B.Stat Admission Test 2006

1. $x^{2/3}+y^{2/3}=a^{2/3} \Rightarrow \frac{2}{3}x^{-1/3}x_1+\frac{2}{3}y^{-1/3}=0 \Rightarrow x^{-1/3}x_1+y^{-1/3}=0 \Rightarrow -x_1=\frac{y^{-1/3}}{x^{-1/3}}$
$=\tan\theta$

$\frac{x^{1/3}}{\sin\theta}=\frac{y^{1/3}}{\cos\theta}=\sqrt{x^{2/3}+y^{2/3}}=\sqrt{a^{2/3}}=a^{1/3}$

$x=a\sin^3\theta;\ y=a\cos^3\theta$ $\qquad y-a\cos^3\theta=\tan\theta(x-a\sin^3\theta)$

$\Rightarrow y\cos\theta-a\cos^4\theta=\sin\theta\,(x-a\sin^3\theta)$

$\Rightarrow y\cos\theta-x\sin\theta = a(\cos^2\theta+\sin^2\theta)(\cos^2\theta-\sin^2\theta)$

$=a\cos 2\theta$

2. a) Then a and b are the roots of the quadratic with rational coefficients $x^2-(a+b)x+ab=0$ thus $a=\frac{a+b}{2}\pm\frac{\sqrt{\Delta}}{2}$

where $\Delta=(a-b)^2$

clearly, $(a-b)^2\neq c^2$ for any rational c, since that would lead to $a\in\mathbb{Q}$

b) i) If $a=r\pm\sqrt{s}$ is a quadratic surd, take $1+r\mp\sqrt{s}$ then

$a+x=1+2r\in\mathbb{Q}$

and $ax=(r+(r^2-s))\pm\sqrt{s}\notin\mathbb{Q}$

If a is not a surd, take $x=-a$

ii) take $y=0$

3. Here n must be odd $n=2k+1$ (let)

now, $n^4+4^n=n^4+4\cdot4^{2k}$
$=n^4+4\cdot(2^k)^4$

from Sophie Germain ~~ine~~quality $\left(a^4+4b^4=(a^2+2b^2+2ab)(a^2+2b^2-2ab)\right)$

we are done

4. In ΔBCE and ΔABE

$BE^2=EC^2+BC^2-2\,EC.BC\cos BCE$
$BE^2=AE^2+AB^2-2AE.AB\cos ABE$

$AB=EC$ because E is the midpoint
$\angle BCF=\angle ABE$

substracting,

$$BC^2 - AB^2 = 2EC\cos\angle BCE\,(BC - AB)$$

$$BC + AB = 2EC\cos\angle BCE$$

In $\triangle CDE$,

$$BC + AB = 2CD$$

$$\Rightarrow BD + CD + AB = 2CD$$

$$\Rightarrow BD + AB = l_1 + l_2$$

Method 2:

let A' be on CB extended, such that $AB = A'B$

EB bisects $\angle ABA'$

$$EA = EA' = EC$$

D is midpoint of $A'D$

Then $AB + BD = A'B + BD$

$$= A'D = \tfrac{1}{2}A'C = DC$$

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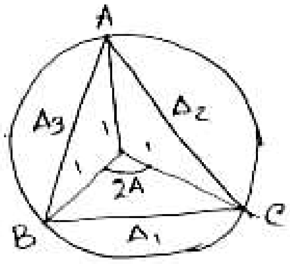

a) $\Delta_1 = \frac{1}{2}\cdot 1\cdot 1\cdot \sin 2A$

$\Delta_2 = \frac{1}{2}\cdot 1\cdot 1\cdot \sin 2B$

$\Delta_3 = \frac{1}{2}\cdot 1\cdot 1\cdot \sin 2C$

$\Delta = \frac{1}{2}(\sin 2A + \sin 2B + \sin 2C)$

b) $A+B+C=\pi \Rightarrow \Delta = \frac{1}{2}(\sin 2A + \sin 2B - \sin 2(A+B))$

$$\Rightarrow \frac{d\Delta}{dA} = \cos 2A - \cos 2(A+B) = 0$$

$$\cos 2A = \cos 2(A+B)$$

$$\Rightarrow 2A = 2\pi - (2A+2B)$$

$$\Rightarrow 2A + B = \pi$$

again we know $A+B+C=\pi$

$$\therefore A = C$$

c) consider isoceles and then prove $c=a$ by calculus

6.[3]

7. Part 1 : $\dfrac{\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2}{n} > \dfrac{\left(\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}\right)^2}{n}$

$\binom{2n}{n} > \dfrac{(2^n)^2}{n} > 2^n$

Part 2 : $\binom{2n}{n} = \dfrac{2^n(1.3\ 5\ .\ (2n-1))}{n!} < \dfrac{2^n.n^n}{n!} = \dfrac{2^n}{n!/n^n} = \dfrac{2^n}{\prod_{i=0}^{n-1}(1-i/n)}$

method 2:

$(n+1) < 2n$; $(n+2) < 2n$; . .

$\therefore \binom{2n}{n} = \dfrac{(n+1)(n+2)\ .\ .\ (n+n)}{n!} < \dfrac{(2n)^n}{n!} = \dfrac{(2n)^n}{(n-0)(n-1)\ .\ (n-n+}$

$= \dfrac{2^n}{\left(\frac{n-0}{n}\right)\cdots\left(\frac{n-n+1}{n}\right)}$

$= \dfrac{2^n}{\prod_{i=0}^{n-1}(1-i/n)}$

8. The point where the tangent to $f(x) = \log_2 x$ pass through $(0,0)$ is $x=e$ and the tangent is the $x/e\ln 2$.

so, the equation $\log_2 x = cx$ has

one solution for $c \leq 0$

two solutions for $c \in (0, 1/e\ln 2)$

one solution for $c = 1/e\ln 2$

no solution for $c > 1/e\ln 2$

hence the answer: $c \in [\infty, 0] \cup \{1/e\ln 2\}$

9. $N = 1000a + 100b + 10c + d$

$M = 1000d + 100c + 10b + a$

now, M and N are both 4 digit number. hence, $M \leq 2499$

again

$1000a + 100b + 10c + d = 4(1000d + 100c + 10b + a)$

$\Rightarrow 1333d + 130c - 20b - 332a = 0$

now, $M \leq 2499$ $d = 1,2$ $c = 1,2,3,4$

now put $d=2$,

$$1333\times 2+130\times c=20b+332a$$

$$\Rightarrow 2666+130c=20b+332a$$

now, $2666+130c-332a=20b$

RHS is a factor of 20 hence LHS must be a factor of 20 and less than 200

then, taking $c=1$, $2666+130-332a=20b$

$$\Rightarrow 2796-332a=20b$$

now, for LHS to be a factor of 20 and value to be less than 200, $a=8$

$$2796-332a=20b \Rightarrow 2796-2656=20b$$

$$\Rightarrow b=7$$

$\therefore N=8712,\ M=2178$

10. $f(n)+f(n-1)=nf(n-1)+(n-1)f(n-2)$

$$f(n)-nf(n-1)=(-1)f(n-1)+f(n-1)f(n-2)$$

$$=(-1)\left(f(n-1)-f(n-1)(n-2)\right)$$

$$=(-1)^2\left(f(n-2)-f(n-2)(n-3)\right)$$

$$\cdots$$

$$=(-1)^{n-1}\left(f(1)-f(0)\right)=(-1)^{n-1}(0-1)=(-1)^n$$

$$\frac{f(n)}{n!}-\frac{f(n-1)}{(n-1)!}=\frac{(-1)^n}{n!}$$

$$\sum_{K=1}^{n}\left[f(n)-\frac{f(n-1)}{(n-1)!}\right]=\sum_{n=1}^{n}\frac{(-1)^n}{n!}$$

$$\Rightarrow \frac{f(n)}{n!}-\frac{f(0)}{0!}=\sum_{K=1}^{n}\frac{(-1)^n}{n!}$$

$$\Rightarrow \frac{f(n)}{n!}=1+\sum_{K=1}^{n}\frac{(-1)^n}{n!}=\sum_{K=0}^{n}\frac{(-1)^n}{n!}$$

$$\Rightarrow f(n)=n!\sum_{K=0}^{n}\frac{(-1)^n}{n!}$$