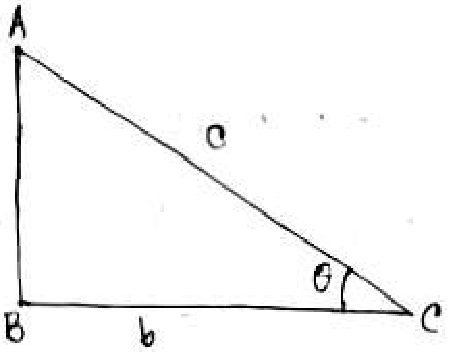
D

B. Stat (Hons) Admission Test 2005

1.



as per our assumption $\angle ACB = \theta$, the smallest angle.

So from the definition of right angle triangle,

$$\angle ACB < \angle BAC < \angle ABC$$

hence,

$$a < b < c$$

and also $\frac{1}{a} > \frac{1}{b} > \frac{1}{c}$

Now from pythagorus,

$$a^2 + b^2 = c^2 \quad \text{—(I)}$$

$$(1/b)^2 + (1/c)^2 = (1/a)^2$$

$$\Rightarrow (b^2 + c^2)a^2 = b^2c^2 \quad \text{—(II)}$$

$$\sin\theta = \frac{a}{c}$$

(I) $\mapsto$ $c^2\sin^2\theta + b^2 = c^2 \Rightarrow b^2 = c^2\cos^2\theta$

(II) $\mapsto$ $(c^2\cos^2\theta + c^2)\,c^2\sin^2\theta = c^2\cos^2\theta \cdot c^2$

$$\Rightarrow (\cos^2\theta + 1)\sin^2\theta = \cos^2\theta$$

$$\Rightarrow (1 - \sin^2\theta + 1)\sin^2\theta = \cancel{\sin^2\theta}\ 1 - \sin^2\theta$$

$$\Rightarrow (2 - \sin^2\theta)\sin^2\theta = \cancel{\sin^2\theta}\ 1 - \sin^2\theta$$

$$\Rightarrow \sin^4\theta - 3\sin^2\theta + 1 = 0$$

$$\Rightarrow \sin^4\theta + 2\sin^2\theta + 1 = 5\sin^2\theta$$

$$\Rightarrow (\sin^2\theta + 1)^2 = (\sqrt{5}\sin\theta)^2$$

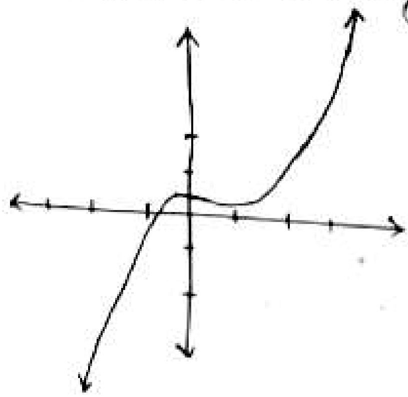
$$\Rightarrow \sin^2\theta - \sqrt{5}\sin\theta + 1 = 0$$

2. $f(x) = \int_0^1 |t-x|\,t\,dt = \int_0^x (x-t)t\,dt + \int_x^1 (t-x)t\,dt$

$$= \left(\frac{xt^2}{2} - \frac{t^3}{3}\right)\Bigg|_0^x + \left(\frac{t^3}{3} - \frac{xt^2}{2}\right)\Bigg|_x^1$$

$$= x^3/3 - x/2 + 1/2$$

After some sketchs, I am drawing the final graph.[1]



3. $f(1,100)$

$= f(1,99) + f(99,100) - 2f(1,99)f(99,100)$

$= f(1,99) + 1/3 - 2/3\, f(1,99)$

$= 1/3 + 1/3\, f(1,99)$

$= 1/3 + 1/3\left(1/3 + 1/3\, f(1,98)\right)$

$= 1/3 + 1/3^2 + 1/3^2\, f(1,98)$

$= 1/3 + 1/3^2 + 1/3^3 + \cdots\cdots + 1/3^{98} + 1/3^{98}\, f(1,2)$

$= 1/3 + 1/3^2 + 1/3^3 + \cdots + 1/3^{99}$

$$= \frac{1/3\left(1/3^{98} - 1\right)}{\left(1/3 - 1\right)} \longrightarrow \frac{1/3}{2/3} \longrightarrow 1/2$$

4. $\sin^5 x + \cos^3 x \geqslant \sin^2 x + \cos^2 x = 1$

equality holds only when

$\sin^5 x = \sin^2 x \Rightarrow \sin^3 x = 1$

$\cos^3 x = \cos x \Rightarrow \cos x = 1$

One can visualize this by sketching a graph.[2]

5. $\angle QCR = 2P$

$\angle QIR = Q + R$

$\angle QOR = P + Q/2 + R/2$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{2P} + \frac{1}{Q+R} + \frac{1}{P + Q/2 + R/2} \quad \text{—— (1)}$$

$1/x$ is a convex function

hence by Jensen's inequality we get,

$$\frac{\frac{1}{2P} + \frac{1}{(Q+R)}}{2} > \frac{1}{\frac{(2P+Q+R)}{2}} \quad \#$$

$$\Rightarrow \left(\frac{1}{2P} + \frac{1}{Q+R}\right) > \frac{4}{(2P+Q+R)}$$

now from (1)

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{2P} + \frac{1}{Q+R} + \frac{1}{P + Q/2 + R/2}$$

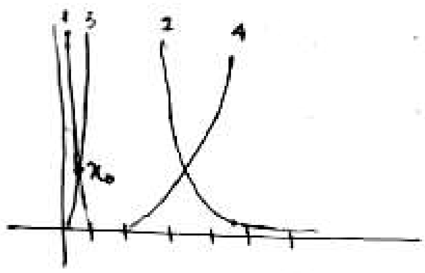
$$> \frac{4}{(2P+Q+R)} + \frac{2}{(2P+Q+R)}$$

$$= \frac{6}{(2P+Q+R)} = \frac{6}{(P+180)} > \frac{6}{(90+180)} = \frac{6}{270} = \frac{1}{45}$$

QED[3]

6. $$h'(x) = \frac{4x^3(1-x)^6 - 6(1-x)^5 x^4}{(1-x)^{12}}$$

$$\cancel{= \frac{4x^3(1-x)^6 + 12(1-x)^5 x^4}{(1-x)}}$$



$1, 2 \to h(x)$

$3, 4 \to \frac{1}{h(x)}$

now one can easily draw,

$$g(x) = f(h(x))$$

$$= h(x) + \frac{1}{h(x)}$$

from the theory of graph sketching[4]

From the theory of graph sketching,

x_0 must be a solution of

$$h(x) + \frac{1}{h(x)} = 0$$

which lies b/w 0 and 1

It is easy to prove $g(x)$ has a root b/w 0 and 1

7. a) let's consider the map f from $A_{m,n}$ to $B_{m+1,n-1}$ as follows

$$f(\alpha_1, \alpha_2, \ldots, \alpha_m) = (\beta_1, \beta_2, \ldots, \beta_{m+1})$$

$\beta_1 = \alpha_1 - 1$;

$\beta_2 = \alpha_2 - \alpha_1$, ...

$\beta_m = \alpha_m - \alpha_{m-1}$;

$\beta_{m+1} = n - \alpha_m$

check it is oneone onto map from $A_{m,n}$ to $B_{m+1,n-1}$

b) let's consider the map g from $A_{m,n}$ to $C_{m,m+n-1}$ as follows

$$g(\alpha_1, \alpha_2, \ldots, \alpha_m) = (\beta_1, \beta_2, \ldots, \beta_m)$$

$\beta_1 = \alpha_1$;

$\beta_2 = \alpha_2 + 1$;

$\beta_3 = \alpha_3 + 2$; .....

$\beta_m = \alpha_m + m - 1$

check it is a oneone onto map from $A_{m,n}$ to $C_{m,n+m-1}$

c) no of elements in $B_{m,n}$ is same as number of non negative integral solution of the system $= \alpha_1 + \alpha_2 + \ldots + \alpha_m = n$

equivalently,

number of distinct permutation of n balls and $(m-1)$ sticks, which is $\dfrac{(m+n-1)!}{n!\,(m-1)!}$

since, $A_{m,n}$ has a one to one correspondence with $B_{m+1,n-1}$. so number of elements in $A_{m,n}$ is same as number of elements in

$$B_{m+1,n-1} = \frac{(m+n-1)!}{(n-1)!\,m!}$$

8. a) let, $g(mn) = 5^{k_1}$

where k_1 is the number of distinct primes that divides mn

now, $g(m) = 5^{k_2}$ where..

$g(n) = 5^{K}$ where.

To be $g(m)g(n) = g(mn)$

it must be $5^{k_1} = 5^{k_2+K}$

but the equality $k_1 = k_2 + K$

will not hold if m and n aren't relatively prime.

here the property thereby hold

b) i) $h(nm) = 0$

then either $h(m) = 0, h(n) = 1$ — A

$h(n) = 0, h(m) = 1$ — B

$h(m) = 0, h(n) = 0$ — C

ii) $h(mn) = 1$

then it must be $h(m) = 1, h(n) = 1$ — D

case C always holds.

OK now consider case A or B. WLOG we consider case A

here,

m is divisible by k^2 $h(m) = 0$

n is not divisible by k^2 $h(n) = 1$

but then,

mn must be divisible by k^2

$h(mn) = 0$

hence it satisfies.

now come to case D

here, m is not divisible by K^2 $h(m)=1$

n is not divisible by K^2 $h(n)=1$

then to hold ~~[illegible]~~ $h(mn)=1$

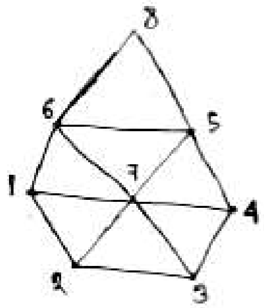
mn is not divisible by K^2

$\Rightarrow mn$ is not divisible by K

$\Rightarrow mn$ donot have any common factor >1 as $K>1$

QED

9. a)



consider a regular hexagon WLOG we can say a point 7 is Red (R).

A ↦ If all of the points 1,2,3,4,5,6 are blue then we get equilateral triangle 264, 135 with vertices of blue colour. contradiction! hence one of them must be (R) red. let WLOG it be the point ~~1~~ 5

B ↦ Now both of the 2,6 must be coloured in different colour i.e blue (B) as if not then we get 167, 127 as our reqd triangle.

C ↦ Now point 4 must be red as we otherwise get 264 as our reqd triangle

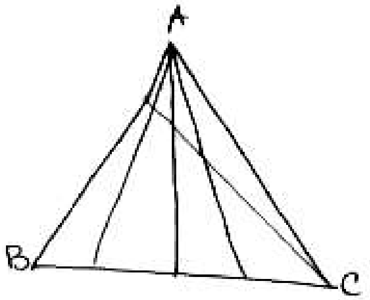
We get three points 1,7,4 as same colour (red). We are done!

b) D ↦ Now point 5 and 3 must be blue as 754 and ~~734~~ ~~or~~ 731 will be then our reqd triangle

Now watch that point 5,6 are (B) blue but point 1,4 are (R) red.

Draw 8 such that 865 construct an equilateral triangle. If point 8 is (B) then our reqd triangle is 568 If red (R) the 1,4,8 is our reqd

10. a)



take an arbitary example if $n=3$ it creates $(n+1)=4$ regions as there are total $(n+2)$ lines now from the other side one line is drawn

the line again partioned it by $(n+1)$ regions a total of $2(n+1)$ regions,

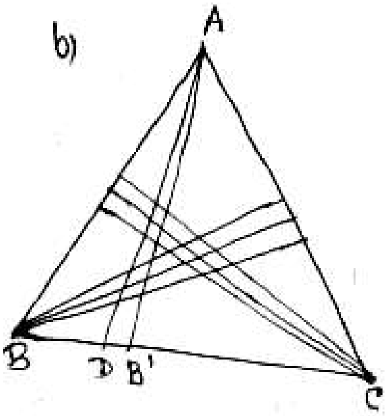
hence,

1 line will create $2(n+1)$ regions

n lines will create $(n+1)(n+1)$ "

$= (n+1)^2$ "

b)



The diagram has to be drawn such that no three straight lines meet at a point other than A

(AB') is drawn. Now ABC is equivalent to ABC in the previous fig 10 a) consists of $(n+1)$ regions

now another line AD is drawn which generate more $(2n+1)$ regions as there are $(2n+2)$ lines. Now generate more $(n-2)$ lines to get a total of $(2n+1)n$ regions

Hence total

$= (n+1)^2 + (2n+1)n$ regions.