Economic Statistics

INDEX NUMBERS

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INDEX NUMBERS

Meaning and Definition :- The Index numbers are intended to show the average percentage changes, in the value of certain product(s) at a specific time, place or situation as compared to any other time, place or situation. Such a study is of great importance in the industry, the management, the business and largely to the Governments for chalking out the wage policy or fixing of prices, import-export policy, etc.

(1) An Index Number is a special type of an average that provides a measurement of relative changes from time to time or place to place.

(2) Index Number is a quantity which by reference to a base period, shows by its variations, the changes in the magnitude over a period of time.

(3) Index Number is a pure number which measures relative changes of price/value/quantity of a set of commodities over two different situations (period of times, places, cities, countries, etc.).

It is sure from these definitions that Index Number is the ratio of two quantities or the same products or variables, with reference to two timings, places or situations. These ratios are usually expressed as percentages which are most suited for comparability.

Index Numbers are mostly given for a time period, in comparison to any earlier time period, which is known as base period or reference period. Hence, for calculations of Index Number, the data are collected for prices and quantities consumed or produced at two different timings, one named as current period and other as base period.

0 : base period

1 : current period

" An index number is a statistical measure designed to show changes in variable or a group of related variable with respect to time, geographical location or other characteristic " — Spiegal.

Notations and Terminology :-

I_{01} : Index for the year '1' as compared to the base year '0'.

P_{01} : Price Index Number

Q_{01} : Quantity Index Number

Price Index Numbers which measure the general changes in the retail or wholesale price level of a particular commodity or group of commodities.

Quantity Index Numbers which measure the change in the quantity of goods manufactured in a factory, e.g. the indices of industrial production or agriculture production.

Binary Commodities are those which are found in both of the years 0 and 1.

Let N_0 and N_1 are respectively the number of items or commodities in the base year and current year. Then N_{01} be the number of items which are common in the base period and the current period, are called binary commodities.

Unique commodities are those which are found in either of the two lists of commodities of the period '0' and '1', but not in both.

Number of Unique commodities $= (N_0 - N_{01}) + (N_1 - N_{01}) = N_0 + N_1 - 2N_{01}$

Value Ratio :- Here, we shall consider only the values for the year 0 and 1, based on a sample of n_{01} items from N_{01} binary commodities.

Define, $$V_{01}(n_{01}) = \frac{\sum_{i=1}^{n_{01}} p_{1i} q_{1i}}{\sum_{i=1}^{n_{01}} p_{0i} q_{0i}}$$ is a ratio of two sum of values.

[p_{0i} : price of the i^{th} binary commodity at base year.
p_{1i} : price of the i^{th} binary commodity at current year.]

Price and Quantity change: The value ratio $V_{01}(n_{01})$ measures the relative change between two sum of values, such as $\sum_{i=1}^{n_{01}} p_{1i} q_{1i}$ and $\sum_{i=1}^{n_{01}} p_{0i} q_{0i}$.

Now, it is conceptually possible to formulate an idea of total price influence and total quantity influence and then to say that they jointly produce the value ratio.

We know, $V_{01} = P_{01} \cdot Q_{01}$. The index number problem will be solved iff when a measure is obtained which isolates the price influence or the quantity influence from the value change in a defined commodity group.

Price and Quantity Variation: Let p_0, p_1, q_0, q_1 denote the price and quantity of a given commodity in the periods '0' and '1', respectively.

(a) Actual Changes: $p_{01} = p_1 - p_0$,
$q_{01} = q_1 - q_0$.

(b) Relative Changes: $\frac{p_1}{p_0}, \frac{q_1}{q_0}$.

Uses of Index Numbers: —

(i) Economic Barometer: " Index numbers are today one of most widely statistical devices ..... they are used to take the pulse of the economy and they have come to be used as indicators of inflationary or deflationary tendencies .... " G. Sompson. The indices of prices, output, trade, import, export, industrial or agricultural production, deposits, exchanges etc. give us a good appraisal of the general trade or activity of the country.

(ii) Formulating Decisions and Policies: Index numbers relating to prices, production, profits, imports and exports, etc are required for any govt. policies and also for decisions of planning and executions.

(iii) Deflating: It means 'making allowance for the effect of changing price levels'. The increase in the prices of consumer goods for a class of people over a period of years means a reduction in the 'purchasing power of money' or 'a measure of real income' for a class of people is obtained on deflating the wage series by dividing each item by an appropriate price index. The 'real income' is also known as 'deflated income'.

(iv) Purchasing Power: The purchasing power of money is the quantity of goods that a given quantity of money will buy. "The reciprocal of a price index number" is used to show the purchasing power of money.

A price index number is the amount of money required to purchase a fixed basket of goods, where as the purchasing power represents the quantity of goods that can be purchased with a fixed amount of money. Here is an illustrative example:

In 1992, the cost of living index number of the industrial workers of Calcutta was 238 with 1982 as base. The purchasing power of the 1982 rupee for the said group was, therefore $\frac{100.0}{238}$ or 0.420 rupee in 1992. This mean that in 1992, the 1982 rupee would purchase only 0.420 of the amounts it could purchase in 1982.

(v) Study Trends and Tendencies: Index Numbers study the relative changes in the level of phenomenon of different situations. It can be useful for studying general trend in time series data. The indices of output, volume of trade, import and export, etc. are extremely useful for studying the changes due to the various components of a time series data — trend, seasonal, cyclical variation and reflect upon the general trend of production and business activity. These can be used to forecast future events.

Stochastic Approach of Index Number : Following Edgeworth, taking an index number of prices for purposes of exposition, and a comparison of prices in base and current years, we have a simple form, some unweighted mean of the price relatives of the selected binary commodities. We may substitute it by a weighted mean with some simple weighting system which has no reference to quantities of commodities bought and/or sold. This is sometimes called 'price relative method'. Edgeworth was the most famous and persistant champion of the median and his strongest defence stated that the AM of relatives is a simple from the total number of possibles and the median is less affected by irregular and unusual relative values than any other averages. The AM bears the thought that the relatives has a symmetric distribution and in particular may be normal whereas the GM bears the thought that the distribution is skewed with minimum as zero and maximum unlimited.

Type of Index Numbers: Index Numbers can be classified into four categories:

(i) Price Index Numbers (Wholesale and Retail),
(ii) Quantity Index Numbers,
(iii) Value Index Numbers,
(iv) Special Purpose Index Numbers.

(A) Wholesale Price Index Numbers :— The wholesale Price Index number measures the change in the general price level from the base period to the current period. Thus the statement "The Wholesale Price Index Number for India during 1989-90 with 1981-82 as the base is 165.7" means that, as compared with the price level during the year 1981-82, the price level during the year 1989-90 increased 1.657 times.

Construction of Whole-sale Price Index Number :→ The various steps are:

⟨1⟩ Purpose of the Index Number: An index number which is properly designed for a purpose can be most useful and powerful tool otherwise it can be equally misleading and dangerous. Thus the first and foremost problem is to determine the purpose of index number without which it is not possible to follow the steps in its construction. Moreover, precise statement of the purpose usually settles some related problems, e.g., if we are to measure the price changes in retail trade, we should use a sample of departmental store sales, not from wholesalers data. Also if the purpose of index number is to measure the changes in the production of steel (say), the problem of selection of commodities is automatically settled.

⟨2⟩ Choice of the Base Period : An ideal base should be :

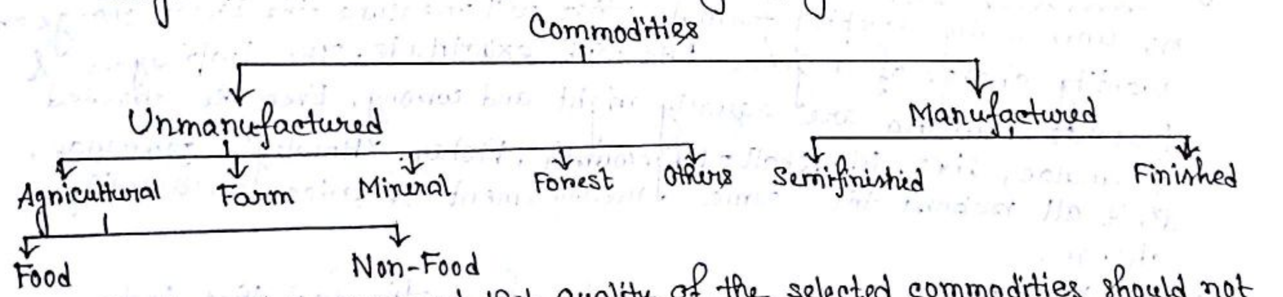
i) The base period should not be too short or too long. It should not be too short, like, a single day or week, because the prices for too short a period are highly unstable and unreliable. Again, it should not be too long because the average prices may eliminate some important fluctuations. Normally, it is not greater than a year, not less than a week.

ii) The base period must be a normal period. A normal period means a period free from all sorts of abnormalities or chance fluctuations (economic boom or depression) in which prices of commodities will be abnormally high. So, if it is not normal period then price relatives will not be of much practical utility.

iii) The base period should be a recent past period. The base period selected should not be too far from the compared or current period as the market conditions (tastes and habits of people) may undergo marked changes, new goods can replace old ones.

iv) The base period should not be pre or post budget year. Since the prices are normally unstable at least for some commodities in pre or post plan year.

⟨3⟩ Choice of Commodities : As time, money and labour are limited, it is impossible and impracticable to include all commodities in the construction. We are to take a suitable sample of commodities satisfying the purpose of construction. The selection of commodities is done by judgement sampling, not by random sampling. Different groups display different patterns of price movements, so commodities are classified into different subgroups showing similar patterns of price fluctuations and judgement sampling from each subgroups are taken or selected. The subgrouping is



Also, it is assumed that quality of the selected commodities should not vary much from period to period and all selected commodities are available in the market at both periods.

⟨4⟩ Collection of Data : The prices of a commodity vary from market to market, within the same year and for different grades. Therefore, we are to collect prices of a commodity from a number of representative markets for a few important grades of the commodity at a particular period of time. We take a random sample of markets and a random sample of shops for each of the selected markets. Then the data are collected from representative shops of the representative wholesale markets.

⟨5⟩ Choice of weights: All items selected in the construction are not of equal importance and further they don't have the same unit of price or are marked in the same pattern. This necessitates attaching weights either to actual prices or to price relatives in a price index. The weights used for the purpose may be either implicit or explicit. The implicit weights are somewhat arbitrary and depend on the economic importance of the items. The explicit weights are rational weights – quantity weights are value weights. A quantity weight relates to amount of commodity produced or distributed/consumed, etc.

If quantities are used as weights, these may either relate to the base year or the current year or a typical year or average of more than one typical year.

⟨6⟩ Method of combining data: Index number involves a comparison of values of a variable or a group of variables over two periods of time or over two different places. Also, price fluctuations of different commodities are reflected in the price relatives. We are to consider some means of combining these individual price fluctuations. It is expected that the pattern of fluctuations is different for different commodities. It has been empirically found that the distribution of price relatives is bell shaped with marketed central tendency and finite dispersion both in base and current periods. So, we take the measures of central tendency (i.e. AM, GM, Median) is combining the different price relatives based on the data in hand with the relative merits and demerits of the method.

Errors in Index Numbers: →

⟨1⟩ Formula Error: This error arises from the fact that there are no universally accepted formula that will measure the price change or quantity change of a given data with exactitude. The Laspeyres & Paasche's Formula are equally right and wrong. Even the crossed formulae, like, Marshall-Edgeworth, Fisher, Bowley's formulae, don't all report the same measurement of price or quantity change.

⟨2⟩ Sampling Error: This error arises from the fact that index numbers are based on a set of n_{01} binary commodities used to represent the whole list of N_{01} commodities. Assuming formula accuracy for the moment, then $P_{01}(N_{01})$ or $Q_{01}(N_{01})$ would give an exact measurement of change in the price levels or quantity levels for the complete list of binary commodities. On the other hand, $P_{01}(n_{01})$ or $Q_{01}(n_{01})$ is based on the data of n_{01} binary commodities rather than the complete list and it is expected that $P_{01}(n_{01})$ will differ in some degree from $P_{01}(N_{01})$. This difference is called sampling Error. The larger the sample, the smaller the sampling error.

⟨3⟩ Homogeneity Error: This error arises from the fact that index numbers are calculated from data on binary commodities, whereas they should be based on all commodities marketed in the base period and current period, including both binary and unique commodities. Since within the passage of time many old commodities will disappear from the market and new commodities will appear. The homogeneity error increases as the gap between the base period and the current period increases.

The making of Index Numbers:

(a) Measuring the total price effect by summing actual prices:

(i) Simple Aggregates Method: → In the construction of Index Numbers of the simple aggregate type, the prices of commodities for the base year as well as the current year are added separately. General price changes are measured by comparing the sums thus obtained for the base year '0' and the current year '1'. Hence symbolically: –

$$P_{01} = \frac{\sum_i p_{1i}}{\sum_i p_{0i}} \times 100, \text{ where},$$

$\sum_i p_{1i}$ = Total of current year prices of various commodities

$\sum_i p_{0i}$ = Total of base year prices of various commodities.

Limitations: – There are two main limitations of the simple aggregative method

(1) The units used in the price or quantity quotations can exert a big influence on the price of the index.

(2) No consideration is given to the relative importance of the commodities.

(ii) Simple Averages of Relative Prices: →

(a) The Arithmatic Mean of the price relatives is $\frac{1}{n}\sum_i \frac{p_{1i}}{p_{0i}}$ for fixed busket of n commodities. The formula for the index number for the current period '1' w.r.t. the base year '0' is given by

$$P_{01} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{p_{1i}}{p_{0i}}\right)$$, the quotient is always multiplied by 100.

(b) The Geometric Mean of Price relatives is $\left\{\prod_{i=1}^{n}\frac{p_{1i}}{p_{0i}}\right\}^{1/n}$ for fixed busket of n commodities. The formula for the index number for the current year '1' w.r.t. the base year '0' is given by

$$P_{01} = \left\{\prod_{i=1}^{n}\frac{p_{1i}}{p_{0i}}\right\}^{1/n}.$$

(c) The Median may also be used in finding the averages of the relative prices, when the frequency distribution of price relatives is skewed, the median is less affected by irregular and unusual relative values that other averages. Median is also appropriate if doubt exists concerning the accuracy of some of the data.

Limitations:- Limitation of this method is the choice of the average to be used. It is true that, though GM is difficult to compute but theoretically a better average than GM. However, because of computational difficulty AM is generally used in practice.

But sometimes GM is preferred because it gives equal importance to equal ratio change. AM is used when frequency distribution is symmetric, but GM is used when frequency distribution of price relatives is skewed.

(iii) Weighted Aggregative Method:- In this method, appropriate weights are assigned to various commodities to reflect their relative importance in the group. Usually, the quantities consumed, sold or marketed in the base year or in a given year or in some typical year are used as weights.

Let q_i refers to the quantity of the i^{th} commodity marketed or produced. So, here q_i is the weight attached to a commodity, then the price index is given by

$$P_{01} = \frac{\sum_i p_{1i} q_i}{\sum_i p_{0i} q_i}.$$

We shall realize now that aggregative index numbers of price measure the changing value of a fixed aggregate of goods, since the total cost or value changes while the components of the aggregate of goods donot, these changes must be due to price changes.

Necessary of weights:- The problem would in no sense be solved if all commodities were produced to a price per unit, for some commodities, such as diamonds, are very costly per pound and yet are not very important in our economic life, while coal, which is tremendously important, is relatively cheaper per pound. So, logical weights must be employed thereafter.

Selection of weights:- By the use of different types of weights, number of formulae have emerged for the construction of index numbers:

(a) Laspeyre's Formula: The Laspeyre's price index is a weighted aggregate price index, where the weights are determined by quantities in the base period. The formula for constructing the index is:

$$L_{01} = P_{01} = \frac{\sum_i p_{1i} q_{0i}}{\sum_i p_{0i} q_{0i}}.$$

(b) Paasche's Formula: The Paasche's price index is a weighted price index in which the weights are determined by quantities in the given year. the formula for constructing the index is:

$$P_{01} = \frac{\sum_i p_{1i} q_{1i}}{\sum_i p_{0i} q_{1i}}.$$

(c) Marshall-Edgeworth Formula: Using the average quantities $\left\{\frac{q_{0i}+q_{1i}}{2}\right\}$ of base and current year, we get the following formula:

$$P_{01} = \frac{\sum_i p_{1i}(q_{0i}+q_{1i})}{\sum_i p_{0i}(q_{0i}+q_{1i})}.$$

(d) Fisher's Formula: Fisher's price index number is given as the geometric mean of Laspeyre's and Paasche's formula. Symbolically,

$$F_{01} = P_{01} = \sqrt{L_{01} \times P_{01}} = \sqrt{\frac{\sum p_{1i}q_{0i}}{\sum p_{0i}q_{0i}} \times \frac{\sum p_{1i}q_{1i}}{\sum p_{0i}q_{1i}}}.$$

(e) Dorbish-Bowley Formula: To take into account the influence of both the base as well as current periods, Bowley suggested the arithmatic average of the Laspeyre's and Paasche's index. The formula is given by

$$B_{01} = P_{01} = \frac{L_{01}+P_{01}}{2}.$$

(iv) Weighted Averages of Relative Prices :-

(a) The weighted AM of price relatives with base year values $(p_{0i}q_{0i})$ as weights is

$$P_{01} = \frac{\sum_i \left(\frac{p_{1i}}{p_{0i}}\right) p_{0i}q_{0i}}{\sum_i p_{0i}q_{0i}} = \frac{\sum_i p_{1i}q_{0i}}{\sum_i p_{0i}q_{0i}} = L_{01}$$

This is same as Laspeyre's formula as a weighted aggregative formula with base year quantities as the weights.

(b) The weighted AM of price relatives with the hypothetical cost of the current year quantities at base year prices, i.e., $(p_{0i}q_{1i})$ as weights is

$$P_{01} = \frac{\sum_i \left(\frac{p_{1i}}{p_{0i}}\right) p_{0i}q_{1i}}{\sum_i p_{0i}q_{1i}} = \frac{\sum_i p_{1i}q_{1i}}{\sum_i p_{0i}q_{1i}} = P_{01}$$

This is same as Paasche's formula as a weighted aggregative formula.

(c) The weighted HM of price relatives with the hypothetical cost of base year quantities at current year prices, i.e., $(p_{1i}q_{0i})$ as weights, is

$$P_{01} = \frac{1}{\left\{\sum_i \frac{1}{\left(\frac{p_{1i}}{p_{0i}}\right)} p_{1i}q_{0i}\right\} \Big/ \sum_i p_{1i}q_{0i}} = \frac{\sum_i p_{1i}q_{0i}}{\sum p_{0i}q_{0i}} = L_{01}$$

(d) A harmonic mean of price relatives by current year values $(p_{1i}q_{1i})$, is

$$P_{01} = \frac{1}{\left\{\sum_i \frac{1}{\left(\frac{p_{1i}}{p_{0i}}\right)} p_{1i}q_{1i}\right\} \Big/ \sum_i p_{1i}q_{1i}} = \frac{\sum_i p_{1i}q_{1i}}{\sum_i p_{0i}q_{1i}} = P_{01}.$$

<u>Criticism of weighting system:</u>– An index number P_{01} measures the relative change in value of a fixed busket of goods in the current year '1' compare to the base year '0'. Using current year quantities as weights we get Paasche's formula $P_{01} = \sum_i p_{1i}q_{1i} / \sum_i p_{0i}q_{1i}$. This formula requires the selection of a new set of weights $\{q_{1i}\}$ for each current year. But frequently it is impossible to obtain current year quantities $\{q_{1i}\}$ and even if they are available, the labour is approximately doubled. Furthermore, each period is directly comparable with the base period. But same criticism is not valid for Liaspeyre's formula, $L_{01} = \sum_i p_{1i}q_{0i} / \sum_i p_{0i}q_{0i}$.

Hence, the weights in Liaspeyre's formula are more meaning-ful compare to Paasche's formula.

<u>Remark:</u>– If $\frac{p_{1i}}{p_{0i}} = \text{constant}\ (K)\ \forall\ i = 1(1)n$, then

$$L = \frac{\sum_i p_{1i}q_{0i}}{\sum_i p_{0i}q_{0i}} = \frac{K.\left(\sum p_{0i}q_{0i}\right)}{\left(\sum_i p_{0i}q_{0i}\right)} = K$$

and $$P = \frac{\sum p_{1i}q_{1i}}{\sum p_{0i}q_{1i}} = \frac{K\left(\sum_i p_{0i}q_{1i}\right)}{\left(\sum_i p_{0i}q_{1i}\right)} = K.$$

Again, if $\frac{q_{1i}}{q_{0i}} = \text{constant}\ (m)\ \forall\ i = 1(1)n$,

then $$L = \frac{\sum_i p_{1i}q_{0i}}{\sum_i p_{0i}q_{0i}} = \frac{\sum_i p_{1i}(q_{1i}/m)}{\sum_i p_{0i}(q_{1i}/m)} = \frac{\sum p_{1i}q_{1i}}{\sum p_{0i}q_{1i}} = P.$$

Hence, if the prices of all the goods change in the same ratio, <u>Liaspeyre's and Paasche's indices will be equal, for then weighting</u> system is irrelevant; or also if the quantities of all the goods change in the same ratio, they will be equal also.

<u>Difference between Liaspeyre's and Paasche's Formula:</u>–

(i) In Liaspeyre's formula, the quantities of the base year are used as weights. But in Paasche's formula, the quantities of the current year are used as weights.

(ii) Liaspeyre's formula represents the cost of maintaining the same rate of consumption or production as in the base year but at current year's price. Where as Paasche's method represents the cost of consumption or production as a whole in the current year as compared with that in the base year.

Question:- Why is the Laspeyre's formula said to have an upward bias and the Paasche's formula have a downward bias?

Solution:-

Laspeyre's Formula tends to overestimate price changes:-

The Laspeyres formula $L_{01} = \sum_i p_{1i} q_{0i} / \sum p_{0i} q_{0i}$ compares the cost in the current year '1' with the cost in the base year '0', of obtaining the base year basket of goods in quantities q_{0i}. Now, the formula assumes that, if their taste doesn't change, people will continue to buy the same amount of goods no matter how great the price rise or fall, while in a free market, there is a shift from those items which are becoming more expensive to those which are becoming cheaper. For example, if $\sum_i p_{0i} q_{0i}$ includes an item of Rs. 500 that purchased sweet potatoes in amount $q_0 = 2$ton at $p_0 =$ Rs. 250 per ton, and if $p_1 =$ Rs. 400 per ton, it is certain that many consumers, being provided with Rs. 800 in the current year '1' to buy 2 ton of sweet potatoes, would shift part of this money to other and more satisfaction in spending some of Rs. 800. Hence $\sum_i p_{1i} q_{0i}$ enables the consumer to raise their standard of base year's economic satisfaction. Since, the cost of obtaining the base year's bill of goods in the current year would be higher than the cost of obtaining the base year's economic satisfaction, the Laspeyres formula overestimates the price changes.

Paasche's Formula tends to underestimate price changes:-

The Paasche's formula $P_{01} = \sum_i p_{1i} q_{1i} / \sum_i p_{0i} q_{1i}$ compares the cost in the current year with the cost in the base year, of obtaining the current year basket of goods in quantities q_{1i}. Now, in a free market, no sensible person would have bought the same goods in the base year as he does now, because the relative prices of good would have been different; there would have been a shift from those items which were expensive to those which were cheaper. The cost $(\sum_i p_{0i} q_{1i})$ of obtaining the present years bill of goods in the base year prices would have been greater than the cost of obtaining the current year's economic satisfactions; the Paasche's formula underestimates the price changes.

Demerits of Fisher's Index Numbers:-

(i) It is hybrid of two index numbers. It is difficult to say what exactly is supposed to measure.

(ii) The ideal index number requires the quantities of both the base and current years. The determination of these quantities is a difficult task.

Question:- Let x_i = price relative for the i^{th} item $= \frac{p_{1i}}{p_{0i}}$,
y_i = quantity relative for the i^{th} item $= \frac{q_{1i}}{q_{0i}}$,
$w_i = p_{0i}q_{0i}$ be the weights of x_i and y_i, $i = 1(1)n$.
Then show that $\frac{L_{01}}{P_{01}} = 1 - \frac{r_{xy} \cdot s_x \cdot s_y}{V_{01}}$, where r_{xy} is the correlation coeffecient between x and y; s_x and s_y are the weighted S.D.'s of x and y. L_{01} = Laspeyre's price index, P_{01} = Paasche's price index, V_{01} = Value Index.

Solution:- We know, $r_{xy} \cdot s_x \cdot s_y = Cov(x, y)$

$$\therefore \; r_{xy} \cdot s_x \cdot s_y = \frac{\sum w_i x_i y_i}{\sum w_i} - \left(\frac{\sum w_i x_i}{\sum w_i}\right)\left(\frac{\sum w_i y_i}{\sum w_i}\right)$$

Now, $r_{xy} \cdot s_x \cdot s_y$

$$= \frac{\left(\sum_{i=1}^{n} p_{0i}q_{0i} \cdot \frac{p_{1i}}{p_{0i}} \cdot \frac{q_{1i}}{q_{0i}}\right)}{\sum_{i=1}^{n} p_{0i}q_{0i}} - \frac{\left(\sum_i p_{0i}q_{0i} \cdot \frac{p_{1i}}{p_{0i}}\right)}{\sum_i p_{0i}q_{0i}} \cdot \frac{\left(\sum_i p_{0i}q_{0i} \cdot \frac{q_{1i}}{q_{0i}}\right)}{\sum_i p_{0i}q_{0i}}$$

$$= \frac{\sum_i p_{1i}q_{1i}}{\sum_i p_{0i}q_{0i}}\left[1 - \frac{\sum_i p_{1i}q_{0i}/\sum_i p_{0i}q_{0i}}{\sum_i p_{1i}q_{1i}/\sum_i p_{0i}q_{1i}}\right] = V_{01}\left[1 - \frac{L_{01}}{P_{01}}\right].$$

Remark:- (i) If $r_{xy} > 0$, we have $L_{01} < P_{01}$ and if $r_{xy} < 0$, then we have $L_{01} > P_{01}$. Hence, if the correlation between price relatives (x) and quantity relatives (y) is positive or negative, then L_{01} is less (or greater) than P_{01}.

If either $r_{xy} = 0$ or if $s_x = 0$ (i.e. x_i = constant) and if $s_y = 0$ (i.e. y_i = constant $\forall i$), then we get $L_{01} = P_{01}$. In other words, if $r_{xy} = 0$ or all price movements are same for all items or all quantity movements are same for all items, then we get $L_{01} = P_{01}$.

(ii) Under normal economic conditions, the correlation between the price and quantity relatives is normally negative because of inverse relation between price and quantity in a demand function (i.e. the consumption (q) of an item is reduced when the price increases highly, is a common phenomenon of the market), then we have $L_{01} > P_{01}$, i.e., in practice, under normal economic conditions, we have $-1 \le r_{xy} < 0$ and consequently $L_{01} > P_{01}$.

(ii) Quantity Index Numbers :- By interchanging the prices (p_{oi} and p_{1i}) and quantities (q_{oi} and q_{1i}) in all above discussed formulae, we get the corresponding formula for calculation of quantity index number which reflect the change in the volume of quantity.

Thus, for example,

$$Q_{01} = \frac{\Sigma q_{1i} p_{oi}}{\Sigma q_{oi} p_{oi}}$$ is the Laspeyre's Formula;

$$Q_{01} = \frac{\Sigma q_{1i} p_{1i}}{\Sigma q_{oi} p_{1i}}$$ is the Paasche's Formula;

$$Q_{01} = E_{01} = \frac{\Sigma q_{1i}(p_{oi}+p_{1i})}{\Sigma q_{oi}(p_{oi}+p_{1i})}$$ is the Edgeworth-Marshall formula for quantity index.

The price index number tells us how much we shall spend in the current year if we buy the same quantity of goods each year, but at varying prices. The quantity index number tells us how much we shall spend in the current year if we buy varying quantities of goods each year, but at the same price.

(iii) Value Index Numbers :- Value Index Numbers are given by the aggregate expenditure for any given year expressed as a percenteye of the same in the base year. Thus,

$$V_{01} = \frac{\Sigma p_{1i} q_{1i}}{\Sigma p_{oi} q_{oi}} \times 100.$$

Question:- Prove that Fisher's Ideal Index number lies between Laspeyre's and Paasche's Index Numbers.

Solution:- Let us consider two real numbers $a>0$, $b>0$.

Let $a<b$
$\Rightarrow a^2 < ab \quad (\because a>0)$
$\Rightarrow a < \sqrt{ab}$

Let $a<b$
$\Rightarrow ab < b^2 \quad (\because b>0)$
$\Rightarrow \sqrt{ab} < b$

So, if $a<b$ then $a < \sqrt{ab} < b$.

Also, we can also show that if $a>b$ then $a > \sqrt{ab} > b$.

Now, consider $a = L_{01}$, $b = P_{01}$, $F_{01} = \sqrt{ab} = \sqrt{L_{01} \cdot P_{01}}$.

If $L<P$, then $L^2 < LP < P^2 \Leftrightarrow L < \sqrt{LP} < P \Leftrightarrow L_{01} < F_{01} < P_{01}$.

If $L>P$, then $L^2 > LP > P^2 \Leftrightarrow L > \sqrt{LP} > P \Leftrightarrow L_{01} > F_{01} > P_{01}$.

In particular, if $L_{01} = P_{01}$ then Lagrange's, Paasche's, Fisher's Indices are all equal.

Question:- Show that Edgeworth - Marshall Index Number lies between Laspeyre's and Paasche's Index Numbers. More specifically,

(a) If $L_{01} < P_{01}$ then $L_{01} < ME_{01} < P_{01}$ and

(b) If $L_{01} > P_{01}$ then $L_{01} > ME_{01} > P_{01}$.

Solution:- Let us consider four positive numbers : a, b, c, d.

If $\frac{a}{b} < \frac{c}{d} \Rightarrow ad < bc$.

$\therefore ad + ab < bc + ab \quad [\because ab > 0]$

$\Rightarrow a(b+d) < b(a+c)$

$\Rightarrow \frac{a}{b} < \frac{a+c}{b+d}$.

Again $\frac{a}{b} < \frac{c}{d} \Rightarrow ad < bc$

$\therefore ad + cd < bc + cd \quad [\because cd > 0]$

$\Rightarrow d(a+c) < c(b+d)$

$\Rightarrow \frac{a+c}{b+d} < \frac{c}{d}$.

So, we have, if $\frac{a}{b} < \frac{c}{d} \Rightarrow \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$ ....(*)

(a) Define, $a = \sum_i p_{1i}q_{0i}$, $b = \sum_i p_{0i}q_{0i}$, $c = \sum_i p_{1i}q_{1i}$, $d = \sum_i p_{0i}q_{1i}$.

Then $\frac{a}{b} = L_{01}$, $\frac{c}{d} = P_{01}$,

and also $\frac{a+c}{b+d} = ME_{01}$.

So, from (*) we get, if $L_{01} < P_{01}$ then $L_{01} < ME_{01} < P_{01}$.

(b) Define, $a = \sum_i p_{1i}q_{1i}$, $b = \sum_i p_{0i}q_{1i}$, $c = \sum_i p_{1i}q_{0i}$, $d = \sum_i p_{0i}q_{0i}$.

Then $\frac{a}{b} = P_{01}$, $\frac{c}{d} = L_{01}$,

and also $\frac{a+c}{b+d} = ME_{01}$.

So, from (*) we get, if $L_{01} > P_{01}$ then $L_{01} > ME_{01} > P_{01}$.

Hence, the proof is complete.

Remark:- Since AM is always greater than GM, so from above two problems, we have -

If $L < P$ then $L < F < ME < P$.

If $L > P$ then $L > ME > F > P$.

Question:- If L_p, L_q, P_p denote, respectively, Laspeyre's Price Index, Laspeyre's Quantity Index and Paasche's Price Index. Show that $L_q(P_p - L_p)$ may be looked upon as a weighted covariance between price relatives and quantity relatives. How will you interpret the result?

Solution:- If L_p and P_p represent Laspeyre's and Paasche's price index numbers and L_q represents Lagrange's Quantity Index Number. Then

$$L_q(P_p - L_p) = \frac{\sum q_{1i}p_{0i}}{\sum q_{0i}p_{0i}}\left(\frac{\sum p_{1i}q_{1i}}{\sum p_{0i}q_{1i}} - \frac{\sum p_{1i}q_{0i}}{\sum p_{0i}q_{0i}}\right)$$

$$= \frac{\sum p_{1i}q_{1i}}{\sum p_{0i}q_{0i}} - \frac{\sum q_{1i}p_{0i}}{\sum q_{0i}p_{0i}} \cdot \frac{\sum p_{1i}q_{0i}}{\sum p_{0i}q_{0i}}$$

$$= \frac{\sum p_{0i}q_{0i}\left(\frac{p_{1i}}{p_{0i}} \times \frac{q_{1i}}{q_{0i}}\right)}{\sum p_{0i}q_{0i}} - L_p \cdot L_q$$

$$= \frac{\sum p_{0i}q_{0i}\left(\frac{p_{1i}}{p_{0i}} - L_p\right)\left(\frac{q_{1i}}{q_{0i}} - L_q\right)}{\sum p_{0i}q_{0i}} \quad \text{.......} (*)$$

which is a weighted correlation coefficient between price relatives $\frac{p_{1i}}{p_{0i}}$ and quantity relatives $\frac{q_{1i}}{q_{0i}}$, when multiplied by their standard deviations, the weights being $p_{0i}q_{0i}$.

Since L_q is positive and the R.H.S. of (*) is positive if the correlation between $\frac{p_{1i}}{p_{0i}}$ and $\frac{q_{1i}}{q_{0i}}$ is positive, in such cases P_p will be larger than L_p. If, on the other hand, the correlation is negative, which is likely to be the case because an increase in price is likely to lead to a decrease in quantity, L_p will be larger than P_p.

Merits of Fisher's Index Number:– The Index is known as "Ideal" due to the following reasons:

(i) It is based on the GM which is theoretically considered to be the best average for constructing index numbers.

(ii) It takes into account both current year as well as base year prices & quantities.

(iii) It satisfies both the Time Reversal test & Factor Reversal Test as suggested by Fisher. (See Later)

(iv) It is free from bias, since the upward bias of Laspeyre's index number is balanced to great extent by the downward bias of Paasche's index number.

How to measure the Errors in Index Numbers :-

<1> Measure of Formula Error :- The formula error arises from from the fact that different formulae report different measurements. According to Fisher, a basis for choosing among formula can be its ability to meet certain mathematical tests of consistency. According to this theory, an index is called 'ideal' if it meets those tests.

Fisher's Reversibility Tests for consistency :-

(i) Time Reversal Test :- The time reversal test can be stated as follows : If the time subscripts of a price (or, quantity) index number formula be interchanged, the resulting price (or, quantity) index formula should be the reciprocal of the original formula, i.e., symbolically, $P_{01} \times P_{10} = 1$.

If the time reversal test is not satisfied, i.e. if $P_{01} \cdot P_{10} \neq 1$, there is a joint error. The measure of this joint error is then

$$E_1 = (P_{10} \cdot P_{01} - 1) .$$

Performance of Different Formulae :-

(a) Laspeyre's Index :-

$$L_{01} = \frac{\Sigma p_{1i} q_{0i}}{\Sigma p_{0i} q_{0i}} \text{ and } L_{10} = \frac{\Sigma p_{0i} q_{1i}}{\Sigma p_{1i} q_{1i}}$$

$$\text{Now, } L_{01} \cdot L_{10} = \frac{\Sigma p_{1i} q_{0i}}{\Sigma p_{0i} q_{0i}} \cdot \frac{\Sigma p_{0i} q_{1i}}{\Sigma p_{1i} q_{1i}} \neq 1.$$

Hence, Laspeyre's formula does not satisfy Time Reversal Test.

(b) Paasche's Index :- $P_{01} = \frac{\Sigma p_{1i} q_{1i}}{\Sigma p_{0i} q_{1i}}$, $P_{10} = \frac{\Sigma p_{0i} q_{0i}}{\Sigma p_{1i} q_{0i}}$.

$$\text{So, } P_{01} \times P_{10} = \frac{\Sigma p_{1i} q_{1i}}{\Sigma p_{0i} q_{1i}} \cdot \frac{\Sigma p_{0i} q_{0i}}{\Sigma p_{1i} q_{0i}} \neq 1.$$

Hence, Paasche's formula does not satisfy Time Reversal Test.

(c) Fisher's Index :-

$$F_{01} = \sqrt{L_{01} \times P_{01}} = \sqrt{\frac{\Sigma p_{1i} q_{0i}}{\Sigma p_{0i} q_{0i}} \times \frac{\Sigma p_{1i} q_{1i}}{\Sigma p_{0i} q_{1i}}} \text{ and}$$

$$F_{10} = \sqrt{L_{10} \times P_{10}} = \sqrt{\frac{\Sigma p_{0i} q_{1i}}{\Sigma p_{1i} q_{1i}} \times \frac{\Sigma p_{0i} q_{0i}}{\Sigma p_{1i} q_{0i}}} = \frac{1}{F_{01}}$$

i.e. $F_{01} \times F_{10} = 1$. Hence Fisher's formula satisfies Time Reversal Test.

(d) Edgeworth - Marshall Index :-

$$E_{01} = \frac{\Sigma p_{1i}(q_{0i} + q_{1i})}{\Sigma p_{0i}(q_{0i} + q_{1i})} , E_{10} = \frac{\Sigma p_{0i}(q_{0i} + q_{1i})}{\Sigma p_{1i}(q_{0i} + q_{1i})}$$

$$\text{So, } E_{01} \times E_{10} = 1.$$

Hence, Edgeworth - Marshall Index satisfies Time Reversal Test.

(e) Median of Price Relatives :–

P_{01} = median of the data $\left\{\frac{p_{1i}}{p_{0i}}, i=1(1)n\right\}$. Let $n=2m+1$, then we can arrange the price relatives such as

$$\underbrace{\frac{p_{11}}{p_{01}} < \frac{p_{12}}{p_{02}} < \cdots < \frac{p_{1\,m-1}}{p_{0\,m-1}} < \frac{p_{1m}}{p_{0m}}}_{m \text{ values}} < \underbrace{\frac{p_{1\,m+1}}{p_{0\,m+1}}}_{\text{mid value}} < \underbrace{\cdots < \frac{p_{1\,2m+1}}{p_{0\,2m+1}}}_{m \text{ values}}.$$

So, $P_{01} = \frac{p_{1\,m+1}}{p_{0\,m+1}}$, Now, P_{10} = median of the value $\left\{\frac{p_{0i}}{p_{1i}}, i=1(1)n\right\}$.

Again, obviously, we get $P_{10} = \frac{p_{0\,m+1}}{p_{1\,m+1}}$. Hence, $P_{01} \times P_{10} = 1$.

Hence, the index number defined by the median of the price relatives satisfies the Time Reversal Test.

(ii) Factor Reversal Test :– The Factor reversal test can be stated as follows: If the p and q factors in a price (or, quantity) index formula be interchanged, so that a quantity (or price) index formula is obtained, the product of two indices should give the value ratio $\left(\frac{\Sigma p_{1i}q_{1i}}{\Sigma p_{0i}q_{0i}}\right)$.

i.e. Price change × Quantity change = Value change.

symbolically, $P_{01} \times Q_{01} = \frac{\Sigma p_{1i}q_{1i}}{\Sigma p_{0i}q_{0i}} = V_{01}$.

The test arises from the argument that a formula which is right as to prices should be equally right as to quantities.

If the test is not satisfied by a formula, then there is a joint error. The measure of joint error, defined by Fisher, is $E_2 = \left(\frac{P_{01}\cdot Q_{01}}{V_{01}} - 1\right)$.

Performance of Different Formulae :–

(a) Fisher's Index :– $F_{01}^{P} = \sqrt{\frac{\Sigma p_{1i}q_{0i}}{\Sigma p_{0i}q_{0i}} \times \frac{\Sigma p_{1i}q_{1i}}{\Sigma p_{0i}q_{1i}}}$ is price index for Fisher's Formula.

And $F_{01}^{Q} = \sqrt{\frac{\Sigma q_{1i}p_{0i}}{\Sigma q_{0i}p_{0i}} \times \frac{\Sigma q_{1i}p_{1i}}{\Sigma q_{0i}p_{1i}}}$ is quantity index for Fisher's formula.

Note that $F_{01}^{P} \times F_{01}^{Q} = \frac{\Sigma p_{1i}q_{1i}}{\Sigma p_{0i}q_{0i}} = V_{01}$.

Hence, Fisher's index satisfies Factor Reversal Test.

[Note that, none of the other formula satisfies the Factor Reversal Test]

(b) Edgeworth – Marshall Index :–

$$E_{01}^{P} = \frac{\Sigma p_{1i}(q_{0i}+q_{1i})}{\Sigma p_{0i}(q_{0i}+q_{1i})}, \quad E_{01}^{Q} = \frac{\Sigma q_{1i}(p_{0i}+p_{1i})}{\Sigma q_{0i}(p_{0i}+p_{1i})}$$

$$\therefore E_{01}^{P} \times E_{01}^{Q} = \frac{\Sigma p_{1i}(q_{0i}+q_{1i})}{\Sigma p_{0i}(q_{0i}+q_{1i})} \cdot \frac{\Sigma q_{1i}(p_{0i}+p_{1i})}{\Sigma q_{0i}(p_{0i}+p_{1i})} \neq \frac{\Sigma p_{1i}q_{1i}}{\Sigma p_{0i}q_{0i}}.$$

Hence, Edgeworth – Marshall Index does not satisfy the Factor Reversal Test.

Remark:– Since Fisher's Index satisfies both Time Reversal Test and Factor Reversal Test, it is termed as "Fisher's Ideal Index Number". Fisher call it 'ideal' not on the grounds of 'Reversibility Tests for consistency' but due to the fact that it measured price or quantity change by utilizing all the data of the two periods compared, i.e., $p_{0i}, p_{1i}, q_{0i}, q_{1i}$.

⟨2⟩ Measure of Sampling Error :– For the purpose of considering this error factor, the price or quantity change based on all N_{01} binary commodities represent complete accuracy, say, $I_{01}^{N_{01}}$ and this value is then without sampling error. This index based on n_{01} sample commodities from this N_{01} commodities is an estimate $\left(I_{01}^{n_{01}}\right)$ of the exact value $I_{01}^{N_{01}}$, and $I_{01}^{n_{01}} - I_{01}^{N_{01}}$ is therefore the sampling error.

In statistical sense, it is a variable, and for all possible samples of size n_{01} from N_{01} binary commodities, there is a frequency distribution of these errors. Here, an index number is an average and we are concerned with the sampling distribution of the average. The error of sampling is the standard deviation of the sampling distribution of the average.

⟨3⟩ Measure of Homogeneity (or Heterogeneity) Error :–

The formal measure of homogeneity error is $P_{01}(N_{01}) - P_{01}(T)$. But there is no existing method of measuring the difference since no one has proposed a means of measuring $P_{01}(T)$.

The R-test of Homogeneity :– Define R for any pair of comparison periods as

$$R = \frac{\text{Number of unique commodities}}{\text{Number of unique and binary commodities}}.$$

Symbolically, the number of unique commodities is $(N_0 - N_{01}) + (N_1 - N_{01})$, and the total number of commodities is $N_0 + N_1$, then

$$R = \frac{N_1 + N_0 - 2N_{01}}{N_1 + N_0},$$ measures the homogeneity in this sense:

⟨i⟩ If $R = 0$, there are no unique commodities
$\Leftrightarrow$ There is complete homogeneity.

⟨ii⟩ If $R = 1$, there are no binary commodities
$\Leftrightarrow$ There is complete heterogeneity.

So, for $R = 0 \Leftrightarrow N_0 = N_1 = N_{01}$ & for $R = 1 \Leftrightarrow N_{01} = 0$. Also, $0 \le R \le 1$.

So, if R is close to 0, our current methods are very close to a solution of the real problem of changing price levels, i.e., $P_{01}(N_{01}) = P_{01}(T)$. It is also clear that the homogeneity error increases as the gap between base period and the current period increases.

Long distance and Series Comparison :– The demand for long-term and series comparison arises because modern students of Economics, find that many of their basic studies of social caution require knowledge of a variable over time.

The difficulties of measurement :– We can, with the help of an index number, know about the price change of two adjacent periods. It is not a matter of direct comparison of distinct periods like 1896 and 1980. When we try to make direct comparison between two distant periods without involving these intermediate years, we have much less solid ground. It is also noted that within that passage of time many new commodities enter to the market and old commodities disappear, also quality of commodities may undergo change. As a result 'homogeneity error' increases because of less number of binary or high number of unique commodities between two periods and the 'formula error' is measured by $D = L - P$ increases because of greater variation between q_{0i} and q_{1i}. Hence, the difficulties of the measurement in the price change between two distant periods, are not limited to lack of realism in these comparison but they are subject to uncertainities or errors in the measurement.

(a) Fixed-base Method of calculating a series of index numbers :–

Suppose we have a series of data for a number of years, with regard to prices and quantities consumed. If we designate the base period as '0' and the successive period which follows as $1, 2, 3, \ldots, K$. We can calculate the price index number for each 'i' with respect to the base period '0', where $i = 1(1)K$, by a direct application of a particular formula.

The full series of price indices are then $P_{01}, P_{02}, \ldots, P_{0K}$ and each index number is calculated with the fixed base period '0'. By this method, we are able to know the gradual change of price level in each subsequent year w.r.t. the same or fixed base year. This method of series comparison is known as the 'Fixed-Base Method'.

(b) Chain Indices :– Instead of calculating the fixed base indices, we calculate the index number $P_{\overline{i-1},i}$ for comparing the prices of period i, with those of the period $\overline{i-1}$, for each $i = 1(1)K$. Thus, we have the indices $P_{01}, P_{12}, P_{23}, \ldots, P_{\overline{K-1},K}$; taking the previous period to any period as base year and these are known as link indices. By multiplying successive links, we obtain the chain indices as shown below:

$$P_{01} = \text{first link},$$
$$P'_{02} = P_{01} \times P_{12},$$
$$P'_{03} = P_{01} \times P_{12} \times P_{23} = P'_{02} \times P_{23},$$
$$\vdots$$
$$P'_{0K} = P_{01} \times P_{12} \times P_{23} \times \ldots \times P_{\overline{K-2},\overline{K-1}} \times P_{\overline{K-1},K} = P_{\overline{K-1},K} \times P'_{0,\overline{K-1}}$$

$$\boxed{\text{Current year F.B.I} = \frac{\text{Current Year CBI} \times \text{Previous Year FBI}}{100}}$$

Should the base be shiftable without error?

Circular Test: Some authorities of the subject demand that an index number formula shall be independent of its base and thus shall be shiftable directly without error. This idea is equivalent to setting up a new test of accuracy of index number formula, the circular test. Formerly the test can be written as

$$P_{01} \cdot P_{12} \cdot P_{23} \cdots P_{\overline{K-1},K} \cdot P_{K0} = 1 \text{.} \quad \text{——— (1)}$$

The time reversal test $P_{01} \cdot P_{10} = 1$ is a particular case of (1). Thus, if a formula satisfies the circular test, then

$$P_{01} \times P_{12} \times P_{23} \times \cdots \times P_{\overline{K-1},K} = \frac{1}{P_{K0}} = P_{0K} \text{.}$$

So, if it satisfies time reversal test, then the chain indices are

$$P_{01} = \text{first link}$$

$$P'_{02} = P_{01} \cdot P_{12} = P_{02}$$

$$P'_{03} = P_{01} \cdot P_{12} \cdot P_{23} = P_{03}$$

$$\vdots$$

$$P'_{0K} = P_{01} \cdot P_{12} \cdot P_{23} \cdots P_{\overline{K-1},K} = P_{0K} \text{;}$$

Therefore, the chain indices are equal to corresponding fixed-base indices. There is no reason to use chain-index method instead of fixed-base method where the formula used in the construction of a chain index meet the circular test as well as time reversal test.

Remarks: (1) The circular test is satisfied only by unweighted or constant-weighted aggregates or geometric averages of relatives.

In case GM of relatives, we have —

$$P_{01} \cdot P_{12} \cdots P_{\overline{K-1},K} \cdot P_{K0} = \left\{\prod_{i=1}^{n} \frac{p_{1i}}{p_{0i}}\right\}^{1/n} \cdot \left\{\prod_{i=1}^{n} \frac{p_{2i}}{p_{1i}}\right\}^{1/n} \cdots \left\{\prod_{i=1}^{n} \frac{p_{Ki}}{p_{\overline{K-1}i}}\right\}^{1/n} \left\{\prod_{i=1}^{n} \frac{p_{0i}}{p_{Ki}}\right\}^{1/n}$$

$$= 1.$$

In case of fixed weighted aggregative formula, we have —

$$P_{01} = \frac{\Sigma p_{1i} q_{ai}}{\Sigma p_{0i} q_{ai}} \text{, we have}$$

$$P_{01} \cdot P_{12} \cdots P_{\overline{K-1},K} \cdot P_{K0} = \left(\frac{\Sigma p_{1i} q_{ai}}{\Sigma p_{0i} q_{ai}}\right)\left(\frac{\Sigma p_{2i} q_{ai}}{\Sigma p_{1i} q_{ai}}\right) \cdots \left(\frac{\Sigma p_{Ki} q_{ai}}{\Sigma p_{\overline{K-1}i} q_{ai}}\right)\left(\frac{\Sigma p_{0i} q_{ai}}{\Sigma p_{Ki} q_{ai}}\right)$$

$$= 1.$$

But remember that unweighted or constant weighted formulae are never right. Since the correct set of weights for periods '0' and '1' are necessarily different from those for the periods '1' and '2' and etc. Therefore, satisfiction of the circular test can't be right if constant weights can't be accepted. Edgeworth-Marshall Index and Fisher's index don't satisfy the circular test, although they satisfy the time reversal test.

⟨2⟩ Fisher says that the circular test is theoretically wrong. The several link indices P_{01}, P_{12}, etc., in going from period 0 to the period K can each be taken will all the accuracy of the best known formula but the P_{K0} represents a backward step that can't be taken, since we can't undo the history inventions, the discoveries of the new resources.

⟨3⟩ The base period can be shifted to any convenient subsequent period if the formula satisfies the circular test, since P_{nk} can be calculated from the following relation, which follows from circular test : $P_{nk} = \frac{P_{0k}}{P_{0n}}$.

⟨4⟩ The practical advantage of a chain index is that the sample of commodities and/or the set of weights may be kept quite up-to-date in any index number. However, any change in the set of commodities or in the set of weights will upset the circular test.

Question:- Distinguish between fixed-base and chain base methods for the construction of index numbers. Discuss their merits.

Solution:- i) We know that the fixed-base index become more and more inaccurate as the distance between the base period and the current period increases. As the chain-base index numbers are based on a number of link indices, each of which is expected to be quite accurate, it is claimed that the chain-base index numbers are more accurate that the fixed base ones; so far as long-term comparison is concerned. Also, a chain index fully utilises the information regarding prices and quantities of all the intervening periods between the base period and the current period, where as a fixed base index makes use of data concerning the base period and current period only.

ii) Some authorities, on the other hand, hold that since a chain index is obtained by multiplying a number of link-indices, it may involve a commutative error, although none has put forward any convincing proof for the existance of such error.

iii) Fixed base index numbers are generally earlier to calculate and more easily understood by users of index numbers than chain-base index numbers.

⇨ Steps in the construction of Chain Indices at a glance:-

⟨i⟩ Express the figures for each period as a percentage of the preceeding period to obtain the Link indices (L.R.).

⟨ii⟩ These link relatives are chained together by successive multiplication to get chain indices (C.I.) by the following formula:

$$\text{Chain Index} = \frac{\text{Current year L.R.} \times \text{Preceeding year C.I.}}{100}.$$

(B) Cost of Living Index Number or Retail Price Index Number :-

It measures the relative change in the amount of money required to keep same producing equal satisfaction in two different situation or periods, in other words, to maintain the same standard of living in both periods. Alternatively, it measures the relative change in amount of money required to buy same basket of foods/items in two different situations. Since the consumption habits of people differ widely from class to class (such as poor, low, middle, high income group) and even within the same class from region to region, age to age, the changes in the level of prices affect different classes differently. CLI are compiled for different classes of people separately basically w.r.t. their income level. The term 'CLI' can be replaced by 'Retail Price Index' or 'Consumer Price Index'. But CLI should not be interpreted as a measure of 'standard of living'.

CLI & Laspeyre's and Paasche's Formulae:- A Cost of Living Index Number may be defined as an index of change in the money required to get equal satisfaction in two different situations.

Let p_{0i}'s and p_{1i}'s denote the consumer prices of a fixed set of goods and services representing the consumption level of a particular section of population, in the base period and in the current period, respectively.

Let $q_1^1, q_2^1, \ldots, q_n^1$ be the of quantities of the fixed set of goods and services while yield equivalent satisfaction in the current period as compared with the base period series $q_{01}, q_{02}, \ldots, q_{0n}$

The CLI number I, for the current period relative to base period, is given by

$$I = \frac{\left(\sum_{i=1}^{n} p_{1i} q_i^1\right)}{\left(\sum_{i=1}^{n} p_{0i} q_{0i}\right)}.$$

This 'I' is called the (true) CLI Number. The formulae Laspeyre's (L) and Paasche's (P) give only the approximate value of the true CLI (I).

Construction of Cost of Living Index:- CLI Number is constructed by the following formulae:

(i) **Aggregate Expenditure or Weighted Aggregative Method :-**

In this method, weights to by assigned to various commodities are provided by the quantities consumed in the base year. Thus, in the usual notations:

$$\text{Cost of Living Index} = \frac{\sum_i p_{1i} q_{0i}}{\sum_i p_{0i} q_{0i}} \times 100$$

$$= \frac{\text{Total Expenditure in current year}}{\text{Total Expenditure in base year}} \times 100$$

Remark:- This is nothing but Laspeyre's Index and is the most popular method for constructing CLI Numbers.

(ii) Family Budget Method or the Method of Weighted Relatives:—

In this method, Cost of Living Index is given by the weighted average of price relatives, the weights being the quantities consumed in the base year. Thus, in the usual notations, if we write

$$P_i = \frac{p_{1i}}{p_{oi}} \times 100 \text{ and } W = p_{oi} q_{oi} \text{ then}$$

$$\therefore \text{Cost of Living Index} = \frac{\sum_i W_i P_i}{\sum_i W_i}.$$

Remark:-

$$\frac{\sum_i W_i P_i}{\sum_i W_i} = \frac{\sum_i p_{oi} q_{oi} \left(\frac{p_{1i}}{p_{oi}}\right)}{\sum_i p_{oi} q_{oi}} \times 100$$

$$= \frac{\sum_i p_{1i} q_{oi}}{\sum_i p_{oi} q_{oi}} \times 100.$$

Question:- What is Family Budget Survey and write down its uses?

Solution:- A survey is conducted among a sample of families from the class of people for whom the index number is intended and scrutinise their budgets in details. This survey is known as 'Family budget enquiry'. The data on the consumption pattern for a short period, say, per month, are collected from each selected family, and the whole enquiry is spread over one full year to account for the seasonal variations, such an enquiry yields data on characteristics like size of the family and it's composition, expenditure, incurred by family members; the data so collected are arranged to yield the average budget of expenditure for all the items

The question of determining the list of items, to be priced and their weights is very important in construction of CLI. This is formed by means of a family budget enquiry. On the basis of this enquiry, a list of items representing the level of living can also be determined. Weights which are proportional to consumption expenditure for items on each group and for the groups as well, are also determined from the family budget enquiry.

Remark:- In construction of CLI, the commodities consumed by the people being classified under the following heads:
Food, Clothing, Fuel and lighting, House and rent and Miscellaneous. Each group should include a representative sample of the items consumed by the people. A seperative index number is to be calculated for each of the major groups and the CLI is constructed by combining the group indices. To give the proper importance on weights to different items, it is necessary to group the similar type of items so that they should enter in the CLI with proper weights.

Main steps on the construction of Cost of Living Index Numbers:-

<1> Scope and Coverage :- At first, we choose a particular group of population or the class for which the index number is intended together with a well defined geographical region such as a city or a particular locality. So, the class should form a homogeneous group of people w.r.t. their income. If a sample is drawn from the class the sample is drawn by stratified sampling with proportional allocation.

<2> Base Period :- It should have all properties of a standard base period. But, it should have the length not greater than a month, normally a week or a fortnight.

<3> Selection of commodities :- The nature, quality and quantity of commodities consumed by the people, the commodities being classified under following heads :

(a) Food ; (b) Clothing ; (c) Fuel and light ; (d) House rent ;
(e) Miscellaneous.

The sample from each of these subgroups are taken satisfying the tastes, habits and customs of the selected class of people. And a list of items representing the level of living can be determined by 'Family Budget Enquiry'.

<4> Collection of Price data :- It is difficult and tedious to obtain retail prices since the retail prices vary from place to place, shop to shop and person to person. The price quotations should be obtained from the 'local markets' where the people reside or from bazaar, fair-price shops and departmental stores from which they usually make their purchases. So, the average prices of the commodities over the shops or markets are called price quotations. It is noted that the prices will be in the uniform units.

<5> Choice of weights :- For the selection of weights for the selected commodities in the construction, we consult 'Family Budget Enquiry'. It is a sample survey conducted by NSSO or CSO or Statistical Bureau selecting a sample of families from the class of people for whom the index is intended and scrutinise their budgets in details. The purpose of the survey is to determine the amount that an average family spends on different items of consumption.

The survey gives the information regarding the weights as —
(a) The nature, quantity and quality of the commodities consumed by the people, (b) The proportion of expenditure on each item bears to the total expenditure on the whole group, (c) The proportion of expenditure on each group bears to the total expenditure on all the groups. In short, "The weight is nothing but the percentage of expenditure on each item of goods and services of the basket, in relation to the total expenditure".

⟨6⟩ Computation of CLI :- The CLI is collected in two steps:
(a) A group index is calculated for each group : Food, Clothing, fuel & lighting, house rent and miscellaneous. A group index is a weighted average of the price relatives of the different items of the group, the weights being proportional to their consumption expenditure. Hence, the j^{th} group index is

$$I_j = \frac{\sum_{i=1}^{n_j} \left(\frac{p_{1i}}{p_{0i}}\right) w_i}{\sum w_i} \times 100$$

(b) The general index is calculated. The general index is the weighted average of the group indices (G_j), the weights being proportional to the consumption expenditure on the different groups. Hence, the general index, i.e., the CLI is,

$$I_{01} = \frac{\sum_j I_j w_j^*}{\sum_j w_j^*}$$, where, j varies over all the groups.

Use of Cost of Living Index Number :-

⟨1⟩ Cost of Living Index Numbers indicate whether the real wages are arising or falling, money wages remaining unchanged. In other words, they are used for the calculation of real wages and for determining the change in the purchasing power of money.

⟨2⟩ Cost of Living Indices are used for the regulation of dearness allowance or the grant of bonus to the workers so as to enable them to meet the increased cost of living.

⟨3⟩ These indices are also used for deflation of income and value series in national accounts.

⟨4⟩ By itself, CLI Numbers don't throw much light on the inflationary or deflationary trend on the soundness of an economy but in conjunction with other tools such as the indices of wholesale prices, wages, profits, production, employment, etc. It serves as an economic indicator for the analysis of price situation.

Question:- Given that CLI for 2004 with 1991 as base year is 250, will Mr. X be satisfied if his income rises to 12000 in 2004 as compared to Rs. 5000 in 1991?

Solution:-

Year	CLI	Income	Relative Income
1991	100	5000	50
2004	250	12000	48

Note that, Relative income $= \frac{\text{Income}}{\text{CLI}}$.

So, Mr. X will not suppose to be satisfied.

For satisfaction, his income needs to be $= 50 \times 250 = 12500$.

Question:- CLI of City A and B are respectively 300 and 400. For a Govt. Employee which city will be preferable and why?

Solution:- In City A, he needs to spend Rs. 300 to get his economic satisfaction to buy some products where as, in City B he needs to pay 400 Rs. for the same.

So, City A is more economical and Govt. Employee should prefer it.

Question:- "Compare to 2003, the purchasing power of the rupee in 2004 was 1·2" — Explain the statement. What effect would this have on Mr. X whose monthly income has remained fixed at Rs. 8000 over the two years?

Solution:- In 2004, $\text{CLI} = \frac{100}{\text{Purchasing power of 2004 compared to 2003}}$

$$= \frac{100}{1 \cdot 2}$$

$$= 83 \cdot 33$$

The above statement conveys that in 2004, the 2003 rupee could purchase 1·2 of the amounts it could purchase in 2003.

If Mr. X used an amount of 8000/- for full monthly satisfaction in 2003, now he has to spend $83 \cdot 33 \times 80 = 6666 \cdot 4$ rupees for getting equal satisfaction. He can save Rs. 1333·6.

LIST OF FORMULAE :-

It is an unit free number by which one can measure relative change in price or quantity or value of a set of commodities over two different situations.

Price Index Number

Wholesale Price Index Number — Consumer Price Index Number.

Notations:- 0 : base period , 1 : current period , i : i^{th} selected binary commodity, $i = 1(1)k$.

p: price , q : quantity .

p_{0i} : price of the i^{th} binary commodity at base period,

q_{0i} : quantity of the i^{th} binary commodity at base period,

w_i : weight corresponding to the i^{th} selected binary commodity.

I_{01} : Price Index number of '1' current year period with respect to '0' base period.

Simple Aggregative Index (P_{01}) : $\frac{\sum p_{1i}}{\sum p_{0i}} \times 100$.

Weighted Aggregative Index (P_{01}) : $\frac{\sum p_{1i} w_i}{\sum p_{0i} w_i} \times 100$.

Laspeyre's Index: $L_{01} = \frac{\sum p_{1i} q_{0i}}{\sum p_{0i} q_{0i}}$, $[w_i = q_{0i}]$

Paasche's Index: $P_{01} = \frac{\sum p_{1i} q_{1i}}{\sum p_{0i} q_{1i}}$, $[w_i = q_{1i}]$

Edgeworth-Marshall Index: $E_{01} = \frac{\sum p_{1i}(q_{0i} + q_{1i})}{\sum p_{0i}(q_{0i} + q_{1i})}$ $\left[w_i = \frac{q_{0i} + q_{1i}}{2}\right]$

Walsch's Index: $W_{01} = \frac{\sum p_{1i}\sqrt{q_{0i} q_{1i}}}{\sum p_{0i}\sqrt{q_{0i} q_{1i}}}$ $\left[w_i = \sqrt{q_{0i} q_{1i}}\right]$

Fisher's Index: $F_{01} = \sqrt{L_{01} \cdot P_{01}}$

Bowley's Index: $B_{01} = \frac{L_{01} + P_{01}}{2}$.

Consumer Price Index:-

(i) Consumer Price Index $= \frac{\sum p_{1i} q_{0i}}{\sum p_{0i} q_{0i}} \times 100$

(ii) Weighted Consumer Price Index $= \frac{\sum PW}{\sum W}$

(iii) Consumer Price Index $= \frac{\sum IW}{\sum W}$.

Test for Index Numbers:-

1. Time Reversal test is satisfied when $P_{01} \times P_{10} = 1$.

2. Factor reversal test is satisfied when $P_{01} \times Q_{01} = \frac{\sum p_{1i} q_{1i}}{\sum p_{0i} q_{0i}} = V_{01}$.

3. Circular test is satisfied when $P_{01} \times P_{12} \times P_{20} = 1$.

DEMAND ANALYSIS

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DEMAND ANALYSIS

__INTRODUCTION__ :- Government formulates economic policies in order to have a balanced economic structure in the country. The demand analysis deals with the following two important aspects of economic statistics:

(i) Demand analysis studies the relationship between market price and demand on the basis of market data (time series data).

(ii) How is the demand affected by gradual changes in 'income level', on the basis of family budget data (also called cross-sectional data)?

__Necessities & Luxuries__ :- The goods or commodities which satisfy our primary needs (e.g., food, cloth, housing, etc.) are called __necessities__. The goods which satisfy luxuries (e.g., ornaments, liquor, cigars, etc) are called __luxuries__. There is an intermediate class of wants, viz., those wants which are not primary wants but are required by convention or habits without which we may not live happily, those goods are termed as __conventional necessities__.

__Demand and Supply__ :- In economics, mere desire for a commodity does not mean demand, unless one can pay and is willing to pay the necessary amount for it. Thus the increase in demand for a particular commodity does not merely mean an increase in the desire for that commodity but it implies that the desire to have it at a given price has increased. __Demand__ for any commodity depends on a number of factors such as price, income, time, price of other commodities, etc.

The functional relationship between consumption of commodity and the factors responsible for the changes in consumption, is defined as the __demand function__ of the commodity. From the demand function, we can find different quantities of a commodity demanded at different prices.

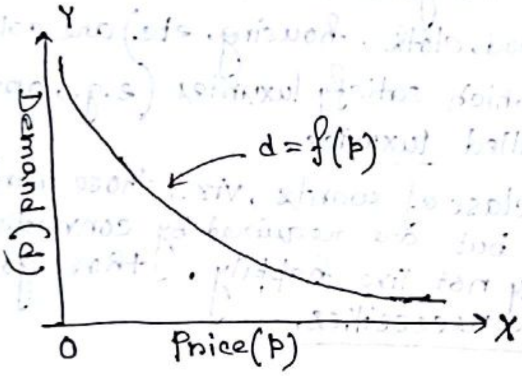
In economics, __supply__ means the amount of commodity available at a given price, which in turn depends on the amount produced which further depends on the price. Supply is a function of price at which commodity is sold in the market.

Laws of supply and demand :-

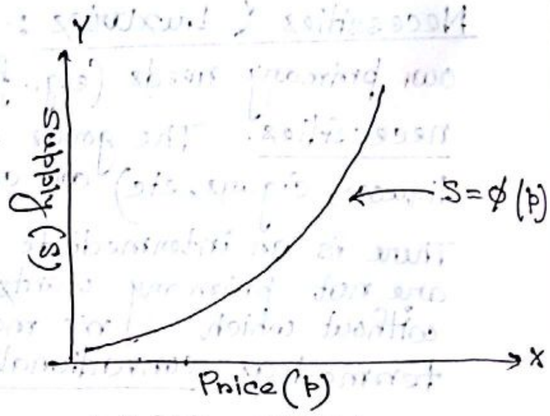
" Demand for a commodity, in general, varies in the direction opposite to that of price whereas supply in general varies in the same direction as price".

According to Cournot, demand is a function of price, i.e., $d = f(p)$, the only assumption regarding $f(p)$ is that it is a diminishing function of p, i.e., $f'(p) < 0$. And also supply is a function of p, i.e., $s = \phi(p)$, where $\phi(p)$ is an increasing function on p, i.e., $\phi'(p) > 0$.

Here is the graphical representation of supply and demand functions:

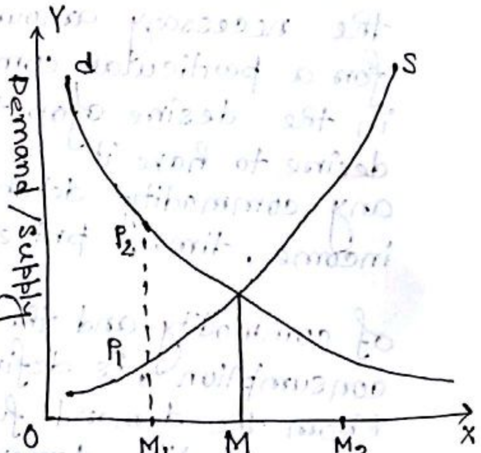


-: Demand curve :-



-: Supply curve :-

These laws further states that the market price settles at a level at which supply and demand are equal and is determined by the point of intersection of the two curves $f(p) = d$ and $\phi(p) = s$. The price OM determined by the point of intersection of the curves $d = f(p)$ and $s = \phi(p)$ is termed as equilibrium price. Thus, the equilibrium price is the solution of the equation $f(p) = \phi(p)$.



-: Demand & supply curve :-

Remark:-

There are some exceptional demand curves which instead of sloping downwards rise upwards. However, Robbert Giffen one such situation in Ireland during 19th century, when as a result of serious fall in the real wage of the workers, even after a sharp rise in the prices of bread, its consumption was on the increase since bread was still cheapest food as compared others. This phenomenon is termed as Giffen's paradox. The demand for Giffens goods, i.e., inferior goods, rises with a rise in price and falls with a fall in price.

WORKED OUT EXAMPLES :-

Ques:- The demand curve and the supply curve of a commodity are given by $D = 19 - 3p - p^2$ and $S = 5p - 1$. Find the equilibrium price and the quantity exchanged.

Solution:- For equilibrium, we have $D = S$.

$$\Rightarrow 19 - 3p - p^2 = 5p - 1$$
$$\Rightarrow p^2 + 8p - 20 = 0$$
$$\Rightarrow (p+10)(p-2) = 0$$
$$\Rightarrow p = 2,\ p = -10 \text{ (rejected)}$$

So, equilibrium price is $p = 2$.
Substituting it in the demand or supply curve, we get $D = S = 9$.

Ques:- The demand function of two commodities A and B are

$$D_A = 10 - p_A - 2p_B,\quad D_B = 6 - p_A - p_B$$

and the corresponding supply functions are

$$S_A = -3 + p_A + p_B;\quad S_B = -2 + p_B$$

where p_A and p_B denote the prices of A and B respectively. Find (i) equilibrium prices (ii) equilibrium quantities exchanged in the market.

Solution:- For equilibrium, we have $D_A = S_A$, $D_B = S_B$.

$$\Rightarrow 10 - p_A - 2p_B = -3 + p_A + p_B \ \&\ 6 - p_A - p_B = -2 + p_B$$
$$\Rightarrow 2p_A + 3p_B - 13 = 0 \text{ and } p_A + 2p_B - 8 = 0$$

solving, we have $p_A = 2$, $p_B = 3$ as equilibrium prices.
Substituting in demand function or supply function, the equilibrium quantities are given by $D_A = S_A = 2$ and $D_B = S_B = 1$.

Ques:- Demand and supply functions for tea are given by

$$x_d = 120 - 2p + 5\frac{dp}{dt} \text{ kgms. per week}$$

$$x_s = 3p - 30 + 50\frac{dp}{dt} \text{ kgms. per week}$$

where p is the price at time t. Find the time path of p for dynamic equilibrium if the initial price is given to be 36 paise per kg.

Solution:- For equilibrium condition, we have $x_d = x_s$.

$$\Rightarrow 120 - 2p + 5\frac{dp}{dt} = 3p - 30 + 50\frac{dp}{dt}$$
$$\Rightarrow \frac{dp}{dt} + \frac{p}{9} = \frac{10}{3}, \text{ which is a linear differential equation}$$

Integrating factor: $e^{\int \frac{1}{9}dt} = e^{t/9}$.

Solution is given by $p \cdot e^{t/9} = \frac{10}{3} \cdot \frac{e^{t/9}}{1/9} + c \Rightarrow p = 30 + ce^{-t/9}$.

Initial price is 36 p/kg, so $c = 6$, when $t = 0$, $p = 36$.

So, $p(t) = 30 + 6e^{-t/9}$.

Price Elasticity of Demand :- One of the most important characteristics of demand function is what is known as its 'elasticity' — according to the law of demand, the changes in price and demand are in opposite direction and it is a common experience that price changes affect the demand for different commodities in different degrees. The quality of demand by virtue of which it extends or contracts with a fall or rise in price is known as '*price elasticity of demand*', a term introduced by Marshall.

Definition:- Price elasticity of demand is defined as the value of the ratio of the relative (or proportionate) change in the demand to the relative (or proportionate) change in price.

Mathematically, let x be the quantity demanded of a commodity 'A' such that the demand function of A is $x = f(p)$, where $f(\cdot)$ is a continuous function. Let the increment in demand x, corresponding to an increment δp in p, be δx. Then elasticity of demand (η_D) is given by

$$\frac{\text{Proportionate change in demand}}{\text{Proportionate change in price}} = \frac{(\delta x)/x}{(\delta p)/p} = \frac{p}{x}\cdot\frac{\delta x}{\delta p}$$

This is the average elasticity of demand over the price change $(p, p+\delta p)$.

The elasticity of demand (η_p) at a particular price level p is given by

$$\eta_p = \lim_{\delta p \to 0} \frac{p}{x}\cdot\frac{\delta x}{\delta p} = \frac{p}{x}\lim_{\delta p\to 0}\frac{\delta x}{\delta p}$$

$$= -\frac{p}{x}\cdot\frac{dx}{dp}$$

[negative sign being taken for the case of demand & price move in opposite direction]

$$= -\frac{p}{f(p)}\cdot\frac{df}{dp}$$

$$= -\frac{d\log f}{d\log p}.$$

Remark:- 1. Since $d = f(p)$ is a decreasing function of p,

$$\frac{df}{dp} < 0 \Rightarrow \eta_p = -\frac{p}{f}\cdot\frac{df}{dp} > 0$$

i.e., price elasticity of demand is always positive.

For $\eta = 1$, elasticity is called normal.

For $\eta > 1$, demand is called overealistic.

For $\eta < 1$, demand is called underealistic.

2. Significance of Elasticity of Demand:– In order to understand the significance of the price elasticity of demand of a commodity for market analysis, we need to consider the total outlay or market turn over for the commodity, viz.,

$$F(p) = p \times d = p \times f(p)$$ is the total expenditure of the population for the purchase of the given commodity.

$$\frac{d}{dp}\left[F(p)\right] = 1 \cdot f(p) + p \cdot f'(p)$$

$$= f(p)\left[1 + p \cdot \frac{f'(p)}{f(p)}\right]$$

$$= f(p)\left[1 - \eta_p\right]$$

(i) If $\eta_p = 1$, then $F(p)$ is constant.

i.e., the money value of the turnover is constant independent of variations in prices of the commodity.

$$F(p) = p \cdot f(p) = c \Rightarrow d = f(p) = \frac{c}{p}.$$

Hence, if $\eta_p = 1$, the demand d is inversely proportional to the price p.

(ii) If $\eta_p < 1$, then $\frac{d}{dp}\left[F(p)\right] > 0$, the

$\Rightarrow F(p)$ is an increasing function of p, i.e., money value of the turnover increases as the prices rise.

(iii) If $\eta_p > 1$, then $\frac{d}{dp}\left[F(p)\right] < 0$,

$\Rightarrow F(p)$ is a decreasing function of p, i.e., the money value of the turnover decreases as the prices rise.

A knowledge of η_p for a given commodity will enable us to determine if the increase (or fall) in the price results in an increase in the turnover and the profit of the monopolist and if so, its extent also.

3. Demand function with constant price elasticity:–

The demand function $d = f(p)$ with constant price elasticity of demand, say, $\eta = \alpha \ (>0)$ at all the points of the curve, is given by

$$-\frac{p}{f} \cdot \frac{df}{dp} = \alpha \Rightarrow \frac{df}{f} = -\frac{dp}{p} \cdot \alpha$$

After integration, $\log f = -\alpha \log p + \log k = \log\left(Kp^{-\alpha}\right)$

$$\Rightarrow d = f(p) = Kp^{-\alpha} \ ; \alpha > 0, K > 0.$$

$$f'(p) = -K\alpha p^{-\alpha-1}.$$

Hence, the price elasticity of demand is given by

$$\eta_p = -\frac{p}{f} \cdot f'(p) = -\frac{p}{Kp^{-\alpha}} \cdot \left(-K\alpha p^{-\alpha-1}\right)$$

Thus, we conclude that the curve of the constant elasticity of demand is a hyperbola whose shape depends on the value of the parameter α

In particular, if we take $\alpha = 1$, the demand curve of constant elasticity becomes $d = f(p) = kp^{-1}$

$\Rightarrow d.p = k$, which is the equation of a rectangular hyperbola ($xy = c$).

Ques :- If the demand function is $p = 4 - 5x^2$, for what values of x the elasticity of demand will be unitary ?

Solution:- $p = 4 - 5x^2$

Differentiating w.r.t. p, we have

$$1 = -10x \cdot \frac{dx}{dp}$$

$$\Rightarrow \frac{dx}{dp} = -\frac{1}{10x}$$

$$\therefore \eta_p = -\frac{p}{x} \cdot \frac{dx}{dp} = \frac{4 - 5x^2}{10x^2}$$

Elasticity of demand will be unitary if $\frac{4-5x^2}{10x^2} = 1 \Rightarrow x = \frac{2}{\sqrt{15}}$

Ques:- If the demand curve is of the form
$$p = ae^{-kx}$$
where, p is the price and x is the demand, prove that the elasticity of demand is $\frac{1}{kx}$. Hence deduce the elasticity of demand on the curve $p = 10e^{-x/2}$.

Solution:- $p = ae^{-kx}$

Differentiating w.r.t. p, we get

$$1 = -ake^{-kx} \cdot \frac{dx}{dp}$$

$$\therefore \eta_p = -\frac{p}{x} \cdot \frac{dx}{dp} = \frac{ae^{-kx}}{x} \cdot \frac{1}{ake^{-kx}} = \frac{1}{kx}$$

(Proved)

Comparing $p = ae^{-kx}$ & $p = 10e^{-x/2}$, we have

$a = 10$, $k = 1/2$.

So, $\eta_p = \frac{2}{x}$.

Ques:- The price elasticity of demand curve $x = f(p)$ is of the form $(a - bp)$, where a and b are given constants. Find the demand curve.

Solution:-

$$\eta_D = -\frac{p}{x}\cdot\frac{dx}{dp} = a - bp$$

$$\therefore \left(\frac{a - bp}{p}\right) dp + \frac{dx}{x} = 0$$

$$\therefore \left(\frac{a}{p} - b\right) dp + \frac{dx}{x} = 0$$

Integrating, we get

$$(a\log p - bp) + \log x = \log c$$

$$\Rightarrow \log\left(p^{\alpha} e^{-bp}\right) + \log x = \log c$$

$$\Rightarrow x p^{\alpha} e^{-bp} = c$$

$$\Rightarrow x = c p^{-\alpha} e^{bp}.$$

Price Elasticity of Supply:- If $s = \phi(p)$ is the supply curve for a commodity 'A' then the price elasticity of supply of A at price p is given by

$$\varepsilon_p = \frac{\text{Relative change in supply of 'A'}}{\text{Relative change in price of 'A'}} = \frac{\delta s/s}{\delta p/p} = \frac{p}{s}\cdot\frac{\delta s}{\delta p}$$

The price elasticity of supply (ε_p) at a particular price level p is given by

$$\varepsilon_p = \lim_{\delta p \to 0} \frac{p}{s}\cdot\frac{\delta s}{\delta p}$$

$$= \frac{p}{s}\cdot\lim_{\delta p\to 0}\frac{\delta s}{\delta p}$$

$$= \frac{p}{s}\cdot\frac{ds}{dp}$$

$$= \frac{d\log s}{d\log p}.$$

[positive sign is taken since price & supply change in the same direction]

Taking $s = \phi(p)$, we get $\varepsilon_p = \frac{p}{s}\cdot\frac{d}{dp}[\phi(p)] = \frac{p}{\phi(p)}\cdot\phi'(p)$

Since, $\phi(p)$ is an increasing function of p, so $\phi'(p) > 0$ & $\varepsilon_p > 0$.

Supply curve with the Constant Price Elasticity :-

The supply function $s = \phi(p)$ with constant price elasticity of supply, $\varepsilon_p = \alpha > 0$, at all points of the curve is given by

$$\frac{p}{\phi(p)} \cdot \phi'(p) = \alpha$$

$$\therefore \quad \frac{\phi'(p)}{\phi(p)} = \frac{\alpha}{p}$$

On integration $\quad \log \phi(p) = \alpha \cdot \log p + \log c = \log(c \cdot p^{\alpha})$

$$\therefore \quad s = \phi(p) = c \cdot p^{\alpha} \; ; \; c > 0, \alpha > 0.$$

Hence $\log s = \alpha \log p + \log c$ represents the curve of constant elasticity of supply which can be graphically drawn on a double logarithmic scale is a straight line.

Income Elasticity of Demand :- In general, the demand function x for any commodity 'A' can be written as

$$x = f(\mu, p_1, p_2, p_3, \ldots, p_n),$$

where μ is income of the people, p is the price of commodity A, $p_1, p_2, \ldots, p_n$ are the prices of related commodities, say, $A_1, A_2, \ldots, A_n$.

Income Elasticity of Demand — Suppose that all prices are assumed to remain constant while income is variable. As income changes prices remaining the same, there will be an income effect and under the influence of the income effect the quantities demanded will change. The quantity demanded will therefore be a function of income only. Let q represent the quantity demanded and y represent income. Then the demand function can be written as $q = q(y)$. It is also known as Engel curve or the income demand curve.

The elasticity of this function is given by $\eta_e = \dfrac{dq/q}{dy/y} = \dfrac{y}{q} \cdot \dfrac{dq}{dy}$

since $y > 0$, $q > 0$, it is clear that if $\dfrac{dq}{dy} > 0$, then $\eta_e > 0$.

The concept of income elasticity of demand can be used for classifying commodities.

Value of income elasticity	Type of goods
$\eta_e > 1$	Normal luxury
$0 < \eta_e < 1$	Normal necessity
$\eta_e < 0$	Inferior necessity

Question:- The demand function for a commodity 'X' is given by

$$x = 300 - 0.5 p_x^2 + 0.02 p_o + 0.05 y$$

where x is the quantity demanded of 'x', p_x is the price of x, p_o is the price of a related commodity and y is the constant income. Compute

(i) The price elasticity of demand for x.

(ii) The income elasticity of demand for x.

Solution:- (i) $\eta_{px} = -\frac{p_x}{x} \cdot \frac{\partial x}{\partial p_x}$

$$= \frac{-p_x}{300 - 0.5 p_x^2 + 0.02 p_o + 0.05 y} \times \{0.05(-2p_x)\}$$

$$= \frac{p_x^2}{300 - 0.5 p_x^2 + 0.02 p_o + 0.05 y}$$

When $p_x = 12$; $p_o = 10$, $y = 200$,

$$\eta_{px} = 0.60 \text{ (approx.)}$$

(ii) $\eta_e = \frac{y}{x} \cdot \frac{\partial x}{\partial y} = \frac{0.05 y}{300 - 0.5 p_x^2 + 0.2 p_o + 0.05 y}$

when $p_x = 12$, $p_o = 10$, $y = 200$,

$$\eta_e = \frac{10}{237.8} = 0.04 \text{ (approx.)}$$

Theorem:- If the proportion of income spent on any commodity remains constant as income increases than the income elasticity of demand for this commodity is equal to unity.

Solution:- Let us suppose that y is the income of the consumer and q is the quantity demanded of any commodity whose price per unit is p. Then total expenditure on this commodity equal to $p.q$ and the proportion of income spent on this commodity $= \frac{p.q}{y}$. Let us suppose that the price of the commodity remains constant. If the proportion remains constant as income increases,

$$\frac{d\left(\frac{p.q}{y}\right)}{dy} = 0 \quad \text{or,} \quad \frac{p\frac{dq}{dy} y - p.q}{y^2} = 0$$

$$\text{or, } p\frac{dq}{dy} \cdot y = p.q$$

$$\text{or, } \frac{dq}{dy} \cdot \frac{y}{q} = 1.$$

or, Income elasticity $= 1$.

Remark:- The theorem provides us with a convenient way of considering whether the income elasticity of demand for any commodity is greater than or less than unity. It has been empirically founded by Engel that an income increases the proportion of income spent on food grains decreases. This is known as Engel's Law.

Engel's Law and Engel's Curve: — A german statistician Ernst Engel after a detailed and systematic study of the family budgets has given the following law:
" As the income grows, the share of income spent on food decreases"
In other words, " The proportion of expenditure on food decreases as household expenditure increases". — Engel's law.

Hence, as income increases the expenditures on different items have changing proportions, and the proportions devoted to urgent needs decrease while for luxuries and semi-luxuries increase.

The graphic representation of the basic relationship between household income and its expenditure on a particular item of consumption is known as Engel's curve.

In general, the demand for any commodity among a class of people may be regarded as depending on the two factors, the price of the commodity and the income of the people, the two factors not necessarily summing up. Thus,

$$d = f(\mu, p),$$

where d is the demand for a commodity, p is its price and μ, the national income.

Since demand is, in general, an increasing function of income μ and decreasing function of price p, we have

$$\frac{\partial}{\partial \mu}(d) > 0 \text{ and } \frac{\partial}{\partial p}(d) < 0$$

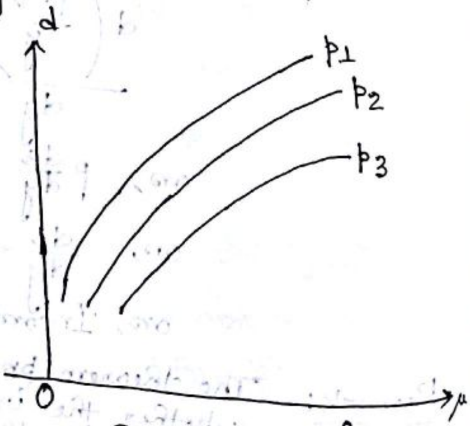
Regarding demand function d as a two parameter function of price (p) and income(μ) it can be represented graphically by a certain surface D in the three-dimensional space, taking the three variables d, p and μ along three rectangular coordinate axes Od, Op and Oμ, O being the origin.

Engel's curves for constant prices: — In particular, if we regard price as fixed constant, $p = p^*$, (say), then the demand function becomes

$$d = f(p^*, \mu) = f_1(\mu),$$

i.e., d becomes a single parameter function of μ. These curves are called Engel's curves for constant prices.

For constant price p, the Engel's curve is concave downwards, i.e., $\frac{\partial^2}{\partial \mu^2}(d) < 0$. This means that as income increases, the rise in demand is slower and slower; a conclusion contained in Engel's law.

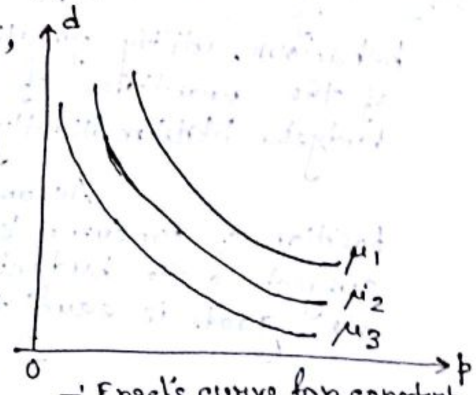


—: Engel's curve for constant prices :—

Engel's curves for constant income:- If we regard income as constant, $\mu = \mu^*$, (say), then the demand function becomes

$$d = f(p; \mu^*) = f_2(p);$$

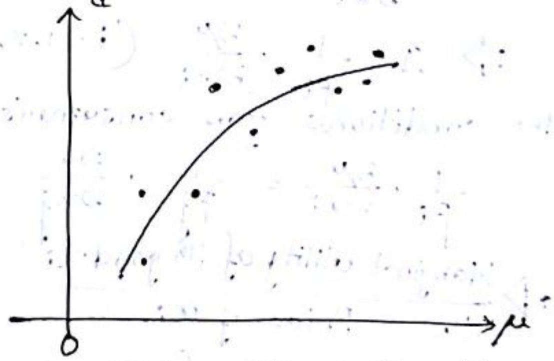
i.e., d becomes a function of the single parameter p (price).
These are called Engel's curve for constant income.



-: Engel's curve for constant incomes: -

Methods of drawing Engel's curves :— (For constant prices)

METHOD 1. The method consists in simultaneously study of the budgets of different families with different income levels. Let d_i be the demand for a commodity at income level μ_i, $i = 1, 2, 3, \ldots, n$. The Engel's curve $d = f(p^*, \mu) = f_1(\mu)$; $p = \text{constant} = p^*$, it then obtained by using the principle of least squares.



This method assumes that the consumption pattern of families at different levels of income is same, an assumption which is far from reality. This drawback may be overcome by stratifying the given population into relative homogeneous groups.

METHOD 2. It consists in comparing the budgets of the same family during different periods of time and studying their consumption patterns on different items of consumption as a consequence of variation in income. The obvious drawback of this method is that the assumption $p = \text{constant} = p^*$ during the periods under consideration is not generally true. Hence this method can be recommended only in the situations when the prices of given commodity and the substitution and complementary goods remain more or less constant during the given period of investigation.

Utility Function:- According to the classical theory of consumer behaviour, utility function (u) can be regarded as a function of the quantities of goods, x_i ($i=1,2,3,\dots,n$) in the consumer's budget. Mathematically, $u = \phi(x_1, x_2, \dots, x_n)$ .... (*)

According to the principle of substitution, the position of consumer's equilibrium is obtained on maximising (*) subject to the budget constraint that aggregate expenditure on all goods is equal to the income, i.e.,

$$\sum_{i=1}^{n} p_i x_i = y \quad \text{——— (1)}$$

where, p_i is the price of the quantity consumed of i^{th} good i.e., x_i and y is the income.

In other words, for consumer's equilibrium, we have to maximise

$$Z = u - \lambda\left[\sum_{i=1}^{n} p_i x_i - y\right]$$

unconditionally, where λ is Lagrange's multiplier.

For extremum, $\dfrac{\partial Z}{\partial x_i} = 0 = \dfrac{\partial u}{\partial x_i} - \lambda p_i$

$$\Rightarrow \lambda = \frac{1}{p_i}\cdot\frac{\partial u}{\partial x_i} \quad (i = 1,2,3,\dots,n)$$

This gives the conditions for consumer's equilibrium as:

$$\frac{1}{p_i}\cdot\frac{\partial u}{\partial x_i} = \frac{1}{p_j}\cdot\frac{\partial u}{\partial x_j} \quad (i \neq j = 1,2,\dots,n)$$

$$\Rightarrow \frac{\text{Marginal utility of } i^{th} \text{ good } x_i}{\text{Price of } x_i} = \frac{\text{Marginal utility of } j^{th} \text{ good } x_j}{\text{Price of } x_j}$$

$$\Rightarrow \frac{\text{Marginal utility of } i\text{th good}}{\text{Marginal utility of } j\text{th good}} = \frac{p_i}{p_j} \quad \text{——— (2)}$$

There are $(n-1)$ such independent marginal utility ratios which provide $(n-1)$ relationships between the quantities of the goods consumed and the relative prices or price ratios. These $(n-1)$ equations along with the budget constraint enable us to solve for x_i's ($i = 1,2,\dots,n$) in terms of the prices p_i's and income y.

Thus, we see that marginal utility equations (2) which are derived from the total utility functions, are used together with the budget equation (1) to get demand functions. Hence the parameters of the utility function will determine the parametric structure of the demand functions.

Ex:- If $u = cx^{\alpha}y^{\beta}$ is an individual's utility function of two goods, show that his demand for the goods is

$$x = \frac{\alpha}{\alpha+\beta}\cdot\frac{\mu}{p_x}, \quad y = \frac{\beta}{\alpha+\beta}\cdot\frac{\mu}{p_y}.$$

where, p_x and p_y are fixed prices and μ is individual's fixed income. Deduce that the elasticity of demand for either good with respect to income or price is equal to unity in absolute value.

Solution:- $\frac{1}{p_x}\cdot\frac{\partial u}{\partial x} = \frac{1}{p_y}\cdot\frac{\partial u}{\partial y}$

$$\Rightarrow \frac{1}{p_x}\cdot c\alpha x^{\alpha-1}y^{\beta} = \frac{1}{p_y}\cdot c\beta x^{\alpha}y^{\beta-1}$$

$$\Rightarrow \frac{\alpha y}{p_x} = \frac{\beta x}{p_y}$$

$$\Rightarrow y\cdot p_y = \frac{\beta x}{\alpha}\cdot p_x \quad \cdots\cdots (1)$$

Since individual's income is fixed, we have

$$xp_x + yp_y = \mu$$

$$\Rightarrow xp_x + \frac{\beta}{\alpha}\cdot xp_x = \mu$$

$$\Rightarrow x = \frac{\alpha}{\alpha+\beta}\cdot\frac{\mu}{p_x}$$

Hence, $y = \frac{\beta}{\alpha+\beta}\cdot\frac{\mu}{p_y}$ [substituting x in (1)]

Income and price elasticities of demand for 'x' are given by

$$\eta_\mu(x) = \frac{\mu}{x}\cdot\frac{\partial x}{\partial \mu}$$

$$= \frac{\mu}{x}\cdot\frac{\alpha}{\alpha+\beta}\cdot\frac{1}{p_x}$$

$$= 1.$$

$$\eta_D(x) = -\frac{p_x}{x}\cdot\frac{\partial x}{\partial p_x} = \frac{p_x}{x}\cdot\frac{\alpha}{\alpha+\beta}\cdot\frac{\mu}{p_x^2} = 1$$

$$\therefore \eta_\mu(x) = \eta_D(x) = 1.$$

Similarly, it can be shown that $\eta_\mu(y) = \eta_D(y) = 1.$

Types of Data Required for Estimating Elasticities :-

The empirical demand analysis is based on the data obtained from two main sources of statistical observation.

(a) Family Budget Data:- Family budget data are collected through sample surveys covering sample of households which are representative of different classes of people w.r.t. income, family size, social class etc., and their expenditure on budget items during a period of a year or over is recorded. In order to study the influence of income level on the expenditure habits of the people, we carry out an experiment consisting of the following main steps:

1. The first step is to select a group of households which are as homogeneous as possible w.r.t. regional environments, social and economic characteristics, family size and other factors that affect the demand, without making any reference to their family income.

2. The next step consists in regulating the family income by allotting the households to different income levels at random, randomisation being resorted to neutralise the effect of factors other than income.

3. Finally a detailed account of the expenditure of each household during a period of a year on various budget items is compiled.

(b) Market Statistics or Time Series Data:- By market statistics we understand time series data relating to the prices of the commodities and their quantities bought or sold at that price at different points of time. The treatment of such data is quite analogous to that of family budget data except that demand is now primarily regarded as a function of price and not of income. The market price of any commodity settles at a level known as the 'equilibrium price' p_1 (say), which is the intersection of the supply and demand curves, $d = f(p)$ and $s = \phi(p)$. A variation in the price of a commodity over time means a shift in either or both of the demand and supply curves. If both the curves $d = f(p)$, $s = \phi(p)$ remain fixed, then the market data remains more or less static and doesn't provide enough number of points for their determination. If both the supply and demand curves shift their positions, then it is unlikely to trace either the supply or the demand functions closely. However, if one of the two curves remains fixed and the other changes its position then the family budget data provide a number of points on the fixed curve and hence the curve, is determined.

Again the demand for any commodity doesnot depend only on its price but also on a number of factors such as income, the price of the substitution (i.e., price of related commodities), etc. Hence for sound statistical analysis of demand, we should either take into account those factors explicitly or eliminate their effect on the demand and the price.

<u>Remark</u>:- (1) We see that both the methods, viz., the family budget (cross-sectional) data and the time series (market statistics) data serve to single out the effect of just one factor, viz., income and price respectively by nutralising the simultaneous effect of other factors that influence the demand.

(2) Taking demand function $d = f(x_1, x_2, \ldots, x_n)$, We know demand function is negatively sloped. Some useful assumptions are:

(i) The shape of the demand curve should remain fixed as the supply curve shifts its position from time to time. <u>e.g.</u> — cournot and Marshall demand curves.

(ii) The demand curve is of constant elasticity.

(iii) Demand functions are of the following forms:

$$d = a_0 + a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

or, $d = a_0 x_1^{a_1} \cdots x_n^{a_n}$; where a_i $(i = 1(1)n)$ are constants.

Methods of Estimating Demand functions :—

(1) <u>Leontief's</u> Method (From Time Series Data) :—

<u>Assumptions</u>: — (i) Each market transaction represents the intersection of instantaneous demand and supply curves which change their position from time to time. This implies that in addition to determining the elasticities of demand and supply we must also study the extent to which the curves have shifted from time to time.

(ii) The shifting of supply and demand curves are independent of each other and donot affect the shape of the curves. This means that a shift of demand curve to the right is just as likely to be associated with a shift of the supply curve to the left as to the right.

(iii) Each of the supply and demand curves is of constant elasticity i.e., the demand and supply curves when plotted on a double logarithmic scale should be straight lines.

If Y_t and X_t are the logarithms of the consumption and price of a commodity at time t, $(t = 1, 2, \ldots, n)$, then we have

$$\text{Demand Curve: } Y_t = \eta_1 X_t + U_t \quad \text{——— (1)}$$

$$\text{Supply Curve: } Y_t = \eta_2 X_t + V_t \quad \text{——— (2)}$$

where, η_1 and η_2 are the elasticities of demand and supply respectively and U_t and V_t are independently distributed with

$$E(U_t) = E(V_t) = 0.$$

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Note that in ① and ②, we have taken Y_t for consumption as well as supply, since for market equilibrium, we have

$$d = S \Rightarrow \log d = \log S = Y_t \text{ (say)}.$$

① and ② can be written as

$$Y_t - \eta_1 X_t = U_t \quad \text{—— ③}$$
$$Y_t - \eta_2 X_t = V_t \quad \text{—— ④}$$

Multiplying ③ and ④, we get

$$Y_t^2 + \eta_1\eta_2 X_t^2 - (\eta_1+\eta_2) X_t Y_t = U_t V_t$$

U_t and V_t are independently distributed with $E(U_t) = 0 = E(V_t) = 0$, such that $Cov(U_t V_t) = 0 \Rightarrow E(U_t V_t) = 0$

$$\text{So, } \sum_t U_t V_t = 0.$$

We get, $\sum Y_t^2 + \eta_1\eta_2 \sum X_t^2 - (\eta_1+\eta_2)\sum X_t Y_t = 0$

Time range $t : [1, n]$ is divided into two equal halves:

$$t_1 : [1, \tfrac{n}{2}] \text{ and } t_2 : [\tfrac{n}{2}+1, n]$$

$$\left.\begin{aligned} &\sum_{t=1}^{n/2} Y_t^2 + \eta_1\eta_2 \sum_{t=1}^{n/2} X_t^2 - (\eta_1+\eta_2)\sum_{t=1}^{n/2} X_t Y_t = 0 \\ \text{and } &\sum_{t=\frac{n}{2}+1}^{n} Y_t^2 + \eta_1\eta_2 \sum_{t=\frac{n}{2}+1}^{n} X_t^2 - (\eta_1+\eta_2)\sum_{t=\frac{n}{2}+1}^{n} X_t Y_t = 0 \end{aligned}\right\} \text{—— (*)}$$

$\sum Y_t^2$, $\sum X_t^2$ and $\sum X_t Y_t$ can be calculated from the time series data and (*) can be solved simultaneously for η_1 & η_2.

Limitations :- (i) The assumption that demand and supply curves can shift in any direction, independent of each other violates the fundamental principle of general theory of equilibrium, viz., " the demand for any one commodity is a function not only of its price but of all the prices ".

(ii) Leontief's assumption that the supply and demand curves shift simultaneously is not a reasonable assumption for agricultural commodity.

(iii) There is no way of testing the validity of the assumption (iii) viz., the elasticities of demand and supply curves are constant.

Criticism:- Apart from a number of economic objections inherent in the assumptions made above, the procedure adopted for estimating η_1 and η_2 is faulty from statistical point of view. The mathematical solution of (*) leads to two curves only when the ellipses of the two scatters into which Leontief breaks up his series are not similar and the corresponding axes are not parallel to one another. But a significant difference between the scatters of the first half and the second half periods indicates that the data are not homogeneous and as such each period needs to be studied separately.

(2) Pigous's Method (From Time Series Data):—

Assumptions:- (i) The demand curve is likely to have a smooth appearance in each interval i.e., the demand curve, for each interval of time, is a curve of constant elasticity, given by

$$d = cp^{-\alpha}$$

$$\Rightarrow \log d = \log c - \alpha \log p$$

$$\Rightarrow \log d = \log c + a \log p \quad (\because a = -\alpha)$$

$$\Rightarrow Y = aX + b;$$

(ii) The demand curve shifts steadily over different periods of time, the rate of shifts being equal in two successive intervals. In other words, Pigou assumed that the rate of shift is such that distance between the ith and (i+1)th position on a logarithmic scale is same as the distance between (i+1)th and (i+2)th position.

Hence according to the assumptions, we have

$$Y_1 = aX_1 + b$$

$$Y_2 = aX_2 + b + r$$

$$\vdots$$

$$Y_i = aX_i + b + (i-1)r$$

$$Y_{i+1} = aX_{i+1} + b + ir$$

$$Y_{i+2} = aX_{i+2} + b + i(r+1)$$

Accordingly, we have

$$(Y_{i+1} - Y_i) - a(X_{i+1} - X_i) = (Y_{i+2} - Y_{i+1}) - a(X_{i+2} - X_{i+1})$$

$$\Rightarrow a = \frac{Y_{i+2} - 2Y_{i+1} + Y_i}{X_{i+2} - 2X_{i+1} + X_i}, \; (i = 1, 2, \ldots) \qquad (*)$$

Pigou's method involves the following steps:

1. Prepare a table of logarithms of time-series values of consumption (Y) and price (X), i.e., log d and log p.

2. Compute a_i from (*). Since demand is diminishing function of price, if for any interval the value of 'a' comes out to be positive it can't be taken as a measure of elasticity of demand for the relevant set of times. On the other hand, if it comes out to be negative it may be regarded as a measure of the elasticity.

3. If negative a_i's exceed the positive a_i's and if all the a_i's are grouped fairly closely about a given value and further if the data are not suspected otherwise then each a_i can be regarded as an observation on the unknown elasticity of demand curve. The mean value of a_i's then taken as a measure of unknown elasticity.

Limitations :- 1. Pigou's method is based on the assumption that the demand curve for the commodity is given by $x = f(p, t)$, where, x is the quantity of the commodity that is demanded, p is the price of the commodity demanded and t is time. This implicitly assumes that the prices of all other related commodities have only a negligible or no effect upon the commodity in question or all the influencing factors are conceived as frozen while studying the variation in x as a result of variations in y. However, in practice it is impossible to freeze all other factors without first taking them into account.

2. Pigou's method breaks down if in the three successive sets of observations, the three price-quantity points are collinear. However, the method can be applied to non-linear functions and the functions which change in directions.

(3) Pigous's Method (From Family Budget Data) :—

This method differs from Pigou's method (from Time Series Data) in that it is derived from the theory of utility and makes use of family budget data which gives us the expenditure of a group of people classified according to their income (wages). Let

$$U_1 = U_1(x) \text{ and } U_2 = U_2(x)$$

be the marginal degree of utility functions (i.e., the rate of change of utility function) for a quantity x of a given commodity 'A' for two neighbouring income groups I and II.

It is assumed that the functions U_1 and U_2 are independent of the quantities of other commodities and hence of the degree of utility of money. If μ_1 and μ_2 denote the degree of utility of money to the two groups respectively then

$$\mu_1 = \frac{U_1(x_1)}{p} \ ; \ \mu_2 = \frac{U_2(x_2)}{p} \quad \ldots\ldots\ldots (1)$$

where x_1 and x_2 are the equilibrium quantities of 'A' consumed by the two groups and p is the price of commodity 'A' which must be same for both the groups.

Hence the two income groups are neighbouring ones i.e., the wage grouping can be taken to be small, Pigou assumed that the tastes and temperament of the people in any two adjacent income groups are approximately the same so that

$$u_1(x) = u_2(x) = u(x), \text{ say} \quad \ldots\ldots\ldots (2)$$

Hence from (1) and (2), we get

$$p = \frac{u(x_1)}{\mu_1} = \frac{u(x_2)}{\mu_2}$$

We have $u(x_2) = u[x_1 + (x_2 - x_1)]$

$$= u x_1 + (x_2 - x_1)\, u'(x_1) \quad \text{[Taylor's expansion]}$$

$$\Rightarrow u'(x_1) = \frac{u(x_2) - u(x_1)}{x_2 - x_1} = \frac{p(\mu_2 - \mu_1)}{x_2 - x_1}$$

$$= \frac{1}{x_2 - x_1} \cdot \frac{\mu_2 - \mu_1}{\mu_1} \cdot u(x_1) \quad \ldots\ldots\ldots (3)$$

By definition, the elasticity of demand (consumption) x_1 w.r.t. utility $u(x_1)$ is

$$\eta_{x_1,u} = \frac{u(x_1)}{x_1} \cdot \frac{dx_1}{d[u(x_1)]} = \frac{u(x_1)}{x_1} \cdot \frac{1}{\frac{d}{dx_1}[u(x_1)]}.$$

$$= \frac{u(x_1)}{x_1 u'(x_1)}.$$

Substituting the value of $u'(x_1)$ from (3), we have

$$\eta_{x_1,u} = \frac{x_2 - x_1}{x_1} \cdot \frac{\mu_1}{\mu_2 - \mu_1} \quad \ldots\ldots\ldots (**)$$

Pigou further assumed that "a small change in the consumption of any ordinary commodity on which a small proportion of man's total income is spent, cannot involve any appreciable change in the marginal utility of money". In other words, he assumed that "the price elasticity of demand curve in respect of any consumption x_1 is equal to the elasticity of the utility curve in respect of that consumption" .... (*)

The elasticity of demand w.r.t. price for the commodity in question viz., 'A' in the lowest wage group when x_1 units of it are consumed is given by

$$\eta_{x_1,p} = \frac{p}{x_1} \cdot \frac{d(x_1)}{dp}$$

$$\therefore \frac{dp}{dx_1} = \frac{u'(x_1)}{\mu_1} \cdot \left[\text{From } \mu_1 = \frac{u(x_1)}{p} \right]$$

So, we get $\eta_{x_1,p} = \frac{p}{x_1} \cdot \frac{\mu_1}{u'(x_1)} = \frac{u(x_1)}{x_1\, u'(x_1)} = \eta_{x_1,u}$

$$= \frac{x_2 - x_1}{x_1} \cdot \frac{\mu_1}{\mu_2 - \mu_1}.$$

Thus Pigou concluded that the price elasticity of demand for the commodity in question in the lowest wage group when x_1 units of it are consumed is equal to the elasticity of the consumption x_1 w.r.t. the utility $u(x_1)$.

Similarly, if the lowest income group consumes y_1 units of commodity 'B', say, then

$$\eta_{y_1,p} = \frac{y_2 - y_1}{y_1} \cdot \frac{\mu_1}{\mu_2 - \mu_1}$$, where notations have usual meanings.

So, $\frac{\eta_{y_1,p}}{\eta_{x_1,p}} = \frac{y_2 - y_1}{y_1} \cdot \frac{x_1}{x_2 - x_1}$

$$\Rightarrow \eta_{y_1,p} = \left[\frac{y_2 - y_1}{y_1} \cdot \frac{x_1}{x_2 - x_1} \right] \cdot \eta_{x_1,p}.$$

If any one of these elasticities of demand η_{y_1} or η_{x_1} is given then other can be obtained from above without any reference to the incomes μ_1 and μ_2.

Limitations :- 1. Even if Pigou's assumptions are granted, his method gives only the ratios of the elasticities of demand of two commodities and not the absolute values of elasticities.

2. However, Pigou's most important assumption viz., " Since a small change .... in respect of that consumption in (*) can't be granted because of basic contradiction in his development. Though the assumption of " constancy of utility of money " is reasonable when income remains fixed and the price of one of the commodities varies, it is not valid when prices are fixed and income varies. This latter assumption implies $\mu_1 = \mu_2$ which, on substituting in (**) gives $\eta_{x_1,u} = \infty$, thus contradicting the assumption of diminishing degree of utility.