PROBABILITY THEORY IV

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SOME PROBABILITY INEQUALITIES

The inequalities which contain probability in either left side or right side or in the both side, are called "Probability Inequalities".

MARKOV'S INEQUALITY :—

Statement :— Let X be a r.v. having finite expectation, i.e. E(X) converges. Then for any non-zero quantity 'a', we have the inequality :

$$P(X \geq a) \leq \frac{E(X)}{a}.$$

Proof :— Let us define a r.v. Y such that

$$Y = \begin{cases} a & \text{if } X \geq a \\ 0 & \text{ow} \end{cases}$$

$$X \geq Y \Rightarrow E(X) \geq E(Y)$$

Now, $E(Y) = a \cdot P(X \geq a) \leq E(X)$

$$\Rightarrow P(X \geq a) \leq \frac{E(X)}{a}.$$

NOTE: Markov inequality holds for any function of r.v. X, i.e. for any real valued function $g(X)$, the markov's inequality is given by

$$P[g(X) \geq a] \leq \frac{E(g(X))}{a}, \quad a \neq 0$$

Proof :— Let us define a function of r.v. Y, $g(Y)$

$$g(Y) = \begin{cases} a & \text{if } g(X) \geq a \\ 0 & \text{ow} \end{cases}$$

$$g(X) \geq g(Y)$$

$$\therefore E(g(X)) \geq E(g(Y))$$

$$\therefore E(g(Y)) = a \cdot P[g(X) \geq a] \leq E(g(X))$$

$$\Rightarrow P[g(X) \geq a] \leq \frac{E(g(X))}{a}, \quad a \neq 0$$

Problem1. If X be any r.v. such that $M(t)=E(e^{tX})$ exists for all t, Show that for any $t>0$,

$$P(tX > s^2 + \ln M(t)) < e^{-s^2}$$

ANS:-
We know that an exponential function is monotonically increasing.

So, $P(tX > s^2 + \ln M(t))$

$= P\left(e^{tX} > e^{s^2 + \ln M(t)}\right)$

$= P\left[e^{tX} > e^{s^2} . e^{\ln M(t)}\right]$

Let $g(X) = e^{tX}$ then by Markov's inequality, we have

$$P\left(e^{tX} > e^{s^2} . e^{\ln M(t)}\right) < \frac{E(e^{tX})}{e^{s^2} . e^{\ln M(t)}} = \frac{M(t)}{M(t) . e^{s^2}}$$

$= e^{-s^2}$ (Proved)

Problem2. For any random variable X, show that,

$$P[|X| > t] \leq \frac{1+t^2}{t^2} E\left(\frac{X^2}{1+X^2}\right) \text{ for any } t > 0.$$

[2002]

ANS:- Here, $P[|X| > t]$

$= P[X^2 > t^2]$

$= P[1+X^2 > 1+t^2]$

$= P\left[\frac{X^2}{1+X^2} > \frac{t^2}{1+t^2}\right]$

Now by Markov's inequality,

$$P\left[\frac{X^2}{1+X^2} > \frac{t^2}{1+t^2}\right] \leq E\left(\frac{X^2}{1+X^2}\right) . \frac{1+t^2}{t^2} \quad \text{[Proved]}$$

C.U. S.T. $P[X > t] < E(e^{aX})/e^{at}$.

ANS:- $P[aX > at] = P[X > t]$

$= P(e^{aX} > e^{at})$ [$\because$ e is monotonically increasing]

By Markov's inequality,

$< \frac{E(e^{aX})}{e^{at}}$, where $E(e^{aX})$ exists where $a > 0$.

Problem 3. A fair die is rolled n times. Find a lower bound to n such that, the probability of at least one six in rolling is $\geq \frac{1}{2}$.

ANS:- Let us define a random variable X representing the number of six by throwing a die n times.

$\therefore X \sim bin(n, \frac{1}{6})$

By Markov's inequality,

$P[X \geq 1] \leq \frac{E(X)}{1}$

$\Rightarrow P[X \geq 1] \leq \frac{n}{6}$ ——— ⟨i⟩

Again it is given that $P[X \geq 1] \geq \frac{1}{2}$

$\therefore$ From ⟨i⟩,

$\frac{n}{6} \geq \frac{1}{2}$

$\Rightarrow n \geq 3$

$\therefore$ the die should be at least thrown 3 times.

Problem 4. $X_1, X_2, \ldots, X_K$ are independent r.v.'s having zero mean and unit variance. Find an upper bound to,

$$P\left[\sum_{i=1}^{K} X_i^2 \geq \lambda K\right], \quad \lambda > 0$$

ANS:-

$X_1, X_2, \ldots, X_K$ are independent r.v.'s with mean 0 and variance 1.

i.e. $E(X_i) = 0 \quad \forall\ i = 1(1)K$.

$V(X_i) = E(X_i^2) - E^2(X_i)$

$\Rightarrow E(X_i^2) = 1 \quad [\because E(X_i) = 0]$

$\Rightarrow \sum_{i=1}^{K} E(X_i^2) = K$

$\Rightarrow E\left(\sum_{i=1}^{K} X_i^2\right) = K \quad [\because X_i\text{'s are independent}]$

Now by Markov's inequality,

$$P\left[\sum_{i=1}^{K} X_i^2 \geq \lambda K\right] \leq \frac{E\left(\sum_{i=1}^{K} X_i^2\right)}{\lambda K} = \frac{K}{\lambda K} = \frac{1}{\lambda}$$

$\therefore$ Required upper bound $= \frac{1}{\lambda}$.

CHEBYSHEV'S INEQUALITY :—

Statement :— For a random variable X having finite mean and variance σ^2, then for any $t > 0$, the chebyshev's inequality is given as follows:

$$P(|X-\mu| \geq t\sigma) \leq \frac{1}{t^2}$$

or

$$P(|X-\mu| \leq t\sigma) \geq 1-\frac{1}{t^2}$$

Proof :— In order to prove chebyshev's inequality, we will first prove Markov's inequality,

Let us define a random variable Z,

$$Z = \begin{cases} a, & Y \geq a \\ 0, & ow \end{cases}$$

where Y is another RV.

From the definition of Z, it is such that,

$$Y \geq Z$$

$$\Rightarrow E(Y) \geq E(Z)$$

$$\Rightarrow E(Y) \geq a . P[Y \geq a]$$

$$\Rightarrow P[Y \geq a] \leq \frac{E(Y)}{a}$$

This is the required Markov's inequality.

Now for the RV X,

$E(X) = \mu < \infty$, $V(X) = \sigma^2 = E(X-\mu)^2 > 0$

Now, $P[|X-\mu| \geq t\sigma] = P[(X-\mu)^2 \geq t^2\sigma^2]$

Now, let us choose $Y = (X-\mu)^2$ and $a = t^2\sigma^2$, then by Markov's inequality, we have,

$$P[(X-\mu)^2 \geq t^2\sigma^2] \leq \frac{E(X-\mu)^2}{t^2\sigma^2} = \frac{\sigma^2}{t^2\sigma^2}$$

$$= \frac{1}{t^2}$$

$$\therefore P[|X-\mu| \geq t\sigma] \leq \frac{1}{t^2} \quad \text{——— } \langle i \rangle$$

Hence proved.

$1 - \langle i \rangle$ gives

$$P[|X-\mu| \leq t\sigma] \geq 1-\frac{1}{t^2} \quad \text{——— } \langle ii \rangle$$

Hence proved.

The Equality Case:— Let us consider a r.v. with probability distribution,

$$P[X=\mu+t\sigma] = P[X=\mu-t\sigma] = \frac{1}{2t^2}, \text{ and}$$

$$P[X=\mu] = 1-\frac{1}{t^2}.$$

where $E(X)=\mu$, and $Var(X)=\sigma^2$.

If $Y=|X-\mu|$, then

$$P[Y=t\sigma] = \frac{1}{t^2} \quad \text{and} \quad P[Y=0] = 1-\frac{1}{t^2}$$

Now,

$$P[Y \geq t\sigma] = P[Y=t\sigma] = \frac{1}{t^2}$$

Therefore, $P[|X-\mu| \geq t\sigma] \leq \frac{1}{t^2}$.

Hence equality holds for Chebyshev's inequality.

Another Proof:— Let $Y=|X-\mu|$,

Then Y is a non-negative random variable with

$$E(Y^2) = E(X-\mu)^2$$
$$= \sigma^2$$

Now for any $t>0$

$$P(Y \geq t\sigma) = P(Y^2 \geq t^2\sigma^2)$$
$$\leq \frac{E(Y^2)}{t^2\sigma^2} \quad [\text{By Markov's inequality}]$$
$$= \frac{1}{t^2}$$

Hence,

$$P\left((X-\mu)^2 \geq t^2\sigma^2\right) \leq \frac{1}{t^2}$$

$$\Rightarrow P(|X-\mu| \geq t\sigma) \leq \frac{1}{t^2} \quad \text{——— (i)}$$

Hence proved,

1 − (i) gives

$$P(|X-\mu| \leq t\sigma) \geq 1-\frac{1}{t^2} \quad \text{——— (ii)}$$

Hence proved.

Problem 5. For a Laplace distribution with PDF

$$f(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < \infty.$$

Find the minimum probability of an observation lying within the mean $\pm$ 3 s.d. interval.

(OR). Compare the value of $P(|X-\mu| \le 3\sigma)$ with the lower bound calculated by Chebyshev's inequality.

ANS:— $P(|X-\mu| \le 3\sigma)$

$$= P(\mu - 3\sigma \le X \le \mu + 3\sigma)$$

$$= \frac{1}{2} \int_{\mu-3\sigma}^{\mu+3\sigma} e^{-|x|} dx$$

$$= \int_{0}^{\mu+3\sigma} e^{-x} dx \quad [\text{ since the integrand is an even functn}]$$

$$= -e^{-x} \Big|_{0}^{\mu+3\sigma}$$

$$= 1 - e^{-(\mu+3\sigma)}$$

$$= .95 \qquad [X \sim \text{Laplace}(0,1)]$$

By Chebyshev's inequality,

$$P[|X-\mu| \ge 3\sigma] \le \frac{1}{3^2}$$

$$\Rightarrow P[|X-\mu| \le 3\sigma] \ge 1 - \frac{1}{9} = \frac{8}{9} = .88$$

Hence, the given probability and the chebyshev's upperbound is nearer to each other.

Problem 6. For the r.v. X having the following PDF

$$f(x) = \frac{e^{-x} \cdot x^{\lambda}}{\Gamma(\lambda+1)}, \quad x > 0$$

S.T. $P(0 < X < 2(\lambda+1)) > \frac{\lambda}{\lambda+1}$

ANS:— $E(X) = (\lambda+1) = V(X)$

From Chebyshev's inequality,

$$1 - \frac{1}{t^2} = \frac{\lambda}{\lambda+1}$$
$$\Rightarrow t = \sqrt{\lambda+1}$$

$$P\left[\left|\frac{X-\mu}{\sigma}\right| < t\right] > 1 - \frac{1}{t^2}$$

$$\Rightarrow P[-\sigma t < (X-\mu) < \sigma t] > 1 - \frac{1}{t^2}$$

$$\Rightarrow P[-(\sqrt{\lambda+1})(\sqrt{\lambda+1}) < (X - \overline{\lambda+1}) < \lambda+1] > 1 - \frac{1}{\lambda+1}$$

$$\Rightarrow P[0 < X < 2(\lambda+1)] > \frac{\lambda}{\lambda+1}$$

Problem 7. Let X be an r.v. with mean μ and variance $\sigma^2 > 0$. If ξ_q denotes the qth quantile of X, show that

$$\mu - \sigma\sqrt{\frac{1-q}{q}} \le \xi_q \le \mu + \sigma\sqrt{\frac{q}{1-q}}.$$

ANS:-

We know that ξ_q satisfies the inequality $P(X \le \xi_q) \ge q$

$$\therefore P\left(\frac{X-\mu}{\sigma} \le \frac{\xi_q - \mu}{\sigma}\right) \ge q$$

If $\xi_q < \mu$, i.e. $\frac{\xi_q - \mu}{\sigma} < 0$, we have from one-sided Chebyshev's inequality,

$$q \le P\left[\frac{X-\mu}{\sigma} \le \frac{\xi_q - \mu}{\sigma}\right] \le \frac{1}{1+\left(\frac{\xi_q-\mu}{\sigma}\right)^2}$$

$$\therefore q \le \frac{1}{1+\left(\frac{\xi_q-\mu}{\sigma}\right)^2}$$

$$\Rightarrow \left(\frac{\xi_q-\mu}{\sigma}\right)^2 \le \frac{1-q}{q}$$

$$\Rightarrow -\sqrt{\frac{1-q}{q}} \le \frac{\xi_q-\mu}{\sigma} \le \sqrt{\frac{1-q}{q}}$$

$$\Rightarrow \mu - \sigma\sqrt{\frac{1-q}{q}} \le \xi_q \le \mu + \sigma\sqrt{\frac{1-q}{q}}. \quad \text{(Proved)}$$

Problem 8. Let g be a non-negative non-decreasing function, Prove that if $E(g(|X-\mu|))$ exists, where $\mu = E(X)$, then

$$P[|X-\mu| > t] < \frac{E(g(|X-\mu|))}{g(t)}$$

ANS:- $$P[g|X-\mu| > g(t)] \le \frac{E\{g|X-\mu|\}}{g(t)}$$

But, $g|X-\mu| > g(t)$

$\Leftrightarrow |X-\mu| > t$ [$\because g$ is non-decreasing & non-negative function]

$$\therefore P[|X-\mu| > t] < \frac{E\{g|X-\mu|\}}{g(t)}. \quad \text{(Proved)}$$

Problem 9. If F be the distribution function of the r.v. X and μ and $\sigma^2 > 0$ are its mean and variance. S.T.

i) $F(x) \le \dfrac{\sigma^2}{\sigma^2 + (x-\mu)^2}$ if $x \le \mu$

ii) $F(x) \geq \dfrac{(x-\mu)^2}{\sigma^2 + (x-\mu)^2}$ if $x \geq \mu$.

ANS:- i) For $x \le \mu$, let us take $x = \mu - t\sigma$, $t = -\dfrac{x-\mu}{\sigma}$.

Then one-sided Chebyshev's inequality gives,—

$$P[-X \geq -\mu + t\sigma] \le \frac{1}{1+t^2}$$

$$\Rightarrow P[X \le \mu - t\sigma] \le \frac{1}{1+t^2}$$

$$\therefore P[X \le x] \le \frac{1}{1 + \left(\frac{x-\mu}{\sigma}\right)^2} = \frac{\sigma^2}{\sigma^2 + (x-\mu)^2} \text{ for } x \le \mu.$$

This result is trivially true for $x = \mu$, since in that case R.H.S $= 1$.

ii) For $x \geq \mu$, let us take $x = \mu + t\sigma$; and $t = \dfrac{x-\mu}{\sigma}$.

Then one-sided Chebyshev's inequality gives,

$$P[-X \geq -\mu - t\sigma] \le \frac{1}{1+t^2}$$

$$\therefore P[X \geq x] \le \frac{\sigma^2}{\sigma^2 + (x-\mu)^2}$$

$$\Rightarrow 1 - F(x-0) \le \frac{\sigma^2}{\sigma^2 + (x-\mu)^2}$$

$$\Rightarrow F(x-0) \geq \frac{(x-\mu)^2}{\sigma^2 + (x-\mu)^2}$$

$$\Rightarrow F(x) \geq F(x-0) \geq \frac{(x-\mu)^2}{\sigma^2 + (x-\mu)^2} \text{ for } x \geq \mu$$

This result is trivially true for $x = \mu$, since then the R.H.S becomes 0.

One sided Chebyshev's Inequality :—

Statement :— For an r.v. having finite mean $E(X)=\mu$ and finite variance σ^2, then we have

$$P[X \geq \mu+\sigma t] \leq \frac{1}{1+t^2}, \quad \text{and}$$

$$P[X \leq \mu-\sigma t] \leq \frac{1}{1+t^2}.$$

Proof :— Define an r.v. $Y = X-\mu$, $E(Y)=0$ as $E(X)=\mu$.

$$\therefore P(Y \geq \sigma t) \leq \frac{\sigma^2}{\sigma^2+\sigma^2 t^2} = \frac{1}{1+t^2} \quad \text{——— (1)}$$

$$\therefore P(X \geq \mu+\sigma t) \leq \frac{1}{1+t^2}.$$

$$P(Y \leq -\sigma t) \leq \frac{\sigma^2}{\sigma^2+\sigma^2 t^2} = \frac{1}{1+t^2} \quad \text{——— (2)}$$

$$\Rightarrow P(X \leq \mu-\sigma t) \leq \frac{1}{1+t^2}.$$

- **Lemma :—** If X be an r.v. with mean zero and finite variance σ^2, then for any $a>0$,

  i) $P(X \geq a) \leq \dfrac{\sigma^2}{\sigma^2+a^2}$

  ii) $P(X \leq -a) \leq \dfrac{\sigma^2}{\sigma^2+a^2}$ [C.U. 1997]

Proof :— For any $b>0$,

$$P(X \geq a) = P(X+b \geq a+b)$$

$$= P[(X+b)^2 \geq (a+b)^2] \leq \frac{E(X+b)^2}{(a+b)^2} \quad \text{[By Markov's inequality]}$$

$$= \frac{E(X^2)+b^2}{(a+b)^2}$$

$$= \frac{\sigma^2+b^2}{(a+b)^2} = G(b), \text{ say,} \quad \text{——— (1)}$$

$$G'(b) = \frac{(a+b)^2 \cdot 2b - (\sigma^2+b^2)\,2(a+b)}{(a+b)^4} = 0$$

$$\Rightarrow ab = \sigma^2$$

$$\Rightarrow b = \frac{\sigma^2}{a} \Rightarrow b_{min} = \frac{\sigma^2}{a}.$$

Thus putting $b_{min} = \frac{\sigma^2}{a}$ in (1), we get,

$$P(X \geq a) \leq \frac{\sigma^2 + \frac{\sigma^4}{a^2}}{\left(a+\frac{\sigma^2}{a}\right)^2} = \frac{\sigma^2}{a^2+\sigma^2}. \quad \text{——— [i) no. proved]}$$

X has mean zero, so $E(-X)=0$ and $V(-X)=\sigma^2$

$\therefore X = -X$

$\Rightarrow P(-X \geq a) \leq \dfrac{\sigma^2}{\sigma^2+a^2}$

$\Rightarrow P(X \leq -a) \leq \dfrac{\sigma^2}{\sigma^2+a^2}$.

Problem 10. For the geometric distribution with PMF

$$f(x) = \begin{cases} \dfrac{1}{2^x}, & \text{if } x = 1, 2, 3, \ldots \\ 0, & \text{ow} \end{cases}$$

Prove that $P[|X-\mu| \leq 2] > \frac{1}{2}$.

ANS:-

$$E(X) = \sum_{x=1}^{\infty} x \cdot \frac{1}{2^x}$$

$$= \frac{1}{2}\left[1 + \frac{1+1}{2} + \frac{1+2}{2^2} + \cdots\right]$$

$$= \frac{1}{2}\left[\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right) + \left(\frac{1}{2}+\frac{2}{2^2}+\cdots\right)\right]$$

$$\Rightarrow E(X) = \frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right) + \frac{1}{2}E(X)$$

$$\Rightarrow \frac{1}{2}E(X) = \frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right)$$

$$\Rightarrow E(X) = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots$$

$$= \frac{1}{1-\frac{1}{2}} = 2$$

$$V(X) = \frac{1/2}{(1/2)^2} = 2$$

Putting $\mu = 2$, and $\sigma = \sqrt{2}$

$\therefore P[|X-\mu| \leq \sqrt{2}\,t] > 1 - \frac{1}{t^2}$ [By Chebyshev's inequality]

Now, putting $t = \sqrt{2}$,

$P[|X-\mu| \leq 2] > \frac{1}{2}$. (Proved)

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CHERNOV'S INEQUALITY :—

Statement:— Let X be an r.v. such that its MGF, $M_X(t) = E(e^{tX})$ exixts. Then

i) $P(X \geq a) \leq \frac{M(t)}{e^{at}}$, $t > 0$

ii) $P(X \leq a) \leq \frac{M(t)}{e^{at}}$, $t < 0$

Proof:— i) for $t > 0$,

$$P(X \geq a)$$
$$= P(tX \geq ta)$$
$$= P(e^{tX} \geq e^{ta}) \leq \frac{E(e^{tX})}{e^{ta}}, \text{ [By Markov's ineauality]}$$
$$= \frac{M_X(t)}{e^{at}}.$$

ii) for $t < 0$,

$$P(X \leq a)$$
$$= P(tX \geq ta)$$
$$= P(e^{tX} \geq e^{ta})$$
$$\leq \frac{E(e^{tX})}{e^{ta}}, \text{ [By Markov's ineauality]}$$
$$= \frac{M_X(t)}{e^{at}}.$$

Result:— Let X be an r.v. such that $a \leq X \leq b$, S.T. $Var(X) \leq \frac{(b-a)^2}{4}$. Justify the case of eauality.

Proof:— $a \leq X \leq b$

$$\Rightarrow a - \frac{a+b}{2} \leq X - \frac{a+b}{2} \leq b - \frac{a+b}{2}$$

$$\Rightarrow \frac{a-b}{2} \leq X - \frac{a+b}{2} \leq \frac{b-a}{2}$$

$$\Rightarrow \left| x - \frac{a+b}{2} \right| \leq \frac{b-a}{2} \quad ; \text{ let } Y = x - \frac{a+b}{2},$$

$$\therefore |Y| \leq \frac{b-a}{2}.$$

$$\Rightarrow E(Y^2) \leq \frac{(b-a)^2}{4}$$

$$\Rightarrow E\left(X - \frac{a+b}{2}\right)^2 \leq \frac{(b-a)^2}{4}$$

$$\therefore Var(Y) = E(Y^2) - E^2(Y) \leq E(Y^2) \leq \frac{(b-a)^2}{4}$$

$$\Rightarrow Var(X) \leq \frac{(b-a)^2}{4}.$$

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CONCEPT OF INCOME DISTRIBUTION

The Distribution of Income :

One of the basic facts of economic observation is that a wide variety of social situations yields a density curve of income distribution that is humped to the left — a skew distribution. A relative frequency distribution of income superimposed upon a free hand graduation of a smooth density is shown in the figure.

In the tabulation, the two extreme classes are open-end classes; negative incomes, though rare, are possible - business losses account for this possibility. Similarly, there is no limit given to the top of the highest income class.

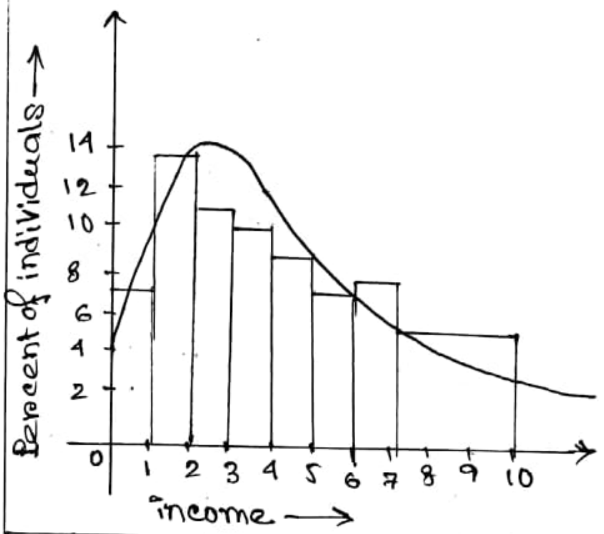


Table: Distribution of Income before Taxes, USA, 1969

Income class	Relative frequency in class (percent)	Cumulative frequency (percent)
under $1000	7	7
1000 - 1999	13	20
2000 - 2999	12	32
3000 - 3999	11	43
4000 - 4999	10	53
5000 - 5999	24	77
6000 - 6999	12	89
$7000 and over	8	97

I) Pareto's Law of Income Distribution :

Several positively skewed distributions are used for fitting income distributions. Of particular importance in this context is the Pareto Distribution. Pareto's law of distn. of incomes can be said in the following statement:

" The logarithm of the proportion of persons, with income y or more is a negatively sloped linear function of the logarithm of that value, i.e. $\log_e y$."

Symbolically, this takes the form $\log P(y) = \log A - \nu \log y$, $\nu > 0$
$\Leftrightarrow P(y) = A y^{-\nu}$, where $P(y)$ = Proportion of units with income $\geq y$, and A, ν are the parameters of the distribution.

This is a cumulative distribution function, but cumulated in a opposite direction (from right to left).

Therefore, $P[Y \geq y] = Ay^{-\nu} \Rightarrow 1 - F_Y(y) = Ay^{-\nu}$, and the density function of Y is $p(y) = \nu A y^{-\nu-1}$.

As income levels tend to zero, the density $p(y)$ approaches to ∞. As income gets larger and larger, the density falls towards zero. This pareto distn. is usually assumed to represent the distn. of incomes at upper levels or atleast, above some low values. In the case of income distn., it does not fit the distn. of low incomes well. We might think of it as a law of the distn. of incomes among taxpayers.

If y_0 is the lower limit, say, the model point, due to truncation then

$1 = P[Y \geq y_0] = Ay_0^{-\nu} \Rightarrow A = y_0^{\nu}$. The range of Y is from y_0 to ∞. With this modification the law can be represented by

$$P(y) = P[Y \geq y] = \begin{cases} 0 & , y < y_0 \\ \left(\frac{y_0}{y}\right)^{\nu} & y \geq y_0 \quad , \text{where } \nu > 0 \end{cases}$$

The density function of the distn is given by

$$p(y) = \begin{cases} \frac{\nu}{y_0}\left(\frac{y_0}{y}\right)^{\nu+1} & , y > y_0 \\ 0 & , \text{OW} \end{cases}$$

<u>Remark</u>:- To determine whether a income distn. follows distn., we plot, on a double logarithmic graph, $\log P(y)$ is plotted against $\log y$.

<u>Aliter</u>:- Statement of Pareto's law of Income Distribution:-

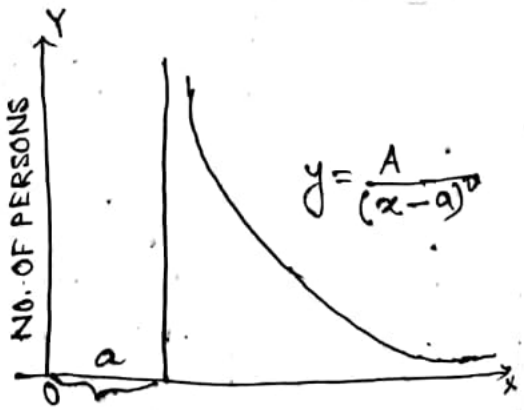
"In all places and at all times, the distribution of income in a stable economy is given by

$$y = A(x-\alpha)^{-\nu}, \text{ where}$$

y = number of people having income x or greater,

a = lowest income at which curve begins,

A and ν are certain parameters.



Shifting origin to $x = a$, Pareto curve becomes $y = Ax^{-\nu}$, taking log both sides, $\log y = \log A - \nu \log x$.

(II) Lognormal Distribution :—

While the Pareto distribution is used for graduating the upper part of the income distribution, the log-normal distn. has been similarly used for the lower part. Furthermore, lognormal distn. is used to describe some related distn.s like household distn. of consumer expenditure.

The r.v. Y is said to follow a lognormal distn. with parameters μ and σ^2 if $\log_e Y \sim N(\mu, \sigma^2)$. The density function of Y is

$$f_Y(y) = \begin{cases} \frac{1}{y\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\log_e y-\mu}{\sigma}\right)^2}, & y>0 \\ 0 & \text{ow} \end{cases}$$

Now, the proportion of persons with income y or more is

$$P(y) = \int_y^\infty \frac{1}{u\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\log_e u-\mu}{\sigma}\right)^2} du = \int_{\frac{\log_e y-\mu}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv$$

$$= 1 - \Phi\left(\frac{\log_e y-\mu}{\sigma}\right)$$

$$\Rightarrow \frac{\log_e y-\mu}{\sigma} = \Phi^{-1}\left(1-P(y)\right) = \eta \text{, say,}$$

where η is called the probit of $P(y)$. This property is often used to identify the log-normal distn..

Curve of Concentration :—

A measurement of inequality which does not require any parametric representation of the distn. was introduced by M.O. Lorenz. The Lorenz curve shows the relationship between the proportions share in population as reckoned from the poorest and the correspond income share.

Let the total frequency of the distn. is N and its total income is I; We write

$$p_x = \frac{\text{Number of persons with income} \le x}{N} = \frac{N_x}{N}$$

$$q_x = \frac{\text{Total income of the persons with income} \le x}{I} = \frac{I_x}{I}$$

Then the graph of $q_x = f(p_x)$ is known as Lorenz curve. The straight line $q_x = p_x$ is known as the 'Line of equal distribution'.

Let us assume p_x follows a Pareto distn. with Pareto coefficient, $\nu > 0$, i.e. $(1 - p_x) = A x^{-\nu}$, if $x \geq x_0$, where $\nu > 1$. The distn. of the Pareto distn. is

$$f(y) = \begin{cases} \nu A \cdot y^{-\nu-1} & \text{if } y > x_0 \\ 0 & \text{ow} \end{cases}$$

. Note that $\boxed{A = x_0^{\nu}}$

Hence, $1 - q_x = \int_x^{\infty} y f(y)\,dy = \frac{\nu}{\nu-1} A x^{-(\nu-1)}$, and

$1 - q_{x_0} = \frac{\nu}{\nu - 1} A x_0^{-(\nu-1)}$, which must be 1,

$$\Leftrightarrow A\left(\frac{\nu}{\nu-1}\right) = x_0^{\nu-1}.$$

Therefore, $(1 - q_x) = \left(\frac{x_0}{x}\right)^{\nu-1}$.

But $1 - p_x = \left(\frac{x_0}{x}\right)^{\nu}$, since $A = x_0^{\nu}$.

$$\Rightarrow (1 - q_x) = \{1 - p_x\}^{\frac{\nu-1}{\nu}}$$

$$\Rightarrow q_x = 1 - (1 - p_x)^{\delta}, \text{ where } \delta = \frac{\nu-1}{\nu}.$$

As a measure of the income inequality, Gini proposed a concentration ratio (ρ):

$$= \frac{\text{area } BDOB}{\Delta BOC} = 1 - \frac{\text{area}(BDOCB)}{\Delta BOC}$$

$$= 1 - 2(\text{area } BDOCB)$$

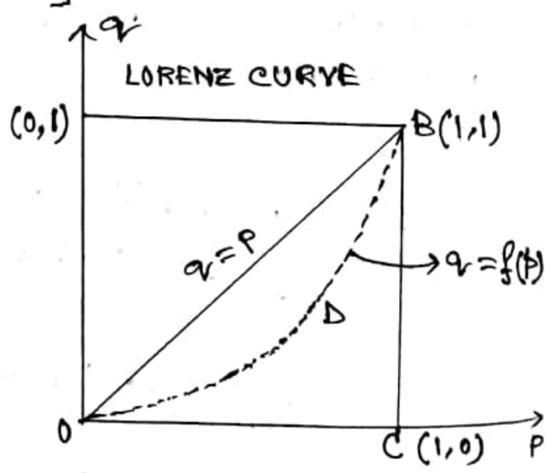
$$= 1 - 2\int_0^1 q\,dp. \quad \left[\because \Delta BOC = \tfrac{1}{2} \times OC \times OB = \tfrac{1}{2}\right]$$

$$= 1 - 2\int_0^1 \{1 - (1-p)^{\delta}\}\,dp$$

$$= 1 - 2\left[p + \frac{(1-p)^{\delta+1}}{\delta+1}\right]_0^1$$

$$= 1 - \frac{2\delta}{\delta+1} \quad \left[\text{putting } \delta = \frac{\nu-1}{\nu}\right]$$

$$= \frac{1}{2\nu+1}, \text{ where } \nu > 1.$$



Lorenz Curve

∴ The graph of this function (known as Lorenz curve) is the design of income concentration and is compared below with the line of equal distribution:—

———— × ————

PROBLEMS ON PROBABILITY

PROBLEMS :

☆ 1. Let X and Y be two random variables with joint p.d.f.

$$f(x,y) = \begin{cases} 1 & \text{if } -y < x < y,\ 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the regression equation of Y on X and that of X on Y. (ISI)

Soln. → Let X & Y be two random variables with joint probability density function,

$$f(x,y) = \begin{cases} 1 & \text{if } -y < x < y,\ 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Here $-y < x < y$ and $0 < y < 1$

$\Rightarrow -1 < x < 1$, which is the marginal range of x.

Again, $y > -x$ and $y > x$

$\therefore y > \max(x, -x)$

$\therefore \max(x,-x) < y < 1$

$\therefore$ marginal PDF of X is given by,

$$f_X(x) = \begin{cases} \displaystyle\int_{\max(x,-x)}^{1} f(x,y)\,dy & , -1 < x < 1 \\ 0 & \text{ow} \end{cases}$$

Case-I : $-1 < x < 0$

$\therefore \max(x, -x) = -x$

$$\therefore f_X(x) = \begin{cases} \int_{-x}^{1} dy, & \text{if } -1 < x < 0 \\ 0 & \text{ow} \end{cases}$$

$$= \begin{cases} 1+x & \text{if } -1 < x < 0 \\ 0 & \text{ow} \end{cases}$$

Case-II : $0 < x < 1$

$\max(x, -x) = x$,

$$\therefore f_X(x) = \begin{cases} \int_{x}^{1} dy & \text{if } 0 < x < 1 \\ 0 & \text{ow} \end{cases}$$

$$= \begin{cases} 1-x & \text{if } 0 < x < 1 \\ 0 & \text{ow} \end{cases}$$

Marginal PDF of Y is given by.

$$f_Y(y) = \begin{cases} \int_{-y}^{y} dx & \text{if } 0 < y < 1 \\ 0 & \text{ow} \end{cases}$$

$$= \begin{cases} 2y & \text{if } 0 < y < 1 \\ 0 & \text{ow} \end{cases}$$

Case I. $-1 < x < 0$,

The conditional distribution of Y given $X = x$ is given by,

$$f_{Y/X}(y) = \frac{f(x,y)}{f_X(x)} = \begin{cases} \frac{1}{1+x} & \text{if } -x < y < 1 \\ 0 & \text{ow} \end{cases}$$

$$\therefore E(Y|X) = \int_{-x}^{1} \frac{y\,dy}{(1+x)}$$

$$= \frac{1}{(1+x)} \cdot \frac{1}{2}(1-x^2)$$

$$= \frac{1-x}{2}.$$

Case 2. $0 < x < 1$,

The conditional distribution of Y given $X=x$ is given by,

$$f_{Y|X}(y) = \frac{f(x,y)}{f(x)} = \begin{cases} \frac{1}{1-x} & \text{if } x<y<1 \\ 0 & \text{ow} \end{cases}$$

Similarly,

$$E(Y|X) = \frac{1+x}{2}.$$

$\therefore$ If $-1<x<1$, then regression equation of Y on X is given by,

$$y = \frac{1-|x|}{2}.$$

The conditional distribution of X given $Y=y$ is given by,

$$f_{X|Y}(x) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{2y} & \text{if } -y<x<y \\ 0 & \text{ow} \end{cases}$$

$$\therefore E(X|Y) = 0$$

$\therefore$ Regression equation of X on Y is given by $x=0$.

☆ 2. Suppose X is the number of heads in 10 tosses of a fair coin. Given $X=5$, what is the probability that the first head occured in the third toss?

Soln. →

Here X be a random variable representing the number of heads in 10 tosses of a fair coin.

$$\therefore X \sim bin\left(10, \frac{1}{2}\right)$$

The pmf of X is given by,

$$f(x) = \begin{cases} \binom{10}{x}\left(\frac{1}{2}\right)^{10} & , x = 0, 1, \ldots, 10 \\ 0 & , ow \end{cases}$$

Now, we have to calculate the following,

P[Head occured in the third toss | X=5]

= P[Head occured in the third toss | 5 heads has occured]

$$= \frac{P\left[\text{Two tails occured in the first two tosses, a head occured in third toss, 4 heads occured in 7 tosses}\right]}{P(X=5)}$$

$$= \frac{P[\text{Tail occured in first toss}] \times P[\text{Tail occured in second toss}] \times P[\text{A head occured in third toss}] \times P[\text{4 heads occured in 7 tosses}]}{P(X=5)}$$

$$= \frac{\left(\frac{1}{2}\right)^3 \binom{7}{4} \left(\frac{1}{2}\right)^7}{\binom{10}{5}\left(\frac{1}{2}\right)^{10}} = \frac{\binom{7}{4}}{\binom{10}{5}} = \frac{5}{36}.$$ [∵ Trials are independent]

☆ 3. Let Y_1, Y_2, Y_3 are iid continuous random variables. For $i=1,2$, define U_i as,

$$U_i = \begin{cases} 1 & \text{if } Y_{i+1} > Y_i \\ 0 & \text{ow} \end{cases}$$

Find the mean & variance of U_1+U_2. (ISI)

Soln.→ We are given that Y_1, Y_2 and Y_3 be iid continuous random variables,

Since Y_i's are contincous,

$$P[Y_i = Y_{i+1}] = 0$$

Now, $P[Y_i > Y_{i+1}] + P[Y_i < Y_{i+1}] + P[Y_i = Y_{i+1}] = 1$

$$\Rightarrow P[Y_i > Y_{i+1}] + P[Y_i < Y_{i+1}] = 1$$

Now,

$$P[Y_i > Y_{i+1}] = \iint_{Y_i > Y_{i+1}} f_{Y_i}(y_i) f_{Y_{i+1}}(y_{i+1})\, dy_i\, dy_{i+1}$$

[∵ Y_i and Y_{i+1} are independent]

$$= \iint_{Y_{i+1} > Y_i} f_{Y_{i+1}}(y_{i+1}) f_{Y_i}(y_i)\, dy_{i+1}\, dy_i$$

$$= P[Y_{i+1} > Y_i]$$

$$\therefore P[Y_{i+1} > Y_i] = \frac{1}{2}$$

Let us define another r.v. U_i as

$$U_i = \begin{cases} 1 & \text{if } Y_{i+1} > Y_i \\ 0 & \text{ow} \end{cases}$$

$$= \begin{cases} 1 & \text{with probability } 1/2 \\ 0 & \text{ow} \end{cases}$$

$$\therefore E(U_i) = \frac{1}{2}$$

$$\therefore E(U_1+U_2) = \frac{1}{2} + \frac{1}{2} = 1$$

Now,

$Var(U_1+U_2) = Var(U_1) + Var(U_2) + 2Cov(U_1, U_2)$

Now,

$Var(U_1) = Var(U_2) = E(U_i^2) - E^2(U_i)$

$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

Now,

$Cov(U_1, U_2) = E(U_1 U_2) - E(U_1)E(U_2)$

Now,

$E(U_1U_2) = 1.P(U_1U_2=1)$

$= P(U_1=1, U_2=1)$

$= P(Y_2 > Y_1, Y_3 > Y_2)$

$= P(Y_3 > Y_2 > Y_1)$

Since, Y_1, Y_2, Y_3 are i.i.d.

$P(Y_3 > Y_2 > Y_1) = P(Y_3 > Y_1 > Y_2) = P(Y_2 > Y_3 > Y_1)$

$= P(Y_2 > Y_1 > Y_3)$

$= P(Y_1 > Y_2 > Y_3)$

$= P(Y_1 > Y_3 > Y_2)$

$= \frac{1}{6}$.

$\therefore E(U_1U_2) = \frac{1}{6}$

$\therefore Cov(U_1U_2) = \frac{1}{6} - \frac{1}{4} = -\frac{1}{12}$

$\therefore Var(U_1+U_2) = \frac{1}{2} - \frac{2}{12}$

$= \frac{1}{3}$.

☆ 4. Suppose a random vector (X, Y) has joint probability density function

$$f(x, y) = 3y$$

on the triangle bounded by the lines $y = 0$, $y = 1 - x$ and $y = 1 + x$. Find the marginal PDF of X and Y. Compute $E\left(Y \mid X \le \frac{1}{2}\right)$. (CU)

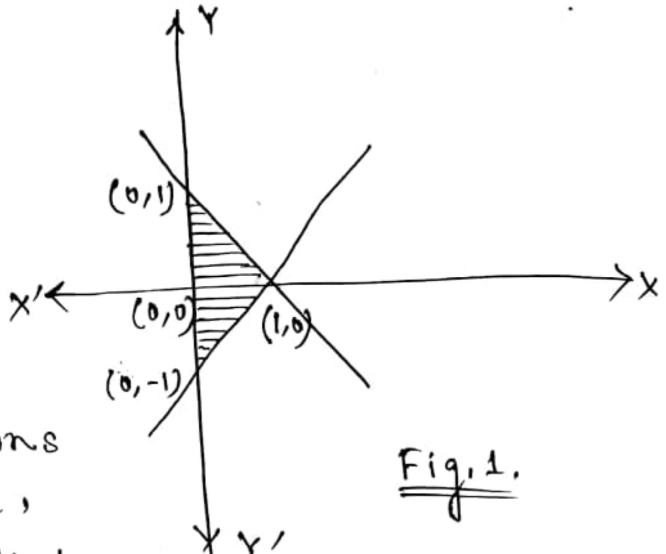


Fig. 1.

Soln. →

The joint PDF of the random vector (X, Y) is given by

$f(x, y) = 3y$, on the shaded triangle of figure 1.

From the figure, the range of the marginal distributions of X and Y are given by,

$0 < x < 1$, $-1 < y < 1$, respectively.

Now, $f(x, y) = 3y$ if $1 - x < y < 1 + x$, $0 < x < 1$

Now, $1 + x > y$

$\Rightarrow x > y - 1$

and $1 - x < y$

$\Rightarrow x - 1 > -y$

$\Rightarrow x > 1 - y$

$\therefore x > \max\{(1-y), (y-1)\}$

$\therefore \max\{(1-y), (y-1)\} < x < 1$

Case I :- $-1 < y < 0$

If $-1 < y < 0$

$\Rightarrow -2 < y - 1 < -1$

and $1 > -y > 0$

$\Rightarrow 2 > 1 - y > 1$

$\therefore \max\{(y-1), (1-y)\} = 1 - y$

$\therefore 1 - y < x < 1$

$\therefore$ Marginal PDF of Y is given by,

$$f_Y(y) = \int_{1-y}^{1} 3y\,dx \quad , \text{if } -1<y<0$$

$$= \begin{cases} 3y^2 & \text{if } -1<y<0 \\ 0 & \text{o.w} \end{cases}$$

Case-II :- if $0<y<1$

$$\max\{(y-1), (1-y)\} = y-1$$

$\therefore$ Marginal PDF of Y is given by,

$$f_Y(y) = \begin{cases} \int_{y-1}^{1} 3y\,dx & , 0<y<1 \\ 0 & , \text{oW} \end{cases}$$

$$= \begin{cases} 3y(2-y) & , 0<y<1 \\ 0 & , \text{oW} \end{cases}$$

Marginal PDF of X is given by,

$$f_X(x) = 3\int_{1-x}^{1+x} y\,dy, \quad 0<x<1$$

$$= \frac{3}{2}\left\{(1+x)^2-(1-x)^2\right\} \quad , 0<x<1$$

$$= \frac{3}{2}\times 4x = 6x, \quad 0<x<1$$

Now if $x \le \frac{1}{2}$

$$\Rightarrow 1-x \geq 1-\frac{1}{2}$$

$$\Rightarrow 1-x \geq \frac{1}{2}$$

and $1+x \le \frac{3}{2}$

$$\therefore 1-x < y < 1+x$$

$$\Rightarrow \frac{1}{2} < y < \frac{3}{2} \quad , \text{if } x \le \frac{1}{2}$$

$\therefore$ Conditional distribution of Y given that $X \le \frac{1}{2}$ is given by,

$$f_{Y|X}(x) = \frac{f(x,y)}{P(X \le \frac{1}{2})} = \begin{cases} \dfrac{3y}{6\int_0^{1/2} x\,dx} & \text{if } \frac{1}{2} < y < \frac{3}{2} \\ 0 & \text{ow} \end{cases}$$

$$= \begin{cases} \dfrac{3y}{\frac{6}{2} \cdot \frac{1}{4}} & \text{if } \frac{1}{2} < y < \frac{3}{2} \\ 0 & \text{ow} \end{cases}$$

$$= \begin{cases} 4y & \text{if } \frac{1}{2} < y < \frac{3}{2} \\ 0 & \text{ow} \end{cases}$$

$$\therefore E\left[Y \middle| X \le \frac{1}{2}\right] = 4\int_{1/2}^{3/2} y^2\,dy$$

$$= 4\left[\frac{y^3}{3}\right]_{1/2}^{3/2}$$

$$= \frac{4}{3}\left[\frac{9}{4} - \frac{1}{4}\right]$$

$$= \frac{8}{3}.$$

☆5. Let X be a continuous random variable with distribution function $F(x)$, which is such that

$$F(a+x) + F(a-x) = 1 \text{ for some fixed } a.$$

i) Show that $E(X) = a$

ii) If Y be an another r.v. defined as

$$Y = \begin{cases} 0 & \text{if } X < a \\ 1 & \text{if } X > a \end{cases}$$

then S.T. Y and $Z = |X-a|$ will be independently distributed.

(ISI)

ANS:-

i) It is given that,

$$F(a+x) + F(a-x) = 1$$

From the above equation it is clear that the distn. of X is symmetric about '**a**',

$$\text{Hence, } E(X-a) = 0$$

$$\Rightarrow E(X) = a$$

ii) It is given that,

$$Y = \begin{cases} 0 & \text{if } X < a \\ 1 & \text{if } X \geq a \end{cases}$$

and $Z = |X-a|$

Now from the equation, $F(x+a) + F(a-x) = 1$,

it is clear that $F(a) = \frac{1}{2}$ [since the distribution is symmetric about 'a']

$$\therefore Y = \begin{cases} 0 & \text{with prob. } \frac{1}{2} \\ 1 & \text{with prob. } \frac{1}{2} \end{cases}$$

Now, for some $z > 0$

$P[Z \leq z, Y = 0]$

$= P[|X-a| \leq z, X < a]$

$= P[-x+a \leq z \leq x-a, X < a]$

$= P[a-z \leq X \leq a+z, X < a]$

$= P[a-z \le X \le \min(a+z, a)]$

$= P[a-z \le X \le a]$

$= F(a) - F(a-z)$

$= \frac{1}{2} - F(a-z)$

$= \frac{1}{2}[F(a+z) - F(a-z)]$

$= \frac{1}{2} P[a-z \le X \le a+z]$

$= \frac{1}{2} P[|X| \le a+z]$

$= \frac{1}{2} P[|X-a| \le z]$, since $a > 0$.

$= \frac{1}{2} P[Z \le z]$

$= P[Z \le z] \cdot F(a)$

$= P[Z \le z] \cdot P[Y=0]$

Similarly, it can be shown that,

$$P[Z \le z, Y=1] = P[Z \le z] \cdot P[Y=1]$$

Hence,

$$P[Z \le z] \cdot P[Y=y] = P[Z \le z, Y=y]$$

Hence, Y and Z are independently distributed.

☆6. A bag contains a coin of value M and a number of other coins whose aggregate value is m. A person draws coins one at a time till he draws the coin of value M. Find the value of his expectation. (ISI)

<u>ANS:-</u> Let the coins be $A, B_1, B_2, \ldots, B_n$.

value of $A = M$ & value of $B_i = m_i$ (say)

such that $\sum_{i=1}^{n} m_i = m$.

Let Y_x be the value of the coins if x drawings are needed, $x = 0, 1, 2, \ldots, n+1$.

and Y is the total value of the coins eventually.

Now, $E(Y) = E\{E(Y_x \mid X = x)\}$

Now, $E(Y_x \mid X=1) = M$

$$E(Y_x \mid X=2) = \frac{M+m_1}{n} + \frac{M+m_2}{n} + \cdots + \frac{M+m_n}{n}$$

$$= M + \frac{m}{n}.$$

$$E(Y_x \mid X=3) = M + \frac{2m}{n}$$

In general, $E(Y_x \mid X=x) = M + \frac{(x-1)m}{n}$.

Now, $$E(Y) = \sum_{x=1}^{n+1} \left[M + (x-1)\frac{m}{n}\right] \cdot P(X=x)$$

Now,

$P(X=x) = P(X$ drawings are required to get the coin A$)$

$$= \frac{n}{n+1} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdots \frac{n-x+1}{n-x+2} \cdot \frac{1}{n-x+1}$$

$$= \frac{1}{n+1}.$$

$$\therefore E(Y) = \sum_{x=1}^{n+1} \left[M + (x-1)\frac{m}{n}\right] \cdot \frac{1}{n+1}$$

$$= \frac{1}{(n+1)}\left[(n+1)M + \frac{m}{n} \cdot \frac{n(n+1)}{2}\right]$$

$$= M + \frac{m}{2}.$$

So the required value of the expectation is

$$\left(M + \frac{m}{2}\right).$$

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SOME IMPORTANT PROBLEMS

☆ 1. A and B have respectively $(n+1)$ and n coins. If they toss their coin simultaneously. What is the probability that, —

i) A will have more heads than B.

ii) A and B will have an equal number of heads.

iii) B will have more heads than A. [1997]

Soln. → Let us define the random variables as follows,

$X =$ No. of heads obtained by A.

$Y =$ No. of heads obtained by B.

$$X \sim \text{bin}\left(n+1, \tfrac{1}{2}\right)$$
$$Y \sim \text{bin}\left(n, \tfrac{1}{2}\right)$$

Then,

$$(n+1-X) \sim \text{bin}\left(n+1, \tfrac{1}{2}\right)$$
$$(n-Y) \sim \text{bin}\left(n, \tfrac{1}{2}\right)$$

i) $P(\text{A will have more heads than B})$

$$= P(X > Y)$$
$$= P(n+1-X > n-Y)$$
$$= P(Y > X-1)$$
$$= P(Y \geq X)$$
$$= 1 - P(X > Y)$$

$$\therefore 2P(X > Y) = 1$$
$$\Rightarrow P(X > Y) = \frac{1}{2}.$$

ii) $P(\text{A and B have equal number of heads})$

$$= P(X = Y)$$
$$= \sum_{i=1}^{n} P(X = i, Y = i)$$
$$= \sum_{i=1}^{n} P(X = i)\, P(Y = i)$$
$$= \sum_{i=1}^{n} \binom{n+1}{i} \left(\frac{1}{2}\right)^{i} \left(\frac{1}{2}\right)^{n+1-i} \cdot \binom{n}{i} \left(\frac{1}{2}\right)^{i} \left(\frac{1}{2}\right)^{n-i}$$
$$= \sum_{i=1}^{n} \binom{n+1}{i} \cdot \binom{n}{i} \left(\frac{1}{2}\right)^{2n+1} = \left(\frac{1}{2}\right)^{2n+1} \sum_{i=1}^{n} \binom{n+1}{i} \binom{n}{i}$$
$$= \left(\frac{1}{2}\right)^{2n+1} \sum_{i=1}^{n} \frac{(n+1)!}{i!\,(n-i+1)!} \cdot \frac{n!}{i!\,(n-i)!}$$
$$= \left(\frac{1}{2}\right)^{2n+1} \binom{2n+1}{n}$$

iii) P(B have more heads than A)

$= P(Y > X)$

$= 1 - P(X \geq Y)$

$= 1 - P(X = Y) - P(X > Y)$

$= 1 - \left(\frac{1}{2}\right)^{2n+1} \binom{2n+1}{n} - \frac{1}{2}$

$= \frac{1}{2}\left[1 - \left(\frac{1}{2}\right)^{2n} \binom{2n+1}{n}\right]$

☆ 2. A book of N pages contains on the average λ misprints per page. Estimate the probability that a page drawn at random contains.

(a) at least one misprints.

(b) More than k misprints.

Soln → Let us define the random variable X as follows,

X = No. of misprints per page,

The book contains λ misprints per page on an average. Since the number of trials i.e. the no. of words is very large and probability ~~that~~ of a misprint is very small, hence according to the definition of Poisson distribution,

$X \sim P(\lambda)$

$$\therefore P(X = x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}, \quad x = 0, 1, 2, \ldots ; \lambda > 0$$

$$= 0, \quad \text{ow}$$

(a) P(at least one misprint)

$= P(X \geq 1)$

$= 1 - P(X < 1)$

$= 1 - P(X = 0)$

$= (1 - e^{-\lambda})$

(b) P(more than k misprints)

$= P(X > k)$

$= P(X \geq k-1)$

$= 1 - P(X \leq k-1)$

$$= 1 - \sum_{x=0}^{k-1} \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

☆ 3. The discrete random variable X has the Power Series distribution with the p.m.f.

$$f(x) = a_x \cdot \frac{\theta^x}{g(\theta)} \quad \text{for } x = 0, 1, 2, \ldots$$

where $g(\theta)$ is a differentiable function. Find mean & variance. [C.U. 2011] [WBSU'11]

Soln. → The given p.m.f. of Power series distribution ~~by~~ is,

$$f(x) = \frac{a_x \theta^x}{g(\theta)}$$

$$\Rightarrow \sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} \frac{a_x \theta^x}{g(\theta)}$$

$$\Rightarrow g(\theta) = \sum_{x=0}^{\infty} a_x \theta^x \quad \left[\because \sum_{x=0}^{\infty} f(x) = 1\right]$$

Now, $$E(X) = \sum_{x=0}^{\infty} x \cdot f(x)$$

$$= \sum_{x=0}^{\infty} \frac{x a_x \cdot \theta^x}{g(\theta)}$$

$$= \frac{\theta \sum_{x=0}^{\infty} a_x \cdot x \cdot \theta^{x-1}}{g(\theta)}$$

$$= \frac{\theta \frac{d}{d\theta}\{g(\theta)\}}{g(\theta)}$$

$$= \theta \frac{g'(\theta)}{g(\theta)}$$

$$= \theta \frac{d}{d\theta} \ln g(\theta)$$

$$\therefore E(X(X-1)) = \sum_{x=0}^{\infty} \frac{x(x-1) a_x \cdot \theta^x}{g(\theta)}$$

$$= \theta^2 \sum_{x=0}^{\infty} \frac{a_x \cdot x(x-1) \theta^{x-2}}{g(\theta)}$$

$$= \theta^2 \frac{g''(\theta)}{g(\theta)}$$

$$\therefore Var(X) = E(X(X-1)) + E(X) - E^2(X)$$

$$= \theta^2 \left[\frac{g''(\theta) g(\theta) - \{g'(\theta)\}^2}{\{g(\theta)\}^2}\right] + E(X)$$

$$= E(X) + \theta^2 \cdot \frac{d^2}{d\theta^2}\left[\frac{g'(\theta)}{g(\theta)}\right]$$

☆4. Consider the following logarithmic distribution with p.m.f., $f(x) = \frac{c\theta^x}{x}$ for $x = 1, 2, 3, \ldots$

where $0 < \theta < 1$. Show that for this distribution,

$$\mu = \frac{c\theta}{1-\theta}, \quad \sigma^2 = \mu\left(\frac{1}{1-\theta} - \mu\right).$$

<u>Soln.</u> :→ The given p.m.f. is

$$f(x) = \frac{c\theta^x}{x}, \quad x = 1, 2, \ldots$$

Now, $$\frac{f(x+1)}{f(x)} = \frac{x}{x+1} \cdot \theta \quad \cdots\cdots \text{(i)}$$

$$\Rightarrow (x+1) f(x+1) = \theta x f(x)$$

$$\Rightarrow \sum_{x=1}^{\infty} (x+1) f(x+1) = \theta \sum_{x=1}^{\infty} x f(x)$$

$$\Rightarrow \sum_{x=2}^{\infty} (x+1) f(x+1) - f(1) = \theta\mu$$

$$\Rightarrow \mu - c\theta = \theta\mu$$

$$\Rightarrow \mu = \frac{c\theta}{1-\theta}.$$

This is the required answer for mean.

From (i),

$$\frac{f(x+1)}{f(x)} = \frac{x}{x+1} \cdot \theta$$

$$\Rightarrow (x+1) f(x+1) = \theta x f(x)$$

$$\Rightarrow (x+1)^2 f(x+1) = \theta \cdot x(x+1) f(x)$$

$$\Rightarrow \sum_{x=1}^{\infty} (x+1)^2 f(x+1) = \theta \sum_{x=1}^{\infty} x^2 f(x) + \theta \sum_{x=1}^{\infty} x f(x)$$

$$\Rightarrow E(X^2) - c\theta = \theta E(X^2) + \theta\mu$$

$$\Rightarrow E(X^2) = \frac{c\theta}{1-\theta} + \frac{\theta\mu}{1-\theta}.$$

$$\therefore \operatorname{Var}(X) = E(X^2) - E^2(X)$$

$$= \mu + \frac{\theta\mu}{1-\theta} - \mu^2$$

$$= \mu\left(\frac{1}{1-\theta} - \mu\right)$$

This is the required answer for variance.

☆5. If $X_1 \sim bin(n_1, p)$ and $X_2 \sim bin(n_2, p)$ then find the distribution of $X_1 | X_1+X_2$; where X_1 & X_2 are independent Random variables.

Sol^n.

As, $X_1 \sim bin(n_1, p)$, so the p.m.f. is

$$P(X_1=x) = \binom{n_1}{x} p^x q^{n_1-x} \; ; \quad p+q=1$$

and similiarly, $X_2 \sim bin(n_2, p)$, so the p.m.f. is

$$P(X_2=x) = \binom{n_2}{x} p^x q^{n_2-x} \; ; \quad p+q=1.$$

Now, we are to find $X_1 | X_1+X_2 \sim ?$

$$P(X_1 | X_1+X_2) = \frac{P(X_1=x \cap X_1+X_2=y)}{P(X_1+X_2=y)}$$

$$= \frac{P(X_1=x, X_2=y-x)}{P(X_1+X_2=y)}$$

$$= \frac{P(X_1=x)P(X_2=y-x)}{P(X_1+X_2=y)} \quad \text{[as } X_1 \text{ \& } X_2 \text{ are independent]}$$

$$= \frac{\binom{n_1}{x} p^x q^{n_1-x} \binom{n_2}{y-x} p^{y-x} q^{n_2-y+x}}{\binom{n_1+n_2}{y} p^y q^{n_1+n_2-y}}$$

$$= \frac{\binom{n_1}{x}\binom{n_2}{y-x}}{\binom{n_1+n_2}{y}}$$

$\therefore X_1 | X_1+X_2 \sim$ Hypergeometric(y, n_1, n_1+n_2)

It is the required answer.

☆ 6. X and Y are independently and identically distributed Poisson random variables with mean λ. Obtain $P(XY = \text{even})$.

<u>Soln</u> →

$$P(XY = \text{even}) = P(X \text{ even or } Y \text{ even})$$
$$= P(X = \text{even}) + P(Y = \text{even}) - P(X \text{ even } Y \text{ even})$$
$$= 2P(X = \text{even}) - P^2(X = \text{even})$$

Now,

$$P(X = \text{even}) = \sum_{i=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^{2i}}{(2i)!} \longrightarrow \langle i \rangle$$
$$= e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^{2i}}{(2i)!}$$
$$= e^{-\lambda} \cdot \left(\frac{e^{\lambda} + e^{-\lambda}}{2} \right) \quad \left[\text{as, } e^{\lambda} + e^{-\lambda} = 2 \sum_{i=0}^{\infty} \frac{\lambda^{2i}}{(2i)!} \right]$$

$$\therefore P(XY = \text{even}) = 2 \cdot e^{-\lambda} \left(\frac{e^{\lambda} + e^{-\lambda}}{2} \right) - e^{-2\lambda} \cdot \frac{(e^{\lambda} + e^{-\lambda})^2}{4}$$
$$= (e^{-2\lambda} + 1) - e^{-2\lambda} \frac{(e^{\lambda} + e^{-\lambda})^2}{4}$$
$$= e^{-2\lambda} \left[1 - \frac{(e^{\lambda} + e^{-\lambda})^2}{4} \right]$$

☆ 7. There are $m+n$ tickets in an urn which are numbered $1, 2, \ldots, m+n$. Suppose n tickets are drawn at random from the urn. Show that the probability that x of the tickets drawn will have numbers exceeding all numbers on the tickets left in the urn is

$$\binom{m+n-x-1}{m-1} \bigg/ \binom{m+n}{m}.$$

Also show that for the corresponding r.v. X,

$$E\left[(m+n-X+r-1)_r \right] = \frac{(m+r-1)_r \binom{m+n+r}{m+r}}{\binom{m+n}{m}}$$

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Soln. → Let us define a random variable X as follows,

X : numbers of tickets drawn will have numbers exceeding all numbers on the tickets left in the urn.

There are $(m+n)$ tickets in the urn out of which n tickets are drawn. Required to find $P[X=x]$

Here, $(n-x)$ tickets are drawn from $(m+n-x-1)$ tickets numbered $1,2,3,\ldots\ldots,m+n-x-1$, and x tickets are drawn from higher numbered tickets $x, m+n-x+1, m+n-x+2, \ldots\ldots, m+n$.

$$\therefore P[X=x] = \frac{\binom{m+n-x-1}{n-x}\binom{x}{x}}{\binom{m+n}{n}} = \frac{\binom{m+n-x-1}{m-1}}{\binom{m+n}{n}} \quad [\text{ANS}]$$

☒ Now,

$E\left[(m+n-X+r-1)_r\right]$

$$= \sum_x (m+n-x+r-1)_r \cdot \frac{\binom{m+n-x-1}{m-1}}{\binom{m+n}{n}}$$

$$= \frac{1}{\binom{m+n}{n}} \sum_x (m+n-x+r-1)_r \binom{m+n-x-1}{m-1}$$

$$= \frac{1}{\binom{m+n}{n}} \sum_x \frac{(m+n-x+r-1)!}{(m-1)!\,(n-x)!}$$

$$= \frac{(m+r-1)!}{(m-1)!\binom{m+n}{m}} \sum_x \binom{m+n-x-1+r}{m+r-1} \quad \text{———} \langle i \rangle$$

Since, $P[X=x]$ is a p.m.f., $\sum_x P[X=x]=1$

$$\Rightarrow \sum_x \binom{m+n-x-1}{m-1} = \binom{m+n}{n}$$

Similarly, $\sum_x \binom{m+n-x-1+r}{m+r-1} = \binom{m+n+r}{n} = \binom{m+n+r}{m+r}$

Then from $\langle i \rangle$, we get, —

$$E\left[(m+n-X+r-1)_r\right] = \frac{(m+r-1)_r}{\binom{m+n}{m}}\binom{m+n+r}{m+r}$$

☆ 8. Banach Match Box Problem: —

A certain mathematician carries two matchboxes in his pocket, each time he wants to use a match, he selects one of boxes at random. Each packet contain N matchsticks.

(a) Find the distribution of the number of sticks in one box, while the other is found empty.

(b) Also find the distribution of the number of sticks remaining in one box becomes empty.

(c) Find mean and variance.

Soln: →

(a) Let us define a r.v. X denoting the number of matchsticks remaining in the match box when the other box is found empty.

Let X_{ij}, $i, j, i \neq j$ denotes the number of matchsticks remaining in the ith box when the jth box is found to be empty.

The mass points of X are $0, 1, \ldots, N$.

For any such mass point x,

$$P[X=x] = P(X_{12}=x) + P(X_{21}=x)$$

We consider the distribution of X_{12}:

The second box will be found empty if the box is chosen for the $(N+1)$th time. At that time the first box contain x matches if $(N-x)$ matches have already taken from it. If the selection of the second box is regarded as success, then the event.

$$P[X_{12}=x] = P\left[(N-x) \text{ failures occur preceding the } (N+1)\text{th success}\right]$$

$$= P[Z = N-x] \quad \text{, where } Z \sim NB\left(N+1, \tfrac{1}{2}\right)$$

$$= \binom{N+1+N-x-1}{N-x} \left(\frac{1}{2}\right)^{N-x} \left(\frac{1}{2}\right)^{N+1}$$

$$= \binom{2N-x}{N-x} \left(\frac{1}{2}\right)^{2N-x+1}$$

Similiarly,

$$P[X_{21}=x] = \binom{2N-x}{N-x}\left(\frac{1}{2}\right)^{2N-x+1}$$

$$\therefore P[X=x] = \binom{2N-x}{N-x}\left(\frac{1}{2}\right)^{2N-x}, \quad x=0,1,2,\ldots,N.$$

(b) Let us define a random variable Y denoting the number of matchsticks remaining in a matchbox when the other matchbox becomes empty.

Let Y_{ij}, $i,j, i\neq j$, denotes the number of matchsticks remaining in the ith box when jth box becomes empty.

The mass points of Y are $0,1,2,\ldots,N$.

For any such mass point y,

$$P[Y=y] = P[Y_{21}=y] + P[Y_{12}=y]$$

Now, $P[Y_{12}=y] = P[Z=N-y]$, $Z \sim N.B(N,\frac{1}{2})$

$$= \binom{N+N-y-1}{N-y}\left(\frac{1}{2}\right)^{N}\left(\frac{1}{2}\right)^{N-y}$$

$$= \binom{2N-y-1}{N-y}\left(\frac{1}{2}\right)^{2N-y}$$

Similarly,

$$P[Y_{21}=y] = \binom{2N-y-1}{N-y}\left(\frac{1}{2}\right)^{2N-y}$$

$$\therefore P[Y=y] = \binom{2N-y-1}{N-y}\left(\frac{1}{2}\right)^{2N-y+1}.$$

☆9. Show that if two binomial distributions of parameters $(n, \frac{1}{2})$ are so superimposed that the kth term of the one coincides with the $(k+1)$st term of the other, then the distribution formed by adding superimposed terms is binomial with parameters $(n+1, \frac{1}{2})$.

Soln. The $(k+1)$th term of the new distribution is,

$$\frac{1}{2}\left\{\binom{n}{k}\frac{1}{2^n} + \binom{n}{k+1}\frac{1}{2^n}\right\}$$

$$= \frac{1}{2^{n+1}}\left\{\binom{n}{k} + \binom{n}{k+1}\right\}$$

$$= \binom{n+1}{k+1}\frac{1}{2^{n+1}}.$$

and now, $\binom{n+1}{0} = 1$ and $\binom{n+1}{n+1} = 1$,

so the p.m.f. of the new distribution is, —

$$f(x) = \binom{n+1}{x}\frac{1}{2^{n+1}}, \quad x = 0, 1, \ldots, n+1.$$

$\therefore X \sim \text{Bin}(n+1, \frac{1}{2})$.

☆10. Let X be a RV with pmf,

$$f_X(x) = \frac{\binom{n}{x}}{2^n}, \quad x = 0, 1, 2, \ldots, n.$$

$$= 0 \quad , \text{ow}$$

Find the MGF of X.

If a RV has the MGF, $M(t) = 2^{-10}(1+e^t)^{10}$, find the p.m.f.?

Soln. → $M_X(t) = E(e^{tX})$

$$= \sum_{x=0}^{\infty} e^{tx} \cdot \binom{n}{x} \cdot \frac{1}{2^n}$$

$$= \frac{1}{2^n}\sum_{x=0}^{n}\binom{n}{x}\cdot(e^t)^x$$

$$= \frac{1}{2^n}(1+e^t)^n, \quad t \in \mathbb{R}$$

Now, note that, $M(t) = 2^{-10}(1+e^t)^{10}$ is in the above form, $n = 10$. So, by uniqueness property of the MGF, the required distribution is, —

$$f(x) = \frac{\binom{10}{x}}{2^{10}}, \quad x = 0, 1, \ldots, 10.$$

$$= 0 \quad , \text{ow}$$

☆11. A RV X has moments about '0' given by,

$$\mu_r' = \frac{K}{K+r}, \quad r=1,2,\ldots\ldots; \text{ where } K>0.$$

Show that the PDF of X is given by,

$$f(x) = \begin{cases} Kx^{K-1}, & \text{if } 0<x<1 \\ 0 & ; \text{OW} \end{cases}$$

Soln. → MGF of X constructed by the sequence $\{\mu_r'\}$ of moments is

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \cdot \frac{K}{K+r}$$

Now, the MGF of the distribution with p.d.f. $f(x)$ is

$$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} \left(\sum_{r=0}^{\infty} t^r \cdot \frac{x^r}{r!} \right) f(x)\,dx$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \left(\int_{-\infty}^{\infty} x^r f(x)\,dx \right)$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \left(\int_{0}^{1} x^r \cdot K \cdot x^{K-1}\,dx \right)$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[K \cdot \frac{x^{K+r}}{K+r} \right]_0^1$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \cdot \frac{K}{K+r}.$$

Here note that,

$$\sum_{r=0}^{\infty} \left| \frac{K}{K+r} \cdot \frac{t^r}{r!} \right| < \sum_{r=0}^{\infty} \frac{|t|^r}{r!} = e^{|t|} < \infty$$

⇒ MGF exists.

The MGF $M_X(t)$ of the RV and MGF $M(t)$ of the probability distribution with PDF $f(x)$ are identical. Hence, by uniqueness of MGF, the RV X has the p.d.f. $f(x)$.

☆12. Determine the probability distribution of the following PGF: probability Generating function = P(t),

$P(t) = \frac{1}{2-t}, \quad |t| \le 1.$

soln. → $P_X(t) = \sum_{x=0}^{\infty} t^x P(X=x), \quad |t| \le 1$

Here, $P_X(t) = (2-t)^{-1}.$

$$= 2^{-1}\left(1-\frac{t}{2}\right)^{-1}$$

$$= \frac{1}{2}\left[1+\frac{t}{4}+\frac{t^2}{8}+\cdots\right]$$

$$= \sum_{x=0}^{\infty} a_x t^x \quad , |t| \le 2$$

Here, $a_x = \frac{1}{2^{x+1}},$

Now, the PMF of the distribution is, — $f(x) = \frac{1}{2^{x+1}}, \; x=0,1,2\ldots$
$= 0 \quad , \text{ow}$

Here, $\sum_{x=0}^{\infty} t^x p(X=x) = \sum_{x=0}^{\infty} t^x \cdot \frac{1}{2^{x+1}}, \quad |t|<2.$

☆13. $M_{X,Y}(t_1,t_2) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{t_1^r}{r!}\cdot\frac{t_2^s}{s!}\cdot \mu'_{r,s}$

Thus $\mu'_{r,s}$ = co-efficient of $\frac{t_1^r}{r!}, \frac{t_2^s}{s!}$ in the expansion of $M_{X,Y}(t_1,t_2)$ and,

$$\mu'_{r,s} = \frac{\partial^{r+s} M_{X,Y}(t_1,t_2)}{\partial t_1^r \cdot \partial t_2^s}.$$

Proof:— $M_{X,Y}(t_1,t_2) = E\left(e^{t_1 x}\cdot e^{t_2 y}\right) = E\left[\left(\sum_{r=0}^{\infty}\frac{t_1^r \cdot x^r}{r!}\right)\left(\sum_{s=0}^{\infty}\frac{t_2^s\cdot y^s}{s!}\right)\right]$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{t_1^r\cdot t_2^s}{r!\,s!}E(X^rY^s)$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{t_1^r\cdot t_2^s}{r!\,s!}\mu'_{r,s}$$

Now, $\frac{\partial^2 M_{X,Y}(t_1,t_2)}{\partial t_1\,\partial t_2} = \frac{\partial^2}{\partial t_1 \partial t_2}E\left(e^{t_1X+t_2Y}\right)$

$$= E\left[\frac{\partial^2}{\partial t_1\partial t_2}\left(e^{t_1 x}\cdot e^{t_2 y}\right)\right]$$

$$= E\left[\frac{\partial}{\partial t_1}\cdot e^{t_1 x}\cdot\frac{\partial}{\partial t_2}\cdot e^{t_2 y}\right]$$

$$= E\left(X\cdot e^{t_1 X}\cdot Y e^{t_2 Y}\right).$$

☆ 14. If X & Y are independent Poisson variable, show that the conditional distribution of X given X+Y is binomial.

Soln. → $X \sim Poi(\lambda_1)$, $Y \sim Poi(\lambda_2)$ $\Rightarrow X+Y \sim Poi(\lambda_1+\lambda_2)$ [proved earlier]

Now,

$$P[X|X+Y] = \frac{P[X=x, X+Y=y]}{P[X+Y=y]}$$

$$= \frac{P[X=x, Y=y-x]}{P[X+Y=y]}$$

$$= \frac{\cancel{e^{-\lambda_1}} \cdot \lambda_1^{x} \cdot \cancel{e^{-\lambda_2}} \cdot \lambda_2^{y-x} \cdot y!}{x!(y-x)! \cdot \cancel{e^{-\lambda_1-\lambda_2}} \cdot (\lambda_1+\lambda_2)^y}$$

$$= \binom{y}{x} \frac{\lambda_1^x \cdot \lambda_2^{y-x}}{(\lambda_1+\lambda_2)^y}$$

$$= \binom{y}{x}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^x \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{y-x}$$

$$= \binom{y}{x}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^x \left(1-\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{y-x}, \quad x = 0, 1, \dots, y$$

$$\therefore X|X+Y \sim Bin\left(y, \frac{\lambda_1}{\lambda_1+\lambda_2}\right).$$

Particular case:—

Now if. $X \sim Poi(\lambda)$, $Y \sim Poi(\lambda)$

then, $X|X+Y \sim Bin(y, \frac{1}{2})$.

☆ 9. Alternative Solution:— $X \sim Bin(n, \frac{1}{2})$, $Y \sim Bin(n, \frac{1}{2})$

$$\therefore \sum_{x=0}^{n} \binom{n}{x}\left(\tfrac{1}{2}\right)^n = 1, \quad \sum_{y=0}^{n} \binom{n}{y}\left(\tfrac{1}{2}\right)^n = 1$$

$$\frac{1}{2^n}\left\{\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{k} + \dots + \binom{n}{n}\right\} = 1$$

$$\frac{1}{2^n}\left\{\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{k+1} + \dots + \binom{n}{n}\right\} = 1$$

$$\frac{1}{2^n} \cdot \left\{\binom{n}{0} + \binom{n+1}{1} + \binom{n+1}{2} + \dots + \binom{n+1}{k+1} + \dots + \binom{n+1}{n} + \binom{n}{n}\right\} = 2$$

$$\Rightarrow \frac{1}{2^{n+1}} \sum_{x=0}^{n+1} \binom{n+1}{x} = 1 \quad \left[\because \binom{n}{0} = \binom{n+1}{0}, \binom{n}{n} = \binom{n+1}{n+1}\right]$$

$$\Rightarrow \sum_{x=0}^{n+1} \binom{n+1}{x} \frac{1}{2^{n+1}} = 1$$

So, the PMF of the new distribution is,

$$f(x) = \binom{n+1}{x} \frac{1}{2^{n+1}} \quad , x = 0, 1, \ldots, n+1$$

$\therefore X \sim Bin(n+1, \frac{1}{2})$

☆15. For a RV X having Power series distribution with $E(X) = Var(X)$, then show that X must have a Poisson distribution.

Soln. → Let,
$$f_X(x) = \begin{cases} \dfrac{a_x \cdot \theta^x}{g(\theta)} & , x = 0, 1, 2, \ldots \\ 0 & , \text{ow} \end{cases}$$

where, $a_x \geq 0$, $g(\theta) = \sum_{x=0}^{\infty} a_x \cdot \theta^x$, $0 < \theta < b$

$$Var(X) = E(X) + \theta^2 \cdot \frac{d^2}{d\theta^2} \{\ln g(\theta)\}$$

$$\Rightarrow \theta^2 \cdot \frac{d^2}{d\theta^2} \{\ln g(\theta)\} = 0.$$

On integration, $\frac{d}{d\theta} \{\ln g(\theta)\} = c$ (constant)

$$\Rightarrow \ln g(\theta) = c\theta + k$$
$$\Rightarrow g(\theta) = e^{c\theta} \cdot e^k$$
$$= k_1 \cdot e^{c\theta}$$

$\therefore g(\theta) = k_1 \cdot e^{c\theta}$

$$\Rightarrow \sum_{x=0}^{\infty} a_x \cdot \theta^x = \sum_{x=0}^{\infty} k_1 \cdot \frac{(c\theta)^x}{x!}$$

$\therefore a_x = k_1 \cdot \frac{c^x}{x!} \quad , x = 0, 1, 2, \ldots$

$$\therefore f_X(x) = \begin{cases} \dfrac{\left(k_1 \cdot \frac{c^x}{x!}\right) \cdot \theta^x}{g(\theta)} & , x = 0, 1, 2, \ldots \\ 0 & , \text{ow} \end{cases}$$

$$= \begin{cases} e^{-c\theta} \cdot \dfrac{(c\theta)^x}{x!} & , x = 0, 1, 2, \ldots \\ 0 & , \text{ow} \end{cases}$$

$\Rightarrow X \sim Poisson(c\theta)$.

☆ 16. Show that the mean of the standard cauchy distribution does not exist.

Soln. $f(x) = \frac{1}{\pi(1+x^2)}$, $x \in \mathbb{R}$

$$E(|X|) = \int_{-\infty}^{\infty} |x| f(x)\, dx$$

Let, $1+x^2 = t$
$x\,dx = \frac{1}{2} dt$

$x=$	$-\infty$	∞
$t=$	0	∞

$$= \int_{-\infty}^{\infty} |x| \cdot \frac{1}{\pi(1+x^2)}\, dx$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{1+x^2}\, dx$$

$$= \frac{1}{2\pi} \int_0^{\infty} \frac{dt}{t}$$

$$= \frac{1}{2\pi} \left[\log t\right]_0^{\infty}$$

$$= \frac{1}{2\pi} \left[\log\infty - \log 0\right]$$

$$= \frac{1}{2\pi} \cdot \infty = \infty.$$

$\therefore$ $E(X)$ does not exist.

Alternative way :—

Let $X \sim C(0,1)$

Then $E(|X|) = \int_{-\infty}^{\infty} |x| \cdot \frac{1}{\pi(1+x^2)}\, dx$

$$= 2 \int_0^{\infty} \frac{x}{\pi(1+x^2)}\, dx$$

$$= \frac{1}{\pi} \lim_{t\to\infty} \int_0^{t} \frac{2x}{1+x^2}\, dx$$

$$= \frac{1}{\pi} \lim_{t\to\infty} \left[\log_e(1+x^2)\right]_0^{t}$$

$$= \frac{1}{\pi} \lim_{t\to\infty} \log_e(1+t^2)$$

$$= +\infty$$

Hence, the mean of the Cauchy distribution does not exist & consequently the higher order moments do not exist.

☆ 17. (LOSS OF MEMORY PROPERTY)
$P(X > s+t / X > s) = P(X > t)$ this relation holds iff X is an exponential distribution. Interpret this.

Soln.

If Part :→ X is an exponential distribution with parameter θ.

$$\therefore P(X > x) = e^{-x/\theta}$$

Now, $P(X > s+t / X > s)$

$$= \frac{P(X > s+t \cap X > s)}{P(X > s)}$$

$$= \frac{P(X > s+t)}{P(X > s)}$$

$$= \frac{e^{-(s+t)/\theta}}{e^{-s/\theta}}$$

$$= e^{-t/\theta}$$

$$= P(X > t)$$

Only if Part :→

Given that, $P(X > s+t / X > s) = P(X > t)$

$$\Rightarrow \frac{P(X > s+t \cap X > s)}{P(X > s)} = P(X > t)$$

$$\Rightarrow P(X > s+t) = P(X > s) P(X > t)$$

Now, $P(X > s+t+u) = P(X > s) P(X > t) P(X > u)$

By induction,

$$P(X > x_1 + x_2 + \cdots + x_n) = P(X > x_1) P(X > x_2) \cdots P(X > x_n)$$

Now, $P(X > x) = P(X > \underbrace{1+1+\cdots+1}_{x})$

$$= (P(X > 1))^x$$

$$= (e^{-1/\theta})^x$$

$$= e^{-x/\theta}$$

$\therefore F(x) = 1 - P(X > x)$
$= 1 - e^{-x/\theta}$

Differentiating $F(x)$ w.r.t. x, we get, —

$$f(x) = \frac{1}{\theta} \cdot e^{-x/\theta}, \quad x > 0, \theta > 0$$

$\therefore X \sim Exp(\theta)$.

Interpretation: —

If X represents the lifetime of a given component, say, an electric bulb, the above result states that the probability that a bulb will be opperative for at least (s+t) units, given that it has already run for (s) units, is the same as it's initial probability for lasting for at least (t) units. This means that the future lifetime often individuals has the same distribution, no matter how old it is at present. This is the 'Lack of Memory Property' of the exponential distribution. Another way of saying this is to say that an 'old' functioning component has the same lifetime distribution as a 'new' functioning component or that component is not subject to failure. If a non-negative Continuous R.V. X have 'Lack of memory Property', then X must have an Exponential distribution.

× — ×

18. DRUNKARD'S STEP PROBLEM: ONE DIMENSIONAL RANDOM WALK

A drunk man performed a random walk over the position $0, \pm 1, \pm 2, \ldots\ldots$. The drunk man starts from the point 0. He takes successive unit steps with probability p at right and probability $(1-p)$ at left. His steps are independent. X be a location of the drunk man after taking n-steps. Find the distribution of $\left(\frac{n+X}{2}\right)$ and find out $E(X)$.

Soln. →

R denotes no. of steps at right after taking n steps.

$\therefore R \sim Bin(n, p)$.

L denotes no. of steps at left after taking n steps.

$\therefore L \sim Bin(n, 1-p)$

Let us define,

X : the position of the drunkard after n steps.

$$R + L = n,$$
$$R - L = X.$$
$$\therefore 2R = n + X \Rightarrow R = \frac{n+X}{2} \sim Bin(n, p)$$

$$\therefore E\left(\frac{n+X}{2}\right) = np$$
$$\Rightarrow E(X) = 2\left[np - \frac{n}{2}\right]$$
$$= 2n\left(p - \frac{1}{2}\right)$$
$$= n(2p - 1).$$

X ——————X

☆ 19. Hazard Function :— The hazard function of a random variable X is defined as $\dfrac{f(x)}{1-F(x)}$, where $f(x) = \dfrac{d}{dx}[F(x)]$.

Result :— The Hazard function of a RV is constant iff X is an exponential r.v.

Proof :—

If Part :→ $X \sim Exp(\theta)$

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, \quad x>0, \theta>0$$

$$F(x) = 1 - e^{-x/\theta}$$

$$\therefore 1-F(x) = e^{-x/\theta}$$

$$\therefore \frac{f(x)}{1-F(x)} = \frac{1}{\theta} \text{ (constant)}$$

Only if Part :→

$$\frac{f(x)}{1-F(x)} = k \text{ (Constant)}$$

Now, $\dfrac{\frac{d}{dx}[F(x)]}{1-F(x)} = k$

or, $-\dfrac{d}{dx}\log(1-F(x)) = k$

or, $d\log(1-F(x)) = -k\,dx$

or, $\log(1-F(x)) = -kx + c$

or, $1-F(x) = e^{-kx+c}$

or, $F(x) = 1 - e^{-kx+c}$

or, $f(x) = \dfrac{d}{dx}\left(1 - e^{-kx+c}\right)$

$= e^{c} \cdot e^{-kx}$

Only if Part :—

OR,

$$\frac{f(x)}{1-F(x)} = k \text{ (constant)}$$

$$\Rightarrow \frac{d/dx\,[F(x)]}{1-F(x)} = k$$

$$\Rightarrow -\frac{d}{dx}\ln(1-F(x)) = k$$

$$\Rightarrow \ln(1-F(x)) = -kx + c$$

$$\Rightarrow 1-F(x) = e^{-kx} \cdot e^{c} = e^{-kx} \cdot \theta$$

$$\Rightarrow F(x) = 1 - \theta e^{-kx}$$

or, $f(x) = \theta k e^{-kx}$

Now, $F(0) = 1 - \theta$

or, $\theta = 1$,

$\therefore f(x) = ke^{-kx}$

$k = \frac{1}{\phi}$, $\therefore f(x) = \frac{1}{\phi} e^{-x/\phi}$

$\therefore X \sim Exp(\phi)$

let, $k = \frac{1}{\theta}$ and let c be so chosen that $e^{c} = \frac{1}{\theta}$,
$\Rightarrow c = \ln\frac{1}{\theta}$,

$\therefore f(x) = \frac{1}{\theta} e^{-x/\theta}$.

$\therefore X \sim Exp(\theta)$.

JOINT DISTRIBUTION OF TWO RANDOM VARIABLES

Two dimensional random Variable
or
Two dimensional random vector :—

Consider a probability space $(\Omega, \mathcal{A}, P)$ arising out of a random experiment. A vector of functions $\underset{\sim}{X} = (X_1, X_2)$ which maps Ω into $\mathbb{R}^2$ is said to be two dimensional random vector, if for each $x_i \in \mathbb{R}$, $i = 1, 2$.

$$\{\omega : X_1(\omega) \le x_1, X_2(\omega) \le x_2\} \in \mathcal{A}$$

- **Example:**— An unbiased coin is tossed twice.

$$\Omega = \{HH, HT, TH, TT\}$$

We take $\mathcal{A}$ as a class of all subsets of Ω.

Let X_1 denotes the number of heads obtained, and X_2 denotes the number of tails obtained.

$$\{\omega : X_1(\omega) \le x_1\} = \begin{cases} \phi & \text{when } x_1 < 0 \\ \{TT\} & \text{when } 0 \le x_1 < 1 \\ \{HH, HT, TT\} & \text{when } 1 \le x_1 < 2 \\ \Omega & \text{when } x_1 \geq 2 \end{cases}$$

$$\{\omega : X_2(\omega) \le x_2\} = \begin{cases} \phi & \text{when } x_2 < 0 \\ \{HH\} & \text{when } 0 \le x_2 < 1 \\ \{HH, TH, TT\} & \text{when } 1 \le x_2 < 2 \\ \Omega & \text{when } x_2 \geq 2 \end{cases}$$

Now, $$\{\omega : X_1(\omega) \le x_1, X_2(\omega) \le x_2\} = \begin{cases} \phi & \text{when } x_1 < 0,\ x_2 \in \mathbb{R} \\ \phi & \text{when } x_2 < 0,\ x_1 \in \mathbb{R} \\ \phi & \text{when } 0 \le x_1 < 1,\ 0 \le x_2 < 1 \\ \phi & \text{when } 0 \le x_1 < 1,\ 1 \le x_2 < 2 \\ \{TT\} & \text{when } 0 \le x_1 < 1,\ x_2 \geq 2 \\ \{HH\} & \text{when } 1 \le x_1 < 2,\ 0 \le x_2 < 1 \\ \{HH, TT\} & \text{when } 1 \le x_1 < 2,\ 1 \le x_2 < 2 \\ \{HH, HT, TT\} & \text{when } 1 \le x_1 < 2,\ x_2 \geq 2 \\ \{HH\} & \text{when } x_1 \geq 0,\ 0 \le x_2 < 1 \\ \{HH, TH, TT\} & \text{when } x_1 \geq 2,\ 1 \le x_2 < 2 \\ \Omega & \text{when } x_1 \geq 2,\ x_2 \geq 2 \end{cases}$$

$\therefore$ $\{\omega : X_1(\omega) \le x_1, X_2(\omega) \le x_2\} \in \mathcal{A} \quad \forall (x_1, x_2) \in \mathbb{R}^2.$

$\therefore$ (X_1, X_2) is a two dimensional random vector.

Distribution function of $\underset{\sim}{X}$:—

Let $\underset{\sim}{X} = (X_1, X_2)$ be a two dimensional random vector. The distribution function of (X_1, X_2) is a function of $F_{\underset{\sim}{X}}(x_1, x_2)$ such that, —

$$F_{X_1,X_2}(x_1,x_2) = P[X_1 \le x_1, X_2 \le x_2] \quad \forall (x_1,x_2) \in \mathbb{R}^2$$

In the above example,

$$F_{X_1,X_2}(x_1,x_2) = \begin{cases} 0 & \text{when } x_1 < 0,\ x_2 \in \mathbb{R} \\ 0 & \text{when } x_2 < 0,\ x_1 \in \mathbb{R} \\ 0 & \text{when } 0 \le x_1 < 1,\ 0 \le x_2 < 1 \\ 0 & \text{when } 0 \le x_1 < 1,\ 1 \le x_2 < 2 \\ 1/4 & \text{when } 0 \le x_1 < 1,\ x_2 \ge 2 \,/ \\ & 1 \le x_1 < 2,\ 0 \le x_2 < 1 \,/ \\ & x_1 \ge 2,\ 0 \le x_2 < 1 \\ 3/4 & \text{when } 1 \le x_1 < 2,\ 1 \le x_2 < 2 \,/ \\ & 1 \le x_2 < 2,\ x_2 \ge 2 \,/ \\ & x_1 \ge 2,\ x_2 \ge 2 \end{cases}$$

$$F_{X_1X_2}(x_1, \infty) = P[X_1 \le x_1]$$
$$= F_{X_1}(x_1) = \text{Marginal Distribution Function of } X_1.$$

$$F_{X_1X_2}(\infty, x_2) = P[X_2 \le x_2]$$
$$= F_{X_2}(x_2) = \text{Marginal Distribution Function of } X_2.$$

$$F_{X_1X_2}(\infty, \infty) = 1$$

$$F_{X_1X_2}(-\infty, x_2) = F_{X_1X_2}(x_1, -\infty)$$
$$= 0.$$

☆ **Problem 1.** If $A = \dfrac{F_{X_1}(x_1) + F_{X_2}(x_2)}{2}$, $G = \sqrt{F_{X_1}(x_1)\, F_{X_2}(x_2)}$

Then show that, $2A - 1 \le F_{X_1X_2}(x_1, x_2) \le G$.

Soln. →
$$2A - 1 = F_{X_1}(x_1) + F_{X_2}(x_2) - 1$$
$$= F_{X_1}(x_1) - P[X_2 > x_2]$$
$$= P[X_1 \le x_1] - P[X_2 > x_2]$$

$F_{X_1X_2}(x_1,x_2) = P[X_1 \le x_1, X_2 \le x_2]$, let $C = X_1 \le x_1$ & $D = X_2 \le x_2$

$$\therefore P(C) + P(D) - 1 \le P(C \cap D)$$
$$\& \quad 2A - 1 \le F_{X_1X_2}(x_1, x_2)$$

$P(C \cap D) \le P(C)$
$P(C \cap D) \le P(D)$ & $\{P(C \cap D)\}^2 \le P(C)\,P(D)$.

$$\therefore F_{X_1X_2}(x_1,x_2) \le \sqrt{F_{X_1}(x_1)\, F_{X_2}(x_2)} = G$$
$$\therefore F_{X_1X_2}(x_1, x_2) \le G.$$
$$\therefore 2A - 1 \le F_{X_1X_2}(x_1, x_2) \le G.$$

■ **Result:– Necessary & sufficient conditions for a function to be Joint distribution function.**

A Function $F_{X_1 X_2}(x_1, x_2)$ is the joint distribution function of some 2-dimensional random variable iff (if and only if)

i) $\Delta_2 F(x_1, x_2) = F(x_1+h_1, x_2+h_2) - F(x_1+h_1, x_2) - F(x_1, x_2+h_2) + F(x_1, x_2) \geq 0$;

ii) $F(-\infty, x_2) = F(x_1, -\infty) = 0$;

iii) $F(\infty, \infty) = 1$;

iv) $F(x_1+0, x_2) = F(x_1, x_2+0) = F(x_1, x_2)$

Proof:– i) Note that,

$$\Delta_2 F(x_1, x_2) = F(x_1+h_1, x_2+h_2) - F(x_1+h_1, x_2) - F(x_1, x_2+h_2) + F(x_1, x_2)$$

$$= P[-\infty < X_1 \leq x_1+h_1, -\infty < X_2 \leq x_2+h_2]$$
$$- P[-\infty < X_1 \leq x_1, -\infty < X_2 \leq x_2+h_2]$$
$$- P[-\infty < X_1 \leq x_1+h_1, -\infty < X_2 \leq x_2]$$
$$+ P[-\infty < X_1 \leq x_1, -\infty < X_2 \leq x_2]$$

Also, note that the probability for the rectangle $\{(X_1, X_2) \mid x_1 < X_1 \leq x_1+h_1, x_2 < X_2 \leq x_2+h_2\}$, which necessarily belongs to $\mathcal{B}^2$, equals the expression on the right hand side. Hence,

$$\Delta_2 F(x_1, x_2) = P[x_1 < X \leq x_1+h_1, x_2 < X_2 \leq x_2+h_2]$$
$$\geq 0$$

ii) Denote by A_{n,x_2} the measurable set $[-\infty < X_1 \leq -n, -\infty < X_2 \leq x_2]$, where n is a positive integer. For fixed x_2, the sequence $\{A_{n,x_2}\}$ is a contracting sequence whose limit is ϕ. It follows that

$$\lim F(-n, x_2) = \lim P(A_{n,x_2}) = P(\lim A_{n,x_2}) = P(\phi) = 0,$$

i.e. $F(-\infty, x_2) = 0$

In a similar way, we have

$$F(x, -\infty) = 0.$$

iii) Consider the sets,

$$A_n = [-\infty < X_1 \le n,\ -\infty < X_2 \le n],$$

for positive integral n, Now, $\{A_n\}$ is an expanding sequence of measurable sets whose limit is Ω. Hence

$$\lim F(n,n) = \lim P(A_n) = P(\lim A_n) = P(\Omega) = 1,$$

i.e. $F(+\infty, +\infty) = 1$.

iv) Let $C_n = [-\infty < X_1 \le x_1 + \frac{1}{n},\ -\infty < X_2 \le x_2]$,

where n is a positive integer. For fixed x_1, x_2; $\{C_n\}$ is a contracting sequence of measurable sets and

$$\lim C_n = [X_1 \le x_1,\ X_2 \le x_2]. \text{ Hence,}$$

$$\lim P(C_n) = P(\lim C_n) = P[X_1 \le x_1,\ X_2 \le x_2],$$

i.e. $\lim\limits_{n\to\infty} F\left(x_1 + \frac{1}{n}, x_2\right) = F(x_1, x_2)$.

But,

$$\lim_{\epsilon \downarrow 0} F(x_1 + \epsilon, x_2) = \lim F\left(x_1 + \frac{1}{n}, x_2\right)$$

and since $F(x_1+0, x_2)$ is, by definition, the same as $\lim\limits_{\epsilon\downarrow 0} F(x_1+\epsilon, x_2)$, we have

$$F(x_1+0, x_2) = F(x_1, x_2).$$

By a similar argument, we get $F(x_1, x_2+0) = F(x_1, x_2)$

- **NOTE:** →

$$P[x_1 < X_1 \le x_1+h_1,\ x_2 < X_2 \le x_2+h_2]$$
$$= F(x_1+h_1, x_2+h_2) - F(x_1+h_1, x_2) - F(x_1, x_2+h_2) + F(x_1, x_2)$$

Proof: →

$$P[x_1 < X_1 \le x_1+h_1,\ x_2 < X_2 \le x_2+h_2]$$
$$= P[x_1 < X_1 \le x_1+h_1,\ X_2 \le x_2+h_2] - P[x_1 < X_1 \le x_1+h_1,\ X_2 \le x_2]$$
$$= P[X_1 \le x_1+h_1,\ X_2 \le x_2+h_2] - P[X_1 \le x_1,\ X_2 \le x_2+h_2]$$
$$- P[X_1 \le x_1+h_1,\ X_2 \le x_2] + P[X_1 \le x_1,\ X_2 \le x_2]$$
$$= F(x_1+h_1, x_2+h_2) - F(x_1, x_2+h_2) - F(x_1+h_1, x_2) + F(x_1, x_2)$$

☆ **Problem 2.** Show that the function $F(x_1, x_2)$ is not a distribution function, where,

$$F(x_1, x_2) = 1 \quad \text{if} \quad x_1+x_2 \le 1$$
$$= 0 \quad \text{if} \quad x_1+x_2 > 1$$

Soln.→ Let us take $(x_1, x_2) = (0,0)$ and $(h_1, h_2) = (1,1)$

$\therefore F(x_1+h_1, x_2+h_2) - F(x_1+h_1, x_2) - F(x_1, h_2+x_2) + F(x_1, x_2)$

$= F(1,1) - F(1,0) - F(0,1) + F(0,0)$

$= 0 - 1 - 1 + 1$

$= -1.$

So, $F(x_1, x_2)$ does not satisfy the property of non-negativity for bivariate distribution.

Hence, $F(x_1, x_2)$ is not a distribution function.

Result:— If $F_1(x_1)$ and $F_2(x_2)$ are univariate distribution functions then the function $F(x_1, x_2)$ defined by,

$$F(x_1, x_2) = F_{X_1}(x_1) F_{X_2}(x_2)\left[1 + \theta\left(1 - F_{X_1}(x_1)\right)\left(1 - F_{X_2}(x_2)\right)\right], \ |\theta| \le 1.$$

is a joint distribution function.

Proof:→ $F(\infty, \infty) = F_{X_1}(\infty) F_{X_2}(\infty)\left[1 + \theta\left(1 - F_{X_1}(\infty)\right)\left(1 - F_{X_2}(\infty)\right)\right]$

$= 1$

$F(-\infty, x_2) = 0 = F(x_1, -\infty)$

Now, let, $G_{X_1}(x_1) = \left(1 - F_{X_1}(x_1)\right) F_{X_1}(x_1)$

$G_{X_2}(x_2) = \left(1 - F_{X_2}(x_2)\right) F_{X_2}(x_2)$

$\therefore F(x_1, x_2) = F_{X_1}(x_1) F_{X_2}(x_2) + \theta G_{X_1}(x_1) G_{X_2}(x_2)$

$\Delta_2 F(x_1, x_2) = F(x_1+h_1, x_2+h_2) - F(x_1+h_1, x_2) - F(x_1, x_2+h_2) + F(x_1, x_2)$

$\Delta_1 F_{X_1}(x_1) = F_{X_1}(x_1+h_1) - F_{X_1}(x_1)$

$\Delta_2 F_{X_2}(x_2) = F_{X_2}(x_2+h_2) - F_{X_2}(x_2)$

$\Delta_2 F_{X_1}(x_1) F_{X_2}(x_2) + \theta \Delta_2 G_{X_1}(x) G_{X_2}(x_2)$ .... (*)

Now, $\Delta_2 F_{X_1}(x_1) F_{X_2}(x_2)$

$= F_{X_1}(x_1+h_1) F_{X_2}(x_2+h_2) + F_{X_1}(x_1) F_{X_2}(x_2) - F_{X_1}(x_1+h_1) F_{X_2}(x_2)$
$\quad - F_{X_1}(x_1) F_{X_2}(x_2+h_2)$

$= F_{X_1}(x_1+h_1)\left\{F_{X_2}(x_2+h_2) - F_{X_2}(x_2)\right\} + F_{X_1}(x_1)\left\{F_{X_2}(x_2) - F_{X_2}(x_2+h_2)\right\}$

$= \Delta_1 F_{X_1}(x_1) \Delta_1 F_{X_2}(x_2)$

$$\Delta_2 G_{X_1}(x_1) G_{X_2}(x_2)$$
$$= \Delta_1 G_{X_1}(x_1) \Delta_1 G_{X_2}(x_2)$$
$$= \Delta_1 F_{X_1}(x_1)\left(1 - F_{X_1}(x_1)\right) \Delta_1 F_{X_2}(x_2)\left(1 - F_{X_2}(x_2)\right)$$

$$\begin{aligned}
\Delta_1 G_{X_1}(x_1) &= G_{X_1}(x_1+h_1) - G_{X_1}(x_1) \\
&= F_{X_1}(x_1+h_1)\left(1 - F_{X_1}(x_1+h_1)\right) - F_{X_1}(x_1)\left(1 - F_{X_1}(x_1)\right) \\
&= F_{X_1}(x_1+h_1) - F_{X_1}(x_1) - \left\{F_{X_1}^2(x_1+h_1) - F_{X_1}^2(x_1)\right\} \\
&= \Delta_1 F_{X_1}(x_1) - \Delta_1 F_{X_1}(x_1)\left\{F_{X_1}(x_1+h_1) + F_{X_1}(x_1)\right\} \\
&= \Delta_1 F_{X_1}(x_1)\left\{1 - F_{X_1}(x_1+h_1) - F_{X_1}(x_1)\right\} \\
&= \left\{F_{X_1}(x_1+h_1) - F_{X_1}(x_1)\right\}\left\{1 - F_{X_1}(x_1+h_1) - F_{X_1}(x_1)\right\}
\end{aligned}$$

$$\Delta_1 G_{X_2}(x_2) = \left\{F_{X_2}(x_2+h_2) - F_{X_2}(x_2)\right\}\left\{1 - F_{X_2}(x_2+h_2) - F_{X_2}(x_2)\right\}$$

$$(*) = \Delta_1 F_{X_1}(x_1) \Delta_1 F_{X_2}(x_2) + \theta \Delta_1 F_{X_1}(x_1) \Delta_1 F_{X_2}(x_2) \left\{1 - F_{X_1}(x_1+h_1) - F_{X_1}(x_1)\right\}\left\{1 - F_{X_2}(x_2+h_2) - F_{X_2}(x_2)\right\}$$

Now, $\Delta_1 F_{X_1}(x_1) \geq 0 \quad [\because F_{X_1} \uparrow x_1]$

$\Delta_1 F_{X_2}(x_2) \geq 0 \quad [\because F_{X_2} \uparrow x_2]$, $|\theta| \leq 1$

$\therefore \ 0 \leq F_{X_1}(x_1) \leq 1, \quad 0 \leq F_{X_1}(x_1+h_1) \leq 1$

$0 \leq F_{X_1}(x_1+h_1) + F_{X_1}(x_1) \leq 2$

$-1 \leq \Delta_1 F_{X_1}(x_1) \leq 1$

$\therefore \ |\Delta_1 F_{X_1}(x_1)| \leq 1$, similarly

$|\Delta_1 F_{X_2}(x_2)| \leq 1$.

$|\theta \Delta_1 F_{X_1}(x_1) \Delta_1 F_{X_2}(x_2)| \leq 1$

$\therefore \ (1+\theta) \Delta_1 F_{X_1}(x_1) \Delta_1 F_{X_2}(x_2) > 0$

Note, marginal distribution function of X_1, X_2 is, —

$P[X_1 \leq x_1] = F(x_1, \infty) = F_{X_1}(x_1)$

$P[X_2 \leq x_2] = F(\infty, x_2) = F_{X_2}(x_2)$

$\therefore$ given two variable marginal distribution functions, we can generate a joint distribution function with same marginals. [it is due to E.J. Gumbel.]

INDEPENDENCE :-

- **Definition :- 1.** The random variables X_1 and X_2 are said to be independent if

$$F_{X_1 X_2}(x_1, x_2) = F_{X_1}(x_1) F_{X_2}(x_2) \quad \forall\ (x_1, x_2) \in \mathbb{R}^2$$

☆ **Problem 3.** Find the marginal distribution function of X_1, X_2 and Check whether they are independent or not.

$$F_{X_1 X_2}(x_1, x_2) = 1 - e^{-x_1} - e^{-x_2} + e^{-x_1 - x_2} \; ; \; x_1, x_2 > 0 .$$

Soln.:- The marginal distribution of X_1 is, —

$$F_{X_1}(x_1) = F_{X_1 X_2}(x_1, \infty) = 1 - e^{-x_1} \quad [\because e^{-\infty} = 0]$$

$\therefore X_1 \sim Exp(0,1)$

The marginal distribution of X_2 is, —

$$F_{X_2}(x_2) = 1 - e^{-x_2} = F_{X_1 X_2}(\infty, x_2)$$

$\therefore X_2 \sim Exp(0,1)$

Two RV's are said to be independently distributed if

$$F_{X_1 X_2}(x_1, x_2) = F_{X_1}(x_1) F_{X_2}(x_2)$$

Here, $F_{X_1}(x_1) = 1 - e^{-x_1}$; $F_{X_2}(x_2) = 1 - e^{-x_2}$

$$\therefore F_{X_1}(x_1) F_{X_2}(x_2) = 1 - e^{-x_1} - e^{-x_2} + e^{-x_1 - x_2}$$
$$= F_{X_1 X_2}(x_1, x_2)$$

$\therefore$ They are independent.

☆ **Problem 4.** $f(x, y) = 2e^{-x-y}$ for $0 < x < y < \infty$
$= 0$ ow

Find the marginal PDF of X & Y.

Soln → The marginal PDF of Y is given by,

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx = 2 \int_0^{\infty} e^{-x-y}\, dx$$
$$= 2e^{-y} \left[-e^{-x}\right]_0^{\infty}$$
$$= 2e^{-y}$$

The marginal PDF of X is given by,

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = 2e^{-x} \int_0^{\infty} e^{-y}\, dy$$
$$= 2e^{-x}$$

- **Definition 2.** The random variables X_1, X_2 are said to be independent iff,

$$f_{X_1 X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) \quad \forall\ (x_1, x_2)$$

ii) Discrete Random Vector: — If a two dimensional random variable (X_1, X_2) takes only finite or countable infinite numbers of pair of values $(x_1, x_2) \in \mathbb{R}^2$, then the random variable is of discrete type. So, $\exists$ a countable set $C \subset \Omega \ni$

$$P[(X_1, X_2) \in C] = 1.$$

Consider a function $f(x_1, x_2) \ni$

$$f(x_1, x_2) = \begin{cases} P[X_1 = x_1, X_2 = x_2] & \text{, where } (x_1, x_2) \text{ is any mass of } (X_1, X_2) \\ 0 & \text{, OW} \end{cases}$$

The function $f(x_1, x_2)$ is called the PMF of (X_1, X_2) if it satisfies the following conditions:

i) $f(x_1, x_2) \geq 0 \ \forall \ (x_1, x_2) \in \mathbb{R}^2$

ii) $\sum_{x_1} \sum_{x_2} f(x_1, x_2) = 1$

where, $F(x_1, x_2) = \sum_{u \leq x_1} \sum_{v \leq x_2} f(u, v)$

Marginal PMF: — The marginal PMF of X_1 is given by,

$$f_{X_1}(x_1) = P[X_1 = x_1]$$
$$= \sum_{x_2} P[X_1 = x_1, X_2 = x_2]$$
$$= \sum_{x_2} f(x_1, x_2)$$

& $f_{X_2}(x_2) = \sum_{x_1} f(x_1, x_2)$

Now,

$P[X_1 = x_1] = \sum_{\substack{\text{conditional} \\ \text{range of } x_2}} P[X_1 = x_1, X_2 = x_2]$

$P[X_2 = x_2] = \sum_{\substack{\text{conditional} \\ \text{range of } x_1}} P[X_1 = x_1, X_2 = x_2]$

- **Example:** —

Consider the bivariate table:

Y \ X	1	2	3	TOTAL
1	0.1	0.2	0.1	0.4
2	0.3	0.1	0.1	0.5
3	0.0	0.1	0.0	0.1
TOTAL	0.4	0.4	0.2	1

Marginal Probability of X: —

$P(X=1) = 0.4$
$P(X=2) = 0.4$
$P(X=3) = 0.2$

Marginal Probability of Y: —

$P(Y=1) = 0.4$
$P(Y=2) = 0.5$
$P(Y=3) = 0.1$

Theorem: — If X_1, X_2 are jointly distributed in the discrete form with $f(x_1, x_2)$ as the joint p.m.f. & if $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$ are their marginal p.m.f.s.

☆ Problem 5. Verify whether the following function is a joint PMF or not, $f(x,y) = \dfrac{e^{-2}}{x!\,(y-x)!}$, $x = 0,1,2,\ldots,y$
$y = 0,1,\ldots,\infty$

Then find the the marginal PMF of X & Y. Check X & Y are independent or not.

Soln. :→

$$f(x,y) \geq 0 \;\forall\, (x,y)$$

$0 \leq x \leq y$
$0 \leq y \leq \infty$
$\therefore 0 \leq x \leq y \leq \infty$
$\therefore 0 \leq x \leq \infty$
$\therefore x \leq y \leq \infty$

$$\sum_{y=0}^{\infty}\sum_{x=0}^{y} \frac{e^{-2}}{x!\,(y-x)!}$$

$$= e^{-2}\sum_{y=0}^{\infty}\frac{1}{y!}\sum_{x=0}^{\infty}\binom{y}{x}$$

$$= e^{-2}\sum_{y=0}^{\infty}\frac{2^y}{y!}$$

$$= e^{-2}.e^{2} = 1$$

OR

$$\sum_{x=0}^{\infty}\sum_{y=x}^{\infty}\frac{e^{-2}}{x!\,(y-x)!}$$

$$= \sum_{x=0}^{\infty}\frac{e^{-2}}{x!}\sum_{y=x}^{\infty}\frac{1}{(y-x)!}$$

let, $(y-x) = t$

$$= \sum_{x=0}^{\infty}\frac{e^{-2}}{x!}\sum_{t=0}^{\infty}\frac{1}{t!}$$

$$= e^{-1}\sum_{x=0}^{\infty}\frac{1}{x!} = e^{-1}.e^{1} = 1.$$

$\therefore f(x,y)$ is a joint PMF.

$$f_X(x) = \sum_{y=x}^{\infty}\frac{e^{-2}}{x!\,(y-x)!}$$

$$= \frac{e^{-2}}{x!}\sum_{y=x}^{\infty}\frac{1}{(y-x)!}$$

$$= \frac{e^{-2}.e}{x!}$$

$$= \frac{e^{-1}}{x!}, \quad x = 0,1,2,\ldots\infty$$

$\therefore X \sim Poi(1)$.

$$f_Y(y) = \sum_{x=0}^{y}\frac{e^{-2}}{x!\,(y-x)!}$$

$$= \frac{e^{-2}}{y!}.2^y, \quad y = 0,1,\ldots,\infty.$$

$\therefore Y \sim Poi(2)$.

Here, $f_X(x)\,f_Y(y) \neq f_{XY}(x,y)$

$\Rightarrow X, Y$ are not independent.

Covariance:– The covariance between X_1 and X_2 is defined by $Cov(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2)$

The correlation coefficient between X_1 & X_2 is defined by,

$$\rho_{X_1X_2} = \frac{Cov(X_1, X_2)}{\sqrt{V(X_1)V(X_2)}}$$

$$\therefore \boxed{-1 \le \rho_{X_1X_2} \le 1.}$$

☆ **Problem 6.** (Continuation of Problem. 5)

Find the $E(X)$ & $E(Y)$, $V(X)$ & $V(Y)$ & $E(XY)$, $Cov(X, Y)$, $\rho_{XY} = ?$

Soln. →

$$E(X) = 1, \quad E(Y) = 2$$

$$V(X) = 1, \quad V(Y) = 2$$

$$E(XY) = \sum_{y=0}^{\infty} \sum_{x=0}^{y} xy \cdot \frac{e^{-2}}{x!\,(y-x)!}$$

$$= e^{-2} \sum_{y=0}^{\infty} \frac{y}{y!} \sum_{x=0}^{y} \binom{y}{x} x$$

$$= e^{-2} \sum_{y=0}^{\infty} \frac{y^2}{y!} \sum_{x=1}^{y} \binom{y-1}{x-1}$$

$$= e^{-2} \sum_{y=0}^{\infty} \frac{y^2}{y!} \cdot 2^{y-1}$$

$$= \frac{1}{2} E(Y^2) \quad , \text{ where } Y \sim Poi(2)$$

$$= \frac{1}{2}\left[2^2 + 2\right]$$

$$= 3.$$

$$Cov(XY) = E(XY) - E(X)E(Y)$$

$$= 3 - 1 \times 2$$

$$= 1$$

$$\therefore \rho_{XY} = \frac{1}{\sqrt{1 \times 2}}$$

$$= \frac{1}{\sqrt{2}}.$$

ii) Continuous Random Vector : ~ A two dimensional random variable X_1, X_2 is said to be continuous if

$F_{X_1X_2}(x_1, x_2)$ is everywhere continuous on $\mathbb{R}^2$.

▣ Absolutely Continuous Random Vector: —

A two dimensional continuous random variable X_1, X_2 is said to be absolutely continuous if $\exists$ a non-negative integrable function, $f_{X_1X_2}(x_1, x_2)$ $\ni$

$$F_{X_1X_2}(x_1, x_2) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} f_{X_1X_2}(u, v)\,du\,dv \quad \forall\ x_1, x_2 \in \mathbb{R}^2$$

in that case,

$$\frac{\partial^2 F_{X_1X_2}(x_1, x_2)}{\partial x_1\,\partial x_2} = f_{X_1X_2}(x_1, x_2)$$

[Provided the derivative exists]

The function $f_{X_1X_2}(x_1, x_2)$ is called the joint PDF of X_1, X_2 if it satisfies the following two conditions:

i) $f_{X_1X_2}(x_1, x_2) \geq 0 \quad \forall\ x_1, x_2 \in \mathbb{R}^2$

ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1X_2}(x_1, x_2)\,dx_1\,\partial x_2 = 1$.

▣ Marginal PDF: — $f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1X_2}(x_1, x_2)\,dx_2 = g(x_1)$

$$f_{X_2}(x_2) = \int_{-\infty}^{\infty} f_{X_1X_2}(x_1, x_2)\,dx_1 = h(x_2)$$

These two are called the marginal PDF of X_1, X_2.

☆ Problem 7. Let g and h be 2 PDFs, with corresponding distribution function G, H. Consider the function

$$f_{XY}(x, y) = g(x)h(y)\left[1 + \alpha\left(2G(x) - 1\right)\left(2H(y) - 1\right)\right]$$

Show that f is a joint PDF with the given marginal PDFs g & h.

(Gumbel)

Soln. →
$$0 < G(x) < 1$$
$$-1 < 2G(x) - 1 < 1$$
$$\therefore |2G(x) - 1| < 1$$
$$|\alpha| \leq 1 \quad \text{and} \quad |2H(y) - 1| < 1$$
$$\therefore -1 \leq \alpha\left(2G(x) - 1\right)\left(2H(y) - 1\right) \leq 1$$
$$\Rightarrow 0 \leq \left[1 + \alpha\left(2G(x) - 1\right)\left(2H(y) - 1\right)\right] \leq 2$$
$$\therefore f_{XY}(x, y) \geq 0$$

Now, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)\,dx\,dy + \alpha\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)\left[2G(x)-1\right]h(y)\left[2H(y)-1\right]dx\,dy$$

$$= 1 + \alpha\int_{-1}^{1}\frac{u}{2}\,du \times \int_{-1}^{1}\frac{v}{2}\,dv$$

$$= 1.$$

let,
$2G(x)-1 = u$
$2g(x)dx = du$
$2H(y)-1 = v$
$2h(y)dy = dv$

Marginal PDFs

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$= \int_{-\infty}^{\infty} g(x)h(y)\,dy + \alpha\int_{-\infty}^{\infty} g(x)\left[2G(x)-1\right]h(y)\left[2H(y)-1\right]$$

$$= g(x)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = h(y).$$

☆ Problem 8. Let X and Y be such that $Y = 2X$. Show that

$$F_{XY}(x,y) = \begin{cases} G_X(x) & \text{for } x \le y/2 \\ G_X\left(\frac{y}{2}\right) & \text{for } x > y/2. \end{cases}$$

Soln. $F_{XY}(x,y) = P\left[X \le x, Y \le y\right]$

$$= P\left[X \le x, 2X \le y\right]$$

$$= P\left[X \le x, X \le y/2\right]$$

Hence, $F_{XY}(x,y) = G_X(x)$ if $x \le y/2$

$= G_X(y/2)$ if $x > y/2$

CONDITIONAL DISTRIBUTION :

- **Discrete case:** – Let (X_1, X_2) be a 2-dimensional discrete random variable with joint PMF $f_{X_1X_2}(x_1,x_2)$, the conditional PMF of X_1 given $X_2 = x_2$ is given by,

$$f_{X_1/X_2}(x_1/x_2) = P[X_1 = x_1 / X_2 = x_2]$$
$$= \frac{P[X_1 = x_1, X_2 = x_2]}{P[X_2 = x_2]}, \text{ if } P[X_2 = x_2] > 0$$
$$= \frac{f_{X_1X_2}(x_1,x_2)}{f_{X_2}(x_2)} \text{ if } f_{X_2}(x_2) > 0$$

Similarly, the conditional PMF of X_2 given $X_1 = x_1$ is given by,

$$f_{X_2/X_1}(x_2/x_1) = \frac{f_{X_1X_2}(x_1,x_2)}{f_{X_1}(x_1)} \text{ if } f_{X_1}(x_1) > 0$$

If X_1 and X_2 are independent,

$$f_{X_1/X_2}(x_1/x_2) = f_{X_1}(x_1) \text{ if } f_{X_2}(x_2) > 0$$
$$f_{X_2/X_1}(x_2/x_1) = f_{X_2}(x_2) \text{ if } f_{X_1}(x_1) > 0$$
$$\therefore f_{X_1X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2).$$

☆ **Problem 9.** (Continuation of Problem 5.)

Find the conditional PMF, $f_{X/Y}(x/y) = ?$ & $f_{Y/X}(y/x) = ?$

Soln. →
$$f_{X|Y}(x/y) = \frac{P[X = x, Y = y]}{P[Y = y]}$$
$$= \frac{e^{-2}/x!(y-x)!}{e^{-2}\frac{2^y}{y!}}$$
$$= \binom{y}{x}\frac{1}{2^y}, \quad y = x, x+1, \dots \infty.$$

$$f_{Y|X}(y/x) = \frac{P[X = x, Y = y]}{P[X = x]}$$
$$= \frac{e^{-2}/x!(y-x)!}{e^{-1}/x!}$$
$$= \frac{e^{-1}}{(y-x)!}, \quad y = x, x+1, \dots \infty.$$

☆ **Problem 10.** A 2-dimensional random vector (X_1, X_2) has PMF,

$$f_{X_1X_2}(x_1, x_2) = \frac{n!\, p_1^{x_1} p_2^{x_2} (1-p_1-p_2)^{n-x_1-x_2}}{x_1!\, x_2!\, (n-x_1-x_2)!}$$

where, $x_1, x_2 = 0(1)n \ni x_1+x_2 \leq n$, [trinomial distribution]

$0 < p_1, p_2 < 1 \ni p_1+p_2 < 1$.

Find the marginal distribution of X_1, X_2 and conditional distn. of $X_1 \mid X_2 = x_2$.

Soln. → The marginal PMF of X_1 is,

$$f_{X_1}(x_1) = \sum_{x_2=0}^{n-x_1} \frac{n!\, p_1^{x_1} p_2^{x_2} (1-p_1-p_2)^{n-x_1-x_2}}{x_1!\, x_2!\, (n-x_1-x_2)!}$$

$$= \frac{p_1^{x_1}\, n!}{(n-x_1)!\, x_1!} \sum_{x_2=0}^{n-x_1} \frac{p_2^{x_2} (1-p_1-p_2)^{(n-x_1)-x_2}}{x_2!\, (n-x_1-x_2)!} \cdot (n-x_1)!$$

$$= \binom{n}{x_1} p_1^{x_1} \left\{ \cancel{p_2} + 1 - p_1 - \cancel{p_2} \right\}^{n-x_1}$$

$$= \binom{n}{x_1} p_1^{x_1} (1-p_1)^{n-x_1}\, I_{x_1}(0, 1, \ldots, n)$$

$\therefore X_1 \sim Bin(n, p_1)$

Similarly,

$$f_{X_2}(x_2) = \binom{n}{x_2} p_2^{x_2} (1-p_2)^{n-x_2}\, I_{x_2}(0, 1, \ldots, n)$$

$\therefore X_2 \sim Bin(n, p_2)$

$\therefore$ Conditional PMF of $X_1 \mid X_2 = x_2$ is,

$$f_{X_1/X_2}(x_1/x_2) = \frac{f_{X_1X_2}(x_1, x_2)}{f_{X_2}(x_2)}$$

$$= \frac{\dfrac{n!\, p_1^{x_1} p_2^{x_2}}{x_1!\, x_2!\, (n-x_1-x_2)!} (1-p_1-p_2)^{n-x_1-x_2}}{\binom{n}{x_2} p_2^{x_2} (1-p_2)^{n-x_2}}$$

$$= \frac{(n-x_2)!}{x_1!\, (n-x_1-x_2)!} \left(\frac{p_1}{1-p_2}\right)^{x_1} \left(1 - \frac{p_1}{1-p_2}\right)^{n-x_1-x_2}$$

$$= \binom{n-x_2}{x_1} \left(\frac{p_1}{1-p_2}\right)^{x_1} \left(1 - \frac{p_1}{1-p_2}\right)^{n-x_1-x_2},$$

where $0 < x_1 < n-x_2$

$\therefore X_1 \mid X_2 = x_2 \sim Bin\left(n-x_2, \frac{p_1}{1-p_2}\right)$. and $0 < \frac{p_1}{1-p_2} < 1$.

- **Absolutely Continuous Case:** — Let $f_{X_1X_2}$ be the joint PDF of X_1 & X_2 and let g_{X_1} be the marginal PDF of X_1, then for h and k, we have for any $x_1 \ni g_{X_1}(x_1) > 0$.

$$P\left[x_2 - \frac{k}{2} \le X_2 \le x_2 + \frac{k}{2} \Big/ x_1 - \frac{h}{2} \le X_1 \le x_1 + \frac{h}{2}\right]$$

$$= \frac{P\left[x_2 - \frac{k}{2} \le X_2 \le x_2 + \frac{k}{2} \cap x_1 - \frac{h}{2} \le X_1 \le x_1 + \frac{h}{2}\right]}{P\left[x_1 - \frac{h}{2} \le X_1 \le x_1 + \frac{h}{2}\right]}$$

$$= \frac{\int_{x_1 - h/2}^{x_1 + h/2} \int_{x_2 - k/2}^{x_2 + k/2} f_{X_1X_2}(u, v)\, du\, dv}{\int_{x_1 - h/2}^{x_1 + h/2} g_{X_1}(u)\, du}$$

$$\to \frac{f_{X_1X_2}(x_1, x_2)\, \cancel{h}k}{f_{X_1}(x_1)\, \cancel{h}} = \frac{f_{X_1X_2}(x_1, x_2)}{g_{X_1}(x_1)}$$

Now, as $h \to 0$, $k \to 0$

$$\therefore \lim_{k \to 0} P\left[x_2 - \frac{k}{2} \le X_2 \le x_2 + \frac{k}{2} \Big/ x_1 - \frac{h}{2} \le X_1 \le x_1 + \frac{h}{2}\right] = \frac{f_{X_1X_2}(x_1, x_2)}{g_{X_1}(x_1)}$$

The L.H.S is denoted by $f_{X_2}(\cdot \mid x_1)$. This behaves like a univariate probability density for $\frac{f_{X_1X_2}(x_1, x_2)}{g_{X_1}(x_1)} \geq 0$ and

$$\int_{-\infty}^{\infty} \frac{f_{X_1X_2}(x_1, x_2)}{g_{X_1}(x_1)}\, dx_2 = \frac{1}{g_{X_1}(x_1)} \int_{-\infty}^{\infty} f_{X_1X_2}(x_1, x_2)\, dx_2$$

$$= \frac{g_{X_1}(x_1)}{g_{X_1}(x_1)} = 1.$$

The function $f_{X_2}(\cdot \mid x_1)$ given by $f_{X_2}(x_2 \mid x_1) = \frac{f_{X_1X_2}(x_1, x_2)}{g_{X_1}(x_1)}$ will therefore be called the conditional PMF $(X_2 \mid X_1 = x_1)$.

SUM LAW OF EXPECTATION: $\longrightarrow$

- Statement: If X_1 and X_2 are jointly distributed discrete random variable $\ni$ $E(X_1)$ and $E(X_2)$ exist, then $E(X_1+X_2)$ exists and
$$E(X_1+X_2)=E(X_1)+E(X_2).$$

Proof: $\rightarrow$

$$E|X_1+X_2|=\sum_{x_1}\sum_{x_2}|x_1+x_2|f_{X_1X_2}(x_1,x_2) \quad [|X+Y|\le|X|+|Y|]$$
$$\le \sum_{x_1}\sum_{x_2}[|x_1|+|x_2|]f_{X_1X_2}(x_1,x_2)$$
$$=\sum_{x_1}|x_1|\sum_{x_2}f_{X_1X_2}(x_1,x_2)+\sum_{x_2}|x_2|\sum_{x_1}f_{X_1X_2}(x_1,x_2)$$
$$=\sum_{x_1}|x_1|f_{X_1}(x_1)+\sum_{x_2}|x_2|f_{X_2}(x_2)$$
$$=E|X_1|+E|X_2|$$
$$\because\ E|X_1|, E|X_2|<\infty$$
$$\therefore\ E|X_1+X_2|\le E|X_1|+E|X_2|<\infty$$

Hence, $E(X_1+X_2)$ exists.

$$E(X_1+X_2)=\sum_{x_1}\sum_{x_2}(x_1+x_2)f_{X_1X_2}(x_1,x_2)$$
$$=\sum_{x_1}x_1\sum_{x_2}f_{X_1X_2}(x_1,x_2)+\sum_{x_2}x_2\sum_{x_1}f_{X_1X_2}(x_1,x_2)$$
$$=E(X_1)+E(X_2).$$

NOTE:- $E(X+Y+Z)=E(X)+E(Y)+E(Z)$ if both of them exists.

PRODUCT LAW OF EXPECTATION: $\longrightarrow$

- Statement: If X_1 and X_2 are independently distributed random variable $\ni$ $E(X_1)$ and $E(X_2)$ exist, then $E(X_1X_2)$ exists and $E(X_1X_2)=E(X_1)E(X_2)$.

Proof:-

$$E|X_1X_2|=\sum_{x_1}\sum_{x_2}|x_1x_2|f_{X_1X_2}(x_1,x_2)$$
$$=\sum_{x_1}\sum_{x_2}|x_1||x_2|f_{X_1}(x_1)f_{X_2}(x_2)$$
$$=E|X_1|E|X_2|<\infty$$

$\therefore$ $E[X_1X_2]$ exists.

$$E(X_1X_2)=\sum_{x_1}\sum_{x_2}x_1x_2f_{X_1X_2}(x_1,x_2)=\sum_{x_1}\sum_{x_2}x_1x_2f_{X_1}(x_1)f_{X_2}(x_2)$$
$$=E(X_1)E(X_2)$$

NOTE:- $E(XYZ)=E(X)E(Y)E(Z)$ if $X, Y \& Z$ are mutually independent.

Remark:– If X_1 and X_2 are independent and $G_1(X_1)$ and $G_2(X_2)$ are functions of X_1 and X_2 $\ni$ $E(G_1(X_1))$ and $E(G_1(X_2))$ exist and expectation of their product exists and

$$E[G_1(X_1)G_2(X_2)] = E[G_1(X_1)]E[G_2(X_2)]$$

Problem 11. If X_1 and X_2 are independent random variables $\ni$

$E(X_i) = \mu_i$, $i = 1, 2$

$V(X_i) = \sigma_i^2$, $i = 1, 2$

$\Rightarrow E(X_i^2) - E^2(X_i) = \sigma_i^2$.

Find, $V(X_1X_2)$ and ρ_{X_1, X_1X_2} = correlation coefficient between X_1 and X_1X_2.

Soln. →

$$V(X_1X_2) = E(X_1X_2)^2 - E^2(X_1X_2)$$
$$= E(X_1^2)E(X_2^2) - E^2(X_1)E^2(X_2)$$
$$= (\sigma_1^2+\mu_1^2)(\sigma_2^2+\mu_2^2) - \mu_1^2\mu_2^2$$
$$= \sigma_1^2\sigma_2^2 + \sigma_1^2\mu_2^2 + \sigma_2^2\mu_1^2$$

$Cov(X_1, X_1X_2)$

$$= E(X_1^2X_2) - E(X_1)E(X_1X_2)$$
$$= E(X_1^2)E(X_2) - E(X_1)E(X_1)E(X_2)$$
$$= \mu_2\sigma_1^2$$

$$\therefore \rho_{X_1, X_1X_2} = \frac{\sigma_1^2\mu_2}{\sigma_1\sqrt{\sigma_1^2\sigma_2^2+\sigma_1^2\mu_2^2+\sigma_2^2\mu_1^2}}$$

Note:→ If X_1, X_2 are independently distributed then they must be uncorrelated but the converse may not be true.

Soln. →

If Part :→ i.e. if $f_{X_1X_2}(x_1, x_2) = f_{X_1}(x_1)f_{X_2}(x_2)$

then, $E(X_1X_2) = E(X_1)E(X_2)$

$$Cov(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2)$$
$$= 0.$$

Only if Part :— Consider the following counter examples :—

1. Let X_1 be a discrete random variable which takes 3 values $-1, 0, 1$ each with probability $\frac{1}{3}$. Define, $X_2 = X_1^2$. Then X_2 take two values $0, 1$ with probability $\frac{1}{3}, \frac{2}{3}$. The Joint distribution of X_1, X_2 is given in the following table :

$X_1 \backslash X_2$	0	1	TOTAL
-1	0	1/3	1/3
0	1/3	0	1/3
1	0	1/3	1/3
TOTAL	1/3	2/3	1

[Uncorrelated but not independent]

$E(X_1) = 0$

$E(X_2) = 0 \times \frac{1}{3} + 1 \times \frac{2}{3} = \frac{2}{3}$.

$E(X_1 X_2) = -\frac{1}{3} + \frac{1}{3} = 0$

$E(X_1^2) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$.

$E(X_2^2) = \frac{2}{3}$.

$V(X_1) = \frac{2}{3}$.

$V(X_2) = \frac{2}{3} - \frac{4}{9} = \frac{2}{9}$.

$\therefore Cov(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2)$
$= 0 - 0 \times 2/3$
$= 0$

$\therefore \rho_{X_1X_2} = 0$

$\therefore$ The random variables are uncorrelated but not independent.

2. Consider the joint distribution of 2 random variables X & Y as follows :

$X \backslash Y$	y_1	y_2	$P[X=x_i]$
x_1	P_{11}	P_{12}	P_{10}
x_2	P_{21}	P_{22}	P_{20}
$P[Y=y_j]$	P_{01}	P_{02}	1

[Uncorrelated but independent]

$x_1 \neq x_2$

$y_1 \neq y_2$

$P[X = x_i, Y = y_j] = P_{ij}$

$P[Y = y_j] = P_{0j}$

$P[X = x_i] = P_{i0}$

$E(X) = x_1 P_{10} + x_2 P_{20}$ $\quad E(XY) = x_1 y_1 P_{11} + x_1 y_2 P_{12} + x_2 y_1 P_{21} + x_2 y_2 P_{22}$

$E(Y) = y_1 P_{01} + y_2 P_{02}$; let X, Y are uncorrelated,

$\therefore E(XY) = E(X)E(Y)$.

$$\Rightarrow x_1 y_1 P_{11} + x_1 y_2 P_{12} + x_2 y_1 P_{21} - (x_1 P_{10} + x_2 P_{20})(y_1 P_{01} + y_2 P_{02}) = 0$$

$$\Rightarrow x_1 y_1 (P_{11} - P_{10} P_{01}) + x_2 y_2 (P_{22} - P_{20} P_{02}) + x_2 y_1 (P_{21} - P_{20} P_{01}) + x_1 y_2 (P_{12} - P_{10} P_{02}) = 0 \quad \text{------ (1)}$$

From the joint distribution table we get,

i) $P_{12} = P_{10} - P_{11}$

$P_{02} = 1 - P_{01}$

$$\therefore P_{12} - P_{10} P_{02} = P_{10} - P_{11} - P_{10}(1 - P_{01})$$
$$= -[P_{11} - P_{10} P_{01}] \quad \text{------- (2)}$$

ii) $P_{21} = P_{01} - P_{11}$

$P_{20} = 1 - P_{10}$

$$\therefore P_{21} - P_{20} P_{01} = P_{01} - P_{11} - P_{01}(1 - P_{10})$$
$$= -[P_{11} - P_{10} P_{01}] \quad \text{------- (3)}$$

iii) $P_{22} - P_{02} P_{20}$

$= P_{11} - P_{10} P_{01}$ ------ (4)

$\therefore$ (1) reduces to $\longrightarrow$

$$(P_{11} - P_{10} P_{01})[x_1 y_1 + x_2 y_2 - x_2 y_1 - x_1 y_2] = 0$$

$$\Rightarrow (P_{11} - P_{10} P_{01})(x_1 - x_2)(y_1 - y_2) = 0$$

$$\Rightarrow P_{11} = P_{10} \cdot P_{01}.$$

From (2), (3), (4) ; we get,

$P_{12} = P_{10} \cdot P_{02}$

$P_{21} = P_{20} \cdot P_{01}$

$P_{22} = P_{20} \cdot P_{02}$

$\therefore P_{ij} = P_{i0} \cdot P_{0j} \quad \forall \; i, j$

$\therefore X$ and Y are independent. [Proved]

☆ **Problem 12.** Let X and Y be independently distributed random variable, $\ni$

$$P[X=i] = P[Y=i] = \frac{1}{n} \quad \forall\ i = 1(1)n.$$

Find, $P[X<Y]$, $P[X=Y]$, $P[X>Y]$.

Soln. →

$$P[X=Y]$$
$$= \sum_{i=1}^{n} P[X=i]\,P[Y=i]$$
$$= n \cdot \frac{1}{n} \cdot \frac{1}{n} = \frac{1}{n}$$

$$P[X<Y] = \sum_{i=1}^{n} P[X<Y, Y=i]$$
$$= \sum_{i=1}^{n} P[X<Y/Y=i]\,P[Y=i]$$
$$= \sum_{i=1}^{n} P[X<i]\,P[Y=i]$$
$$= \sum_{i=1}^{n} \left(\frac{i-1}{n}\right) P[Y=i]$$
$$= \frac{1}{n}\left[\sum_{i=1}^{n} i \cdot \frac{1}{n} - \sum_{i=1}^{n} \frac{1}{n}\right]$$
$$= \frac{1}{n}\left[\frac{n+1}{2} - 1\right]$$
$$= \frac{n-1}{2n}.$$

$$P[X>Y] = 1 - P[X=Y] - P[X<Y]$$
$$= 1 - \frac{1}{n} - \frac{n-1}{2n}$$
$$= \frac{2n-2-n+1}{2n}$$
$$= \frac{n-1}{2n}.$$

Moments of the Conditional Distribution :—

The conditional mean of $X_1/X_2=x_2$ is defined by,

$$E[X_1/X_2=x_2]=\sum_{x_1} x_1 f_{X_1/X_2}(x_1/x_2) \quad \text{[Provided it exists]}$$

$$=\eta_{x_2}, \text{ say}$$

The conditional mean of $X_2/X_1=x_1$ is defined by,

$$E[X_2/X_1=x_1]=\sum_{x_2} x_2 f_{X_2/X_1}(x_2/x_1) \quad \text{[Provided it exists]}$$

$$=\eta_{x_1}, \text{ say}$$

Conditional variance of $X_1/X_2=x_2$ is,

$$V(X_1/X_2=x_2)=E\{(X_1-\eta_{x_2})^2/X_2=x_2\}$$

$$=E[X_1^2/X_2=x_2]-\eta_{x_2}^2 \quad \text{[Provided it exists]}$$

Similarly,

$$V(X_2|X_1=x_1)=E[X_2^2/X_1=x_1]-\eta_{x_1}^2 \quad \text{[Provided it exists]}$$

☆ **Problem 13.** A random vector (X,Y) has PMF

$$f_{XY}(x,y)=\frac{(x+y+k-1)!}{x!\,y!\,(k-1)!}p_1^x p_2^y(1-p_1-p_2)^k, \quad 0<x,y<\infty,\ k>1, \text{ is an integer.}$$

where $0<p_1,p_2<1 \ni p_1+p_2<1$.

Find marginal and conditional distribution.

Soln :→ $$f_X(x)=\sum_{y=0}^{\infty} f_{XY}(x,y)$$

$$=\sum_{y=0}^{\infty}\frac{(x+y+k-1)!}{x!\,y!\,(k-1)!}p_1^x p_2^y(1-p_1-p_2)^k$$

$$=\frac{p_1^x}{x!}\cdot\frac{(1-p_1-p_2)^k}{(k-1)!}\cdot(x+k-1)!\sum_{y=0}^{\infty}\frac{(x+y+k-1)!}{y!\,(x+k-1)!}p_2^y$$

$$=\binom{x+k-1}{x}p_1^x(1-p_1-p_2)^k\cdot\frac{1}{(1-p_2)^{x+k}}$$

$$=\binom{x+k-1}{x}\left(\frac{p_1}{1-p_2}\right)^x\left(1-\frac{p_1}{1-p_2}\right)^k, \quad 0\le x<\infty;\ 0<\frac{p_1}{1-p_2}<1,\ k>1, \text{ is an integer.}$$

$$\therefore X\sim NB\left(k,\frac{p_1}{1-p_2}\right).$$

Similarly,

$$f_Y(y)=\binom{y+k-1}{y}\left(\frac{p_2}{1-p_1}\right)^y\left(1-\frac{p_2}{1-p_1}\right)^k, \quad 0\le y<\infty,\ 0<\frac{p_2}{1-p_1}<1,\ k>1, \text{ is an integer.}$$

$$f_{Y/X}(y/x) = \frac{\dfrac{(x+y+k-1)!}{x!\,y!\,(k-1)!}\, p_1^{\,x}\, p_2^{\,y}\,(1-p_1-p_2)^k}{\dfrac{(x+k-1)!}{x!\,(k-1)!}\left(\dfrac{p_1}{1-p_2}\right)^x \left(\dfrac{1-p_1-p_2}{1-p_2}\right)^k}$$

$$= \frac{(x+y+k-1)\, p_2^{\,y}\,(1-p_2)^{x+k}}{y!\ (x+k-1)!}$$

$$= \binom{x+y+k-1}{y} p_2^{\,y}\,(1-p_2)^{x+k}, \quad \begin{array}{l} 0 \le y < \infty \\ 0 < p_2 < 1 \end{array}$$

Similarly,

$$f_{X/Y}(x/y) = \binom{x+y+k-1}{x} p_1^{\,x}\,(1-p_1)^{y+k}, \quad \begin{array}{l} 0 \le x < \infty \\ 0 < p_1 < 1 \end{array}$$

▨ **Result:-** If $E(X_1/X_2 = x_2)$ exists for almost all values of x_2 (i.e. for all values of x_2) $\ni f_{X_2}(x_2) > 0$ then,

$$E(X_1) = E\left[E(X_1/X_2)\right] = E\left[\eta_{X_2}\right]$$

Soln. →

$$E(X_1) = \sum_{x_1} x_1 f_{X_1}(x_1)$$

$$= \sum_{x_1} x_1 \left(\sum_{x_2} f_{X_1X_2}(x_1, x_2)\right)$$

$$= \sum_{x_1}\sum_{x_2} x_1 f_{X_1X_2}(x_1, x_2)$$

$$= \sum_{x_1}\sum_{x_2 \in A} x_1 \cdot f_{X_2}(x_2)\, f_{X_1/X_2}(x_1/x_2)$$

$$= \sum_{x_2 \in A}\left[\sum_{x_1} x_1 f_{X_1/X_2}(x_1/x_2)\right] f_{X_2}(x_2) \quad \left\{ \text{where, } A = \{x_2 : f_{X_2}(x_2) > \right.$$

$$= \sum_{x_2 \in A}\left[\eta_{x_2}\right] f_{X_2}(x_2)$$

$$= E\left[\eta_{x_2}\right]$$

Result: — If $E[X_1/X_2=x_2]$ and $V[X_1/X_2=x_2]$ exist for almost all values of x_2 then,

$$V(X_1) = E[V(X_1/X_2)] + V[E(X_1/X_2)]$$

Soln. →

R.H.S.

$$E[V(X_1/X_2)] = E[E(X_1^2/X_2) - \eta_{x_2}^2]$$
$$= E[E(X_1^2/X_2)] - E(\eta_{x_2}^2)$$
$$= E(X_1^2) - E(\eta_{x_2}^2)$$

$$V[E(X_1/X_2)] = V(\eta_{x_2})$$
$$= E[\eta_{x_2}^2] - E^2(\eta_{x_2})$$
$$= E[\eta_{x_2}^2] - E^2(X_1)$$

$\therefore$ $E[V(X_1/X_2)] + V[E(X_1/X_2)]$
$$= E(X_1^2) - E^2(X_1)$$
$$= V(X_1) = \text{L.H.S.}$$

Note: → If $E(X_2/X_1=x_1)$ and $V(X_2/X_1=x_1)$ exist for all values of x_1, $V(X_2) = E(V(X_2/X_1)) + V(E(X_2/X_1))$

☆ **Problem 14.** $X \sim P(\mu)$ and $Y/X=x \sim Poi(\lambda x)$. S.T.

$$\rho_{XY} = \sqrt{\frac{\lambda}{1+\lambda}}.$$

Soln. → $f_X(x) = \dfrac{e^{-\mu}.\mu^x}{x!} I_x(0,1,\ldots,\infty)$

$E(X) = \mu$,
$V(X) = \mu$.

$E(Y|X=x) = \lambda x$, $V(Y/X=x) = \lambda x$

$$E(Y) = E[E(Y|X)]$$
$$= E(\lambda X) = \lambda\mu$$
$$V(Y) = E(\lambda X) + V(\lambda X)$$
$$= \lambda\mu + \lambda^2\mu.$$
$$E(XY) = E[X E(Y/X)]$$
$$= E[X.\lambda X]$$
$$= \lambda[\mu^2+\mu]$$

$$\therefore \rho_{XY} = \frac{\lambda[\mu^2+\mu] - \lambda\mu^2}{\sqrt{\mu(\lambda\mu+\lambda^2\mu)}}$$
$$= \sqrt{\frac{\lambda}{1+\lambda}}.$$

☆ Problem 15. Let (X, Y) be a bivariate discrete random variable with joint PMF

$$f_{XY}(x, y) = \begin{cases} \dfrac{1}{\binom{m+1}{2}} & \text{if } \begin{array}{l} y = 1, 2, \dots, x \\ x = 1, 2, \dots, m \end{array} \\ 0 & \text{ow} \end{cases}$$

m is a positive integer >1.

find $E(X)$ and $E(X/Y)$ and $E[E(X/Y)]$

Soln. → $f_X(x) = \sum_{y=1}^{x} \dfrac{1}{\binom{m+1}{2}} = \dfrac{x}{\binom{m+1}{2}}$.

$$E(X) = \sum_{x=1}^{m} x^2 \cdot \frac{1}{\binom{m+1}{2}} = \frac{1}{\binom{m+1}{2}} \cdot \frac{m(m+1)(2m+1)}{6}$$

$$= \frac{2m+1}{3}.$$

$$f_Y(y) = \sum_{x=y}^{m} f_{XY}(x, y) = \frac{m-y+1}{\binom{m+1}{2}}.$$

$$f_{X/Y}(x/y) = \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{1/\binom{m+1}{2}}{\frac{m-y+1}{\binom{m+1}{2}}} = \frac{1}{m-y+1}, \quad x = y, \dots, m$$

$$E[X/Y=y] = \sum_{x=y}^{m} x \cdot \frac{1}{m-y+1}$$

$$= \frac{1}{m-y+1} \cdot (m-y+1)\left[\frac{m+y}{2}\right]$$

$$= \frac{m+y}{2}.$$

Now, $E[E(X/Y)] = E\left[\dfrac{m+Y}{2}\right]$

$$= \frac{m}{2} + \frac{1}{2} E(Y)$$

$$= \frac{m}{2} + \frac{1}{2}\left\{\sum_{y=1}^{m} y \cdot \frac{m-y+1}{\binom{m+1}{2}}\right\}$$

$$= \frac{m}{2} + \frac{1}{2} \cdot \frac{1}{\binom{m+1}{2}}\left[m \cdot \frac{m(m+1)}{2} + \frac{m(m+1)}{2} - \frac{m(m+1)(2m+1)}{6}\right]$$

$$= \frac{m}{2} + \left[\frac{m}{2} + \frac{1}{2} - \frac{2m+1}{6}\right]$$

$$E[E(X/Y)] = \frac{m}{2} + \frac{3m+3-2m-1}{6}$$
$$= \frac{3m+3m+3-2m-1}{6}$$
$$= \frac{2m+1}{3} = E(X).$$

☆ Problem 16. A random variable X has PMF

$$f(x) = \begin{cases} \frac{1}{3}, & x = 1, 2. \\ 0 & \text{OW} \end{cases}$$

and $Y/X=x \sim Bin(x, \frac{1}{2})$.

Find the correlation coefficient between X and Y.

Soln. →

$$E(X) = 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} = 1$$
$$V(X) = 1 \cdot \frac{1}{3} + 4 \cdot \frac{1}{3} - 1 = \frac{2}{3}$$
$$E[Y/X=x] = \frac{x}{2}.$$
$$V[Y/X=x] = \frac{x}{4}.$$

$$\therefore E(Y) = E[E(Y/X)]$$
$$= E[\tfrac{X}{2}]$$
$$= \frac{1}{2}$$

$$\therefore V(Y) = E[V(Y/X)] + V[E(Y/X)]$$
$$= E(\tfrac{X}{4}) + V[\tfrac{X}{2}]$$
$$= \frac{1}{4} + \frac{2}{12}$$
$$= \frac{5}{12}$$

$$E(XY) = E[E(XY/X)] = E[X E(Y/X)]$$
$$= E[X \cdot \tfrac{X}{2}]$$
$$= \frac{1}{2}[5/3]$$
$$= \frac{5}{6}$$

$$\therefore \rho_{XY} = \frac{\frac{3}{2} - \frac{5}{3} \cdot \frac{5}{6}}{\sqrt{\frac{2}{9} \cdot \frac{17}{36}}}$$
$$= \sqrt{\frac{2}{17}}.$$

$$\rho_{XY} = \frac{E(XY) - E(X)E(Y)}{\sqrt{V(X)V(Y)}}$$
$$= \frac{5/6 - 1/2}{\sqrt{2/3 \times 5/12}} = \frac{1/3}{\frac{\sqrt{5}}{3\sqrt{2}}}$$
$$= \frac{\sqrt{2}}{\sqrt{5}}$$

Joint MGF :—

- **Discrete case:** Let (X_1, X_2) be a two-dimensional random variable with PMF $f_{X_1X_2}(x_1, x_2)$; the joint MGF of (X_1, X_2) denoted by, $M(t_1, t_2)$ & is defined by,

$$M(t_1, t_2) = E(e^{t_1X_1 + t_2X_2})$$, provided the expectation exists for all (t_1, t_2) ∋ $i = 1, 2.$ $|t_i| < h_i$ for $h_i > 0\ \forall i$

$$= \sum_{x_1}\sum_{x_2} e^{t_1x_1 + t_2x_2} f_{X_1X_2}(x_1, x_2)$$

- **Continuous case:** Let (X_1, X_2) be a two-dimensional absolutely continuous random vector with joint PDF $f(x_1, x_2)$, the joint MGF of (X_1, X_2) denoted by, $M(t_1, t_2)$ is defined by,

$$M(t_1, t_2) = E(e^{t_1X_1 + t_2X_2})$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x_1 + t_2x_2} f(x_1, x_2)\, dx_1\, dx_2$$ [Provided the expectation exists $\forall\ |t_i| < h_i$ for $h_i > 0\ \forall\ i = 1, 2$]

Calculation of Moments: —

$$M(t_1, t_2) = E(e^{t_1X_1 + t_2X_2})$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{t_1^r t_2^s}{r!\, s!} E(X_1^r X_2^s)$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{t_1^r t_2^s}{r!\, s!} \mu'_{rs}$$

μ'_{rs} = coefficient of $\frac{t_1^r t_2^s}{r!\, s!}$ in $M(t_1, t_2)$

$$M(t_1, t_2) = E(e^{t_1X_1 + t_2X_2})$$

$$\frac{\partial M}{\partial t_1} = E[X_1 \cdot e^{t_1X_1 + t_2X_2}]$$

$$\frac{\partial^2 M}{\partial t_1^2} = E[X_1^2 \cdot e^{t_1X_1 + t_2X_2}], \quad \frac{\partial^2 M}{\partial t_1 \partial t_2} = E[X_1X_2 \cdot e^{t_1X_1 + t_2X_2}]$$

$$\therefore \left.\frac{\partial M}{\partial t_1}\right|_{\underset{\sim}{t} = \underset{\sim}{0}} = E(X_1)$$

$$\left.\frac{\partial^2 M}{\partial t_1^2}\right|_{\underset{\sim}{t} = \underset{\sim}{0}} = E(X_1^2)$$

$$\left.\frac{\partial^2 M}{\partial t_1 \partial t_2}\right|_{\underset{\sim}{t} = \underset{\sim}{0}} = E(X_1X_2).$$

Example:-

☆ 1) If X, Y has PMF $f_{XY}(x,y) = \frac{e^{-2}}{x!\,(y-x)!}$, $x = 0,1,\ldots,y$
$y = 0,1,\ldots,\infty$

Find $M(t_1,t_2)$?

Soln. → $M(t_1,t_2) = \sum_{y=0}^{\infty}\sum_{x=0}^{y} e^{t_1 x + t_2 y} \cdot \frac{e^{-2}}{x!\,(y-x)!}$

$$= \sum_{y=0}^{\infty} \frac{1}{y!} \cdot e^{-2} \cdot e^{t_2 y} \sum_{x=0}^{y} e^{t_1 x} \binom{y}{x}$$

$$= \sum_{y=0}^{\infty} \frac{\{e^{t_2}(1+e^{t_1})\}^y}{y!} \cdot e^{-2}$$

$$= e^{-2} \cdot e^{e^{t_2}(1+e^{t_1})}$$

☆ 2) If $X_i \sim$ i.i.d. $N(0,1)$, find joint mgf of $Y_1 = X_1 + X_2$, $Y_2 = X_1^2 + X_2^2$. Hence S.T. Y_1 & Y_2 are uncorrelated but not independent.

Soln. → $M(t_1,t_2) = E\left[e^{t_1 Y_1 + t_2 Y_2}\right]$

$$= E\left[e^{t_1 X_1 + t_1 X_2 + t_2 X_1^2 + t_2 X_2^2}\right]$$

$$= E^2\left[e^{(t_1 + t_2 X_1) X_1}\right] \quad \text{as } X_1, X_2 \text{ are i.i.d.}$$

$\because X \sim N(0,1)$

$E\left[e^{t_1 X_1 + t_2 X_1^2}\right]$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[x_1^2(1-2t_2) - 2t_1 x_1\right]} dx_1$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1-2t_2}{2}\left[x_1^2 - \frac{2t_1 x_1}{1-2t_2} + \left(\frac{t_1}{1-2t_2}\right)^2 - \left(\frac{t_1}{1-2t_2}\right)^2\right]} dx_1$$

$$= e^{\frac{t_1^2}{2(1-2t_2)}} \sqrt{\frac{1}{1-2t_2}} \qquad \left[\because |t_2| < \tfrac{1}{2}\right]$$

$$\therefore M(t_1,t_2) = \frac{1}{1-2t_2} \cdot e^{t_1^2/(1-2t_2)}.$$

Putting $t_2 = 0$, the marginal MGF of Y_1,

$M_{Y_1}(t_1) = e^{t_1^2}$

Putting $t_1 = 0$, the marginal MGF of Y_2,

$M_{Y_2}(t_2) = \frac{1}{1-2t_2}$.

$\therefore M(t_1,t_2) \neq M_{Y_1}(t_1)\, M_{Y_2}(t_2)$

$\therefore Y_1$ and Y_2 are not independent.

$E(Y_1) = 0$, $E(Y_2) = E(X_1^2) + E(X_2^2) = 2$

$E(Y_1 Y_2) = E\left[(X_1 + X_2)(X_1^2 + X_2^2)\right] = 0$

$\therefore \operatorname{cov}(Y_1, Y_2) = 0$

$\therefore \rho_{Y_1 Y_2} = 0 \quad \therefore Y_1$ & Y_2 are uncorrelated.

Bivariate Normal Distribution :

An absolutely continuous random vector (X_1, X_2) is said to follow bivariate normal distn with parameters $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ if the joint PDF of (X_1, X_2) is

$$f(x_1, x_2) = \frac{1}{\sigma_1\sigma_2 . 2\pi\sqrt{1-\rho^2}} \cdot e^{-\frac{1}{2}Q(x_1, x_2)} \quad ; \quad \begin{array}{l} x_1, x_2 \in \mathbb{R} \\ \sigma_1, \sigma_2 \in \mathbb{R} \\ \mu_1, \mu_2 \in \mathbb{R} \\ |\rho| < 1 \end{array}$$

$$Q(x_1, x_2) = \frac{1}{1-\rho^2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]$$

We write, $(X_1, X_2) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$.

The marginal PDF of X_1 is

$$\int_{-\infty}^{\infty} f(x_1, x_2)\,dx_2 = \int_{-\infty}^{\infty} \frac{1}{\sigma_1\sigma_2 2\pi\sqrt{1-\rho^2}}\, e^{-\frac{1}{2}Q(x_1, x_2)}\, dx_2$$

where, $Q(x_1, x_2) = \frac{1}{1-\rho^2}\left[\left(\frac{x_2-\mu_2}{\sigma_2}\right) - \rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\right]^2 + \left(\frac{x_1-\mu_1}{\sigma_1}\right)^2$

$$\therefore f_{X_1}(x_1) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho)^2}\left[\left(\frac{x_2-\mu_2}{\sigma_2}\right) - \rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\right]^2 - \frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2} dx_2$$

$$= \frac{1}{\sigma_1\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2}, \quad -\infty < x_1 < \infty$$

[as, the integrand is in the form of a normal PDF]

Similarly, it can be shown that,

$$f_{X_2}(x_2) = \frac{1}{\sigma_2\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}, \quad -\infty < x_2 < \infty.$$

Conditional Distribution of $X_2 / X_1 = x_1$: $\longrightarrow$

$$f_{X_2/X_1}(x_2/x_1) = \frac{f_{X_1X_2}(x_1, x_2)}{f_{X_1}(x_1)}$$

$$= \frac{\frac{1}{\sigma_1\sigma_2 2\pi\sqrt{1-\rho^2}}\, e^{-\frac{1}{2}Q(x_1, x_2)}}{\frac{1}{\sigma_1\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2}}$$

$$= \frac{\dfrac{1}{\sigma_2\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{1}{2}\cdot\frac{1}{\sigma_2^2(1-\rho^2)}\left(x_2-\mu_2-\rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1)\right)^2} \cdot \cancel{e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2}}}{\cancel{e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2}}}$$

$$= \frac{1}{\sigma_2\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{1}{2}\cdot\frac{1}{\sigma_2^2(1-\rho^2)}\left(x_2-\mu_2-\rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1)\right)^2}, \quad -\infty < x_2 < \infty$$

$\therefore \quad X_2 / X_1 = x_1 \sim N\left[\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1),\ \sigma_2^2(1-\rho^2)\right]$

$\therefore \quad X_1 / X_2 = x_2 \sim N\left[\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2-\mu_2),\ \sigma_1^2(1-\rho^2)\right]$

Moments: $\rightarrow$ $E(X_1) = \mu_1$, $V(X_1) = \sigma_1^2$

$E(X_2) = \mu_2$, $V(X_2) = \sigma_2^2$

$$E(X_1X_2) = E\left[E(X_2X_1/X_1)\right] = E\left[X_1 E(X_2/X_1)\right]$$

$$= E\left[X_1\left(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1)\right)\right]$$

$$= \mu_2\mu_1 + \rho\frac{\sigma_2}{\sigma_1}\left[E(X_1^2) - \mu_1^2\right]$$

$$= \mu_1\mu_2 + \rho\sigma_1\sigma_2.$$

$$\Rightarrow \rho = \frac{E(X_1X_2) - \mu_1\mu_2}{\sigma_1\sigma_2}.$$

☆ **Problem:-** If $(X_1, X_2) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, X_1 and X_2 are independent iff $\rho = 0$.

ANS:-

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left\{\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right\}}, \quad -\infty < x_1, x_2 < \infty$$

$$= f_{X_1}(x_1) f_{X_2}(x_2).$$

Now, if X_1, X_2 are independent,

$E(X_1X_2) = E(X_1)E(X_2)$

$\therefore \mathrm{Cov}(X_1, X_2) = 0,$

$\therefore \rho = 0.$

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MGF of Bivariate Normal Distribution: —

Joint MGF of (X_1, X_2) is

$M(t_1, t_2)$

$= E\left[e^{t_1X_1 + t_2X_2}\right]$

$= E\left[e^{t_1X_1} E\left[e^{t_2X_2}/X_1\right]\right]$

$= E\left[e^{t_1X_1} e^{t_2\left(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(X_1-\mu_1)\right) + \frac{t_2^2\sigma_2^2(1-\rho^2)}{2}}\right]$

$= e^{\mu_2t_2 + t_2^2\sigma_2^2\left(\frac{1-\rho^2}{2}\right) - \mu_1t_2\rho\frac{\sigma_2}{\sigma_1}} E\left[e^{t_1X_1 + t_2\rho\frac{\sigma_2}{\sigma_1}X_1}\right]$

$= \exp\left[\mu_2t_2 + t_2^2\sigma_2^2\left(\frac{1-\rho^2}{2}\right) - \cancel{\mu_1t_2\rho\frac{\sigma_2}{\sigma_1}} + \mu_1t_1 + \cancel{\mu_1t_2\rho\frac{\sigma_2}{\sigma_1}} + \frac{\sigma_1^2}{2}\left(t_1 + t_2\rho\frac{\sigma_2}{\sigma_1}\right)^2\right]$

$= \exp\left[\mu_1t_1 + \mu_2t_2 + \frac{1}{2}\left[t_1^2\sigma_1^2 + t_2^2\sigma_2^2 + 2\rho\sigma_1\sigma_2t_1t_2\right]\right]$

$\therefore M(t_1, t_2) = e^{\mu_1t_1 + \mu_2t_2 + \frac{1}{2}(t_1^2\sigma_1^2 + t_2^2\sigma_2^2 + 2\rho\sigma_1\sigma_2t_1t_2)}$

[expectation exists $\forall\, t_1, t_2$]

☆ **Problem 2.** If $(X_1, X_2) \sim BN(0, 0, \sigma_1^2, \sigma_2^2, \rho)$

Find the correlation coefficient between X_1^2, X_2^2.

Ans: — $E(X_1^2) = \sigma_1^2 + \mu_1^2 = \sigma_1^2$

$E(X_2^2) = \sigma_2^2.$

$E(X_1^2X_2^2)$

$= E\left[X_1^2 E(X_2^2/X_1)\right]$

$= E\left[X_1^2\left\{\sigma_2^2(1-\rho^2) + \rho^2\frac{\sigma_2^2}{\sigma_1^2}X_1^2\right\}\right]$

$= \sigma_2^2(1-\rho^2)\sigma_1^2 + \rho^2\frac{\sigma_2^2}{\sigma_1^2}3\sigma_1^4$

$= \sigma_2^2(1-\rho^2)\sigma_1^2 + 3\rho^2\sigma_1^2\sigma_2^2$

$\therefore V(X_1^2) = 3\sigma_1^4 - \sigma_1^4 = 2\sigma_1^4$

$V(X_2^2) = 2\sigma_2^4$

$\therefore \rho_{X_1^2X_2^2} = \frac{\sigma_1^2\sigma_2^2(1-\rho^2) - \sigma_1^2\sigma_2^2 + 3\rho^2\sigma_1^2\sigma_2^2}{2\sigma_1^2\sigma_2^2}$

$= \rho^2.$

☆ **Problem 3.** Give an example of a joint distribution of a 2-dimensional random vectors $(X_1, X_2) \ni$ X_1^2, X_2^2 are independent but X_1 and X_2 are not.

ANS:— Consider the joint PDF

$$f(x_1, x_2) = \frac{1}{4}(1 + x_1 x_2) \quad , \quad |x_1| \le 1, \; |x_2| \le 1$$

$$f(x_1) = \frac{1}{4}\int_{-1}^{1}(1 + x_1 x_2)\, dx_2 = \frac{1}{4}[1+1+0] = \frac{1}{2}.$$

Let $U = X_1^2$, $V = X_2^2$

The joint distribution function of U and V is

$$F(U, V) = P[U \le u, V \le v]$$

$$= P\left[|X_1| \le \sqrt{u}, |X_2| \le \sqrt{v}\right]$$

$$= \int_{-\sqrt{u}}^{\sqrt{u}} \int_{-\sqrt{v}}^{\sqrt{v}} \frac{1}{4}(1 + x_1 x_2)\, dx_1 dx_2$$

$$= \frac{1}{4}\left[2(\sqrt{u}.\sqrt{v})2\right]$$

$$= \sqrt{u}\sqrt{v}$$

$$\therefore P[U \le u] = \frac{1}{2}\int_{-\sqrt{u}}^{\sqrt{u}} du = \sqrt{u}$$

$$\therefore P[V \le v] = \frac{1}{2}\int_{-\sqrt{v}}^{\sqrt{v}} dv = \sqrt{v}$$

$$\therefore F(u, v) = F(u)F(v)$$

$\therefore X_1^2$ and X_2^2 are independent but X and X_2 are not.

Note:—

$$(X_1, X_2) \sim BN(0, 0, 1, 1, \rho)$$

$$X_1/X_2 \sim N\left(\rho x_2, \sqrt{1-\rho^2}\right).$$

$$E[X_1 X_2^{2r}] = E\left[E\left(X_1 X_2^{2r}/X_2\right)\right]$$

$$= E\left[X_2^{2r} E(X_1/X_2)\right]$$

$$= E\left[X_2^{2r} \rho X_2\right] = \rho \times 0$$

☆ Problem 4. If $(X_1, X_2) \sim N_2(0,0,1,1,\rho)$, then show that —

i) $P[X_1 > 0, X_2 > 0] = \frac{1}{4} + \frac{1}{2\pi} \sin^{-1}(\rho)$

ii) $P[X_1 < 0, X_2 < 0] = \frac{1}{4} + \frac{1}{2\pi} \sin^{-1}(\rho)$

iii) $P[X_1 > 0, X_2 < 0] = \frac{1}{4} - \frac{1}{2\pi} \sin^{-1}(\rho)$

iv) $P[X_1 < 0, X_2 > 0] = \frac{1}{4} - \frac{1}{2\pi} \sin^{-1}(\rho)$

Soln. →

i) The joint PDF of (X_1, X_2) is

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x_1^2 + x_2^2 - 2\rho x_1 x_2)}, \quad |\rho| < 1, \quad -\infty < x_1, x_2 < \infty,$$

$$P[X_1 > 0, X_2 > 0] = \int_0^\infty \int_0^\infty \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x_1^2 + x_2^2 - 2\rho x_1 x_2)} dx_1 dx_2$$

Let us consider the transformation,

$$x_1 = r\cos\theta, \quad r > 0$$
$$x_2 = r\sin\theta, \quad 0 < \theta < \pi/2$$
$$\therefore x_1^2 + x_2^2 = r^2$$

Jacobian of the transformation is,

$$J\left(\frac{x_1, x_2}{r, \theta}\right) = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\therefore |J| = r \quad [\because r > 0]$$

$$\therefore P[X_1 > 0, X_2 > 0] = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_0^\infty \int_0^{\pi/2} e^{-\frac{1}{2(1-\rho^2)}(r^2 - \rho r^2 \sin 2\theta)} \cdot r\,dr\,d\theta$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_0^\infty \int_0^{\pi/2} e^{-\frac{r^2}{2(1-\rho^2)}(1-\rho\sin 2\theta)} \cdot r\,dr\,d\theta$$

Put, $\frac{r^2(1-\rho\sin 2\theta)}{2(1-\rho^2)} = z$

$$\Rightarrow r\,dr = \frac{1-\rho^2}{1-\rho\sin 2\theta} dz$$

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$$\therefore P[X_1>0, X_2>0] = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_0^{\pi/2}\int_0^{\infty} \frac{1-\rho^2}{1-\rho\sin 2\theta} e^{-z}\, dz\, d\theta$$

$$= \frac{\sqrt{1-\rho^2}}{2\pi}\int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{1+\tan^2\theta - 2\rho\tan\theta} = \frac{\sqrt{1-\rho^2}}{2\pi}\int_0^{\pi/2} \frac{d\theta}{1-\rho\sin 2\theta}$$

Let, $\tan\theta = y$
$\therefore \sec^2\theta\, d\theta = dy$

$$= \frac{\sqrt{1-\rho^2}}{2\pi}\int_0^{\infty} \frac{dy}{1+y^2-2\rho y}$$

$$= \frac{\sqrt{1-\rho^2}}{2\pi}\int_0^{\infty} \frac{dy}{(y-\rho)^2+(1-\rho)^2}$$

$$= \frac{\cancel{\sqrt{1-\rho^2}}}{2\pi} \cdot \frac{1}{\cancel{\sqrt{1-\rho^2}}} \tan^{-1}\left(\frac{y-\rho}{\sqrt{1-\rho^2}}\right)\Bigg|_0^{\infty}$$

$$= \frac{1}{2\pi}\left(\frac{\pi}{2} + \tan^{-1}\frac{\rho}{\sqrt{1-\rho^2}}\right)$$

$$= \frac{1}{4} + \frac{1}{2\pi}\tan^{-1}\left(\frac{\rho}{\sqrt{1-\rho^2}}\right)$$

$$= \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}(\rho).$$

Let $A = \{X_1>0\}$ and $B = \{X_2>0\}$

ii) $P[X_1>0, X_2>0] = P[A\cap B] = P(A) - P(A\cap B^c)$
$= P(A) - P(B) + P[A^c \cap B^c]$

$$= \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{2}} + P[X_1<0, X_2<0]$$

$$\therefore P[X_1<0, X_2<0] = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}(\rho).$$

iii) $P[X_1>0, X_2<0] = P[X_1>0] - P[X_1>0, X_2>0]$

$$= \frac{1}{2} - \frac{1}{4} - \frac{1}{2\pi}\sin^{-1}(\rho)$$

$$= \frac{1}{4} - \frac{1}{2\pi}\sin^{-1}(\rho)$$

iv) $P[X_1<0, X_2>0] = P[X_1>0, X_2<0]$

$$= \frac{1}{4} - \frac{1}{2\pi}\sin^{-1}(\rho).$$

☆ **Continuation:—** If $q = P[X_1X_2<0]$ then S.T. $\rho = \cos(\pi q)$.

ANS:— $q = P[X_1X_2<0] = P[X_1<0, X_2>0] + P[X_1>0, X_2<0]$

$$= \frac{1}{2} - \frac{1}{\pi}\sin^{-1}(\rho)$$

or, $\rho = \sin\left(\frac{\pi}{2} - \pi q\right) = \cos(\pi q)$, where,
$(X_1, X_2) \sim N_2(0,0,1,1,\rho)$.

NOTE :→

$$V(Y) = E[V(Y/X)] + V[E(Y/X)]$$

$$= E[\sigma_y^2 (1-\rho^2)] + V\left[\mu_y + \rho \frac{\sigma_y}{\sigma_x}(x-\mu_x)\right]$$

$$= \sigma_y^2 (1-\rho^2).$$

$$E(XY) = E[E(XY/X)] = E\left[X\left\{\mu_y + \rho\frac{\sigma_y}{\sigma_x}(x-\mu_x)\right\}\right]$$

$$= \mu_y E(X) + \rho\frac{\sigma_y}{\sigma_x}E(X^2) - \rho\frac{\sigma_y}{\sigma_x}\mu_x E(X)$$

$$Cov(X,Y) = \cancel{\sigma_Y E(X)} + \rho\frac{\sigma_Y}{\sigma_X}E(X^2) - \rho\frac{\sigma_Y}{\sigma_X}\mu_X E(X) - \cancel{\mu_Y E(X)}$$

$$= \rho\frac{\sigma_y}{\sigma_x}V(X).$$

$$\therefore \rho_{XY} = \frac{\rho\sigma_Y V(X)}{\sigma_X\sqrt{V(X)}\sqrt{\sigma_Y^2(1-\rho^2)+\rho^2\sigma_Y^2 V(X)}}$$

$$= \frac{1}{\sqrt{1+\frac{\sigma_X^2(1-\rho^2)}{\rho^2 V(X)}}}$$

☆ **Problem 5.** Let $f_1(x,y)$; $f_2(x,y)$ be two bivariate normal density functions with zero means, unit variances and correlation coefficients ρ_1, ρ_2, respectively ($\rho_1 \neq \rho_2$). Show that the density function

$f(x,y) = \frac{1}{2}f_1(x,y) + \frac{1}{2}f_2(x,y)$ is not normal. But its density functions are normal.

Soln. →

$$f_1(x,y) = \frac{1}{2\pi\sqrt{1-\rho_1^2}} e^{-\frac{1}{2(1-\rho_1^2)}(x^2+y^2-2\rho_1 xy)}$$

$$f_2(x,y) = \frac{1}{2\pi\sqrt{1-\rho_2^2}} e^{-\frac{1}{2(1-\rho_2^2)}(x^2+y^2-2\rho_2 xy)}$$

$$\therefore f(x,y) = \frac{1}{4\pi}\left\{\frac{1}{\sqrt{1-\rho_1^2}} e^{-\frac{1}{2(1-\rho_1^2)}(x^2+y^2-2\rho_1 xy)} + \frac{1}{\sqrt{1-\rho_2^2}} e^{-\frac{1}{2(1-\rho_2^2)}(x^2+y^2-2\rho_2 xy)}\right\}$$

$$\because f(x) = \frac{1}{4\pi\sqrt{1-\rho_1^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho_1^2)}(x^2+y^2-2\rho_1 xy)}dy + \frac{1}{4\pi\sqrt{1-\rho_2^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho_2^2)}(x^2+y^2-2\rho_2 xy)}dy$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}}.$$ ∴ Marginally, $X \sim N(0,1)$

similarly, $Y \sim N(0,1)$

Mode: — If $(X_1, X_2) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, find the value of (x_1, x_2) for which $f(x_1, x_2)$ is maximum.

Soln →
$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2}Q(x_1, x_2)}$$

∴ f is maximum when Q is maximum.

$$Q(x_1, x_2) = \frac{1}{\sqrt{1-\rho^2}}\left\{\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right\}$$

$$= \left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 \left\{\frac{x_2-\mu_2-\rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1)}{\sigma_1\sqrt{1-\rho^2}}\right\}^2$$

$$\therefore \frac{x_1-\mu_1}{\sigma_1} = 0 \Rightarrow x_1 = \mu_1$$

$$\Rightarrow x_2 = \mu_2$$

∴ mode is at (μ_1, μ_2).

☆ **Problem 6.** If $X_1 \sim N(\mu, \sigma^2)$ & $X_2 / X_1 = x_1 \sim N(x_1, \sigma^2)$, S.T. the joint distn. of (X_1, X_2) is bivariate normal, Obtain the parameters of this distribution.

Soln. →

$$f(x_1, x_2) = f(x_1) f(x_2/x_1)$$
$$= \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}\left(\frac{x_1-\mu}{\sigma}\right)^2 - \frac{1}{2}\left(\frac{x_2-x_1}{\sigma}\right)^2}$$
$$= \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\{Q(x_1, x_2)\}}$$

Now,
$$Q(x_1, x_2) = (x_1-\mu_1)^2 + (x_2-x_1)^2$$
$$= (x_1-\mu_1)^2 + \{(x_2-\mu) - (x_1-\mu)\}^2$$
$$= 2(x_1-\mu)^2 + (x_2-\mu)^2 - 2(x_1-\mu)(x_2-\mu)$$

Hence if the dispersion matrix is p.d.

$$\det\begin{bmatrix} 2 & -1/2 \\ -1/2 & 1 \end{bmatrix} = \frac{9}{4} > 0$$

so, the joint distribution of $(X_1, X_2) \sim$ Bivariate normal distribution.

$$\mu = E(X_2) = E[E(X_2/X_1)] = E(X_1) = \mu$$
$$V(X_2) = E[V(X_2/X_1)] + V[E(X_2/X_1)]$$
$$= E(\sigma^2) + V(X_1)$$
$$= 2\sigma^2$$
$$E(X_1X_2) = E[E(X_1X_2/X_1)] = E[X_1 E(X_2/X_1)] = E(X_1^2)$$
$$= \sigma^2 + \mu^2.$$

$\therefore \text{Cov}(X_1, X_2) = \sigma^2 + \cancel{\mu^2} - \cancel{\mu^2} = \sigma^2$

$\therefore \rho = \frac{\cancel{\sigma^2}}{\sqrt{2}\cancel{\sigma^2}} = \frac{1}{\sqrt{2}}$.

$\therefore (X_1, X_2) \sim BN\left(\mu, \mu, \sigma^2, 2\sigma^2, \frac{1}{\sqrt{2}}\right)$.

☆ Problem 7. Show that for the distn. $N_2(0, 0, 1, 1, \rho)$, the moments obey the recurrence relation

$$\mu_{rs} = (r+s-1)\rho\mu_{r-1,s-1} + (r-1)(s-1)(1-\rho^2)\mu_{r-2,s-2}.$$

Hence, or otherwise, show that

$\mu_{rs} = 0$ if $r+s$ is odd; [1998]

$\mu_{13} = \mu_{31} = 3\rho$; $\mu_{22} = 1+\rho^2$.

Soln → $\mu_{rs} = E(X^r Y^s)$

$$= E(X^r Y^{s-1}\{Y - \rho X + \rho X\})$$

$$= E[X^r Y^{s-1}(Y - \rho X)] + \rho E(X^{r+1} Y^{s-1})$$

$$= E[X^r Y^{s-1}(Y - \rho X)] + \rho\mu_{r+1,s-1}.$$

Now,

$$E(X^r Y^{s-1}(Y - \rho X)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^r y^{s-1}(y - \rho x)\frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2}(x^2+y^2-2\rho xy)\frac{1}{1-\rho^2}}\, dx\,dy$$

$$= \int_{-\infty}^{\infty}\frac{x^r e^{-x^2/2}}{\sqrt{2\pi}}\left\{\int_{-\infty}^{\infty} y^{s-1}(y - \rho x)\frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(y-\rho x)^2} dy\right\} dx$$

$$= \int_{-\infty}^{\infty}\frac{x^r e^{-x^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(s-1) y^{s-2}\frac{1}{\sqrt{2\pi}}\sqrt{1-\rho^2}\, e^{-\frac{1}{2}\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)^2} dy\,dx$$

$$= (s-1)(1-\rho^2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{y^{s-2}x^r}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)} dx\,dy$$

$$= (s-1)(1-\rho^2)\mu_{r,s-2}.$$

$\therefore \mu_{r,s} = (s-1)(1-\rho^2)\mu_{r,s-2} + \rho\mu_{r+1,s-1}$ ----- ①

Interchanging the rolls of r, and s, it can be similarly shown that,

$\mu_{r,s} = (r-1)(1-\rho^2)\mu_{r-2,s} + \rho\mu_{r-1,s+1}$ ----- ②

Using ②, $\mu_{r,s-2} = (r-1)(1-\rho^2)\mu_{r-2,s-2} + \rho\mu_{r-1,s-1}$ --- ③

Using ③, $\mu_{r+1,s-1} = r(1-\rho^2)\mu_{r-1,s-1} + \rho\mu_{r,s}$ ---- ④

Thus, $\mu_{rs} = (s-1)(1-\rho^2)\left[(r-1)(1-\rho^2)\mu_{r-2,s-2} + \rho\mu_{r-1,s-1}\right]$
$+ \rho\left[r(1-\rho^2)\mu_{r-1,s-1} + \rho\mu_{r,s}\right]$

$\Rightarrow \mu_{rs} = (s-1)(r-1)(1-\rho^2)\mu_{r-2,s-2} + \rho(r+s-1)\mu_{r-1,s-1}$

if $r+s$ is odd, $\therefore r \neq s$

set, $r > s$

now, $\mu_{rs} = c_1\mu_{r-2,s-2} + c_2\mu_{r-1,s-1}$

$= c_1\left[c_3\mu_{r-4,s-4} + c_4\mu_{r-3,s-3}\right]$
$+ c_2\left[c_5\mu_{r-3,s-3} + c_6\mu_{r-2,s-2}\right]$

$= k_1\mu_{r-4,s-4} + k_2\mu_{r-3,s-3}$

$\vdots$

$= k_3\mu_{r-s,0} + k_4\mu_{r-s+1,1}$.

$\therefore \mu_{r-s} = 0$ since $r-s$ is odd,

and $\mu_{r-s+1,1} = E\left[X^{r-s+1}Y\right] = E\left[X^{r-s+1}E(Y/X)\right]$
$= E\left[X^{r-s+2}\right]$
$= 0 \quad [r-s+2 = \text{odd}]$

Similarly, it can be shown that,
if $r < s$, $\mu_{r,s} = 0$ if $r+s = $ odd.

Regression:— Consider two variables X and Y where Y is the study of variance and X is the auxilary variable. Our problem is to predict Y, when X is known. Let $\eta_x = E[Y/X=x]$ provided it exists. The regression curve of Y on X is defined by the locus of the point (x, η_x) when x varies, same properties $E[Y-g(X)]^2$ is minimum when, $g(x) = \eta_x$ where $g(x)$ is any predictor of Y based on X

$E[Y-g(X)]^2 = E\left[(Y-\eta_x) - (g(X)-\eta_x)\right]^2$
$= E[Y-\eta_x]^2 + E[g(X)-\eta_x]^2 - 2E\left[[Y-\eta_x][g(X)-\eta_x]\right]$
$= E[Y-\eta_x]^2 + E[g(X)-\eta_x]^2$
$\geq E[Y-\eta_x]^2 \quad [\because E[g(X)-\eta_x]^2 \geq 0]$

$\therefore E(Y-g(X))^2$ is minimum when, $g(X) = \eta_x$.

Now,
$E\left[(Y-\eta_x)(g(X)-\eta_x)\right]$
$= E\left[(g(X)-\eta_x)E\{(Y-\eta_x)/X\}\right]$
$= E\left[(g(X)-\eta_x)\times 0\right] = 0$

Correlation Coefficient between Y and η_X :—

$$\text{Cov}(Y, \eta_X) = E[(Y - E(Y))(\eta_X - E(\eta_X))]$$
$$= E[(Y - E(Y))(\eta_X - E(Y))]$$
$$= E[(\eta_X - E(Y)) E[(Y - E(Y))/X]]$$
$$= E[\eta_X - E(Y)]^2$$
$$= V(\eta_X)$$

$$\therefore \rho_{Y,\eta_X} = \frac{V(\eta_X)}{\sqrt{V(Y)V(\eta_X)}} = \sqrt{\frac{V(\eta_X)}{V(Y)}}$$

$\therefore \rho_{Y,\eta_X} \geq |\rho_{Y,g(x)}|$ for any other predictor $g(x)$

$$\text{Cov}(Y, g(X)) = E[(Y - E(Y))(g(X) - E(g(X)))]$$
$$= E[(g(X) - E(g(X))) E\{(Y - E(Y))/X\}]$$
$$= \text{cov}(\eta_X, g(X)).$$

$$\therefore \rho^2_{Y,g(X)} = \frac{\text{Cov}^2(Y, g(X))}{V(Y)V(g(X))} = \frac{V^2(\eta_X)}{V(Y)V(\eta_X)} \times \frac{\text{cov}^2(\eta_X, g(X))}{V(\eta_X) \cdot V(g(X))}$$
$$= \rho^2_{Y,\eta_X} \, \rho^2_{\eta_X, g(X)}$$

$$\therefore \rho_{Y,g(X)} \leq \rho_{Y,\eta_X}$$

'=' holds when η_X and $g(X)$ have linear relationship.

Note:— The square of the maximum correlation attd is called the correlation ratio of Y on X and is usually denoted by

$$\eta^2_{YX} = \frac{V(\eta_X)}{V(Y)}, \quad 0 \leq \eta^2_{YX} \leq 1 \quad [\text{To show}]$$

$$V(Y) = V[E(Y/X)] + E[V(Y/X)]$$
$$\geq V(\eta_X) \quad [\because E(V(Y/X)) \geq 0]$$

$$\therefore \frac{V(\eta_X)}{V(Y)} \leq 1 \Rightarrow \eta^2_{YX} \leq 1$$

Determination of regression equation or, regression curve :

Suppose the regression equation of Y on X is linear.

Let, $\eta_X = a + bX$

$$E(\eta_X) = a + bE(X)$$

$$\therefore \mu_y = a + b\mu_x \cdots\cdots (1)$$

$$E[X\eta_X] = a\mu_x + bE(X^2)$$

$$E(XY) = a\mu_x + b(\sigma_x^2 + \mu_x^2)$$

$$\text{cov}(X,Y) = a\mu_x + b\sigma_x^2 + b\mu_x^2 - \mu_x(a + b\mu_x) = b\sigma_x^2$$

$$\therefore b = \frac{\text{cov}(X,Y)}{\sigma_x^2}$$

$$\therefore \eta_x = \mu_y + \frac{\text{cov}(X,Y)}{\sigma_x^2}(x - \mu_x).$$

x ———— x

Introduction of Bivariate Distribution : —

In many experiments an observation is expressable not as a single numerical quantity but as a family of several seperate quantities. If a pair of distinguishable dice are thrown, the outcome (x, y) is a pair, where x denotes the face value of the 1st dice and y denotes the face value of 2nd dice.

Similarly, to record the height and weight of every person in a certain community we need a pair (x, y), where the components respectively denotes the heights and weights,

DISTRIBUTION OF TWO JOINTLY DISTRIBUTED R.V.'s

Let X and Y be two jointly distributed r.v. with joint distribution as shown in the following table:

X \ Y	y_1	y_2	$\cdots$	y_j	$\cdots$	y_l	
x_1	p_{11}	p_{12}	$\cdots$		$\cdots$	p_{1l}	p_{10}
x_2	p_{21}	p_{22}	$\cdots$		$\cdots$	p_{2l}	p_{20}
$\vdots$	$\vdots$			$\vdots$		$\vdots$	$\vdots$
x_i			$\cdots$	p_{ij}	$\cdots$		p_{i0}
$\vdots$	$\vdots$					$\vdots$	$\vdots$
x_k	p_{k1}	p_{k2}	$\cdots$		$\cdots$	p_{kl}	p_{k0}
	p_{01}	p_{02}	$\cdots$	p_{0j}	$\cdots$	p_{0l}	

$p_{ij} = P[X = x_i, Y = y_j] \quad \forall\ i = 1(1)k,\ j = 1(1)l$

$$p_{i0} = \sum_{j=1}^{l} p_{ij}, \quad p_{0j} = \sum_{i=1}^{k} p_{ij}$$

$\therefore p_{i0} = P[X = x_i] \quad \therefore p_{0j} = P[Y = y_j]$

$$P(Y = y_j / X = x_i) = \frac{p_{ij}}{p_{i0}}$$ = conditional probability that $Y = y_j$ given $X = x_i$.

X \ Y	-1	0	1	
-1	$1/6$	0	$1/6$	$2/6$
0	0	$2/6$	0	$2/6$
1	$1/6$	0	$1/6$	$2/6$
	$2/6$	$2/6$	$2/6$	1

$$E(X) = -\frac{2}{6} + \frac{2}{6} = 0$$

$$E(Y) = -\frac{2}{6} + \frac{2}{6} = 0$$

$$E(XY) = \frac{1}{6} - \frac{1}{6} - \frac{1}{6} + \frac{1}{6} = 0$$

$r_{xy} = 0, \quad Cov(x, y) = E(XY) - E(X)E(Y) = 0$

independent if

$p_{ij} = P[X = x_i, Y = y_j] = P[X = x_i]\, P[Y = y_j] = p_{i0}\, p_{0j} \quad \forall (i, j)$

But here,

$P[X = x_i, Y = y_j] \neq P[X = x_i]\, P[Y = y_j]$ {for at least one case} for two jointly distributed discrete random variable X and Y the joint probability mass function may be denoted by,

$$f_{XY}(x,y) = P[X=x, Y=y]$$, and their joint distribution defined by $F_{XY}(x,y)$ gives the probability $P[X \le x, Y \le y]$.

N.T. the marginal PMF of Y is

$$f_Y(y) = P[Y=y] = \sum_x f_{XY}(x,y).$$

The conditional PMF of Y for given $X=x$ is,

$$f_{Y/X}(y/x) = P[Y=y/X=x] = \frac{f_{XY}(x,y)}{f_X(x)}.$$

If X and Y are two jointly distributed continuous r.v.s then their joint d.f. is,

$$F_{XY}(x,y) = P[X \le x, Y \le y] = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(x,y)\,dx\,dy$$

if we denote the joint PMF of X and Y by $f_{XY}(x,y)$ and the marginal PMF of X (Y) by $f_X(x)$ ($f_Y(y)$).

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx, \ .$$

and the conditional PDF of Y for given $X=x$ is,

$$f_{Y/X}(y/x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{f_{XY}(x,y)}{\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy}.$$

Defn. :→ If $\exists$ a function $F_{XY}(x,y)$ on $\mathbb{R}^2$ satisfying

i) $F_{XY}(x,-\infty) = F_{XY}(-\infty,y) = 0$

ii) $F_{XY}(+\infty,+\infty) = 1$

iii) $F_{XY}(x+0,y) = F_{XY}(x,y+0) = F_{XY}(x,y)$

iv) for any $h>0$, $k>0$

$$\Delta F_{XY}(x,y) = F_{XY}(x+h,y+k) - F_{XY}(x+h,y) - F_{XY}(x,y+k) + F_{XY}(x,y) \ge 0$$

then $\exists$ a pair of r.v.s X, Y having unique distribution with distribution function given by, — $F_{XY}(x,y)$.

Theorem :— Let $F_{XY}(x,y)$ be the distribution function of the jointly distributed random variables X, Y; then

i) $F_{XY}(x,-\infty) = F_{XY}(-\infty, y) = 0$

ii) $F_{XY}(\infty,\infty) = 1$

iii) $F_{XY}(x+0, y) = F_{XY}(x, y+0) = F_{XY}(x,y)$

iv) for any $h>0$ & $k>0$,

$$\Delta F_{XY}(x,y) = F_{XY}(x+h, y+k) - F_{XY}(x+h, y) - F_{XY}(x, y+k) + F_{XY}(x,y) \geqslant 0$$

Proof :— i) Let $A_n(x) = \{(X,Y) : X \le x, Y \le -n\}$

$A_{n+1}(x) \subseteq A_n(x)$

i.e. $\{A_n\}$ is monotone decreasing sequence of sets. So,

$$\lim_{n\to\infty} A_n = \phi$$

Now, $F_{XY}(x,-\infty) = \lim_{n\to\infty} P[X \le x, Y \le -n]$

$= \lim_{n\to\infty} P[A_n(x)]$ [by continuity theorem of probability]

$= P[\lim_{n\to\infty} A_n(x)]$

$= P(\phi) = 0.$

Similarly, it can be shown that,

$F_{XY}(-\infty, y) = P[X \le -\infty, Y \le y] = 0$

ii) $A_n = \{(x,y) : -\infty < x < n, -\infty < y < n\}$

$A_n \subseteq A_{n+1}$

i.e. $\{A_n\}$ is a monotone expanding sequence of sets.

$$\lim_{n\to\infty} A_n = \mathbb{R}^2$$

$F_{XY}(+\infty,+\infty) = \lim_{n\to\infty} P[X \le n, Y \le n] = \lim_{n\to\infty} P(A_n) = P[\lim_{n\to\infty} A_n]$

$= P[\mathbb{R}^2]$ [By continuity theorem of probability]

$= 1$

$\therefore F_{XY}(+\infty,+\infty) = 1.$

iii) $F_{XY}(x+0, y)$

$= \lim_{\epsilon \to 0} F_{XY}(x+\epsilon, y) = \lim_{n\to\infty} F_{XY}\left(x+\frac{1}{n}, y\right)$

let, $A_n = \{(x,y) : -\infty < X < x+\frac{1}{n}, -\infty < Y \le y\}$

$A_n \supseteq A_{n+1}$

$\lim_{n\to\infty} A_n = \{(x,y) : -\infty < X \le x, -\infty < Y \le y\}$

so, $\lim_{n\to\infty} F_{XY}\left(x+\frac{1}{n}, y\right) = \lim_{n\to\infty} P\left[-\infty < X < x+\frac{1}{n}, -\infty < Y < y\right]$

$= \lim_{n\to\infty} P(A_n)$

$= P\left[\lim_{n\to\infty} A_n\right]$

$= P\left[-\infty < X \le x, -\infty < Y \le y\right]$

$= F_{XY}(x, y)$

$\therefore F_{XY}(x+0, y) = F_{XY}(x, y)$

similarly, it can be shown that,

$F_{XY}(x, y+0) = F_{XY}(x, y)$

$\therefore F_{XY}(x, y+0) = F_{XY}(x+0, y) = F_{XY}(x, y)$.

iv) for every $h>0, k>0$

$\Delta F_{XY}(x,y) = F_{XY}(x+h, y+k) - F_{XY}(x+h, y) - F_{XY}(x, y+k) + F_{XY}(x,y)$

≥ 0

<u>Proof:</u>—

$\Delta F_{XY}(x,y) = \{F_{XY}(x+h, y+k) - F_{XY}(x+h, y)\} - \{F_{XY}(x, y+k) - F_{XY}(x,y)\}$

$= \{P[X \le x+h, Y \le y+k] - P[X \le x+h, Y \le y]\}$

$- \{P[X \le x, Y \le y+k] - P[X \le x, Y \le y]\}$

$= P[X \le x+h, y < Y \le y+k] - P[X \le x, y < Y \le y+k]$

$= P[x < X \le x+h, y < Y \le y+k] \geq 0$

since $h>0, k>0$.

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Definition of Marginal Distribution: — If X and Y are two jointly distributed random variable with joint distribution function $F_{XY}(x, y)$ then the marginal distribution of X is given by,

$$F_X(x) = F_{XY}(x, \infty)$$

Similarly, the marginal distribution of Y is

$$F_Y(y) = F_{XY}(\infty, y).$$

Definition of Conditional Distribution: — If X and Y be the two jointly distributed discrete random variables then the conditional distribution of Y for given $X = x$, when $P[X = x] > 0$ is given by, —

$$F_{Y/X}(y/x) = \sum_{\{t \le y\}} P[Y = t / X = x]$$

$$= \sum_{\{t \le y\}} \frac{P[X = x, Y = t]}{P[X = x]}$$

if X and Y are two jointly distributed continuous random variable, then for given x, the conditional distribution of Y for $X = x$ is given by, —

$$F_{Y/X}(y/x) = \lim_{\epsilon \to 0} P[Y \le y / x - \epsilon < X \le x + \epsilon],$$

provided the limit exists.

i.e. $$F_{Y/X}(y/x) = \lim_{\epsilon \to 0} \frac{P[x - \epsilon < X \le x + \epsilon, Y \le y]}{P[x - \epsilon < X \le x + \epsilon]}$$

if $\exists$ a non-negative function $f_{Y/X}(y/x)$ such that

$$F_{Y/X}(y/x) = \int_{-\infty}^{x} f_{Y/X}(t/x)\, dt$$

then $f_{Y/X}(t/x)$ is called the conditional PDF of Y for given $X = x$.

Note:— The conditional expectation is called the regression of Y on X.

Examples:—

☆ **Problem 1.** Let X & Y be two jointly distributed continuous random variable with joint PDF, —

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left\{x^2 - 2\rho xy + y^2\right\}\right], \quad x \in \mathbb{R},\ y \in \mathbb{R}$$

i) Find the marginal PDF of X

ii) Find the conditional PDF of Y for given $X = x$.

Solution:—

i)
$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2(1-\rho^2)}\left\{x^2 - 2\rho xy + y^2\right\}\right] dy$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2(1-\rho^2)}\left\{(y-\rho x)^2 + (1-\rho^2)x^2\right\}\right] dy$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-x^2/2} \int_{-\infty}^{\infty} \exp\left[-\frac{(y-\rho x)^2}{2(1-\rho^2)}\right] dy$$

since,
$$\frac{1}{\sqrt{2\pi}\,\sigma_Y} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma_Y^2}(y-\mu_Y)^2\right] = 1$$

Here, $\mu_Y = \rho x$ and $\sigma_Y^2 = (1-\rho^2)$

$$\therefore \frac{1}{\sqrt{2\pi}\sqrt{(1-\rho^2)}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2(1-\rho^2)}(y-\rho x)^2\right] dy = 1$$

$$\therefore \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2(1-\rho^2)}(y-\rho x)^2\right] dy = \sqrt{2\pi(1-\rho^2)}$$

$$\therefore f_X(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, \quad x \in \mathbb{R},$$

$\therefore X \sim N(0,1)$.

ii) Conditional PDF of Y for given $X = x$ is

$$f_{Y/X=x}(y/x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{\frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-x^2/2} \cdot e^{-\frac{(y-\rho x)^2}{2(1-\rho^2)}}}{\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}}$$

$$= \frac{1}{\sqrt{2\pi(1-\rho^2)}}\, e^{-\frac{(y-\rho x)^2}{2(1-\rho^2)}},$$

i.e. $Y/X = x \sim N\left(\rho x, (\sqrt{1-\rho^2})^2\right)$, $-\infty < y < \infty$

$$E[Y/X=x] = \int_{-\infty}^{\infty} y\, f_{Y/X}(y/x)\,dy = \rho x$$

$$E(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\,dy.$$

☆ Problem 2. $f(x,y) = \begin{cases} e^{-y}(1-e^{-x}) & \text{if } 0<x<y \text{ and } 0<y<\infty \\ e^{-x}(1-e^{-y}) & \text{if } 0<y<x \text{ and } 0<x<\infty \end{cases}$

i) find marginal PDF of X,
ii) find conditional PDF of Y for given X = x and also $E(Y/X=x)$.

Solution:— The marginal PDF of X is

$$f_X(x) = \int_0^\infty f_{XY}(x,y)\,dy$$

$$= \int_0^x f_{XY}(x,y)\,dy + \int_x^\infty f_{XY}(x,y)\,dy$$

$$= \int_0^x e^{-x}(1-e^{-y})\,dy + \int_x^\infty e^{-y}(1-e^{-x})\,dy$$

$$= e^{-x}[x+e^{-x}-1] + e^{-x}(1-e^{-x})$$

$$= xe^{-x}, \quad 0<x<\infty$$

for some x, where x > 0,

$$f_{Y/X}(y/x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

$$= \begin{cases} \dfrac{e^{-y}(1-e^{-x})}{xe^{-x}} = \dfrac{e^{-(y-x)}(1-e^{-x})}{x}, & x<y<\infty \\ \dfrac{1-e^{-y}}{x}, & 0<y<x \end{cases}$$

$$\int_0^\infty f_{Y/X}(y/x)\,dy = \int_0^x \frac{1-e^{-y}}{x}\,dy + \int_x^\infty \frac{1-e^{-x}}{x} e^{-(y-x)}\,dy$$

$$= \frac{1}{x}[x+e^{-x}-1] + \frac{1}{x}[1-e^{-x}]$$

$$= 1.$$

$$E[Y/X=x] = \int_0^\infty y\, f_{Y/X}(y/x)\,dy$$

☆ **Problem 3.** Let X and Y be two jointly distributed continuous random variable with PDF

$$f_{XY}(x,y) = \frac{1}{8}(y^2 - x^2)e^{-y} \quad \text{when, } 0<y<\infty, \ |x|<y$$

i.e. $-y < x < y$.

Solution:—

$$f_X(x) = \begin{cases} \int_x^\infty f_{XY}(x,y)\,dy & \text{for } x>0 \\ \int_{-x}^\infty f_{XY}(x,y)\,dy & \text{for } x<0 \end{cases}$$

when $x>0$,

$$f_X(x) = \frac{1}{8}\int_x^\infty (y^2 - x^2)e^{-y}\,dy$$

$$= \frac{1}{8}\int_x^\infty y^2 e^{-y}\,dy - \frac{x^2}{8}\int_x^\infty e^{-y}\,dy$$

$$= \frac{1}{8}\left[-y^2 e^{-y}\Big|_x^\infty + 2\int_x^\infty y e^{-y}\,dy + \frac{x^2}{8} e^{-y}\Big|_x^\infty\right]$$

$$= \frac{1}{8}\left[-y^2 e^{-y}\right]_x^\infty + 2\cdot\frac{1}{8}\left[-e^{-y}\cdot y\right]_x^\infty + \frac{1}{4}\left[\int_x^\infty e^{-y}\right] - \frac{x^2}{8}e^{-x}$$

$$= \frac{1}{8}x^2 e^{-x} + \frac{1}{4}e^{-x}\cdot x + \frac{1}{4}e^{-x} - \frac{x^2}{8}e^{-x}$$

when $x<0$

$$f_X(x) = \frac{1}{8}\int_{-x}^{\infty}(y^2 - x^2)e^{-y}\,dy$$

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☆ Example 4. Let X and Y be two jointly distributed r.v. with joint PDF

$$f_{XY}(x,y) = \begin{cases} x^2 + \frac{xy}{3}, & \text{if } \begin{matrix} 0<x<1 \\ 0<y<2 \end{matrix} \\ 0, & \text{ow} \end{cases}$$

Find i) the marginal PDF of X.
ii) Marginal PDF of Y
iii) Compute $P[X>\frac{1}{2}]$, $P[Y<X]$, $P[Y<\frac{1}{2}/X<\frac{1}{2}]$

Solution: —

i) Marginal PDF of X is $f_X(x) = \int_0^2 f_{XY}(x,y)\,dy = \int_0^2 (x^2 + \frac{xy}{3})\,dy$

$$= 2x^2 + \frac{2x}{3}, \quad 0<x<1.$$

$$\therefore \int_0^1 f_X(x)\,dx = \int_0^1 \left(2x^2 + \frac{2x}{3}\right) dx = 1.$$

ii) Marginal PDF of Y is $f_Y(y) = \int_0^1 \left(x^2 + \frac{xy}{3}\right) dx = \frac{1}{3} + \frac{y}{6}$

$$\therefore \int_0^2 f_Y(y)\,dy = \int_0^2 \left(\frac{1}{3} + \frac{y}{6}\right) dy = 1.$$

iii) $P[Y<X] = \int_0^1 \left[\int_0^x f_{XY}(x,y)\,dy\right] dx = \int_0^1 \left[x^3 + \frac{x^3}{6}\right] dx$

$$= \frac{7}{24}.$$

$$P[Y<\tfrac{1}{2}/X<\tfrac{1}{2}] = \frac{P[X<\frac{1}{2}, Y<\frac{1}{2}]}{P[X<\frac{1}{2}]}$$

$$= \frac{\int_0^{1/2}\int_0^{1/2} f_{XY}(x,y)\,dx\,dy}{\int_0^{1/2} f_X(x)\,dx}$$

$$= \frac{\int_0^{1/2}\int_0^{1/2} \left(x^2 + \frac{xy}{3}\right) dx\,dy}{\int_0^{1/2} \left(2x^2 + \frac{2x}{3}\right) dx}$$

$$=$$

☆ **Problem 5.** Let X and Y be two jointly distributed r.v.s ∋ the marginal PDF of X is

$$f_X(x) = \begin{cases} 1 & \text{if } -\frac{1}{2} < x < \frac{1}{2} \\ 0 & \text{ow} \end{cases}$$

also let the conditional PDF of Y give $X = x$ be given by

$$f_Y(y/x) = \begin{cases} 1 & \text{if } x < y < x+1 \\ 0 & \text{ow} \end{cases}$$

for $-\frac{1}{2} < x < 0$ and by

$$f_Y(y/x) = \begin{cases} 1 & \text{if } -x < y < 1-x \\ 0 & \text{ow} \end{cases}$$

for $0 < x < \frac{1}{2}$,

show that X and Y are uncorrelated. Are they independent?

Solution:— The joint PDF of X and Y is

$$f_{XY}(x,y) = f_Y(y/x)\, f_X(x)$$

$$= \begin{cases} 1, & -\frac{1}{2} < x < 0 \text{ and } x < y < x+1 \\ & \text{or, } \\ & 0 < x < \frac{1}{2} \text{ and } -x < y < 1-x \\ 0, & \text{ow} \end{cases}$$

Note that, $\displaystyle\int_{-1/2}^{0}\int_{x}^{x+1} f_{XY}(x,y)\,dy\,dx + \int_{0}^{1/2}\int_{-x}^{1-x} f_{XY}(x,y)\,dy\,dx$

$= 1.$

$$E(XY) = \int_{-1/2}^{0}\int_{x}^{x+1} xy\, f_{XY}(x,y)\,dy\,dx + \int_{0}^{1/2}\int_{-x}^{1-x} xy\, f_{XY}(x,y)\,dy\,dx$$

$= 0$

$$E(X) = \int_{-1/2}^{1/2} x f_X(x)\,dx = 0$$

Hence, X & Y are uncorrelated.

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☆ Problem 6. Let X and Y have the circular normal distn. with zero mean, i.e. $X \& Y \sim N_2(0,0,\sigma^2,\sigma^2,0)$. Consider a circle C and a square S of equal area both with abc (0,0).

Prove that, $P[(X,Y) \in C] > P[(X,Y) \in S]$ (2008)

Solution:— The joint PDF of X & Y is given by

$$f(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(x^2+y^2)}, \quad x \in \mathbb{R},\ y \in \mathbb{R},\ \sigma > 0$$

Let us consider a square S, with vertices $(a,-a), (a,a), (-a,a), (-a,-a)$.

The area of the square $= 4a^2 = S$

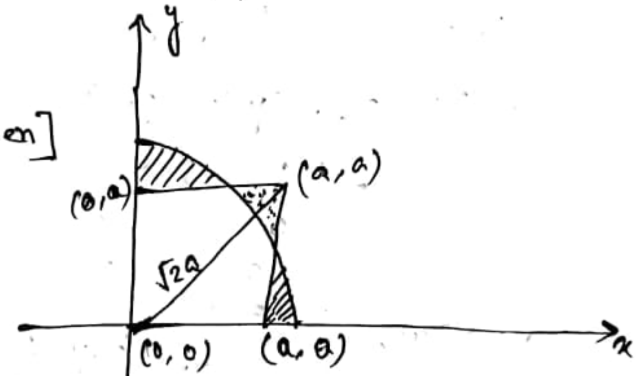
Consider a circle C with radius $= r$, and the centre at (0,0),

Area of $C = \pi r^2$

Hence, $\pi r^2 = 4a^2$ [given]

$$\Rightarrow r = \frac{2a}{\sqrt{\pi}}$$

Therefore, $a < r < \sqrt{2}a$



Now, $$P[XY \in S] = \iint_{x,y \in S} f(x,y)\,dx\,dy$$

$$= 4\int_0^a \int_0^a f(x,y)\,dx\,dy \quad \text{[By symmetry]}$$

$$P[XY \in C] = \iint_{x,y \in C} f(x,y)\,dx\,dy$$

Now, in the first quadrant,

$P[X,Y \in C] - P[X,Y \in S]$

$$= \iint_{x,y \in A} f(x,y)\,dx\,dy - \iint_{x,y \in B} f(x,y)\,dx\,dy$$ [From the figure cancelling the common region]

A = shaded region.

B = dotted region.

Now if $(X, Y) \in A$, then,

$$x^2 + y^2 < r^2$$

$$\Rightarrow -\frac{(x^2+y^2)}{2\sigma^2} > -\frac{r^2}{2\sigma^2}$$

$$\Rightarrow f(x,y) > \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \text{ ———— (i)}$$

If $(X, Y) \in B$

$$x^2 + y^2 > r^2$$

$$\Rightarrow f(x,y) < \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \text{ ———— (ii)}$$

From (i) & (ii) we get,

$$\iint_{x,y \in A} f(x,y)\,dx\,dy > \iint_{x,y \in B} f(x,y)\,dx\,dy$$

$$\therefore P[X, Y \in C] > P[X, Y \in S]$$

This inequality similarly holds for the other quadrants.

☆ Problem 7. Show that, —

$$\frac{1}{\sqrt{2\pi}} \int_0^a e^{-x^2/2}\,dx < \frac{1}{2}\sqrt{1 - e^{-\frac{2a^2}{\pi}}}$$

Solution:- $P(X, Y \in S) = 4 \int_0^a \int_0^a \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}(x^2+y^2)}\,dx\,dy$

$$= \frac{4}{2\pi}\left[\int_0^a e^{-\frac{x^2}{2}}\,dx\right]^2$$

$$P(X, Y \in C) = 4 \int_0^{\pi/2} \int_0^r \frac{1}{2\pi} e^{-\frac{1}{2\sigma^2}R^2} \cdot R\,dR\,d\theta$$

$$= \frac{\cancel{4}}{\cancel{4}}\left(1 - e^{-r^2/2}\right)$$

$$\therefore P(X, Y \in C) > P(X, Y \in S)$$

$$\Rightarrow \frac{1}{4}\left(1 - e^{-r^2/2}\right) > \frac{1}{2\pi}\left[\int_0^a e^{-x^2/2}\,dx\right]^2$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_0^a e^{-x^2/2}\,dx < \frac{1}{2}\left(1 - e^{-r^2/2}\right)^{1/2}$$

☆ **Problem 8.** Let X and Y be two r.v.'s with mean zero, variance unity and correlation coefficient ρ, then S.T.

$$E\left[\text{Max}(X^2, Y^2)\right] \le 1 + \sqrt{1-\rho^2}$$ (C.U. 2010) [WBSU'11]

Solution:-

$$\text{Max}(X^2, Y^2) + \text{Min}(X^2, Y^2) = X^2 + Y^2$$
$$\text{Max}(X^2, Y^2) - \text{Min}(X^2, Y^2) = |X^2 - Y^2|$$

$$\text{Max}(X^2, Y^2) = \frac{1}{2}\left[(X^2+Y^2) + |X^2 - Y^2|\right]$$

$$E\left[\text{Max}(X^2, Y^2)\right] = \frac{1}{2}\left[E(X^2) + E(Y^2) + E|(X+Y)(X-Y)|\right]$$
$$= \frac{1}{2}\left[1 + 1 + E|(X+Y)(X-Y)|\right]$$

By C-S inequality,

$$E^2\left[|(X+Y)(X-Y)|\right] \le E(X+Y)^2 E(X-Y)^2$$

i.e. $E^2|X^2 - Y^2| \le \left(2 + 2E(XY)\right)\left(2 - 2E(XY)\right)$

$$\Rightarrow E|X^2 - Y^2| \le 2\sqrt{1-\rho^2}$$

$$\therefore E\left[\max(X^2, Y^2)\right] \le 1 + \frac{1}{2} \cdot 2\sqrt{1-\rho^2}$$
$$\le 1 + \sqrt{1-\rho^2}$$

NOTE:- If one asks to compute mean of a two-dimensional random vector then we obtain a two component vector called "mean vector" $\underset{\sim}{\mu}$ and if one asks to compute the variance then we obtain a 2×2 mtx called Dispersion matrix Σ.

Let $\underset{\sim}{X} = (X, Y)$, $E(\underset{\sim}{X}) = \underset{\sim}{\mu}$

$$\text{Var}(\underset{\sim}{X}) = \Sigma = \begin{pmatrix} \text{var}(X) & \text{cov}(X,Y) \\ \text{Cov}(X,Y) & \text{var}(Y) \end{pmatrix}.$$

Note:- If η_x is a linear function of X and if in addition the conditional distribution is homoscedastic, i.e. the conditional variance $V(Y/X)$, denoted by $\sigma_{Y.X}^2$ is algebrically independent of X, then the correlation coefficient may be given a more concrete meaning.

Theorem:- If the regression of Y on X is linear and variance of Y/X is algebrically independent of X, then

$$V(Y|X) = \sigma_Y^2 (1-\rho^2).$$

Proof:-

$$V(Y|X) = E[(Y-\eta_x)^2/X]$$

$$= E[(Y-\alpha-\beta X)^2/X] \quad [\because \text{Regression is linear}]$$

$$= E[(Y-\mu_Y)-\beta(X-\mu_X)]^2 \quad [\text{since it is independent of } x]$$

$$= E\left[\left\{(Y-\mu_Y) - \rho\frac{\sigma_Y}{\sigma_X}(X-\mu_X)\right\}^2\right]$$

$$= E(Y-\mu_Y)^2 - 2\rho\frac{\sigma_Y}{\sigma_X}E(Y-\mu_Y)(X-\mu_X) + \frac{\rho^2\sigma_Y^2}{\sigma_X^2}E(X-\mu_X)^2$$

$$= \sigma_Y^2 - 2\rho\frac{\sigma_Y}{\sigma_X}\times\rho\sigma_X\sigma_Y + \rho^2\frac{\sigma_Y^2}{\sigma_X^2}\times\sigma_X^2$$

$$= \sigma_Y^2 - 2\rho^2\sigma_Y^2 + \rho^2\sigma_Y^2$$

$$= \sigma_Y^2(1-\rho^2)$$

[Proved]

Note:- In case of Bivariate normal distribution, $\rho^2 = 1 \Rightarrow$ the PDF is undefined. Then for $\rho = 1$, the distn. is called singular Bivariate distribution.

Correlation index and Correlation ratio :

When a regression curve is fitted to the conditional mean, we may denote by Y_x, the estimated value of η_x and by ϵ_x, the residual $Y - Y_x$. The variance of ϵ_x may be taken as an index of the usefullness of the fitted curve. Now in case, where a linear regression equation is fitted on η_x.

Then $V(\epsilon_x) = V(Y - Y_x) = V(Y - \alpha - \beta X)$

$$= V\left(Y - \mu_Y - \rho \frac{\sigma_Y}{\sigma_X}(X - \mu_X)\right)$$

$$= V\left[(Y - \mu_Y)\right] + \rho^2 \frac{\sigma_Y^2}{\sigma_X^2} V(X - \mu_X)$$

$$- 2\rho \frac{\sigma_Y}{\sigma_X} \operatorname{Cov}\left[(X - \mu_X), (Y - \mu_Y)\right]$$

$$= \sigma_Y^2 + \rho^2 \frac{\sigma_Y^2}{\sigma_X^2} \cdot \sigma_X^2 - 2\rho \frac{\sigma_Y}{\sigma_X} \cdot \rho \sigma_Y \sigma_X$$

$$= \sigma_Y^2 (1 - \rho^2)$$

Now, $\operatorname{Cov}(Y, \epsilon_x) = 0$ [from the normal equation]

$\therefore \operatorname{Cov}(Y, Y - Y_x) = 0$

$\therefore \operatorname{Cov}(Y, Y_x) = V(Y)$

&

$$\boxed{V(\epsilon_x) = V(Y) - V(Y_x)}$$

Now,

$$\rho^2 = 1 - \frac{V(\epsilon_x)}{\sigma_Y^2} \quad \text{or,} \quad \rho^2 = 1 - \frac{V(\epsilon_x)}{V(Y)}$$

$$= \frac{V(Y_x)}{V(Y)}.$$

LIMIT THEOREMS

MODES OF CONVERGENCE :—

[A]. Convergence in Distribution or in Law :—

• **Definition:**—

(I) Let $\{F_n(x)\}$ be a sequence of D.F.s. If there exists a D.F. $F(x)$ such that, as $n \to \infty$, $F_n(x) \to F(x)$ at every point x at which $F(x)$ is continuous, we say that $\{F_n(x)\}$ converges in distribution or in law to $F(x)$.

Then we write $F_n(x) \xrightarrow{D} F(x)$

or, $F_n(x) \xrightarrow{w} F(x)$

(II) If $\{X_n\}$ be a sequence of RV's and $\{F_n(x)\}$ is the corresponding sequence of DF's, we say that $\{X_n\}$ converges in distribution or in law to X if $\exists$ a RV X with D.F. $F(x)$ such that, as $n \to \infty$, $F_n(x) \to F(x)$ at every point x at which $F(x)$ is continuous.

Then we write $X_n \xrightarrow{D} X$

or, $X_n \xrightarrow{L} X$

• **Example:**— 1. Let $F_n(x)$ be a sequence of D.F.s, where

$$F_n(x) = \begin{cases} 0 & , x < 0 \\ 1 - \frac{1}{n} & , 0 \le x < n \\ 1 & , x \geq n \end{cases}$$

Does $\{F_n(x)\}$ converge in distribution?

Soln. → Here, $\lim_{n\to\infty} F_n(x) = 0 \quad , x<0$
$\qquad = 1 \quad , 0 \le x < \infty$

[Note that x is a real numbers, $-\infty < x < \infty$ and we ignore the point $x \geq \infty$.]

Now, $F(x)$ is a dF of a RV degenerated at $x=0$.

Hence $\{F_n(x)\}$ converges 'in distribution' or 'weakly' to $F(x)$.

Ex. 2. Let $\{F_n(x)\}$ be a sequence of DF.s where

$$F_n(x) = \begin{cases} 0 & , x \le -n \\ \frac{x+n}{2n} & , -n < x < n \\ 1 & , x \geq n \end{cases}$$

Does $\{F_n(x)\}$ converge in distribution?

Soln. $\lim_{n\to\infty} F_n(x) = \frac{1}{2} \quad , -\infty < x < \infty$
$\qquad = F(x)$, say,

Clearly, $F(x)$ is a D.F., as $F(-\infty) = \frac{1}{2} \neq 0$ and $F(\infty) = \frac{1}{2} \neq 1$

Hence, $\{F_n(x)\}$ does not converge to a D.F.,

$\Leftrightarrow F_n(x) \stackrel{w}{\not\longrightarrow} F(x)$

Remark: – It is important to realise that it is quite possible for a sequence of D.F.s to converge to a function that is not a D.F.

Ex. 3. Let $\{X_n\}$ be a sequence of i.i.d. R.V.'s following $R(0,\theta)$. Does $X_{(n)} = \max_{i=1(1)n}\{X_i\}$ converge in law?

Soln. → DF of $X_{(n)}$ is $F_n(x)$

$$= P[X_{(n)} \le x]$$

$$= P[X_1 \le x, X_2 \le x, \dots, X_n \le x]$$

$$= \{P[X_1 \le x]\}^n \quad \text{as } X_i\text{'s are i.i.d.}$$

$$= \begin{cases} 0, & x \le 0 \\ \left(\int_0^x \frac{1}{\theta}\,dt\right)^n, & 0 < x < \theta \\ 1, & x \ge \theta \end{cases}$$

$$= \begin{cases} 0, & x \le 0 \\ \left(\frac{x}{\theta}\right)^n, & 0 < x < \theta \\ 1, & x \ge \theta \end{cases}$$

Now, $\lim_{n\to\infty} F_n(x) = \begin{cases} 0, & x < \theta \\ 1, & x \ge \theta \end{cases} = F(x)$ (let)

$\left[\text{For } 0 < x < \theta,\ \left(\frac{x}{\theta}\right)^n \to 0 \text{ as } n \to \infty\right]$

$F(x)$ which is the DF of a R.V. X degenerate at $x = \theta$.

Hence, $X_{(n)} \xrightarrow{L} X$, where X is degenerated at $x = \theta$.

Ex. 4. Let $\{X_n\}$ be a sequence of i.i.d. RV's following $R(0,\theta)$. Find the limiting distribution of $Y_n = n(\theta - X_{(n)})$.

Soln. → D.F. of Y_n is $G_n(y) = P[Y_n \le y]$

$$= P[n(\theta - X_{(n)}) \le y]$$

$$= P\left[X_{(n)} \ge \theta - \frac{y}{n}\right]$$

$$= 1 - F_{X_{(n)}}\left(\theta - \frac{y}{n}\right)$$

$$= \begin{cases} 0 & \text{if } y \le 0 \\ 1 - \left(1 - \frac{y}{\theta n}\right)^n & \text{if } 0 < y < n\theta \\ 1 & \text{if } y \ge n\theta \end{cases}$$

, where $\theta > 0$

Now, $\lim\limits_{n\to\infty} G_n(y) = \begin{cases} 0 & , y \le 0 \\ 1 - e^{-y/\theta} & , 0 < y < \infty \end{cases}$

$= G(y)$ which is the D.F. of an exponential distribution with mean θ.

$\left[\lim\limits_{n\to\infty} \left(1 + \frac{-y/\theta}{n}\right)^n = e^{-y/\theta} \right]$

Hence, $Y_n \xrightarrow{L} Y$, where $Y \sim$ Exp distn. with mean θ.

Ex. 5. Let $\{X_n\}$ be a sequence of i.i.d. $N(\mu, \sigma^2)$ R.V.s. Find the limiting distribution of $\overline{X}_n$.

Soln. → D.F. of $\overline{X}_n$ is

$$F_{\overline{X}_n}(x) = P[\overline{X}_n \le x]$$

$$= P\left[\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \le \frac{x-\mu}{\sigma/\sqrt{n}} \right] \text{, where } \overline{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$= \Phi\left(\frac{\sqrt{n}(x-\mu)}{\sigma} \right)$$

Now, $\lim\limits_{n\to\infty} F_{\overline{X}_n}(x) = \begin{cases} \Phi(-\infty) & , \text{ if } x < \mu \\ \Phi(0) & , \text{ if } x = \mu \\ \Phi(\infty) & , \text{ if } x > \mu \end{cases}$

$= \begin{cases} 0 & , \text{ if } x < \mu \\ 1/n & , \text{ if } x = \mu \\ 1 & , \text{ if } x > \mu \end{cases}$

Consider a D.F., $F(x) = \begin{cases} 0 & , x < \mu \\ 1 & , x \ge \mu \end{cases}$

Hence, $\lim\limits_{n\to\infty} F_{\overline{X}_n}(x) = F(x)$

at every point x at which $F(x)$ is continuous.

[To be sure, $\lim\limits_{n\to\infty} F_{\overline{X}_n}(\mu) = \frac{1}{2} \ne 1 = F(\mu)$, but $F(x)$ is discontinuous at $x = \mu$]

Hence, $\overline{X}_n \xrightarrow{L} X$, where X is a R.V. degenerate at $X = \mu$.

Ex. 6. Convergence in distribution doesnot imply the convergence in moments.

ANS: → Counter example: —

Let $F_n(x) = \begin{cases} 0 & , x < 0 \\ 1 - \frac{1}{n} & , 0 \leq x < n \\ 1 & , x \geq n \end{cases}$

be a sequence of DF's, $n \in \mathbb{N}$

Here, $\lim_{n\to\infty} F_n(x) = \begin{cases} 0 & , x < 0 \\ 1 & , x \geq 0 \end{cases}$

$= F(x)$, which is the DF of a R.v. X

degenerate at $x = 0$.

Hence, $F_n(x) \xrightarrow{D} F(x)$

The PMF X_n with D.F. $F_n(x)$ is given by

$P[X_n = 0] = 1 - \frac{1}{n}$

$P[X_n = n] = 1/n$

$\left[\begin{array}{l} P[X_n = 0] = (1 - \frac{1}{n}) - 0 \\ P[X_n = n] = 1 - (1 - \frac{1}{n}) \end{array} \right]$

$E(X_n^r) = 0^r \left(1 - \frac{1}{n}\right) + n^r \cdot \frac{1}{n} = n^{r-1} \longrightarrow \infty$ as $n \to \infty$ for $r > 1$

PMF of X is $P[X = 0] = 1$

$E(X^r) = 0^r \cdot P[X = 0] = 0$.

Clearly, $E(X_n^r) \not\longrightarrow E(X^r)$; $r > 1$

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[B]. Convergence in Probability:

- **Definition:** Let $\{X_n\}$ be a sequence of RV's defined on some probability space $(\Omega, \mathcal{A}, P)$, we say that the sequence $\{X_n\}$ converges in probability to the R.V. X, if for every $\epsilon > 0$,

$$P[|X_n - X| > \epsilon] \to 0 \text{ as } n \to \infty$$

or, $P[|X_n - X| < \epsilon] \to 1$ as $n \to \infty$

We write, $X_n \xrightarrow{P} X$.

- **Example:** 1. Let $\{X_n\}$ be a sequence of R.V.'s with PMF

$P[X_n = 0] = 1 - \frac{1}{n}$,

$P[X_n = 1] = 1/n$.

Does $\{X_n\}$ converge in probability to some R.V. X?

ANS: $\left[\text{Note that, } \begin{matrix} P[X_n = 0] \to 1, \text{ and} \\ P[X_n = 1] \to 0 \end{matrix} \quad \text{as } n \to \infty\right]$

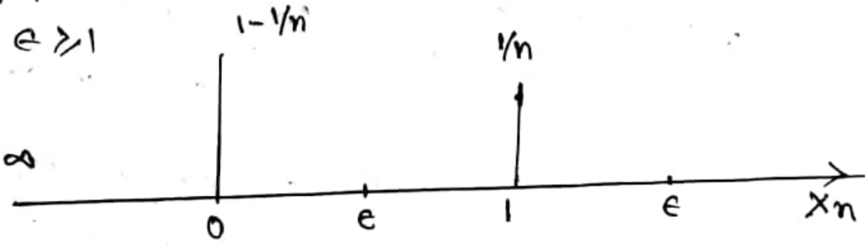
Now, $P[|X_n - 0| > \epsilon]$

$= P[X_n > \epsilon]$

$= \begin{cases} P[X_n = 1] & , 0 < \epsilon < 1 \\ 0 & , \epsilon \geq 1 \end{cases}$

$= \begin{cases} 1/n & , 0 < \epsilon < 1 \\ 0 & , \epsilon \geq 1 \end{cases}$

$\to 0$ as $n \to \infty$



for every $\epsilon > 0$.

Hence, for every $\epsilon > 0$,

$P[|X_n - 0| > \epsilon] \to 0$ as $n \to \infty$

$\Rightarrow P[|X_n - X| > \epsilon] \to 0$ as $n \to \infty$

, where X is a R.V. degenerate at $x = 0$.

Therefore, $X_n \xrightarrow{P} X$.

Ex. 2. Let $\{X_n\}$ be a sequence of i.i.d. $R(0,\theta)$ R.V.'s. Show that $X_{(n)} = \max_{i=1(1)n} \{X_i\}$ converges in probability to 'θ'.

Soln. → For any $\epsilon > 0$,

$$P[|X_{(n)} - \theta| < \epsilon] = P[\theta - \epsilon < X_{(n)} < \theta + \epsilon]$$
$$= F_{X_{(n)}}(\theta+\epsilon) - F_{X_{(n)}}(\theta-\epsilon)$$
$$= \begin{cases} 1 - \left(\frac{\theta-\epsilon}{\theta}\right)^n, & \text{if } 0 < \epsilon < \theta \\ 1 - 0, & \text{if } \epsilon \geq \theta \end{cases}$$
$$\longrightarrow 1 \text{ as } n \to \infty.$$

Hence, $X_{(n)} \xrightarrow{P} \theta$.

$$\left[\because F_{X_{(n)}}(x) = \begin{cases} 0, & x \leq 0 \\ \left(\frac{x}{\theta}\right)^n, & 0 < x < \theta \\ 1, & x \geq \theta \end{cases} \right.$$

As $0 < \epsilon < \theta$, $0 < \frac{\theta - \epsilon}{\theta} < 1$ and $\lim_{n\to\infty} \left(\frac{\theta-\epsilon}{\theta}\right)^n = 0$]

Theorem: → $X_n \xrightarrow{P} X$
$\Rightarrow X_n \xrightarrow{L} X$

Proof: → Let $F_n(\cdot)$ and $F(\cdot)$ be the CDFs of X_n and X, we have,

$$\{\omega : X_n(\omega) \leq x\} = \{\omega : X(\omega) \leq x', X_n(\omega) \leq x\} \cup \{\omega : X(\omega) > x', X_n(\omega) \leq x\}$$
$$\leq \{\omega : X(\omega) \leq x'\} \cup \{\omega : X(\omega) > x', X_n(\omega) \leq x\}$$

Hence, $F_n(x) \leq F(x') + P[X > x', X_n \leq x]$

As $X_n \xrightarrow{P} X$, we have for $x' > x$,

$$0 \leq P[X_n > x', X \leq x] \leq P[|X_n - X| > x' - x] \longrightarrow 0 \text{ as } n \to \infty.$$

Therefore, $\overline{\lim_{n\to\infty}} F_n(x) \leq F(x')$, $x < x'$

Similarly, by interchanging Role X_n and X, we get

$$F(x'') \leq \underline{\lim_{n\to\infty}} F_n(x) \text{ for } x'' < x,$$

Hence for $x'' < x < x'$,

$$F(x'') \leq \underline{\lim_{n\to\infty}} F_n(x) \leq \overline{\lim_{n\to\infty}} F_n(x) \leq F(x')$$

As $F(\cdot)$ has only countable number of discontinuity points, we choose x to be a point of continuity of F, and letting $x' \downarrow x$ and $x'' \uparrow x$, we have,

$$F(x) \leq \underline{\lim}_{n\to\infty} F_n(x) \leq \overline{\lim}_{n\to\infty} F_n(x) \leq F(x)$$

$$\Rightarrow \underline{\lim}_{n\to\infty} F_n(x) = \overline{\lim}_{n\to\infty} F_n(x) = F(x)$$

$$\Rightarrow \lim_{n\to\infty} F_n(x) = F(x), \text{ at all continuity points of } F(x).$$

Con. $X_n \xrightarrow{L} X \not\Rightarrow X_n \xrightarrow{P} X$

['the covergence in law' does not imply 'the convergence in Probability'. The converse of the theorem is not true.]

Proof:-

Counter Example:— Let $\{X_n\}$ be a sequence of identically distributed RV's and let (X, X_n) has the following distribution;

X_n \ X	0	1	TOTAL
0	0	1/2	1/2
1	1/2	0	1/2
TOTAL	1/2	1/2	1

Clearly, $X_n \xrightarrow{L} X$, but,

$$P\{|X_n - X| > \tfrac{1}{2}\} = P\{|X_n - X| = 1\}$$
$$= P\{X_n = 0, X = 1\} + P\{X_n = 1, X = 0\}$$
$$= 1 \not\to 0.$$

Hence, $X_n \overset{P}{\not\to} X$, but $X_n \xrightarrow{L} X$.

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Ex. 3. Let $X \sim N(0,1)$,

Define, $X_n = \begin{cases} X, & \text{if } n = 2m-1 \\ -X, & \text{if } n = 2m \end{cases}$, where $m \in \mathbb{N}$

Show that $\rightarrow$ $X_n \xrightarrow{L} X$, but $X_n \overset{P}{\not\rightarrow} X$

Soln. $\rightarrow$ As $X \sim N(0,1)$ and X is symmetric about '0', X and $-X$ have the same distribution.

Hence, $X_n \sim N(0,1) \ \forall\ n \in \mathbb{N}$

and $X_n \xrightarrow{D} X$,

For any $\epsilon > 0$, when $n = 2m$,

$$P[|X_n - X| < \epsilon] = P\left[|X| < \frac{\epsilon}{2}\right]$$
$$= 2\Phi\left(\frac{\epsilon}{2}\right) - 1$$
$$\not\rightarrow 1 \text{ as } n \rightarrow \infty.$$

Hence, $X_n \overset{P}{\not\rightarrow} X$.

C. rth mean Convergence :

- **Definition** : $\rightarrow$ Let $\{X_n\}$ be a sequence of RV's such that $E|X_n|^r < \infty$, for $r > 0$, We say that $\{X_n\}$ converges in the rth mean to a R.V. X if $E|X|^r < \infty$ and $E|X_n - X|^r \rightarrow 0$ as $n \rightarrow \infty$.

- **Example** :- 1. Let $\{X_n\}$ be a sequence of R.V.'s such that $P[X_n = 0] = 1 - \frac{1}{n}$, $P[X_n = 1] = \frac{1}{n}$, $n \in \mathbb{N}$. Show that $\{X_n\}$ converges in 2nd mean to some R.V. X.

Soln. $\rightarrow$ $E|X_n - 0|^2 = E(X_n^2)$

$$= 0^2 \cdot \left(1 - \frac{1}{n}\right) + 1^2 \cdot \frac{1}{n} = \frac{1}{n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence, $E|X_n - X|^2 \rightarrow 0$ as $n \rightarrow \infty$

where X is a R.V. degenerate at $x = 0$.

$\Rightarrow \{X_n\}$ converges in 2nd mean to X.

Theorem: → If $E|X_n - X|^r \to 0$ as $n \to \infty$ then $X_n \xrightarrow{P} X$.

Proof: → For any $\epsilon > 0$,

$$P[|X_n - X| > \epsilon] = P[|X_n - X|^r > \epsilon^r] \quad , r > 0$$

$$< \frac{E|X_n - X|^r}{\epsilon^r}, \text{ by Markov's inequality.}$$

For $\epsilon > 0$,

$$0 \le P[|X_n - X| > \epsilon] < \frac{E|X_n - X|^r}{\epsilon^r} \to 0 \text{ as } n \to \infty.$$

$$\Rightarrow P[|X_n - X| > \epsilon] \to 0 \text{ as } n \to \infty.$$

$$\Rightarrow X_n \xrightarrow{P} X$$

Remark: —

1) The rth mean convergence
$\Rightarrow$ the convergence in probability.
$\Rightarrow$ the convergence in law.

2) The convergence in rth mean
$\Rightarrow$ the convergence in probability but the converse is not true.

Counter Example: — Let $\{X_n\}$ be a sequence of R.V.'s such that,

$$P[X_n = 0] = 1 - \frac{1}{n}$$
$$P[X_n = n] = \frac{1}{n}, \quad n \in \mathbb{N}$$

For $\epsilon > 0$, $P[|X_n - 0| > \epsilon] = P[X_n > \epsilon] = P[X_n = n]$

$$= \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Hence, $X_n \xrightarrow{P} 0$

$\Leftrightarrow X_n \to X$, where X is a RV degenerate at $x = 0$.

But, $E|X_n - X|^2 = E|X_n - 0|^2 = E(X_n^2)$

$$= 0^2 \cdot \left(1 - \frac{1}{n}\right) + n^2 \cdot \frac{1}{n}$$

$$= n \to \infty \quad \text{as } n \to \infty$$

i.e. $\not\to 0$.

Hence, $X_n \xrightarrow{P} X$ but $\{X_n\}$ does not converge in 2nd mean to X.

WEAK LAW OF LARGE NUMBERS (WLLN) :—

Let $\{X_n\}$ be a sequence of R.V.s. Let $S_n = \sum_{k=1}^{n} X_k$, $n \in \mathbb{N}$. We say that $\{X_n\}$ obeys the weak law of large numbers (WLLN) with respect to the sequence $\{b_n\}$, $b_n > 0$ and $b_n \uparrow \infty$, if $\exists$ a sequence $\{a_n\}$ of real numbers such that

$$\frac{S_n - a_n}{b_n} \xrightarrow{P} 0 .$$

Here, a_n is called the centering constant and b_n is called the norming constant.

Chebyshev's WLLN :—

Let $\{X_n\}$ be a sequence of independent R.V.'s such that $E(X_n) = \mu_n$ and $Var(X_n) = \sigma_n^2 < \infty$ then

$$\lim_{n\to\infty} \left(\frac{1}{n^2} \sum_{k=1}^{n} \sigma_k^2 \right) = 0$$

$$\Rightarrow \bar{X}_n - \bar{\mu}_n \xrightarrow{P} 0$$

where $\bar{X}_n = \frac{1}{n} \sum_{k=1}^{n} X_k$, $\bar{\mu}_n = \frac{1}{n} \sum_{k=1}^{n} \mu_k$

$\boxed{\text{OR}}$ $\{X_n\}$ obeys WLLN.

Soln. $\Rightarrow$ Proof:-

$$E(\bar{X}_n) = \frac{1}{n} \sum_{k=1}^{n} E(X_k) = \frac{1}{n} \sum_{k=1}^{n} \mu_k = \bar{\mu}_n .$$

$$Var(\bar{X}_n) = Var\left(\frac{1}{n} \sum_{k=1}^{n} X_k \right) = \frac{1}{n^2} \sum_{k=1}^{n} Var(X_k) = \frac{1}{n^2} \sum_{k=1}^{n} \sigma_k^2 .$$

as X_k's are independent.

For every $\epsilon > 0$,

$$0 \le P\left[|\bar{X}_n - \bar{\mu}_n| > \epsilon \right] < \frac{Var(\bar{X}_n)}{\epsilon^2} = \frac{\frac{1}{n^2} \sum_{k=1}^{n} \sigma_k^2}{\epsilon^2} \to 0$$

as $n \to \infty$

provided $\frac{1}{n^2} \sum_{k=1}^{n} \sigma_k^2 \to 0$ as $n \to \infty$.

Hence, $\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} \sigma_k^2 = 0$

$$\Rightarrow P\left[|\bar{X}_n - \bar{\mu}_n| > \epsilon \right] \to 0 \text{ as } n \to \infty \text{ for every } \epsilon > 0$$

$$\Rightarrow \bar{X}_n - \bar{\mu}_n \xrightarrow{P} 0$$

$$\Leftrightarrow \frac{S_n}{n} - \frac{\sum_{k=1}^{n} \mu_k}{n} \xrightarrow{P} 0$$

$$\Rightarrow \frac{S_n - a_n}{b_n} \xrightarrow{P} 0$$ where $a_n = \sum_{k=1}^{n} \mu_k$, $b_n = n > 0$ and $b_n \uparrow \infty$,

$\Leftrightarrow$ $\{X_n\}$ obeys WLLN w.r.t. the sequence $\{b_n\}$.

Example : 1. Examine whether the WLLN holds for the following sequences $\{X_n\}$ of independent R.V. s :

i) $P[X_n = -2^n] = 2^{-2n-1} = P[X_n = 2^n]$

$P[X_n = 0] = 1 - 2^{-2n}$.

ii) $P[X_n = -\frac{1}{n}] = \frac{1}{2} = P[X_n = \frac{1}{n}]$

Soln. → i) $\mu_k = E(X_k) = (-2^k) \cdot 2^{-2k-1} + (2^k) \cdot 2^{-2k-1} + 0 \cdot (1 - 2^{-2k})$

$= 0$

and $Var(X_k) = \sigma_k^2 = E(X_k^2)$

$= (-2^k)^2 \cdot 2^{-2k-1} + (2^k)^2 \cdot 2^{-2k-1} + 0$

$= 1, \; k \in \mathbb{N}$

Now, $\frac{1}{n^2} \sum_{k=1}^{n} \sigma_k^2 = \frac{1}{n^2} \sum_{k=1}^{n} 1 = \frac{1}{n} \to 0$ as $n \to \infty$.

Hence, $\{X_n\}$ obeys WLLN, by Chebyshev's WLLN.

ii) Here $\mu_k = 0$ and $\sigma_k^2 = V(X_k) = E(X_k^2) = \frac{1}{k^2}, \; n \in \mathbb{N}$

Now, $\frac{1}{n^2} \sum_{k=1}^{n} \sigma_k^2 = \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{k^2} < \frac{c}{n^2}$

[$\sum_{k=1}^{n} \frac{1}{k^2}$ is a convergent p-series,

$\Rightarrow \sum_{k=1}^{n} \frac{1}{k^2} < \sum_{k=1}^{\infty} \frac{1}{k^2} = c$, a finite quantity]

Hence, $\{X_n\}$ obeys WLLN, by Chebyshev's WLLN.

Ex. 2. Let $P[X_n = -n^p] = \frac{1}{2} = P[X_n = n^p]$

Show that WLLN holds for the sequence $\{X_n\}$ of independent R.V.'s if $p < \frac{1}{2}$.

Soln. → Here, $\mu_k = E(X_k) = 0$,

$\sigma_k^2 = V(X_k) = E(X_k^2) = (-k^p)^2 \cdot \frac{1}{2} + (k^p)^2 \cdot \frac{1}{2}$

$= k^{2p}, \; k \in \mathbb{N}$

Now, $\frac{1}{n^2} \sum_{k=1}^{n} \sigma_k^2 = \frac{1}{n^2} \sum_{k=1}^{n} k^{2p} < \frac{1}{n^2} \int_1^n x^{2p} \, dx$

$= \frac{n^{2p+1} - 1}{n^2 (2p+1)}$

Now,
$$0 \le \frac{1}{n^2}\sum_{k=1}^{n}\sigma_k^2 < \frac{n^{2p+1}-1}{n^2(2p+1)} < \frac{n^{2p-1}}{2p+1} \longrightarrow 0 \text{ as } n\to\infty$$
if $2p-1<0$

if $p<\frac{1}{2}$

$\Rightarrow$ if $p<\frac{1}{2}$, $\frac{1}{n^2}\sum_{k=1}^{n}\sigma_k^2 \longrightarrow 0$ as $n\to\infty$.

Hence, $\{X_n\}$ obeys WLLN if $p<\frac{1}{2}$.

Ex. 3. Decide whether WLLN holds for the sequence $\{X_n\}$ of independent R.V.'s :

i) $P[X_n = \pm 2^{-n}] = \frac{1}{2}$

$\Leftrightarrow P[X_n = -2^{-n}] = \frac{1}{2} = P[X_n = 2^{-n}]$

ii) $P[X_n = n] = P[X_n = -n] = \frac{1}{2\sqrt{n}}$

$P[X_n = 0] = 1 - \frac{1}{\sqrt{n}}$.

Soln. $\rightarrow$ i) Here, $\mu_k = 0$,

and $\sigma_k^2 = V(X_k) = E(X_k^2) = 2^{-2k}$, $k \in \mathbb{N}$

Now,
$$\frac{1}{n^2}\sum_{k=1}^{n}\sigma_k^2 = \frac{1}{n^2}\sum_{k=1}^{n}2^{-2k} = \frac{1}{n^2}\cdot\frac{\frac{1}{4}\left\{1-\left(\frac{1}{4}\right)^n\right\}}{1-\frac{1}{4}}$$
$$= \frac{1}{3}\cdot\frac{1}{n^2}\left\{1-\left(\frac{1}{4}\right)^n\right\} < \frac{1}{3n^2} \longrightarrow 0 \text{ as } n\to\infty.$$

Hence, $\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\sigma_k^2 = 0$

$\Rightarrow \{X_n\}$ obeys WLLN by Chebyshev's WLLN.

ii) $\mu_k = 0$

and
$$\sigma_k^2 = E(X_k^2) = (-k)^2\cdot\frac{1}{2\sqrt{k}} + (k)^2\cdot\frac{1}{2\sqrt{k}} + 0$$
$$= k^{3/2}.$$

Now, $\frac{1}{n^2}\sum_{k=1}^{n}\sigma_k^2 = \frac{1}{n^2}\sum_{k=1}^{n}k^{3/2}$

For large n, $\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{3/2} \simeq \int_0^1 x^{3/2}dx = \frac{2}{5}$

$$\Rightarrow \sum k^{3/2} \simeq \frac{2n^{5/2}}{5}$$

For large n, $\frac{1}{n^2}\sum_{k=1}^{n}k^{3/2} \simeq \frac{1}{n^2}\cdot\frac{2n^{5/2}}{5} = \frac{2}{5}\sqrt{n} \not\to 0$, $\longrightarrow \infty$ as $n\to\infty$.

As $\frac{1}{n^2}\sum_{k=1}^{n}\sigma_k^2 \not\longrightarrow 0$ as $n\to\infty$.

We cannot draw any conclusion by Chebyshev's WLLN, whether WLLN holds or not.

Markov's WLLN: — Let $\{X_n\}$ be a sequence of RV's such that $E(X_n) = \mu_n$ and $V(X_n) = \sigma_n^2 < \infty$, $n \in \mathbb{N}$, then

$\bar{X}_n - \bar{\mu}_n \xrightarrow{P} 0$, provided $\frac{1}{n^2}\left(\text{Var}\left(\sum_{k=1}^{n} X_k\right)\right) \longrightarrow 0$ as $n \to \infty$.

i.e. $\{X_n\}$ obeys WLLN, provided $\frac{1}{n^2}\text{Var}\left(\sum_{k=1}^{n} X_k\right) \longrightarrow 0$ as $n \to \infty$.

Proof: — For any $\epsilon > 0$,

$$0 \le P\left[|\bar{X}_n - \bar{\mu}_n| > \epsilon\right] \le \frac{E|\bar{X}_n - \bar{\mu}_n|^2}{\epsilon^2}$$

$$= \frac{V(\bar{X}_n)}{\epsilon^2} = \frac{\frac{1}{n}V\left(\sum_{k=1}^{n} X_k\right)}{\epsilon^2} \to 0 \text{ as } n \to \infty$$

provided $\frac{1}{n^2}\text{Var}\left(\sum_{k=1}^{n} X_k\right) \to 0$ as $n \to \infty$.

Remark: — Chebyshev's WLLN is a particular case of Markov's WLLN.

Ex. 4. Let $\{X_n\}$ be a sequence of RV's with common finite variance σ^2. Suppose $\rho_{x_i x_j} \le 0$ $\forall\, i \ne j$. Prove that WLLN holds for $\{X_n\}$.

Soln. → Note that,

$$\frac{1}{n^2}\text{Var}\left(\sum_{k=1}^{n} X_k\right) = \frac{1}{n^2}\left\{\sum_{k=1}^{n} V(X_k) + \sum\sum_{k \ne j} \text{Cov}(X_k, X_j)\right\}$$

$$\le \frac{1}{n^2}\sum_{k=1}^{n} V(X_k) \text{ as } \rho_{X_k, X_j} \le 0,\ k \ne j$$

Hence, $$0 \le \frac{V\left(\sum_{k=1}^{n} X_k\right)}{n^2} \le \frac{1}{n^2}\sum_{k=1}^{n}\sigma^2 = \frac{\sigma^2}{n} \longrightarrow 0 \text{ as } n \to \infty$$

Hence, for every $\epsilon > 0$,

$$0 \le P\left[|\bar{X}_n - \bar{\mu}_n| > \epsilon\right] < \frac{V(\bar{X}_n)}{\epsilon^2} = \frac{\frac{1}{n^2}V\left(\sum_{k=1}^{n} X_k\right)}{\epsilon^2} \longrightarrow 0 \text{ as } n \to \infty$$

$\Rightarrow P\left[|\bar{X}_n - \bar{\mu}_n| > \epsilon\right] \longrightarrow 0$ as $n \to \infty$

$\Rightarrow \bar{X}_n - \bar{\mu}_n \xrightarrow{P} 0$

$\Rightarrow \{X_n\}$ obeys WLLN.

Bernoulli's Law of Large numbers: — Let 'f' be the number of occurrences of an event A in 'n' independent trials and 'p' be the probability of occurences of an event A in each trial, then for every $\epsilon > 0$; $P\left[\left|\frac{f}{n} - p\right| < \epsilon\right] \to 1$, as $n \to \infty$.

In otherwords, the sequence of relative frequency of the event A, $\left\{\frac{f}{n}\right\}$ converges in probability to 'p'.

Proof: ⇒ Consider the occurrence of the event A as a success. Then f, the number of successes in n independent Bernoulli trials, follows $Bin(n, p)$.

Hence, $E(f) = np$, $V(f) = np(1-p)$.

$$\Rightarrow E\left(\frac{f}{n}\right) = p, \quad V\left(\frac{f}{n}\right) = \frac{p(1-p)}{n}.$$

For every $\epsilon > 0$,

$$P\left[\left|\frac{f}{n} - p\right| \geq \epsilon\right] \leq \frac{E\left(\frac{f}{n} - p\right)^2}{\epsilon^2} = \frac{V\left(\frac{f}{n}\right)}{\epsilon^2} = \frac{p(1-p)}{n\epsilon^2}$$

Hence, for every $\epsilon > 0$,

$$0 \leq P\left[\left|\frac{f}{n} - p\right| \geq \epsilon\right] \leq \frac{p(1-p)}{n\epsilon^2} \leq \frac{1}{4n\epsilon^2} \to 0 \text{ as } n \to \infty$$

$\Rightarrow P\left[\left|\frac{f}{n} - p\right| < \epsilon\right] \to 1$ as $n \to \infty$ $\left[\because \frac{p + (1-p)}{2} \geq \sqrt{p(1-p)}\right.$ by AM ≥ GM,

$\Rightarrow \frac{1}{2} \geq \sqrt{P(1-P)}$

$\left.\Rightarrow \frac{1}{4} \geq P(1-P)\right]$

$\Rightarrow \frac{f}{n} \xrightarrow{P} p$.

Remark: — We have $P\left[\left|\frac{f}{n} - p\right| < \epsilon\right] \to 1$, as $n \to \infty$, for every $\epsilon > 0$,

⇒ for large n, the values of $\frac{f}{n}$ are very close to 'p' with probability $\simeq 1$.

⇒ for large n, $p \simeq \frac{f}{n}$, which is nothing but statistical definition of probability.

Therefore, the Bernoulli's Law of Large number is the foundation of statistical definition of probability.

Theorem:— A necessary & sufficient condition for WLLN.

Let $\{X_n\}$ be a sequence of RV's, Define,

$$Y_n = \frac{1}{n}\sum_{k=1}^{n} X_k$$

A necessary & sufficient condition for the sequence $\{X_n\}$ to satisfy the WLLN is that

$$E\left(\frac{Y_n^2}{1+Y_n^2}\right) \longrightarrow 0 \text{ as } n\to\infty.$$

Proof:→ Note that, $\frac{y^2}{1+y^2} > \frac{\epsilon^2}{1+\epsilon^2}$

$$\Rightarrow y^2 > \epsilon^2 \Rightarrow |y| > \epsilon$$

For $\epsilon > 0$,

$$P[|Y_n| > \epsilon] = P\left[\frac{Y_n^2}{1+Y_n^2} > \frac{\epsilon^2}{1+\epsilon^2}\right]$$

$$< \frac{E\left(\frac{Y_n^2}{1+Y_n^2}\right)}{\frac{\epsilon^2}{1+\epsilon^2}}, \text{ by Markov's inequality.}$$

If $E\left(\frac{Y_n^2}{1+Y_n^2}\right) \to 0$ as $n\to\infty$,

$P[|Y_n| > \epsilon] \to 0$ as $n\to\infty$

$\Rightarrow Y_n \xrightarrow{P} 0$, i.e. $\overline{X}_n \xrightarrow{P} 0$

i.e. $\{X_n\}$ obeys WLLN.

Now,
$$E\left(\frac{Y_n^2}{1+Y_n^2}\right) = \int_{-\infty}^{\infty} \frac{y^2}{1+y^2}\, dF_n(y)$$

$$= \int_{|y|\le\epsilon} \frac{y^2}{1+y^2}\, dF_n(y) + \int_{|y|>\epsilon} \frac{y^2}{1+y^2}\, dF_n(y)$$

$$< \epsilon^2 \cdot \int_{|y|\le\epsilon} dF_n(y) + \int_{|y|>\epsilon} dF_n(y)$$

$$\left[\frac{y^2}{1+y^2} < 1 \;\forall\, y \text{ and } \frac{y^2}{1+y^2} < y^2 \le \epsilon^2 \text{ for } |y|\le\epsilon\right]$$

Hence, $E\left(\frac{Y_n^2}{1+Y_n^2}\right) < \epsilon^2 \cdot 1 + P[|Y_n| > \epsilon]$

If $\{X_n\}$ obeys WLLN, then for every $\epsilon > 0$,

$P[|Y_n| > \epsilon] \to 0$, as $n\to\infty$

$\Rightarrow E\left(\frac{Y_n^2}{1+Y_n^2}\right) < \epsilon^2$, for large n.

$\Rightarrow E\left(\frac{Y_n^2}{1+Y_n^2}\right) \to 0$, as $n\to\infty$.

Ex. 5. Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of i.i.d. $C(\mu, 1)$ R.V.'s. Show that $\{X_n\}$ does not obey WLLN, i.e. $\overline{X}_n \overset{P}{\not\to} \mu$.

Soln. →

[**Result:-** If $X_1, X_2, \ldots, X_n$ are independent $C(\mu_i, \sigma_i)$ then $\sum_{i=1}^{n} X_i \sim C\left(\sum_{i=1}^{n}\mu_i, \sum_{i=1}^{n}\sigma_i\right)$.

In particular,

(a) if $X_1, \ldots, X_n$ are i.i.d. $C(\mu, \sigma)$, then $\sum_{i=1}^{n} X_i \sim C(n\mu, n\sigma)$.

$$\Rightarrow \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim C(\mu, \sigma).$$

(b) if $X_1, \ldots, X_n$ are i.i.d. $C(0,1)$ then $\overline{X} \sim C(0,1)$.]

Here $X_1, \ldots, X_n, \ldots$ be i.i.d. $C(\mu, 1)$

$$\Rightarrow \overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \sim C(\mu, 1)$$

Define, $Y_n = \overline{X}_n - \mu$

Clearly, $Y_n \sim C(0,1)$.

Note that,
$$E\left(\frac{Y_n^2}{1+Y_n^2}\right) = \int_{-\infty}^{\infty} \frac{y^2}{1+y^2} \cdot \frac{1}{\pi(1+y^2)}\, dy$$

$$= \frac{2}{\pi}\int_0^{\infty} \frac{y^2}{(1+y^2)^2}\, dy \qquad y^2 = z \Rightarrow 2y\,dy = dz$$

$$= \frac{\cancel{2}}{\pi \times \cancel{2}}\int_0^{\infty} \frac{\sqrt{z}\, dz}{(1+z)^2}$$

$$= \frac{1}{\pi}\int_0^{\infty} \frac{z^{3/2-1}}{(1+z)^{3/2+1/2}}\, dz$$

$$= \frac{1}{\pi} \cdot B\left(\frac{3}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{\pi} \cdot \frac{\Gamma(3/2)\Gamma(\frac{1}{2})}{\Gamma(2)}$$

$$= \frac{1}{\pi} \cdot \frac{\frac{1}{2}\left[\Gamma(\frac{1}{2})\right]^2}{\Gamma(2)}$$

$$= \frac{1}{2} \not\to 0 \text{ as } n \to \infty$$

Hence $\{X_n\}$ does not obey WLLN,

$$\left[\Leftrightarrow Y_n \overset{P}{\not\to} 0 \Leftrightarrow \overline{X}_n \overset{P}{\not\to} \mu\right]$$

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CENTRAL LIMIT THEOREM :

For a sequence $\{X_n\}$ of i.i.d. R.V.'s, we have $\overline{X}_n \xrightarrow{P} \mu$, provided μ exists. Hence, WLLN holds for i.i.d. sequence $\{X_n\}$ of RV's, provided mean $E(X_i) = \mu$ exists. But this gives no idea as to how the distribution of $\overline{X}_n$ can be approximated in large samples. Here, we consider the condition under which the distribution of $S_n = \sum_{k=1}^{n} X_k$ or $\overline{X}_n$ converges to normal distribution.

- **Definition:** — If the distribution of a R.V. Y_n depends on a parameter n, and if there exists two quantities a_n and b_n (which may or may not depend on n) such that

$$\lim_{n\to\infty} P\left[\frac{Y_n - a_n}{b_n} \le y\right] = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \cdot e^{-t^2/2} dt, \text{ for all } y \in \mathbb{R}$$

then we say that Y_n is asymptotically normally distributed with mean a_n and variance b_n^2. We also say that $\frac{Y_n - a_n}{b_n}$ follows the central limit theorum or normal convergence.

Notation: — $\frac{Y_n - a_n}{b_n} \xrightarrow{L} X \sim N(0,1)$.

or, $Y_n \overset{a}{\sim} N(a_n, b_n^2)$

$\Leftrightarrow Y_n \sim AN(a_n, b_n^2)$.

Lindeberg - Lavy CLT [i.i.d. case]:

Let $\{X_n\}$ be a sequence of i.i.d. R.V.s with common mean μ and finite variance σ^2. Let, $S_n = \sum_{k=1}^{n} X_k$, $n \in \mathbb{N}$

Then, for any $x \in \mathbb{R}$,

$$P\left[\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \le x\right] \longrightarrow \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \cdot e^{-t^2/2} dt, \text{ as } n\to\infty.$$

$$\Leftrightarrow P\left[\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \le x\right] \longrightarrow \Phi(x), \text{ as } n \to \infty$$

Remark: — For a sequence $\{X_n\}$ of i.i.d. RV's, by Lindeberg - Lavy CLT,

we have, $\frac{S_n - E(S_n)}{\sqrt{V(S_n)}} \xrightarrow{L} X \sim N(0,1)$

or, $\frac{\overline{X}_n - E(\overline{X}_n)}{\sqrt{V(\overline{X}_n)}} \xrightarrow{L} X \sim N(0,1)$

Here, $Y_n = S_n$ or $\overline{X}_n$, $a_n = E(Y_n)$, $b_n = \sqrt{V(Y_n)}$.

Ex.1. Let Y_n be the numbers of independent trials necessary to get rth success, where 'p' is the probability of success in each trial. Find $\lim_{n\to\infty} P\left[\frac{pY_n - n}{\sqrt{nq}} \le y\right]$.

Soln. → Let $\{X_n\}$ be a sequence of i.i.d. R.V.'s where each X_n denotes the number of trials required to get the 1st success in a sequence of independent Bernoulli trials. Clearly, $X_n \sim Geo(p)$, $n \in \mathbb{N}$

Then, $Y_n = \sum_{k=1}^{n} X_k$ = Numbers of trials required to get the rth success.

Note that, $E(X_n) = \frac{1}{p}$, $V(X_n) = \frac{q}{p^2}$, a finite quantity, $n \in \mathbb{N}$

Applying Lindeberg - Lavy for the sequence $\{X_n\}$ of i.i.d. R.V.'s with finite variance,

$$P\left[\frac{Y_n - E(Y_n)}{\sqrt{V(Y_n)}} \le y\right] \longrightarrow \Phi(y), \text{ as } n\to\infty,\ y \in \mathbb{R}$$

$$\Rightarrow P\left[\frac{Y_n - \frac{n}{p}}{\sqrt{\frac{nq}{p^2}}} \le y\right] \longrightarrow \Phi(y), \text{ as } n\to\infty$$

$$\Rightarrow \lim_{n\to\infty} P\left[\frac{pY_n - n}{\sqrt{nq}} \le y\right] = \Phi(y)$$

Ex.2. Using CLT, evaluate the limit $\lim_{n\to\infty} \sum_{k=0}^{n} e^{-n} \cdot \frac{n^k}{k!}$

Soln. → Let $\{X_n\}$ be a sequence of i.i.d. Poisson variables with mean $\lambda = 1$.

Note that, $E(X_n) = V(X_n) = 1$, finite, $n \in \mathbb{N}$.

Also, $S_n = \sum_{k=1}^{n} X_k \sim P(n)$.

By Lindeberg-Lavy CLT,

$$\lim_{n\to\infty} P\left[\frac{S_n - E(S_n)}{\sqrt{V(S_n)}} \le x\right] = \Phi(x) \quad \forall\ x \in \mathbb{R}$$

$$\Rightarrow \lim_{n\to\infty} P\left[\frac{S_n - n}{\sqrt{n}} \le x\right] = \Phi(x), \quad \forall\ x \in \mathbb{R}$$

For, $x = 0$,

$$\lim_{n\to\infty} P[S_n \le n] = \Phi(0).$$

$$\Rightarrow \lim_{n\to\infty} \sum_{k=0}^{n} P[S_n = k] = \frac{1}{2}$$

$$\Rightarrow \lim_{n\to\infty} \sum_{k=0}^{n} e^{-n} \cdot \frac{n^k}{k!} = \frac{1}{2}.$$

Ex. 3. Using CLT, evaluate $\lim_{n\to\infty} \int_0^n \frac{e^{-x} \cdot x^{n-1}}{\Gamma(n)} dx$.

Soln. → Let $\{X_n\}$ be a sequence of i.i.d. exponential R.V.'s with mean $= 1$,
then $S_n = \sum_{k=1}^{n} X_k \sim \text{Gamma}(n)$.

~~Ex. 4. Let $\{X_n\}$ be a sequence of i.i.d. R.V.'s with Gamma$(\frac{1}{2}$~~

Ex. 4. Evaluate: $\lim_{n\to\infty} \frac{1}{2^{n/2}\Gamma(\frac{n}{2})} \cdot \int_{n+\sqrt{2n}}^{\infty} e^{-t/2}\, t^{n/2-1}\, dt$.

Soln. → Let $\{X_n\}$ be a sequence of i.i.d. R.V.'s following Gamma$\left(\frac{1}{2}, \frac{1}{2}\right)$. $f(x) = \begin{cases} \dfrac{e^{-x/2} \cdot x^{\frac{1}{2}-1}}{2^{\frac{1}{2}}\,\Gamma(\frac{1}{2})} & , \text{if } x > 0 \\ 0 & , \text{ow} \end{cases}$

Then, $S_n = \sum_{k=1}^{n} X_k \sim \text{Gamma}\left(\frac{1}{2}, \frac{n}{2}\right)$

$E(S_n) = n$, $V(S_n) = 2n$. By Lindeberg-Lévy CLT, —

$$\lim_{n\to\infty} P\left[\frac{S_n - E(S_n)}{\sqrt{V(S_n)}} \le x\right] = \Phi(x)$$

$$\Rightarrow \lim_{n\to\infty} P\left[\frac{S_n - n}{\sqrt{2n}} \le 1\right] = \Phi(1)$$

$$\Rightarrow \lim_{n\to\infty} P\left[S_n \le n + \sqrt{2n}\right] = \Phi(1)$$

$$\Rightarrow \lim_{n\to\infty} P\left[S_n > n + \sqrt{2n}\right] = 1 - \Phi(1)$$

$$\Rightarrow \lim_{n\to\infty} \left\{\frac{1}{2^{n/2}\Gamma(\frac{1}{2})} \int_{n+\sqrt{2n}}^{\infty} e^{-t/2}\, t^{n/2-1}\, dt\right\} = \Phi(-1).$$

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Ex. 5. Let $\{X_n\}$ be a sequence of i.i.d. R.V.'s with common mean μ and finite variance σ^2. Show that $(S_n - n\mu) \overset{P}{\not\longrightarrow} 0$ but $a_n(S_n - n\mu) \xrightarrow{P} 0$, provided $a_n\sqrt{n} \to 0$ as $n \to \infty$.

Soln. → Here $\{X_n\}$ is a sequence of i.i.d. RV's with finite variance. Then by Lindeberg Liavy CLT,

$$\frac{S_n - E(S_n)}{\sqrt{V(S_n)}} \overset{a}{\sim} N(0,1), \text{ as } n \to \infty$$

$$\Rightarrow \frac{S_n - n\mu}{\sqrt{n\sigma^2}} \overset{a}{\sim} N(0,1), \text{ as } n \to \infty$$

For every $\epsilon > 0$,

$$P[|S_n - n\mu| < \epsilon] = P\left[\left|\frac{S_n - n\mu}{\sigma\sqrt{n}}\right| < \frac{\epsilon}{\sigma\sqrt{n}}\right]$$

$$\simeq P\left[|Z| < \frac{\epsilon}{\sigma\sqrt{n}}\right], \text{ for large } n, \text{ where } Z \sim N(0,1)$$

$$= 2\Phi\left(\frac{\epsilon}{\sigma\sqrt{n}}\right) - 1 \longrightarrow 2\Phi(0) - 1 = 0, \text{ as } n \to \infty$$

Hence, $P[|S_n - n\mu| < \epsilon] \not\longrightarrow 1$ as $n \to \infty$.

$\Rightarrow (S_n - n\mu) \overset{P}{\not\longrightarrow} 0$

For every $\epsilon > 0$,

$$P[|a_n(S_n - n\mu)| < \epsilon] = P\left[\left|\frac{S_n - n\mu}{\sigma\sqrt{n}}\right| < \frac{\epsilon}{(a_n\cdot\sqrt{n})\sigma}\right]$$

$$\simeq P\left[|Z| < \frac{\epsilon}{\sigma(a_n\cdot\sqrt{n})}\right]$$

$$= 2\Phi\left(\frac{\epsilon}{\sigma\cdot(a_n\cdot\sqrt{n})}\right) - 1.$$

$$\longrightarrow 2\Phi(+\infty) - 1 = 1 \text{ as } n \to \infty$$

provided, $a_n\sqrt{n} \to 0$ as $n \to \infty$.

Remark: → $\{X_n\}$ obeys WLLN, w.r.t. $\{b_n\}$, where $b_n > 0$, $b_n \uparrow \infty$, if $\frac{S_n - a_n}{b_n} \xrightarrow{P} 0$.

Note that, $S_n - n\mu \overset{P}{\not\longrightarrow} 0$, with the choice of $b_n = \frac{1}{a_n}$

Here $a_n = n\mu$, $b_n = 1$, but where $a_n\sqrt{n} \to 0$ as $n \to \infty$,

$\Rightarrow \frac{\sqrt{n}}{b_n} \to 0$ as $n \to \infty$.

$\therefore a_n(S_n - n\mu) \xrightarrow{P} 0$.

In particular, we may take,

$b_n = n$, then $\frac{1}{n}(S_n - n\mu) = (\bar{X}_n - \mu) \xrightarrow{P} 0$

De-Moivre and Laplace Limit theorem :

If $\{X_n\}$ be a sequence of i.i.d. Bernoulli R.V.'s with probability 'p' of success in each trial, then

$$\frac{S_n - np}{\sqrt{np(1-p)}} \overset{a}{\sim} N(0,1) \text{ as } n \to \infty, \text{ where } S_n = \sum_{k=1}^{n} X_k .$$

Proof:→ Let $Z_n = \dfrac{S_n - np}{\sqrt{np(1-p)}}$

MGF of Z_n is $M_{Z_n}(t) = E[e^{t \cdot Z_n}] = E\left[e^{t\left(\frac{S_n - np}{\sqrt{npq}}\right)}\right]$

$$= e^{-\frac{tnp}{\sqrt{npq}}} E\left[e^{\frac{t}{\sqrt{npq}} \cdot S_n}\right]$$

$$= e^{-\frac{tnp}{\sqrt{npq}}} M_{S_n}\left(\frac{t}{\sqrt{npq}}\right)$$

$$= e^{-\frac{tnp}{\sqrt{npq}}} \left\{q + pe^{\frac{t}{\sqrt{npq}}}\right\}^n$$

$$= \left\{q \cdot e^{-\frac{tp}{\sqrt{npq}}} + p \cdot e^{\frac{tq}{\sqrt{npq}}}\right\}^n$$

$$= \left[q\left\{1 - \frac{tp}{\sqrt{npq}} + \frac{1}{2}\frac{t^2p^2}{npq} + O\left(n^{-3/2}\right)\right\} + p\left\{1 + \frac{tq}{\sqrt{npq}} + \frac{1}{2}\frac{t^2q^2}{npq} + O\left(n^{-3/2}\right)\right\}\right]^n$$

$$= \left[(p+q) + \frac{1}{2}\frac{t^2pq}{npq}(p+q) + O\left(n^{-3/2}\right)\right]^n$$

[where, $O(n^{-3/2})$ represents terms involving $n^{-3/2}$ and higher powers of 'n' in the denominator.]

$$= \left[1 + \frac{t^2}{2n} + O\left(n^{-3/2}\right)\right]^n$$

Now, $\ln M_{Z_n}(t) = n\log_e\left\{1 + \frac{t^2}{2n} + O\left(n^{-3/2}\right)\right\}$

$$= n\left\{\frac{t^2}{2n} + O\left(n^{-3/2}\right)\right\}$$

$$= \frac{t^2}{2} + O\left(n^{-1/2}\right) \longrightarrow \frac{t^2}{2} \text{ as } n \to \infty.$$

Hence, $\lim_{n\to\infty} M_{Z_n}(t) = e^{t^2/2} = M(t)$, which is the MGF of $N(0,1)$.

By uniqueness of MGF,

$$Z_n = \frac{S_n - np}{\sqrt{np(1-p)}} \overset{a}{\sim} N(0,1) \text{ as } n \to \infty.$$

Normal Approximation to Poisson :

[CLT for a sequence of i.i.d. Poisson R.V.'s]

If $\{X_n\}$ be a sequence of i.i.d. RV's each following $P(\lambda)$ distribution, then, $\frac{S_n - n\lambda}{\sqrt{n\lambda}} \xrightarrow{L} Z \sim N(0,1)$

Proof: → Let, $Z_n = \frac{S_n - n\lambda}{\sqrt{n\lambda}}$.

$$M_{Z_n}(t) = E\left[e^{tZ_n}\right]$$

$$= e^{-t\sqrt{n\lambda}}\, E\left[e^{\frac{t}{\sqrt{n\lambda}} \cdot S_n}\right]$$

$$= e^{-t\sqrt{n\lambda}} \cdot \left[M_{S_n}\left(\frac{t}{\sqrt{n\lambda}}\right)\right]$$

$$= e^{-t\sqrt{n\lambda}} \cdot e^{n\lambda\left(e^{t/\sqrt{n\lambda}} - 1\right)} \quad \text{as } S_n \sim P(n\lambda)$$

$$= e^{-t\sqrt{n\lambda} + n\lambda\left\{\frac{t}{\sqrt{n\lambda}} + \frac{1}{2}\cdot\frac{t^2}{n\lambda} + o(n^{-3/2})\right\}}$$

$$= e^{\frac{1}{2}t^2 + o(n^{-1/2})} \longrightarrow e^{t^2/2} \text{ as } n \to \infty$$

Hence, $\lim_{n\to\infty} M_{Z_n}(t) = e^{t^2/2} = M(t)$, which is the mgf of $N(0,1)$ distribution.

By uniqueness of MGF, $Z_n = \frac{S_n - n\lambda}{\sqrt{n\lambda}} \xrightarrow{L} Z \sim N(0,1)$

[CLT for a sequence of i.i.d. Gamma $(\theta, 1)$, i.e. $\text{Exp}(\theta)$ RV's.]

If $\{X_n\}$ be a sequence of i.i.d. Exponential RV's with mean θ, then $\frac{S_n - n\theta}{\sqrt{n\theta^2}} \overset{a}{\sim} N(0,1)$ as $n \to \infty$.

Proof: → Let, $Z_n = \frac{S_n - n\theta}{\sqrt{n\theta^2}}$

$$M_{Z_n}(t) = e^{-t\sqrt{n}}\, E\left(e^{t/\theta\sqrt{n} \cdot S_n}\right)$$

$$= e^{-t\sqrt{n}} \cdot M_{S_n}\left(\frac{t}{\theta\sqrt{n}}\right)$$

$$= e^{-t\sqrt{n}}\left(1 - \theta \cdot \frac{t}{\theta\sqrt{n}}\right)^{-n}$$

$$= e^{-t\sqrt{n}}\left(1 - \frac{t}{\sqrt{n}}\right)^{-n}$$

[Here, $S_n = \sum_{k=1}^{n} X_k \sim \text{Gamma}(\theta, n)$;
$M_{S_n}(t) = (1-\theta t)^{-n}$, if $t < \frac{1}{\theta}$]

Now, $\ln M_{Z_n}(t) = -t\sqrt{n} - n\ln\left(1-\frac{t}{\sqrt{n}}\right)$

$$= -t\sqrt{n} - n\left\{\frac{t}{\sqrt{n}} + \frac{t^2}{2n} + 0\left(n^{-3/2}\right)\right\}$$

$$= -\cancel{t\sqrt{n}} + \cancel{t\sqrt{n}} + \frac{t^2}{2} + 0\left(n^{-3/2}\right)$$

$$= \frac{t^2}{2} + 0\left(n^{-3/2}\right) \longrightarrow t^2/2 \text{ as } n\to\infty.$$

Hence, $\lim_{n\to\infty} M_{Z_n}(t) = e^{t^2/2} = M_Z(t)$, which is the MGF of $Z \sim N(0,1)$.

By uniqueness of MGF; $Z_n = \frac{S_n - n\lambda}{\sqrt{n\lambda}} \xrightarrow{L} Z \sim N(0,1)$

Relationship between CLT and WLLN :-

CLT is a generalization of WLLN for a sequence of i.i.d. R.V.'s with finite variance.

Let $\{X_n\}$ be a sequence of i.i.d. RV's with common mean μ and variance $\sigma^2 (<\infty)$.

By Lindeberg - Levy CLT,

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{L} Z \sim N(0,1)$$

Now, $P\left[|\overline{X}_n - \mu| < \epsilon\right] = P\left[\left|\frac{\overline{X}_n-\mu}{\sigma/\sqrt{n}}\right| < \frac{\epsilon\sqrt{n}}{\sigma}\right]$

$$\simeq P\left[|Z| < \frac{\epsilon\sqrt{n}}{\sigma}\right], \text{ for large } n.$$

$$= 2\Phi\left(\frac{\epsilon\sqrt{n}}{\sigma}\right) - 1 \longrightarrow 2\Phi(+\infty) - 1 = 1.$$

as $n\to\infty$, for every $\epsilon > 0$,

Hence, $\overline{X}_n \xrightarrow{P} \mu$

$\Leftrightarrow \{X_n\}$ obeys WLLN.

Hence, CLT is stronger than WLLN for a sequence of i.i.d. R.V.'s with finite variance.

But for the sequence $\{X_n\}$ of independent R.V.'s, CLT may hold but the WLLN may not hold.

Further WLLN and CLT :

A. Kinchin's WLLN : $\rightarrow$ If $\{X_n\}$ is a sequence i.i.d. R.V.'s, then $\mu = E(X_i)$ exists.

$\Rightarrow \overline{X}_n \xrightarrow{P} \mu$, i.e. $\{X_n\}$ obeys WLLN.

Ex. 1) Check whether WLLN holds for the following sequence of i.i.d. R.V.'s (a) with PDF, $f(x) = \begin{cases} \frac{1+\delta}{x^{2+\delta}}, & x>1 \\ 0 & , \text{ow} \end{cases}$

, where $\delta > 0$.

(b) with PMF $P[X = (-1)^{k-1} \cdot k] = c \cdot \frac{1}{k^3}$, $k=1,2,\ldots$

(c) with PMF $P[X = 2^{k-2\log k}] = 2^{-k}$, $k=1,2,\ldots$

Soln. $\rightarrow$

(a) $$E|X_1| = \int_1^\infty |x| \cdot \frac{1+\delta}{x^{2+\delta}}\,dx$$

$$= (1+\delta)\int_1^\infty \frac{1}{x^{1+\delta}}\,dx$$

$$= (1+\delta)\lim_{a\to\infty}\int_1^a x^{-1-\delta}\,dx$$

$$= (1+\delta)\lim_{a\to\infty}\left[\frac{x^{-\delta}}{-\delta}\right]_1^a$$

$$= (1+\delta)\lim_{a\to\infty}\left[\frac{1-a^{-\delta}}{\delta}\right]$$

$$= \frac{1+\delta}{\delta}, \quad \delta>0$$

Hence $E(X)$ exists and $E(X_1) = \frac{1+\delta}{\delta}$.

By Kinchin's WLLN,

$$\overline{X}_n \xrightarrow{P} \frac{1+\delta}{\delta} = E(X_1)$$

i.e. $\{X_n\}$ obeys WLLN.

(c) $E|X_1| = E(X_1)$ as X_1 is non-negative.

$$= \sum_{k=1}^{\infty} 2^{k-2\log k} \cdot 2^{-k} = \sum_{k=1}^{\infty} 2^{-2\log k}$$

$$= \sum_{k=1}^{\infty} \frac{1}{4^{\log k}}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k^{\log 4}}$$, which is p-series with $p = \log_e 4 > 1$

[$z = 2^{2\log k}$
$\Rightarrow \log z = 2\log k \cdot \log 2$
$= 2\log 2 \cdot \log k$
$= \log 2^2 \cdot \log k$
$= \log 4 \cdot \log k$
$\Rightarrow z = 4^{\log k}$
$= k^{\log 4}$]

Hence, $E(X_1) = \mu$ exists.

By Khinchin's WLLN, $\bar{X}_n \xrightarrow{P} E(X)$,

i.e. $\{X_n\}$ obeys WLLN.

B. Liapounov's CLT : $\rightarrow$

Let $\{X_n\}$ be a sequence of independent R.V.'s with $E(X_i) = \mu_i$, $V(X_i) = \sigma_i^2$ and,

$$\rho_i^3 = E|X_i - \mu_i|^3 < \infty, \ \forall i$$

Define, $\rho^3 = \sum_{i=1}^{n} \rho_i^3$ and $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$.

If $\lim_{n\to\infty} \frac{\rho}{\sigma} = 0$, then

$$\frac{S_n - \sum_{i=1}^{n} \mu_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}} \overset{a}{\sim} N(0,1)$$

[$\because$ Let $\{X_n\}$ be a sequence of independent R.V.'s with $E(X_i) = \mu_i$, $V(X_i) = \sigma_i^2$, and

$$\rho_i^{2+\delta} = E|X_i - \mu_i|^{2+\delta} < \infty,$$

for some $\delta > 0$, $\forall i$, Define, $\rho^{2+\delta} = \sum_{i=1}^{n} \rho_i^{2+\delta}$,

and $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$;

If $\lim_{n\to\infty} \frac{\rho}{\sigma} = 0$, then

$$\frac{S_n - \sum_{i=1}^{n} \mu_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}} \overset{a}{\sim} N(0,1).$$]

Ex. 1. Examine if the CLT and the WLLN holds for the sequence of independent R.V.'s $\{X_n\}$, where $P[X_n = -n] = P[X_n = n] = \frac{1}{2\sqrt{n}}$,

$P[X_n = 0] = 1 - \frac{1}{\sqrt{n}}$.

Soln. →

$$\mu = E(X_K) = 0$$

$$\sigma_X^2 = K^{3/2}$$

$$\rho_K^3 = E|X_K - \mu_K|^3 = K^{5/2},$$

Then, $\sigma^2 = \sum_{K=1}^{n} K^{3/2}$

$$\rho^3 = \sum_{K=1}^{n} \rho_K^3 = \sum_{K=1}^{n} K^{5/2}$$

$$\therefore \lim_{n\to\infty} \left(\frac{\sum_{K=1}^{n} \sigma_K^2}{n^2} \right) = \lim_{n\to\infty} \frac{\sum_{K=1}^{n} K^{3/2}}{n^2}$$

$$= \lim_{n\to\infty} \left(\frac{\int_0^n x^{3/2} dx}{n^2} \right)$$

$$= \lim_{n\to\infty} \frac{2n^{5/2}}{5n^2}$$

$$= \lim_{n\to\infty} \frac{2}{5}\sqrt{n}$$

$$= +\infty$$

Chebyshev's condition does not hold.

⇒ Can not draw any conclusion whether WLLN holds or not.

Now, $\lim_{n\to\infty} \frac{\rho}{\sigma} = \lim_{n\to\infty} \frac{\left(\sum_{K=1}^{n} K^{5/2}\right)^{1/3}}{\left(\sum_{K=1}^{n} K^{3/2}\right)^{1/2}}$

$$= \lim_{n\to\infty} \frac{\left(\int_0^n x^{5/2} dx\right)^{1/3}}{\left(\int_0^n x^{3/2} dx\right)^{1/2}}$$

$$= c. \lim_{n\to\infty} \frac{n^{7/6}}{n^{5/4}}$$

$$= c. \lim_{n\to\infty} \frac{1}{\sqrt[12]{n}} = 0$$

By Liapounov's CLT,

$$\frac{S_n - \sum_{k=1}^{n} \mu_k}{\left(\sum_{k=1}^{n} \sigma_k\right)^{1/2}} \overset{a}{\sim} N(0,1)$$

$$\Rightarrow S_n \overset{a}{\sim} N\left(0, \frac{2}{5} n^{3/2}\right).$$

Ex.2. $\{X_n\}$ is a sequence of independent R.V.'s. Determine if they obey WLLN and/or CLT

i) $P[X_k = 2^k] = \frac{1}{2} = P[X_k = -2^k]$

ii) $P[X_k = 2^{-k}] = \frac{1}{2} = P[X_k = -2^{-k}]$

[C]. Theorem:→ Let $X_n \xrightarrow{P} X$, $Y_n \xrightarrow{P} Y$. Then

i) $aX_n \xrightarrow{P} aX$

ii) $X_n + Y_n \xrightarrow{P} X + Y$

iii) $X_n Y_n \xrightarrow{P} X.Y$

iv) $\frac{X_n}{Y_n} \xrightarrow{P} \frac{X}{Y}$ if $P[Y_n = 0] = 0 = P[Y = 0]$

v) $f(X_n) \xrightarrow{P} f(X)$, if $f(\cdot)$ is continuous on $\mathbb{R}$.

Proof:→ For $\epsilon > 0$

$$P[|X_n - X| < \epsilon] \to 1$$

and $P[|Y_n - Y| < \epsilon] \to 1$ as $n \to \infty$

i) For every $\epsilon > 0$,

$$P[|aX_n - aX| < \epsilon] = P\left[|X_n - X| < \frac{\epsilon}{|a|}\right] \to 1 \text{ as } n \to \infty.$$

$\therefore aX_n \xrightarrow{P} aX$.

ii) Note that,

$$\left\{|X_n - X| < \frac{\epsilon}{2}\right\} \cap \left\{|Y_n - Y| < \frac{\epsilon}{2}\right\} \subseteq \left\{|X_n + Y_n - X - Y| < \epsilon\right\}$$

$$\Rightarrow \left\{|X_n - X| > \frac{\epsilon}{2}\right\} \cup \left\{|Y_n - Y| > \frac{\epsilon}{2}\right\} \supseteq \left\{|X_n + Y_n - X - Y| > \epsilon\right\}$$

$$\Rightarrow P[|X_n + Y_n - X - Y| > \epsilon] \le P\left[|X_n - X| > \frac{\epsilon}{2}\right] P\left[|Y_n - Y| > \frac{\epsilon}{2}\right]$$

$$\to 0 \text{ as } n \to \infty$$

Hence, $X_n + Y_n \xrightarrow{P} X + Y$.

v) $f(\cdot)$ is continuous.

$\Rightarrow |f(x_n) - f(x)| < \delta$

whenever $|X_n - X| < \epsilon$, for sufficiently large n.

Hence, $P[|X_n - X| < \epsilon] \leq P[|f(X_n) - f(X)| < \delta]$

$\Rightarrow P[|f(x_n) - f(x)| > \delta] \geq P[|X_n - X| < \epsilon]$

$\longrightarrow 1$ as $n \to \infty$,

for $\epsilon > 0$, $\delta > 0$.

$\Rightarrow P[|f(x_n) - f(x)| < \delta] \to 1$ as $n \to \infty$.

$\Rightarrow f(X_n) \xrightarrow{P} f(X)$

Remark:—

i) $X_n \xrightarrow{P} X$

$\Rightarrow \frac{1}{X_n} \xrightarrow{P} \frac{1}{X}$, provided $P[X_n = 0] = P[X = 0] = 0$.

ii) $X_n \xrightarrow{P} X \Rightarrow X_n^2 \xrightarrow{P} X^2$

iii) $X_n Y_n = \dfrac{(X_n + Y_n)^2 - (X_n - Y_n)^2}{4}$

$\xrightarrow{P} \dfrac{(X+Y)^2 - (X-Y)^2}{4} = XY$

Theorem:— [Slutsky Theorem]

Let $\{X_n, Y_n\}$ be a sequence of pair of R.V.'s and $c \in \mathbb{R}$. Then

i) $X_n \xrightarrow{L} X$, $Y_n \xrightarrow{P} c$

$\Rightarrow X_n + Y_n \xrightarrow{L} X + c$

ii) $X_n \xrightarrow{L} X$, $Y_n \xrightarrow{P} c$

$\Rightarrow X_n Y_n \xrightarrow{L} cX$.

* Important Example:—

If $\{X_n\}$ is a sequence of i.i.d. $C(0,1)$ R.V.'s, then it can be shown that $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \sim C(0,1)$

i.e. $\overline{X}_n$ is not asymptotically normal.

Hence, $\overline{X}_n$ does not follow CLT, where $X_n \overset{i.i.d.}{\sim} C(0,1)$; $n \in \mathbb{N}$

×——×

BASIC PROBABILITY

Q.1. In a town of 221 people, a person tells a rumour to a second person, who in turn repeats it to a third person and so on. At each step the receipt of the rumour is chosen at random from 220 people available. Find the probability that the rumour will be told 3 times without being repeated to any person.

Solution:- Total no of possible outcomes $= 220^3$

Total no. of favourable outcomes $= (220)\times(219)\times(217)$.

$= (220)_3$ · since it is performed without repetition.

$\therefore$ Required probability $= \dfrac{(220)_3}{220^3}$

$\simeq 0.98$.

Q.2. A light bulb is considered bad if it lasts less than 700 hrs of use. For a particular brand of light bulbs, suppose that the probability of a randomly chosen bulb turning out bad is 0.10. Also the probability that a randomly chosen bulb lasts at least 600 hrs (of use) is 0.98. Suppose T denotes the lifetime of a randomly chosen bulb.

(a) What's the prob. that T is at least 700 hours?

(b) What's the prob. that a randomly chosen bulb is not good but T is at least 600 hrs?

Solution:- T : lifetime of a randomly chosen bulb.

$P(T < 700) = 0.10$

$P(T \geq 600) = 0.98$

(a) $P(T \geq 700) = 1 - P(T < 700) = 0.90$

(b) $P(600 \leq T < 700) = P(T < 700) - P(T < 600)$

$= 0.10 - 0.02$

$= 0.08$

Q.3. A balanced die is rolled 3 times. If it is known that the face appeared at least once what is the probability that it appeared exactly once?

Solution:- A : the face appeared exactly once

B : the face appeared at least once

$P(A|B) = P(A \cap B)/P(B)$

$= \dfrac{3\cdot\left(\frac{5^2}{6^3}\right)}{1-\left(\frac{5}{6}\right)^3}$

$= \dfrac{75}{91}$.

Q.4. Suppose there are 3 chests each having 2 drawers. The first chest has a gold coin in each drawer, the second chest has a gold coin in one drawer and a silver coin in the other drawer, and the third chest has a silver coin in each drawer. A chest is chosen at random and a drawer is opened. If that drawer contains a gold coin what is the prob. that the second drawer has a silver coin?

Solution:-

$$\frac{1/3}{2/3} = 1/2$$

Q.5. Suppose a factory has two machines A and B that make 60% and 40% of the total production, respectively. 3% of A's output and 2% of B's output is defective.

(a) If a product is selected at random from the output what's the probability that it was produced by machine B?

(b) If a product is selected at random from the output what's the probability that it is defective?

(c) Given a defective product, what is the probability that it was produced by machine B?

Solution:- $P(A) = 0.6$

$P(B) = 0.4$

$P(A|D) = 0.03$ $P(D|A) = 0.03$

$P(B|D) = 0.02$ $P(D|B) = 0.02$

(a) $P(B) = 0.4$

(b) $P(D) = P(A)\,P(D|A) + P(B)\,P(D|B) = 0.6 \times 0.03 + 0.4 \times 0.02$

$= 0.026$

(c) $$P(B|D) = \frac{P(B)\,P(D|B)}{P(D)}$$

$= 0.308$

Q.6. Student population of a college consists of 70% men and 30% women. Assume that 20% of men and 5% of women smoke here. What's the probability that a student observed smoking is a man?

Solution:- $P(\text{choosing a man}) = P(M) = 0.7$

$P(\text{choosing a woman}) = P(W) = 0.3$

$P(\text{smoking} \mid M) = P(S|M) = 0.2$

$P(S|F) = 0.05$

$$P(M|S) = \frac{P(M \cap S)}{P(S)}$$

$$= \frac{P(M)\,P(S|M)}{P(M)\,P(S|M) + P(W)\,P(S|W)} \quad ; \text{By Baye's Theorum.}$$

$$= \frac{0.2 \times 0.7}{0.2 \times 0.7 + 0.05 \times 0.3}$$

$$= 0.903$$

Q.7. Suppose that there is a test for cancer with the property that 90% of those with cancer react positively whereas 5% of those without cancer also react positively. Assuming that 1% of the patients in a hospital have cancer. What is the probability that a patient selected at random who reacts positively to this test actually has cancer?

Solution:- $P(\text{Having cancer}) = P(C) = 0.01$

$P(E) = P(\text{the result is positive})$

$P(E|C^c) = 0.05$

$P(E|C) = 0.9$

$$P(C|E) = \frac{P(C)\,P(E|C)}{P(C)\,P(E|C) + P(C^c)\,P(E|C^c)}$$

$$= \frac{0.01 \times 0.9}{0.01 \times 0.9 + 0.99 \times 0.05}$$

$$= \frac{2}{13}.$$

Q. 8. A student is taking a multiple choice exam in which each question has 4 possible answer, exactly one of which is correct. If the student knows the answer she selects the correct answer. Otherwise s/he selects one answer at random from 4 alternatives (there is no negative marking). Suppose that the student knows the answer to 70% of the questions.

(a) What's the probability that on a given question the student gets the correct answer?

(b) If the student gets the correct answer to a question, what's the probability that s/he knows the answer?

Solution:- B : A student answers correctly

A_1 : A student knows the answer

A_2 : A student guesses the answer

$$P(A_1) = 0.7 \qquad P(A_2) = 0.3$$

$$P(B|A_1) = 1, \qquad P(B|A_2) = 1/4.$$

(a) $$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2)$$

$$= 7/10 \times 1 + 3/10 \times 1/4$$

$$= \frac{31}{40} = 0.775$$

(b) $$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2)}$$

$$= \frac{7/10 \times 1}{P(B)}$$

$$= \frac{7/10}{31/40}$$

$$= \frac{28}{31}$$

$$= 0.903$$

DISCRETE DISTRIBUTIONS

Q.1. Let $X \sim \text{Binomial}(10, p)$. Finally the probability distribution of (i) $Y = \frac{X}{2} + 5$, (ii) $Z = 100 - X^2$.

Solution:-

(i) $P(Y = y) = P\left(\frac{X}{2} + 5 = y\right)$

$P\left(Y = \frac{x}{2} + 5\right)$

$= P$

$$= P(X = 2y - 10)$$

$$= \binom{10}{2y-10} p^{2y-10} (1-p)^{20-2y} \; ; \quad y = 5, 5.5, 6, \ldots, 10.$$

(ii) $P(Z = z) = P(100 - X^2 = z)$

$$= P(X = \sqrt{100 - z})$$

$$= \binom{100}{\sqrt{100-z}} p^{\sqrt{100-z}} (1-p)^{10 - \sqrt{100-z}} \; ; \quad z = 0, 19, \ldots, 99, 100.$$

Q.2. An urn contains 8 red marbles and 2 blue marbles. Draw 2 marbles at random from the urn. Let X = number of red marbles in the sample. Find the prob. distn. of X if the sampling is done using (i) SRSWR (ii) SRSWOR.

Solution:- X : No. of Marbles in the sample.
X takes values $0, 1, 2, \ldots$

(i) SRSWR:-

$$f(x) = \begin{cases} \frac{4}{100}, & x = 0 \\ \frac{32}{100}, & x = 1 \\ \frac{64}{100}, & x = 2 \end{cases}$$

$\therefore X \sim \text{Bin}\left(2, \frac{8}{10}\right)$

$$F(x) = \begin{cases} \left(\frac{2}{10}\right)^2, & x < 0 \leq x < 1 \\ \left(\frac{2}{10}\right)^2\left(\frac{8}{10}\right) + \left(\frac{2}{10}\right)^2, & 1 \leq x < 2 \\ \left(\frac{8}{10}\right)^2 + \left(\frac{1}{5}\right), & 2 \leq x < 3 \end{cases}$$

(ii) SRSWOR:-

$$F(x) = \begin{cases} \left(\frac{2}{10}\right)\left(\frac{1}{9}\right), & 0 \leq x < 1 \\ \left(\frac{2}{90}\right) + \left(\frac{2}{10} \times \frac{8}{9}\right), & 1 \leq x < 2 \\ \left(\frac{2}{90} + \frac{16}{90}\right) + \left(\frac{8}{10} \times \frac{7}{9}\right), & 2 \leq x < 3 \end{cases}$$

$$f(x) = \begin{cases} \frac{2}{90}, & x = 0 \\ \frac{32}{90}, & x = 1 \\ \frac{56}{90}, & x = 2 \end{cases}$$

$\therefore X \sim \text{Hypergeometric}(20, 8, 2)$.

Q.3. Let $f(x)=\begin{cases} c\,2^{x} & \text{if } x=1,2,\ldots 100. \\ 0 & \text{otherwise.} \end{cases}$

Find the value of c for which $f(x)$ is a p.m.f.

<u>Solution:-</u> $\sum_{1}^{100} f(x) = 1 \Rightarrow c.\ \frac{2(2^{100}-1)}{2-1} = 1$

$\Rightarrow c = \frac{1}{2^{100}-2}.$

Q.4. Let m be a positive integer and $X \sim$ Geometric(p). Define

$$Y = \begin{cases} X & \text{if } X < m \\ m & \text{if } X \geq m \end{cases}$$

Find the pmf of Y.

<u>Solution:-</u> $f(x) = \begin{cases} p(1-p)^{x} & , x=0,1,2,\ldots \\ 0 & , \text{ow} \end{cases}$

$$P(Y=y) = \begin{cases} P(X=y), & y=0,1,2,\ldots,m-1 \\ P(X \geq m), & y=m \end{cases}$$

$$= \begin{cases} p(1-p)^{y}, & y=0,1,2,\ldots,m-1 \\ (1-p)^{m}, & y=m \end{cases}$$

Q.5. A student is taking a multiple choice exam in which each question has 4 possible answers, exactly one of which is correct. If the student knows the answer s/he selects the correct answer. Otherwise s/he selects one answer at random from the 4 possible answers. (There is no negative marking). Suppose that there are 100 questions carrying 1 mark each in the exam and the student knows answer to 70% of the questions. Let X denotes the total marks that the student will get in this exam. What's the probability distribution of X?

<u>Solution:-</u> X : Total marks obtained by a student in the exam.

$$P(X=x) = \begin{cases} \binom{30}{x-70}\left(\frac{1}{4}\right)^{x-70}\left(\frac{3}{4}\right)^{100-x} & ; x=70,\ldots,100. \\ 0 & ; \text{ow} \end{cases}$$

Q.6. If the number of accidents occuring on a highway each day is a Poisson RV with parameter $\lambda=3$. What's the prob. that no accidents occur there today?

<u>Solution:-</u> X : # of accidents occuring on a highway each day

$X \sim$ Poi(3)

$f(x) = \begin{cases} e^{-3}.\frac{3^{x}}{x!} & , x=0,1,2,\ldots \\ 0 & , \text{ow} \end{cases}$ $\therefore P(X=0) = e^{-3}$ = Ans.

Q.7. If $X \sim$ Geometric(p), with $p = 0.7$. Find (a) $P(X>3)$. (b) $P(2 < X \le 4$ or $X > 5)$

Solution:-

$$P(X>x) = (1-p)^{x+1} \; ; \; P(X=x) = p(1-p)^{x} ; \; p = 0.7$$

(a) $P(X>3) = (1-0.7)^{3+1}$

(b) $P(2 < X \le 4) + P(X>5)$

$= P(X=3) + P(X=4) + P(X>5)$

$= 0.025$.

Q.8. Roll a balanced die n times independently, $n \ge 1$.

(a) What's the joint probability distribution of (X,Y), where X is the number of times 6 dots show up, Y is the number of times 5 dots show up?

(b) What is the prob. distn of the number of times either 6 dots or 5 dots show up?

Solution:-

(a) X : # times 6 dots show up $\quad X \sim Bin(n, \frac{1}{6})$

Y : # times 5 dots show up. $\quad Y \sim Bin(n, \frac{1}{6})$

$$P(X=x, Y=y) = \begin{cases} \binom{n}{x}\binom{n-x}{y}\left(\frac{1}{6}\right)^{x+y}\left(\frac{4}{6}\right)^{n-x-y} & ; x,y = 0(1)n \\ & \quad 0 \le x+y \le n \\ 0 & ; \text{ow} \end{cases}$$

(b) Z : # times either 6 dots or 5 dots show up.

$Z \sim Bin(n, \frac{1}{3})$

$$f(z) = \begin{cases} \binom{n}{z}\left(\frac{1}{3}\right)^{z}\left(\frac{2}{3}\right)^{n-z} & , z = 0, 1, \dots, n \\ 0 & , \text{ow} \end{cases}$$

Q.9. Let X and Y be independent r.v.s with common variance σ^2. Find the $cov(U,V)$, where $U = X+Y$, $V = X-Y$.

Solution:-

$$Cov(X+Y, X-Y) = E(X^2-Y^2) - [E(X+Y)E(X-Y)]$$

$$= E(X^2) - E(Y^2) - [E(X^2) - E(Y^2)]$$

$$= V(X) - V(Y)$$

$$= \sigma^2 - \sigma^2 = 0.$$

Q.10. Let $X \sim Pois(\lambda_1)$, $Y \sim Pois(\lambda_2)$ independently. Define $Z = X+Y$. Find $Cov(X,Z)$ and $Cor(X,Z)$?

Solution:-

$$Z \sim Poi(\lambda_1+\lambda_2)$$

$$Cov(X,Z) = cov(X,X) + cov(X,Y) = V(X) + Cov(X,Y)$$
$$= V(X)$$
$$= \lambda_1.$$

$$Cor(X,Z) = \frac{V(X)}{\sqrt{V(X)V(Z)}} = \sqrt{\frac{\lambda_1}{\lambda_1+\lambda_2}}.$$

Q.11. Let X be uniformly distributed on $\{0,1,\ldots,N\}$, where N is an integer and $N \geq 1$. Find $E(X)$, $V(X)$?

Solution:-

$$f(x) = \begin{cases} \frac{1}{N+1}, & x = \{0,1,\ldots,N\}, N \geq 1 \\ 0, & ow \end{cases}$$

$$E(X) = \sum_{x=0}^{N} x \cdot \frac{1}{N+1} = \frac{N(N+1)}{2(N+1)} = \frac{N}{2}.$$

$$V(X) = \sum_{x=0}^{N} x^2 \cdot \frac{1}{N+1} - \frac{N^2}{4} = \frac{N^2+2N}{12}.$$

Q.12. Let $X_1,\ldots,X_n$ be i.i.d. with mean μ and variance σ^2. Define the sample mean and variance as

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$$

(a) Show that $E(\bar{X}) = \mu$, $V(\bar{X}) = \sigma^2/n$.

(b) Show that $\sum_{i=1}^{n}(X_i-\bar{X})^2 = \sum_{i=1}^{n}(X_i-\mu)^2 - n(\bar{X}-\mu)^2$

(c) Show that $E(S^2) = \sigma^2$.

Solution:-

(a) $$E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\cdot n\mu = \mu$$

$$V(\bar{X}) = \frac{1}{n^2} V\left(\sum_1^n X_i\right) = \frac{1}{n^2}\cdot n\sigma^2 = \frac{\sigma^2}{n}.$$

(b) $$\sum_{i=1}^{n}(X_i-\bar{X})^2 = \sum_{i=1}^{n}(X_i-\mu-\bar{X}+\mu)^2$$
$$= \sum_1^n (X_i-\mu)^2 - 2\sum_1^n (X_i-\mu)(\bar{X}-\mu) + \sum_1^n(\bar{X}-\mu)$$
$$= \sum_1^n (X_i-\mu)^2 - 2n(\bar{X}-\mu)^2 + n(\bar{X}-\mu)^2$$

(c) $$E\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2\right) = \frac{1}{n}E\left[\sum_1^n (X_i-\mu)^2 - n(\bar{X}-\mu)^2\right]$$
$$= \frac{1}{n}\left\{\sum_1^n V(X_i) - nV(\bar{X})\right\}$$
$$= \frac{1}{n}\left\{n\sigma^2 - \frac{n\sigma^2}{n}\right\} = \frac{n-1}{n}\sigma^2$$

$$\therefore E\left(\frac{1}{n-1}\sum(X_i-\bar{X})^2\right) = \sigma^2$$

Q.13. Suppose $X \sim$ Hypergeometric (m, p, n)

(a) Find $Var(X)$ arguing as follows:

$$Y_i = \begin{cases} 1 & \text{if the } i\text{th trial is a success} \\ 0 & \text{Otherwise} \end{cases}$$

$X = \sum_{i=1}^{n} Y_i$. Then

$$\begin{aligned} Cov(Y_i, Y_j) &= E(Y_i Y_j) - E(Y_i)E(Y_j) \\ &= P(Y_i = 1, Y_j = 1) - p^2 \\ &= P(Y_i = 1)\, P(Y_j = 1 \mid Y_i = 1) - p^2 \end{aligned}$$

<u>Solution:-</u>

$$Cov(Y_i, Y_j) = p \cdot \frac{mp-1}{m-1} - p^2 \qquad Y_i \sim \text{Bernoulli}(p)$$

$$= \frac{pq}{m-1}.$$

$$V(X) = V\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} V(Y_i) + 2\sum_{i<j} Cov(Y_i, Y_j) = \sum_{i=1}^{n} p(1-p)$$

$$+ 2 \cdot \frac{n(n+1)}{2} \cdot Cov(Y_i, Y_j)$$

$$= npq + \frac{n(n+1)pq}{m-1}$$

$$= npq(m-n)$$

$$\boxed{\text{CHEBYSHEV'S INEQUALITY}}$$

Q.1. Let $X_1, X_2, \dots,$ be a sequence of independent Bernoulli (p) RVs. Let $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ for $n = 1, 2, \dots\dots$ Show that for any $\epsilon > 0$

$$P(|Y_n - p| > \epsilon) \le \frac{1}{4n\epsilon^2}$$

<u>Solution:-</u>

$X_i \sim Ber(p)$

$\sum X_i \sim Bin(n, p)$

$E(\sum X_i) = np$

$E\left(\frac{1}{n}\sum X_i\right) = p.$

$$V\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n^2} V(\sum X_i) = \frac{1}{n^2} \cdot np(1-p) = \frac{p(1-p)}{n}$$

$$P(|Y_n - p| > \epsilon) \le \frac{p(1-p)}{n\epsilon^2} \le \frac{1}{4n\epsilon^2}, \text{ since } p(1-p) \le \frac{1}{4}.$$

$$\left[\begin{array}{l} \text{Markov's Inequality:- } P(X \geq \epsilon) \leq \frac{E(X)}{\epsilon}. \\ \text{Chebyshev's Inequality:- } P(|X-\mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2} = \frac{V(X)}{\epsilon^2}. \\ \quad \underline{\text{Alt:-}}\ P(|X-\mu| \geq \epsilon\sigma) \leq \frac{1}{\epsilon^2}. \end{array}\right]$$

Q.2. For any RV X, show that

$$P[|X|>t] \leq \frac{1+t^2}{t^2} E\left(\frac{X^2}{1+X^2}\right) \text{ for any } t>0.$$

Solution:-

$$P[X^2>t^2] = P[1+X^2 > 1+t^2]$$
$$= P\left[\frac{X^2}{1+X^2} > \frac{t^2}{1+t^2}\right]$$
$$\leq E\left(\frac{X^2}{1+X^2}\right) \cdot \frac{1+t^2}{t^2}. \quad \text{[By Markov's Inequality]}$$

Q.3. For the geometric distribution with PMF

$$f(x) = \begin{cases} \frac{1}{2^x} & \text{if } x=1,2,3,\ldots \\ 0 & \text{o/w} \end{cases}$$

Prove that $P[|X-\mu| \leq 2] > \frac{1}{2}$.

Solution:-

$$E(X) = 2$$
$$V(X) = 2, \quad \sigma = \sqrt{2}$$

$\therefore\ P[|X-\mu| \leq \sigma t] \geq 1-\frac{1}{t^2}$, by Chebyshev's inequality.

Take $t=\sqrt{2}$.

$$\therefore P[|X-\mu| \leq 2] \geq \frac{1}{2}.$$

Q.4. A fair die is rolled n times. Find a lower bound to n such that the prob. of at least one six in rolling is $\geq \frac{1}{2}$.

Solution:- X : # six by throwing a die n times

$$X \sim Bin\left(n, \tfrac{1}{6}\right)$$

$P[X \geq 1] \leq E(X)$, by Markov's inequality.

$$\Rightarrow P[X \geq 1] \leq \frac{n}{6}.$$

Given $\frac{n}{6} \geq \frac{1}{2}$, i.e., $n \geq 3$.

CONTINUOUS DISTRIBUTION

Q.1. Let X be a continuous RV with pdf

$$f(x) = \begin{cases} c_1 x & \text{for } 0 \le x < 1/3 \\ c_2(1-x) & \text{for } 1/3 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where, c_1 and c_2 are constants which make f a continuous function.
(a) Find c_1 and c_2
(b) Find cdf of X.
(c) Find $P(X \geq 2/3)$ and $P(1/3 < X \leq 2/3)$.

Solution:- (a) $$\int_0^{1/3} c_1 x\,dx + \int_{1/3}^{1} c_2(1-x)\,dx = 1$$

$$\Rightarrow \frac{c_1}{18} + \frac{2c_2}{9} = 1$$

$$\Rightarrow c_1 + 4c_2 = 18$$

$$\lim_{x \to 1/3^-} c_1 x = \lim_{x \to 1/3^+} c_2(1-x) \Rightarrow \frac{c_1}{3} = \frac{2}{3}c_2$$

$$\therefore c_1 - 2c_2 = 0$$

$$\therefore c_2 = 3,\ c_1 = 6.$$

(b) $$F(x) = \begin{cases} 0 & , x < 0 \\ c_1 \cdot \frac{x^2}{2} = 3x^2 & , 0 \le x < 1/3 \\ \int_0^{1/3} c_1 x\,dx + c_2 \int_{1/3}^{x} (1-t)\,dt = 3x - \frac{1}{2}(3x^2+1) & , \frac{1}{3} \le x < 1. \\ 1 & , x \geq 1 \end{cases}$$

(c) $$P(X \geq 2/3) = 1 - F(2/3) = 1 - 2 + \frac{1}{2}\left(\frac{2}{3}+1\right) = \frac{1}{6}$$

$$P(1/3 < X \leq 2/3) = F(2/3) - F(1/3) = 5/6 - 1 + 2/3 = 1/2.$$

Q.2. Let $U \sim U(0,1)$. Set $X = 0$ if $U \leq 1/2$
$= 1$ if $U > 1/2$

(a) What's the probability distribution of X?
(b) Explain how this result can be generalised to generate observations from Binomial(n,p) for arbitrary n and $0 < p < 1$ from a random sample of $U(0,1)$.

Solution:- (a) $P(X=0) = P(U \leq 1/2) = 1/2$
$P(X=1) = P(U > 1/2) = 1/2$
$X_i \sim$ Bernoulli$(1-p)$.

(b) $Z = \sum_{i=1}^{n} X_i \sim$ ~~Bernoulli~~ Binomial$(n, 1-p)$.

Q.3. Suppose that test scores in a subject are normally distributed with mean 50 and variance 25.
(a) Find the proportion of students who scored below 40.
(b) Find the median score, lower quartile and the 95th percentile score.

Solution:- $X \sim N(\mu, \sigma^2) = N(50, 5^2)$

(a) $P(X<40) = P\left(\frac{X-\mu}{\sigma} < \frac{40-\mu}{\sigma}\right)$

$= P(Z < -2)$

$= \Phi(-2) = 0.02275$

(b) $P(X \le \xi_{1/2}) = 1/2$ $P(X \le \xi_p) = p$

$\therefore \xi_{1/2} = \mu = 50$ $\Rightarrow \Phi\left(\frac{\xi_p - \mu}{\sigma}\right) = p.$

$\therefore$ Median $= 50$

$P(X \le \xi_{1/4}) = 0.25$

[Calculations done using Normal Table]

$\Rightarrow \xi_{1/4} = 47.$

$P(X \le \xi_{0.95}) = 0.95$

$\Rightarrow \xi_{0.95} = 58.25$

Q.4. Let X denote the lifetime of a product, and assume that it is a continuous RV having the pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $E(X) = 1/\lambda$ and $V(X) = 1/\lambda^2$
(b) Find the pdf of $Y = X^{1/\alpha}$ where $\alpha > 0$ is a constant.

Solution:- $X \sim Exp(\lambda)$

(a) $E(X) = \int_0^\infty x\lambda e^{-\lambda x} dx = \int_0^\infty (\lambda x)^{2-1} e^{-\lambda x} dx$

$= \frac{1}{\lambda}\int_0^\infty u e^{-u} du = \frac{1}{\lambda}$, $u = \lambda x$

$E(X^2) = \lambda\int_0^\infty x^2 e^{-\lambda x} = \frac{\Gamma(3)}{\lambda^2} = \frac{2}{\lambda^2}$; $V(X) = \frac{1}{\lambda^2}$.

(b) $P(Y \le y) = P(X^{1/\alpha} \le y) = P(X \le y^\alpha)$, $\alpha > 0$

$= F(y^\alpha)$

$f(y) = \alpha y^{\alpha-1} f(y^\alpha) = \alpha y^{\alpha-1} . \lambda e^{-\lambda e^{\alpha}}$

$= \lambda\alpha y^{\alpha-1} e^{-\lambda y^\alpha}$, $y > 0$

Q.5. Let (X,Y) have the joint pdf

$$f(x,y) = \begin{cases} \lambda^2 \exp(-\lambda y), & \text{if } \infty > y > x > 0; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal prob. distn. of X and Y.
(b) Find $E(Y)$ and $Var(Y)$.
(c) Find the mean & variance of χ_K^2.

Solution:- (a) $f(x) = \lambda^2 \int_x^\infty e^{-\lambda y} dy = -\lambda\left[e^{-\lambda y}\right]_x^\infty = \lambda e^{-\lambda x}, \; x > 0$

$\therefore X \sim Exp(\lambda)$.

$f(y) = \lambda^2 \int_0^y e^{-\lambda y} dx = \frac{\lambda^2}{\Gamma 2} y e^{-\lambda y}, \; \lambda > 0, y > 0$

$\therefore Y \sim Gamma(2, \lambda)$.

(b) $$E(Y^r) = \int_0^\infty \frac{y^{r+2-1} e^{-\lambda y}}{\Gamma 2} \lambda^2 dy$$

$$= \frac{1}{\lambda^r} \cdot \frac{\Gamma(2+r)}{\Gamma(2)}$$

$$E(Y) = \frac{1}{\lambda} \cdot \frac{\Gamma(3)}{\Gamma(2)} = \frac{2}{\lambda}.$$

$$E(Y^2) = \frac{1}{\lambda^2} \cdot \frac{\Gamma(4)}{\Gamma(2)} = \frac{6}{\lambda^2}$$

$$\therefore V(Y) = E(Y^2) - E^2(Y) = \frac{2}{\lambda^2}$$

(c) $$f_{\chi^2}(x) = \begin{cases} \frac{1}{2^{K/2}\Gamma\left(\frac{K}{2}\right)} e^{-x/2} x^{K/2-1}, & 0 < x < \infty, K > 0 \\ 0 & \text{ow} \end{cases}$$

$$E(\chi^2) = \int_0^\infty \frac{z e^{-z/2} z^{K/2-1}}{2^{K/2} \Gamma\frac{K}{2}} dz = \frac{\Gamma\left(\frac{K}{2}+1\right) \cdot 2^{K/2+1}}{\Gamma\frac{K}{2} \cdot 2^{K/2}} = K$$

$$V(\chi^2) = \int_0^\infty \frac{z^2 e^{-z/2} z^{K/2-1}}{\Gamma\frac{K}{2} \cdot 2^{K/2}} dz - K^2$$

$$= \frac{\Gamma\left(\frac{K}{2}+2\right) \cdot 2^{K/2+2}}{\Gamma\frac{K}{2} \cdot 2^{K/2}} - K^2$$

$$= (K+2)K - K^2$$

$$= 2K.$$

CENTRAL LIMIT THEOREM

Q.1. Let $X \sim \text{Binomial}(100, 0.1)$. Using the normal approximation
(a) find $P(X=10)$ (b) find $P(3 < X \leq 16)$

Solution:- $np = 10$, $np(1-p) = 9$

(a) $P(9.5 \leq X \leq 10.5)$

$$= P\left(\frac{9.5-10}{3} \leq Z \leq \frac{10.5-10}{3}\right)$$

$$= \Phi\left(\frac{1}{6}\right) - \Phi\left(-1/6\right)$$

$=$

(b) $P(3 < X \leq 16)$

$$= P\left(\frac{2.5-10}{3} \leq Z \leq \frac{16.5-10}{3}\right)$$

$$= \Phi(2.16) - \Phi(-2.5)$$

Q.2. Let $U_1, U_2, \ldots$ be an i.i.d. sequence of $U(0,1)$ random variables and set $S_n = \sum_{i=1}^{n} U_i$. Find (approximately) $P(145 \leq S_{300} \leq 160)$.

Solution:- $E(S_n) = E\left(\sum_{i=1}^{n} U_i\right) = \frac{n}{2}$,

$$V(S_n) = V\left(\sum_{i=1}^{n} U_i\right) = \frac{n}{12}$$

$$E(S_{300}) = 150, \quad V(S_{300}) = 25$$

$$P(145 \leq S_{300} \leq 160) = P\left(\frac{145-150}{5} \leq \frac{S_{300}-150}{5} \leq \frac{160-150}{5}\right)$$

$$= P(-1 \leq Z \leq 2)$$

$$= \Phi(2) - \Phi(-1) \text{ ; by CLT, } \frac{S_n - E(S_n)}{\sqrt{V(S_n)}} \overset{a}{\sim} N(0,1)$$

$$= 0.81859$$

ORDER STATISTIC

Q.1. Let X denote the waiting time at a service counter. Suppose $X \sim Exp(\lambda)$. Consider the waiting times of n randomly selected customers at this service counter, and let Y_1 denote the shortest waiting time and Y_2 denote the longest waiting time.

Find (a) the joint distribution of (Y_1, Y_2);

(b) Show that $Y_1 \sim Exp(n\lambda)$;

(c) the distribution of $Y_2 - Y_1$.

<u>Solution:-</u> (a)

$$f_{X_{(1)}, X_{(2)}}(x, y) = n(n-1)\left[F(y) - F(x)\right]^{n-2} f(x) f(y)$$

$$f(y_1, y_2) = n(n-1)\left[e^{-\lambda y_1} + e^{-\lambda y_2}\right]^{n-2} \cdot \lambda^2 e^{-\lambda y_1 - \lambda y_2}, \quad y_1 \leq y_2$$

(b)
$$f(y_1) = n\left(1 - F(y)\right)^{n-1} f(y_1)$$

$$= n\left[e^{-\lambda y_1}\right]^{n-1} \cdot \lambda e^{-\lambda y_1}$$

$$= n\lambda e^{-n\lambda y_1}, \quad y_1 \geq 0$$

$\therefore Y_1 \sim Exp(n\lambda)$.

(c) $R = Y_2 - Y_1$.