots$$

Then $A_1, A_2, \ldots$ is a sequence of disjoint event from $\mathcal{A}$. By the axiom of countable additivity of the probability function $P(\cdot)$, we have $P\left(\sum_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

$$\text{or, } P(\phi) = P(\phi) + P(\phi) + \cdots$$

$$\Rightarrow P(\phi) = 0$$

(ii) $P(\cdot)$ is monotone, i.e., $A \subseteq B$, then $P(A) \leq P(B)$.

Proof:- $B = A + A^c \cap B$

$P(B) = P(A) + P(A^c \cap B)$

$\therefore P(B) \geq P(A)$. since $P(A^c \cap B) \geq 0$.

(iii) $P(\cdot)$ is subtractive, i.e., $A \subseteq B$, then $P(B-A) = P(B) - P(A)$

Proof:- Since $A \subseteq B$,

$\therefore B = A + B - A$

$\therefore P(B) = P(A) + P(B-A)$

$\therefore P(B-A) = P(B) - P(A)$.

(iv) $P(\cdot)$ is finitely additive, i.e., $P\left(\sum_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$ for disjoint events $A_1, \ldots, A_n$ from $\mathcal{Q}$.

Proof:- Let us define $A_i = \phi \ \forall \ i = n+1, n+2, \ldots$

Thus $A_1, A_2, \ldots, A_n, A_{n+1}, \ldots$ is a sequence of disjoint events from $\mathcal{Q}$.

$\therefore$ By the axiom of countable additivity, we have

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \text{ or } P\left(\sum_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

But $\sum_{i=1}^{\infty} A_i = \sum_{i=1}^{n} A_i + \sum_{i=n+1}^{\infty} A_i = \sum_{i=1}^{n} A_i$

Hence the proof is done.

Cor: 1. If $A_1, A_2, \ldots$ are mutually exclusive and exhaustive events of Ω. Show that $\sum P(A_i) = 1$.

Proof:- Since $A_1, A_2, \ldots$ are mutually exclusive and exhaustive,

$\sum A_i = \Omega$.

$\Rightarrow P(\sum A_i) = P(\Omega) = 1$ [By the axiom of unit norm of $P(\cdot)$]

Furth $P(\sum A_i) = \sum P(A_i)$ [By the principle of finite additivity of $P(\cdot)$]

Hence, $\sum P(A_i) = 1$.

Cor. 2. $P(A^c) = 1 - P(A)$

Sol. Since A and A^c are mutually exclusive and exhaustive

so, $P(A^c) + P(A) = 1$.

so, $P(A^c) = 1 - P(A)$.

σ- ALGEBRA

1) Explain the concept of σ-algebra of events.

Sol^n →

Class of sets (events) means a collection of some sets closed under one or more set operations.

σ field of events : → A non-empty class C of sets is said to be a σ-field of sets if —

i) C is closed under complementation.

ii) C is closed under countable unions (or intersections)

i.e. C is a σ- field if —

i) $A \in C \Rightarrow A^c \in C$.

ii) $A_1, A_2, \ldots \in C \Rightarrow \bigcup_i A_i \in C$.

A σ- field is also closed under the formation of finite unions.

By definition, $\bigcup_{i=1}^{\infty} A_i \in C$.

Now, consider the events ; $A_1, A_2, \ldots, A_n \in C$.

$A_{n+1} = A_{n+2} = A_{n+3} = \ldots\ldots = A_n \in C$.

Now, $$\bigcup_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{n} A_i\right) \cup \left(\bigcup_{i=n+1}^{\infty} A_i\right)$$

$$= \left(\bigcup_{i=1}^{n} A_i\right) \cup A_n = \bigcup_{i=1}^{n} A_i$$

$\therefore \bigcup_{i=1}^{n} A_i \in C$.

since a σ-field is also a field and it has all the properties of a and it contains ϕ and Ω.

However, it is also noted that a field containing infinite number of sets may not be a σ-field.

Example :—

(i) $\{\phi, A, A^c, \Omega\}$ is the smallest σ-field containing $A \subset \Omega$.

(ii) $\{\phi, A, A^c, B, B^c, A \cup B, A \cap B, A^c \cup B^c, A^c \cap B^c, A \cup B^c, B \cup A^c, A \cap B^c, B \cap A^c, (A \cap B^c) \cap (A^c \cup B), (A \cap B^c) \cup (B \cup A^c), \Omega\}$ is a σ-field.

2> Discuss briefly its importance in probability theory.

Ans: In probability theory, mainly in axiomatic approach, probability is a set function. Ω be the sample space $\mathcal{Q}$ is a σ-field of events of Ω. Whenever we told about probability of any event, we have to choose those events from $\mathcal{Q}$. Probability axioms and all the theorem of probability which generally deals with class or sequence of events one must choose those class of sets from σ-field. σ-field is a monotone class. Sigma field is a field too. The σ-field generated by a class of sets of the Real line $\mathbb{R}'$. i.e. minimal sigma field of the Real line $\mathbb{R}'$ is called the Borel σ-field. For defining probabilistic variable we consider a probability space $(\Omega, \mathcal{Q}, P)$ and random variable is defined as a Borel measurable function with respect to a sigma field. It's the importance of σ-field in probability theory.

3> Give an example of a set of events which is not a σ-algebra.

Sol^n →

(a) A σ-field is non-empty and closed under the formation of countable unions and complementations.

So, $\{\phi, A, \Omega\}$ and $\{\phi, A, B, A^c, B^c, \Omega\}$ are not σ-field.

(b) Let Ω be a set. Call a subset A of Ω cofinite if A^c is finite then the set of all subsets of Ω which are either finite or cofinite is a field on Ω. If Ω is infinite, then this field is not a σ-field.

4> Define monotone class with example.

Sol^n → A class C of sets is said to be a monotone class of it is closed under monotone operation:

i.e. for a monotone sequence $\{A_n\}$ of sets $\in C$.

$$\Rightarrow \lim_{n\to\infty} A_n \in C.$$

Ex. σ-field is a monotone class (since σ-field is closed under countable unions or intersections).

CLASSICAL DEFINITION

1) A coin is tossed until a head appears. Write down its sample space.

Sol^n → Here the sample space, $\Omega = \{H, TH, TTH, TTTH, \dots\}$

2) What do you mean by 'event' and 'elementary event'? Describe with example.

Sol^n →

Event :- An event is a subset of sample space. Clearly, $A \subseteq \Omega$ is an event and A contains some or no sample points.

Elementary Events :- An event which can't be further decomposed into smaller events is called an elementary event.

Example :- considering a random experiment of 'tossing a coin twice.'

$$\Omega = \{HH, HT, TH, TT\}.$$

Then event i) 'At least one head' is $A = \{HH, HT, TH\} \subseteq \Omega$

Here $A \subseteq \Omega$. Here A is a composite event.

The event ii) 'No heads' is $B = \{TT\}$.

Here B is an elementary event.

3) Write down the classical definition of probability and its limitations.

Sol^n

Classical Def^n : If a random experiment can result in N (finite) mutually exclusive, exhaustive and equally likely cases and N(A) of them are favourable to the occurance of the event A, then the probability of occurance of A is —

$$P(A) = \frac{N(A)}{N}.$$

Limitations :

i) It is assumed here that all the cases are equally likely. The definition of Probability is found useful when applied to the outcomes of the game of chance. If the outcomes of a random experiment are not equally likely then this definition is not applicable

ii) This definition breaks down if the no. of all possible outcomes is infinite.

iii) It assumes equal probability of occurance of all elementary events. But in real life, it is not easy to identify the outcomes as equally likely.

4) A number is selected from a set of N natural numbers. What is the probability that it's a multiplier of 3.

Sol^n

Here the sample space contain N points.
Let A be the event that the outcome is a multiplier of 3.

Then, $A = \{3m : 1 \leq m \leq [N/3]\}$, where $[N/3]$ is the greatest integer less than or equal to N/3 (to compute $[N/3]$, just divide N by 3 and round down). So $N(A) = [N/3]$. Therefore, the probability that a random natural number between 1 to N is a multiplier of 3 is equals to $\frac{[N/3]}{N}$

$$\therefore P(A) = \frac{[N/3]}{N}$$

FREQUENCY DEFINITION

1) Describe the intuitive idea of probability in terms of relative frequency. Give the statistical definition or relative frequency definition of Probability. Discuss the convergence of "Irregular sequence" of relative frequencies. Also discuss the limitations of this definition.

Sol^n →

Statistical or Empirical approach of probability :–

Suppose A is an event of a random experiment.

Suppose it is possible to repeat the experiment a large number of times under essentially similar conditions. Denote by $f_n(A)$, the number of occurance of A in 'n' repetations, $f_n(A)$ is called the frequency of A and $\frac{f_n(A)}{n}$ is called the relative frequency of A. A kind of regularity is observed when a large number of repetitions is considered. It is an observed fact that the relative frequencies stabilize to a certain value as 'n' becomes large. This tendency seems to be inherent in the nature of a random experiment, and stability of relative frequencies for the large value of n constitutes the basis of statistical definition of probability. This kind of regularity in a random experiment is known as '<u>statistical regularity</u>'. The limiting value of $\frac{f_n(A)}{n}$ as $n \to \infty$ is called the probability of A, provided that the limit exists.

Relative frequency Definition of Probability :→

<u>Definition By Richard Von Mises</u> : Suppose a random experiment can be repeated indefinitely under identical condition. Then the probability of the event A, denoted by P(A) will be given by

$$P(A) = \lim_{n \to \infty} \frac{f_n(A)}{n} .$$

where,
n = No. of repetitions of the experiment &
$f_n(A)$ = frequency of the occurance of the event A in the first n repetations of the experiment.

<u>Irregular collectives</u> : The above definition of probability has to do with infinite sequences of observations which is termed as "irregular collectives". Each collectives satisfy two conditions:

(a) <u>Existance of limit</u> :- The limit of relative frequency $\left\{\frac{f_n(A)}{n}\right\}$ of events with particulars attributes within the collectives exists.

In fact, the limit of such sequences always exists as necessity. In other words, it is a regular phenomenon.

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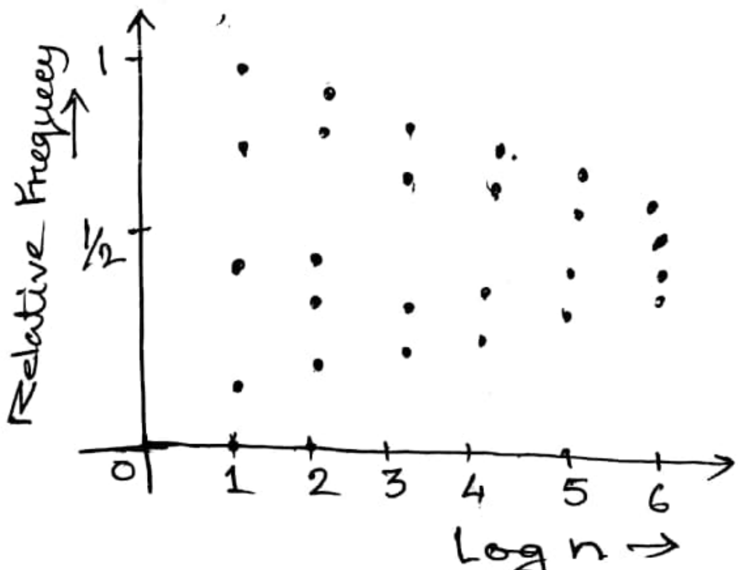
This the limit of such sequences exists and is unique. This irregular sequences of relative frequencies converges to 'statistical regularity'.

(b) The Principle of randomness :- The limits are invariant with respect to the choice of any subsequence of the collectives, which is independent of the attribute under consideration.

Example : Consider a set of n repetations of the experiment of tossing for each of $n = 1, 10, 100, 1000, \ldots\ldots$. If we plot the values of the relative frequency of 'head', This particular feature of relative frequency is called 'statistical regularity'. as the number of tossing increases the relative frequenciys tends to stabilize at 0.5.



Limitations :

i) If an experiment is repeated a number of times, the experimental conditions may not remain identical or homogeneous.

ii) The $\lim\limits_{n \to \infty} \frac{f_n(A)}{n}$ may not be unique.

Axiomatic Approach

1) Explain the concept of Kolmogorov's Axiomatic def^n of probability. Using this show that —

i) $P(\phi) = 0$, when ϕ is null set.

ii) $P(A) \leq 1$, for any event A.

Sol^n

Axiomatic Definition :— Let Ω be the sample space of a random experiment and $\mathcal{A}$ be a σ-field of events of Ω. A set function $P(\cdot)$ defined on $\mathcal{A}$ is called a probability measure if it satisfies the following conditions;

Axiom I (Axiom of non-negativity): $P(A) \geq 0 \; \forall A \in \mathcal{A}$.

Axiom II (Axiom of unit-norm): $P(\Omega) = 1$.

Axiom III (Axiom of countable additivity): If $A_i, i = 1(1)\infty$ be a disjoint sequence of events in $\mathcal{A}$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

In axiomatic approach probability is regarded as a set function

i) Let $A_1, A_2, \ldots$ be events in $\mathcal{A} \ni A_i = \phi, \forall i$. Then $\bigcup_{i=1}^{\infty} A_i = \phi$ and since $A_i \cap A_j = \phi \cap \phi = \phi, \forall i \neq j$.

Then A_i's are also mutually exclusive (i.e disjoint)

$\therefore$ By the axiom of countable additivity, we have

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

$$\text{or, } P(\phi) = P(\phi) + P(\phi) + P(\phi) + \cdots$$

But this can happen if either $P(\phi) = 0$ or, $P(\phi) = \infty$ or $-\infty$.
But since P is a finite real valued function, so $P(\phi) = \infty$ or $-\infty$ is not possible.

So, $P(\phi) = 0$. (Proved)

ii) As $A \subset \Omega$ for each $A \in \mathcal{A}$.

$\Rightarrow P(A) \leq P(\Omega)$.

Now, from the axiom of unit norm, we know $P(\Omega) = 1$.

So, we get $\rightarrow P(A) \leq 1$ for any event A.

(OR)

$A \cup A^c = \Omega$, $A \cap A^c = \phi$ $\cdot A \subset A^c$

by finite additivity of $P[\cdot]$,

$$P[A \cup A^c] = P[A] + P[A^c] - P[A \cap A^c]$$

$$\therefore P(\Omega) = P(A) + P(A^c)$$

$$0 \le P(A^c) = 1 - P(A), \text{ by Axiom I}.$$

$$\therefore P[A] \le 1.$$

2) (a) Let $A_1, \ldots, A_n$ be n events $\ni P(A_i) = 1$ $\forall i = 1(1)n$.

(b) Let $A_1, A_2, \ldots$ be the events $\ni P(A_i) = 0$, $\forall i = 1, 2, \ldots$ then show that $P\left(\bigcup_{i=1}^{\infty} A_i\right) = 0$.

(c) If the events A_i's are mutually exclusive and exhaustive events of Ω, $i = 1, 2, \ldots$

S.T. $\sum_i P(A_i) = 1$.

Sol^n →

(a) If A_i, $i = 1(1)n$ be events in $\mathcal{A}$, then Bonferroni inequality gives—

$$P\left(\bigcap_{i=1}^{n} A_i\right) \ge \sum_{i=1}^{n} P(A_i) - n + 1 \quad \text{—— (i)}$$

From the axiom of unit norm, $P(\Omega) = 1$.

As $A \subset \Omega$, $\forall A \in \mathcal{A}$.

$\therefore P(A) \le P(\Omega) = 1$ $\therefore P(A) \le 1$ —— (ii)

Here, $P(A_i) = 1$ $\forall i = 1(1)n$ —— (iii)

So, From (i), (ii), (iii) we get $P\left(\bigcap_{i=1}^{n} A_i\right) = 1$.

(b) If $P(A_i) = 0$, we know from Boole's inequality

$$P\left(\bigcup_{i=1} A_i\right) \le \sum_{i=1}^{n} P(A_i) \text{ and } P(A) \ge 0.$$

So, if $P(A_i) = 0$ $\forall i \ge 1$, we get, $P\left(\bigcup_{i=1}^{n} A_i\right) = 0$.

Hence the result is proved.

(c) Since, A_i's are exhaustive events, then $\bigcup_i A_i = \Omega$.

$\therefore P\left(\bigcup_i A_i\right) = P(\Omega) = 1$.

Again A_i's are mutually exclusive, $P\left(\bigcup_i A_i\right) = P\left(\sum_i (A_i)\right)$

$\therefore P\left(\sum_i A_i\right) = 1$ i.e. $\sum_i P(A_i) = 1$. [By the principle of countable additivity of $P(\cdot)$]

INDEPENDENCE OF EVENTS

1) i) Define mutually exclusive, exhaustive and mutually independent events. Let the two events be mutually exclusive are they mutually independent.

ii) Show by an example that pairwise independence does not necessarily imply mutual independence.

iii) Distinguish between pairwise and mutual independence of a finite set of events.

iv) Show that if A_1, A_2, A_3 are mutually independent then A_1^c, A_2^c, A_3^c are also mutually where A_i^c is the complement of A.

Solⁿ → i) Mutually exclusive events : Several events $A_1, A_2, \ldots, A_n$ in relation to a random experiment are said to be mutually exclusive (or, disjoint) if any two of them can't occur simultaneously. Everytime the experiment is performed is $A_i \cap A_j = \phi \ \forall \ i \neq j, \ i<j = 1(1)n$.

Exhaustive Events : several events in relation to a random experiment are said to be exhaustive events if at least one of them necessarily occurs. Thus the events $A_1, A_2, \ldots, A_n$ or $A_1, A_2, \ldots$ are exhaustive if

$$\bigcup_{i=1}^{\infty} A_i = \Omega.$$

Pairwise Independence of a set of events :

A set of events $\{A_1, A_2, \ldots, A_n\}$ is said to be pairwise independent if

$$P(A_i \cap A_j) = P(A_i)P(A_j), \quad i \neq j, \ i<j.$$

Here we have $\binom{n}{2}$ restrictions.

Mutually independence of a set of events :

A set of events $\{A_1, A_2, \ldots, A_n\}$ is said to be mutually independent if

$$P(A_i \cap A_j) = P(A_i)P(A_j), \quad i \neq j$$

$$P(A_i \cap A_j \cap A_k) = P(A_i)P(A_j)P(A_k), \quad i<j<k.$$

$$\vdots$$

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2)\cdots P(A_n).$$

i.e. $P\left[\bigcap_{i=1}^{n} A_i\right] = \prod_{i=1}^{n} P[A_i]$

The idea of mutual independent emerges from the following fact.

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1)\, P(A_2|A_1)\, P(A_3|A_1 \cap A_2) \dots P\left(A_n \,\middle|\, \bigcap_{i=1}^{n-1} A_i\right).$$

Under statistical independence if all the conditional probabilities become equal to the respective unconditional probabilities, then we get—

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1)\, P(A_2) \dots P(A_n)$$

Here we have $(2^n - 1 - n)$ restrictions.

If two events are mutually exclusive then they will not be mutually independent.

A fair coin is tossed twice, $\Omega = \{HH, TH, HT, TT\}$

A: Two head appears $= \{HH\}$

B: One head & one tail appear $= \{HT, TH\}$

("exactly one head appears")

This two events are mutually exclusive.

$A \cap B = \{\phi\}$

$\therefore P(A \cap B) = 0.$

$P(A) = \frac{1}{4}$, $P(B) = \frac{2}{4} = \frac{1}{2}$.

$\therefore P(A \cap B) \neq P(A)\, P(B) = \frac{1}{8}$.

They are not mutually independent

<u>Note</u>: → Mutually exclusive events in general are not independent and also independent events are not in general mutually exclusive.

<u>ii) & iii)</u>

<u>Distinction between Pairwise Independence and Mutually Independence:</u>

$A_1, A_2, \dots, A_n$ are pairwise independent if $P(A_i \cap A_j) = P(A_i)\, P(A_j)$ $\forall\, i, j\ (i \leq j)$, but for mutually independence it is necessary that all of the $(2^n - n - 1)$ equation hold as mentioned earlier. It is evident that mutually independence implies pairwise independence but the converse may not be true. <u>An example to show that pairwise independence does not imply mutually independence.</u>

Suppose a fair coin is tossed twice

Let A: the first toss gives a head.
B: the second toss gives a head.
C: both give the same outcome.

$\Omega = \{HH, HT, TH, TT\}$

$A = \{HH, HT\}$ $\quad B \cap C = \{HH\}$ $\quad P(A) = P(B) = P(C) = \frac{1}{2}$

$B = \{HH, TH\}$ $\quad A \cap C = \{HH\}$

$C = \{HH, TT\}$ $\quad A \cap B \cap C = \{HH\}$

$A \cap B = \{HH\}$

$\therefore P(A \cap B) = \frac{1}{4} = P(A)P(B)$; $P(B \cap C) = \frac{1}{4} = P(B)P(C)$;

$P(A \cap C) = \frac{1}{4} = P(A)P(C)$.

$\therefore$ A, B, C are pairwise independent.

$\therefore P(A \cap B \cap C) = \frac{1}{4} \neq P(A)P(B)P(C)$.

$\therefore$ A, B, C are not mutually independent.

iv) We know that mutually independence does necessarily imply pairwise independence.

So, A_1, A_2, A_3 are both mutually and pairwise independent.

$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$; $P(A_1 \cap A_2) = P(A_1)P(A_2)$.

$P(A_2 \cap A_3) = P(A_2)P(A_3)$; $P(A_1 \cap A_3) = P(A_1)P(A_3)$.

$$
\begin{aligned}
P(A_1^c \cap A_2^c \cap A_3^c) &= 1 - P(A_1 \cup A_2 \cup A_3) \\
&= 1 - [P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) \\
&\quad - P(A \cap C) + P(A \cap B \cap C)] \\
&= \{1 - P(A)\} - P(B)(1 - P(A)) - P(C)\{1 - P(A)\} \\
&\quad + P(B)P(C)\{1 - P(A)\} \\
&= \{1 - P(A)\}\{1 - P(B) - P(C) + P(B)P(C)\} \\
&= \{1 - P(A)\}\{(1 - P(B)) - P(C)\{1 - P(B)\}\} \\
&= \{1 - P(A_1)\}\{1 - P(A_2)\}\{1 - P(A_3)\} \\
&= P(A_1^c)P(A_2^c)P(A_3^c)
\end{aligned}
$$

$\therefore A_1^c, A_2^c, A_3^c$ are mutually independent if A_1, A_2, A_3 are mutually independent.

2) In a sample space of 8 equally likely points find the following:

i) Three events that are pairwise independent but not mutually independent.

ii) Three events that are mutually independent.

Sol^n → i) Consider a random experiment of a ~~certain~~ coin is thrown thrice.

Sample space is, $\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

and $P[\omega] = \frac{1}{8}$; $\omega \in \Omega$.

Define, A_1 : At least two heads $= \{HHH, HHT, HTH, THH\}$

$A_2 : \{HHH, HHT, HTT, THT\}$

$A_3 : \{HHH, HHT, TTH, TTT\}$.

$P(A_i) = \frac{1}{2} \quad \forall\, i = 1, 2, 3$.

Now, $P(A_1 \cap A_2) = P(\{HHH, HHT\}) = \frac{2}{8} = \frac{1}{4}$

$= P(A_1)P(A_2)$

$P(A_2 \cap A_3) = P(\{HHH, HHT\}) = \frac{1}{4}$

$= P(A_2)P(A_3)$

$\therefore A_1, A_2, A_3$ are pairwise independent.

Now, $P[A_1 \cap A_2 \cap A_3] = P[\{HHH, HHT\}]$

$= \frac{1}{4} \neq \frac{1}{8} = P(A_1)P(A_2)P(A_3)$.

ii) $A : \{HHH, THT, HTH, HHT\}$

$B : \{HHH, THT, HTT, THH\}$

$C : \{HHH, HTT, HTH, TTT\}$

$P(A) = P(B) = P(C) = \frac{1}{2}$

$P(A \cap B) = \frac{1}{4} = P(A)P(B)$; $P(B \cap C) = \frac{1}{4} = P(B)P(C)$

$P(A \cap C) = \frac{1}{4} = P(A)P(C)$.

$P(A \cap B \cap C) = \frac{1}{8} = P(A)P(B)P(C)$.

$\therefore A, B, C$ are pairwise independent as well as mutual independent.

IMPORTANT THEOREMS

1) Define conditional probability. Show that it satisfy all the axioms of probability.

Solⁿ

• Conditional Probability :-

Classical Defⁿ : Conditional probability of the occurance of the event B given that A has already been occured, denoted by $P(B|A)$, is defined as,

$$P(B|A) = \frac{N(A\cap B)}{N(A)}, \text{ provided } N(A)>0.$$

where, $N(A)$ is the no. of cases favorable to the event A, $N(A\cap B)$ is the no. of cases favorable to the simultaneous occurance of A and B.

If N be the total no. of equally likely elementary cases then

$$P(B|A) = \frac{N(A\cap B)/N}{N(A)/N} = \frac{P(A\cap B)}{P(A)}, \quad P(A)>0.$$

$$\Rightarrow P(A\cap B) = P(B|A)\,P(A).$$

Axiomatic Defⁿ : Consider the probability space $(\Omega, \mathcal{A}, P)$ where Ω is the sample space, $\mathcal{A}$ is the σ-field of the subspace of Ω and P is the probability function defined on $\mathcal{A}$.

Let $A \in \mathcal{A} \ni P(A)>0$, then conditional probability of occurance of any event B belonging to $\mathcal{A}$ given that A has already been occured is defined as.

$$P(B|A) = \frac{P(A\cap B)}{P(A)}.$$

• Conditional Probability satisfies all the axioms of Probability :

i) We have $P(A\cap B) \geq 0 \;\forall B$ and (B/A), and $P(A)>0$.

So, $\frac{P(A\cap B)}{P(A)} \geq 0 \;\forall B$.

i.e. $P(B|A) \geq 0$ for any $B \in \mathcal{A}$.

$\Rightarrow$ Axiom I of probability.

ii) Since, $(\Omega \cap A) = A$.

$$\therefore P(\Omega | A) = \frac{P(\Omega \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1 \quad (\because P(A) > 0).$$

$\Rightarrow$ Axiom II of probability.

iii) Let us consider a sequence of disjoint events $\{C_n\}$.

$C_n \in \mathcal{B} \; \forall n$.

Now, $$P\left[\bigcup_{n=1}^{\infty} C_n | A\right] = \frac{P\left[\left(\bigcup_{n=1}^{\infty} C_n\right) \cap A\right]}{P(A)}, \quad P(A) > 0.$$

$$= \frac{P\left[\bigcup_{n=1}^{\infty} (C_n \cap A)\right]}{P(A)}$$

where $\{C_n \cap A\}$ is also a sequence of disjoint event, as $\{C_n\}$ is so, and $(C_n \cap A) \in \mathcal{B} \; \forall n$.

$$= \frac{\sum_{n=1}^{\infty} P(C_n \cap A)}{P(A)}$$

[by the axiom of unconditional prob.]

$$= \sum_{n=1}^{\infty} \frac{P(C_n \cap A)}{P(A)} = \sum_{n=1}^{\infty} P(C_n | A)$$

$\Rightarrow$ Axiom III of probability

Hence the proof.

2) What do you mean by Stochastic independence of events?

Sol^n $\rightarrow$ The event A is said to be Stochastically independent of the event B if occurrance of A does not depend upon the occurance or non-occurrance of B, i.e.

$$P(A|B) = P(A), \quad P(B) > 0.$$

$$\Rightarrow \frac{P(A \cap B)}{P(B)} = P(A).$$

$$\Rightarrow P(A \cap B) = P(A)\,P(B) \quad \text{--- (i)}$$

Similarly B is said to be Stochastically independent of the event A if.

$$P(B|A) = P(B), \quad P(A) > 0.$$

$$\Rightarrow P(A \cap B) = P(A)\,P(B) \quad \text{--- (ii)}$$

Note that the expression (i) is symmetric in A & B, Hence instead of saying A is independent of B or B is independent of A, one must say A & B are independent of each other.

Remark: If two events are mutually exclusive then they will not be stochastically independent of each other.

3) State and proof Compound Probability Theorem.

Sol^n

Statement :→ (Compound Probability) The probability of simultaneous occurance of A and B is given by the product of the unconditional probability of the event A by the conditional probability of B, supposing that A actually occured. In other words

$$P(A \cap B) = P(A) P(B|A).$$

Proof :→

Let there be n no. of all possible outcomes, of these

n_A = no. of outcomes favorable to A.

n_B = no. of outcomes favorable to B.

n_{AB} = no. of outcomes favorable to A and B.

Then, $P(A) = \frac{n_A}{n}$, $P(A \cap B) = \frac{n_{AB}}{n}$ and $P(B|A) = \frac{n_{AB}}{n_A}$.

$$P(A \cap B) = \frac{n_{AB}}{n}$$

$$= \frac{n_A}{n} \times \frac{n_{AB}}{n_A}$$ [It is supposed that A has actually been occured, i.e., $P(A) > 0$ and hence $n_A > 0$]

$$= P(A) P(B/A).$$

Hence the theorem is proved.

In general case, if $A_1, \ldots, A_n$ be any events in $\mathcal{B}$, then by induction

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1) P(A_2|A_1) P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap \cdots \cap A_{n-1}),$$

provided $P(A_1 \cap \cdots \cap A_{n-1}) > 0$.

— This is called 'Law of Multiplication'.

4) State & proof the theorem of total probability.

Sol^n →

Total Probability

Theorem: Let $(\Omega, \mathcal{A}, P)$ be the probability space, suppose $\{H_n\}$ is a sequence of mutually exclusive and exhaustive events such that $P(H_n) > 0 \ \forall n$, $H_n \in \mathcal{A} \ \forall n$.

Then the probability of any event $B \in \mathcal{A}$ is given by
$$P(B) = \sum_{n=1}^{\infty} P(H_n) P(B|H_n)$$

Proof: Since $\{H_n\}$ is a sequence of mutually exclusive and exhaustive events,
$$\therefore \bigcup_{n=1}^{\infty} H_n = \Omega.$$

Now, $B = B \cap \Omega$.
$$\therefore P(B) = P\left[B \cap \left(\bigcup_{n=1}^{\infty} H_n\right)\right] = P\left[\bigcup_n (B \cap H_n)\right]$$

Note that $H_i \cap H_j = \phi \ \forall i \neq j$
$$\Rightarrow (B \cap H_i) \cap (B \cap H_j) = \phi \ \forall i \neq j.$$

Clearly, $\{B \cap H_n\}$ is also a sequence of mutually disjoint events $\in \mathcal{A}$.

Hence by Axiom-III, we have → $P\left(\bigcup_n (B \cap H_n)\right) = \sum_n P(B \cap H_n)$.

Thus, $P(B) = \sum_n P(B \cap H_n)$.

So, $P(B) = \sum_n P(H_n) P(B|H_n)$ [From the axiom of compound probability]

Hence the proof.

Implication :- The implication of this result is that the unconditional probability of the event B can be obtained as the weighted average of the conditional probabilities.

5) State and proof Bayes theorem.

Solⁿ

Statement: (Bayes' Theorem)

For a sequence of mutually exclusive and exhaustive events $A_1, A_2, \ldots \in \mathcal{A}$ with $P(A_i) > 0$ $\forall\, i = 1, 2, \ldots$

$$P(A_j|B) = \frac{P(A_j)\,P(B|A_j)}{\sum_{i=1}^{\infty} P(A_i)\,P(B|A_i)}$$, where B is any other event.

Proof: Since $A_1, A_2, \ldots$ are mutually exclusive and exhaustive events, $P(A_i) > 0$,

$$\therefore \sum_{i=1}^{\infty} P(A_i) = P\left(\sum_{i=1}^{\infty} A_i\right) = P(\Omega) = 1.$$

$$B = B \cap \Omega = B \cap \left(\sum_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} (B \cap A_i)$$

$$\therefore P(B) = P\left(\sum_{i=1}^{\infty} (B \cap A_i)\right)$$ [$(B \cap A_i)$ is a sequence of mutually disjoint events $\in \mathcal{A}$, applying Axiom III].

$$= \sum_{i=1}^{\infty} P(B \cap A_i)$$

$$= \sum_{i=1}^{\infty} P(A_i)\,P(B|A_i) \quad \text{—— (i)}$$

Now, $$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)}$$

$$= \frac{P(A_j)\,P(B|A_j)}{\sum_{i=1}^{\infty} P(A_j)\,P(B|A_j)}$$, by (i) and since $P(B) > 0$.

Hence the theorem is proved.

(24)

6) What is the probability of getting s points when n die is rolled.

Solⁿ:⇒ Here, no. of all possible cases is 6^n.
Since, the dice are fair, so all possible cases are equally likely.

The no. of favorable cases of 'getting a sum s',

= The no. of solutions $(r_1, r_2, \ldots, r_n)$ of the equation
$\{r_1+r_2+\cdots+r_n = s, \; r_i = 1(1)6$.

= The coefficient of x^s in the expansion of
$(x^1+x^2+\cdots+x^6)^n$.

= The coefficient of x^s in $\left\{\frac{x(1-x^6)}{1-x}\right\}^n$.

= The coefficient of x^{s-n} in the expansion of
$\{(1-x^6)^n (1-x)^{-n}\}$.

= The coefficient of x^{s-n} in $\left\{\sum_{i=0}^{n} \binom{n}{i}(-x^6)^i\right\}$
$\left\{\sum_{j=0}^{\infty} \binom{n+j-1}{j} x^j\right\}$.

$$= \sum_{i=0}^{\left[\frac{s-n}{6}\right]} (-1)^i \binom{n}{i} \binom{n+s-n-6i-1}{s-n-6i} \quad \left[\begin{array}{l}\text{Here } 0 \le 6i \le s-n \\ \Rightarrow 0 \le i \le \left[\frac{s-n}{6}\right]\end{array}\right]$$

$$= \sum_{i=0}^{\left[\frac{s-n}{6}\right]} (-1)^i \binom{n}{i} \binom{s-6i-1}{s-n-6i}.$$

So, the required probability is $= \frac{1}{6^n} \sum_{i=0}^{\left[\frac{s-n}{6}\right]} (-1)^i \binom{n}{i} \binom{s-6i-1}{n-1}$

Cor. Show that the prob. of getting a total of s with n die is same as the prob. of throwing $(7n-s)$.

Sol. The co-efficient of x^s in $(x+x^2+\cdots+x^6)^n$

= " " " $\left(\frac{1}{x}\right)^s$ " $\left(\frac{1}{x}+\frac{1}{x^2}+\cdots+\frac{1}{x^6}\right)^n = \left\{\frac{x^{-1}(1-x^{-6})}{1-x^{-1}}\right\}^n$

= " " " x^{7n-s} " $x^{7n}\left\{\frac{x(x^6-1)}{x^7(x-1)}\right\}^n = \left\{\frac{x(1-x^6)}{1-x}\right\}^n$

= " " " x^{7n-s} " $(x+x^2+\cdots+x^6)^n$

7) State & proof Boole's inequality.

Sol^n Statement: (Boole's Inequality) If A_i $(i=1(1)n)$ be any events in $\mathcal{A}$, then.

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

Proof :→ Consider first A_1 and A_2. Now

$$A_1 \cup A_2 = A_1 + (A_2 - A_1),$$

the events $A_1, (A_2 - A_1)$ being disjoint. Hence

$$P(A_1 \cup A_2) = P(A_1) + P(A_2 - A_1)$$
$$\leq P(A_1) + P(A_2).$$

Since $(A_2 - A_1) \subset A_2$. Thus the inequality is proved for $n=2$. Now

$$\bigcup_{i=1}^{n} A_i = \left(\bigcup_{i=1}^{n-1} A_i\right) \cup A_n$$

$$\Rightarrow P\left(\bigcup_{i=1}^{n} A_i\right) \leq P\left(\bigcup_{i=1}^{n-1} A_i\right) + P(A_n)$$
$$\leq P\left(\bigcup_{i=1}^{n-2} A_i\right) + P(A_{n-1}) + P(A_n).$$
$$\cdots\cdots\cdots$$
$$\leq \sum_{i=1}^{n} P(A_i)$$

'=' holds iff $A_i \cap A_j = \phi \ \forall \ i<j = 1(1)n$.

8) State and proof Bonferroni's inequality.

Soln. Statement: (Bonferroni's Inequality)

If A_i, $i = 1(1)k$ be events in $\mathscr{A}$, then

$$P\left(\bigcap_{i=1}^{K} A_i\right) \geq \sum_{i=1}^{K} P(A_i) - (K-1).$$

Proof :→ We have $\left(\bigcap_{i=1}^{K} A_i\right) = \left(\bigcup_{i=1}^{K} A_i^c\right)^c$

$$\therefore P\left(\bigcap_{i=1}^{K} A_i\right) = 1 - P\left(\bigcup_{i=1}^{K} A_i^c\right) \quad \text{——} \quad (1)$$

By Boole's inequality,

$$P\left(\bigcup_{i=1}^{K} A_i^c\right) \leq \sum_{i=1}^{K} P(A_i^c) = \sum_{i=1}^{K} [1 - P(A_i)]$$

$$= K - \sum_{i=1}^{K} P(A_i).$$

$$\Rightarrow 1 - P\left(\bigcap_{i=1}^{K} A_i\right) \leq K - \sum_{i=1}^{K} P(A_i) \quad [\text{Applying } (1)]$$

$$\Rightarrow P\left(\bigcap_{i=1}^{K} A_i\right) \geq \sum_{i=1}^{K} P(A_i) - (K-1)$$

'=' holds iff $A_i \cap A_j = \phi \ \forall \ i < j = 1(1)K$.

i.e. if chosen events from $\mathscr{A}$ are mutually exclusive.

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9) State & proof Poincare's Theorem :-

Sol.ⁿ

Statement :- (Poincare's Theorem) For any $n(>1)$ events $A_1, A_2, \cdots, A_n$ not necessarily mutually exclusive. The probability of occurance of atleast one of these events will be given by

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j=1}^{n}\sum P(A_i \cap A_j) + \sum\sum_{i<j<k=1}^{n}\sum P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n-1} P(A_1 \cap \cdots \cap A_n).$$

Proof :→ First consider two events A_1 and A_2.

since $A_1 \cup A_2 = A_1 + (A_2 - A_1 \cap A_2)]$

$\therefore P(A_1 \cup A_2) = P(A_1) + P[A_2 - (A_1 \cap A_2)]$

since $A_1 \cap A_2 \subset A_2$

$\therefore P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

The result is true for $n=2$

Now, assumed that the result is true for $n=m$

so, $P\left(\bigcup_{i=1}^{m} A_i\right) = \sum_{i=1}^{m} P(A_i) - \sum_{i<j=1}^{m}\sum P(A_i \cap A_j) + \cdots + (-1)^{m-1} P(A_1 \cap \cdots \cap A_m)$

Now, Let us include one more event $A_{m+1} \in G$.

Now, $\bigcup_{i=1}^{m+1} A_i = \left(\bigcup_{i=1}^{m} A_i\right) \cup A_{m+1}$

$$\therefore P\left(\bigcup_{i=1}^{m+1} A_i\right) = P\left(\bigcup_{i=1}^{m} A_i\right) + P(A_{m+1}) - P\left[\left(\bigcup_{i=1}^{m} A_i\right) \cap A_{m+1}\right]$$

$$= P\left(\bigcup_{i=1}^{m} A_i\right) + P(A_{m+1}) - P\left[\bigcup_{i=1}^{m} (A_i \cap A_{m+1})\right]$$

$$= \left[\sum_{i=1}^{m} P(A_i) - \sum_{i<j=1}^{m}\sum P(A_i \cap A_j) + \cdots + (-1)^{m-1} P\left(\bigcap_{i=1}^{m} A_i\right)\right] + P(A_{m+1}) - \left[\sum_{i=1}^{m} P(A_i \cap A_{m+1}) - \sum_{i<j=1}^{m}\sum P(A_i \cap A_j \cap A_{m+1}) + \cdots + (-1)^{m-1} P\left(\bigcap_{i=1}^{m+1} A_i\right)\right]$$

$$= \sum_{i=1}^{m+1} P(A_i) - \sum_{i<j=1}^{m+1}\sum P(A_i \cap A_j) + \sum\sum_{i<j<k=1}^{m+1}\sum P(A_i \cap A_j \cap A_{k+m+1}) + \cdots + (-1)^{m} P\left[\bigcap_{i=1}^{m+1} A_i\right]$$

∴ The theorem is true for $n = m+1$, when it is true for $n=m$. Hence by induction, result follows.

State & Proof Hunt's Theorem.

Sol^n

Statement: (Hunt's Theorem) For any $n(>1)$ events not necessarily mutually exclusive, $A_i \in \mathcal{A}$, $A_1, \ldots, A_n$, $i=1(1)n$.

$$P\left(\bigcap_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j=1}^{n}\sum P(A_i \cup A_j) + \cdots + (-1)^{n-1} P\left(\bigcup_{i=1}^{n} A_i\right).$$

Proof: → We know $P(A^c) = 1 - P(A)$, using this repeatedly.

$$P\left(\bigcap_{i=1}^{n} A_i\right) = 1 - P\left(\bigcup_{i=1}^{n} A_i^c\right).$$

$$= 1 - \sum_{i=1}^{n} P(A_i^c) + \sum_{i<j=1}^{n}\sum P(A_i^c \cap A_j^c)$$

$$+ \cdots + (-1)^n P(A_1^c \cap \cdots \cap A_n^c).$$

$$= 1 - \left\{n - \sum_{i=1}^{n} P(A_i)\right\} + \left\{\binom{n}{2} - \sum_{i<j=1}^{n}\sum P(A_i \cup A_j)\right\}$$

$$+ \cdots + (-1)^n \left\{\binom{n}{n} - P(A_1 \cup A_2 \cup \cdots \cup A_n)\right\}$$

$$= \sum_{i=1}^{n} P(A_i) - \sum_{i<j=1}^{n}\sum P(A_i \cup A_j) + \cdots + (-1)^{n-1} P(A_1 \cup \cdots \cup A_n)$$

Since $1 - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = (1-1)^n = 0.$

10) State and proof Continuity Theorem.

Sol^n:→ Proof: First consider $\{A_n\}$ to be monotonically non-decreasing. Then $A_1 \subset A_2 \subset A_3 \subset \cdots$.

$$\therefore \lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n .$$

$$= \sum_{n=1}^{\infty} \left(A_n - \bigcup_{i=1}^{n-1} A_i\right).$$

$$= \sum_{n=1}^{\infty} (A_n - A_{n-1}),$$ since A_i's are monotonically non-decreasing, so. $\bigcup_{i=1}^{n-1} A_i = A_{n-1}$.

$$\therefore P\left(\lim_{n\to\infty} A_n\right) = P\left[\sum_{n=1}^{\infty} (A_n - A_{n-1})\right]$$

$$= \sum_{n=1}^{\infty} P(A_n - A_{n-1})$$ [as $(A_n - A_{n-1})$ are disjoint events so by the axiom of countable additivity of $P(\cdot)$].

$$= \sum_{n=1}^{\infty} [P(A_n) - P(A_{n-1})]$$ [$\because A_{n-1} \subset A_n$, so therefore $P(\cdot)$ is subtractive]

$$= \lim_{n\to\infty} \sum_{i=1}^{n} [P(A_i) - P(A_{i-1})]$$

$$= \lim_{n\to\infty} P(A_n).$$

Now consider $\{A_n\}$ to be monotonically non-~~decreasing~~ increasing. Then, $\{A_n^c\}$ will be monotonically non-decreasing and each will belong to $\mathcal{G}$.

$$\therefore P\left(\lim_{n\to\infty} A_n^c\right) = \lim_{n\to\infty} P(A_n^c)$$

$$\Rightarrow P\left(\lim_{n\to\infty} A_n\right)^c = \lim_{n\to\infty} (1 - P(A_n)).$$

$$\Rightarrow 1 - P\left(\lim_{n\to\infty} A_n\right) = 1 - \lim_{n\to\infty} P(A_n)$$

i.e. $$P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

∴ Hence the result is proved.

Statement:- (Continuity theorem)

If $\{A_n\}$ is a monotone sequence of events belonging to $\mathcal{G}$, then —

$$P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

11) Derive an expression for the probability of realization of exactly 'm' out of 'n' events. (Jordon's Theorem).

Sol^n

Statement : Probability that exactly m of the events A_i, $i=1(1)n$, will occur is.

$$P_{[m]} = S_m - \binom{m+1}{m} S_{m+1} + \binom{m+2}{m} S_{m+2} - \cdots + (-1)^{n-m} \binom{n}{m} S_n .$$

where $S_1 = \sum_{i=1}^{n} P(A_i)$, $S_2 = \sum\sum_{i<j=1}^{n} P(A_i \cap A_j)$ and so on.

Proof :→ Consider first the probability that just m specified events among A_i, $i=1(1)n$, will occur, say the events $A_1, A_2, \ldots, A_m$. The probability is..

$$P(A_1 \cap A_2 \cap \cdots \cap A_m \cap A_{m+1}^c \cap A_{m+2}^c \cap \cdots \cap A_n^c) .$$

Now, take $A_1 \cap A_2 \cap \cdots \cap A_m = B$.

The above equals to $P\left[B \cap \left(\bigcup_{i=m+1}^{n} A_i^c\right)\right]$

$$= P(B) - P\left[B \cap \left(\bigcup_{i=m+1}^{n} A_i\right)\right]$$

$$= P(B) - P\left[\bigcup_{i=m+1}^{n} (B \cap A_i)\right]$$

$$= P\left(\bigcap_{i=1}^{m} A_i\right) - \sum_i P(B \cap A_i) + \sum\sum_{i<j} P(B \cap A_i \cap A_j)$$

$$- \cdots + (-1)^{n-m} P\left(\bigcap_{i=1}^{n} A_i\right) \quad \text{———} (*)$$

We may choose m specified events out of n events in $\binom{n}{m}$ mutually exclusive ways. So, the required probability is the sum of $\binom{n}{m}$ such terms. Again, each such probability has $\binom{n-m}{i}$ terms included in S_{m+i} and sign attached to it is $(-1)^i$ while the total number of terms in S_{m+i} is $\binom{n}{m+i}$. Hence, the coefficient of S_{m+i} in the expression $(*)$ is

$$(-1)^i \frac{\binom{n}{m}\binom{n-m}{i}}{\binom{n}{m+i}}$$

Therefore,

$$P_{[m]} = S_m - \binom{m+1}{m} S_{m+1} + \binom{m+2}{m} S_{m+2} - \cdots\cdots + (-1)^{n-m}\binom{n}{m} S_n.$$

THEOREMS

Problem-1

Discuss the method of determination of the probability that $m(\geq 2)$ or more of the events occur simultaneously out n events $A_1, A_2, \cdots, A_n$. Illustrate with an example.

Sol^n:→ At least m events occur iff exactly $m+i$ events occur, $i=0,1,2,\ldots,n-m$.

$$\therefore P_m = P_{[m]} + P_{[m+1]} + \cdots + P_{[n]}.$$

$$= S_m - \left\{\binom{m+1}{m} - \binom{m+1}{m+1}\right\} S_{m+1} + \cdots + (-1)^{n-m}\left\{\binom{n}{m} - \binom{n}{m+1} + \cdots + (-1)^{n-m}\binom{n}{n}\right\} S_n.$$

[Using Jordon's first theorem].

coefficient of S_{m+i} in the R.H.S. is

$$= (-1)^i \sum_{j=0}^{i} (-1)^j \binom{m+i}{m+j}.$$

$$= \sum_{j=0}^{i} (-1)^j \left\{\binom{m+i-1}{m+j} + \binom{m+i-1}{m+j-1}\right\}.$$

$$= \left\{\binom{m+i-1}{m} + \binom{m+i-1}{m-1}\right\} - \left\{\binom{m+i-1}{m+1} + \binom{m+i-1}{m}\right\} + \left\{\binom{m+i-1}{m+2} + \binom{m+i-1}{m+1}\right\} + \cdots + (-1)^i\binom{m+i-1}{m+i-1}$$

$$= \binom{m+i-1}{m-1}$$

Problem-2

Let $B_1, B_2, \ldots$ be a partition of the sample space and $P(C) > 0$. Show that —

$$P(A|C) = \sum_{j=1}^{\infty} P(B_j|C)\, P(A|B_j^c).$$

Sol^n: →

$$P(A|C) = \frac{P(A \cap C)}{P(C)}$$

$$= \frac{P\left[\bigcup_{j=1}^{\infty} A C B_j\right]}{P(C)}$$ [as B_j's are the partition of the sample space]

$$= \frac{\sum_{j=1}^{\infty} P(A C B_j)}{P(C)}$$ [By Axiom III].

$$= \sum_{j=1}^{\infty} \frac{P(A|CB_j)\, P(B_j C)}{P(C)}$$ [Applying compound probability law]

$$= \sum_{j=1}^{\infty} P(A|B_j C)\, P(B_j|C)$$

[Applying compound probability law for $P(B_j C) = P(B_j|C) \cdot P(C)$]

(Proved)

Note:- 1. For events A and B,

$$B = B \cap \Omega = B \cap (A \cup A^c)$$
$$= (B \cap A) \cup (B \cap A^c)$$

So, $P(B) = P(B \cap A) + P(B \cap A^c)$

If $A \subset B$, $A \cap B$, so $P(B) = P(A) + P(B \cap A^c) \geq P(A)$.

2. Law of addition:- For any two events A and B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof:- $A \cup B = A \cup (B \cap A^c)$

$P(A \cup B) = P(A) + P(B \cap A^c)$, since $A \cap (B \cap A^c) = \phi$ —— (1)

$B = B \cap \Omega = B \cap (A \cup A^c)$

$= (B \cap A) \cup (B \cap A^c)$ —— (2)

So, $P(B) = P(B \cap A) + P(B \cap A^c)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Ex. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$.

Problem-3

Construct four events A, B, C and D in a random experiment such that —

(i) $0 < P(A) < P(A|B) < 1$ and

(ii) $0 < P(C|D) < P(C) < 1$ and comment.

Sol^n Let there be 5 black and 5 white balls, balls are drawn one by one WOR at random.

Suppose, W_i is the event denoting that ith ball drawn is white.

Note that —

$$P(W_2) = P(W_2|W_1)P(W_1) + P(W_2|W_1^c)P(W_1^c)$$

$$= \frac{4}{9}\cdot\frac{1}{2} + \frac{5}{9}\cdot\frac{1}{2}$$

$$= \frac{1}{2}.$$

Choose, $A = C = W_2$

$B = W_1^c$

$D = W_1$

$0 < P(A) = \frac{1}{2} < P(A|B) = \frac{5}{9} < 1.$

$0 < P(C|D) = \frac{4}{9} < P(C) = \frac{1}{2} < 1.$

(OR)

Consider a random experiment of "tossing a fair coin thrice."

$\Omega = \{HHH, HHT, HTH, HTT, THT, TTH, TTT, THH\}.$

Let $A = \{HHH\}$ = 'Three heads'.

$B = \{HHH, HHT, HTH, THH\}$ = 'atleast two heads'.

$P(A) = \frac{1}{8}$, $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{1/8}{4/8} = \frac{1}{4}.$

$\therefore 0 < P(A) < P(A|B) < 1.$

Let $C = \{HTT, THT, TTH, TTT\}$ = 'atmost one head'.

and $D = \{HHT, HTH, THH, HTT, THT\}$

$P[C] = \frac{4}{8} = \frac{1}{2}$, $P(C|D) = \frac{P(C\cap D)}{P(D)} = \frac{2/8}{5/8} = \frac{2}{5}.$

$\therefore 0 < P(C|D) < P(C) < 1.$

Problem-4

For any two events A and B, show that —

$\max\{0, P(A)+P(B)-1\} \le P(A\cap B) \le \min[P(A), P(B)]$
$\le \max\{P(A), P(B)\} \le P(A\cup B) \le \min\{P(A)+P(B), 1\}$

Sol^n :→ We have $P(A\cap B) \geq 0$ and also

$P(A\cap B) \geq P(A)+P(B)-1$.

$\Rightarrow \max[0, P(A)+P(B)-1] \le P(A\cap B)$;

$A\cap B \subset A$, ~~A∩B⊂B~~ $A\cap B \subset B$.

$\Rightarrow P(A\cap B) \le \min[P(A), P(B)]$;

$A\cup B \supset A$, $A\cup B \supset B$.

$\Rightarrow P(A\cup B) \geq \max[P(A), P(B)]$;

$P(A\cup B) \le P(A)+P(B)$ by Boole's inequality & $P(A\cup B) \le 1$.

$\Rightarrow P(A\cup B) \le \min\{P(A)+P(B), 1\}$.

Finite Probability Models:- In a finite probability model, the sample space Ω has a finite number of outcomes, say, $\omega_1, \omega_2, \ldots, \omega_M$

$\therefore \Omega = \{\omega_1, \omega_2, \ldots, \omega_M\}$.

Define simple events : $S_i = \{\omega_i\}$ $\forall i = 1(1)M$.

$\bigcup_{i=1}^{M} S_i = \Omega$, $S_i \cap S_j = \phi$ for $i \neq j$

Ex. Toss a coin twice, $\Omega = \{\underset{\omega_1}{HH}, \underset{\omega_2}{HT}, \underset{\omega_3}{TH}, \underset{\omega_4}{TT}\}$, $S_1 = \{HH\}$

$E = \{$exactly one head turns up$\} = \{\omega_2, \omega_3\} = S_2 \cup S_3$.

Facts:- (i) Any two different simple events are disjoint if $S_i \cap S_j = \phi$ for $i \neq j$.

(ii) Any event E is the union of simple events S_i corresponding to outcomes contained in E.

$\therefore$ Probability of any event is a finite probability model is the sum of probabilities of simple events whose union is E.

SOME SOLVED EXAMPLES :-

Q.1. Two players A and B has respectively $(n+1)$ and n coins. If they toss their coins simultaneously what's the prob. that

(a) A will have more heads than B

(b) A will have as many heads as B

(c) A will have fewer heads than B

Solution:- X: Number of heads obtained by A

Y: Number of heads obtained by B

Assuming the coin is fair, so

P(A will have x heads and B will have y heads)

$$= \binom{n+1}{x}\left(\frac{1}{2}\right)^{n+1} \cdot \binom{n}{y}\left(\frac{1}{2}\right)^{n} = \frac{1}{2^{2n+1}}\binom{n+1}{x}\binom{n}{y}.$$

(a)
$$P(X>Y) = \sum_{x>y} \frac{1}{2^{2n+1}}\binom{n+1}{x}\binom{n}{y}$$

$$P(X\le Y) = \frac{1}{2^{2n+1}}\sum_{x\le y}\binom{n+1}{x}\binom{n}{y} = \frac{1}{2^{2n+1}}\sum_{x\le y}\binom{n+1}{n+1-x}\binom{n}{n-y}$$

$$= \frac{1}{2^{2n+1}}\sum_{x'>y'}\binom{n+1}{x'}\binom{n}{y'}$$

$$= P(X>Y)$$

So, $P(X>Y) = \frac{1}{2}$.

(b)
$$P(X=Y) = \frac{1}{2^{2n+1}}\sum_{x}\binom{n+1}{x}\binom{n}{x}$$

$$= \frac{1}{2^{2n+1}}\sum_{x}\left[\binom{n+1}{x}\cdot\binom{n}{n-x}\right]$$

$$= \frac{1}{2^{2n+1}}\sum_{x}\left[\text{coefficient of } t^{x} \text{ in } (1+t)^{n+1} \times \text{coefficient of } t^{n-x} \text{ in } (1+t)^{n}\right]$$

$$= \frac{1}{2^{2n+1}} \times \text{coefficient of } t^{n} \text{ in } (1+t)^{2n+1}$$

$$= \frac{1}{2^{2n+1}}\binom{2n+1}{n}.$$

(c)
$$P(X<Y) = 1 - P(X\ge Y) = 1 - P(X>Y) - P(X=Y)$$

$$= 1 - \frac{1}{2} - \frac{1}{2^{2n+1}}\binom{2n+1}{n}.$$

Q.2. In a game called 'odd men out', n persons toss coin to determine one person who will buy refreshments for the whole group. If there is a person in the group whose outcome (head/tail) is different from that of any other member in the group, then that person is an odd man. What's the probability that —

(a) In a game there will be an odd man.

(b) r plays will be required to conclude the game.

Solution:- The odd man will be found if

(a) i. all the other $(n-1)$ members will get a Head and the remaining one gets tail and

ii. other $(n-1)$ members will get a Tail and the remaining will get a Head.

$$P(i^{th} \text{ part}) = \binom{n}{n-1}\left(\frac{1}{2}\right)^n$$

$$P(ii^{th} \text{ part}) = \binom{n}{n-1}\left(\frac{1}{2}\right)^n$$

$$P(\text{There will be an odd man in a game}) = 2\binom{n}{n-1}\left(\frac{1}{2}\right)^n$$

$$= \frac{n}{2^{n-1}}.$$

(b) r plays will be required if the first $(r-1)$ plays will not give an odd man and the rth one will give an odd man. Since the plays are independent,

$$\text{Required probability} = \left\{1 - \frac{n}{2^{n-1}}\right\}^{r-1} \cdot \frac{n}{2^{n-1}}.$$

Q.3. A fair die is thrown 7 times. What is the probability of getting a total of 30 points.

Solution:- Each throw of the die will give any of the numbers $1, 2, 3, \ldots, 6$.

Therefore the total no. of elementary events in 7 throws is $= 6^7$.

The die is fair implies that each of these 6^7 elementary events will be equally likely.

Let A denote the event that the sum of the points is 30. Then the number of elementary events favourable to A is the number of solutions of $x_1 + x_2 + \cdots + x_7 = 30$

The number of solutions is the same as the coefficient of t^{30} in the expansion of $(t + t^2 + \cdots + t^6)^7$

$$= t^7 (1 + t + t^2 + \cdots + t^5)^7$$

$$= \frac{t^7 (1 - t^6)^7}{(1 - t)^7}, \quad |t| < 1$$

This is the same as the coefficient of t^{23} in

$$\left(1 - 7t^6 + \binom{7}{2} t^{12} - \binom{7}{3} t^{18} - \cdots - t^{42}\right)\left(1 + 7t + \frac{7.8}{2} t^2 + \frac{7.8.9}{3!} t^3 + \cdots\right)$$

This is again equal to

$$\left(\frac{7.8.9\ldots29}{23!}\right) - 7\left(\frac{7.8.9.\ldots23}{17!}\right) + \binom{7}{2}\left(\frac{7.8.9\ldots17}{11!}\right) - \binom{7}{3}\left(\frac{7.8.9\ldots11}{5!}\right) = n_1 \text{ (say)}$$

So, required prob. $= \dfrac{n_1}{6^7}$.

Q.4. A man addresses in n envelops and write n cheques in payment of n bills. If the n bills are placed at random in the n envelopes. What is the prob. that each bill will be placed in a wrong envelopes.

Solution:-

A_i : The event that i^{th} bill goes to the i^{th} envelope, $i=1(1)n$

Then required probability is $= P\left(\bigcap_{i=1}^{n} A_i^c\right)$

$$= P\left(\bigcup_{i=1}^{n} A_i\right)^c$$

$$= 1 - P\left(\bigcup_{i=1}^{n} A_i\right)$$

$$= 1 - S_1 + S_2 - S_3 + \cdots + (-1)^n S_n$$

where, $S_1 = \sum_i P(A_i)$; $S_2 = \sum\sum_{i<j} P(A_i \cap A_j)$, ...,

$S_n = P(A_1 \cap A_2 \cap \cdots \cap A_n)$.

Now n bills can be placed in n envelopes in $n!$ ways. These placements are made at random means that these $n!$ arrangements are ~~all~~ equally likely. Now in order that A_i has to occur, the i^{th} bill will go to the i^{th} envelope and the remaining $(n-1)$ bills can be placed among the $(n-1)$ envelopes in $(n-1)!$ ways.

$$\therefore P(A_i) = \frac{(n-1)!}{n!} = \frac{1}{n}.$$

By a similar argument, $P(A_i \cap A_j) = \frac{1}{n(n-1)}$ for $i<j$.

$P(A_i \cap A_j \cap A_k) = \frac{1}{n(n-1)(n-2)}$ for $i<j<k$

$\therefore$ Require probability is equal to

$$1 - \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n(n-1)} - \cdots\cdots + (-1)^n \binom{n}{n}\frac{1}{(n)_n}$$

$$= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}$$

$$\rightarrow e^{-1} \text{ when } n \to \infty.$$

Ex. 5. Three prisoners whom we may call A, B and C, are informed by the jailer that one of them has been chosen at random to be executed and the other two are to be freed. Prisoner A who has studied probability theory, then reasons to himself that he has probability 1/3 of being executed. He then asks the jailer to tell him privately which of his fellow prisoners will be free, claiming that there would not be any harm in divulging this information, since he already knows that at least one will go. The jailer being an ethical fellow refuses to reply to this question, pointing out that if A knew which of his fellows were to be set free, then his prob. of being executed would increase to $\frac{1}{2}$, since he would then be one of two prisoners one of whom is to be executed. Show that the probability that A will be executed is still 1/3, even if the jailer were to answer his question, assuming that in the event that A is to be executed, the jailer is as likely to say that B is to be set free as he is to say that C is to be set free.

Solution:- E_i : the event that prisoner i will be executed, $i = A, B, C$.

F_i : the event that prisoner i will be set free, $i = A, B, C$.

$P(E_i) = 1/3$.

Compute,

$P(E_A)$, $P(E_A \mid F_B)$, $P(E_A \mid F_C)$.

$P(E_A) = 1/3$.

$P(F_B \mid E_A) = \frac{1}{2}$ ∵ since if the decision is to execute A then the jailer is as likely to say B will be set free as to say C will be set free.

$P(F_B \mid E_B) = 0$, since the jailer can't tell a lie.

$P(F_B \mid E_C) = 1$, since the jailer can't tell A that he will be set free.

By Bayes theorem,

$$P(E_A \mid F_B) = \frac{P(E_A)\,P(F_B \mid E_A)}{P(E_A)\,P(F_B \mid E_A) + P(E_B)\,P(F_B \mid E_B) + P(E_C)\,P(F_B \mid E_C)}$$

$$= \frac{1/3 \times 1/2}{1/3 \times 1/2 + 0 \times 1/3 + 1 \times 1/3} = 1/3.$$

Similarly, $P(E_A \mid F_C) = 1/3$.

∴ Probability remains $1/3$.

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Ex.6. There are N coupons numbered $1, 2, \ldots, N$ in a box. If n coupons drawn at random, then what's the probability that the highest number on coupon drawn is m.

Ans. A_i : the event that all coupons drawn have numbers not exceeding i.

Case-I:- (Drawing WR)

Required prob. $= P(A_m) - P(A_{m-1})$

$$= \frac{m^n - (m-1)^n}{N^n}$$

Case-II:- Required prob. $= P(A_m) - P(A_{m-1})$

$$= \frac{(m)_n - (m-1)_n}{(N)_n}.$$

Ex.7. A fair die is thrown n times. What's the prob. that each of the six numbers $1, 2, 3, \ldots, 6$ will appear at least once.

Solution:- Let A_i denote the event that the ith variety will appear at least once, $i = 1(1)6$.

Then the probability we require is $P\left(\bigcap_{i=1}^{6} A_i\right)$

$$= P\left(\bigcup_{i=1}^{6} A_i^c\right)^c$$

$$= 1 - P\left(\bigcup_{i=1}^{6} A_i^c\right)$$

$$= 1 - \left\{\sum_{i=1}^{6} P(A_i^c) - \sum_{i<j}\sum P(A_i^c \cap A_j^c) + \cdots + \sum_{i_1<i_2<\cdots<i_5}\sum\cdots\sum P(A_{i_1}^c \cap \cdots \cap A_{i_5}^c)\right\}$$

[By Poincare's theorem]

In a single throw of die, prob. that the number i will not appear is $\frac{5}{6}$. $P(A_i^c) = \left(\frac{5}{6}\right)^n$, since throws are independent.

Similarly, $P(A_i^c \cap A_j^c) = \left(\frac{4}{6}\right)^n$.

$\vdots$

$$P(A_{i_1}^c \cap A_{i_2}^c \cap \cdots \cap A_{i_5}^c) = \left(\frac{1}{6}\right)^n$$

So, required prob. is

$$= 1 - \binom{6}{1}\left(\frac{5}{6}\right)^n + \binom{6}{2}\left(\frac{4}{6}\right)^n - \binom{6}{3}\left(\frac{3}{6}\right)^n + \binom{6}{4}\left(\frac{2}{6}\right)^n - \binom{6}{5}\left(\frac{1}{6}\right)^n.$$

SELECTED PROBLEMS :- (Application of Jordon's Theorem)

(1) (Matching Problem) :- n letters are placed at random into similarly addressed n envelopes. Find the probability of exactly m matches? Also

Solution :- Jordon's theorem states,

$$P(\text{occurance of exactly } m \text{ of } n \text{ events}) = P_{[m]} = \sum_{i=0}^{n-m} (-1)^{i} \binom{m+i}{m} \times S_{m+i}$$

where $S_K = \sum_{1 \le i_1 < i_2 < \cdots < i_K \le n} P(A_{i_1}, A_{i_2}, \ldots, A_{i_K})$

Now, $P(A_i) = \dfrac{(n-1)!}{n!} = \dfrac{1}{n} \;\forall\, i$

$$\Rightarrow S_1 = \sum_{1}^{n} P(A_i) = n \cdot \frac{(n-1)!}{n!}$$

$$P(A_i \cap A_j) = \frac{(n-2)!}{n!} \;\forall\, i \ne j$$

$$\Rightarrow S_2 = \sum_{1 \le i < j \le n} P(A_i \cap A_j) = \binom{n}{2} \cdot \frac{(n-2)!}{n!}$$

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$$\Rightarrow S_K = \binom{n}{K} \frac{(n-K)!}{n!} = \frac{1}{K!}$$

$$\Rightarrow S_{m+i} = \frac{1}{(m+i)!}$$

$$\therefore P(\text{exactly } m \text{ matches}) = \sum_{i=0}^{n-m} (-1)^i \binom{m+i}{m} \frac{1}{(m+i)!}$$

$$= \frac{1}{m!} \sum_{i=0}^{n-m} \frac{(-1)^i}{i!}$$

$$= \frac{e^{-1}}{m!}, \text{ since } n \uparrow \infty.$$

$$\therefore P(\text{at least } m \text{ matches}) = \sum_{i=0}^{n-m} (-1)^i \binom{m+i-1}{m-1} S_{m+i}$$

$$= \sum_{i=0}^{n-m} (-1)^i \binom{m+i-1}{m-1} \frac{1}{(m+i)!}$$

$$= \frac{1}{(m-1)!} \sum_{i=0}^{n-m} \frac{(-1)^i}{(m+i) \cdot i!}$$

(2) A fair die is rolled 10 times. Find the prob. that exactly 3 of the face value will occur.

Solution:- A_i : 'i' does not turn up. $i = 1, 2, \ldots, 6$.

$$P(\text{exactly } 3 \text{ of } A_1, A_2, \ldots, A_6 \text{ occur}) = S_3 - \binom{4}{3} S_4 + \binom{5}{3} S_5 - \binom{6}{3} S_6$$

$$= \binom{6}{3}\left(\frac{3}{6}\right)^{10} - \binom{6}{4}\left(\frac{2}{6}\right)^{10} - \binom{6}{5}\left(\frac{1}{6}\right)^{10}.$$

(3) n distinguishable balls are placed at random into n cells. Find the prob. that m cells remain occupied.

Solution:- A_i : 'i' th cell remains empty.

$P(\text{exactly } n-m \text{ of } A_1, \ldots, A_n \text{ occur})$

$$= S_{n-m} - \binom{n-m+1}{n-m} S_{n-m+1} + \binom{n-m+2}{n-m} S_{n-m+2} - \cdots + (-1)^m \binom{n}{n-m}$$

where, $$S_K = \sum_{1 \le i_1 < i_2 < \cdots < i_K \le n} P(A_{i_1} A_{i_2} \cdots A_{i_K})$$

$$\therefore S_K = \binom{n}{K} \cdot \frac{(n-K)^n}{n^n}$$

Total Probability Theorem :—

Theorem:— For a sequence of mutually exclusive and exhaustive events $A_1, A_2, \ldots$ with $P(A_i) > 0 \ \forall i = 1, 2, \ldots$

$$P(B) = \sum_{i=1}^{\infty} P(A_i) P(B|A_i),$$ where B be any event with $P(B) > 0$.

Proof:- Since $A_1, A_2, \ldots$ are mutually exclusive and exhaustive events

$$\sum_{i=1}^{\infty} A_i = \Omega \quad \therefore P\left(\sum_{i=1}^{\infty} A_i\right) = P(\Omega) = 1 = \sum_{i=1}^{\infty} P(A_i)$$

$$B = B \cap \Omega = B \cap \sum_{i=1}^{\infty} A_i = \sum_{i=1}^{\infty} (B \cap A_i)$$

$$\therefore P(B) = P\left(\sum_{i=1}^{\infty} (B \cap A_i)\right)$$

$$= \sum_{i=1}^{\infty} P(B \cap A_i)$$

$$= \sum_{i=1}^{\infty} P(A_i) P(B|A_i) \quad \text{[Proved]}$$

Implication of this result is that the unconditional probability of the event B can be obtained as the weighted average of the conditional probabilities.

Baye's Theorem:- For a sequence of mutually exclusive and exhaustive events $A_1, A_2, \ldots$ with $P(A_i) > 0 \ \forall\, i = 1, 2, \ldots$

$$P(A_j|B) = \frac{P(A_j)\,P(B|A_j)}{\sum_{i=1}^{\infty} P(A_i)\,P(B|A_i)}, \text{ where } P(B) > 0.$$

Proof:- Since $A_1, A_2, \ldots$ are m.e. and exhaustive events.

$$\therefore \sum_{i=1}^{\infty} P(A_i) = P\left(\sum_{i=1}^{\infty} A_i\right) = P(\Omega) = 1.$$

$$\therefore B = B \cap \Omega = B \cap \sum_{i=1}^{\infty} A_i$$

$$= \sum_{i=1}^{\infty} (B \cap A_i)$$

$$\therefore P(B) = P\left(\sum_{i=1}^{\infty} (B \cap A_i)\right) = \sum_{i=1}^{\infty} P(B \cap A_i) = \sum_{i=1}^{\infty} P(A_i)\,P(B|A_i)$$

$$\therefore P(A_j|B) = \frac{P(A_j \cap B)}{P(B)}, \ P(B) > 0.$$

$$= \frac{P(A_j)\,P(B|A_j)}{\sum_{i=1}^{\infty} P(A_i)\,P(B|A_i)}.$$

[Here $P(A_j)$ is prior probability and $P(A_j|B)$ is called posterior probability]

Extended Baye's Theorem:- Let C be the event in $\mathcal{A}$, then under the conditions of Bayes theorem together with the condition $P(A_i \cap B) > 0 \ \forall\, i = 1, 2, \ldots$

$$P(C|B) = \frac{\sum_{i=1}^{\infty} P(A_i)\,P(B|A_i)\,P(C|A_i \cap B)}{\sum_{i=1}^{\infty} P(A_i)\,P(B|A_i)}$$

Proof:- Since $A_1, A_2, \ldots$ are m.e. and exhaustive events,

$$P(C|B) = \sum_{i=1}^{\infty} P(A_i \cap C|B)$$

$$= \sum_{i=1}^{\infty} P(A_i|B)\,P(C|A_i \cap B)$$

$$= \frac{\sum_{i=1}^{\infty} P(A_i)\,P(B|A_i)\,P(C|A_i \cap B)}{\sum_{i=1}^{\infty} P(A_i)\,P(B|A_i)}$$

Hence the theorem is proved.

Application of Baye's Theorem:-

(1) In answering a question on a multiple-choice test, an examinee either knows the answer (with probability p) or he guesses (with probability $1-p$). Let the probability of answering the question correctly be 1 for an examinee who knows the answer and $1/m$ for one who guesses (m being the number of multiple choice alternatives). Supposing an examinee answers a question correctly. What's the prob. that he really knows the answer?

Solution:- Let A_1 be the event that an examinee knows the answer
A_2 " " " " " guesses " "
Now B be the event that the answer is correct.

$P(A_1) = p$, $P(A_2) = 1-p$, since A_1, A_2 are m.e. & exhaustive events.
$P(B|A_1) = 1$ and $P(B|A_2) = \frac{1}{m}$.

By Bayes theorem, $$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1)+P(A_2)P(B|A_2)}$$

$$= \frac{p.1}{p.1+(1-p)\frac{1}{m}}$$

$$= \frac{mp}{1+(m-1)p}.$$

(2) There are two drawers in each of three boxes that are identical in appearence. The first one contains a gold coin in each drawer, the second contains a silver coin in each drawer, but the third contains a gold coin in one drawer and a silver coin in the other. A box is chosen, one of its drawers is opened and a gold coin is found. What's the probability that the other drawer too will have a gold coin?

Solution:- Let A_1 be the event that the first box is chosen.
A_2 " " " " " second " " ".
A_3 " " " " " third " " ".
and B be the event that the second one is a gold coin.

$P(B|A_1) = 1$, $P(B|A_2) = 0$, $P(B|A_3) = \frac{1}{2}$.

$P(A_1) = P(A_2) = P(A_3) = \frac{1}{3}$.

By Bayes' theorem,
$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{\sum_{i=1}^{3} P(A_i)P(B|A_i)}$$

$$= \frac{1/3 \times 1}{1/3\times 1 + 1/3 \times 0 + 1/3 \times \frac{1}{2}}$$

$$= \frac{2}{3}.$$

(3) In a doll factory, machines M_1, M_2 and M_3 manufacture respectively 45, 25 and 30 percent of the total output. Of their output, 6, 8, 3 percent respectively are defective. What is the probability that it was manufactured by M_1?

Sol. B_i : Chosen ball is manufactured by machine M_i
A : Chosen doll is defective

By Baye's theorem,

$$P(B_i|A) = \frac{P(B_i)\,P(A|B_i)}{\sum_{i=1}^{3} P(B_i)\,P(A|B_i)}, \quad i=1,2,3$$

$$P(M_1|A) = \frac{45\times 6}{45\times 6 + 25\times 8 + 30\times 3} = \frac{27}{56}.$$

(4) An urn containing 5 balls has been filled up by taking 5 balls from another urn containing 5W & 5B balls. A ball is taken at random from urn 1 and it happens to be black. What's the prob. of drawing a white ball from the remaining?

Sol. Let B_i denotes that among 5 balls kept in urn 1, exactly ith are white.
A : the first ball taken from Urn 1 is black.
C : the second ball drawn from Urn 1 is white.

$$P(C|A) = \frac{\sum_{i=0}^{4} P(B_i)\,P(A|B_i)\,P(C|A\cap B_i)}{\sum_{i=0}^{5} P(B_i)\,P(A|B_i)}, \text{ by Extended Baye's theorem.}$$

$$P(B_i) = \binom{5}{i}\binom{5}{5-i}\Big/\binom{10}{5}, \quad i=0(1)5$$

$$P(A|B_i) = \frac{5-i}{5}, \quad P(C|A\cap B_i) = \frac{i}{4}, \quad i=0(1)4.$$

i	$P(B_i)$	$P(A\|B_i)$	$P(C\|A\cap B_i)$
0	$1/\binom{10}{5}$	1	0
1	$25/\binom{10}{5}$	4/5	1/4
2	$100/\binom{10}{5}$	3/5	1/2
3	$100/\binom{10}{5}$	2/5	3/4
4	$25/\binom{10}{5}$	1/5	1
5	$1/\binom{10}{5}$	0	

$$\therefore P(C|A) = \frac{5/18}{1/2} = \frac{5}{9}.$$

Application of Total Probability Theorem:-

1. A box has 12 red and 6 black balls. A ball is selected from the box. If it is red, it is returned to box. If the ball is black, it and 2 additional balls are added to the box. Find the probability that a second ball drawn from the box is
(i) red (ii) black.

Sol. Let R_i and B_i respectively be the event that the i^{th} ball drawn is red and that the i^{th} ball drawn is black for $i = 1, 2$.

$$P(R_1) = \frac{12}{18}, \quad P(B_1) = \frac{6}{18}$$

$$P(R_2|R_1) = \frac{12}{18}, \quad P(R_2|B_1) = \frac{12}{20}$$

$$P(B_2|R_1) = \frac{6}{18}, \quad P(B_2|B_1) = \frac{8}{20}$$

(i) $P(R_2) = P(R_1)\,P(R_2|R_1) + P(B_1)\,P(R_2|B_1)$

$$= \frac{12}{18} \times \frac{12}{18} + \frac{6}{18} \times \frac{12}{20}$$

$$= \frac{29}{45}.$$

(ii) $P(B_2) = P(R_1)\,P(B_2|R_1) + P(B_1)\,P(B_2|B_1)$

$$= \frac{12}{18} \times \frac{6}{18} + \frac{6}{18} \times \frac{8}{20}$$

$$= \frac{16}{45}.$$

2. Let the probability p_n that a family has n children be αp^n when $n \geq 1$ and let $p_0 = 1 - \alpha p(1 + p + p^2 + \cdots)$. Suppose that a child is as likely to be a male as to be a female. Show that for $K \geq 1$, the prob. that a family contains exactly K boys is $\dfrac{2\alpha p^K}{(2-p)^{K+1}}$.

Sol. Let B_n denote the event that the family contains n children and A_K denote the event that it has K boys. The probability we require is $P(A_K)$.

Note that $B_n\ (n=0,1,2,\dots)$ are exhaustive as well as mutually exclusive. So we apply the theorem of total probability to get

$$P(A_K) = \sum_{n=0}^{\infty} P(B_n)\, P(A_K \mid B_n)$$

Given that $P(B_n) = \alpha p^n$, where $p_0 = 1 - \alpha p(1+p+p^2+\dots)$

$$P(A_K \mid B_n) = P(K \text{ boys} \mid n \text{ children}) = \begin{cases} \binom{n}{K}\left(\frac{1}{2}\right)^n & \text{for } n \geq K \\ 0 & \text{for } n < K \end{cases}$$

Hence by Total Probability theorem,

$$P(A_K) = \sum_{n=0}^{\infty} P(B_n)\, P(A_K \mid B_n)$$

$$= \sum_{n=K}^{\infty} P(B_n)\, P(A_K \mid B_n)$$

$$= \sum_{n=K}^{\infty} \alpha p^n \binom{n}{K}\left(\frac{1}{2}\right)^n$$

$$= \alpha\left(\frac{p}{2}\right)^K \sum_{n=K}^{\infty} \binom{n}{K}\left(\frac{p}{2}\right)^{n-K}$$

$$= \alpha\left(\frac{p}{2}\right)^K \left(1-\frac{p}{2}\right)^{-(K+1)}$$

$$= \frac{2\alpha p^K}{(2-p)^{K+1}}.$$

Reading Material: (Extra)

1. Outline of Statistics by Gun Gupta Dasgupta (All Exercises)
2. Problems in Probability by Gun Roy.
3. Probability theory by Hoel Port Stone.

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RANDOM VARIABLES

RANDOM VARIABLES

Variables

- Random/stochastic/Probabilistic
  - Discrete
  - Continuous
- Non-random/De-generate random variables

In any probability problem, we may associate with each outcome (elementary event) of the experiment of a finite real number. In many cases the outcome themselves are finite real numbers. This will be the case in tossing a die. In other cases, the numbers are artificially introduced. Thus for example, in tossing a coin thrice, the outcomes are not numbers but we may be interested in the number of heads obtained from the three tosses.

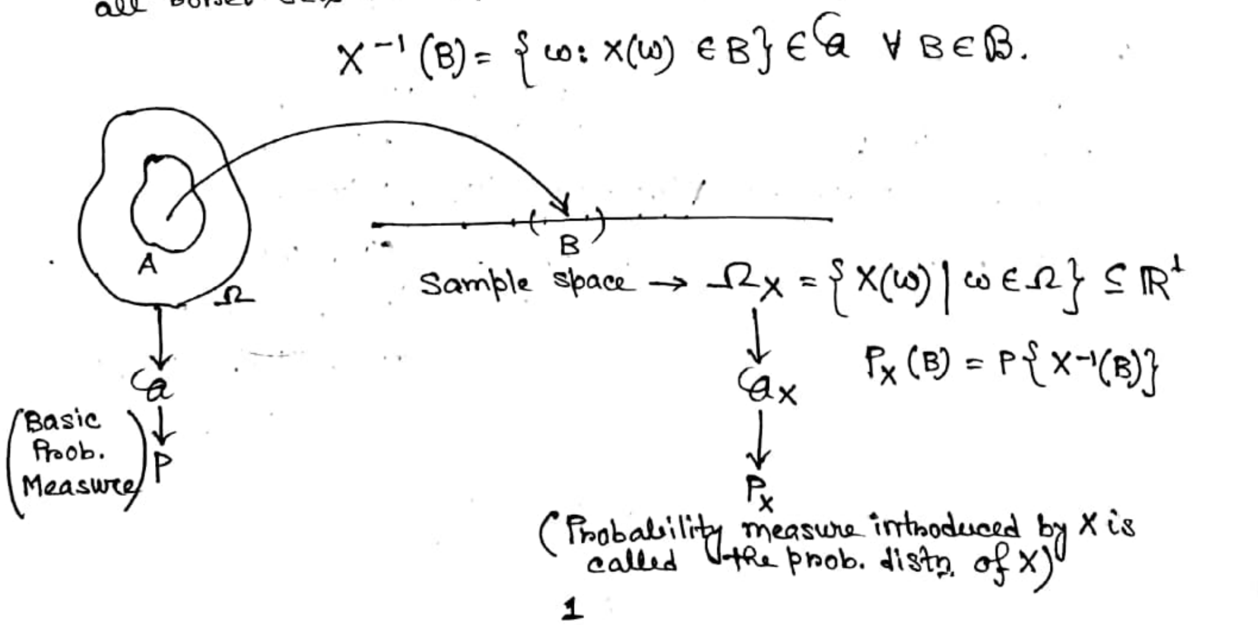
Definitions of Random Variables:-

(1) Let $(\Omega, \mathcal{A}, P)$ be a given probability space. Then a random variable is defined as a (Borel-measurable) function X w.r.t. $\mathcal{A}$, i.e., a random variable X is a function defined on the sample space Ω such that for every $x \in \mathbb{R}^1$, the inverse image $X^{-1}\{(-\infty, x]\} = \{\omega \mid -\infty < X(\omega) \le x\}$ of the Borel set $(-\infty, x]$ under X is measurable w.r.t. $\mathcal{A}$ (i.e. belongs to $\mathcal{A}$).

(2) Let $(\Omega, \mathcal{A}, P)$ be a sample space of a random experiment. A real valued function $X(\omega)$ defined on Ω is called a Random Variable if $\{\omega : X(\omega) \le x\} \in \mathcal{A} \ \forall x \in \mathbb{R}$.

(3) Let $(\Omega, \mathcal{A}, P)$ be a given probability space of a random experiment. A finite single-valued function X that maps Ω into $\mathbb{R}^1$ is called a random variable if the inverse image under X of all Borel sets in $\mathbb{R}^1$ are events, i.e. if

$$X^{-1}(B) = \{\omega : X(\omega) \in B\} \in \mathcal{A} \quad \forall B \in \mathcal{B}.$$



Although the induced probability measure $P_X(\cdot)$ characterises the distribution of probability for X but this is a self-dependent concept and therefore not easy to understand. Let us, therefore, see in the following has a pointwise characterization of the distribution of probability for X can be developed.

Let us consider the Borel set $(-\infty, x]$ for $x \in \mathbb{R}^1$ instead of B and also let X is a random variable defined on a given probability space $(\Omega, \mathcal{A}, P)$ introduces the probability measure $P_X(\cdot)$. Now since $\{\omega \mid -\infty < X(\omega) \leq x\} = X^{-1}\{(-\infty, x]\} \; \forall x \in \mathbb{R}^1$.

$$\therefore P_X\{(-\infty, x]\} = P[\omega \mid -\infty < X(\omega) \leq x] = F_X(x), \; x \in \mathbb{R}^1.$$

Thus, for varying values of $x \in \mathbb{R}^1$, the (point) function $F_X(x)$ characterizes the same as the (set) for $P_X\{(-\infty, x]\}$ does and accordingly is called the (cumulative) distribution function (d.f.) of the probability distribution of X.

Remark:- (1) The notation of probability doesn't enter into the definition of a random variable.

(2) If X is a random variable, the sets $\{X = x\}$, $\{a < X < b\}$, $\{X < x\}$, $\{a \leq X < b\}$, $\{a < X \leq b\}$, $\{a \leq X \leq b\}$, etc. are all events. Moreover, we could have used any of these events to define a r.v.

Example of R.V. :-

(1) Let E: tossing of a fair coin.
Then the sample space is: $\Omega = \{H, T\}$.
Let us define $X(H) = 1$, $X(T) = 0$. Then

$$X^{-1}(-\infty, x] = \{\omega : -\infty < X(\omega) \leq x\} = \begin{cases} \phi, & \text{if } x < 0 \\ \{T\}, & \text{if } 0 \leq x < 1 \\ \{H, T\}, & \text{if } 1 \leq x. \end{cases}$$

(2) Let E: tossing a coin twice.
Then the sample space is $\Omega = \{HH, TH, HT, TT\}$.
Define $X(\omega)$: the number of heads in ω, $\omega \in \Omega$.
Therefore, $X(HH) = 2$, $X(TH) = 1 = X(HT)$, $X(TT) = 0$.

$$X^{-1}(\infty, x] = \{\omega : -\infty < X(\omega) \leq x\} = \begin{cases} \phi, & x < 0 \\ \{TT\}, & 0 \leq x < 1 \\ \{TT\}, \{TH\}, \{HT\}, & 1 \leq x < 2 \\ \Omega, & x \geq 2. \end{cases}$$

Hence, $X(\omega)$ is a random variable defined on Ω.

(3) Let E: tossing a coin thrice.

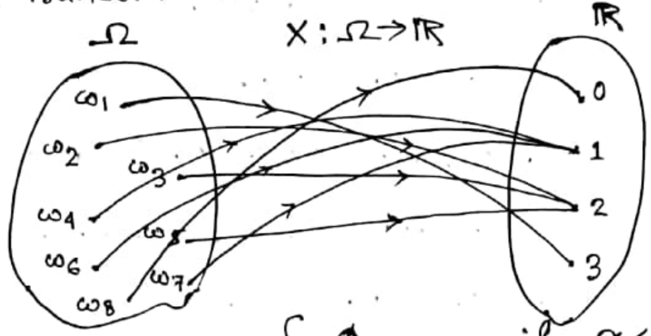
$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

Define $X(\omega)$: the number of heads in ω, $\omega \in \Omega$.

Then $X(HHH) = 3$, $X(HHT) = X(THH) = X(HTH) = 2$, $X(TTT) = 0$,

$X(HTT) = X(THT) = X(TTH) = 1$.

$\therefore$ X is a random variable with domain Ω and range $\{0,1,2,3\}$



Here, $X^{-1}(-\infty, x] = \begin{cases} \phi & \text{if } x < 0 \\ \{TTT\} & \text{if } 0 \le x < 1 \\ \{HTT, THT, TTH\} & \text{if } 1 \le x < 2 \\ \{HHT, HTH, THH\} & \text{if } 2 \le x < 3 \\ \{HHH\} & \text{if } 3 \le x < 4 \\ \Omega & \text{if } 4 \le x. \end{cases}$

Thus X is a random variable here.

Here Values of $X = \{3, 2, 2, 1, 2, 1, 1, 0\}$.

$$X(\omega_i) = \begin{cases} 0, & i = 8 \\ 1, & i = 4, 6, 7 \\ 2, & i = 2, 3, 5 \\ 3, & i = 1. \end{cases}$$

For any particular event $\{X \le 2.75\}$, the event space is

$\{HHT, HTT, HTH, THT, TTH, THH, TTT\}$.

If $\{0.5 \le X \le 1.72\}$, then event space $= \{HTT, THT, TTH\}$.

(4) Let E: a coin is tossed until a head appears.

X: Number of tosses required.

Here $\Omega = \{H, TH, TTH, \ldots\}$ and X assumes countably infinite number of values $1, 2, 3, \ldots$ with $X(\omega_1) = 1$, $X(\omega_2) = 2$, etc.

Here $X^{-1}(-\infty, x] = \begin{cases} \phi, & \text{if } x < 1 \\ \{H\}, & \text{if } 1 \le x < 2 \\ \{TH\}, & \text{if } 2 \le x < 3 \\ \{TTH\}, & \text{if } 3 \le x < 4 \\ \ldots\ldots \end{cases}$

Thus, X is a random variable.

Problem: 1. Let X be a random variable, then
(a) Is $|X|$ also a random variable?
(b) Is X^2 also a random variable?

Solution:- Let X be an r.v. defined on $(\Omega, \mathcal{A})$.
Then $\{\omega : X(\omega) \le x\} \in \mathcal{A} \ \forall \ x \in \mathbb{R}^1$.

(a) Now, $|X(\omega)|$ is a real valued function defined on $(\Omega, \mathcal{A})$,
$\{\omega : |X(\omega)| \le x\} = \phi$ if $x<0$, and

Note that
$$\{\omega : |X(\omega)| \le x\}$$
$$= \{\omega : -x \le X(\omega) \le x\} \text{ if } x \ge 0$$
$$= \{\omega : X(\omega) \le x\} \cap \{x : X(\omega) < -x\}^c.$$

Hence, $\{\omega : |X(\omega)| \le x\} \in \mathcal{A} \ \forall \ x$.
So, $|X|$ is also an r.v defined on $(\Omega, \mathcal{A})$.

(b) Clearly, $X^2(\omega)$ is a real valued function on $(\Omega, \mathcal{A})$.
Note that,
$$\{\omega : X^2(\omega) \le x\}$$
$$= \begin{cases} \phi & \text{if } x<0 \\ \{\omega : -\sqrt{x} \le X(\omega) \le \sqrt{x}\} & \text{if } x \ge 0 \end{cases}$$
$$= \begin{cases} \phi & \text{if } x<0 \\ \{\omega : X(\omega) \le \sqrt{x}\} \cap \{\omega : X(\omega) < \sqrt{-x}\}^c, & \text{if } x \ge 0 \end{cases}$$
$$\in \mathcal{A}$$

Hence, $X^2(\omega)$ is a random variable defined on $(\Omega, \mathcal{A})$.

Problem: 2. If $X(\omega)$ is a random variable on $(\Omega, \mathcal{A})$, then show that $cX(\omega)$ is also a random variable on $(\Omega, \mathcal{A})$.

Proof:- Let x be any arbitrary but fixed real number.
Then $(-\infty, x] \in \mathcal{B}$.
For $c>0$,
$$(cX)^{-1}(-\infty, x] = \{\omega : cX(\omega) \le x\} = \{\omega : X(\omega) \le \tfrac{x}{c}\}$$
$$= X^{-1}\left(-\infty, \frac{x}{c}\right] \in \mathcal{A}$$
$(\because X \text{ is an r.v.})$

So, $cX(\omega)$ is also a random variable.

⇨ DISTRIBUTION FUNCTION / CUMULATIVE DISTRIBUTION FUNCTION :—

Definition:- Let X be a random variable defined on $(\Omega, \mathcal{A}, P)$. Define a point function $F(\cdot)$ on $\mathbb{R}'$ by

$$F(x) = P\{\omega : X(\omega) \leq x\}$$, for all $x \in \mathbb{R}'$, is called the distribution function of R.V. X.

Properties:- (Alternative Definition*)

A real valued function $F(x)$ defined on $\mathbb{R}$ [or $(-\infty, \infty)$] which satisfies the following properties:

⟨i⟩ $x_1 < x_2 \Rightarrow F(x_1) \leq F(x_2) \ \forall \ x_1, x_2 \in \mathbb{R}'$.
i.e. $F(x)$ is monotonically non-decreasing.

⟨ii⟩ $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$

⟨iii⟩ $F(+\infty) = \lim_{x \to +\infty} F(x) = 1$.

⟨iv⟩ $F(x+0) = \lim_{h \to 0+} F(x+h) = F(x) \ \forall \ x \in \mathbb{R}'$.
i.e. $F(x)$ is right continuous, is called a distribution function of X.

Proof of the properties of distribution function:-

⟨i⟩ $x_1 < x_2$

$\Rightarrow \{X \leq x_1\} \subseteq \{X \leq x_2\}$

so, by the monotonicity theorem of probability,

$$P(X \leq x_1) \leq P(X \leq x_2)$$

i.e. $F(x_1) \leq F(x_2)$

⟨ii⟩ Let us take a sequence of events $B_n = \{X \leq -n\}$, $n = 1, 2, \ldots$

$\therefore B_n$ is a contracting sequence of events, i.e., monotonically decreasing. Hence, by continuity theorem,

$$\lim_{n\to\infty} P(B_n) = P(\lim_{n\to\infty} B_n)$$

$$\lim_{n\to\infty} P(X \leq -n) = P(\lim_{n\to\infty} \{X \leq -n\})$$

or, $\lim_{n\to\infty} P(X \leq -n) = P(\phi)$

or, $\lim_{n\to\infty} F(-n) = 0 \Rightarrow F(-\infty) = 0$.

⟨iii⟩ Let us take a sequence $A_n = \{X \leq n\}$

$\therefore A_n$ is an expanding sequence of events, i.e., monotonically increasing.

Hence, by continuity theorem,

$$\lim_{n\to\infty} P(A_n) = P(\lim_{n\to\infty} A_n)$$

$$\lim_{n\to\infty} P(X \leq n) = P(\lim_{n\to\infty} (X \leq n))$$

or, $\lim_{n\to\infty} P(X \leq n) = P(\Omega)$

or, $\lim_{n\to\infty} F(n) = 1$.

or, $F(\infty) = 1$.

(iv) Let us take a sequence of events $C_n = \{X \le x + \frac{1}{n}\}$, $n = 1, 2, \ldots$

$\therefore C_n$ is a contracting sequence of events, i.e., monotonically decreasing. Hence, by continuity theorem,

$$P(\lim_{n\to\infty} C_n) = \lim_{n\to\infty} P(C_n)$$

$$\therefore P\left(\lim_{n\to\infty}\{X \le x + \tfrac{1}{n}\}\right) = \lim_{n\to\infty} P\left(X \le x + \tfrac{1}{n}\right)$$

i.e. $P(X \le x) = \lim_{n\to\infty} P\left(X \le x + \frac{1}{n}\right)$

i.e. $F(x) = \lim_{n\to\infty} F\left(x + \frac{1}{n}\right)$

Take, $\frac{1}{n} = h$, as $n \to \infty$, $h \to 0$.

$\lim_{n\to\infty} F(x+h) = F(x)$ or, $F(x+0) = F(x)$.

Remark:- (1) $F(x)$ is not necessary continuous to the left.

Justification:- Define, $D_n = \{\omega : X(\omega) \le x - \frac{1}{n}\}$, $n \in \mathbb{N}$

Note that, $\lim_{n\to\infty} D_n = \lim_{n\to\infty}\{\omega : X(\omega) \le x - \frac{1}{n}\}$

$$= \{\omega : X(\omega) < x\}$$

By continuity theorem of probability,

$$\lim_{n\to\infty} P[D_n] = P[\lim_{n\to\infty} D_n]$$

$\Rightarrow \lim_{n\to\infty} P[\{\omega : X(\omega) \le x - \frac{1}{n}\}] = P[\{\omega : X(\omega) < x\}]$

$\Rightarrow \lim_{n\to\infty} F\left(x - \frac{1}{n}\right) = P[\{\omega : X(\omega) \le x\}] - P[\{\omega : X(\omega) = x\}]$

$\Rightarrow \lim_{h\to 0+} F(x-h) = F(x) - P[X = x]$

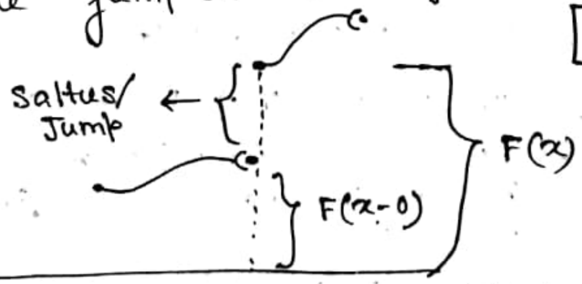
$\Rightarrow F(x) - F(x-0) = P[X = x] \ge 0$

Hence, $F(x-0)$ is not necessary equal to $F(x)$, i.e., $F(x)$ is not necessarily continuous to the left.

(2) Jump or Saltus of a distribution function:-

If $P[X = a] = 0$, then $F(a-0) = F(a)$ and $F(x)$ is continuous at $x = a$.

If $P[X = a] > 0$, then the quantity $F(a) - F(a-0) = P(X=a)$ is called the jump or saltus of the d.f. $F(x)$ at $x = a$.



[$F(x) - F(x-0)$
$= P[X \le x] - P[X \le x-0]$
$= P[X \le x] - P[X < x]$
$= P[X = x]$.]

If $P[X = a] > 0$, then $F(x)$ has discontinuity at $X = a$ with saltus $P[X = a]$. So that the jump of a distribution function F at $X = x$ equals to the probability mass situated or concentrated at $X = x$.

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(3) A necessary and sufficient condition for the r.v. X or its d.f. F to be continuous at $X=x$ is $P[X=x]=0$.

<u>Proof</u>:- Let $P[X=x]=0$

Then $F(x)-F(x-0)=0$

i.e. $F(x)=F(x-0)$ ....(1)

Further since, F is d.f., $\therefore F(x)=F(x+0)\ \forall\ x\in\mathbb{R}^1$. ....(2).

From (1) and (2), we have

$$F(x)=F(x-0)=F(x+0)$$

i.e. F is continuous at $X=x$.

<u>(Necessary)</u>:- F is continuous at $X=x$.

$\therefore F(x-0)=F(x)=F(x+0)$

$\Rightarrow F(x)-F(x-0)=0$

i.e. $P[X=x]=0$.

$\therefore$ The condition is necessary.

Ex.1. Let X be the R.V. denoting "the number of heads in tossing a fair coin thrice". Find the cdf.

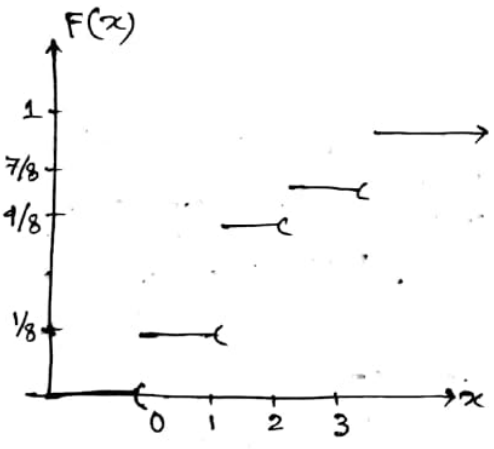
<u>Solution</u>:-

$$\Omega=\{\underset{\omega_1}{HHH},\ \underset{\omega_2}{HTT},\ \underset{\omega_3}{HHT},\ \underset{\omega_4}{HTH},\ \underset{\omega_5}{THH},\ \underset{\omega_6}{TTH},\ \underset{\omega_7}{THT},\ \underset{\omega_8}{TTT}\}$$

Note that
$$X(\omega_i)=\begin{cases}3 & \text{if } i=1\\ 2 & \text{if } i=3,4,5\\ 1 & \text{if } i=2,6,7\\ 0 & \text{if } i=8.\end{cases}$$

Since the coin is fair, hence $P[\omega_i]=\frac{1}{8}$, $\omega_i\in\Omega$.

The CDF of X is $F(x)=P[\{\omega: X(\omega)\le x\}]$



$$=\begin{cases}P[\phi] & \text{if } x<0\\ P[\{TTT\}] & \text{if } 0\le x<1\\ P[\{HTT,THT,TTH,TTT\}] & \text{if } 1\le x<2\\ P[\Omega-\{HHH\}] & \text{if } 2\le x<3\\ P[\Omega] & \text{if } x\ge 3.\end{cases}$$

$$=\begin{cases}0 & \text{if } x<0\\ 1/8 & \text{if } 0\le x<1\\ 4/8 & \text{if } 1\le x<2\\ 7/8 & \text{if } 2\le x<3\\ 1 & \text{if } x\ge 3\end{cases}$$

<u>Note</u>:- The set of values of X together with their corresponding probabilities is called the d.f. of X.

Ex.2. Suppose that 3 cards are drawn from a deck of 52 cards, one by one, at random and with replacement. Let X be the number of spades drawn; then X is a random variable. If an outcome of spades is denoted by s, and other outcomes are represented by t, then X is a real-valued function defined on the sample space

$$\Omega = \{\underset{\omega_1}{sss}, \underset{\omega_2}{tss}, \underset{\omega_3}{sts}, \underset{\omega_4}{sst}, \underset{\omega_5}{stt}, \underset{\omega_6}{tst}, \underset{\omega_7}{tts}, \underset{\omega_8}{ttt}\}$$

Here $X(\omega_i) = \begin{cases} 0 & \text{if } i=8 \\ 1 & \text{if } i=5,6,7 \\ 2 & \text{if } i=2,3,4 \\ 3 & \text{if } i=1. \end{cases}$

Now, we must determine the values that X assumes and the probabilities that are associated with them. Clearly, X can take values $0,1,2,3$. The probabilities associated with these values are calculated as follows:

$$P(X=0) = P(ttt) = \frac{39}{52} \times \frac{39}{52} \times \frac{39}{52} = \frac{27}{64}.$$

$$P(X=1) = P\{(stt), (tst), (tts)\}$$
$$= 3\left(\frac{1}{4} \times \frac{3}{4} \times \frac{3}{4}\right) = \frac{27}{64}.$$

$$P(X=2) = P\{(sst), (sts), (tss)\}$$
$$= 3\left(\frac{1}{4} \times \frac{1}{4} \times \frac{3}{4}\right) = \frac{9}{64}.$$

$$P(X=3) = P(sss)$$
$$= \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{64}.$$

If the cards are drawn without replacement, the probabilities associated with the values $0,1,2$ and 3 are

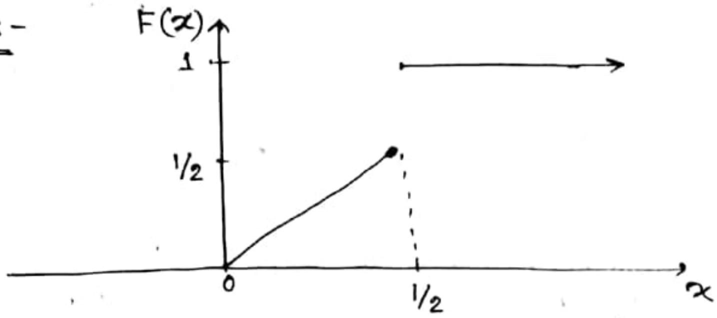
$$P(X=0) = \frac{\binom{39}{3}}{\binom{52}{3}}, \quad P(X=1) = \frac{\binom{13}{1}\binom{39}{2}}{\binom{52}{3}}, \quad P(X=2) = \frac{\binom{13}{2}\binom{39}{1}}{\binom{52}{3}},$$

$$P(X=3) = \frac{\binom{13}{3}}{\binom{52}{3}}, \quad P(X=i) = \frac{\binom{13}{i}\binom{39}{3-i}}{\binom{52}{3}}, \quad i=0,1,2,3.$$

Problem:- 1. Check whether the following function are distribution function or not:

$$F(x) = \begin{cases} 0 & \text{if } x<0 \\ x & \text{if } 0 \le x < 1/2 \\ 1 & \text{if } x \geq 1/2 \end{cases}$$

Solution:-



(i) From the graph it is clear that the function is non-decreasing.

(ii) $F(-\infty) = \lim_{x\to-\infty} F(x) = \lim_{x\to-\infty}(0) = 0$.

(iii) $F(\infty) = \lim_{x\to\infty} F(x) = \lim_{x\to\infty}(1) = 1$.

(iv) $\lim_{h\to 0} F(0+h) = \lim_{h\to 0} h = 0 = F(0)$

$\lim_{h\to 0} F\left(\frac{1}{2}+h\right) = \lim_{h\to 0}(1) = 1 = F(1/2)$

So, $F(x)$ is right - continuous.

So, $F(x)$ is a cdf here..

Problem 2. Is the following function cdf or not?

$$F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{x^2}{2} & \text{if } 0 < x \le 1 \\ \frac{1}{2} + \frac{(x-1)^3}{3} & \text{if } 1 < x \le 2 \\ \frac{6}{7} + \frac{1}{7}(x-2)^4 & \text{if } 2 < x \le 3 \\ 1 & \text{if } x > 3. \end{cases}$$

Solution:- (i) It is non-decreasing.

(ii) $F(-\infty) = 0$ if $x \le 0$ (iii) $F(\infty) = 1$ if $x > 3$.

(iv) $F(x+0) = \lim_{h\to 0+} F(0+h) = \lim_{h\to 0+} \frac{h^2}{2} = 0 = F(0)$.

$\Rightarrow F(x)$ is continuous to the right at $x=0$.

$F(1+0) = \lim_{h\to 0+} F(1+h) = \lim_{h\to 0+}\left\{\frac{1}{2} + \frac{h^3}{3}\right\} = \frac{1}{2} = F(1)$.

$\Rightarrow F(x)$ is continuous to the right at $x=1$.

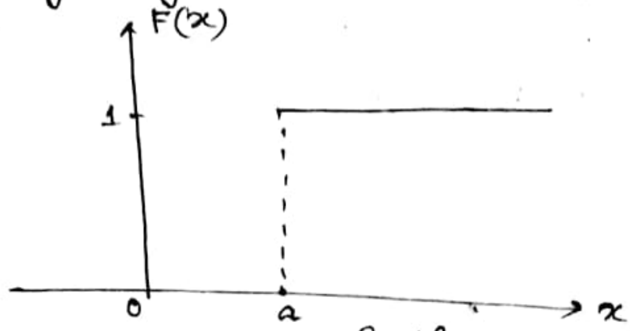
$F(2+0) = \operatorname{lt}_{h\to 0+} F(2+h) = \lim_{h\to 0+}\left\{\frac{6}{7} + \frac{h^4}{7}\right\} = \frac{6}{7} \neq \frac{5}{6} = F(2)$.

$\Rightarrow F(x)$ is not right continuous at $x=2$.

$\therefore F(x)$ can't be a CDF.

Problem:- 3. The random variable X assumes the value 'a' with probability unity. Sketch its d.f.

Solution:-



-: Graph of cdf :-

$P(X=a)=1$

$P(X<a)=0$

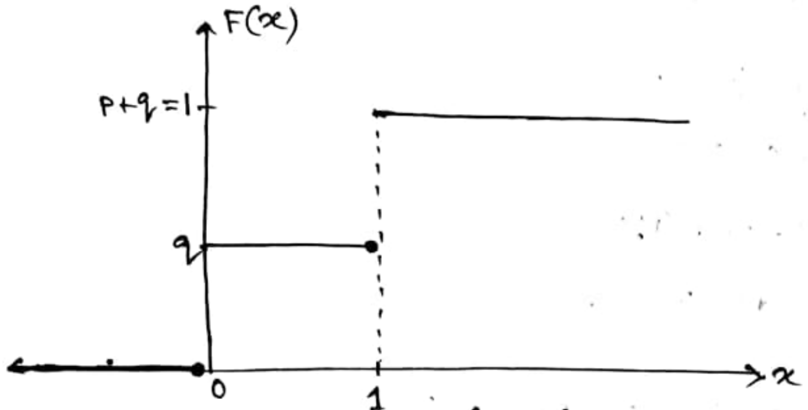
$P(X\le a)=1$

$P(X\le x)=1\ \forall\, x\ge a$

$$F(x)=\begin{cases}0 & \text{if } x<a\\ 1 & \text{if } x\ge a\end{cases}$$

Problem:- 4 The random variable X assumes the value 1 with probability p and the value 0 with probability q. Sketch the d.f.

Solution:-



-: Graph of cdf :-

$P[X=0]=q$

$P[X=1]=p.$

So, $P(X\le x)=F(x)$

$$F(x)=\begin{cases}0, & \text{if } x<0\\ q, & \text{if } 0\le x<1\\ 1, & \text{if } x\ge 1\end{cases}$$

Since $F(0)=P(X\le 0)$

$=q.$

$F(1)=P(X\le 1)$

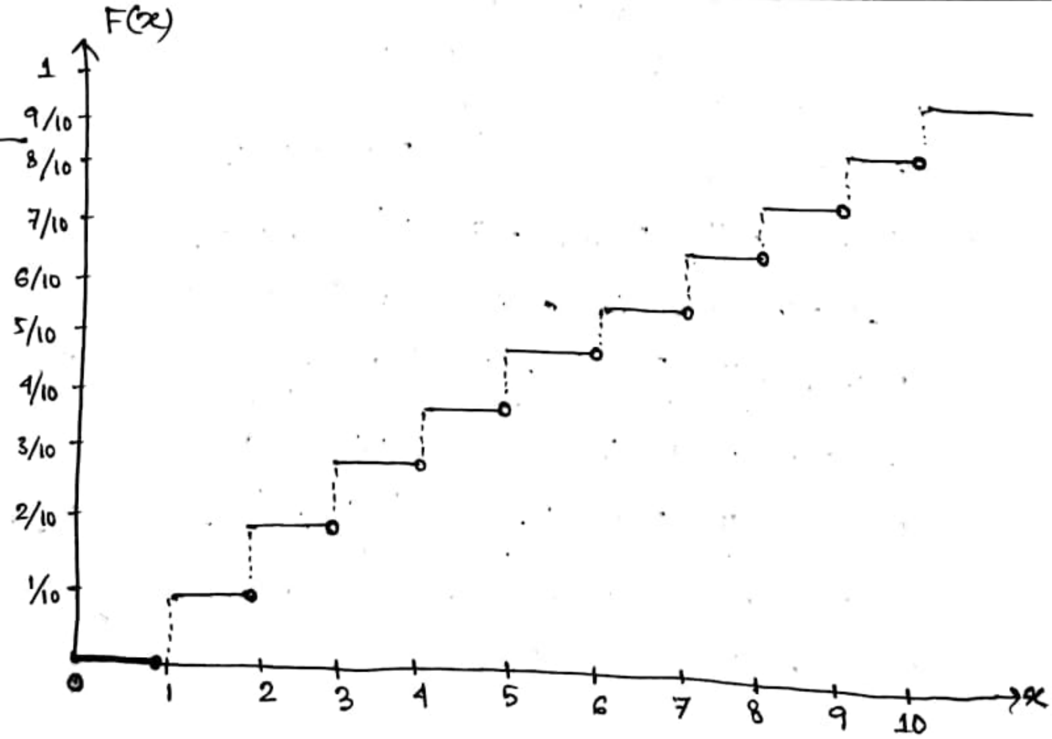
$=P(X=0)+P(X=1)$

$=p+q=1.$

Problem:- 5. A whole number is chosen at random between 1 and 10. Sketch the df of the related random variable.

Solution:-

x	$P(X=x)$	$F(x)$
1	1/10	1/10
2	1/10	2/10
3	1/10	3/10
4	1/10	4/10
⋮	⋮	⋮
10	1/10	1



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Event concerning X	Probability of the event in terms of F	Event Concerning X	Probability of the event in terms of F
$X \le a$	$F(a)$	$a < X \le b$	$F(b) - F(a)$
$X > a$	$1 - F(a)$	$a < X < b$	$F(b^-) - F(a)$
$X < a$	$F(a^-)$	$a \le X \le b$	$F(b) - F(a^-)$
$X \geqslant a$	$1 - F(a^-)$	$a \le X < b$	$F(b^-) - F(a^-)$
$X = a$	$F(a) - F(a^-)$		

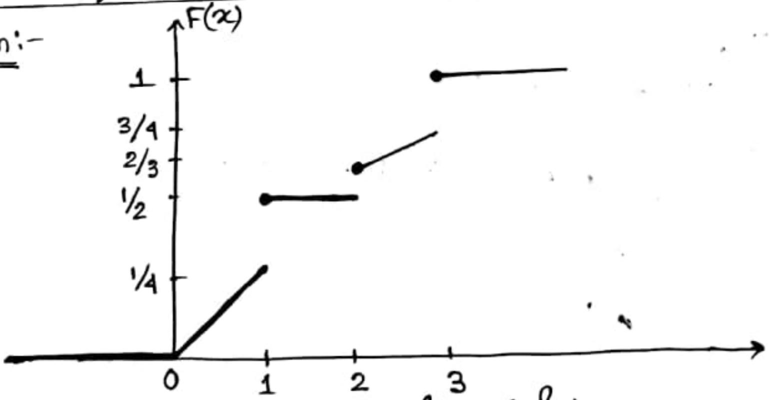
Ex. 1. The d.f. of an r.v. X is given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x/4 & \text{if } 0 \le x < 1 \\ 1/2 & \text{if } 1 \le x < 2 \\ \frac{1}{12}x + \frac{1}{2} & \text{if } 2 \le x < 3 \\ 1 & \text{if } x \geqslant 3 \end{cases}$$

Sketch its d.f. and compute the followings:

(a) $P(X < 2)$, (b) $P(X = 2)$, (c) $P(1 \le X < 3)$,
(d) $P(X > 3/2)$, (e) $P(X = \frac{5}{2})$, (f) $P(2 < X \le 7)$.

Solution:-



—: sketch of the d.f. :—

(a) $P(X < 2) = F(2^-) = 1/2$

(b) $P(X = 2) = F(2) - F(2^-) = \left(\frac{2}{12} + \frac{1}{2}\right) - \frac{1}{2} = \frac{1}{6}$.

(c) $P(1 \le X < 3) = P(X < 3) - P(X < 1)$
$= F(3^-) - F(1^-)$
$= (3/12 + 1/2) - 1/4 = 1/2$.

(d) $P(X > 3/2) = 1 - F(3/2) = 1 - 1/2 = 1/2$.

(e) $P(X = 5/2) = 0$, since F is cont. at $\frac{5}{2}$.

(f) $P(2 < X \le 7) = F(7) - F(2)$
$= 1 - (2/12 + 1/2)$
$= 1/3$.

Ex.2. For the experiment of flipping a fair coin twice, let X be the number of tails and calculate $F(t)$, the distn. function of X, and then sketch its graph.

Solution:- Since, X assumes only the values 0, 1 and 2.

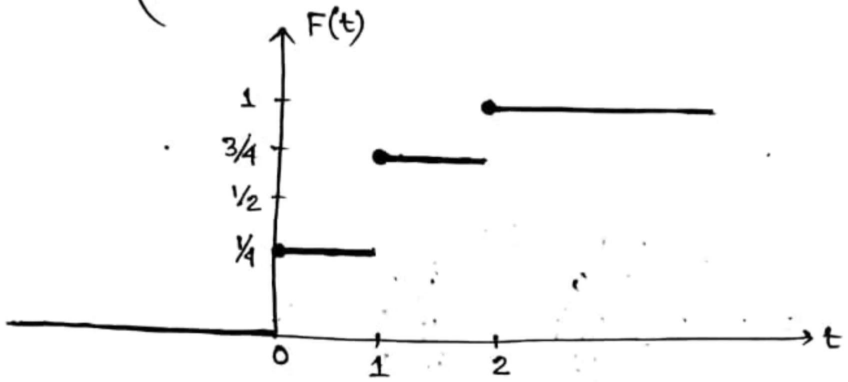
We have $F(t) = P(X \le t) = 0$, if $t < 0$.

$$= P(X=0) = P(HH) = \frac{1}{4}, \text{ if } 0 \le t < 1$$

$$= P(X=0 \text{ or } X=1) = P\{(HH, HT, TH)\} = \frac{3}{4}, \text{ if } 1 \le t < 2.$$

$$= 1, \text{ if } t \geq 2.$$

$$F(t) = \begin{cases} 0 & t < 0 \\ 1/4 & 0 \le t < 1 \\ 3/4 & 1 \le t < 2 \\ 1 & t \geq 2 \end{cases}$$



Sketch of the distribution function (F(t))

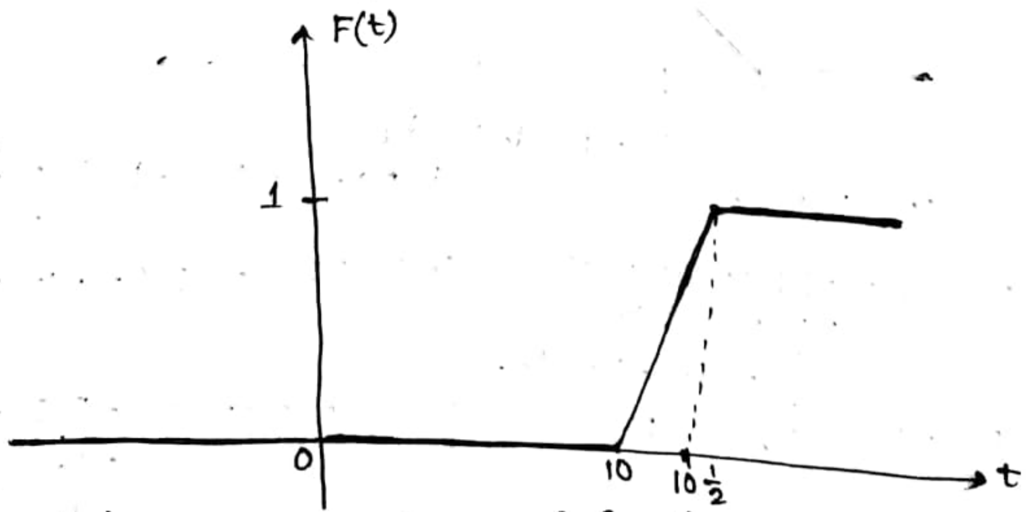
Ex.3. Suppose that a bus arrives at a station everyday between 10.00 AM and 10.30 AM at random. Let X be the arrival time; find the d.f. of X and sketch its graph.

Solution:- The bus arrives at the station at random between 10 and 10.30 AM.

$$F(t) = P(X \le t) = 0, \text{ if } t \le 10$$

$$= \frac{t-10}{10\frac{1}{2} - 10} = 2(t-10); \text{ if } 10 < t < 10\frac{1}{2}$$

$$= 1, \text{ if } t \geq 10\frac{1}{2}$$



-: sketch of the d.f. (F(t)):-

Ex. 4. The sales of a convenience store on a randomly selected day are X thousand dollars, where X is a random variable with a d.f. of the following form:

$$F(t) = \begin{cases} 0 & \text{if } t<0 \\ \frac{1}{2}t^2 & \text{if } 0 \le t < 1 \\ K(4t - t^2) & \text{if } 1 \le t < 2 \\ 1 & \text{if } t \geq 2. \end{cases}$$

Suppose that this convenience store's total sales on any given day are less than \$2000.

(a) Find the value of K.

(b) Let A and B be the events that tomorrow the store's total sales are between 500 and 1500 dollars, and over 1000 dollars, respectively. Find P(A) and P(B).

(c) Are A and B independent events?

Solution:-

(a) Since $X < 2$, we have that $P(X<2) = 1$,

So, $F(2^-) = 1$

$\Rightarrow K(4t - t^2) = 1$, at $t = 2$, we have

$K(8-4) = 1 \Rightarrow K = \frac{1}{4}$.

(b) $P(A) = P\left(\frac{1}{2} \le X \le \frac{3}{2}\right) = F\left(\frac{3}{2}\right) - F\left(\frac{1}{2}^-\right)$

$= F\left(\frac{3}{2}\right) - F\left(\frac{1}{2}\right)$

$= \frac{15}{16} - \frac{1}{8}$

$= \frac{13}{16}$.

$P(B) = P(X > 1) = 1 - F(1) = 1 - P[X \le 1]$

$= 1 - \frac{3}{4}$

$= \frac{1}{4}$.

(c) $P(A \cap B) = P\left(1 < X \le \frac{3}{2}\right) = F\left(\frac{3}{2}\right) - F(1)$

$= \frac{15}{16} - \frac{3}{4}$

$= \frac{3}{16}$

$P(A \cap B) \neq P(A) \cdot P(B)$.

So, A and B are not independent.

Note:- The RV's here in neither discrete nor continuous type.

Ex. 5. Suppose $G(x) = \begin{cases} 0 & \text{if } x < -1 \\ 1/2 & \text{if } -1 \le x \le 0 \\ a + b e^{-x^2/2} & \text{if } x > 0 \end{cases}$.

Determine the values of a and b so that $G(x)$ is a distribution function.

Solution:- To be a d.f., $G(x)$ needs to satisfy four properties.

(i) $G(-\infty) = 0$

(ii) $G(\infty) = 1 \Rightarrow \lim_{x \to \infty} \left(a + b e^{-x^2/2}\right) = 1$

$\Rightarrow a + b.0 = 1 \Rightarrow a = 1.$

(iii) $G(x)$ is non-decreasing.

(iv) $G(x)$ is right continuous. So, we have

$$G(0) = \frac{1}{2} = \lim_{h \to 0+} G(0+h)$$

$$= \lim_{h \to 0+} \left(a + b e^{-h^2/2}\right)$$

$$= a + b$$

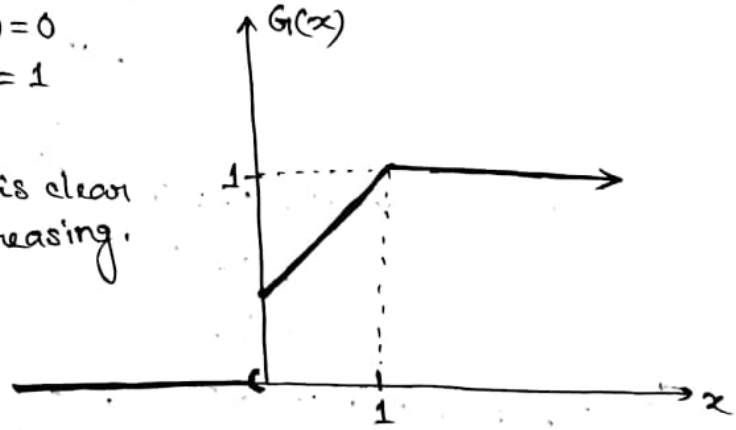
So, $b = -1/2$.

Ex. 6. Verify whether the following function $G(x)$ is a c.d.f. or not:

$$G(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/2 & \text{if } x = 0 \\ 1/2 + x/2 & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

Solution:- (i) $G(-\infty) = 0$

(ii) $G(\infty) = 1$



(iii) From Graph it is clear that $G(x)$ is increasing.

(iv) $G(x+0) = \lim_{h \to 0+} F(0+h)$

$$= \lim_{h \to 0} \left(\frac{1}{2} + \frac{h}{2}\right)$$

$$= \frac{1}{2} = G(0)$$

$G(x+1) = \lim_{h \to 0} G(1+h)$

$$= \lim_{h \to 0} (1)$$

$$= 1 = G(1).$$

$\therefore$ $G(x)$ is right continuous.

So, $G(x)$ is a c.d.f.

Ex. 7. Let F_1 and F_2 be two d.f.s. If a and b are non-negative integers whose sum is unity then show that $aF_1 + bF_2$ is also d.f.s.

Solution:-

(i) $F = aF_1 + bF_2$.

Let $x_1 < x_2$

Then since F_1 and F_2 are d.f.s, so, we have

$F_1(x_1) \leq F_1(x_2)$ & $F_2(x_1) \leq F_2(x_2)$.

Since a and b are non-negative integers, so

$$aF_1(x_1) + bF_2(x_1) \leq aF_1(x_2) + bF_2(x_2)$$

$$\Rightarrow (aF_1 + bF_2)(x_1) \leq (aF_1 + bF_2)(x_2)$$

$\Rightarrow F(x_1) \leq F(x_2)$. So, F is non-decreasing.

(ii) $F_1(-\infty) = 0$, $F_2(-\infty) = 0$.

$$F(-\infty) = (aF_1 + bF_2)(-\infty)$$
$$= aF_1(-\infty) + bF_2(-\infty)$$
$$= 0.$$

(iii) $F_1(\infty) = 1$, $F_2(\infty) = 1$.

$$F(\infty) = (aF_1 + bF_2)(\infty)$$
$$= aF_1(\infty) + bF_2(\infty)$$
$$= a + b = 1.$$

(iv) Now, $F_1(x+0) = \lim_{h \to 0} F_1(x+h) = F_1(x) \ \forall x$.

And, $F_2(x+0) = \lim_{h \to 0} F_2(x+h) = F_2(x) \ \forall x$.

Now, $F(x+0) = \lim_{h \to 0} (aF_1 + bF_2)(x+h)$

$$= \lim_{h \to 0} aF_1(x+h) + \lim_{h \to 0} bF_2(x+h)$$
$$= aF_1(x) + bF_2(x)$$
$$= F(x) \ \forall x.$$

Hence, $F(x)$ is right continuous.

So, F is a d.f.

Ex. 8. $F(x)$ is a d.f. Then show that $G(x)$ is also a d.f., where

$$G(x) = \left[\,1-(1-F(x))^n\,\right], \quad n \in \mathbb{N}.$$

Solution:- (i) Let $x<y$

$$F(x) \leq F(y)$$

$$\Rightarrow 1-F(x) \geq 1-F(y)$$

$$\Rightarrow (1-F(x))^n \geq (1-F(y))^n$$

$$\Rightarrow 1-(1-F(x))^n \leq 1-(1-F(y))^n$$

$$\Rightarrow G(x) \leq G(y).$$

(ii) $G(-\infty) = \lim_{x\to-\infty} G(x)$

$$= \lim_{x\to-\infty} \left\{1-(1-F(x))^n\right\}$$

$$= 1-\lim_{x\to-\infty}(1-F(x))^n$$

$$= 1-\left\{\lim_{x\to-\infty}(1-F(x))\right\}^n$$

$$= 1-\left(1-\lim_{x\to-\infty}F(x)\right)^n$$

$$= 1-(1-F(-\infty))^n$$

$$= 1-1 = 0.$$

(iii) $G(\infty) = \lim_{x\to\infty} G(x)$

$$= \lim_{x\to\infty}\left\{1-(1-F(x))^n\right\}$$

$$= 1-\left(1-\lim_{x\to\infty}F(x)\right)^n$$

$$= 1-(1-F(\infty))^n$$

$$= 1.$$

(iv) $\lim_{h\to 0} G(x+h)$

$$= \lim_{h\to 0}\left\{1-(1-F(x+h))^n\right\}$$

$$= 1-\left(1-\lim_{h\to 0}F(x+h)\right)^n$$

$$= 1-(1-F(x))^n$$

$$= G(x).$$

$\therefore$ $G(x)$ is also a d.f.

Ex. 9. Show that every d.f. F has the following properties:

(a) $\lim_{x\to\infty} x\int_x^\infty \frac{1}{z}\,dF(z)=0$ (b) $\lim_{x\to+0} x\int_x^\infty \frac{1}{z}\,dF(z)=0$

(c) $\lim_{x\to-\infty} x\int_{-\infty}^x \frac{1}{z}\,dF(z)=0$ (d) $\lim_{x\to-0} x\int_{-\infty}^x \frac{1}{z}\,dF(z)=0$

Solution:-

(a) $x>0$

$$0\le x\int_x^\infty \frac{1}{z}\,dF(z)\le x\int_x^\infty \frac{1}{x}\,dF(z)$$

$\left[\begin{array}{l} x<z<\infty \\ 0<\frac{1}{z}<\frac{1}{x}\end{array}\right]$

$$= x\cdot\frac{1}{x}\int_x^\infty dF(z)$$

$$=1-F(x).$$

Take limit both sides,

$$0\le \lim_{x\to\infty} x\int_x^\infty \frac{1}{z}\,dF(z)\le \lim_{x\to\infty}\left(1-F(x)\right)$$

$$=0.$$

$$\therefore \lim_{x\to\infty} x\int_x^\infty \frac{1}{z}\,dF(z)=0.$$

(b) Let x be a positive proper fraction.

$$0\le x\int_x^\infty \frac{1}{z}\,dF(z)=x\int_x^{\sqrt{x}}\frac{1}{z}\,dF(z)+x\int_{\sqrt{x}}^\infty \frac{1}{z}\,dF(z)$$

For, $x\to 0+$, first part $\to 0$.

$\sqrt{x}<z<\infty \Rightarrow 0<\frac{1}{z}<\frac{1}{\sqrt{x}}$.

$$0\le x\int_x^\infty \frac{1}{z}\,dF(z)\le x\int_{\sqrt{x}}^\infty \frac{1}{z}\,dF(z)$$

$$\le x\int_{\sqrt{x}}^\infty \frac{1}{\sqrt{x}}\,dF(z)$$

$$\le \sqrt{x}\quad \left[\because 1-F(\sqrt{x})<1\right]$$

$$\therefore \lim_{x\to 0+} x\int_x^\infty \frac{1}{z}\,dF(z)=0.$$

(c) $x<0$ and $0\le z\le x$.

$\therefore$ z and x are both negative so

$$\frac{x}{z}\le 1.$$

$$0\le x\int_{-\infty}^x \frac{1}{z}\,dF(z)\le \int_{-\infty}^x dF(z)=F(x)$$

$$\therefore 0\le \lim_{x\to-\infty} x\int_{-\infty}^x \frac{1}{z}\,dF(z)\le \lim_{x\to-\infty} F(x)$$

$$=0.$$

$$\therefore \lim_{x\to-\infty} x\int_{-\infty}^x \frac{1}{z}\,dF(z)=0.$$

(d) Taking x to be a negative proper fraction.

$$0\ge x\int_{-\infty}^x \frac{1}{z}\,dF(z)$$

$$=x\int_{-\infty}^{\sqrt{x}}\frac{1}{z}\,dF(z)+x\int_{\sqrt{x}}^x \frac{1}{z}\,dF(z)$$

Ex. 10. Show that if F is d.f., then G and H defined, for some $h>0$, by

(a) $G(x) = \frac{1}{h}\int_x^{x+h} F(u)du$, (b) $H(x) = \frac{1}{2h}\int_{x-h}^{x+h} F(u)du$ are d.f.s.

Solution:- (a) ⟨i⟩ For $x_1 < x_2$

$$F(x_1+u) \le F(x_2+u)$$

$$\Rightarrow \int_0^h F(x_1+u)du \le \int_0^h F(x_2+u)du$$

$$\Rightarrow \frac{1}{h}\int_{x_1}^{x_1+h} F(u)\,du \le \frac{1}{h}\int_{x_2}^{x_2+h} F(u)du$$

$$\Rightarrow G(x_1) \le G(x_2)$$

$\therefore$ G is non-decreasing.

⟨ii⟩ & ⟨iii⟩

$$G(x) = \frac{1}{h}\int_x^{x+h} F(u)du \le \frac{1}{h}\int_x^{x+h} F(x+h)du$$

$$= F(x+h) \geq F(x)$$

and similarly, $G(x) = \frac{1}{h}\int_x^{x+h} F(u)du \geq F(x)$

$$F(x) \le G(x) \le F(x+h)$$

$$\therefore 0 = \lim_{x\to-\infty} F(x) \le \lim_{x\to-\infty} G(x) \le \lim_{x\to-\infty} F(x+h) = 0$$

$$\Rightarrow \lim_{x\to-\infty} G(x) = 0 \Rightarrow G(-\infty) = 0.$$

And $1 = \lim_{x\to\infty} F(x) \le \lim_{x\to\infty} G(x) \le \lim_{x\to\infty} F(x+h) = 1$

$$\Rightarrow \lim_{x\to\infty} G(x) = 1 \Rightarrow G(\infty) = 1.$$

(iv) $\lim_{K\to0+} G(x+K) = \lim_{K\to0+} \frac{1}{h}\int_{x+K}^{x+K+h} F(u)du$

$$= \frac{1}{h}\lim_{K\to0+}\int_x^{x+h} F(u+K)du$$

$$= \frac{1}{h}\int_x^{x+h} \lim_{K\to0+} F(u+K)\,du$$

[Interchanging of limit & integration sign is possible since $F(x)$ is bounded]

$$= \frac{1}{h}\int_x^{x+h} F(u)du = G(x).$$

So, $G(x)$ is also a d.f.

(b) $$H(x) = \frac{1}{2h}\int_{x-h}^{x+h} F(u)\,du = \frac{1}{2h}\int_{-h}^{h} F(x+t) \quad [\, u \to t \ni u = x+t \,]$$

(i) $x < y$

$$F(x) \le F(y) \Rightarrow F(x+t) \le F(y+t) \quad \forall t$$

$$\Rightarrow \frac{1}{2h}\int_{-h}^{h} F(x+t)\,dt \le \frac{1}{2h}\int_{-h}^{h} F(y+t)\,dt$$

$$\Rightarrow H(x) \le H(y).$$

(ii) $$H(-\infty) = \lim_{u\to-\infty} H(u)$$

$$= \lim_{x\to-\infty} \frac{1}{2h}\int_{-h}^{h} F(x+t)\,dt = \frac{1}{2h}\int_{-h}^{h} \lim_{x\to-\infty} F(x+t)\,dt$$

$$= \frac{1}{2h}\int_{-h}^{h} F(-\infty)\,dt = 0 \text{ as } F(-\infty) = 0.$$

(iii) $$H(\infty) = \lim_{u\to\infty} H(u) = \lim_{x\to\infty} \frac{1}{2h}\int_{-h}^{h} F(x+t)\,dt$$

$$= \frac{1}{2h}\int_{-h}^{h} \lim_{x\to\infty} F(x+t)\,dt = \frac{1}{2h}\int_{-h}^{h} F(\infty)\,dt$$

$$= 1. \quad [\because F(\infty) = 1]$$

(iv) $$\lim_{K\to0} H(x+K) = \lim_{K\to0} \frac{1}{2h}\int_{-h}^{h} F(x+K+t)\,dt = \lim_{K\to0}\int_{x+K-h}^{x+K+h} F(u)\,du$$

$$= \frac{1}{2h}\int_{-h}^{h} \lim_{K\to0} F(x+K+t)\,dt$$

$$= \frac{1}{2h}\int_{-h}^{h} F(x+t)\,dt$$

$$= G(x).$$

$\therefore$ $G(x)$ is also a d.f.

Ex. 10. If X_1 and X_2 are independently and identically distributed r.v.s. Prove that —

(i) $P\{|X_1 - X_2| > t\} \le 2P\{|X_1| > t/2\}$.

(ii) If $a \geq 0$ such that $P(X_1 \geq a) \le 1-p$, $P(X_1 \le -a) \le 1-p$, then $P\{|X_1 - X_2| \geq t\} \geq p \cdot P\{|X_1| > a+t\}$, for $t > 0$.

Solution:- (i)

$$\left\{|X_1| > \frac{t}{2}\right\} \cup \left\{|X_2| > \frac{t}{2}\right\} \supset \{|X_1 - X_2| > t\}$$

$$\therefore P\{|X_1 - X_2| > t\} \le P\left\{|X_1| > \frac{t}{2}\right\} + P\left\{|X_2| > \frac{t}{2}\right\}$$

$$= 2P\left\{|X_1| > \frac{t}{2}\right\}.$$

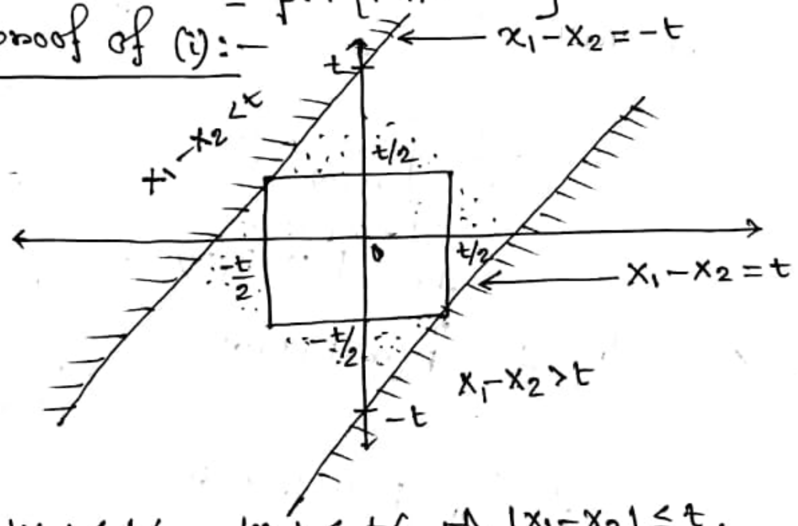
(ii) $\{X_1 \geq a+t, X_2 \le a\} \cup \{X_1 \le -a-t, X_2 \geq -a\} \subset \{|X_1 - X_2| \geq t\}$

$$\therefore P\{|X_1 - X_2| \geq t\} \geq P\{X_1 > a+t, X_2 \le a\} + P\{X_1 < -a-t, X_2 \geq -a\}$$

$$\geq p \cdot P\{X_1 \geq a+t\} + p \cdot P\{X_1 \le -a-t\}$$

$$= p \cdot P\{|X_1| > a+t\}.$$

Alternative proof of (i):-



Note that, $|X_1| \le t/2$, $|X_2| \le t/2 \Rightarrow |X_1 - X_2| \le t$.

$$\therefore \{|X_1| \le t/2, |X_2| \le t/2\} \subseteq \{|X_1 - X_2| \le t\}$$

From the monotonicity theorem of Probability,

$$P(|X_1| \le t/2, |X_2| \le t/2) \le P(|X_1 - X_2| \le t)$$

$$\Rightarrow P(|X_1 - X_2| > t) \le P(|X_1| > t/2 \text{ or } |X_2| > t/2)$$

$$\le P(|X_1| > t/2) + P(|X_2| > t/2)$$

[From Boole's Inequality]

But X and Y are i.i.d. r.v.s, so $P[|X_1| > t/2] = P[|X_2| > t/2]$

So, $P(|X_1 - X_2| > t) \le 2P(|X_1| > t/2)$.

DISCRETE & CONTINUOUS RANDOM VARIABLES

A. Discrete Random Variable :-

Definition:— A random variable X takes only a countable (finite or infinite) number of isolated values $x_1, x_2, \ldots, x_n, \ldots$ with $P[X=x_i]>0$ $\forall i$, is called a discrete random variable.

The points $x_1, x_2, \ldots$ that have positive probabilities of occurance are called the jump or mass points of the r.v. X.

Probability Mass Function:— (PMF) Let X be a discrete R.V. with mass points $x_1, x_2, \ldots$ Then $\Omega = \bigcup_{i=1}^{\infty} \{\omega : X(\omega) = x_i\}$ and

$$1 = P(\Omega) = \sum_{i=1}^{\infty} P[\{\omega : X(\omega) = x_i\}] \quad [\text{By countable additivity of } P[\cdot]]$$

$$= \sum_{i=1}^{\infty} P[X = x_i].$$

Definition:- Let X be a discrete RV with mass points $\{x_1, x_2, \ldots\}$: Then the function

$$f(x) = \begin{cases} P[X=x_i] & \text{if } x = x_i,\ i=1,2,\ldots \\ 0 & \text{if } x \neq x_i, \end{cases}$$

is called the PMF of the RV X.

Theorem:- A function $f(x)$ is said to be a PMF of some discrete RV X if

(i) $f(x) \geq 0 \ \forall x \in \mathbb{R}$ (ii) $\sum_x f(x) = 1$.

Alternative definition:- The probability mass function $f(x)$ of a RV X whose set of possible values are $\{x_1, x_2, \ldots\}$ is a function from $\mathbb{R}$ to $\mathbb{R}$ that satisfies the following properties:

(i) $f(x) = 0$ if $x \neq x_i$

(ii) $f(x) = P[X = x_i]$ if $x = x_i$, $i = 1, 2, \ldots$

(iii) $\sum_x f(x) = 1$.

Given the pmf of a discrete distribution, we can get the distribution function by successive addition, i.e.,

$$F(x) = f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n), \text{ where}$$

$$x_1 < x_2 < x_3 < \cdots < x_n < x < x_{n+1} < \cdots$$

On the other hand, given the df, we can get the pmf by successive subtraction, i.e.,

$$f(x_i) = F(x_i) - F(x_{i-1}) = P[X \leq x_i] - P[X < x_i]$$

is the probability at the point x_i.

Ex.1. For what values of θ and c is the function f given by

$$f(x) = \begin{cases} c.\dfrac{\theta^x}{x} & , x = 1,2,3,\ldots \\ 0 & , \text{otherwise} \end{cases}$$

a PMF ?

Solution:- (i) As $f(x) > 0 \; \forall \; x = 1,2,3,\ldots$

Hence, $c > 0$, $\theta > 0$.

(ii)
$$1 = \sum_x f(x) = \sum_{x=1}^{\infty} c.\frac{\theta^x}{x}$$
$$= c\sum_{x=1}^{\infty} \frac{\theta^x}{x}$$
$$= c\left\{-\log_e(1-\theta)\right\} \text{, if } 0<\theta<1.$$

$$\therefore \; c = -\frac{1}{\log_e(1-\theta)} \text{ and } 0<\theta<1.$$

Ex.2. Let $f(x) = \begin{cases} pq^x & , x = 0,1,2,3,\ldots ; \; p+q=1, \; 0<p<1. \\ 0 & , \text{OW} \end{cases}$

Does $f(x)$ define a PMF of some RV X? What is the DF of x? Find $P[n \le X \le m]$, $n, m \in \mathbb{N}$.

Solution:- (i) $0 < p < 1$

$\Rightarrow 1-p > 0$

$\Rightarrow (1-p)^x > 0 \quad [\because x = 0,1,2,3,\ldots]$

$\Rightarrow p(1-p)^x > 0 \quad [\because 0<p<1]$

$\therefore f(x) \geq 0.$

(ii) $$\sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} pq^x = p\sum_{x=0}^{\infty} q^x = \frac{p}{1-q} = \frac{p}{p} = 1.$$

$\therefore f(x)$ defines a PMF of some RV X.

(iii) $$F(x) = P[X \le x] = p + pq + pq^2 + \cdots + pq^x$$
$$= p\left[1 + q + q^2 + \cdots + q^x\right]$$
$$= p.\frac{1-q^{x+1}}{1-q} \quad [x = 0,1,2,\ldots]$$
$$= 1 - q^{x+1};$$

(iv) $$P[n \le X \le m] = P[X \le m] - P[X < n]$$
$$= P[X \le m] - P[X \le n-1]$$
$$= F(m) - F(n-1) = \{1 - q^{m+1}\} - \{1 - q^n\}$$
$$= (q)^n - q^{m+1}.$$

Ex.3. Find the PMF of the RV X whose DF is

$$F(x) = \begin{cases} 0 & , x < 0 \\ \dfrac{i(i+1)}{n(n+1)} & , i \le x \le i+1,\ i = 0, 1, \ldots, (n-1), \\ 1 & , x \geq n. \end{cases}$$

Solution:- Note that $i = 1, 2, 3, \ldots, n$.

$$P[X=i] = P[X \le i] - P[X < i]$$
$$= F(i) - F(i-0)$$
$$= \frac{i(i+1)}{n(n+1)} - \frac{i(i-1)}{n(n+1)}$$
$$= \frac{2i}{n(n+1)}.$$

$$F(i-0) = P[X<i]$$
$$= P[X \le i-1]$$
$$= F(i-1)$$
$$= \frac{(i-1)i}{n(n+1)}.$$

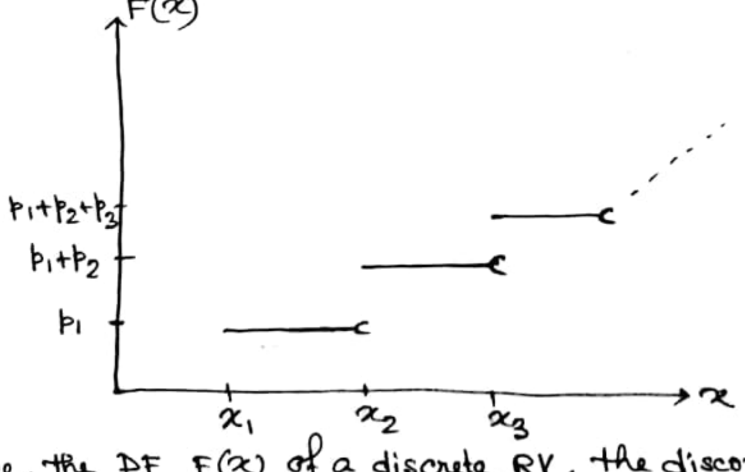
The PMF of X is $f(x) = \begin{cases} \dfrac{2x}{n(n+1)} & , x = 1(1)n, \\ 0 & , \text{OW}. \end{cases}$

Distribution Function of Discrete Random Variables :-

Let X be a discrete R.V. with mass points $x_1 < x_2 < \ldots\ldots$ Then the D.F. is

$$F(x) = P[X \le x] = \begin{cases} 0 & , x < x_1 \\ P[X = x_1] & , x_1 \le x < x_2 \\ \sum_{i=1}^{2} P[X = x_i] & , x_2 \le x < x_3 \\ \vdots & \end{cases}$$

$$= \begin{cases} 0 & , x < x_1 \\ \sum_{i=1}^{K} P[X = x_i] & , x_K \le x < x_{K+1},\ K = 1, 2, 3, \ldots \end{cases}$$



Hence, the DF $F(x)$ of a discrete RV, the discontinuity points are the mass points of the RV. The number of discontinuity points is the same as the no. of mass points.

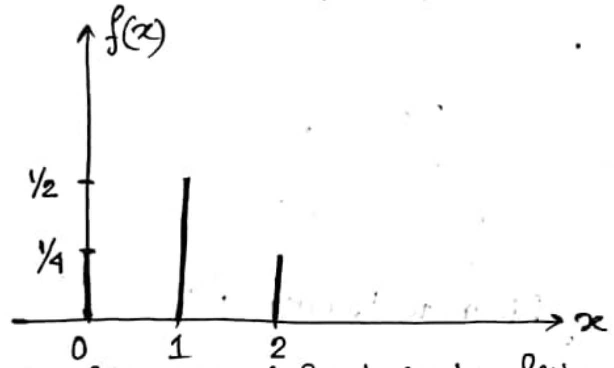
Ex.4. Construct an R.V. whose cdf is discontinuous at three points.

Solution:- Consider the random experiment of tossing a fair coin twice.
Then $\Omega = \{HH, HT, TH, TT\}$. Define, $X(\omega) =$ No. of heads in ω, $\omega \in \Omega$.

$$\therefore X(\omega) = \begin{cases} 0 & \text{if } \omega = TT \\ 1 & \text{if } \omega = HT, TH \\ 2 & \text{if } \omega = HH. \end{cases}$$

Since the coin is fair, all the sample points are equally likely,
i.e., $P[\{\omega\}] = \frac{1}{4}$, $\forall\, \omega \in \Omega$.

Hence,
$$P[X=0] = P[\{TT\}] = \frac{1}{4} = f(0).$$
$$P[X=1] = P[\{HT, TH\}] = \frac{1}{2} = f(1).$$
$$P[X=2] = P[\{HH\}] = \frac{1}{4} = f(2).$$

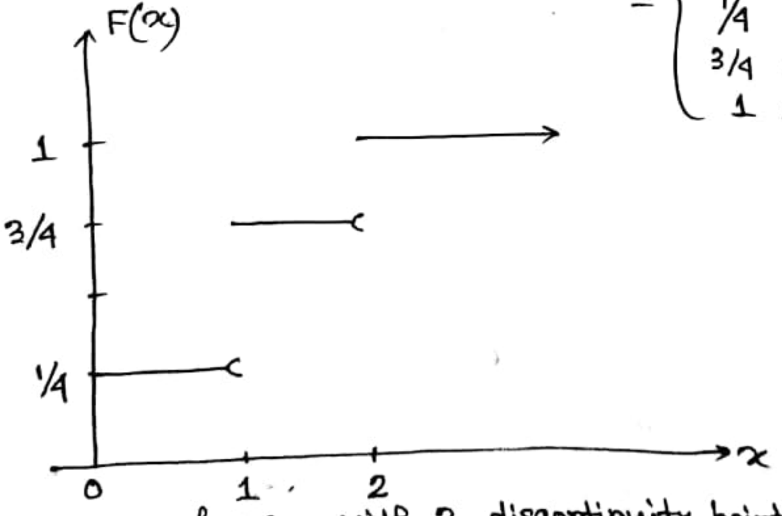


-: Graph of the no. of heads in two flips of a fair coin :-

The distribution function of X is $F(x) = P[X \le x]$

$$= \begin{cases} 0 & , x<0 \\ P[X=0] & , 0 \le x < 1 \\ P[X=0]+P[X=1] & , 1 \le x < 2 \\ \sum_{i=0}^{2} P[X=i] = 1 & , x \geq 2 \end{cases}$$

$$= \begin{cases} 0 & , x<0 \\ 1/4 & , 0 \le x < 1 \\ 3/4 & , 1 \le x < 2 \\ 1 & , x \geq 2 \end{cases}$$



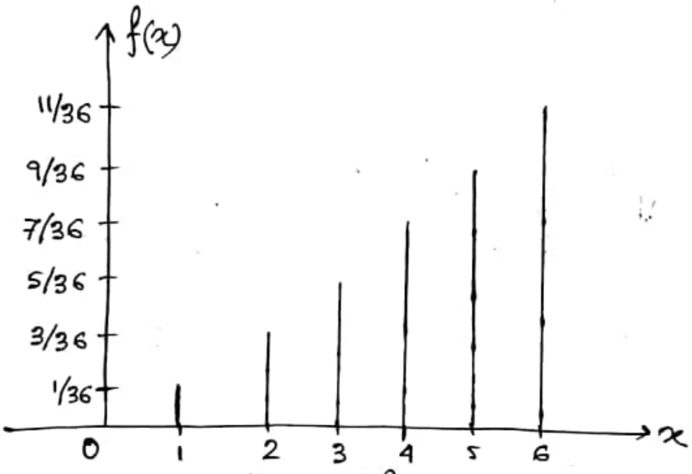
-: Graph of F(x) with 3 discontinuity points :-

Ex.5. In the experiment of rolling a balanced die twice, let X be the maximum of the two numbers obtained. Determine & sketch the PMF & DF of X.

Solution:- The possible values of X are 1, 2, 3, 4, 5 and 6. The sample space of this experiment consists of 36 equally likely outcomes. Hence, the probability of any of them is $\frac{1}{36}$. Thus,

$$f(1) = P(X=1) = P(\{1,1\}) = \frac{1}{36},$$
$$f(2) = P(X=2) = P(\{1,2\}, \{2,2\}, \{2,1\}) = \frac{3}{36},$$
$$f(3) = P(X=3) = P(\{\{1,3\}, \{2,3\}, \{3,3\}, \{3,2\}, \{3,1\}\}) = \frac{5}{36},$$

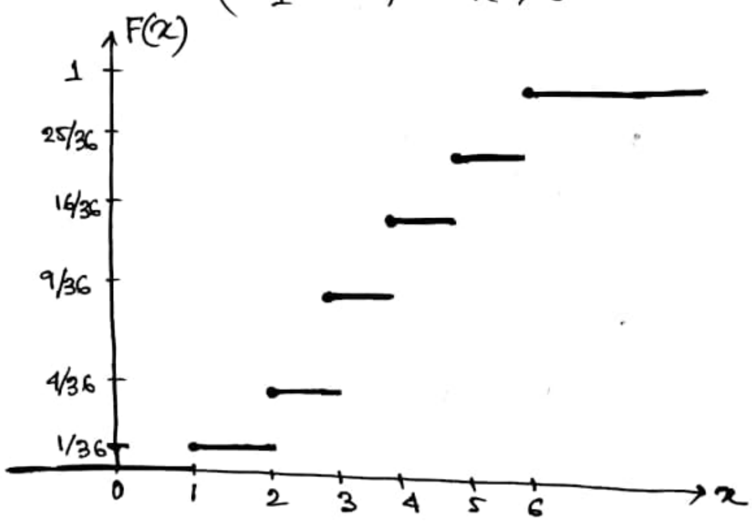
Similarly, $f(4) = \frac{7}{36}$, $f(5) = \frac{9}{36}$, $f(6) = \frac{11}{36}$ and $f(0) = 0$.



-: Graph of PMF of X :-

The distribution of X is F(x), is as follows:

$$F(x) = \begin{cases} 0, & x < 1 \\ 1/36, & 1 \le x < 2 \\ 4/36, & 2 \le x < 3 \\ 9/36, & 3 \le x < 4 \\ 16/36, & 4 \le x < 5 \\ 25/36, & 5 \le x < 6 \\ 1, & x \geq 6 \end{cases}$$



Ex.6. Can a function of the form $f(x) = \begin{cases} c\left(\frac{2}{3}\right)^x & , x = 1, 2, 3, \ldots \\ 0 & , \text{otherwise} \end{cases}$

be a probability mass function (PMF)?

Solution:-

Note that, here $f(x) \geq 0$ if $c \geq 0$.

And, to be a p.d.f, the below condition also needs to be satisfied

$$\sum_{i=1}^{\infty} c\left(\frac{2}{3}\right)^i = 1$$

$$\Rightarrow c \cdot \frac{2/3}{1 - 2/3} = 1$$

$$\Rightarrow c = 1/2 .$$

Thus, only for $c = 1/2$, $f(x)$ can be a PMF.

Ex.7. Let X be the number of births in a hospital until the first girl is born. Assume that the probability is $1/2$ that a baby born is a girl. Determine the PMF and DF of X.

Solution:- X is an r.v. that can assume any positive integer i, $f(i) = P(X = i)$, and X_i : i occurs if the first $i-1$ births are all boys and the i^{th} birth is a girl.

Thus $f(i) = \left(\frac{1}{2}\right)^{i-1}\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^i$ for $i = 1, 2, 3, \ldots$

and $f(x) = 0$ if $x \neq 1, 2, 3, \ldots$

$$F(t) = \begin{cases} 0 & \text{if } t < 1 \\ 1/2 & \text{if } 1 \leq t < 2 \\ 1/2 + 1/4 & \text{if } 2 \leq t < 3 \\ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} & \text{if } 3 \leq t < 4 \\ \vdots & \\ \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}, & \text{if } (n-1) \leq t < n \end{cases}$$

So, $$F(t) = \begin{cases} 0 & \text{if } t < 1 \\ \sum_{i=1}^{n-1}\left(\frac{1}{2}\right)^i & \text{if } \overline{n-1} \leq t < n, \; n = 2, 3, 4, \ldots \end{cases}$$

$$= \begin{cases} 0 & \text{if } t < 1 \\ 1 - \left(\frac{1}{2}\right)^{n-1} & \text{if } \overline{n-1} \leq t < n \;\; \forall \; n = 2, 3, 4, \ldots \end{cases}$$

B. Continuous Random Variable :-

Definition:- A random variable X is said to be a continuous RV if it takes any value within its range of variation.

For a continuous RV X, $P[X=x]=0 \ \forall x$,

By construction or axiomatic definition,

$$F(x) - F(x-0) = P[X=x] = 0 \ \forall x.$$

$\Rightarrow F(x)$ is continuous everywhere.

If $F(x)$ is continuous everywhere, then the associated R.V. X is known as Continuous Random Variable.

Absolutely continuous Random Variable:- An R.V. X with D.F. $F(x)$ is said to be an absolutely continuous RV, if $\exists$ a non-negative function $f(\cdot)$ such that

$$F(x) = \int_{-\infty}^{x} f(t)\,dt, \ \forall x \in \mathbb{R}.$$

where $F(x) = P[X \le x]$ is the distribution function of the RV X.

It may be noted that ——

(i) $F(-\infty) = \lim\limits_{x \to -\infty} F(x) = \lim\limits_{x \to -\infty} \int_{-\infty}^{x} f(x)\,dx = 0.$

(ii) $F(\infty) = \lim\limits_{x \to \infty} F(x) = \lim\limits_{x \to \infty} \int_{-\infty}^{x} f(x)\,dx = 1.$

(iii) $P[a < X \le b] = F(b) - F(a)$

$$= \int_{-\infty}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx$$

$$= \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx$$

$$= \int_{a}^{b} f(x)\,dx = P[a < X < b] = P[a \le X < b] = P[a \le X \le b]$$

And the function $f(x)$ is called the probability density function (pdf).

Theorem:- A function $f(x)$ is said to be a PDF of some absolutely continuous R.V. X if it satisfies

(i) $f(x) \geq 0 \ \forall x$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

Result:- If $F(x)$ is absolutely continuous and $f(x)$ is continuous at $X = x$, then $F'(x) = \frac{dF(x)}{dx} = f(x)$.

Proof:- Newton-Leibnitz Formula:-

$$I(\theta) = \int_{a(\theta)}^{b(\theta)} f(x,\theta)\,dx \text{, then } I'(\theta) \text{ is defined as}$$

$$I'(\theta) = \frac{dI(\theta)}{d\theta} = \int_{a(\theta)}^{b(\theta)} \frac{df}{d\theta}\,.\,dx + \frac{db(\theta)}{d\theta} f(b(\theta),\theta) - \frac{da(\theta)}{d\theta} f(a(\theta),\theta)$$

Here, $$F'(x) = \frac{dF(x)}{dx} = \frac{d}{dx}\int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} \frac{df(x)}{dx}\,dx + 1.f(x) - 0.f(-\infty)$$

$$= \int_{-\infty}^{x} 0.\,dx + f(x) = f(x).$$

Probability Density Function (PDF):- For an absolutely continuous RV X with D.F. $F(x)$, note that

$$\frac{d}{dx}[F(x)] = f(x),$$

$$\Rightarrow f(x) = \lim_{h\to 0+} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0+} \frac{P[x < X \leq x+h]}{h},$$

since $P[X = x] = 0$.

For small $h(>0)$, $f(x) \simeq \frac{P[x<X\leq x+h]}{h}$, which is the ratio of the probability contained in $(x, x+h]$ for the distribution and the length of the interval, i.e., $f(x) \simeq \frac{P[x<X\leq x+h]}{h}$ is the probability contained for the distribution per unit length in the interval $(x, x+h]$, where $h>0$ is small.

That is why, the quantity $f(x)$ is known as the probability density at the point x and the function $f(x)$ is called pdf of RV X.

Definition:- If X is an absolutely continuous RV X with D.F. $F(x)$, then $\exists$ a non-negative function $f(x) \ni$

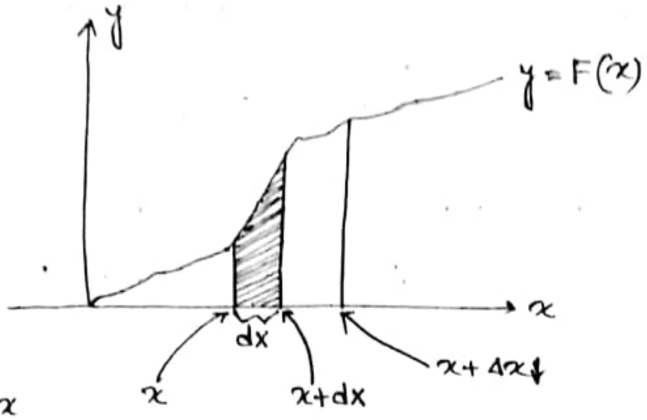
$$F(x) = \int_{-\infty}^{x} f(t)\,dt \;\forall\, x\in\mathbb{R}$$ and then the function $f(x)$ is called the PDF of X. It satisfies the properties:

(i) $f(x) \geq 0 \;\forall\, x\in\mathbb{R}$

(ii) $\int_{-\infty}^{\infty} f(t)\,dt = 1$.

Question:- Why $f(x)$ is called a pdf? Justify your answer.

Solution:-



$$\int_{x}^{x+dx} f(x)\,dx = \int_{x}^{x+dx} F'(x)\,dx = \Big[F(x)\Big]_{x}^{x+dx}$$

$$= F(x+dx) - F(x)$$

$$= P[x < X \le x+dx]$$

dP = Elementary probability lies within the interval $(x, x+dx]$.

$$= P[x < X \le x+dx]$$

$$= \lim_{\Delta x \to 0} P[x < X \le x+\Delta x]$$

$$= \lim_{\Delta x \to 0} \{F(x+\Delta x) - F(x)\}$$

$$= \lim_{\Delta x \to 0} \left\{\frac{F(x+\Delta x) - F(x)}{\Delta x} \cdot \Delta x\right\}$$

$$= \left\{\lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x}\right\}\left\{\lim_{\Delta x \to 0} \Delta x\right\}$$

$$= F'(x)\,dx$$

[Remark:-
* The geometrical interpretation of $\int_{-\infty}^{\infty} f(x)\,dx = 1$ is that the area bounded by the curve $y = f(x)$ and x-axis is unity.]

$\therefore$ $\underbrace{dP}_{(M)} = f(x)\underbrace{dx}_{(V)}$, where $f(x)$ is the PDF of X.

(M) , (V).

In probability theory, 'probability' has been regarded as "mass" and "dx" is the one-dimensional analogue of "volume", so that "dP" plays the role of "mass (M)" and 'dx' plays the role of "volume (V)" in the relation $M = \rho V$ of elementary physics.

Therefore, the function $f(x)$ is playing the role of "density" and that is why $f(x)$ is called the probability density function within an infinitisimal interval about $X = x$.

- **Note:-** For non-absolutely continuous RV, the PDF doesn't exist.

Remark:- (1) The PDF of an R.V. X is not a probability function?

Solution:- Note that, the probability $P[\cdot]$ is a set function defined on a σ-field $\mathcal{A}$ of events, where as $f(x)$ is a point function defined on $\mathbb{R}$. Again, $f(x)$ can take values >1, for some x, but $P[\cdot]$ can't exceed 1.

(2) Is PMF of an R.V. X a probability function?

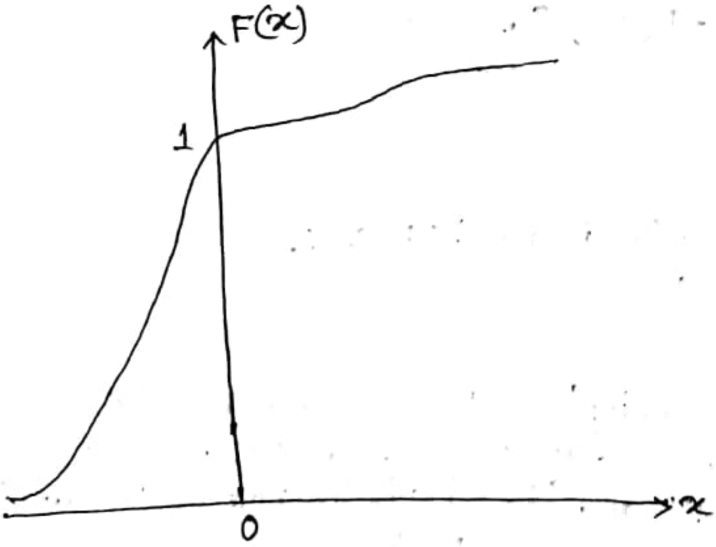
Solution:- Note that, PMF of X is $f(x) = \begin{cases} P[X=x_i] & , x = x_i, i=1,2,3,\ldots \\ 0 & , \text{ow} \end{cases}$

can't exceed 1 like probability function. But $f(x)$ is a point function defined on $\mathbb{R}$ unlike probability function which is a set function defined on a σ-field $\mathcal{A}$ of events.

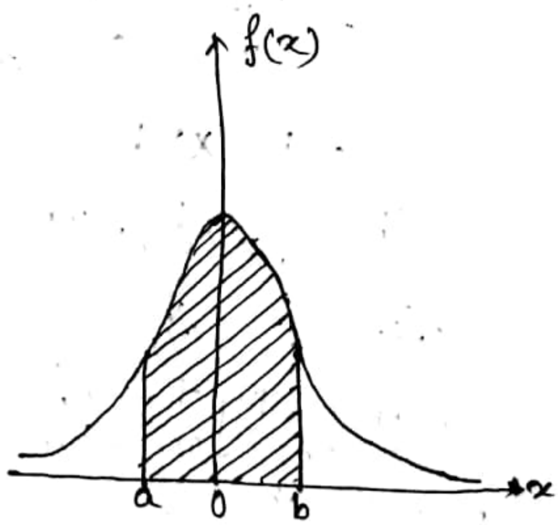
(3) $\displaystyle\int_a^b f(t)\,dt = P(a<X<b) = P(a \le X < b) = P(a < X \le b) = P(a \le X \le b)$

⇨ Implication of $P[X=a]=0 \ \forall\ a \in \mathbb{R}$ is that the value of the density function $f(x)$ at no point represents a probability. The probabilistic significance of $f(x)$ is that its integral over any subset of real numbers B gives that X lies in B. In particular —

" The area over an integral I under the graph of $f(x)$ represents the probability that an RV X will belong to I. The area under $f(x)$ to the left of a given point t is $F(t)$, the value of the DF of X at t."



-: d.f. of a continuous RV X :-



-: Curve of pdf of a continuous X. The shaded area under $f(x)$ is the probability that $X \in I = (a, b)$:-

Ex.1. Verify that the function $f(x)$ can be looked upon as the PDF of a continuous random variable.

$$f(x) = \begin{cases} x/2 & , 0 < x \le 1 \\ 1/2 & , 1 < x \le 2 \\ \frac{3-x}{2} & , 2 < x \le 3 \\ 0 & , 3 < x \le 4 \end{cases}$$

Obtain the Distribution function.

Solution:- Clearly, $f(x) \geq 0 \ \forall x \in \mathbb{R}$.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0.dx + \int_{0}^{1} \frac{x}{2}dx + \int_{1}^{2} \frac{1}{2}dx + \int_{2}^{3} \frac{3-x}{2}dx + \int_{3}^{\infty} 0.dx$$

$= 1.$ Hence, $f(x)$ is a pdf.

The D.F. is $F(x) = \int_{-\infty}^{x} f(t)\,dt$

$$= \begin{cases} \int_{-\infty}^{x} 0.dt = 0 & , \text{if } x \le 0 \\ \int_{-\infty}^{0} 0.dt + \int_{0}^{x} \frac{t}{2}dt = \frac{x^2}{4} & , \text{if } 0 < x \le 1 \\ \int_{-\infty}^{1} f(t)dt + \int_{1}^{x} \frac{1}{2}dt = F(1) + \frac{x-1}{2} = \frac{2x-1}{4} & , \text{if } 1 < x \le 2 \\ \int_{-\infty}^{2} f(t)dt + \int_{2}^{x} \frac{3-t}{2}dt = F(2) + \int_{2}^{x} \frac{3-t}{2}dt = \frac{6x - x^2 - 5}{4} & , \text{if } 2 < x \le 3 \\ \int_{-\infty}^{3} f(t)dt + \int_{3}^{\infty} 0.dt = F(3) = 1 & , \text{if } x > 3 \end{cases}$$

Ex.2. Let $f(x) = \begin{cases} k & , 0 < x < 1/2 \\ 0 & , \text{ow} \end{cases}$ be a pdf of X. Find the constant k.

Solution:- $f(x) \geq 0 \Rightarrow k \geq 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{1/2} k.dx = \frac{k}{2} = 1 \Rightarrow k = 2.$$

Ex. 3. Suppose that $P[X \geq x]$ is given for an absolutely continuous RV X. How will you find the corresponding PDF?

$$P[X \geq x] = \begin{cases} 1 & \text{if } x \leq 0 \\ e^{-\lambda x} & \text{if } x > 0 \end{cases}, \text{ where } \lambda > 0.$$

Solution:-

$$F(x) = P[X \leq x]$$
$$= 1 - P[X \geq x] \quad , \text{ as } P[X = x] = 0.$$

PDF of X is given by,

$$f(x) = \frac{d}{dx}\left(F(x)\right) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{d}{dx}\left(1 - e^{-\lambda x}\right) & \text{if } x > 0, \lambda > 0 \end{cases}$$

$$= \begin{cases} 0 & \text{if } x < 0 \\ \lambda e^{-\lambda x} & \text{if } x > 0 \end{cases}$$

Ex. 4. Let X be a continuous RV with pdf given by

$$f(x) = \begin{cases} ax & , \ 0 \leq x < 1 \\ a & , \ 1 \leq x < 2 \\ -ax + 3a & , \ 2 \leq x < 3 \\ 0 & , \ 3 < x \end{cases}$$

Determine 'a' and $F(x)$.

Solution:-

$$\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow a\int_0^1 x dx + a\int_1^2 dx + \int_2^3 (3a - ax)dx = 1$$
$$\Rightarrow a = 2.$$

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2\int_0^x t dt = x^2 & \text{if } 0 \leq x < 1 \\ 2\int_0^1 x dx + 2\int_1^x dt = 2x - 1 & \text{if } 1 \leq x < 2 \\ 2\int_0^1 x dx + 2\int_1^2 dx + \int_2^x (6 - 2t)dt = 6x - x^2 - 5 & \text{if } 2 \leq x < 3 \\ 1 & \text{if } x \geq 3 \end{cases}$$

- **Degenerate Random Variable:—** An r.v. X is said to be degenerate at c if
$$F(x) = \begin{cases} 0 & \text{if } x < c \\ 1 & \text{if } x \geq c \end{cases}$$

Let us define $\epsilon(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

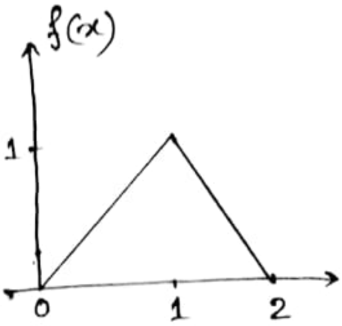
Then for an r.v. X degenerate at c, $F(x) = \epsilon(x - c)$.

Ex.5. Let X have the triangular pdf $f(x) = \begin{cases} x, & 0 \le x < 1 \\ 2-x, & 1 \le x < 2 \\ 0, & \text{ow} \end{cases}$

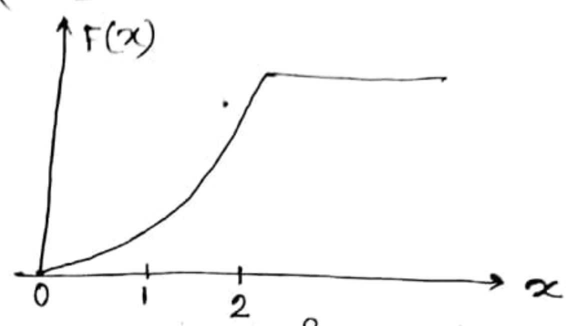
Sketch its pdf, dF.

Solution:

$$F(x) = \begin{cases} 0, & x \le 0 \\ \int_0^x t\,dt = \frac{x^2}{2}, & 0 < x \le 1 \\ \int_0^1 t\,dt + \int_1^x (2-t)\,dt = 2x - \frac{x^2}{2} - 1, & 1 \le x < 2 \\ 1, & x \geq 2 \end{cases}$$



-: Graph of pdf :-

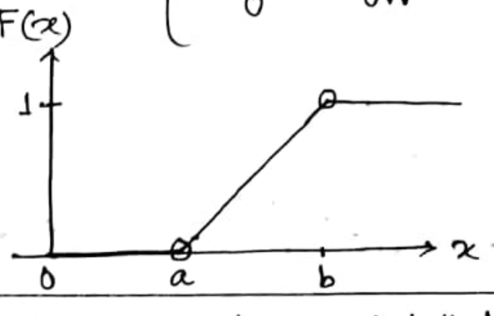


-: Graph of D.F. :-

Ex.6. Find the pdf of X if its distribution function is

$$F(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } x > b \end{cases}$$

Solution:- $f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{ow} \end{cases}$



Ex.7. If X has the continuous distribution with pdf $f(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{ow} \end{cases}$

Obtain the distribution function.

Solution:-

$$F(x) = \begin{cases} \int_{-\infty}^{x} f(t)\,dt = 0 & \text{if } x < 0 \\ \int_{-\infty}^{0} f(t)\,dt + \int_0^x f(t)\,dt = 1 - e^{-x} & \text{if } x > 0 \end{cases}$$

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Ex.8. If X has the continuous distribution with pdf

$$f(x) = \begin{cases} \frac{x^2}{2!} e^{-x} & \text{if } x > 0 \\ 0 & \text{ow} \end{cases}$$

Find its DF.

Solution:-

$$F(x) = \begin{cases} \int_{-\infty}^{x} f(t)\,dt = 0 & \text{for } x \leq 0 \\ \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{x} f(t)\,dt = \int_{0}^{x} \frac{t^2}{2!} e^{-t}\,dt & \text{for } x > 0 \end{cases}$$

$$= \begin{cases} 1 - e^{-x} \sum_{i=1}^{2} \frac{x^i}{i!} & \text{, for } x > 0 \\ 0 & \text{, ow} \end{cases}$$

Ex.9. Experience has shown that while walking a certain park, the time X, in minutes, between seeing two people smoking has a density function of the form

$$f(x) = \begin{cases} \lambda x e^{-x} & , x > 0 \\ 0 & \text{ow} \end{cases}$$

(a) Calculate the value of λ, (b) Find the DF of X.

(c) What's the probability that Mr. X, who has just seen a person smoking will see another person smoking in 2 to 5 minutes? In at least 7 minutes?

Solution:- (a) $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{0}^{\infty} \lambda x e^{-x}\,dx = 1 \Rightarrow \lambda = 1.$

(b) $F(t) = \begin{cases} \int_{-\infty}^{t} f(x)\,dx = \left[-(x+1)e^{-x}\right]_0^t = -(t+1)e^{-t} + 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$

(c) $P[2 < X < 5] = F(5) - F(2) = (1 - 6e^{-5}) - (1 - 3e^{-2}) \approx 0.37.$

$P[X \geq 7] = 1 - P[X \leq 7] = 1 - F(7) = 8e^{-7} \approx 0.007.$

Ex.10. If X has an absolutely continuous distribution with pdfs as shown below, then find their df?

(i) $f(x) = \frac{\theta}{2}\exp(-\theta|x-\alpha|)$, where $\theta > 0$

(ii) $f(x) = \frac{a}{\pi[a^2+(x-\theta)^2]}$, where $a > 0$.

Solution:- (i)

$$f(x) = \begin{cases} \frac{\theta}{2} e^{-\theta(x-\alpha)} & \text{if } x > \alpha \\ \frac{\theta}{2} e^{-\theta(\alpha-x)} & \text{if } x \leq \alpha \end{cases}$$

Now,
$$F(x) = \begin{cases} \int_{-\infty}^{x} \frac{\theta}{2} e^{-\theta(\alpha-t)} dt & \text{if } x \leq \alpha \\ \int_{-\infty}^{\alpha} f(t)\,dt + \int_{-\infty}^{x} f(t)\,dt & \text{if } x > \alpha \end{cases}$$

$$= \begin{cases} \frac{1}{2} e^{-\theta|x-\alpha|} & \text{if } x \leq \alpha \\ 1 - \frac{1}{2} e^{-\theta|x-\alpha|} & \text{if } x > \alpha \end{cases}$$

(ii)
$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \frac{a}{\pi}\int_{-\infty}^{x} \frac{1}{[a^2+(t-\theta)^2]}\,dt$$

$$= \frac{a}{\pi}\cdot\frac{1}{a}\left[\tan^{-1}\frac{x-\theta}{a} - \tan^{-1}(-\infty)\right]$$

$$= \frac{1}{\pi}\left[\tan^{-1}\left(\frac{x-\theta}{a}\right) + \frac{\pi}{2}\right] \text{ if } a > 0.$$

So,
$$F(x) = \begin{cases} \frac{1}{\pi}\tan^{-1}\left(\frac{x-\theta}{a}\right) + \frac{1}{2} & \text{if } a > 0 \\ 0. & \text{ow} \end{cases}$$

Ex. 11. (a) Sketch the graph of the function

$$f(x) = \begin{cases} \frac{1}{2} - \frac{1}{4}|x-3| & , \text{if } 1 \le x \le 5 \\ 0 & , \text{Otherwise} \end{cases}$$

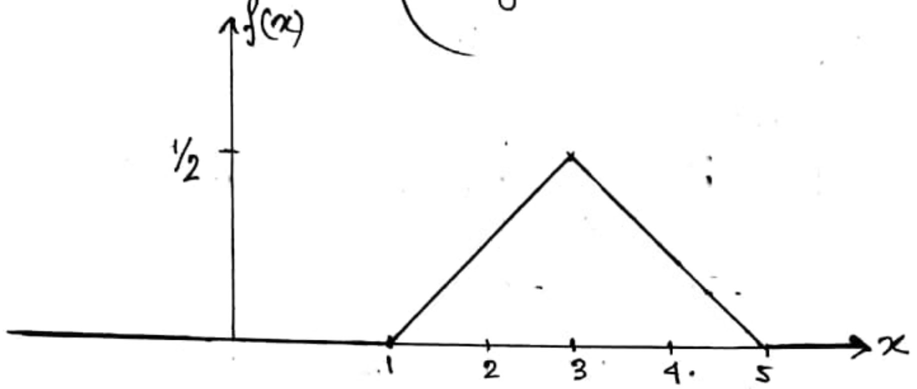
and show that it is the PDF of an RV X.

(b) Find F, the d.f. of X, and show that it is continuous.

(c) Sketch the graph of F.

Solution: – (a)

$$f(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{1}{2} + \frac{1}{4}(x-3) & \text{if } 1 \le x < 3 \\ \frac{1}{2} - \frac{1}{4}(x-3) & \text{if } 3 \le x < 5 \\ 0 & \text{if } x \ge 5 \end{cases}$$



–: Graph of PDF :–

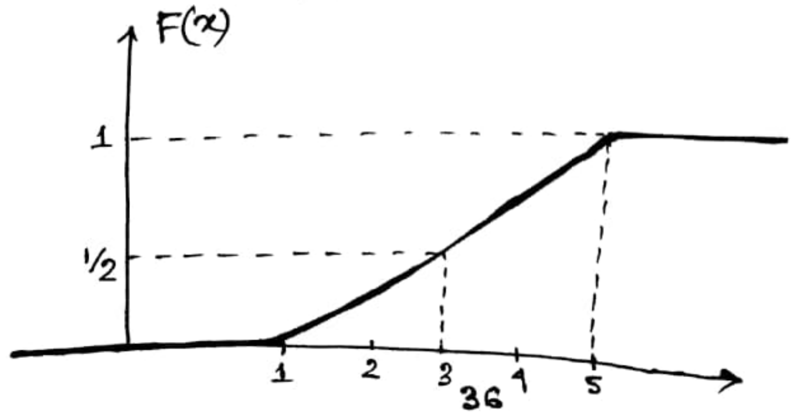
Since $f(x) \geq 0$, and the are under f from 1 to 5, being the area of the triangle ABC is $\frac{1}{2} \times 4 \times \frac{1}{2} = 1$. So, f is a density function of some random variable X.

(b)

$$F(t) = \begin{cases} \int_{-\infty}^{t} 0 . dx = 0 & ; \text{if } t < 1 \\ \int_{1}^{t} [\frac{1}{2} + \frac{1}{4}(x-3)] dx = \frac{1}{8}t^2 - \frac{1}{4}t + \frac{1}{8} & ; \text{if } 1 \le t < 3 \\ \int_{1}^{3} [\frac{1}{2} + \frac{1}{4}(x-3)] dx + \int_{3}^{t} [\frac{1}{2} - \frac{1}{4}(x-3)] dx & \\ = -\frac{1}{8}t^2 + \frac{5}{4}t - \frac{17}{8} & ; \text{if } 3 \le t < 5 \\ \int_{1}^{3} [\frac{1}{2} + \frac{1}{4}(x-3)] dx + \int_{3}^{5} [\frac{1}{2} - \frac{1}{4}(x-3)] dx = 1 & ; \text{if } t \ge 5 \end{cases}$$

F is continuous because $\lim_{t \to 1^-} F(t) = 0 = F(1)$, $\lim_{t \to 3^-} F(t) = \frac{1}{8}(3)^2 - \frac{1}{4}(3) + \frac{1}{8} = F(3)$.

(c)



Ex.12. Let X be a positive random variables having probability density function f. If $f(x) \le c$, show that, for $a > 0$, $P[X > a] \geq 1 - ac$.

Solution:– $P[X > a] = 1 - P[X \le a] = 1 - \int_0^a f(x)\,dx$ [$\because$ X is +ve r.v.]

$f(x) \le c$; so, $P(X > a) \geq 1 - \int_0^a c\,dx = 1 - ac$.

Ex.13. The r.v. X is continuously distributed with a density function f which is symmetrical about '0', so that $f(x) = f(-x)\ \forall x$, show that
(a) $F(0) = 1/2$; (b) $P[X > a] = \frac{1}{2} - \int_0^a f(x)\,dx$, $a > 0$; (c) $P[-a < X < a] = 2F(a) - 1$

Solution:– (a) $F(0) = \int_{-\infty}^{0} f(x)\,dx$ Put $x = -x$.

$$= -\int_{\infty}^{0} f(-x)\,dx$$

$$= \int_0^{\infty} f(-x)\,dx = \int_0^{\infty} f(x)\,dx, \text{ since } f(x) = f(-x).$$

We know, $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx = 1 \Rightarrow F(0) = \frac{1}{2}$.

(b) $P(X > a) = \int_a^{\infty} f(x)\,dx$, $a > 0$

$$= \int_{-\infty}^{\infty} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx$$

$$= 1 - \left[\int_{-\infty}^{0} f(x)\,dx + \int_0^a f(x)\,dx\right] = \frac{1}{2} - \int_0^a f(x)\,dx.$$

(c) $P[-a < X < a] = \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$

$$= 2\int_0^a f(x)\,dx$$

$$= 2\left[\int_{-\infty}^{a} f(x)\,dx - \int_{-\infty}^{0} f(x)\,dx\right]$$

$$= 2[F(a) - F(0)]$$

$$= 2\left[F(a) - \frac{1}{2}\right]$$

$$= 2F(a) - 1.$$

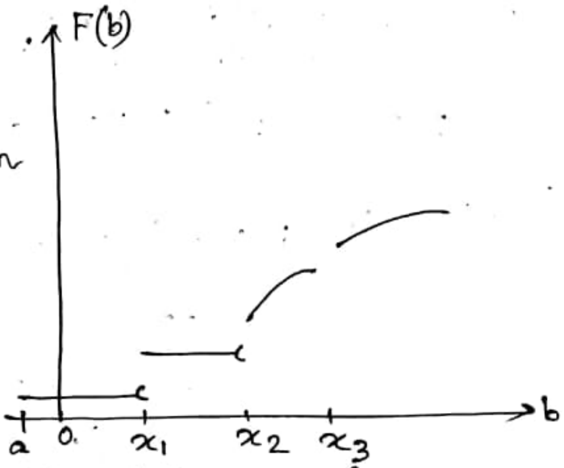
Ex. 14. Show that the set of discontinuity points of a distribution function is at most countable.

Solution:- Let $(a,b]$ be a finite interval with atleast n discontinuity points: $a < x_1 < x_2 < \cdots < x_n \le b$.

Then, $F(a) \le F(x_1 - 0) < F(x_1) \le \cdots \le F(x_n - 0) < F(x_n) \le F(b)$.

Let, $p_K = F(x_K) - F(x_K - 0)$, $K = 1(1)n$.

Hence, $F(b) - F(a) \geq \sum_{K=1}^{n} p_K$.



Note that, the number of points m in $(a,b]$ with jump $p_K > \epsilon > 0$ is given by $F(b) - F(a) \geq \sum_{K=1}^{n} p_K > m\epsilon$

$$\Rightarrow m < \frac{1}{\epsilon}\{F(b) - F(a)\}$$

Thus, for every integer N, the number of discontinuity points with jump $> \frac{1}{N}$ is less than $N\{F(b) - F(a)\}$, i.e. is finite.

Hence, there are no more than a countable number of discontinuity points in every finite interval $(a,b]$. Since R is a countable union of finite intervals, then the proof is complete.

Ex. 15. Define $G(t) = P(X < t)$, $t \in (-\infty, \infty)$, when X is a random variable. Show that G is non-decreasing and left continuous.

Solution:- (i) $G(t) = P(X < t)$

For $t_1 < t_2$,

$\{X < t_2\} = \{X < t_1\} \cup \{t_1 \le X < t_2\}$.

$\therefore P[X < t_2] = P[X < t_1] + P[t_1 \le X < t_2]$ [By Finite additivity]

$\therefore G(t_2) - G(t_1) = P[t_1 \le X < t_2] \geq 0$ [Since $P[\cdot]$ is non-negative]

So, G is non-decreasing.

(ii) Define, $A_n = \{X < t - \frac{1}{n}\}$, $n \in \mathbb{N}$

$\therefore \lim_{n\to\infty} A_n = \{X < t\}$

By continuity theorem of probability,

$$\lim_{n\to\infty} P[A_n] = P[\lim_{n\to\infty} A_n]$$

$$\Rightarrow \lim_{n\to\infty} P[X < t - \tfrac{1}{n}] = P[X < t]$$

$$\Rightarrow \lim_{n\to\infty} G(t - \tfrac{1}{n}) = G(t)$$

$$\Rightarrow G(t - 0) = G(t).$$

So, G is left-continuous.

Remark:-

Probability Distribution:- Distribution of total probability, i.e., unity over a partition of S.

A probability distribution is measured by distribution function defined as

$$F(x) = P(X \le x).$$

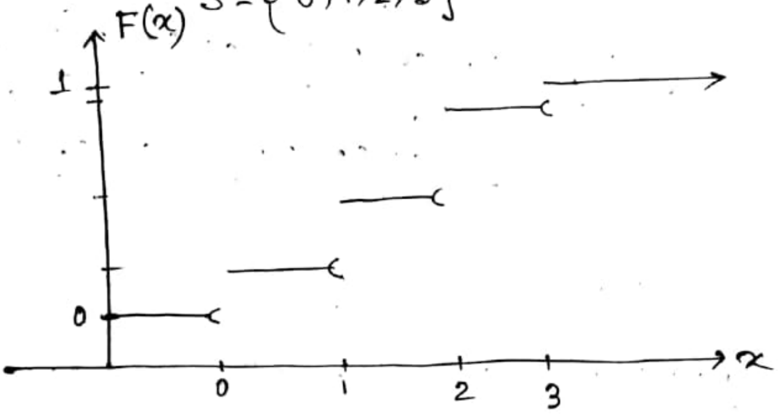
S : Countable $\Leftrightarrow$ X is a discrete Random Variable

S : non-degenerate inf. $\Leftrightarrow$ X is a continuous Random Variable.

⟨i⟩ Discrete Random Variable:-

X : No. of heads obtained by tossing a fair coin twice.

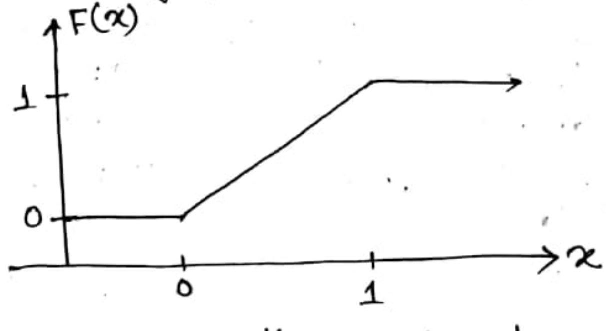
$S = \{0, 1, 2, 3\}$



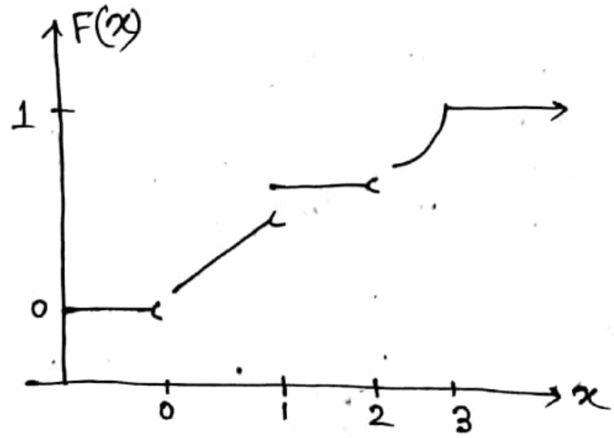
⟨ii⟩ Continuous Random Variable:- A clerk goes out for lunch at any time point between 12PM to 1PM.

X : time point.

$S = (0, 1)$ [with suitable change of location]



⟨iii⟩ Mixture distribution:- A mixture of ⟨i⟩ and ⟨ii⟩ situations:-



Ex. 16 A fair die is thrown repeatedly till every face has appeared at least once. Let X denote the number of throws made. Find the distribution of X.

Solution:- $P(X=x) = P(X \leq x) - P(X \leq x-1)$

= (The probability that in x throws all varieties will appear) − (The probability that in $(x-1)$ throws all varieties will appear)

$$= \left\{1 - \binom{6}{1}\left(\frac{5}{6}\right)^x + \binom{6}{2}\left(\frac{4}{6}\right)^x - \binom{6}{3}\left(\frac{3}{6}\right)^x + \binom{6}{4}\left(\frac{2}{6}\right)^x - \binom{6}{5}\left(\frac{1}{6}\right)^x\right\}$$

$$- \left\{1 - \binom{6}{1}\left(\frac{5}{6}\right)^{x-1} + \binom{6}{2}\left(\frac{4}{6}\right)^{x-1} - \binom{6}{3}\left(\frac{3}{6}\right)^{x-1} + \binom{6}{4}\left(\frac{2}{6}\right)^{x-1} - \binom{6}{5}\left(\frac{1}{6}\right)^{x-1}\right\}$$

Ex. 17. An urn contains N cards labelled from 1 to N. If n drawings are made at random without replacements from the urn, let X denotes the least number drawn. Find the distribution of the random variable X.
If Y denotes the highest number drawn, obtain the distn. of Y.

Solution:- X denotes the least number drawn. The mass points of X are $1, 2, \ldots, \overline{N-n+1}$. Now n cards can be drawn out of N cards WOR in $\binom{N}{n}$ ways and in order that the least number drawn is x, we have to select the $n-1$ cards other than the card numbered x from the set $x+1, x+2, \ldots, n$, i.e., from $N-x$ cards and this can be done in $\binom{N-x}{n-1}$ ways.

Hence, $$P(X=x) = \frac{\binom{N-x}{n-1}}{\binom{N}{n}}, \quad x = 1, 2, \ldots, \overline{N-n+1}.$$

Y denotes the highest number drawn. The mass points of Y are $n, n+1, \ldots, N$.
Total number of ways in which n cards can be drawn from N item is $\binom{N}{n}$. In order that the highest number drawn is y, we have to select remaining $(n-1)$ cards from the card numbered y from the set $1, 2, 3, \ldots, \overline{y-1}$, i.e. from $(y-1)$ cards and this can be done in $\binom{y-1}{n-1}$ ways.

Hence, $$P(Y=y) = \frac{\binom{y-1}{n-1}}{\binom{N}{n}}, \quad y = n, n+1, \ldots, N.$$

Ex. 18. A man wants to open his door and has n keys, only one of which fits the door. For some reason which can only be surmised, he tries the keys independently and at random. Find a probability distribution of the number of attempts needed to be made by the man

(a) if unsuccessful keys are not eliminated from further selections

(b) if they are.

Solution:- X: Number of attemps needed to open the door.

Case: (a) :- In this case X can take values $1, 2, 3, \ldots, \infty$.

$\therefore P(X=x) = P($wrong key at the first $x-1$ trials attempt and right key at the x^{th} attempt$)$

$$= \left(\frac{n-1}{n}\right)^{x-1} \cdot \frac{1}{n} \quad \text{[since attempts are made independently]}$$

Case: (b) :- In this can x can take values $1, 2, 3, \ldots, n$.

$\therefore P(X=x) = P($wrong key at the first $x-1$ trials attempt and right key at x^{th} attempt$)$

$$= \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \frac{n-3}{n-2} \cdots \frac{n-x}{n-x+1}\left(\frac{1}{n-x}\right)$$

$$= \frac{1}{n}.$$

Ex. 19. There are n tickets in a jar numbered $1, 2, 3, \ldots, n$. Tickets are drawn at random and with replacements from the jar and their numbers are noted; the operation stops as soon as a ticket drawn appears for the second time. If X be the total number of drawings made, then find its PMF.

Solution:- The mass points of x are $2, 3, \ldots, \overline{n+1}$.

$P(X=x) = P($The first $x-1$ drawings will give $x-1$ distinct tickets and the x^{th} drawing will give a repetition$)$

$=$

$=$

$$= \frac{{}^nP_{x-1}}{n^{x-1}} \times \frac{x-1}{n}$$

$$= (x-1)!\binom{n}{x-1} \cdot \frac{(x-1)}{n^x} \quad ; \; x = 2, 3, \ldots, \overline{n+1}.$$

Ex. 20. The duration (in minutes) of long-distance telephone calls made from a certain city has the distribution function F, given by

$$F(x) = \begin{cases} 0 & \text{if } x<0 \\ 1-\frac{1}{2}e^{-x/3}-\frac{1}{2}e^{-[x/3]} & \text{if } x \geq 0 \end{cases}$$

What is the probability that a telephone call lasts for
(i) more than six minutes?
(ii) less than four minutes?
(iii) Exactly three minutes?
What is the conditional probability that the duration of a call is,
(iv) less than nine minutes, given that more than five minutes.

Solution:– (i)

$$P(X>6) = 1-P(X\leq 6) = 1-F(6) = e^{-2};$$

(ii)

$$P(X>4) = F(4-0) = 1-\frac{1}{2}\left(e^{-4/3}+e^{-1}\right);$$

(iii)

$$P(X=3) = P(X\leq 3)-P(X<3) = F(3)-F(3-0) = \frac{(1-e^{-1})}{2};$$

(iv)

$$P(X<9 \mid X>5) = \frac{P(5<X<9)}{P(X>5)} = \frac{P(X<9)-P(X\leq 5)}{1-P(X\leq 5)} = \frac{F(9-0)-F(5)}{1-F(5)} = 1-\frac{e^{-2}+e^{-3}}{e^{-1}+e^{-5/3}};$$

Remark:– Note that the distribution given above is a Mixed distn., i.e., the weighted average of a discrete and a absolutely continuous distribution. Here $F = \frac{1}{2}F_d + \frac{1}{2}F_c$; where

$$F_c(x) = \begin{cases} 1-e^{-x/3}, & x\geq 0 \\ 0 & \text{ow} \end{cases} \qquad F_d(x) = \begin{cases} 1-e^{-[x/3]}, & x\geq 0 \\ 0 & ,\text{ow} \end{cases}$$

This is explained by Decomposition theorem, see later.

• **Decomposition Theorem:-** There may be a distribution whose d.f. is neither discrete nor continuous (absolutely). Such a distribution is called purely singular.

Every distribution function $F(x)$ can be decomposed into two parts according to $F(x) = \alpha F_d(x) + (1-\alpha) F_c(x)$, where $0 \le \alpha \le 1$, and $F_d(x)$, $F_c(x)$ are the DF of discrete and continuous RVs, respectively.

Note that, for $\alpha = 0$, $F(x) = F_c(x)$ is purely continuous. For $\alpha = 1$, $F(x) = F_d(x)$ is purely discrete, for $0 < \alpha < 1$, then $F(x) = \alpha F_d(x) + (1-\alpha) F_c(x)$ is neither absolutely continuous nor purely discrete and it is called mixed distribution.

Further $F = \alpha F_d + \beta F_c + \gamma F_s$, where, F_d is discrete, F_c is absolutely continuous and F_s is singular, ($\alpha \ge 0, \beta \ge 0, \gamma \ge 0, \alpha+\beta+\gamma = 1$).

Ex.1. Let for an r.v. X,
$$F(x) = \begin{cases} 0 & , x < 0 \\ \frac{x}{2} & , 0 \le x < 1 \\ \frac{3}{4} & , 1 \le x < 2 \\ 1 & , 2 \le x \end{cases}$$
Show that $F(x)$ can be written as a mixture of two distribution functions.

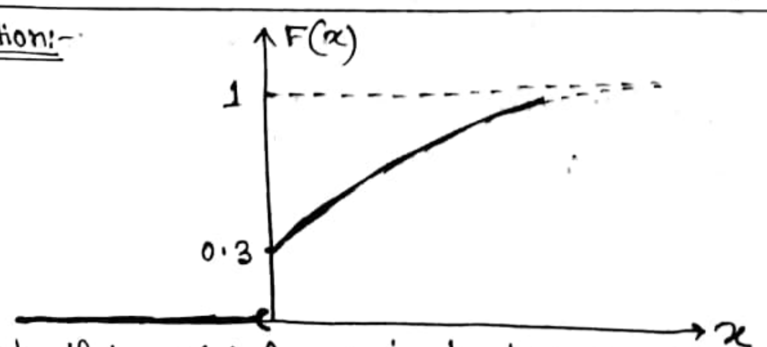
Solution:- $F(x) = \frac{1}{2}\left(G(x) + H(x)\right)$, where

$$G(x) = \begin{cases} 0 & , x < 1 \\ 1/2 & , 1 \le x < 2 \\ 1 & , 2 \le x \end{cases} \qquad H(x) = \begin{cases} 0 & , x < 0 \\ x & , 0 \le x < 1 \\ 1 & , x > 1 \end{cases}$$

Then X has a mixed distribution.

Ex.2. An R.V. has the DF
$$F(x) = \begin{cases} 1 - 0.7e^{-x} & , \text{if } x \ge 0 \\ 0 & , \text{otherwise} \end{cases}$$
Sketch the D.F. Give an e.g. of an R.V. that may be supposed to have the distribution. Also, find
(a) $P[X=0]$, (b) $P[X \le 4]$, (c) $P[3 < X \le 5]$.

Solution:-



Note that, $F(x)$ has a jump at $x=0$ and $F(x)$ is continuous on $(0, \infty)$.

Hence, $F(x)$ is the DF of a RV X, i.e., neither purely discrete nor purely continuous; i.e., the RV X has an isolated value at $x=0$ and takes any value in $(0,\infty)$ as a continuous R.V..

Write, $$F(x) = 0.3 \times F_d(x) + (1-0.3)\, F_c(x)\text{; where}$$

$$F_d(x) = \begin{cases} 0 & \text{if } x<0 \\ 1 & \text{if } x \geq 0 \end{cases} \quad \text{and} \quad F_c(x) = \begin{cases} 0 & \text{if } x<0 \\ 1-e^{-x} & \text{if } x \geq 0 \end{cases}$$

Now, (a) $P[X=0] = P[X \leq 0] - P[X<0]$
$= F(0) - F(0-0)$
$= (1 - 0.7e^{-0}) - 0$
$= 0.3$

(b) $P[X \leq 4] = F(4) = 1 - 0.7e^{-4}$

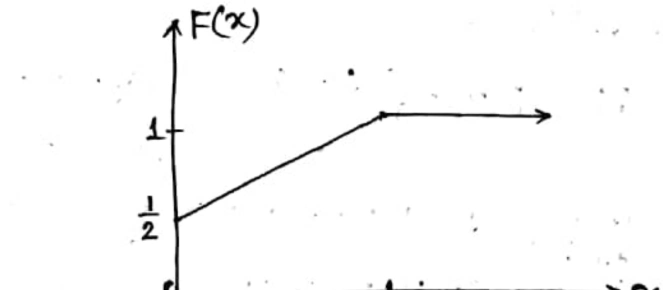
(c) $P[3<X<5] = P[X \leq 5] - P[X \leq 3]$
$= F(5) - F(3)$
$= \{1 - 0.7e^{-5}\} - \{1 - 0.7e^{-3}\}$
$= 0.7(e^{-3} - e^{-5})$

Ex. 3. Let X be an RV with DF

$$F(x) = \begin{cases} 0 & , x<0 \\ 1/2 & , x=0 \\ 1/2 + x/2 & , 0<x<1 \\ 1 & , x \geq 1 \end{cases}$$

Sketch the DF. Give an example of an RV that may be supposed to have the distribution. Also express $F(x)$ as $\alpha F_d(x) + (1-\alpha) F_c(x)$.

Solution:-



Clearly, $F(x)$ has a jump at $x=0$ and it is continuous on $(0,\infty)$. So, $F(x)$ is a dF of mixed-type.

Hence, $F(x)$ is the D.F. of an RV which can take the isolated value $x=0$ and can take any value between $(0,1)$ as a continuous RV. We have only one discontinuity point, say $x=0$, of $F(x)$.

Define, $$F_d(x) = \begin{cases} 0 & , x<0 \\ 1 & , x \geq 0 \end{cases}$$

$$F(x) = \alpha F_d(x) + (1-\alpha) F_c(x)$$

When $x=0$, $F(0) = \alpha F_d(0) + (1-\alpha) F_c(0)$

$\Rightarrow \frac{1}{2} = \alpha.1 + (1-\alpha).0$, as $F_c(x)$ is continuous at $x=0$, and $F(x)=0, x<0$.

$\Rightarrow \alpha = \frac{1}{2}$,

Now, for $0<x<1$, $F(x) = \frac{1}{2} F_d(x) + \frac{1}{2} F_c(x)$

$\Rightarrow \frac{1}{2} + \frac{x}{2} = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot F_c(x)$

$\Rightarrow F_c(x) = x$.

For $x \geq 1$, $F(x) = \frac{1}{2} F_d(x) + \frac{1}{2} F_c(x)$

$\Rightarrow 1 = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot F_c(x)$

$\Rightarrow F_c(x) = 1$.

So, $F_d(x) = \begin{cases} 0 & , x<0 \\ 1 & , x \geq 0 \end{cases}$ and $F_c(x) = \begin{cases} 0 & , x \leq 0 \\ x & , 0<x<1 \\ 1 & , x \geq 1. \end{cases}$

with $\alpha = \frac{1}{2}$.

GEOMETRIC PROBABILITY

Let Ω be a given region and A be a subset of Ω. We are interested in the probability that a randomly chosen point in Ω falls in A or not. Here, randomly chosen means that a point may be any point of Ω and that the probability of its falling in some subset A of Ω is proportional to the measure of A (independent of the location of the shape of A). Then the probability that a randomly chosen point in Ω falls in A, is defined as

$$P[A] = \frac{\text{Measures of } A}{\text{Measures of } \Omega}.$$

Remark:- Let X be a randomly chosen point in Ω. As the point X is chosen randomly, then the total probability unity is uniformly distributed over Ω, i.e.,

$$f(x) = \begin{cases} K \text{ (constant)}, & x \in \Omega \\ 0, & \text{otherwise} \end{cases}$$

Ex.1. A point is picked at random from a unit square $\Omega = \{(x,y) : 0 \le x, y \le 1\}$. Find the probability that a randomly chosen point in Ω falls in

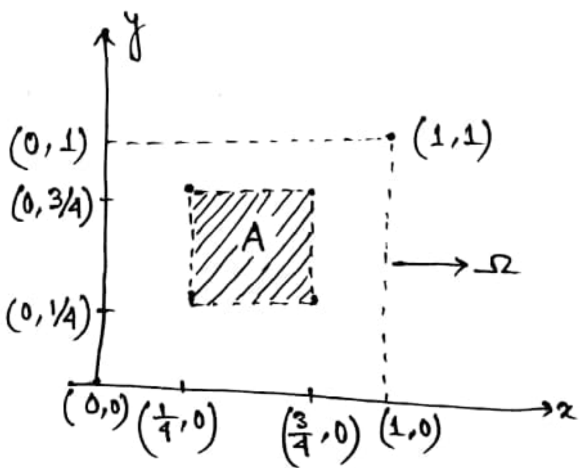
(a) $A = \left\{(x,y) : \frac{1}{4} \le x \le \frac{3}{4}, \frac{1}{4} \le y \le \frac{3}{4}\right\}$.

(b) $B = \left\{(x,y) : \left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{1}{4}\right\}$

Solution:- (a)

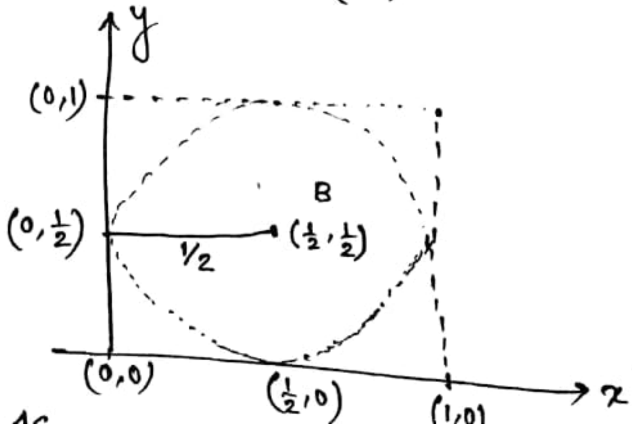
Required probability is

$$P[A] = \frac{\text{Area of } A}{\text{Area of } \Omega} = \frac{1/2 \times 1/2}{1 \times 1} = \frac{1}{4}.$$



(b) Required probability is

$$P[A] = \frac{\text{Area of } B}{\text{Area of } \Omega} = \frac{\pi\left(\frac{1}{2}\right)^2}{1 \times 1} = \frac{\pi}{4}.$$



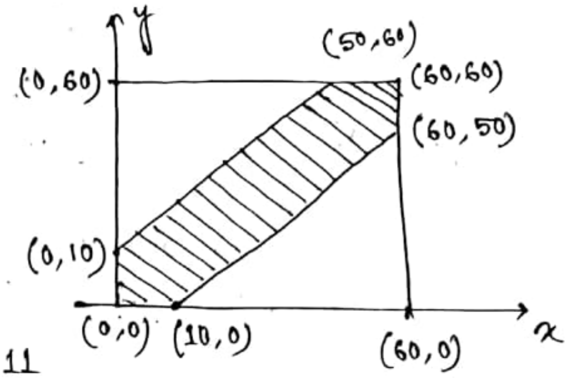
Ex.2. Two persons Amal and Bimal come to the club at random points of time between 6 PM. and 7 PM. and stays for 10 minutes each. What is the chance that they will meet? [ISS'2012]

Solution:-

Let X : The time when Amal come to the club between 6 to 7 PM.
Y : The time when Bimal come to the club between 6 to 7 PM.

Then, $0 \le x, y \le 60$, since 1hr. = 60 minutes.

The Required probability is



$$P[|X-Y| \le 10]$$

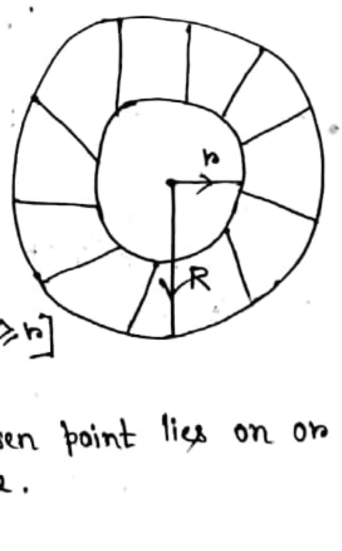
$$= \frac{\text{Area of the shaded region}}{\text{Area of the square}}$$

$$= \frac{60 \times 60 - 50 \times 50}{60 \times 60} = 1 - \frac{25}{36} = \frac{11}{36}.$$

Remark:- If they wait for 20 minutes and then they leave the place, then the required probability will be $\frac{60 \times 60 - 40 \times 40}{60 \times 60} = \frac{5}{9}$, provided each of them comes at random to the spot during the specified time and their times of arriving are independent.

Ex.3. 'n' points are chosen at random and independently of one another inside a sphere of radius R. Find the probability that the distance from the centre of the sphere to the nearest point is not less than $r\ (<R)$.

Solution:- Let $X_1, X_2, \ldots X_n$ be the distances of the chosen n points from the centre of the sphere. Here, $X_1, X_2, \ldots, X_n$ are independently distributed and have the same probability distribution.



Required probability $= P[X_1 \ge r, X_2 \ge r, \ldots, X_n \ge r]$

$$= P[X_1 \ge r] \cdot P[X_2 \ge r] \cdots P[X_n \ge r]$$

$$= \{P[X_1 \ge r]\}^n$$

Here, $P[X_1 \ge r]$ = The probability that a single chosen point lies on or outside the smaller sphere.

$$= \frac{\text{Volume of the shaded region}}{\text{Volume of the sphere of radius } R}$$

$$= \frac{\frac{4}{3}\pi (R^3 - r^3)}{\frac{4}{3}\pi R^3} = \left(1 - \frac{r^3}{R^3}\right).$$

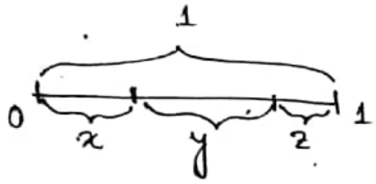
Hence, the required probability $= \left(1 - \frac{r^3}{R^3}\right)^n$.

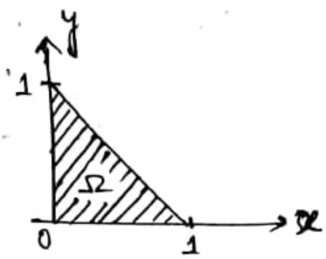
Ex. 4. A bar of unit length is broken into three parts x, y, z. Find the probability that a triangle can be from the resulting parts.

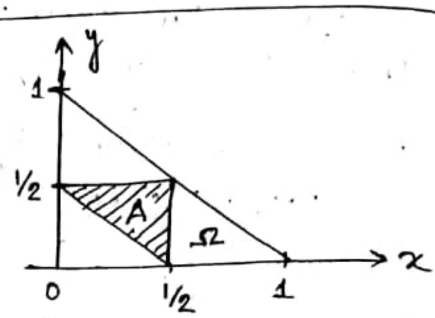
(OR)

Two points are chosen at random from a line segment. Show that the probability that the 3 parts obtained this way form a triangle is $\frac{1}{4}$.

Solution:-







Three parts are: $x, y, z(=1-(x+y))$.

The conditions $x>0, y>0, x+y<1$ are imposed on the quantities x and y; so, the sample space is the interior of a right triangle with unit legs, so, Area of $\Omega = \frac{1}{2}$.

The condition A requiring that a triangle could be formed from the segments $x, y, 1-(x+y)$ reduces to the following conditions: (1) The sum of any two sides is greater than the third side; (2) The difference between any two sides is smaller than third side.

This condition is associated with the triangle domain A.

So, Area of $A = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$.

$$P[A] = \frac{\text{Area of } A}{\text{Area of } \Omega} = \frac{1}{4}.$$

REPEATED TRIALS

- **Bernoulli Trial:-** Repeated independent trials are said to be Bernoulli trials if each trial results in two outcomes, i.e., a success and a failure and the probability of success $p\ (0<p<1)$ remains same throughout the trials.

Example:- Suppose a random experiment is repeated n times independently. Then the occurrance of the event A may be termed as occurance of a success and these repetitions constitute Bernoulli Trials.

Corresponding n Bernoulli trials, one may define n independent random variables $X_1, X_2, \ldots, X_n \ni P(X_i=1) = p = 1-P(X_i=0)\ \forall\ i$.

Here $X_1, X_2, \ldots, X_n$ are called independent Bernoulli RV.

Clearly, PMF of X_i is given by

$$f(x) = p^x(1-p)^{1-x} \quad ; \quad 0<x<1,\ 0<p<1.$$

Ex.1. Two persons A and B toss a fair coin $(n+1)$ times and n times respectively. Find the probability that
(i) A will have as many heads as B (ii) A will have more heads than B.

Solution:- Let X and Y denote the number of heads obtained by A and B, respectively.

(i) $P(X=Y) = P\left(\bigcup_{i=0}^{n} (X=i, Y=i)\right) = P\left[\bigcup_{i=0}^{n} A_i\right]$, since $A_i \cap A_j = \phi$ $\forall i \neq j$

$$= \sum_{i=0}^{n} P(A_i) = \sum_{i=0}^{n} P(X=i)\,P(Y=i)$$

$$= \sum_{i=1}^{n} \frac{\binom{n+1}{i}\binom{n}{i}}{2^{n+1}\cdot 2^{n}} = \frac{\binom{2n+1}{n}}{2^{2n+1}}.$$

(ii) $X = U + V$

U: No of heads in 1st n tosses by A

V: 1 or 0 according as the $(n+1)^{th}$ toss by A results in heads or not.

Y and Y are i.i.d. random variable.

$$P[X > Y]$$

$$= P[U + V > Y]$$

$$= P[U+V>Y \mid V=1]\,P[V=1] + P[U+V>Y \mid V=0]\,P[Y=0]$$

$$= \frac{1}{2}\left\{P[U > Y-1] + P[U > Y]\right\}$$

$$= \frac{1}{2}\left[P(U \geq Y) + P(U < Y)\right]$$

$$= \frac{1}{2}.$$

Ex.2. An urn contains a white balls and b black balls.
(i) n balls are chosen at random WR / WOR.
(ii) balls are drawn one by one WR/WOR till the n white balls being produced.

Solution:-

X : Number of white balls drawn

Y : Number of black balls preceeding the n^{th} white balls.

Z : Number of drawings required to produce n white balls.

Clearly, $Z = n + Y$.

(i) WR:- $$P(X=x) = \binom{n}{x}\left(\frac{a}{a+b}\right)^x\left(\frac{b}{a+b}\right)^{n-x}, \quad x=0,1,2,\dots,n.$$

$$= \binom{n}{x}\frac{a^x\, b^{n-x}}{(a+b)^n}.$$

WOR:- $$P(X=x) = \binom{n}{x} P(W_1)P(W_2|W_1)\cdots\cdots P(W_n^c|W_1\cap W_2\cap\cdots)$$

$$= \binom{n}{x}\cdot\frac{a}{a+b}\cdot\frac{a-b}{a+b-1}\cdots\cdots\frac{a-(x-1)}{a+b-(x-1)}\cdot\frac{b}{a+b-x}\cdot\frac{b-\overline{n-x-}}{a+b-n}$$

$$= \binom{n}{x}\frac{(a)_x\,(b)_{n-x}}{(a+b)_n}$$

$$= \frac{\binom{a}{x}\binom{b}{n-x}}{\binom{a+b}{n}}$$

(ii) WR:- $$P(Y=y) = \binom{n-1+y}{y}\frac{a^n\, b^y}{(a+b)^{n+y}}, \quad y=0,1,2,\dots$$

WOR:- $$P(Y=y) = \binom{n-1+y}{y}\frac{(a)_n\,(b)_y}{(a+b)_{n+y}}, \quad y=0,1,2,\dots,b$$

WR:- $$P(Z=z) = \binom{z-1}{n-1}\frac{a^n\, b^{z-n}}{(a+b)^z}, \quad z=n, n+1, \dots\dots$$

WOR:- $$P(Z=z) = \binom{z-1}{n-1}\frac{(a)_n\,(b)_{z-n}}{(a+b)_z}, \quad z=n, n+1, \dots\dots, b+n.$$

EXPECTATION

Expectation or Mean:– Let X be an RV defined on $(\Omega, \mathcal{A}, P)$.
The expectation or mean of the R.V. X, denoted by $E(X)$, is defined by

$$E(X) = \begin{cases} \sum_{i=1}^{\infty} x_i P[X=x_i], & \text{provided } \sum_{i=1}^{\infty} |x_i| P[X=x_i] \text{ converges,} \\ & \text{if } X \text{ is a discrete R.V. with mass points } x_1, x_2, \ldots \\ \int_{-\infty}^{\infty} x f_n(x)\,dx, & \text{provided } \int_{-\infty}^{\infty} |x| f_n(x)\,dx \text{ converges,} \\ & \text{if } X \text{ is a continuous R.V. with PDF } f_n(x) \end{cases}$$

Remark:– (1) Let $\{(x_i, f_i) : i = 1(1)k, \sum_{i=1}^{n} f_i = n\}$ be a sample of size n from the RV X (discrete).
The sample mean is given by, $\bar{x} = \sum_{i=1}^{k} x_i \cdot \frac{f_i}{n} \rightarrow \sum_{i=1}^{k} x_i P[X=x_i]$,
if $n \to \infty$, by statistical definition of probability, $\bar{x} \to E(X)$ as $n \to \infty$.

(2) Consider a discrete random variable X which takes countable infinite number of values x_i with positive probabilities p_i, $i = 1, 2, \ldots$. If $\sum_{i=1}^{\infty} x_i p_i$ converges conditionally, then the series for different re-arrangements of the terms takes different values of $x_i p_i$. If the mean $E(X)$ is to serve as a measure of central tendency of a distribution, then the order of terms should have nothing to do with it. To make the series $\sum_{i=1}^{\infty} x_i p_i$ independent of the order of the terms, we require that the series to be absolutely convergent, i.e., $\sum_{i=1}^{\infty} |x_i| p_i < \infty$, i.e., $E(X)$ exists. Note that $\sum_{i=1}^{\infty} x_i p_i$ may converge but $\sum_{i=1}^{\infty} |x_i| p_i$ may not; in that case we say that $E(X)$ doesn't exist.

Case-I :– Discrete Distribution with a finite number of mass points

Suppose X is an RV having a discrete distribution with the PMF f, if the mass points of X are $x_1, x_2, \ldots, x_n$. Then by definition, the expected value of X is $\sum_{i=1}^{n} x_i f(x_i)$. It is denoted by $E(X)$ or μ. Thus $E(X)$ is the sum of the products of the mass points by their respective probabilities. **Ex.** Let X denotes the points obtained in throwing a fair die, then $E(X) = \frac{1}{6}(1+2+3+4+5+6) = 3.5$.

Case II :– Discrete Distribution with countably many mass points

Let f be the PMF of X and the mass points be $x_1, x_2, \ldots$; we may like to take the sum of the series $\sum_{i=1}^{\infty} x_i f(x_i)$ as the expected value of X. The $E(X)$ is said to exist if the series $\sum_{i=1}^{\infty} x_i f(x_i)$ is absolutely convergent, i.e., $\sum_{i=1}^{\infty} |x_i| f(x_i) < \infty$. It is defined by

$$E(X) = \sum_{i=1}^{\infty} x_i f(x_i).$$

Ex. 1. Let X be the number of trials required to get the first success in a series of Bernoulli Trials with probability of success p. Then find the expected value of X.

Solution:- $f(x) = \begin{cases} pq^{x-1} & , \text{if } x = 1, 2, \dots \\ 0 & , \text{Otherwise} \end{cases}$

$$\sum_{i=1}^{\infty} |x_i| f(x_i) = p\left[1 + 2q + 3q^2 + \cdots\right] \text{, the series converges since } 0<q<1.$$

$$= \frac{p}{(1-q)^2}$$

$$E(X) = p(1-q)^{-2} = \frac{p}{p^2} = \frac{1}{p}.$$

Ex. 2. A coin is tossed until a head appears. Let X be the number of tosses required. Calculate the value of the number of trials required (including the last toss in which a head has to appear)

Solution:- $P(X=i) = \frac{1}{2^i}, \; i = 1, 2, 3, \dots$

$$E(X) = \sum_{i=1}^{\infty} i \cdot \frac{1}{2^i} = \frac{1}{2} + 2 \cdot \frac{1}{2^2} + 3 \cdot \frac{1}{2^3} + \cdots = 2.$$

Indicator Random Variable:- For an event associated with a random experiment we define an RV $I_A(\omega)$ on each point ω in the sample space $\Omega \ni$

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

$I_A(\omega)$ is called the indicator RV (function) of the set A.

$$\therefore E\{I_A(\omega)\} = 1\sum_{\omega \in A} P(\omega) + 0\sum_{\omega \notin A} P(\omega) = \sum_{\omega \in A} P(\omega) = P(A).$$

The probability of an event A is the expectation of its indicator RV $I_A(\omega)$.

Case III :- Absolutely Continuous distribution

Suppose X has the absolutely continuous distribution with pdf f. The $E(X)$ is said to exist if the integral $\int_{-\infty}^{\infty} x.f(x)dx$ is absolutely convergent, i.e. $\int_{-\infty}^{\infty} |x| f(x) dx < \infty$.

In case, it exists, then expectation is defined as

$$E(X) = \int_{-\infty}^{\infty} x.f(x)dx.$$

Ex.3. Let X has the pdf $f(x) = \begin{cases} \frac{1}{\theta} & \text{if } 0<x<\theta \\ 0 & \text{Otherwise} \end{cases}$. Find $E(X)$?

Solution:-

$$\int_{-\infty}^{\infty} |x| f(x)dx = \int_{-\infty}^{0} |x| f(x)dx + \int_{0}^{\theta} |x| f(x)dx + \int_{\theta}^{\infty} |x| f(x)dx$$

$$= \int_{0}^{\theta} |x| f(x)\,dx$$

$$= \frac{1}{\theta} \int_{0}^{\theta} |x|dx = \frac{1}{\theta}\int_{0}^{\theta} x\,dx$$

Hence, the integral is convergent. Hence $E(X)$ exists and equal to

$$E(X) = \frac{1}{\theta}\left[\frac{x^2}{2}\right]_0^{\theta} = \frac{\theta}{2}.$$

Ex.4. Let X be an RV takes the values $x_i = (-1)^{i-1}(i+1)$, with probability $p_i = \frac{1}{i(i+1)}$, $i \in \mathbb{N}$. Does $E(X)$ exists?

Solution:- Note that, $\sum_{i=1}^{\infty} x_i P[X=x_i] = \sum_{i=1}^{\infty} x_i p_i = \sum_{i=1}^{\infty} (-1)^{i-1} \cdot (i+1) \cdot \frac{1}{(i+1)i}$

$$= \sum_{i=1}^{\infty} (-1)^{i-1} \cdot \frac{1}{i},$$

here, $\sum_{i=1}^{\infty} |x_i| p_i = \sum_{i=1}^{\infty} |x_i| p_i = \sum_{i=1}^{\infty} \frac{1}{i}$ diverges but

$\sum_{i=1}^{\infty} x_i p_i = \sum_{i=1}^{\infty} (-1)^{i-1} \cdot \frac{1}{i}$ converges conditionally.

As, $\sum_{i=1}^{\infty} |x_i| p_i$ diverges, $E(X)$ doesn't exist; although, $\sum_{i=1}^{\infty} x_i p_i$ converges conditionally.

Ex.5. Let X be an RV with PDF $f(x) = \begin{cases} \frac{1}{x^2} &, 1<x<\infty \\ 0 &, \text{Otherwise} \end{cases}$. Show that, $E(X)$ doesn't exist.

Solution:-

$$\int_{-\infty}^{\infty} |x| f(x)dx = \int_{-\infty}^{1} |x| \cdot 0\,dx + \int_{1}^{\infty} |x| \cdot \frac{1}{x^2}dx = \int_{1}^{\infty} \frac{1}{x}dx = \infty.$$

$\therefore E(X)$ doesnot exist.

Ex. 6. X has a continuous distribution pdf $f(x) = \frac{1}{\pi(1+x^2)}, x \in \mathbb{R}$. Show that $E(X)$ doesnot exist.

Solution:-

Note that $\displaystyle\int_{-\infty}^{\infty} |x| f(x)\,dx = \int_{-\infty}^{\infty} |x| \cdot \frac{1}{\pi(1+x^2)} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$

$$= \frac{1}{2\pi}\int_0^{\infty} \frac{dt}{t} \qquad \left[\begin{array}{l} 1+x^2 = t \\ x\,dx = dt/2 \\ \begin{array}{c|c|c} x & -\infty & \infty \\ \hline t & 0 & \infty \end{array} \end{array}\right]$$

$$= \frac{1}{2\pi}\left[\log t\right]_0^{\infty}$$

$$= \infty.$$

$\therefore E(X)$ does not exist.

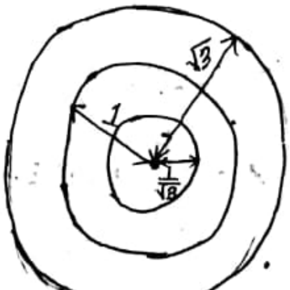
Ex. 7. A target is made of 3 concentric circles of radii $\frac{1}{\sqrt{3}}$, 1 and $\sqrt{3}$ feet. Shots within the inner circle count 4 points, within the next ring 3 points and within the third ring 2 points. Shots outside the target count zero. Let X be the distance of the hit from the centre (in feet) and the pdf of X be

$$f(x) = \begin{cases} \frac{2}{\pi(1+x^2)}, & x > 0 \\ 0, & \text{o.w} \end{cases}$$

What will be the expected value of the score in (a) one shot; (b) a set of 5 shots?

Solution:- X: the hit from centre, Y: the score.

Mass points of Y	Corresponding probabilities
4	$P(Y=4) = P\left(X \in \left(0, \frac{1}{\sqrt{3}}\right)\right)$
3	$P(Y=3) = P\left(X \in \left(\frac{1}{\sqrt{3}}, 1\right)\right)$
2	$P(Y=2) = P\left(X \in (1, \sqrt{3})\right)$
0	$P(Y=0) = P\left(X \in (\sqrt{3}, \infty)\right)$



(a) $E(Y) = 4 \cdot P[Y=4] + 3 \cdot P[Y=3] + 2 \cdot P[Y=2] + 0 \cdot P(Y=0)$.

$= 4 \cdot P\left[0 < X \le \frac{1}{\sqrt{3}}\right] + 3 \cdot P\left[\frac{1}{\sqrt{3}} < X \le 1\right] + 2 \cdot P\left[1 < X \le \sqrt{3}\right]$

Now, $\displaystyle P[a < X \le b] = F(b) - F(a) = \int_a^b f(x)\,dx = \int_a^b \frac{2}{\pi(1+x^2)}\,dx$

$$= \frac{2}{\pi}\left[\tan^{-1} x\right]_a^b = \frac{2}{\pi}\left[\tan^{-1} b - \tan^{-1} a\right]$$

so, $E(Y) = \frac{2}{\pi}\left\{4\left(\frac{\pi}{6} - 0\right) + 3\left(\frac{\pi}{4} - \frac{\pi}{6}\right) + 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\right\} = \frac{13}{6}$.

(b) Let Y_i denote the score in the i^{th} shot, $i = 1(1)5$. Then, total score in 5 shots is $Z = \sum_{i=1}^{5} E(Y_i) = 5 \cdot E(Y_1) = 5 \cdot \frac{13}{6} = \frac{65}{6}$.

THEOREMS ON EXPECTATION :-

Theorem. 1. If the RV $X = c$, a finite real number with probability 1, i.e. $X = C$ almost everywhere, then $E(X) = c$.

Proof:- $E(X) = \int X dP$

$$= \int_{X=C} x dP = c \cdot P(X=c) = c.$$

Theorem. 2. If c is a finite real number and $E(X)$ exists then $E(cX)$ also exists and equals to $c \cdot E(X)$.

Proof:- Since $E(X)$ exists, so $\int |X| dP < \infty$.

Now $\int |cX| dP = |c| \int |X| dP$, since c is finite, so $\int |cX| dP < \infty$.

So, $E(cX)$ exists and equals $c \int X dP = cE(X)$.

Theorem. 3. If X and Y are both R.V.s, and $E(X)$ and $E(Y)$ exist, then $E(X+Y)$ exists and equals to $E(X) + E(Y)$

Sol. $E(X)$ & $E(Y)$ exist. $\Rightarrow \int |X| dP < \infty$, $\int |Y| dP < \infty$.

Now, $\int |X+Y| dP \le \int |X| dP + \int |Y| dP < \infty$.

So, $E(X+Y)$ exist. $\int (X+Y) dP = \int X dP + \int Y dP = E(X) + E(Y)$.

Theorem. 4. If $E(X)$ exists, then $|E(X)| \le E|X|$.

Proof:- $\left| \int X dP \right| \le \int |X| dP \Rightarrow |E(X)| \le E|X|$, since $E(X)$ exists, so X is integrable w.r.t. P.

Theorem. 5. If $E(X)$ exists and a and b are real numbers $\ni a \le X \le b$, then $a \le E(X) \le b$.

Sol. $a \le X \le b$

$$\int a dP \le \int X dP \le \int b dP$$

$\therefore a \le E(X) \le b$.

Theorem. 6. If $E(X)$ exists, then for every real number a and b, $E(a+bX)$ exists and equals to $a+bE(X)$.

Proof:- As $E(X)$ exists, so $\int |X| dP < \infty$.

$$\int |a+bX| dP \le \int \{|a| + |b||X|\} dP$$
$$= \int |a| dP + \int |b||X| dP$$
$< \infty$, since a and b are both finite.

So, $E(a+bX)$ exists and equals $\int (a+bX) dP = \int a dP + \int bX dP$
$$= a + bE(X).$$

Corollary:- Put $a = -E(X)$ and $b = 1$, then $E[X - E(X)] = -E(X) + E(X)$
$$= 0.$$

Theorem. 7. Let X be a bounded RV, show that $E(X)$ exists.

Proof:- Since X is bounded, $\exists$ a real number $M \ni P[|X| \le M] = 1$.

Now, $E|X| = \int_{-\infty}^{\infty} |x| dF(x) \le M \int_{-\infty}^{\infty} dF(x) = M < \infty$.

$\Rightarrow E(X)$ exists.

In general, $E|X^r| = \int_{-\infty}^{\infty} |x|^r dF(x) \le M^r \int_{-\infty}^{\infty} dF(x) = M^r < \infty$.

$\therefore \mu_r' = E(X^r)$ also exists.

Theorem. 8. If $E(X)$ exists and $X \geq 0$ almost everywhere with prob. 1, will be greater than equal to 0.

Proof:- $E(X)$ exists and X is integrable w.r.t. P.

Now, $\int X dP = \int X^+ dP + \int X^- dP$, where $X^+ = \begin{cases} X & \text{if } X > 0 \\ 0 & \text{if } X < 0 \end{cases}$

and $X^- = \begin{cases} X & \text{if } X < 0 \\ 0 & \text{if } X > 0 \end{cases}$

$\because X > 0$ almost everywhere, so,

$$\int X^- dP = 0 \text{ and } \int X^+ dP \geq 0.$$

$$\therefore \int X dP \geq 0 \Rightarrow E(X) \geq 0.$$

Theorem. 9. If $E(X)$ and $E(Y)$ both exist and $X \geq Y$ almost everywhere, then $E(X) \geq E(Y)$.

Proof:- Let $X - Y = Z$, $X \geq Y$ a.e., then $X - Y \geq 0$ a.e.
i.e. $Z \geq 0$ a.e.

So, from previous theorem, $E(Z) \geq 0 \Leftrightarrow E(X) \geq E(Y)$.

[**Note:-** Define the RV XI_A such that $XI_{A}(\omega) = X(\omega)$ if $\omega \in A$
$= 0$ if $\omega \notin A$.

Define, A is a measurable set, If $P(A) > 0$ and $E(XI_A)$ exists, then $\frac{E(XI_A)}{P(A)}$ is called the conditional expectation of X given A.

$$E(X|A) = \frac{E(XI_A)}{P(A)}.$$

Now, $E(XI_A)$ exists if $E(X)$ exists, since $|XI_A| \leq |X|$.]

Theorem. 10. Let $\{A\}$ be a measurable partition of the sample space $\ni$ $P(A_i) > 0$ $\forall i$ and let $E(X)$ exists then $E(X) = \sum_i P(A_i) E(X|A_i)$, $\bigcup_i A_i = \Omega$.

Proof:-
$$E(X) = \int_{\Omega} x\,dP = \int_{\cup A_i} x\,dP = \sum_i \int_{A_i} x\,dP = \sum_i \int_{A_i} XI_{A_i}\,dP = \sum_i E(XI_{A_i})$$
$$= \sum_i P(A_i) E(X|A_i).$$

Problem:- 1. An urn is filled with N balls by a random mechanism so that the no. of white balls in the urn is an RV, whose expectation is M. If a ball is drawn at random then what is the probability that it is white?

Solution:- Let B_i : Event that there are i white balls $(i = 1(1)N)$.
B_i's are mutually exclusive as well as exhaustive.
A : Event that the ball drawn is white.

Then from the theorem of total probability, $P(A) = \sum_{i=1}^{N} P(B_i) P(A|B_i)$;

Now, $P(A|B_i) = \frac{i}{N}$ (since if there are i white balls, then the prob. of drawing a white ball is $\frac{i}{N}$).

So, $P(A) = \sum_{i=1}^{N} \frac{i}{N} . P(B_i) = \frac{1}{N} \sum_{i=1}^{N} i P(B_i)$.

But, $\sum_{i=1}^{N} i P(B_i)$ is the expected no. of white balls $= M$.

Hence, $P(A) = \frac{M}{N}$.

Problem:- 2. The RV X takes non-negative integer values. Show that $E(X)=\sum_{K=0}^{\infty} P(X>K)$, provided the series on the right hand side converges.

Solution:-

$$E(X)=\sum_{x=0}^{\infty} x P[X=x]$$

$$= P(1)+2P(2)+3P(3)+\cdots\cdots$$

$$=\{P(1)+P(2)+P(3)+\cdots\}+\{P(2)+P(3)+\cdots\}+\{P(3)+P(4)+\cdots\}+\cdots$$

[By the rearrangement of the terms]

$$=P[X>0]+P[X>1]+\cdots\cdots$$

$$=\sum_{K=0}^{\infty} P(X>K).$$

Problem:- 3. Find the mean of the truncated Poisson distribution with pmf

$$f(x)=\begin{cases}\dfrac{e^{-\lambda}}{1-e^{-\lambda}}\cdot\dfrac{\lambda^x}{x!} & \text{if } x=1,2,3,\ldots\ldots\\ 0 & \text{ow}\end{cases}$$

Solution:-

$$E(X)=\sum_{x=1}^{\infty} x\cdot\frac{e^{-\lambda}}{1-e^{-\lambda}}\cdot\frac{\lambda^x}{x!}$$

$$=\frac{\lambda}{1-e^{-\lambda}}\sum_{x=1}^{\infty}\frac{e^{-\lambda}\cdot\lambda^{x-1}}{(x-1)!}=\frac{\lambda}{1-e^{-\lambda}}.$$

Problem:- 4. Show that for triangular distribution with pdf

$$f(x)=\begin{cases}\dfrac{1}{\alpha}\left[1-\dfrac{|x-\theta|}{\alpha}\right] & \text{if } |x-\theta|\le\alpha\\ 0 & \text{ow}\end{cases}$$

the mean is equal to θ.

Solution:-

$$E(X)=E(X-\theta+\theta)=E(X-\theta)+E(\theta)$$

$$=\int_{\theta-\alpha}^{\theta+\alpha}(x-\theta)f(x)\,dx+\theta$$

$x-\theta=z$

$dx=dz$

x	θ	$\theta-\alpha$	$\theta+\alpha$
z	0	$-\alpha$	α

$$=\theta+\int_{\theta-\alpha}^{\theta}(x-\theta)\left[1+\frac{(x-\theta)}{\alpha}\right]dx+\int_{\theta}^{\theta+\alpha}(x-\theta)\left[1-\frac{(x-\theta)}{\alpha}\right]dx$$

$$=\theta+\int_{-\alpha}^{0}z\,dz+\int_{-\alpha}^{0}z^2\,dz+\int_{0}^{\alpha}z\,dz-\int_{0}^{\alpha}z^2\,dz$$

$$=\theta.$$

Expectation of a function of Random Variable :- If X is an RV defined on $(\Omega, \mathcal{A}, P)$ and $g(x)$ be a function of X, then $g(x)$ is also an RV defined on $(\Omega, \mathcal{A}, P)$ and the expected value of $g(x)$ is defined as

$$E[g(X)] = \begin{cases} \sum_{i=1}^{\infty} g(x_i) P[X = x_i], & \text{provided } \sum_i |g(x_i)| P[X = x_i] \text{ converges, if } X \text{ is of discrete type.} \\ \int_{-\infty}^{\infty} g(x_i) f_X(x) dx, & \text{provided } \int_{-\infty}^{\infty} |g(x_i)| f_X(x) dx \text{ converges, if } X \text{ is of continuous type with pdf } f_X(x). \end{cases}$$

Alternative Definition:- Let $g(x)$ be a function of x which is itself an RV then expectation of $g(X)$ is said to exist if $\int_{-\infty}^{\infty} |g(x)| dF(x) < \infty$.

Then $E(g(X)) = \int_{-\infty}^{\infty} g(x) dF(x)$.

Problem:- 5. From an urn with a white and b black balls, c balls are taken at random and transferred to another which contains α white and β black balls. Show that the probability of getting a white ball from the 2nd urn after the transfer is

$$\frac{\frac{ca}{a+b} + \alpha}{\alpha + \beta + c}.$$

Solution:- X : No. of white balls transferred to the 2nd urn.

Let A : Event that a white ball from 2nd urn is obtained.

B_x : Event that x white balls are transfered, $x = 0, 1, 2, \ldots, c$.

From the theorem of total probability,

$$P(A) = \sum_x P(B_x) P(A|B_x) = \sum_x P(B_x) \cdot \frac{\alpha + x}{\alpha + \beta + c}$$

$$= \frac{\alpha}{\alpha+\beta+c} \sum_x P(B_x) + \sum_x \frac{x}{\alpha+\beta+c} P(B_x)$$

$$= \frac{\alpha}{\alpha+\beta+c} \cdot 1 + \frac{E(X)}{\alpha+\beta+c}$$

$$= \frac{\alpha}{\alpha+\beta+c} + \frac{\frac{ca}{a+b}}{\alpha+\beta+c}$$

$$= \frac{\frac{ca}{a+b} + \alpha}{\alpha+\beta+c}.$$

$$P(B_x) = \frac{\binom{a}{x}\binom{b}{c-x}}{\binom{a+b}{c}}$$

$x = 0, 1, 2, \ldots, c$.

$$E(X) = \sum_x x P(B_x) = \frac{ca}{a+b}$$

= Mean of Hypergeometric Distribution.

Result :- 1. If X is a non-negative integer valued random variable, then show that $E(X) = \sum_{x=0}^{\infty} \{1-F(x)\}$, provided $E(X)$ exists.

Proof:- $E(X) = \sum_{x=0}^{\infty} x p_x$, where $p_x = P[X=x]$.

$= 0.p_0 + 1.p_1 + 2.p_2 + \cdots\cdots$

$= (p_1 + p_2 + \cdots) + (p_2 + p_3 + \cdots) + (p_3 + p_4 + \cdots) + \cdots\cdots$

[Rearranging of the terms of the series is possible since $E(X)$ exists.]

$= P[X>0] + P[X>1] + P[X>2] + \cdots\cdots$

$= \sum_{x=0}^{\infty} P[X>x] = \sum_{x=0}^{\infty} [1 - P[X \le x]] = \sum_{x=0}^{\infty} \{1-F(x)\}$.

Result :- 2. Suppose X is a non-negative RV whose mean exists and equals μ. Prove that —

(a) $\lim_{x\to\infty} x[1-F(x)] = 0$,

and hence in case X is absolutely continuous, then

(b) $\int_0^{\infty} [1-F(x)]dx = \mu$.

Proof:- (a) $E(X)$ exists and equals to μ, Note that

$$\lim_{x\to\infty} \int_x^{\infty} u\,dF(u) = 0.$$

Again, $\int_x^{\infty} u\,dF(u) \geq x\int_x^{\infty} dF(u) = x[1-F(x)] \geq 0$ [$\because X$ is non-negative]

$$\therefore \int_x^{\infty} u\,dF(u) \geq x[1-F(x)] \geq 0.$$

$$\therefore \lim_{x\to\infty} \int_x^{\infty} u\,dF(u) \geq \lim_{x\to\infty} x[1-F(x)] \geq 0$$

$$\therefore 0 \geq \lim_{x\to\infty} x[1-F(x)] \geq 0$$

$$\therefore \lim_{x\to\infty} x[1-F(x)] = 0.$$

(b) $\mu = \int_0^\infty x f(x)\,dx$

$$= \int_0^\infty x\left[-d(1-F(x))\right], \quad \text{since } f(x) = -\frac{d}{dx}\left[1-F(x)\right]$$

$$= \left[-x(1-F(x))\right]_0^\infty + \int_0^\infty [1-F(x)]\,dx$$

$$= \lim_{u\to\infty}\left[-x(1-F(x))\right]_0^u + \int_0^\infty [1-F(x)]\,dx$$

$$= \lim_{u\to\infty}\left[-u(1-F(u))\right] + \int_0^\infty [1-F(x)]\,dx$$

$$= \int_0^\infty [1-F(x)]\,dx \quad [\text{From part (a)}]$$

Also, see —

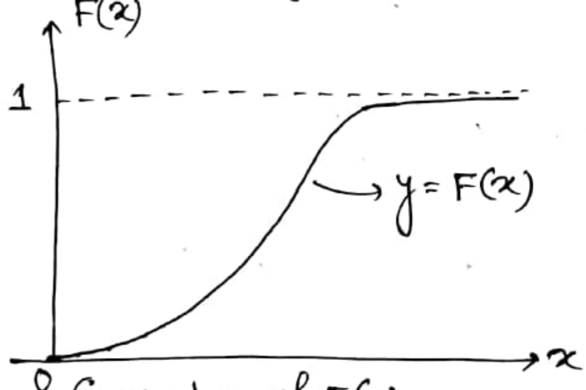
$$E(X^2) = \int_0^\infty x^2 f(x)\,dx$$

$$= \int_0^\infty x^2\left[-d(1-F(x))\right]$$

$$= \left[-x^2(1-F(x))\right]_0^\infty + 2\int_0^\infty x[1-F(x)]\,dx$$

$$= \int_0^\infty 2x[1-F(x)]\,dx.$$

Remark:- 1. $E(X) = \int_0^\infty [1-F(x)]\,dx.$



∴ Geometry of E(X):—

2. $$E(X) = \int_0^\infty x f(x)\,dx = \int_0^t x f(x)\,dx + \int_t^\infty x f(x)\,dx$$

Hence $E(X)$ exists iff $\int_0^\infty x f(x)\,dx$ converges,

iff $\int_t^\infty x f(x)\,dx$ converges.

Again, $$E(X) = \int_0^\infty [1-F(x)]\,dx = \int_0^t [1-F(x)]\,dx + \int_t^\infty [1-F(x)]\,dx$$

∴ $E(X)$ exists iff $\int_t^\infty [1-F(x)]\,dx$ converges

iff $\int_t^\infty P[X>x]\,dx$ converges.

Result:-3. (a) If X is any RV $\ni E(X)$ exists, then show that

$$E(X) = \int_0^{\infty} [1 - F(x) - F(-x)]\,dx,$$

(b) Hence show that $E(X^2) = \int_0^{\infty} 2x\,[1 - F(x) - F(-x)]\,dx.$

Proof:- (a)

$$\int_0^{\infty} [1 - F(x) - F(-x)]\,dx$$

$$= \int_0^{\infty} [1 - F(x)]\,dx - \int_0^{\infty} F(-x)\,dx$$

$$= \int_0^{\infty} x f(x)\,dx - \int_{-\infty}^{0} (-x) f(x)\,dx$$

[From Result 2(b)]

$$= \int_{-\infty}^{0} x f(x)\,dx + \int_0^{\infty} x f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x f(x)\,dx = E(X).$$

(b)

$$\int_0^{\infty} 2x\,[1 - F(x) - F(-x)]\,dx$$

$$= \int_0^{\infty} 2x\,[1 - F(x)]\,dx - \int_0^{\infty} 2x F(-x)\,dx$$

$$= \int_0^{\infty} x^2 f(x)\,dx + \int_{-\infty}^{0} x^2 f(x)\,dx$$

[From Result.2(b)]

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= E(X^2).$$

Ex. 1. Evaluate $E(X)$ for the RV X with DF

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1-(1-x)^n & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

Solution:- For a non-negative continuous RV X with DF $F(x)$,

$$E(X) = \int_0^\infty [1 - F(x)]\,dx$$

$$= \int_0^1 [1-F(x)]\,dx + \int_1^\infty [1-F(x)]\,dx$$

$$= \int_0^1 (1-x)^n\,dx + \int_1^\infty 0 \cdot dx$$

$$= \int_0^1 (1-x)^n\,dx = \left[-\frac{(1-x)^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}.$$

Ex. 2. Let X and Y be two non-negative continuous RV with respective DF $F(x)$ and $G(x)$ $\ni$ $F(x) \geq G(x)$ $\forall\, x \geq 0$. If $E(X)$ and $E(Y)$ exist, then show that $E(X) \leq E(Y)$.

Solution:- As X and Y are two non-negative continuous RV's, then

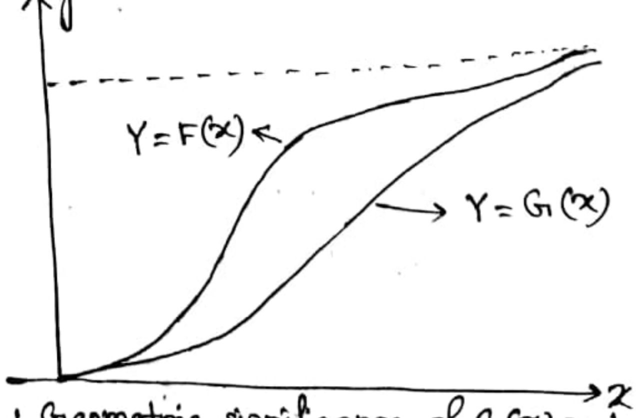
$$E(X) = \int_0^\infty \{1 - F(x)\}\,dx, \quad E(Y) = \int_0^\infty \{1 - G(x)\}\,dx.$$

Hence, $F(x) \geq G(x)$ $\forall\, x \geq 0$

$$1 - F(x) \leq 1 - G(x) \quad \forall\, x \geq 0$$

$$\Rightarrow \int_0^\infty \{1-F(x)\}\,dx \leq \int_0^\infty \{1-G(x)\}\,dx, \quad x \geq 0$$

$$\Rightarrow E(X) \leq E(Y)$$



-: Geometric significance of $G(x)$ and $F(x)$:-

Ex.3. Find the expected number of throws of a fair dice until a six is obtained.

Solution:- Consider 'getting a six' in a throw of a fair dice as success.

Let, X be the number of throws required to get first success.

Then $P[X=x] = P$[The first $(x-1)$ throws result in failures and a success occurs, at the x^{th} throw]

$$= P[\underbrace{F\,F\,\ldots\ldots F}_{x-1 \text{ times}}\,S]$$

$= \{P[F]\}^{x-1}\{P[S]\}$, due to independence.

$$= \left(\frac{5}{6}\right)^{x-1}\left(\frac{1}{6}\right), x = 1, 2, 3, \ldots$$

$$\therefore E(X) = \sum_{x=1}^{\infty} x.P[X=x] = \sum_{x=1}^{\infty} x\left(\frac{5}{6}\right)^{x-1}\left(\frac{1}{6}\right)$$

$$= \frac{1}{6}\sum_{x=1}^{\infty} x\left(\frac{5}{6}\right)^{x-1}$$

$$= \frac{1}{6}\cdot\left(1-\frac{5}{6}\right)^{-2} = 6.$$

Ex.4. Balls are taken one by one with replacement out of an urn containing 'a' white balls and 'b' black balls until the first white ball is drawn. What is the expectation of the number of black balls preceeding the first white ball.

Solution:- $P[\text{A white ball is drawn}] = \frac{a}{a+b} = p$,

$P[\text{A black ball is drawn}] = \frac{b}{a+b} = q$.

X: An RV denoting the number of black balls drawn,

P[No. of black balls drawn preceeding the first white ball]

$= pq^x$; $x = 0, 1, 2, \ldots$

So, $f(x) = \begin{cases} pq^x, & x = 0, 1, 2, \ldots \\ 0, & ow \end{cases}$

Now, $\sum f(x) = 1$

$\Rightarrow \sum_{x=0}^{\infty} p(1-p)^x = 1 \ldots\ldots (*)$

Differentiating (*) w.r.t. p,

$$\sum_x q^x - \sum_x x.pq^{x-1} = 0$$

$$\Rightarrow \sum xpq^{x-1} = \frac{1}{1-q}$$

$$\Rightarrow \frac{1}{q}\sum_x xpq^x = \frac{1}{p}$$

$$\Rightarrow E(X) = \frac{q}{p} = \frac{\frac{b}{a+b}}{\frac{a}{a+b}} = \frac{b}{a} \quad [\because a+b \neq 0]$$

Ex. 5. From an urn containing N identical tickets numbered 1 to N, n tickets are drawn with replacement. Let X be the largest number drawn. Hence find $E(X)$. Also, show that, for large N,

$$E(X) \simeq \frac{nN}{n+1}.$$

Solution:- Note that, $P[X \le x]$ = Probability that the largest number in n drawn tickets with replacement is less than or equal to x.

= Prob. that each of n drawn ticket is less than equal to x.

$$= \frac{x^n}{N^n}, \quad x = 1(1)N.$$

Hence, $F_X(x) = \dfrac{x^n}{N^n}$, $x = 1(1)N$.

For non-negative integer valued R.V. X, we have

$$E(X) = \sum_{x=1}^{\infty} [1 - F_X(x)]$$

$$= \sum_{x=1}^{N-1} \left\{1 - \frac{x^n}{N^n}\right\} + \sum_{x=N}^{\infty} (1-1)$$

$$= N - \frac{1}{N^n} \sum_{x=0}^{N-1} x^n.$$

For large N,

$$\frac{1}{N} \sum_{x=0}^{N-1} \left(\frac{x}{N}\right)^n \simeq \int_0^1 y^n \, dy, \text{ for large } N.$$

$$= \frac{1}{n+1}.$$

$$\Rightarrow \frac{1}{N^n} \sum_{x=0}^{N-1} x^n = \frac{N}{n+1}, \text{ for large } N.$$

Now, $$E(X) = N - \frac{1}{N^n} \sum_{x=0}^{N-1} x^n$$

$$= N - \frac{N}{n+1}$$

$$= \frac{nN}{n+1}, \text{ for large } N.$$

Moments :~

If we take $g(X) = X^r$, $r \in N$, then $\mu_r' = E(X^r)$, if it exists, is called the r^{th} order raw moment of X.

[For a sample data, $\{(x_i, f_i) : i = 1(1)k, \sum_{i=1}^{k} f_i = n\}$, the r^{th} order sample raw moment, $m_r' = \frac{1}{n}\sum_i x_i^r f_i$. Now,

$$m_r' = \sum_{i=1}^{k} x_i^r \cdot \frac{f_i}{n} \longrightarrow \sum_{i=1}^{k} x_i^r \cdot P[X = x_i] \text{ as } n \to \infty,$$ by statistical definition of probability]

Here, $\mu_1' = E(X)$ is the mean of X.

Now, take $g(X) = \{X - E(X)\}^r$, then

$\mu_r = E(X - E(X))^r$, if it exists, is called the r^{th} order central moment of X.

Variance of X :~

The 2^{nd} order central moment of X, $\mu_2 = E(X - E(X))^2$, is called the variance of X and it is denoted by $Var(X)$ or σ_X^2.

Now,
$$\begin{aligned}
\sigma_X^2 = Var(X) &= E\{X - E(X)\}^2 \\
&= E(X - \mu)^2, \quad \mu = E(X). \\
&= E(X^2 - 2\mu X + \mu^2) \\
&= E(X^2) - 2\mu E(X) + \mu^2 \\
&= E(X^2) - \mu^2 \\
&= E(X^2) - E^2(X).
\end{aligned}$$

- **Ex.1.** If X is a non-negative R.V., then $E(X) \geq 0$.

Sol. Let X be a discrete RV with mass points $x_1, x_2, \ldots\ldots$.
As X is non-negative RV, $\therefore x_i \geq 0 \;\; \forall i = 1, 2, \ldots$

Now, $E(X) = \sum_{i=1}^{\infty} x_i \cdot P[X = x_i] \geq 0$ as $P[X = x_i] > 0 \;\forall i$.

- **Ex.2.** If X is a non-negative R.V. and $E(X) = 0$, then $P[X = 0] = 1$.

Sol. $E(X) = 0 = \sum_{i=1}^{\infty} x_i P[X = x_i]$, where $x_i \geq 0$, $P[X = x_i] > 0 \;\forall i$

So, $x_i \cdot P[X = x_i] = 0 \;\; \forall i$

$\Rightarrow x_i = 0 \;\; \forall i$ as $P[X = x_i] > 0$.

$\Rightarrow X = 0$ with probability 1.

i.e. $P[X = 0] = 1$.

Problem:- Show that if X is an RV such that $P[a \le X \le b] = 1$, then $E(X)$ and $Var(X)$ exist and $a \le E(X) \le b$ and $Var(X) \le \frac{(b-a)^2}{4}$.

Solution:- Let X be an R.V. of discrete type.

Then $a \le x_i \le b \;\; \forall i$ as, $P[a \le X \le b] = 1$.

$\Rightarrow aP[X = x_i] \le x_i P[X = x_i] \le bP[X = x_i]$

$\Rightarrow a \le E(X) \le b$.

Now, $Var(X) = E\{X - E(X)\}^2 \le E\left(X - \frac{a+b}{2}\right)^2$.

[$\because$ Variance is least mean square deviation]

Now, $a \le x_i \le b$

$$a - \frac{a+b}{2} \le x_i - \frac{a+b}{2} \le b - \frac{a+b}{2}$$

$$\Rightarrow \left(x_i - \frac{a+b}{2}\right)^2 \le \left(\frac{b-a}{2}\right)^2$$

$$\Rightarrow \sum_{i=1}^{\infty}\left(x_i - \frac{a+b}{2}\right)^2 P[X = x_i] \le \left(\frac{b-a}{2}\right)^2 \sum_{i=1}^{\infty} P[X = x_i]$$

$$\Rightarrow E\left(X - \frac{a+b}{2}\right)^2 \le \left(\frac{b-a}{2}\right)^2 \Rightarrow Var(X) \le \frac{(b-a)^2}{4}.$$

Result:- 1. Show that $E(X-a)^2$ is minimum when $a = E(X)$;

$Var(X) \le E(X-a)^2$.

Solution:-

$$E(X-a)^2 = E\{X - E(X) + E(X) - a\}^2$$
$$= E\{X - E(X)\}^2 + \{E(X) - a\}^2$$
$$= Var(X) + (E(X) - a)^2$$

i.e. $E(X-a)^2 \ge Var(X)$, since $(E(X)-a)^2 \ge 0$.

'$=$' holds when $E(X) = a$. So, $E(X-a)^2$ is minimum when $E(X)=a$

Result: 2. Suppose that for the variable X, the 2^{nd} order moment exists, and $\mu_2'(A) \ge \mu_2$.

[Standard deviation is the least RMS deviation]

Proof:-

$$\mu_2'(A) = E(X-A)^2$$
$$= E[(X-\mu) + (\mu - A)]^2$$
$$= E(X-\mu)^2 + (\mu - A)^2 + 2(\mu - A)E(X-\mu)$$
$$= \mu_2 + (\mu - A)^2 \quad [\because E(X-\mu) = 0]$$

$\therefore \mu_2'(A) \ge \mu_2$.

'$=$' sign holds if $A = \mu$.

Theorem:- If the moment of order r exists for an R.V. X, then moment of order $s (< r)$ also exists.

Proof:- Let X be a continuous R.V. with PDF $f(x)$.

Note that
$$\int_{-\infty}^{\infty} |x|^s f(x)\,dx$$
$$= \int_{|x|\le 1} |x|^s f(x)\,dx + \int_{|x|>1} |x|^s f(x)\,dx$$
$$\le \int_{|x|\le 1} 1 \cdot f(x)\,dx + \int_{|x|>1} |x|^s f(x)\,dx$$
$$\le P[|X| \le 1] + \int_{|x|>1} |x|^r f(x)\,dx$$

[For $|x| \le 1$, $|x|^s \le 1$, and for $|x| > 1$, $|x|^s < |x|^r$ as $s < r$]

$$\int_{-\infty}^{\infty} |x|^s f(x)\,dx \le P[|X| \le 1] + \int_{|x|>1} |x|^r f(x)\,dx$$
$$\le 1 + \int_{-\infty}^{\infty} |x|^r f(x)\,dx < \infty, \text{ as } E(X^r) \text{ exists.}$$

Hence, $E(X^s)$ exists, provided $E(X^r)$ exists, for $s < r$.

Ex.1. Give an example of discrete distribution whose mean exists but variance does not.

Sol.:- Let X be a discrete RV with pmf
$$P[X=i] = \begin{cases} K \cdot \frac{1}{i^3}, & i = 1, 2, 3, \ldots \\ 0, & \text{Otherwise} \end{cases}$$

where, $K = \dfrac{1}{\sum_{i=1}^{\infty} \frac{1}{i^3}}$.

Note that, $\sum_i |i| P[X=i] = \sum_i i \cdot \frac{K}{i^3} = K \sum_{i=1} \frac{1}{i^2}$, converges.

[$\because \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$]

But, $\sum_{i=1}^{\infty} |i^2| \cdot P[X=i] = \sum_{i=1}^{\infty} i^2 \cdot \frac{K}{i^3} = K \sum_{i=1}^{\infty} \frac{1}{i}$, diverges.

Hence, $E(X)$ exists but $E(X^2)$ does not.

$\Rightarrow$ $E(X)$ exists but $Var(X)$ does not exist.

Ex.2. Give an example of a continuous distribution where mean exists but variance does not.

Sol.:- Let X be a continuous RV with PDF

$$f(x) = \begin{cases} \dfrac{2}{x^3}, & \text{if } x > 1 \\ 0, & \text{ow} \end{cases}$$

Note that, $$\int_{-\infty}^{\infty} |x| f(x)\,dx = \int_1^{\infty} x \cdot \frac{2}{x^3}\,dx = \lim_{t\to\infty} \int_1^t \frac{2}{x^2}\,dx$$

$$= 2\lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t$$

$$= 2\lim_{t\to\infty} \left(1 - \frac{1}{t}\right)$$

$$= 2.$$

But, $$\int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_1^{\infty} x^2 \cdot \frac{2}{x^3}\,dx = \int_1^{\infty} \frac{2}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{2}{x}\,dx$$

$$= 2\lim_{t\to\infty} \left[\ln x\right]_1^t$$

$$= 2\lim_{t\to\infty} \ln t$$

$$= +\infty.$$

Hence, $E(X)$ exists but $E(X^2)$ or $Var(X)$ does not.

Remark:- 1. Consider the PMF $$P[X = x] = \begin{cases} \dfrac{K}{x^{p+2}}, & x = 1, 2, 3, \ldots \\ 0, & \text{ow} \end{cases}$$

Note that $E(X^p) = \mu'_p$ exists but $E(X^{p+1}) = \mu'_{p+1}$ does not.

2. Consider the PDF $$f(x) = \begin{cases} \dfrac{p+1}{x^{p+2}}, & x > 1 \\ 0, & \text{ow} \end{cases}$$

Note that μ'_p exists but μ'_{p+1} does not.

Problem:- 1. Assume that n random variables $X_1, X_2, \ldots, X_n$ are independent and each takes the values $+1$ and -1 with probabilities p and $1-p$, respectively. Find the expectation and variance of the product of the random variables.

Solution:- Let $X = X_1 X_2 \cdots X_n$

$\therefore\ E(X) = E(X_1)E(X_2)\cdots E(X_n)$ [Due to independence of X_i's]

$$E(X) = \prod_{i=1}^{n} E(X_i)$$

$$= \prod_{i=1}^{n} \{(+1)p + (-1)(1-p)\}$$

$$= (2p-1)^n.$$

$$Var(X) = E(X^2) - E^2(X)$$

$$= E(X_1^2 X_2^2 \cdots X_n^2) - (2p-1)^{2n}$$

$$= \prod_{i=1}^{n} E(X_i^2) - (2p-1)^{2n}, \text{ by product law of}$$

$$= \prod_{i=1}^{n} \{(+1)^2 p + (-1)^2(1-p)\} - (2p-1)^{2n}$$

$$= 1 - (2p-1)^{2n}$$

Problem:- 2. If X is a discrete R.V. and $E(X^2) = 0$, show that $P(X=0) = 1$. Deduce that if $Var(X) = 0$ then $P(X=\mu) = 1$, where $\mu = E(X)$.

Solution:- (i) $E(X^2) = 0$

$$\Rightarrow \sum_x x^2 p(x) = 0$$

$$\Rightarrow 0.P(X=0) + 1.P(X=1) + 4.P(X=2) + \cdots = 0 \quad \cdots\cdots (*)$$

From $(*)$, we get $P(X=j) = 0 \ \forall\ j = 1, 2, 3, \ldots$

$$\sum_x p(x) = 1 \Rightarrow P(X=0) = 1.$$

(ii) $E(X) = \mu$.

$Var(X) = 0$

$$\Rightarrow E(X^2) - E^2(X) = 0$$

$$\Rightarrow E(X^2) = \mu^2.$$

$$\Rightarrow \sum_x x^2 P(X=x) = \mu^2$$

$$\Rightarrow \underbrace{0.P(X=0) + 1.P(X=1) + 2^2.P(X=2) + \cdots + \mu^2 P(X=\mu) + \cdots}_{=\mu^2}$$

$$\Rightarrow \mu^2.P(X=\mu) = \mu^2, \text{ where } P(X=j) = 0 \ \forall\ j \neq \mu, j > 0.$$

$$\Rightarrow P(X=\mu) = 1.$$

Problem. 3. Consider the distribution of an R.V. X with pdf

$$f(x) = \begin{cases} \dfrac{\beta\alpha^{\beta}}{x^{\beta+1}} & \text{if } x \geq \alpha \\ 0 & \text{if } x < \alpha \end{cases}$$

where, both α and β are positive. Show that the moment of order n exists iff $n < \beta$. Assuming $\beta > 2$, find the mean and variance of the distribution.

Solution:- $E(X^n) = \displaystyle\int_{\alpha}^{\infty} \frac{\beta\alpha^{\beta} \cdot x^n}{x^{\beta+1}}\,dx = \int_{0}^{\infty} \beta\alpha^{\beta} x^{n-\beta-1}\,dx$

$$= \beta\alpha^{\beta} \cdot \left[\frac{x^{n-\beta}}{n-\beta}\right]_{\alpha}^{\infty}$$

$$= \frac{\beta\alpha^{\beta}}{n-\beta}\left(0 - \alpha^{n-\beta}\right), \text{ when } n<\beta.$$

$$= \frac{\beta\alpha^{n}}{\beta-n}.$$

So, $E(X) = \dfrac{\beta\alpha}{\beta-1}$; $E(X^2) = \dfrac{\alpha^2\beta}{\beta-2}$.

$\therefore$ $Var(X) = E(X^2) - E^2(X) = \dfrac{\alpha^2\beta}{\beta-2} - \dfrac{\beta^2\alpha^2}{(\beta-1)^2}$.

Problem. 4. Let X be an R.V. with PMF $p(x) = \begin{cases} c\cdot\binom{2N-x}{N} 2^{x}, & x = 0,1,\dots,N \\ 0 & \text{, Otherwise.} \end{cases}$

(i) Find the constant c, (ii) find $\dfrac{p(x+1)}{p(x)}$ and $E(X)$?

Solution:- (i) $\displaystyle\sum_{x} p(x) = 1 \Rightarrow \sum_{x=0}^{N} c\binom{2N-x}{N} 2^{x} = 1$

$$\Rightarrow c\left[\binom{2N}{N} + \binom{2N-1}{N}\cdot 2 + \cdots\cdots\right] = 1.$$

(ii) $$\frac{p(x+1)}{p(x)} = \frac{\binom{2N-x-1}{N}}{\binom{2N-x}{N}}\cdot 2 = \frac{2(N-x)}{(2N-x)}.$$

Now, $(2N-x)\,p(x+1) = 2(N-x)\,p(x)$

$$\Rightarrow \sum_{x=0}^{\infty}(2N-x)p(x+1) = 2\sum_{x=0}^{\infty}(N-x)p(x)$$

$$\Rightarrow \sum_{x=0}^{\infty}\{(2N+1)-(x+1)\}\,p(x+1) = 2\sum_{x=0}^{\infty}(N-x)p(x)$$

$$\Rightarrow (2N+1) - E(X) = 2N - 2E(X)$$

$$\Rightarrow E(X) = -1.$$

Sum and Product Laws of Expectations :-

- **Sum Law :-** If X and Y are two discrete R.V.s, then $E(X+Y) = E(X)+$

Proof:- Let X takes the values $x_1, x_2, \ldots, x_i, \ldots$ and
Y takes the values $y_1, y_2, \ldots, y_j, \ldots$

Define, $\{A_i\}$ and $\{B_j\}$ are two partitions of Ω.

Now,
$$E(X+Y) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(x_i+y_j)\, P[\{\omega: X(\omega)=x_i, Y(\omega)=y_j\}]$$
$$= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(x_i+y_j)\, P[A_i \cap B_j]$$
$$= \sum_i\sum_j x_i . P[A_i\cap B_j] + \sum_i\sum_j y_j P[A_i\cap B_j]$$
$$= \sum_{i=1}^{\infty} x_i \sum_{j=1}^{\infty} P[A_i\cap B_j] + \sum_{j=1}^{\infty} y_j \sum_{i=1}^{\infty} P[A_i\cap B_j]$$
$$= \sum_{i=1}^{\infty} x_i P[A_i] + \sum_{j=1}^{\infty} y_j P[B_j] \quad \text{[By Total Probability Theorem]}$$
$$= \sum_{i=1}^{\infty} x_i P[X=x_i] + \sum_{j=1}^{\infty} y_j P[Y=y_j]$$
$$= E(X) + E(Y).$$

Independence of two Random Variables :- Consider two discrete RVs X and Y. Let $\{x_1, x_2, \ldots\}$ and $\{y_1, y_2, \ldots\}$ be the sets of mass points of X and Y, respectively.
Then, define $A_i = \{\omega: X(\omega)=x_i\}$ and $B_j = \{\omega: Y(\omega) = y_j\}$.
Here $\{A_i\}$ and $\{B_j\}$ are two partitions of Ω.

Definition:- The discrete R.V.s X and Y are said to be independent iff $\{A_i\}$ and $\{B_j\}$ are two independent partitions of Ω.

iff $P[A_i\cap B_j] = P[A_i]\cdot P[B_j]$ $\forall (i,j)$

i.e. $P[X=x_i, Y=y_j] = P[X=x_i]P[Y=y_j]$ $\forall$ i,j.

- **Product Law :-** If X and Y are independent discrete R.V.s, then
$$E(XY) = E(X)E(Y).$$

Proof:- Let $\{x_1, x_2, \ldots\}$ and $\{y_1, y_2, \ldots\}$ be the sets of mass points of X and Y, respectively.
Then define, $A_i = \{\omega: X(\omega)=x_i\}$ and $B_j = \{\omega: Y(\omega)=y_j\}$.
Hence, $\{A_i\}$ and $\{B_j\}$ are two partitions of Ω.

Since X and Y are independent;

$$P[A_i \cap B_j] = P[A_i]\,P[B_j] \quad \forall (i,j).$$

Now, $$E(XY) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_i y_j \cdot P[\{\omega : X(\omega) = x_i,\ Y(\omega) = y_j\}]$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_i y_j\, P[A_i \cap B_j]$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_i y_j\, P[A_i]\, P[B_j], \text{ due to independence.}$$

$$= \left\{\sum_{i=1}^{\infty} x_i P[A_i]\right\}\left\{\sum_{j=1}^{\infty} y_j P[B_j]\right\}$$

$$= \left\{\sum_{i=1}^{\infty} x_i P[X = x_i]\right\}\left\{\sum_{j=1}^{\infty} y_j P[Y = y_j]\right\}$$

$$= E(X)\cdot E(Y).$$

Remark:- If $E(XY) = E(X)\cdot E(Y)$, then

$$\sum_i\sum_j x_i y_j P[X = x_i, Y = y_j] = \left\{\sum_i x_i P[X = x_i]\right\}\left\{\sum_j y_j P[Y = y_j]\right\}$$

$$= \sum_i\sum_j x_i y_j P[X = x_i]\,P[Y = y_j]$$

This don't necessarily imply that

$$P[X = x_i, Y = y_j] = P[X = x_i]\,P[Y = y_j] \quad \forall\, i, j.$$

$\Rightarrow$ X and Y are independent.

Ex. Consider an R.V. X with $P[X = -1] = \frac{1}{3}$, $P[X = 0] = \frac{1}{3}$, $P[X = 1] = \frac{1}{3}$. Define $Y = X^2$. Then show that $E(XY) = E(X)E(Y)$ but X and Y are not independent.

Solution:- $P[Y = 0] = P[X^2 = 0] = P[X = 0] = \frac{1}{3}$,

$P[Y = 1] = P[X^2 = 1] = P[X = \pm 1] = \frac{2}{3}$.

Y \ X	−1	0	1
0	0	1/3	0
1	1/3	0	1/3

$E(X) = (-1)\cdot\frac{1}{3} + (+1)\cdot\frac{1}{3} = 0.$

$E(XY) = (-1)\cdot 1\cdot\frac{1}{3} + 0 + 1\cdot 1\cdot\frac{1}{3} = 0$

$E(XY) = 0 = E(X)E(Y)$.

But, $P[X = -1, Y = 1] = P[X = -1] = \frac{1}{3}$

— : All entries give the probabilities $P[X = x, Y = y]$:—

$$\neq P[X = -1]\,P[Y = 1] = \frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9}.$$

$\Rightarrow$ X and Y are not independent.

Quantiles:→ A number ξ_p satisfying $P[X \le \xi_p] \ge p$ and $P[X \ge \xi_p] \ge 1-p$, $0<p<1$, is called a quantile of order 'p' or p^{th} order quantile of R.V. X.

If ξ_p is a p^{th} order quantile of an R.V. X with DF $F(x)$, then

$$F(\xi_p) \ge p \text{ and } 1 - F(\xi_p - 0) \ge 1-p,$$

$$\text{i.e. } p \le F(\xi_p) \text{ and } p \ge F(\xi_p - 0)$$

$$\text{i.e. } F(\xi_p - 0) \le p \le F(\xi_p).$$

If X is continuous R.V., then $P[X = \xi_p] = 0$, i.e., $F(\xi_p - 0) = F(\xi_p)$ and $F(\xi_p) = p$ and ξ_p is the solution of the equation $F(x) = p$.

Ex.1. Let X be an R.V. with PMF
$P[X=-2] = P[X=0] = \frac{1}{4}$, $P[X=1] = \frac{1}{3}$, $P[X=2] = \frac{1}{6}$.
(i) Find Median ($\xi_{1/2}$)? (ii) Find a quantile of order $p = 0.2$ of the R.V. X?

Solution:- (i) $P[X \le 0] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$$P[X \ge 0] = 1 - P[X=-2] = \frac{3}{4} > 1 - \frac{1}{2}.$$

Also, note that for any $x \in (0,1)$.

$$P[X \le x] = \frac{1}{2} \text{ and } P[X \ge x] = \frac{1}{2} = 1 - \frac{1}{2}.$$

It follows that every x, $0 \le x < 1$ is a median of x.

(ii) $P[X \le -2] = \frac{1}{4} > 0.2$ and $P[X \ge -2] = 1 > 1 - 0.2$

Hence, $p = 0.2^{th}$ quantile of X is -2.

Ex.2. Consider an RV. X with PDF

$$f(x) = \begin{cases} \theta . e^{-\theta(x-a)}, & \text{if } x > a \\ 0, & \text{Otherwise} \end{cases}$$

Find $\xi_{1/2}(X)$. Also find $E|X - \xi_{1/2}(x)|$?

Solution:- Since X is a continuous RV, the median $\xi_{1/2}$ is a solution of $F_X(x) = \frac{1}{2}$.

$$F_X(x) = \int_{-\infty}^{x} f(t)dt = \frac{1}{2} = \int_a^x \theta . e^{-\theta(t-a)} dt = 1 - e^{-\theta(x-a)}.$$

$$\Rightarrow -\theta(x-a) = -\ln 2$$

$$\Rightarrow x = a + \frac{1}{\theta}\ln 2.$$

Hence, $\xi_{1/2} = a + \frac{1}{\theta}\ln 2$.

Now, $E|X-\xi_{1/2}| = \int_a^\infty \left|x - a - \frac{\ln 2}{\theta}\right| . \theta e^{-\theta(x-a)}\, dx$

$$= \int_{-\frac{\ln 2}{\theta}}^\infty |y| . \theta . e^{-\theta\left(y+\frac{\ln 2}{\theta}\right)} dy \quad \left[\text{Take } y = x-a-\frac{\ln 2}{\theta}\right]$$

$$= \frac{\theta}{2}\int_{-\frac{\ln 2}{\theta}}^\infty |y| . e^{-\theta y} dy \quad \left[\because e^{-\theta . \frac{\ln 2}{\theta}} = e^{-\ln 2} = e^{\log_e 1/2} = 1/2\right]$$

$$= \frac{1}{2\theta}\int_{-\ln 2}^\infty |z| e^{-z} dz \quad [\because \theta y = z,\ \theta dy = dz]$$

$$= \frac{1}{2\theta}\left[\int_{-\ln 2}^0 -ze^{-z}dz + \int_0^\infty ze^{-z}dz\right]$$

$$= \frac{1}{2\theta}\left\{\left[(z+1)e^{-z}\right]_{-\ln 2}^0 + \Gamma(2)\right\}$$

$$= \frac{1}{2\theta}\left[\left\{-(-\ln 2+1)e^{\ln 2}\right\} + 1\right]$$

$$= \frac{1}{2\theta}\{1 + 2\ln 2 - 2 + 1\} = \frac{\ln 2}{\theta} = \frac{\log_2 e}{\theta}$$

$$= (\xi_{1/2} - a).$$

Measures of Central Tendency :- $\mu = E(X)$, $\xi_{1/2}(X)$ are the measures of central tendency of the distribution of X.

Mode :- If X is a discrete (continuous) R.V., then the value x for which the PMF (or PDF) $f_X(x)$ is maximum, is called the mode of the distribution of the R.V. X.

Harmonic Mean :- HM of a non-zero RV is given by $HM = \frac{1}{E\left(\frac{1}{X}\right)}$, provided the expectation exists.

Geometric Mean :- GM of a positive RV X is denoted by G and it is given by $\log G = E(\log_e X)$.

Measure of Dispersion :- $SD(X) = \sigma_X = \sqrt{V(X)}$; $QD = \frac{\xi_{3/4} - \xi_{1/4}}{2}$;

$C.V. = \frac{\sqrt{V(X)}}{E(X)}$, provided $E(X) > 0$.

Measure of Skewness and Kurtosis :- $\gamma_1 = \frac{\mu_3}{\mu_2^{3/2}}$ is a measure of skewness.

$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$. $\quad \gamma_2 = \beta_2 - 3 = \frac{\mu_4}{\mu_2^2} - 3$ is a measure of excess of kurtosis of X.

Ex. Let $F(t)$ be the probability that a system fails by time t and let $\gamma(t).\Delta t + o(\Delta t)$ be the probability of failure in the interval $(t, t+\Delta t)$ given that it has survived up to 't'. S.T. $F(t)$ satisfies the differential equation $\frac{F'(t)}{1-F(t)} = \gamma(t)$, leading to the functional form $F(t) = 1 - \exp\left[\int_0^t \gamma(x)dx\right]$. In particular, if $\gamma(t) = 0$, find $F(t)$.

Solution:- Let T be the lifetime of the system. Hence, $F(t) = P[T \le t]$

By problem, $P[t < T < t+\Delta t / T > t] = \gamma(t).\Delta t + o(\Delta t)$

$$\Rightarrow \frac{P[t < T < t+\Delta t]}{P[T > t]} = \gamma(t).\Delta t + o(\Delta t)$$

$$\Rightarrow \frac{F(t+\Delta t) - F(t)}{1-F(t)} = \gamma(t).\Delta t + o(\Delta t)$$

$$\Rightarrow \frac{\lim\limits_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t}}{1-F(t)} = \gamma(t) + \lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t}$$

$$\Rightarrow \frac{F'(x)}{1-F(x)} = \gamma(x), \text{ for } x > 0.$$

$$\therefore \int_0^t \frac{F'(x)}{1-F(x)} = \int_0^t \gamma(x)dx$$

$$\Rightarrow \left[-\log_e(1-F(x))\right]_0^t = \int_0^t \gamma(x)dx$$

$$\Rightarrow -\log_e\{1-F(t)\} + \log_e\{1-F(0)\} = \int_0^t \gamma(x)dx$$

$$\Rightarrow \log_e\{1-F(t)\} = -\int_0^t \gamma(x)dx \quad \left[\text{As, } T \text{ is a non-negative RV, } F(0) = 0\right]$$

$$\Rightarrow F(t) = 1 - e^{-\int_0^t \gamma(x)dx}$$

In particular, let $\gamma(x) = \theta \; \forall \; x > 0$

$$\Rightarrow F(t) = 1 - e^{-\int_0^t \theta dx} = 1 - e^{-\theta t}, \; t > 0.$$

Remark:- $\lim \frac{P[t < T < t+\Delta t / T > 0]}{\Delta t}$

$$= \frac{\lim\limits_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t}}{1-F(t)}$$

$= \frac{F'(t)}{1-F(t)}$ is called the instantaneous failure rate or Hazard rate at time point 't'.

Symmetric Distribution :—

Definition:- An R.V. X is said to be symmetrically distributed about 'a' if $P[X \le a-x] = P[X \ge a+x]$ $\forall x$.

iff $F_X(a-x) = 1 - F_X(a+x) + P[X = a+x]$ $\forall x$.

For a discrete R.V. X with PMF $f(x)$, X is said to be symmetric about 'a' if $f(a-x) = f(a+x)$ $\forall x$.

For a continuous R.V. X with PDF $f(x)$, we have

$$F_X(a-x) = 1 - F_X(a+x) \text{ as } P[X = a+x] = 0.$$

$$\Rightarrow -f(a-x) = -f(a+x) \;\forall x.$$

$$\Rightarrow f(a+x) = f(a-x) \;\forall x.$$

Result:- Show that for a symmetric R.V., all odd order central moments are zero, provided they exist. Also show that the mean of the RV is the point of symmetry.

Proof:- Let X be a continuous RV with DF $F(x)$ and let X be symmetric about a point 'a'. By definition, $f(a-x) = f(a+x)$ $\forall x$.

Now, $$E(X-a)^{2n-1} = \int_{-\infty}^{\infty} (x-a)^{2n-1} f(x)\,dx$$

$$= \int_{-\infty}^{a} (x-a)^{2n-1} f(x)\,dx + \int_{a}^{\infty} (x-a)^{2n-1} f(x)\,dx$$

$$= \int_{-\infty}^{0} y^{2n-1} f(a+y)\,dy + \int_{0}^{\infty} y^{2n-1} f(a+y)\,dy \quad [\because y = x-a]$$

$$= \int_{\infty}^{0} (-u)^{2n-1} f(a-u)(-du) + \int_{0}^{\infty} y^{2n-1} f(a+y)\,dy \quad [\text{Put } y=-u \text{ in the } 1^{st} \text{ integral}]$$

$$= -\int_{0}^{\infty} u^{2n-1} f(a+u)\,du + \int_{0}^{\infty} u^{2n-1} f(a+u)\,du \quad [\because f(a-u) = f(a+u)]$$

$$= 0, \quad \forall n \in \mathbb{N}.$$

For $n=1$, $E(X-a) = 0 \Rightarrow E(X) = a$.

Hence, $\mu_{2n-1} = E\{X - E(X)\}^{2n-1} = E(X-a)^{2n-1} = 0$ $\forall n \in \mathbb{N}$.

Result:- If X is symmetrically distributed about 'a', show that median of X is 'a'.

Proof:- By definition of symmetric distribution about 'a',

$$P[X \le a-x] = P[X \ge a+x] \;\forall x.$$

For $x=0$, we have $P[X \le a] = P[X \ge a]$.

Problem: (Banach's Match Box Problem) A certain mathematician carries two match boxes in his pocket. Each time he wants to use a match, he selects one of the boxes at random. Find the probability that when the mathematician discovers that one box is empty, the other box contains r matches, $r = 0,1,2,\ldots,n$; where n is the no. of matches initially contained in each box. Find the expected no. of matches.

Solution:- Since there is one match-box in his two pockets, the probability of selecting at random to have a match is $p=\frac{1}{2}$. Let us identify 'success' with the choice of the left pocket. The left pocket will be found empty at a moment when the right pocket contains exactly r matches iff exactly $(N-r)$ failures preceding the $(N+1)^{th}$ success. The prob. of this event is

$$P_L = \binom{2n-r}{n-r}\left(\frac{1}{2}\right)^{n}\left(\frac{1}{2}\right)^{n-r}\cdot\frac{1}{2} = \binom{2n-r}{n-r}\cdot\frac{1}{2^{2n-r+1}}.$$

[The required event is that $(2n-r+1)$ trials is needed to get the $(n+1)^{th}$ success. For the occurance of the events we must have a success at the last trial and there are $(N-r)$ failure in the first $(2N-r)$ trials].

The same argument applied to the right pocket and then the corresponding probability is

$$P_R = \binom{2n-r}{n-r}\cdot\frac{1}{2^{2n-r+1}}.$$

Hence, the probability that there are r matches in one box when the other box is found empty is $P_L + P_R = 2\binom{2n-r}{n-r}\frac{1}{2^{2n-r+1}}$

$$= \binom{2n-r}{n-r}\frac{1}{2^{2n-r}}.$$

Note that, if R be the expected no. of matches, then we have

$$n - E(R) = n - \sum_{r=0}^{n} r.P[R=r]$$

$$= \sum_{r=0}^{n-1}(n-r)\cdot\binom{2n-r}{n-r}\cdot\frac{1}{2^{2n-r}}$$

$$= \sum_{r=0}^{n-1}(2n-r)\binom{2n-r-1}{n-r-1}\cdot\frac{1}{2^{2n-r}}.$$

$$= \sum_{r=0}^{n-1}\left\{\frac{(2n+1)-(r+1)}{2}\right\}\binom{2n-r-1}{n-r-1}\cdot\frac{1}{2^{2n-r-1}}.$$

$$= \frac{2n+1}{2}\sum_{r=0}^{n-1}P[R=r+1] - \frac{1}{2}\sum_{r=0}^{n-1}(r+1)P[R=r+1]$$

$$= \frac{2n+1}{2}\left\{1 - P[R=0]\right\} - \frac{1}{2}E(X) = \frac{2n+1}{2}\left\{1-\binom{2n}{n}\frac{1}{2^{2n}}\right\} - \frac{E(X)}{2}$$

$$\therefore \frac{1}{2}E(R) = \frac{1}{2}\left\{(2n+1)\binom{2n}{n}\frac{1}{2^{2n}} - 1\right\} \quad [\because E(R) = (2n+1)P[R=0]-1]$$

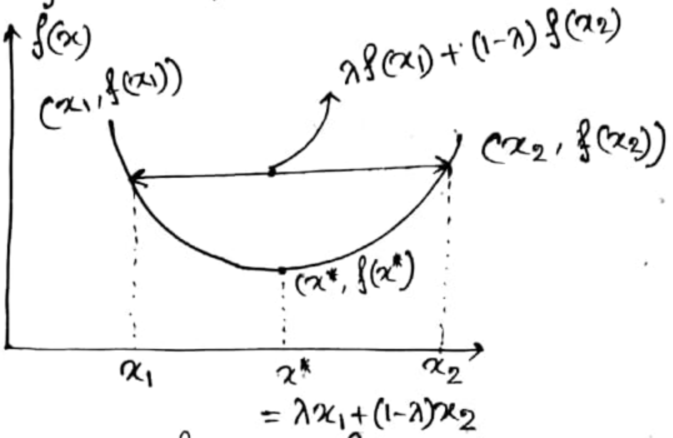
$$\therefore E(R) = (2n+1)\binom{2n}{n}\frac{1}{2^{2n}} - 1.$$

☒ JENSEN'S INEQUALITY:-

Convex Function:- (I) If $f(x)$ is twice differentiable, i.e. $f''(x)$ exists and $f''(x) \geq 0 \ \forall x \in I$, then $f(x)$ is convex in the interval I.

(II) A function $f(x)$ is said to be convex on an interval I if for $x_1, x_2 \in I$,

$$f\{\lambda x_1 + (1-\lambda)x_2\} \leq \lambda f(x_1) + (1-\lambda) f(x_2) \ ; \ 0 \leq \lambda \leq 1.$$



—: Graph of convex form: —

Jensen's Inequality:- If $f(x)$ is continuous and convex function on I and X is an R.V. such that $P[X \in I] = 1$, then $E\{f(X)\} \geq f\{E(X)\}$.

Proof:- Assuming $f(x)$ is twice differentiable. By Taylor's theorem,

$$f(x) = f(\mu) + (x-\mu) f'(\mu) + \frac{(x-\mu)^2}{2!} f''(\mu^*)$$, where μ^* lies between μ and x ; $\mu = E(X)$.

Note that, as $f(x)$ is convex, then $f''(x) \geq 0 \ \forall x$.

$$\Rightarrow f''(\mu^*) \geq 0$$

$$\therefore f(x) \geq f(\mu) + (x-\mu) f'(\mu) \ ; \ \forall x > 0$$

For an RV X,

$$f(X) \geq f(\mu) + (X-\mu) f'(\mu)$$

$$\Rightarrow E(f(X)) \geq f(\mu) + f'(\mu) E(X-\mu) = f(\mu)$$

$$\Rightarrow E(f(X)) \geq f(E(X)).$$

Remark:- A function $f(x)$ is concave on I iff $-f(x)$ is convex on I.

Jensen's Inequality:- For a concave function $f(x)$ on I, $E(f(X)) \leq f(E(X))$

Proof:- $f(x)$ is concave.

$\Rightarrow -f(x)$ is convex.

$$\Rightarrow E(-f(x)) \geq -f(E(X))$$

$$\Rightarrow E(f(X)) \leq f(E(X)).$$

Ex. For an RV X which assumes only positive values, show that (i) $E\left(\frac{1}{X}\right) \geq \frac{1}{E(X)}$, (ii) $E(\log_e X) \leq \log_e E(X)$.

Solution:-

(i) Let $f(x) = \frac{1}{x}$, $x > 0$

By Taylor's theorem, $f(x) = f(\mu) + (x-\mu) f'(\mu) + \frac{(x-\mu)^2}{2!} f''(\mu^*)$.

$\mu = E(X)$, μ^* lies between μ and x.

$f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3} > 0 \ \forall x > 0$.

Hence, $f(x) \geq f(\mu) + (X-\mu) f'(\mu)$

$\Rightarrow E(f(x)) \geq f(\mu)$

$\therefore E\left(\frac{1}{X}\right) \geq \frac{1}{E(X)}$.

(ii) Take $f(x) = \log_e x$, $x > 0$, $E(X) = \mu$.

$f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2} < 0$.

Jensen's inequality states that $E(f(x)) \leq f(E(X)) = f(\mu)$

$\Rightarrow E(\log_e X) \leq \log_e E(X)$.

CAUCHY-SCHWARTZ INEQUALITY:-

$E(g^2(X))\, E(h^2(X)) \geq E^2(g(X)h(X))$, provided $E(g(X))$ and $E(h(X))$ both exist.

Proof:- For any real λ, $E(g(X) + \lambda h(X))^2 \geq 0$

$\Rightarrow E(g^2(X)) + \lambda^2 E(h^2(X)) + 2\lambda E(g(X)h(X)) \geq 0$

$\Rightarrow E(g^2(X)) + 2\lambda E(g(X)h(X)) + \lambda^2 E(h^2(X)) \geq 0$

$\Rightarrow a\lambda^2 + b\lambda + c \geq 0$ [Take $a = E(h^2(X))$, $b = 2E(g(X)h(X))$, $c = E(g^2(X))$.]

$\Rightarrow \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} \geq 0$ [$\because a > 0$]

$\Rightarrow \left(\lambda + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} \geq 0$

choose $\lambda = -\frac{b}{2a}$,

$\therefore b^2 - 4ac \geq 0$.

$\Rightarrow E(h^2(X))E(g^2(X)) \geq E^2(h(X)g(X))$

Note:- In general, for two jointly distributed RVs X and Y, $E(g^2(X))E(h^2(Y)) \geq E^2(g(X)h(Y))$, provided $E(g^2(X))$ and $E(h^2(Y))$ both exist.

'=' holds in CS Inequality iff $E(g(X) + \lambda h(X))^2 = 0$

$\Rightarrow g(X) + \lambda(h(X)) = 0$ almost everywhere

i.e. $g(X) \propto h(X)$.

Application of CS Inequality :—

(a) $E(g^2(X))\,E(h^2(X)) \geq E^2(g(X)\,h(X))$

Choose $g(X)=X$, $h(X)=1$, almost everywhere.

$\boxed{E(X^2) \geq E^2(X)}$ $\Rightarrow E(X^2)-E^2(X) \geq 0 \Rightarrow Var(X) \geq 0.$

(b) Replace 'X' by $|X-E(X)|$ in (a),

$E(X-E(X))^2 \geq (E|X-E(X)|)^2$

$\Rightarrow \sqrt{E(X-E(X))^2} \geq E|X-E(X)|$

$\Rightarrow \boxed{SD(X) \geq MD(X)}$

(c) Replace 'X' by $\sqrt{X}$ in (a), where $P(X>0)=1$.

$\therefore E(X) \geq E^2(\sqrt{X})$

$\Rightarrow \boxed{\sqrt{E(X)} \geq E(\sqrt{X})}$

(d) Let $g(X)=\sqrt{X}$, $h(X)=\frac{1}{\sqrt{X}}$, $P[X>0]=1$.

$\therefore E(X)\,E\left(\frac{1}{X}\right) \geq 1$

$\Rightarrow \boxed{E(X) \geq \frac{1}{E\left(\frac{1}{X}\right)}}$ $\Rightarrow$ AM $\geq$ HM.

(e) $g(X)=X-E(X)$, $h(Y)=Y-E(Y)$.

Then $E(X-E(X))^2\,E(Y-E(Y))^2 \geq \{E^2(X-E(X))(Y-E(Y))\}$

i.e. $V(X)\,V(Y) \geq cov^2(X,Y)$

$\Rightarrow \boxed{|r_{XY}| \leq 1}$

Note:— Let X be an RV with mean μ and variance $\sigma^2\,(>0)$.

Define, $Z=\frac{X-\mu}{\sigma}$. Note that,

$$E(Z^r) = \frac{E(X-\mu)^r}{\sigma^r} = \frac{\mu_r}{\mu_2^{r/2}}.$$

Problem:- show the followings: —

(i) $\beta_2 \geq 1$ (ii) $\beta_2 > \beta_1$ (iii) $\beta_2 \geq \beta_1 + 1$.

Solution:- (i) Take, $g(Z) = Z^2$, $h(Z) = 1$.

$\therefore E(Z^4) \geq E^2(Z^2)$, provided $E(Z^4)$ exists, i.e. $\frac{\mu_4}{\mu_2^2}$ exists.

Now, $E(Z^4) = \frac{\mu_4}{\mu_2^2}$, $E(Z^2) = \frac{\mu_2}{\mu_2} = 1$.

$\therefore \frac{\mu_4}{\mu_2^2} \geq 1 \Rightarrow \beta_2 \geq 1$.

'=' holds iff $Z^2 = C$ almost everywhere.

$\Rightarrow Z = K$ a.e.

i.e. $\left|\frac{X-\mu}{\sigma}\right| = K$, $\Rightarrow X = \mu \pm K\sigma$.

i.e. X assumes two distinct values with equal probabilities.

(ii) $g(Z) = Z^2$, $h(Z) = Z$.

$\therefore E(Z^4)E(Z) \geq E^2(Z^3)$

$$\frac{\mu_4}{\mu_2^2} \geq \frac{\mu_3^2}{\mu_2^3}$$

i.e. $\beta_2 \geq \beta_1$.

'=' holds iff $g(Z) = Kh(Z)$ a.e.

i.e. Z is degenerate R.V.

i.e. X is also degenerate R.V.

in that case, μ_2 vanishes.

Thus, equality will not hold good, i.e., $\beta_2 > \beta_1$.

(iii) $g(Z) = Z^2 - 1$, $h(Z) = Z$,

$\therefore E(Z^4 - 2Z^2 + 1)E(Z^2) \geq E(Z^3 - Z)$

$$\therefore \frac{\mu_4}{\mu_2^2} - 1 \geq \frac{\mu_3^2}{\mu_2^3}$$

$\Rightarrow \beta_2 \geq \beta_1 + 1$.

'=' holds iff $g(X) = kh(X)$ a.e.

$\Rightarrow Z^2 - 1 = hZ$

$\Rightarrow Z$ assumes two distinct values

$\Rightarrow X$ assumes two distinct values not necessarily symmetrically placed w.r.t. μ.