1. M is the junction, C is the station, AM = MB is the train length

$\Rightarrow$ CM is the C-median of ΔABC.

$\theta = \angle AMS$ is acute $\Rightarrow \angle A > \angle B$, in agreement with $\angle ACM = \alpha > \beta = \angle MCB$

$(0, R)$ is circumcircle of ΔABC and CF its C-altitude, $\parallel$ to AB through C cuts (0) again at Z.

$$\tan\theta = CF/MF = 2CF/CZ = 4[ABC]/AB.CZ = \frac{BC.CA}{R.CZ} = \frac{2\sin A\sin B}{\sin(A-B)} = \frac{2\sin\alpha\sin\beta}{\sin(\alpha-\beta)}^*$$

* let ABC be a triangle $A \neq B$ and let $P \in (AB)$ be a point for which denote $m(\widehat{ACP}) = x$ and $m(\widehat{BCP}) = y$ Prove that

$$\frac{\sin A\sin B}{\sin(A-B)} = \frac{\sin x\sin y}{\sin(x-y)} \text{ if and only if } PA = PB$$

$\Rightarrow$ suppose WLOG $A > B$ and hence $x > y$. Denote the circumcentre ω of the ΔABC and the second point

$M \in CP \cap \omega$. construct the points $X \in \omega$, $Y \in \omega$ for which $MX \parallel AB \parallel CY$. Observe that $m(\widehat{CMY}) = A - B$

$m(\widehat{XYM}) = x - y$ and $PC/PM = CB.CA/MA.MB = \frac{\sin A\sin B}{\sin x\sin y}$

Observe that,

$$CY/MX = \sin(A-B)/\sin(x-y)$$

$$\text{hence } \frac{\sin A\sin B}{\sin(A-B)} = \frac{\sin x\sin y}{\sin(x-y)}$$

$$\Rightarrow PC/PM = CY/MX$$

$$\Rightarrow P \in XY \Rightarrow PA = PB$$

2. $$\int_0^{2\pi} f(x)\cos x\,dx = f(x)\sin x\Big|_0^{2\pi} - \int_0^{2\pi} f'(x)\sin x\,dx$$

$$= -\int_0^{2\pi} f'(x)\sin x\,dx$$

$$= -\left[\int_0^{\pi} f'(x)\sin x\,dx + \int_{\pi}^{2\pi} f'(x)\sin x\,dx\right]$$

$$= -\left[\int_0^{\pi} f'(x)\sin x\,dx + \int_0^{\pi} f'(x+\pi)\sin(x+\pi)\,dx\right]$$

$$= -\left[\int_0^{\pi} f'(x)\sin x\,dx - \int_0^{\pi} f'(x+\pi)\sin x\,dx\right]$$

$$= \int_0^{\pi}(f'(x+\pi) - f'(x))\sin x\,dx \qquad \text{here } f'(x) \geq 0$$

$f'(x)$ is increasing

3. Assume that, by symmetry P is closer to A than to B. Then area of APQ is less than BPR. So we disregard APQ.

If P moves away from A the area of PQCR increases while the area BPR decreases, therefore the minimum of their maximum occurs when the two areas are equal in size, which is when

$$CR/_{RB} = 1/2$$

$$\text{so, } CR/_1 = 1/3 \qquad \text{Thus } M = 2/9$$

4. $A_1 = 2, 7, 14, 23, 34$

$\Delta_2 = 5, 7, 9, 11$ $\qquad T_r = r^3/3 + r^2/2 - 11r/6 + 5$

$\Delta_3 = 2, 2, 2$

5. Consider the circle with radius $= 30$ cm

volume of the max. rectangle enclosed within it $(\sqrt{2}r)^2 = 2r^2$
$= 1800\text{ cm}^2$

max height $= 60$ cm (as lid on)

hence maximum volume $= (1800 \times 60)\text{ cm}^3$

6. $\log x = g(x) = xf(x)$

$\Rightarrow (-1)^2 x^{-1} = g'(x) = xf'(x) + f(x)$

$\Rightarrow (-1)^3 x^{-2}\, 1! = g''(x) = xf''(x) + 2f'(x)$

$\Rightarrow (-1)^4 x^{-3}\, 2! = g'''(x) = xf'''(x) + 3f''(x)$

$\Rightarrow (-1)^{n+1}(n-1)!\, x^{-n} = g^n(x) = xf^n(x) + nf^{n-1}x$

evaluate at 1 and we get

$$(-1)^{n+1}(n-1)! = f^{(n)}(1) + nf^{(n-1)}(1)$$

from induction—

$$f^n(1) = (-1)^{n+1} (n-1)! - nf^{n-1}(1)$$
$$= (-1)^{n+1} n! \, \frac{1}{n}$$

7. let A, B, C, D, E be the vertices of the pentagon and let F, G, H, I and J be the midpoint of the sides AB, BC, CD, DE and EA respectively.

Also let O be the centre of the circumscribed circle, since they are radii of the same circle OA, OB, OC, OD are congruent.

Also, since the pentagon is regular AB, BC, CD, DE and EA are congruent. Thus $\Delta AOB, \Delta BOC, \Delta COB, \Delta DOE, \Delta EOA$ are congruent, isoceles triangles. since their vertices add upto 360 degrees, each vertex measures to 72 degrees.

Now for ease, I consider only ΔAOB. since it is isoceles, the median from the vertex is same as the angle bisector and the altitude.

Thus OF is also the angle ~~b~~bisector, and $\angle AOF = \angle BOF = 36$ degrees

Also ΔAFO is a right triangle with right angle at F. Using trigonometric functions, $AO = AF \operatorname{cosec} \angle AOF$
$= \frac{x}{2} \operatorname{cosec} 36$

8. let the number of ways be $f(n)$

$$f(n+1) = f(n) + \binom{4n}{2} + 2\times\binom{2n}{1} + 1\times\binom{2n}{1} + 1$$

and $f(1) = 1$

and $f(n+1) = f(n) + 8n^2 + 4n + 1$

$$f(n) - f(1) = \frac{4}{3}(n-1)(n)(2n-1) + 2(n-1)n + (n-1)$$

$$f(n) = \frac{n}{3}(8n^3 - 6n + 1) \quad \forall\, n \in \mathbb{N}$$

method 2:

brute force $\mapsto$ for each case

0-0-0 : $\binom{n}{3}$ 1-1-2 : $\binom{n}{2} \cdot n$

0-1-3 : $(n)(n)(n)$ 2-3-3 : $n \cdot \binom{n}{2}$

0-2-2 : $n \cdot \binom{n}{2}$

$$\text{Total} = \frac{n(n-1)(n-2)}{6} + n^3 + \frac{3n^2(n-1)}{2}$$

$$= \frac{n}{3}(8n^2-6n+1) \quad \forall n \in N$$

9. The region we want is bounded by the $\perp$ bisectors of segments P_0P_i $i=1,2,3,4,5$

P_0P_1 is bisected by $y=2$

P_0P_2 is bisected by $x=2$

P_0P_3 is bisected by $x+y=-2$

P_0P_4 is bisected by $x+y=3$

P_0P_5 is bisected by $x+y=5$

clearly we can ignore $x+y=5$ because the bound provided by $x+y=3$ is stronger.

So the region we seek has vertices $(2,-4)$, $(-4,2)$, $(2,1)$, $(1,2)$

The perimeter, hence, $10+7\sqrt{2}$

10. There are $K^2-(K-1)^2-1 = 2(K-1)$ non square integers b/w $m\ K^m$ and $m\ (K+1)^n$ square integers.

So there are $2(1+2+3+\dots+(K-1)) = K(K-1)$ non square integers preceeding the K^m square integer

we want such that

$$K(K-1) = n \Rightarrow K = \frac{1+\sqrt{4n-1}}{2}$$

so the n^{th} nonsquare integer is

$$\left(\frac{1+\sqrt{4n-1}}{2}\right)^2 - 1 \quad \text{or} \quad \left[\left(\frac{1+\sqrt{4n+1}}{2}\right)^2\right]$$

where $[\cdot]$ is G.I.F

$$\text{i.e } n+\left(\frac{\sqrt{4n+1}-1}{2}\right) \quad \text{or} \quad \left[n+\left(\frac{\sqrt{4n+1}+1}{2}\right)\right]$$

it suffices to show that

$$\langle\sqrt{n}\rangle = \begin{cases} \dfrac{\sqrt{4n+1}-1}{2} & \text{if it is an Integer} \quad \text{— (I)} \\ \left[\dfrac{\sqrt{4n+1}+1}{2}\right] & \text{if not} \quad \text{— (II)} \end{cases}$$

now (I) is an integer if and only if n is of form $m(m-1)$ for some $m \in \mathbb{Z}$*

also it is equal to $(m-1)$

so to prove 1, we need to show $\langle \sqrt{m(m-1)} \rangle = m$

Indeed

$$(m-1) - \tfrac{1}{2} < \sqrt{m(m-1)} < (m-1) + \tfrac{1}{2}$$

so (1) is true

for (2),

we observe

$$\left[\sqrt{n + \tfrac{1}{4}} + \tfrac{1}{2}\right] = \left[\sqrt{n} + \tfrac{1}{2}\right] \text{ for } n \in \mathbb{Z} \text{ and } n > 1$$

also the last expression is nothing but $\langle\sqrt{n}\rangle$

* proof of the assertion follows —

we want $4n+1 = I^2$ for some $I \in \mathbb{Z}$

so $4n = (I+1)(I-1)$

since LHS is even hence RHS must be even

so I must be of form $(2m+1)$ for some $m \in \mathbb{Z}$

i.e. $4n = (2m+2)(2m)$

$\Rightarrow n = (m+1)m$