B Stat Admission Test 2007

1. $a^4+a^3+a^2+a+1=0$

$\Rightarrow a^5-1=0$

hence, a is the fifth root of unity $a=\left(\cos\frac{2k\pi}{5}+i\sin\frac{2k\pi}{5}\right)$

$a^{2m}+a^m+1/a^m+1/a^{2m}=5$

If m is a multiple of 5 $a^{4m}+a^{3m}+a^{2m}+a^m=-1$

QED

2.

3. $\int_1^x [u]([u]+1)f(u)du = \int_1^2 1.2 f(u)du + \int_2^3 2.3 f(u)du + \dots + \int_{[x]}^x [x]([x]+1)du$

$2\sum_{i=1}^{[x]} i\int_i^x f(u)du = 2\left(1\int_1^x f(u)du + 2\int_2^x f(u)du + \dots + [x]\int_{[x]}^x \cancel{([x]+1)} f(u)d\right.$

$= 2\left\{\left(1\int_1^2 f(u)du + 1\int_2^3 f(u)du + \dots + 1\int_{[x]}^x f(u)du\right) + \left(2\int_2^3 f(u)du + \dots + 2\int_{[x]}^x f(u)du\right) + \dots + [x]\int_{[x]}^x f(u)du\right\}$

$= 2\left\{\left(1\int_1^2 f(u)du\right) + (1+2)\int_2^3 f(u)du + \dots + (1+2+\dots+[x])\int_{[x]}^x f(u)du\right.$

$= 2\left\{1\int_1^2 f(u)du + 3\int_2^3 f(u)du + \dots + \frac{[x]([x]+1)}{2}\int_{[x]}^x f(u)du\right\}$

$= 1.2\int_1^2 f(u)du + 2.3\int_2^3 f(u)du + \dots + [x]([x]+1)\int_{[x]}^x f(u)du$

Proved

4. using barycentric coordinates,

moment of inertia of A,B,C wrt centroid G is

$4/9(m_a^2+m_b^2+m_c^2) = 1/3(a^2+b^2+c^2) > 4/9(1/9\,a^2+4/9\,a^2+16/25\,c^2)$

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5. $y = \cos\theta(\sin\theta + \sqrt{\sin^2\theta + 3})$

$\Rightarrow y\sec\theta = (\sin\theta + \sqrt{\sin^2\theta + 3})$

$\Rightarrow (y\sec\theta)^2 + \sin^2\theta - 2y\tan\theta = 3 + \sin^2\theta$

$\Rightarrow (y\tan\theta)^2 - 2y\tan\theta + y^2 - 3 = 0$

$D = (2y)^2 - 4y^2(y^2-3) \geq 0 \Rightarrow 4 - 4(y^2-3) \geq 0$

$\Rightarrow y^2 \leq 4$

$\therefore -2 \leq y \leq 2$

6. Arrange values of f in $n \times n$ matrix. Each row is a permutation of S (ii)

$\Rightarrow$ Every k from S shows n times, odd times, in the matrix

Suppose k is not on the main diagonal

$(f(r,r)$: r belong to S $\langle S)$

Since matrix is symmetric (i), k shows in the matrix even number of times.

Contradiction!

7. Let, a, b, h be the three edges meeting at the particular vertex, such that a, b are sides of a base triangle, having an angle θ b/w them and h is the height of the prism, we have the prism's volume

$V = hab\sin\theta$

$K = \frac{1}{2}ab\sin\theta + ah + bh$

$\geq 3\sqrt[3]{\frac{1}{2}ab\sin\theta \cdot ah \cdot bh} = \sqrt[3]{\frac{54\,(hab\sin\theta)^2}{\sin\theta}}$

$\geq \sqrt[3]{54V^2}$

hence $V \leq \sqrt{K^3/\cancel{\sin\theta}\,54}$

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equality holds if $\sin\theta = 1$ and $\frac{1}{2}ab\sin\theta = ah = bh$

or, $a = b = 2h$

in that case, $K = 6h^2 \Rightarrow h = \sqrt{K/6}$ for the largest prism

method 2:

If you put 4 such prisms together the question becomes: considering a prism with parallelogram base and total surface 4K show that the maximum possible volume is $4\sqrt{K^3/54}$

The prism times 4 with maximum volume is obviously a cube

$V(h,b,c,a) = \frac{1}{2}hbc\sin A$ subject to constraint

$K(h,b,c,a) = \frac{1}{2}hbc\sin A + hb + hc$

$F(h,b,c,A) = V(h,b,c,A) - \lambda K(h,b,c,A)$

V was maximum for ——

$\partial F/\partial h = \frac{1}{2}bc\sin A - \lambda(b+c) = 0$

$\partial F/\partial b = \frac{1}{2}hc\sin A - \lambda(h + \frac{1}{2}c\sin A) = 0$

$\partial F/\partial c = \frac{1}{2}hb\sin A - \lambda(h + \frac{1}{2}b\sin A) = 0$

$\partial F/\partial A = \frac{1}{2}hbc\cos A - \lambda(\frac{1}{2}bc\cos A) = 0$

last question gives

either $\lambda = h$ or $\cos A = 0$

Trying $\lambda = h$ in 2nd/3rd equation leads to $h^2 = 0$ which is not maximum and not acceptable

$\cos A = 0$

and $\sin A = 1$

substracting from 2nd and 3rd equation then gives

$h(c-b) = \lambda(c-b)$ since $\lambda \neq h$ $\therefore b = c$

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First solution gives, $\lambda = b/4$

and using this in the 2nd/3rd equation yields $h = b/2$

Consequently,

for max. volume

$$v^2/K^3 = \left(b^3/4\right)^2 \left(2/3\, b^2\right) = 1/54$$

6. Alternative:

Take $n \times n$ grid. Enter $f(r,s)$ in the cell $f(r,s)$ you can see the total picture is symmetric w.r.t one of its diagonal. If any element of S is not present on the diagonal then it should be even number of times in the grid since the figure is symmetric wrt the diagonal.

But each element of S is presently exactly n times in the grid so this gives a contradiction since n is odd.

7. Alternative:

Suppose the area of the face is S. The two sides be a and b. The height is h.

So the volume is Sh.

and $K = S + (a+b)h$

$$S \leq \frac{ab}{2}$$

$$\frac{S + \frac{2(a+b)h}{2}}{2} \geq \left(\frac{Sh^2(a+b)^2}{4}\right)^{1/2} \geq (Sabh^2)^{1/3} \geq (2S^2h^2)^{1/3}$$

$$\therefore Sh \leq \sqrt{K^3/54}$$

8.

A	B	C
D	E	F
G	H	I

$G \times D \times A \times H \times I \times E \times B \times F \times C$

$9 \times 6 \times 3 \times 6 \times 3 \times 4 \times 2 \times 2 \times 1$ ways

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9 ~~of~~ If (x,y) is any of (a_1,b_1) and (a_2,b_2) then nothing to prove,
so, suppose not

then suppose (x,y) doesnot lie on the straight line passing through (a_1,b_1) and (a_2,b_2) then the triangular region formed by these 3 points ~~continued~~ * contained ~~by~~ in X by the condition (iii)

~~So~~ But given any nondegenerate triangle we can always find points interior in the triangular region (x_1,y_1) and (x_2,y_2) s.t $x_1 \neq x_2$ and $y_1 = y_2$ but they belong to X and donot satisfy (i), hence contradiction.

So (x,y) lie on the st. line passing through (a_1,b_1) and (a_2,b_2) now as $a_1 \leq x \leq a_2$ and $b_1 \leq y \leq b_2$

— hence the result

10. There are $n_1, n_2, \ldots, n_k$ such that $(n_1, n_2, \ldots, n_k) = 1$. let $\sum_i a_i n_i = 1$ If given $n = \sum_i (a_i n) n_i$. If $\sum_i b_i n_i = 0$ then we had another solution $n = \sum_i (a_i n + b_i) n_i$

because we can choose $(b_i) = ((0, \ldots, n_j), (0, \ldots, -n_i), (0, \ldots 0))$

and make $(a_i n + b) \geq 0$

for any $n > (s - n_i - k + 1) n_i$ were $s = \sum_i n_i$

alternative:

1) case 1: If $(n_1, n_2) = 1$ then as $n_1 x + n_2 y = 1$ has infinitely many solutions in $x, y \in \mathbb{Z}$

so we get $c_1, c_2 > 0$ integers such that $c_1 n_1 + c_2 n_2 = 1$

let WLOG $c_1 < c_2$ so, $c_2(n_1 + n_2) - (c_2 - c_1) n_1 = 1$

set $m_2 = c_2(n_1 + n_2)$; $m_1 = (c_2 - c_1) n_1$

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case 2: If $(n_1, n_3) = d > 1$ then by condition (II)
there is $n_3 < n_4$ s.t $(n_3, n_4) = d_1$ and
d don't divide d_1 let $e = (d, d_1)$ then by the argument
given earlier we have
$p_1, q_1, p_2, q_2 \in A$ s.t
$p_2 - p_1 = d$ and $q_2 - q_1 = d$ then
since $(dx + d_1 y) = e$ has infinitely many solutions
we get $(p_2 - p_1)x + (q_2 - q_1)y = e$ where $x, y \in \mathbb{Z}^+$
set $m_2 = p_2 x + q_2 y$; $m_1 = p_1 x + q_1 y$
then $m_2 - m_1 = e < d \leq K = n_2 - n_1$

II) If $n_2 - n_1 = K_1 > 1$ then by this method we get $K_1 > K_2 > 0$
since going on like this we eventually get ~~$K_2 = 1$~~
$K_l = 1$ in the l_{th} step.
so, $m_2 - m_1 = 1$, hence we get two consecutive
integers.

III) any number from $n_0^2, \ldots, n_0^2 + n_0$ can be obtained by
$(n_0 - k)n_0 + k(n_0 + 1)$ varying k from 0 to $n_0 - 1$
any number from $n_0^2 + n_0, \ldots, n_0^2 + 2n_0$ can be obtained
by $(n_0 - K)n_0 + K(n_0 + 1) + n_0$ varying K from 0
to $(n_0 - 1)$ and like this way —

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