1. There are 8 balls numbered 1, 2, ..., 8 and 8 boxes numbered 1, 2, ..., 8. The number of ways one can put these balls in the boxes so that each box gets one ball and exactly 4 balls go in their corresponding numbered boxes is
(a) $3 \times \binom{8}{4}$
(b) $6 \times \binom{8}{4}$
(c) $9 \times \binom{8}{4}$
(d) $12 \times \binom{8}{4}$

Solution: (C)

We can choose 4 balls out of 8 balls in $\binom{8}{4}$ ways.

Let us say boxes 5, 6, 7, 8 has got the balls numbered 5, 6, 7, 8 respectively. So we are done with exactly 4 balls in their corresponding number. Now we will find the number of ways in which we can put rest of the balls in different numbered boxes. So, we have 1, 2, 3, 4 numbered boxes and 1, 2, 3, 4 numbered balls.

The sample space looks like below figure,
So, we can put the balls in the boxes with given condition in $9 \times 8^C_4$ ways.

2. Let $\alpha$ and $\beta$ be two positive real numbers. For every integer $n > 0$, define $a_n = \int \{\alpha/u(u^\alpha + 2 + u^{-\alpha})\}du$ (integration runs from $\beta$ to $n$). Then $\lim(a_n)$ as $n \rightarrow \infty$ is equal to

(a) $1/(1 + \beta^\alpha)$
(b) $\beta^\alpha/(1 + \beta^{-\alpha})$
(c) $\beta^\alpha/(1 + \beta^\alpha)$
(d) $\beta^{-\alpha}/(1 + \beta^\alpha)$

Solution: (A)

We have, $a_n = \int \{\alpha/u(u^\alpha + 2 + u^{-\alpha})\}du$ (integration runs from $\beta$ to $n$)
\[ a_n = \int \{\alpha u^{\alpha-1}/(u^\alpha + 1)^2\}du \] (integration runs from $\beta$ to $n$)
Substitute, $u^\alpha + 1 = z$
So, $\alpha u^{\alpha-1}du = dz$
\[ a_n = \int dz/z^2 \] (integration runs from $1 + \beta^\alpha$ to $1 + n^\alpha$)
\[ a_n = [-1/z] (z \text{ running from } 1 + \beta^\alpha \text{ to } 1 + n^\alpha) \]
\[ \lim(a_n) \text{ as } n \rightarrow \infty = 1/(1 + \beta^\alpha) \]

3. Let $f : \mathbb{R} \rightarrow \mathbb{R}^2$ be a function given by $f(x) = (x^m, x^n)$, where $x \in \mathbb{R}$ and $m, n$ are fixed positive integers. Suppose that $f$ is one-one. Then

(a) Both $n$ and $m$ must be odd
(b) At least one of $m$ and $n$ must be odd
(c) Exactly one of $m$ and $n$ must be odd
(d) Neither $m$ nor $n$ can be odd.

Solution: (B)

Let us take $x = 2$ and $x = -2$.
\[ f(2) = (2^m, 2^n) \text{ and } f(-2) = \{(-2)^m, (-2)^n\} \]
Clearly $m, n$ both cannot be even otherwise $f$ is not one-one.
Clearly if at least one of them is odd then $f$ is one-one.
4. \( \lim\left[\frac{(e^{x^2} - e^{2x})}{(x - 2)e^{2x}}\right] \) as \( x \to 2 \) equals
(a) 0
(b) 1
(c) 2
(d) 3

Solution: (C)

\[
\lim\left[\frac{(e^{x^2} - e^{2x})}{(x - 2)e^{2x}}\right] \text{ as } x \to 2
\]
\[
= \lim\left[\frac{e^{2x}(e^{x^2 - 2x} - 1)}{(x - 2)e^{2x}}\right] \text{ as } x \to 2
\]
\[
= \lim\left[\frac{e^{(x^2 - 2x)} - 1}{(x - 2)}\right] \text{ as } x \to 2
\]
\[
= \lim\left[\frac{(2x - 2)e^{(x^2 - 2x)}}{1}\right] \text{ as } x \to 2 \text{ (Applying L'Hospital rule)}
\]
\[
= (2\times2 - 2)e^0/1 = 2
\]

5. A circle is inscribed in a triangle with sides 8, 15, 17 cms. The radius of the circle in cms is
(a) 3
(b) \( \frac{22}{7} \)
(c) 4
(d) None of the above.

Solution: (A)

Area of triangle ABC
\[
= \sqrt[\{(15 + 17 + 8)/2\} \{15 + 17 - 8)/2\} \{(15 + 8 - 17)/2\} \{(17 + 8 - 15)/2\}
\]
\[
= \sqrt{(20\times12\times3\times5)}
\]
\[
= 5\times3\times4
\]
Now, area of triangle ABC
= area of triangle AOB + area of triangle BOC + area of triangle COA
= \frac{1}{2} \times r \times 8 + \frac{1}{2} \times r \times 17 + \frac{1}{2} \times r \times 15
= \frac{1}{2} \times (8 + 17 + 15) \times r
= 20 \times r

Now, 20 \times r = 5 \times 3 \times 4

So, r = 3.

6. Let α, β and γ be the angles of an acute angled triangle. Then the quantity \( \tan \alpha \tan \beta \tan \gamma \)

(a) Can have any real value
(b) Is \( \leq 3\sqrt{3} \)
(c) Is \( \geq 3\sqrt{3} \)
(d) None of the above.

Solution: (C)

Now, \( \tan(\alpha + \beta) = (\tan \alpha + \tan \beta)/(1 - \tan \alpha \tan \beta) \)
\Rightarrow \( \tan(\pi - \gamma) = (\tan \alpha + \tan \beta)/(1 - \tan \alpha \tan \beta) \) (As \( \alpha + \beta + \gamma = \pi \))
\Rightarrow \(-\tan \gamma (1 - \tan \alpha \tan \beta) = (\tan \alpha + \tan \beta)\)

Now, \( \tan \alpha \tan \beta \tan \gamma \)
= \( \tan \gamma - \tan \gamma + \tan \alpha \tan \beta \tan \gamma \)
= \( \tan \gamma - \tan \gamma (1 - \tan \alpha \tan \beta) \)
= \( \tan \gamma + \tan \alpha + \tan \beta \) (From above)

So, we have, \( \tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma = x \) (say)

Now, AM \( \geq \) GM

\( \frac{\tan \alpha + \tan \beta + \tan \gamma}{3} \geq (\tan \alpha \tan \beta \tan \gamma)^{1/3} \)

\( \frac{x}{3} \geq x^{1/3} \)

\( x^{2/3} \geq 3 \)

\( x \geq 3\sqrt{3} \)
7. Let \( f(x) = |x|\sin x + |x - \pi|\cos x \) for \( x \in \mathbb{R} \). Then
   (a) \( f \) is differentiable at \( x = 0 \) and \( x = \pi \)
   (b) \( f \) is not differentiable at \( x = 0 \) and \( x = \pi \)
   (c) \( f \) is differentiable at \( x = 0 \) but not differentiable at \( x = \pi \)
   (d) \( f \) is not differentiable at \( x = 0 \) but differentiable at \( x = \pi \)

Solution: (C)

Now, \( f(x) = |x|\sin x + |x - \pi|\cos x \)

\[ f(x) = -x\sin x - (x - \pi)\cos x \quad \text{for} \quad x \leq 0 \]

\[ f(x) = x\sin x - (x - \pi)\cos x \quad \text{for} \quad 0 < x \leq \pi \]

\[ f(x) = x\sin x + (x - \pi)\cos x \quad \text{for} \quad x > \pi \]

Now, \( \lim (f(x) - f(0))/(x - 0) \) as \( x \to 0^- \)

\[ = \lim \{-x\sin x - (x - \pi)\cos x - \pi\}/x \quad \text{as} \quad x \to 0^- \]

\[ = \lim \{-\sin x - x\cos x - \cos x + (x - \pi)\sin x\}/1 \quad \text{as} \quad x \to 0^- = -1. \]

(Applying L’Hospital rule)

Now, \( \lim (f(x) - f(0))/(x - 0) \) as \( x \to 0^+ \)

\[ = \lim \{x\sin x - (x - \pi)\cos x - \pi\}/x \quad \text{as} \quad x \to 0^+ \]

\[ = \lim \{\sin x + x\cos x - \cos x + (x - \pi)\sin x\}/1 \quad \text{as} \quad x \to 0^+ = -1. \]

(Applying L’Hospital rule)

So, we have \( f \) is differentiable at \( x = 0 \) as both the limit are same.

Now, \( \lim (f(x) - f(\pi))/(x - \pi) \) as \( x \to \pi^- \)

\[ = \lim \{x\sin x - (x - \pi)\cos x - 0\}/(x - \pi) \quad \text{as} \quad x \to \pi^- \]

\[ = \lim \{\sin x + x\cos x - \cos x + (x - \pi)\sin x\}/1 \quad \text{as} \quad x \to \pi^- = \pi\cos \pi - \cos \pi \]

\[ = -\pi + 1 \]

(Applying L’Hospital rule)

Now, \( \lim (f(x) - f(\pi))/(x - \pi) \) as \( x \to \pi^+ \)

\[ = \lim \{x\sin x + (x - \pi)\cos x - 0\}/(x - \pi) \quad \text{as} \quad x \to \pi^+ \]
\[ \lim_{x \to n^+} \frac{\sin x + x \cos x + \cos x - (x - n)\sin x}{1} = n \cos n + \cos n \]
\[ = -n - 1 \]

(Applying L’Hospital rule)

As both the limits are different so we have, \( f(x) \) is not differentiable at \( x = n \).

8. Consider a rectangular cardboard box of eight 3, breadth 4 and length 10 units. There is a lizard in one corner A of the box and an insect in the corner B which is farthest from A. The length of the shortest path between the lizard and the insect along the surface of the box is

(a) \( \sqrt{5^2 + 10^2} \)
(b) \( \sqrt{7^2 + 10^2} \)
(c) \( 4 + \sqrt{3^2 + 10^2} \)
(d) \( 3 + \sqrt{4^2 + 10^2} \)

Solution: (B)

Clearly the shortest path is \( \sqrt{7^2 + 10^2} \).
9. Recall that, for any non-zero complex number \( w \) which does not lie on the negative real axis, \( \arg(w) \) denotes the unique real number \( \theta \) in \((-\pi, \pi)\) such that \( w = |w|(\cos \theta + i \sin \theta) \). Let \( z \) be any complex number such that its real and imaginary parts are both non-zero. Further, suppose that \( z \) satisfies the relations \( \arg(z) > \arg(z + 1) \) and \( \arg(z) > \arg(z + i) \). Then \( \cos(\arg(z)) \) can take

(a) Any value in the set \((-1/2, 0) \cup (0, 1/2)\) but none from outside
(b) Any value in the interval \((-1, 0)\) but none from outside
(c) Any value in the interval \((0, 1)\) but none from outside
(d) Any value in the set \((-1, 0) \cup (0, 1)\) but none from outside.

Solution: (B)

Now, \( z + 1 = \cos \theta + i \sin \theta + 1 = (1 + \cos \theta) + i \sin \theta \)
\[
\arg(z+1) = \tan^{-1}\left\{\frac{\sin \theta}{1 + \cos \theta}\right\} = \tan^{-1}\left\{\frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}\right\} = \tan^{-1}\left\{\tan(\theta/2)\right\} = \theta/2
\]

Now, \( \arg(z) > \arg(z + 1) \)
\(\Rightarrow\) \( \theta > \theta/2 \)
\(\Rightarrow\) \( \theta \) is positive.

Now, \( z + i = \cos \theta + i \sin \theta + i = \cos \theta + i(1 + \sin \theta) \)

Now, \( \arg(z+i) = \tan^{-1}\{(1 + \sin \theta)/\cos \theta\} \)
\(= \tan^{-1}\{(\cos \theta/2 + \sin \theta/2)^2/(\cos \theta/2 + \sin \theta/2)(\cos \theta/2 - \sin \theta/2)\} \)
\(= \tan^{-1}\{(\cos \theta/2 + \sin \theta/2)/(\cos \theta/2 - \sin \theta/2)\} \)
\(= \tan^{-1}(1 + \tan(\theta/2))/(1 - \tan(\theta/2)) \)
\(= \tan^{-1}\tan(\pi/4 + \theta/2) \)
\(= \pi/4 + \theta/2 \).

Now, \( \arg(z) > \arg(z + i) \)
\(\Rightarrow\) \( \theta > \pi/4 + \theta/2 \)
\(\Rightarrow\) \( \theta > \pi/2 \)

For \( \theta > \pi/2 \) \( \cos \theta \) is negative and can take any value in the interval \((-1, 0)\).
10. An aeroplane $P$ is moving in the air along a straight line path which passes through the points $P_1$ and $P_2$, and makes an angle $\alpha$ with the ground. Let $O$ be the position of an observer as shown in the figure below. When the plane is at the position $P_1$ its angle of elevation is $30^\circ$ and when it is at $P_2$ its angle of elevation is $60^\circ$ from the position of the observer. Moreover, the distances of the observer from the points $P_1$ and $P_2$ respectively are $100$ metres and $500/3$ metres.

Then $\alpha$ is equal to

(a) $\tan^{-1}\{(2 - \sqrt{3})/(2\sqrt{3} - 1)\}$
(b) $\tan^{-1}\{(2\sqrt{3} - 3)/(4 - 2\sqrt{3})\}$
(c) $\tan^{-1}\{(2\sqrt{3} - 2)/(5 - \sqrt{3})\}$
(d) $\tan^{-1}\{(6 - \sqrt{3})/(6\sqrt{3} - 1)\}$

Solution: (D)

Angle $AOP_2 = 120^\circ$

$\Rightarrow$ Angle $OP_2P_1 = 120^\circ + \alpha$

$\Rightarrow$ Angle $OP_1P_2 = 180^\circ - (120^\circ + \alpha + 30^\circ) = 30^\circ - \alpha$ (As angle $P_1OP_2 = 60^\circ - 30^\circ = 30^\circ$)

Now, in triangle $OP_1P_2$, we have, $POP_1/sin(OP_2P_1) = OP_2/sin(OP_1P_2)$

$\Rightarrow$ $1000/sin(120^\circ + \alpha) = 500/3sin(30^\circ - \alpha)$

$\Rightarrow$ $2 \times 3sin(30^\circ - \alpha) = sin(120^\circ + \alpha)$

$\Rightarrow$ $6(sin30^\circ cos\alpha - cos30^\circ sin\alpha) = sin120^\circ cos\alpha + cos120^\circ sin\alpha$

$\Rightarrow$ $3cos\alpha - 3\sqrt{3}sin\alpha = (\sqrt{3}/2)cos\alpha - (1/2)sin\alpha$

$\Rightarrow$ $(6 - \sqrt{3})cos\alpha = (6\sqrt{3} - 1)sin\alpha$

$\Rightarrow$ $tan\alpha = (6 - \sqrt{3})/(6\sqrt{3} - 1)$

$\Rightarrow$ $\alpha = tan^{-1}\{(6 - \sqrt{3})/(6\sqrt{3} - 1)\}$
11. The sum of all even positive divisors of 1000 is
(a) 2170
(b) 2184
(c) 2325
(d) 2340

Solution: (B)

Now, 1000 = 2^3 \cdot 5^3
Sum of all the divisors of 1000 (including 1 and 1000)
= \{(2^4 - 1)/(2 - 1)\} \times \{(5^4 - 1)/(5 - 1)\} = 15 \times 156.
Sum of all the divisors of 5^3 (including 1 and 125) = (5^4 - 1)/(5 - 1) = 156.
So, sum of all even positive divisors of 1000 = 15 \times 156 - 156
= 156(15 - 1) = 156 \times 14 = 2184.

12. The equation \(x^2 + (b/a)x + (c/a) = 0\) has two real roots \(\alpha\) and \(\beta\). If \(a > 0\), then the area under the curve \(f(x) = x^2 + (b/a)x + (c/a)\) between \(\alpha\) and \(\beta\) is
(a) \((b^2 - 4ac)/2a\)
(b) \((b^2 - 4ac)^{3/2}/6a^3\)
(c) \(-(b^2 - 4ac)^{3/2}/6a^3\)
(d) \(-(b^2 - 4ac)/2a\)

Solution: (C)

Now, \(\alpha, \beta\) are roots of the equation \(x^2 + (b/a)x + (c/a)\)
\[\Rightarrow \alpha + \beta = -(b/a) \quad \text{and} \quad \alpha\beta = c/a\]
\[\Rightarrow \beta - \alpha = \sqrt{\{(\beta + \alpha)^2 - 4\alpha\beta\}} = \sqrt{\{(b^2/a^2) - 4c/a\}} = \sqrt{b^2 - 4ac}/a\]
Area under curve \(f(x)\) between \(\alpha\) and \(\beta\) = \(\int f(x)dx\) (integration running from \(\alpha\) to \(\beta\))
= \(\int \{x^2 + (b/a)x + (c/a)\}dx\) (integration running from \(\alpha\) to \(\beta\))
= \[\{(x^3/3) + (b/a)(x^2/2) + (c/a)x\}\] (from \(\alpha\) to \(\beta\))
= \{(\beta^3/3) + (b/a)(\beta^2/2) + (c/a)\beta\} - \{(\alpha^3/3) + (b/a)(\alpha^2/2) + (c/a)\alpha\}
= (\beta^3 - \alpha^3)/3 + (b/a)(\beta^2 - \alpha^2)/2 + (c/a)(\beta - \alpha)
= (\beta - \alpha)(\beta^2 + \alpha\beta + \alpha^2)/3 + (b/a)(\beta - \alpha)(\beta + \alpha)/2 + (c/a)(\beta - \alpha)
\[
= (\beta - \alpha)[((\beta + \alpha)^2 - \alpha \beta)/3 + (b/a)(\beta + \alpha)/2 + (c/a)]
\]
\[
= \{\sqrt{(b^2 - 4ac)/a}\}[(b^2/a^2) - (c/a)]/3 - (b/a)(b/a)/2 + (c/a)]
\]
\[
= \{\sqrt{(b^2 - 4ac)/a}\}(-b^2/a^2)/6 + (2/3)(c/a)]
\]
\[
= \{\sqrt{(b^2 - 4ac)/a}\}(b^2 - 4ac)/6a
\]
\[
= -(b^2 - 4ac)^{3/2}/6a^2.
\]

13. The minimum value of \(x_1^2 + x_2^2 + x_3^2 + x_4^2\) subject to \(x_1 + x_2 + x_3 + x_4 = a\) and \(x_1 - x_2 + x_3 - x_4 = b\) is

(a) \((a^2 + b^2)/4\)
(b) \((a^2 + b^2)/2\)
(c) \((a + b)^2/4\)
(d) \((a + b)^2/2\)

Solution: (A)

Now, \(x_1 + x_3 = (a + b)/2\) and \(x_2 + x_4 = (a - b)/2\)

Now, \(x_1^2 + x_3^2)/2 \geq ((x_1 + x_3)/2)^2\)

\(\Rightarrow (x_1^2 + x_3^2) \geq (a + b)^2/8\)

Similarly, \((x_2^2 + x_4^2) \geq (a - b)^2/8\)

Adding the above two inequalities we get, \(x_1^2 + x_2^2 + x_3^2 + x_4^2 \geq ((a + b)^2 + (a - b)^2)/8 = 2(a^2 + b^2)/8 = (a^2 + b^2)/4\)

14. The value of \(\lim\left[\sum\binom{2n}{2r}x^{3r}\right]/\sum\binom{2n}{2r+1}x^{3r}\) (numerator summation running from \(r = 0\) to \(r = n\), denominator summation is running from \(r = 0\) to \(r = n-1\) as \(n \to \infty\) is

(a) \(0\)
(b) \(1\)
(c) \(\sqrt{3}\)
(d) \((\sqrt{3} - 1)/(\sqrt{3} + 1)\)

Solution: (C)

\((1 + x)^{2n} = \binom{2n}{0} + \binom{2n}{1}x + \binom{2n}{2}x^2 + \binom{2n}{3}x^3 + \ldots + \binom{2n}{2n-1}x^{2n-1} + \binom{2n}{2n}x^{2n}\)

\((1 - x)^{2n} = \binom{2n}{0} - \binom{2n}{1}x + \binom{2n}{2}x^2 - \binom{2n}{3}x^3 + \ldots - \binom{2n}{2n-1}x^{2n-1} + \binom{2n}{2n}x^{2n}\)
Adding we get, \((1 + x)^{2n} + (1 - x)^{2n} = 2[2^nC_0 + 2^nC_2x^2 + 2^nC_4x^4 + \ldots + 2^nC_{2n}x^{2n}]\)

Subtracting we get, \((1 + x)^{2n} - (1 - x)^{2n} = 2x[2^nC_1 + 2^nC_3x^2 + 2^nC_5x^4 + \ldots + 2^nC_{2n-1}x^{2n-2}]\)

Now, putting \(x = \sqrt{3}\) we get, \(\{(1 + \sqrt{3})^{2n} + (1 - \sqrt{3})^{2n}\}/2 = 2^nC_0 + 2^nC_2*3 + 2^nC_4*3^2 + \ldots + 2^nC_{2n}*3^n\) = Numerator.

\(\Rightarrow\) Numerator = \(\{(1 + \sqrt{3})^{2n} + (1 - \sqrt{3})^{2n}\}/2\)

Similarly putting \(x = \sqrt{3}\) in the 2\(^{nd}\) equation we get,

Denominator = \(\{(1 + \sqrt{3})^{2n} - (1 - \sqrt{3})^{2n}\}/2\sqrt{3}\)

Therefore, the given expression = \(\lim_{n \to \infty}\frac{\{(1 + \sqrt{3})^{2n} + (1 - \sqrt{3})^{2n}\}/2}{\{(1 + \sqrt{3})^{2n} - (1 - \sqrt{3})^{2n}\}/2\sqrt{3}}\) as \(n \to \infty\)

= \(\sqrt{3}\lim_{n \to \infty}\frac{[1 + ((\sqrt{3} - 1)/\sqrt{3 + 1})2n]}{[1 - ((\sqrt{3} - 1)/\sqrt{3 + 1})2n]}\) as \(n \to \infty\)

= \(\sqrt{3}(1 - 0)/(1 - 0)\)

= \(\sqrt{3}\)

15. For any real number \(x\) \(\tan^{-1}(x)\) denote the unique real number \(\theta\) in \((-\pi/2, \pi/2)\) such that \(\tan\theta = x\). Then \(\lim_{n \to \infty}\sum\tan^{-1}\{1/(1 + m + m^2)\}\) where \(m\) runs from 1 to \(n\) as \(n \to \infty\)

(a) Is equal to \(n/2\)
(b) Is equal to \(n/4\)
(c) Does not exist
(d) None of the above.

Solution: (B)

Now, \(\tan^{-1}\{1/(1 + m + m^2)\} = \tan^{-1}[\{(m + 1) - m\}/\{1 + m(m+1)\}]\)

= \(\tan^{-1}(m + 1) - \tan^{-1}(m)\)

\(\tan^{-1}\{1/(1 + 1 + 1^2)\} = \tan^{-1}(2) - \tan^{-1}(1)\)

\(\tan^{-1}\{1/(1 + 2 + 2^2)\} = \tan^{-1}(3) - \tan^{-1}(2)\)

\(\tan^{-1}\{1/(1 + 3 + 3^2)\} = \tan^{-1}(4) - \tan^{-1}(3)\)

…..
\[
\tan^{-1}\left\{\frac{1}{1 + n + n^2}\right\} = \tan^{-1}(n + 1) - \tan^{-1}(n)
\]

\[
\sum\tan^{-1}\left\{\frac{1}{1 + m + m^2}\right\} = \tan^{-1}(n + 1) - \tan^{-1}(1) \quad \text{where } m \text{ runs from } 1 \text{ to } n.
\]

The given expression becomes,

\[
\lim\{\tan^{-1}(n + 1) - \tan^{-1}(1)\} \quad \text{as } n \to \infty
\]

\[
= \tan^{-1}(\infty) - \tan^{-1}(1)
\]

\[
= (\pi/2) - (\pi/4)
\]

\[
= \pi/4
\]

16. Let \( n \) be an integer. The number of primes which divide both \( n^2 - 1 \) and \( (n + 1)^2 - 1 \) is

(a) At most one.
(b) Exactly one.
(c) Exactly two.
(d) None of the above.

Solution: (A)

Now, \( n^2 - 1 = (n + 1)(n - 1) \) and \( (n + 1)^2 - 1 = n(n + 2) \)

Now, \( n + 1 \) is relatively prime with both \( n \) and \( n + 2 \) as they are consecutive integers.

So, only possible case is a prime may divide \( n - 1 \) and \( n + 2 \) as \( n - 1 \) and \( n \) are also relatively prime.

We see that if 3 divides \( n - 1 \) then 3 divides \( n + 2 \).

We also see that if 3 divides \( n \) or \( n + 1 \) then 3 doesn’t divide the rest three.

So, at most one prime (3) can divide both \( n^2 - 1 \) and \( (n + 1)^2 - 1 \).

17. The value of \( \lim[\sum\{6n/(9n^2 - r^2)\}] \) as \( n \to \infty \) is

(a) 0
(b) \( \log(3/2) \)
(c) \( \log(2/3) \)
(d) \( \log(2) \)

Solution: (D)
18. A person X standing at a point P on a flat plane starts walking. At each step, he walks exactly 1 foot in one of the directions North, South, East or West. Suppose that after 6 steps X comes to the original position P. Then the number of distinct paths that X can take is
(a) 196
(b) 256
(c) 344
(d) 400

Solution: (D)

Total number of cases = 100×4 = 400.

19. Consider the branch of the rectangular hyperbola xy = 1 in the first quadrant. Let P be a fixed point on this curve. The locus of the mid-point of the line segment joining P and an arbitrary point Q on the curve is part of
(a) A hyperbola
(b) A parabola
(c) An ellipse
(d) None of the above.

Solution: (A)

Let co-ordinate of point P is (a, b)
Let co-ordinate of point Q is (h, k)
Co-ordinate of mid-point of P and Q is {(h + a)/2, (k + b)/2}

Now, P and Q lies on the hyperbola xy = 1

So, we have, ab = 1 and hk = 1.

Putting (h + a)/2 = x and (k + b)/2 = y i.e. h = 2x – a and k = 2y – b we get,

\[(2x - a)(2y - b) = 1 \Rightarrow 4xy - 2ay - 2bx + ab = 1\]
\[4xy - 2ay - 2bx = 0 \text{ (As } ab = 1\]
\[xy - bx/2 - ay/2 = 0\]

Which is the equation of a hyperbola.
20. The digit at the unit place of 
\[(1! - 2! + 3! - ...... + 25!) (1! - 2! + 3! - .... + 25!)\] is

(a) 0  
(b) 1  
(c) 5  
(d) 9

Solution: (B)

Now, \[1! - 2! + 3! - 4! + 5! = 1 - 2 + 6 - 24 + 120 = 101\]

Now, last digit of \[-6! + 7! - .... + 25! Is zero.\]

\[\Rightarrow 1! - 2! + 3! - .... + 25! Has the last digit as 1.\]

\[\Rightarrow The last digit of (1! - 2! + 3! - .... + 25!) (1! - 2! + 3! - .... + 25!) is 1.\]

21. Let \(A_1, A_2, ...., A_n\) be the vertices of a regular polygon and \(A_1A_2, A_2A_3, ...., A_{n-1}A_n, A_nA_1\) be its n sides. If \((1/A_1A_2) - (1/A_1A_4) = (1/A_1A_3)\), then the value of n is

(a) 5  
(b) 6  
(c) 7  
(d) 8

Solution: (C)
Let, the side of the regular polygon is \( a \).

Now, angle \( A_1A_2A_3 = (n - 2)\pi/n \)

In triangle \( A_1A_2A_3 \), \( A_1A_2 = A_2A_3 \)

\[ \Rightarrow \quad \text{Angle } A_2A_1A_3 = \text{Angle } A_2A_3A_1 \]

\[ \Rightarrow \quad \text{Angle } A_2A_3A_1 = (1/2)[n - (n - 2)\pi/n] = \pi/n \]

In triangle \( A_1A_2A_3 \), \( A_1A_3/\sin \{(n - 2)\pi/n\} = a/\sin(\pi/n) \)

\[ \Rightarrow \quad A_1A_3 = \sin(n - 2n)/\sin(\pi/n) = \sin(2n)/\sin(\pi/n) = 2\cos(n) \]

Let, Angle \( A_3A_1A_4 = A \)

Now, angle \( A_1A_3A_4 = (n - 2)\pi/n - \pi/n = n - 3\pi/n \)

Angle \( A_1A_4A_3 = n - A - \pi + 3\pi/n = 3\pi/n - A \)

Now, in triangle \( A_1A_3A_4 \) we have,

\[ A_1A_4/\sin(n - 3\pi/n) = a/\sin A = A_1A_3/\sin(3\pi/n - A) \]

From, \( a/\sin A = A_1A_3/\sin(3\pi/n - A) \) we get,

\[ \cot A = \{2\cos(\pi/n) + \cos(3\pi/n)\}/\sin(3\pi/n) \text{ (using } \sin(3\pi/n - A) \text{) } \]

\[ = \sin(3\pi/n)\cos A - \cos(3\pi/n)\sin A \]

\[ \Rightarrow \quad \sin A = \sin(3\pi/n)/\sqrt[2]{[1 + 4\cos(\pi/n)\cos(3\pi/n) + 4\cos^2(\pi/n)]} \]

Now, from the relation, \( A_1A_4/\sin(n - 3\pi/n) = a/\sin A \) we get,

\[ A_1A_4/\sin(3\pi/n) = a\sqrt[2]{[1 + 4\cos(\pi/n)\cos(3\pi/n) + 4\cos^2(\pi/n)]}/\sin(3\pi/n) \]

\[ \Rightarrow \quad A_1A_4 = a\sqrt[2]{[1 + 4\cos(\pi/n)(4\cos^3(\pi/n) - 3\cos(\pi/n))]} + 4\cos^2(\pi/n)] \]

\[ \Rightarrow \quad A_1A_4 = a(4\cos^2(\pi/n) - 1) \]

Now, given \( 1/A_1A_2 - 1/A_1A_3 = 1/A_1A_4 \)

\[ \Rightarrow \quad (1/a) - 1/2\cos(n) = 1/a(4\cos^2(\pi/n) - 1) \]

\[ \Rightarrow \quad 1 - 1/2\cos(n) = 1/(4\cos^2(n) - 1) \]

This equation is satisfied by \( n = 7 \).
22. Suppose that α and β are two distinct numbers in the interval 
(0, π). If \( \sin \alpha + \sin \beta = \sqrt{3}(\cos \alpha - \cos \beta) \) then the value of \( \sin 3\alpha + \sin 3\beta \) is

(a) 0
(b) \( 2\sin\{3(\alpha + \beta)/2\} \)
(c) \( 2\cos\{3(\alpha - \beta)/2\} \)
(d) \( \cos\{3(\alpha - \beta)/2\} \)

Solution: (A)

Now, \( \sin \alpha + \sin \beta = \sqrt{3}(\cos \alpha - \cos \beta) \)

\[ \Rightarrow 2\sin\{(\alpha + \beta)/2\}\cos\{(\alpha - \beta)/2\} = \sqrt{3}*2\sin\{(\alpha + \beta)/2\}\sin\{(\beta - \alpha)/2\} \]

\[ \Rightarrow \cos\{(\alpha - \beta)/2\} = -\sqrt{3}\sin\{(\alpha - \beta)/2\} \]

\[ \Rightarrow \tan\{(\alpha - \beta)/2\} = \tan\left(\pi - \frac{\pi}{6}\right) \]

\[ \Rightarrow (\alpha - \beta)/2 = \pi - \pi/6 \]

Now, \( \sin 3\alpha + \sin 3\beta \)

\[ = 2\sin\{3(\alpha + \beta)/2\}\cos\{3(\alpha - \beta)/2\} \]

\[ = 2\sin\{3(\alpha + \beta)/2\}\cos\{3(\pi - \pi/6)\} \]

\[ = 2\sin\{3(\alpha + \beta)/2\}\cos(3\pi - \pi/2) \]

\[ = 2\sin\{3(\alpha + \beta)/2\}\{-\cos(\pi/2)\} \]

\[ = 0 \text{ (As } \cos(\pi/2) = 0) \]

23. Consider the function \( h(x) = x^2 - 2x + 2 + 4/(x^2 - 2x + 2), x \in \mathbb{R} \). Then \( h(x) - 5 = 0 \) has

(a) No solution
(b) Only one solution
(c) Exactly two solutions
(d) Exactly three solutions.

Solution: (D)

Now, \( h(x) - 5 = 0 \)

\[ \Rightarrow x^2 - 2x + 2 + 4/(x^2 - 2x + 2) - 5 = 0 \]
(x^2 - 2x + 2)^2 - 5(x^2 - 2x + 2) + 4 = 0 (As x^2 - 2x + 2 ≠ 0 because roots are imaginary)

(x^2 - 2x + 2)^2 - 4(x^2 - 2x + 2) - (x^2 - 2x + 2) + 4 = 0

(x^2 - 2x + 2)(x^2 - 2x + 2 - 4) - (x^2 - 2x + 2 - 4) = 0

(x^2 - 2x + 2)(x^2 - 2x - 2) - (x^2 - 2x - 2) = 0

(x^2 - 2x - 2)(x^2 - 2x + 2 - 1) = 0

(x^2 - 2x - 2)(x^2 - 2x + 1) = 0

(x^2 - 2x - 2)(x - 1)^2 = 0

So, x = 1 is a solution and x^2 - 2x - 2 = 0

x = \{2 ± \sqrt{4 + 8}\}/2

x = 1 ± \sqrt{3}

So, the equation has 3 solutions viz. x = 1, 1 + \sqrt{3} and 1 - \sqrt{3}

24. Consider the quadratic equation x^2 + bx + c = 0. The number of pairs (b, c) which the equation has solutions of the form cosα and sinα for some α is

(a) 0
(b) 1
(c) 2
(d) Infinite.

Solution: (D)

25. Let θ_1 = 2\pi/3, θ_2 = 4\pi/7, θ_3 = 7\pi/12. Then

(a) sinθ_1\sinθ_1 < sinθ_2\sinθ_2 < sinθ_3\sinθ_3
(b) sinθ_2\sinθ_2 < sinθ_1\sinθ_1 < sinθ_3\sinθ_3
(c) sinθ_3\sinθ_3 < sinθ_1\sinθ_1 < sinθ_2\sinθ_2
(d) sinθ_1\sinθ_1 < sinθ_3\sinθ_3 < sinθ_2\sinθ_2

Solution: (D)
26. Consider the following two curves on the interval (0, 1):
   \( C_1 : y = 1 - x^4 \) and \( C_2 : y = \sqrt{1 - x^2} \). Then on (0, 1)
   (a) \( C_1 \) lies above \( C_2 \)
   (b) \( C_2 \) lies above \( C_1 \)
   (c) \( C_1 \) and \( C_2 \) intersect at exactly one point
   (d) None of the above.

Solution: (C)

If we look for the point of intersection of \( C_1 \) and \( C_2 \) we get,
\[
1 - x^4 = \sqrt{1 - x^2}
\]
\[
(1 + x^2)(1 - x^2) = \sqrt{1 - x^2}
\]
\[
(1 + x^2)\sqrt{1 - x^2} = 0
\]
\[
1 - x^2 = 0 \quad (\text{As } 1 + x^2 \text{ cannot be equal to 0})
\]
\[
x = 1 \quad (\text{As } -1 \text{ doesn’t fall in the interval } (0, 1))
\]
\[
C_1 \text{ and } C_2 \text{ intersect at exactly one point.}
\]

27. Let \( f \) be a real valued function on \( \mathbb{R} \) such that \( f \) is twice differentiable. Suppose that \( f' \) vanishes only at 0 and \( f'' \) is everywhere negative. Define a function \( h \) by \( h(x) = (x - a)^2 - f(x) \), where \( a > 0 \). Then
   (a) \( h \) has a local minima in \((0, a)\)
   (b) \( h \) has a local maxima in \((0, a)\)
   (c) \( h \) is monotonically increasing in \((0, a)\)
   (d) \( h \) is monotonically decreasing in \((0, a)\).

Solution: (A)

Now, \( h(x) = (x - a)^2 - f(x) \)
\[
\Rightarrow h'(x) = 2(x - a) - f'(x)
\]
\[
h''(x) = 2 - f''(x)
\]
Now, \( f''(x) \) is negative everywhere.
\[
\Rightarrow h''(x) > 0 \text{ everywhere.}
\]
\[
h(x) \text{ has a local minima.}
\]
Now, \( h'(x) = 0 \) gives, \( x = a + f'(x)/2 \)
28. Consider the triangle with vertices (1, 2), (-5, -1) and (3, -2). Let Δ denote the region enclosed by the above triangle. Consider the function \( f : \Delta \to \mathbb{R} \) defined by \( f(x, y) = |10x - 3y| \). Then the range of \( f \) is the interval

(a) \([0, 36]\)  
(b) \([0, 47]\)  
(c) \([4, 47]\)  
(d) \([36, 47]\)

Solution: (B)

Clearly (0, 0) is inside the triangle.

Hence, \( f(0, 0) = 0 \) in \( \mathbb{R} \). So, 0 is in the range of \( f \).

Option (c) and (d) cannot be true.

Now, (-5, -1) is in \( \Delta \)

\[
f(-5, -1) = |10 \times (-5) - 3 \times (-1)| = 47
\]

So, 47 is in range of \( f \).

Option (a) cannot be true.
29. For every positive integer $n$, let $<n>$ denote the integer closest to $\sqrt{n}$. Let $A_k = \{n > 0 : <n> = k\}$. The number of elements in $A_{49}$ is

(a) 97  
(b) 98  
(c) 99  
(d) 100

Solution: (B)

$49.5^2 = 2450.25$ and $48.5^2 = 2352.25$

Number of elements in $A_{49} = 2450 - 2353 + 1 = 98$.

30. Consider a square $ABCD$ inscribed in a circle of radius 1. Let $A'$ and $C'$ be two points on the (smaller) arcs $AD$ and $CD$ respectively, such that $A'ABCC'$ is a pentagon in which $AA' = CC'$. If $P$ denotes the area of the pentagon $A'ABCC'$ then

(a) $P$ cannot be equal to 2.  
(b) $P$ lies in the interval $(1, 2]$.  
(c) $P$ is greater than or equal to 2.  
(d) None of the above.

Solution: (D)
Clearly from the picture none of option (a), (b), (c) always correct because $A'$ and $C'$ can move from $A$ to $D$ and $C$ to $D$ respectively. When $A'$ and $C'$ very close to $D$ i.e. $A'D = C'D \to 0$ then area of the trapezium $A'ACC'$ is near to $1$ and when $A'$ and $C'$ are close to $A$ and $C$ respectively i.e. $AA'$ and $CC'$ both $\to 0$ then area of trapezium $A'ACC'$ is near to $0$. 

Also, clearly, area of triangle $AA'E = area$ of triangle $CC'F$

If area of triangle $AA'E > (area$ of triangle $DEF)/2$ then area of the pentagon $A'ABCC'$ greater than $2$ as area of triangle $ADC = 1$

When $A'$ is at position such that $E$ is the mid-point of $AD$. Then area of triangle $DEF = (1/2)\times(1/\sqrt{2})\times(1/\sqrt{2}) = \frac{1}{4}$

Then, $AE = 1/\sqrt{2}$ and perpendicular from $A'$ to $AE = 1 - 1/\sqrt{2}$

So, area of triangle $A'AE = (1/2)\times(1/\sqrt{2})\times(1 - 1/\sqrt{2}) = (\sqrt{2} - 1)/4$

$$\Rightarrow\quad area\ of\ triangle\ A'AE\ +\ area\ of\ triangle\ C'CF = (\sqrt{2} - 1)/2$$

Now, $(\sqrt{2} - 1)/2 - \frac{1}{4} = (2\sqrt{2} - 3)/4 > 0$

So, Area of the pentagon can be $> 2$ and also from above discussion we have area of pentagon can be $< 2$ and so it can be equal to $2$. 