B.Stat. Admission Test 2010 Solution Paper

1. **There are 8 balls numbered 1,2,...,8 and 8 boxes numbered 1,2,....,8. The number of ways one can put these balls in the boxes so that each box gets one ball and exactly 4 balls go in their corresponding numbered boxes is**
   (a) $3\times{}^{8}C_4$
   (b) $6\times{}^{8}C_4$
   (c) $9\times{}^{8}C_4$
   (d) $12\times{}^{8}C_4$

Solution: (C)

We can choose 4 balls out of 8 balls in ${}^{8}C_4$ ways.

Let us say boxes 5, 6, 7, 8 has got the balls numbered 5, 6, 7, 8 respectively. So we are done with exactly 4 balls in their corresponding number. Now we will find the number of ways in which we can put rest of the balls in different numbered boxes. So, we have 1, 2, 3, 4 numbered boxes and 1, 2, 3, 4 numbered balls.

The sample space looks like below figure,

	1	2	3	4
1.	4	1	2	3
2.	2	1	4	3
3.	3	1	4	2
4.	4	3	1	2
5.	3	4	1	2
6.	2	4	1	3
7.	4	3	2	1
8.	3	4	2	1
9.	2	3	4	1

So, we can put the balls in the boxes with given condition in $9\times{}^{8}C_4$ ways.

2. Let α and β be two positive real numbers. For every integer n > 0, define $a_n = \int\{\alpha/u(u^{\alpha} + 2 + u^{-\alpha})\}du$ (integration runs from β to n). Then $\lim(a_n)$ as n -> ∞ is equal to

(a) $1/(1 + \beta^{\alpha})$
(b) $\beta^{\alpha}/(1 + \beta^{-\alpha})$
(c) $\beta^{\alpha}/(1 + \beta^{\alpha})$
(d) $\beta^{-\alpha}/(1 + \beta^{\alpha})$

Solution: (A)

We have, $a_n = \int\{\alpha/u(u^{\alpha} + 2 + u^{-\alpha})\}du$ (integration runs from β to n)

⇨ $a_n = \int\{\alpha u^{\alpha}/u(u^{2\alpha} + 2u^{\alpha} + 1)\}du$ (integration runs from β to n)

⇨ $a_n = \int\{\alpha u^{\alpha-1}/(u^{\alpha} + 1)^2\}du$ (integration runs from β to n)

Substitute, $u^{\alpha} + 1 = z$

So, $\alpha u^{\alpha-1}du = dz$

$a_n = \int dz/z^2$ (integration runs from $1 + \beta^{\alpha}$ to $1 + n^{\alpha}$)

$a_n = [-1/z]$ (z running from $1 + \beta^{\alpha}$ to $1 + n^{\alpha}$)

$a_n = -1/(1 + n^{\alpha}) + 1/(1 + \beta^{\alpha})$

$\lim(a_n)$ as n -> ∞ = $1/(1 + \beta^{\alpha})$

3. Let f : R -> R^2 be a function given by $f(x) = (x^m, x^n)$, where x Є R and m, n are fixed positive integers. Suppose that f is one-one. Then

(a) Both n and m must be odd
(b) At least one of m and n must be odd
(c) Exactly one of m and n must be odd
(d) Neither m nor n can be odd.

Solution: (B)

Let us take x = 2 and x = -2.

$f(2) = (2^m, 2^n)$ and $f(-2) = \{(-2)^m, (-2)^n\}$

Clearly m, n both cannot be even otherwise f is not one-one.

Clearly if at least one of them is odd then f is one-one.

4. $\lim[(e^{x^2} - e^{2x})/\{(x - 2)e^{2x}\}]$ as x -> 2 equals

(a) 0
(b) 1
(c) 2
(d) 3

Solution: (C)

$\lim[(e^{x^2} - e^{2x})/\{(x - 2)e^{2x}\}]$ as x -> 2

$= \lim[e^{2x}\{e^{(x^2 - 2x)} - 1\}/\{(x - 2)e^{2x}\}]$ as x -> 2

$= \lim[\{e^{(x^2 - 2x)} - 1\}/(x - 2)]$ as x -> 2

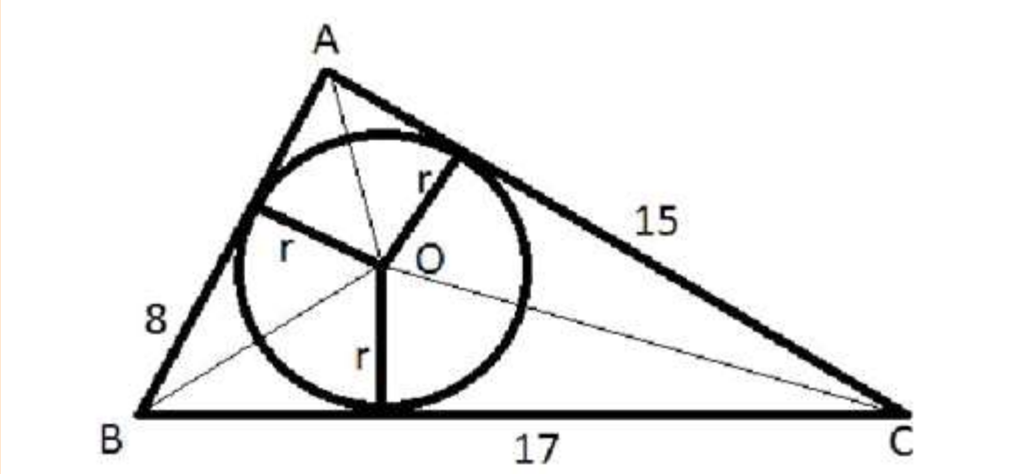
$= \lim\{(2x - 2)e^{(x^2 - 2x)}\}/1$ as x -> 2 (Applying L'Hospital rule)

$= (2\times2 - 2)e^0/1 = 2$

5. A circle is inscribed in a triangle with sides 8, 15, 17 cms. The radius of the circle in cms is

(a) 3
(b) 22/7
(c) 4
(d) None of the above.

Solution: (A)



Area of triangle ABC

$= \sqrt{[\{(15 + 17 + 8)/2\}\{(15 + 17 - 8)/2\}\{(15 + 8 - 17)/2\}\{(17 + 8 - 15)/2\}}$
$= \sqrt{(20\times12\times3\times5)}$
$= 5\times3\times4$

Now, area of triangle ABC
= area of triangle AOB + area of triangle BOC + area of triangle COA
= $(1/2)\times r\times 8 + (1/2)\times r\times 17 + (1/2)\times r\times 15$
= $(1/2)\times(8 + 17 + 15)\times r$
= $20\times r$

Now, $20\times r = 5\times 3\times 4$

So, $r = 3$.

6. Let α, β and γ be the angles of an acute angled triangle. Then the quantity tanαtanβtanγ
(a) Can have any real value
(b) Is ≤ 3√3
(c) Is ≥ 3√3
(d) None of the above.

Solution: (C)

Now, $\tan(\alpha + \beta) = (\tan\alpha + \tan\beta)/(1 - \tan\alpha\tan\beta)$

⇨ $\tan(\pi - \gamma) = (\tan\alpha + \tan\beta)/(1 - \tan\alpha\tan\beta)$ (As $\alpha + \beta + \gamma = \pi$)

⇨ $-\tan\gamma(1 - \tan\alpha\tan\beta) = (\tan\alpha + \tan\beta)$

Now, $\tan\alpha\tan\beta\tan\gamma$

$= \tan\gamma - \tan\gamma + \tan\alpha\tan\beta\tan\gamma$

$= \tan\gamma - \tan\gamma(1 - \tan\alpha\tan\beta)$

$= \tan\gamma + \tan\alpha + \tan\beta$ (From above)

So, we have, $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma = x$ (say)

Now, AM ≥ GM

$(\tan\alpha + \tan\beta + \tan\gamma)/3 \geq (\tan\alpha\tan\beta\tan\gamma)^{1/3}$

$x/3 \geq x^{1/3}$

$x^{2/3} \geq 3$

$x \geq 3\sqrt{3}$

7. Let f(x) = |x|sinx + |x – π|cosx for x Є R. Then

- **(a) f is differentiable at x = 0 and x = π**
- **(b) f is not differentiable at x = 0 and x = π**
- **(c) f is differentiable at x = 0 but not differentiable at x = π**
- **(d) f is not differentiable at x = 0 but differentiable at x = π**

Solution: (C)

Now, $f(x) = |x| \sin x + |x - \pi| \cos x$

$f(x) = -x\sin x - (x - \pi)\cos x$ for $x \le 0$

$f(x) = x\sin x - (x - \pi)\cos x$ for $0 < x \le \pi$

$f(x) = x\sin x + (x - \pi)\cos x$ for $x > \pi$

Now, $\lim(f(x) - f(0))/(x - 0)$ as $x \to 0-$

$= \lim\{-x\sin x - (x - \pi)\cos x - \pi\}/x$ as $x \to 0-$

$= \lim\{-\sin x - x\cos x - \cos x + (x - \pi)\sin x\}/1$ as $x \to 0- = -1$.

(Applying L'Hospital rule)

Now, $\lim(f(x) - f(0))/(x - 0)$ as $x \to 0+$

$= \lim\{x\sin x - (x - \pi)\cos x - \pi\}/x$ as $x \to 0+$

$= \lim\{\sin x + x\cos x - \cos x + (x - \pi)\sin x\}/1$ as $x \to 0+ = -1$.

(Applying L'Hospital rule)

So, we have f is differentiable at x = 0 as both the limit are same.

Now, $\lim(f(x) - f(\pi))/(x - \pi)$ as $x \to \pi-$

$= \lim\{x\sin x - (x - \pi)\cos x - 0\}/(x - \pi)$ as $x \to \pi-$

$= \lim\{\sin x + x\cos x - \cos x + (x - \pi)\sin x\}/1$ as $x \to \pi- = \pi\cos\pi - \cos\pi$

$= -\pi + 1$

(Applying L'Hospital rule)

Now, $\lim(f(x) - f(\pi))/(x - \pi)$ as $x \to \pi+$

$= \lim\{x\sin x + (x - \pi)\cos x - 0\}/(x - \pi)$ as $x \to \pi+$

$= \lim\{\sin x + x\cos x + \cos x - (x - \pi)\sin x\}/1$ as $x \to \pi+$

$= \pi\cos\pi + \cos\pi$

$= -\pi - 1$

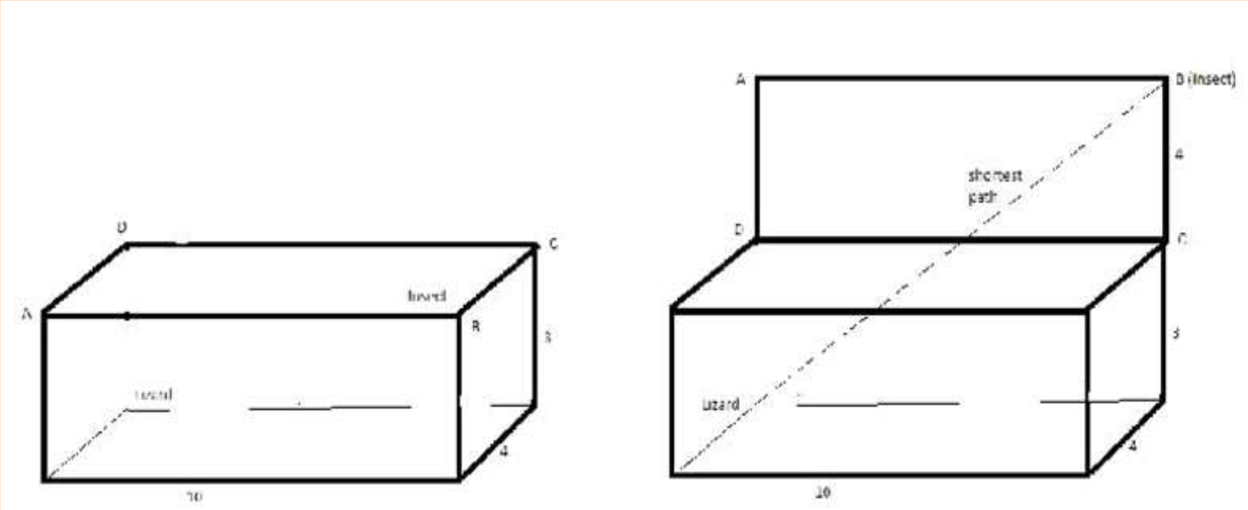
(Applying L'Hospital rule)

As both the limits are different so we have, f(x) is not differentiable at $x = \pi$.

8. Consider a rectangular cardboard box of eight 3, breadth 4 and length 10 units. There is a lizard in one corner A of the box and an insect in the corner B which is farthest from A. The length of the shortest path between the lizard and the insect along the surface of the box is

(a) $\sqrt{(5^2 + 10^2)}$
(b) $\sqrt{(7^2 + 10^2)}$
(c) $4 + \sqrt{(3^2 + 10^2)}$
(d) $3 + \sqrt{(4^2 + 10^2)}$

Solution: (B)



Clearly the shortest path is $\sqrt{(7^2 + 10^2)}$.

9. Recall that, for any non-zero complex number w which does not lie on the negative real axis, arg(w) denotes the unique real number Θ in $(-\pi, \pi)$ such that $w = |w|(\cos\Theta + i\sin\Theta)$. Let z be any complex number such that its real and imaginary parts are both non-zero. Further, suppose that z satisfies the relations arg(z) > arg(z + 1) and arg(z) > arg(z+ i). Then cos(arg(z)) can take

- **(a) Any value in the set $(-1/2, 0)\cup(0, ½)$ but none from outside**
- **(b) Any value in the interval (-1, 0) but none from outside**
- **(c) Any value in the interval (0, 1) but none from outside**
- **(d) Any value in the set $(-1, 0)\cup(0, 1)$ but none from outside.**

Solution: (B)

Now, $z + 1 = \cos\Theta + i\sin\Theta + 1 = (1 + \cos\Theta) + i\sin\Theta$

$\arg(z+1)=\tan^{-1}\{(\sin\Theta/(1+ \cos\Theta)\} = \tan^{-1}[\{2\sin(\Theta/2)\cos(\Theta/2)\}/2\cos^2(\Theta/2)]$

$= \tan^{-1}\{\tan(\Theta/2)\} = \Theta/2$

Now, $\arg(z) > \arg(z + 1)$

⇨ $\Theta > \Theta/2$

⇨ Θ is positive.

Now, $z + i = \cos\Theta + i\sin\Theta + i = \cos\Theta + i(1 + \sin\Theta)$

Now, $\arg(z+i) = \tan^{-1}\{(1 + \sin\Theta)/\cos\Theta\}$

$= \tan^{-1}\{(\cos\Theta/2 + \sin\Theta/2)^2/(\cos\Theta/2 + \sin\Theta/2)(\cos\Theta/2 - \sin\Theta/2)\}$

$= \tan^{-1}\{(\cos \Theta/2 + \sin\Theta/2)/(\cos\Theta/2 - \sin\Theta/2)\}$

$= \tan^{-1}\{(1 + \tan\Theta/2)/(1 - \tan\Theta/2)\}$

$= \tan^{-1}\tan(\pi/4 + \Theta/2)$

$= \pi/4 + \Theta/2.$

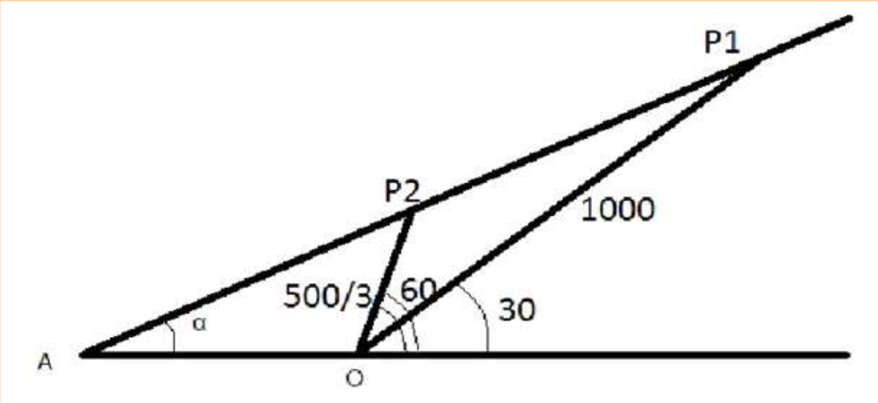
Now, arg(z) > arz(z + i)

⇨ $\Theta > \pi/4 + \Theta/2$

⇨ $\Theta > \pi/2$

For $\Theta > \pi/2$ $\cos\Theta$ is negative and can take any value in the interval (-1, 0).

10. An aeroplane P is moving in the air along a straight line path which passes through the points P_1 and P_2, and makes an angle α with the ground. Let O be the position of an observer as shown in the figure below. When the plane is at the position P_1 its angle of elevation is 30° and when it is at P_2its angle of elevation is 60° from the position of the observer. Moreover, the distances of the observer from the points P_1 and P_2 respectively are 100 metres and 500/3 metres.



Then α is equal to

(a) $\tan^{-1}\{(2 - \sqrt{3})/(2\sqrt{3} - 1)\}$
(b) $\tan^{-1}\{(2\sqrt{3} - 3)/(4 - 2\sqrt{3})\}$
(c) $\tan^{-1}\{(2\sqrt{3} - 2)/(5 - \sqrt{3})\}$
(d) $\tan^{-1}\{(6 - \sqrt{3})/(6\sqrt{3} - 1)$

Solution: (D)

Angle $AOP_2 = 120°$

⇨ Angle $OP_2P_1 = 120° + α$

⇨ Angle $OP_1P_2 = 180° - (120° + α + 30°) = 30° - α$ (As angle P_1OP_2 $= 60° - 30° = 30°$)

Now, in triangle OP_1P_2, we have, $POP_1/\sin(OP_2P_1) = OP_2/\sin(OP_1P_2)$

⇨ $1000/\sin(120° + α) = 500/3\sin(30° - α)$

⇨ $2×3\sin(30° - α) = \sin(120° + α)$

⇨ $6(\sin 30° \cos α – \cos 30° \sin α) = \sin 120° \cos α + \cos 120° \sin α$

⇨ $3\cos α - 3\sqrt{3}\sin α = (\sqrt{3}/2)\cos α – (1/2)\sin α$

⇨ $(6 - \sqrt{3})\cos α = (6\sqrt{3} – 1)\sin α$

⇨ $\tan α = (6 - \sqrt{3})/(6\sqrt{3} – 1)$

⇨ $α = \tan^{-1}\{(6 - \sqrt{3})/(6\sqrt{3} – 1)\}$

11. The sum of all even positive divisors of 1000 is

(a) 2170
(b) 2184
(c) 2325
(d) 2340

Solution: (B)

Now, $1000 = 2^3 5^3$

Sum of all the divisors of 1000 (including 1 and 1000)

$= \{(2^4 - 1)/(2 - 1)\}\times\{(5^4 - 1)/(5 - 1)\} = 15\times156.$

Sum of all the divisors of 5^3 (including 1 and 125) $= (5^4 - 1)/(5 - 1) = 156.$

So, sum of all even positive divisors of $1000 = 15\times156 - 156$

$= 156(15 - 1) = 156\times14 = 2184.$

12. The equation $x^2 + (b/a)x + (c/a) = 0$ has two real roots α and β. If $a > 0$, then the area under the curve $f(x) = x^2 + (b/a)x + (c/a)$ between α and β is

(a) $(b^2 - 4ac)/2a$
(b) $(b^2 - 4ac)^{3/2}/6a^3$
(c) $-(b^2 - 4ac)^{3/2}/6a^3$
(d) $-(b^2 - 4ac)/2a$

Solution: (C)

Now, α, β are roots of the equation $x^2 + (b/a)x + (c/a)$

⇨ $\alpha + \beta = -(b/a)$ and $\alpha\beta = c/a$

⇨ $\beta - \alpha = \sqrt{\{(\beta + \alpha)^2 - 4\alpha\beta\}} = \sqrt{\{(b^2/a^2) - 4c/a\}} = \sqrt{(b^2 - 4ac)}/a$

Area under curve f(x) between α and β = $\int f(x)dx$ (integration running from α to β)

$= \int\{x^2 + (b/a)x + (c/a)\}dx$ (integration running from α to β)

$= [\{(x^3/3) + (b/a)(x^2/2) + (c/a)x]$ (from α to β)

$= \{(\beta^3/3) + (b/a)(\beta^2/2) + (c/a)\beta\} - \{(\alpha^3/3) + (b/a)(\alpha^2/2) + (c/a)\alpha\}$

$= (\beta^3 - \alpha^3)/3 + (b/a)(\beta^2 - \alpha^2)/2 + (c/a)(\beta - \alpha)$

$= (\beta - \alpha)(\beta^2 + \alpha\beta + \alpha^2)/3 + (b/a)(\beta - \alpha)(\beta + \alpha)/2 + (c/a)(\beta - \alpha)$

$= (\beta - \alpha)[\{(\beta + \alpha)^2 - \alpha\beta\}/3 + (b/a)(\beta + \alpha)/2 + (c/a)]$

$= \{\surd(b^2 - 4ac)/a\}[\{(b^2/a^2) -(c/a)\}/3 - (b/a)(b/a)/2 + (c/a)]$

$= \{\surd(b^2 - 4ac)/a\}\{-(b^2/a^2)/6 + (2/3)(c/a)\}$

$= \{\surd(b^2 - 4ac)/a\}(b^2 - 4ac)/6a$

$= - (b^2 - 4ac)^{3/2}/6a^2$.

13. The minimum value of $x_1^2 + x_2^2 + x_3^2 + x_4^2$ subject to $x_1 + x_2 + x_3 + x_4 = a$ and $x_1 - x_2 + x_3 - x_4 = b$ is

(a) $(a^2 + b^2)/4$
(b) $(a^2 + b^2)/2$
(c) $(a + b)^2/4$
(d) $(a + b)^2/2$

Solution: (A)

Now, $x_1 + x_3 = (a + b)/2$ and $x_2 + x_4 = (a - b)/2$

Now, $(x_1^2 + x_3^2)/2 \geq \{(x_1 + x_3)/2\}^2$

⇨ $(x_1^2 + x_3^2) \geq (a + b)^2/8$

Similarly, $(x_2^2 + x_4^2) \geq (a - b)^2/8$

Adding the above two inequalities we get, $x_1^2 + x_2^2 + x_3^2 + x_4^2 \geq \{(a + b)^2 + (a - b)^2\}/8 = 2(a^2 + b^2)/8 = (a^2 + b^2)/4$

14. The value of $\lim[\{\Sigma({}^{2n}C_{2r}\times 3^r)\}/\{\Sigma({}^{2n}C_{2r+1}\times 3^r)\}]$ (numerator summation running from r = 0 to r = n, denominator summation is running from r = 0 to r n-1) as n -> ∞ is

(a) 0
(b) 1
(c) $\surd 3$
(d) $(\surd 3 - 1)/(\surd 3 + 1)$

Solution: (C)

$(1 + x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1x + {}^{2n}C_2x^2 + {}^{2n}C_3x^3 + \ldots.. + {}^{2n}C_{2n-1}x^{2n-1} + {}^{2n}C_{2n}x^{2n}$
$(1 - x)^{2n} = {}^{2n}C_0 - {}^{2n}C_1x + {}^{2n}C_2x^2 - {}^{2n}C_3x^3 + \ldots.. - {}^{2n}C_{2n-1}x^{2n-1} + {}^{2n}C_{2n}x^{2n}$

Adding we get, $(1 + x)^{2n} + (1 - x)^{2n} = 2[{}^{2n}C_0 + {}^{2n}C_2*x^2 + {}^{2n}C_4*x^4 + ... + {}^{2n}C_{2n}*x^{2n}$

Subtracting we get, $(1 + x)^{2n} - (1 - x)^{2n} = 2x[{}^{2n}C_1 + {}^{2n}C_3*x^2 + {}^{2n}C_5*x^4 + ... + {}^{2n}C_{2n-1}*x^{2n-2})$

Now, putting $x = \sqrt{3}$ we get, $\{(1 + \sqrt{3})^{2n} + (1 - \sqrt{3})^{2n}\}/2 = {}^{2n}C_0 + {}^{2n}C_2*3 + {}^{2n}C_4*3^2 +....+ {}^{2n}C_{2n}*3^n$ = Numerator.

⇨ Numerator = $\{(1 + \sqrt{3})^{2n} + (1 - \sqrt{3})^{2n}\}/2$

Similarly putting $x = \sqrt{3}$ in the 2nd equation we get,

Denominator = $\{(1 + \sqrt{3})^{2n} - (1 - \sqrt{3})^{2n}\}/2\sqrt{3}$

Therefore, the given expression = $\frac{\lim[\{(1+\sqrt{3})2n + (1-\sqrt{3})2n\}/2]}{\lim[\{(1+\sqrt{3})2n - (1-\sqrt{3})2n\}/2\sqrt{3}]}$ as n -> ∞

= $\sqrt{3} \lim \frac{[1 + \{(\sqrt{3}-1)/(\sqrt{3}+1)\}2n]}{[1 - \{(\sqrt{3}-1)/(\sqrt{3}+1)\}2n]}$ as n - > ∞

= $\sqrt{3}(1 - 0)/(1 - 0)$

= $\sqrt{3}$

15. For any real number x l $\tan^{-1}(x)$ denote the unique real number θ in (-π/2, π/2) such that tanθ = x. Then $\lim[\Sigma\tan^{-1}\{1/(1 + m + m^2)\}]$ where m runs from 1 to n as n -> ∞

(a) Is equal to π/2

(b) Is equal to π/4

(c) Does not exist

(d) None of the above.

Solution: (B)

Now, $\tan^{-1}\{1/(1 + m + m^2)\} = \tan^{-1}[\{(m + 1) - m\}/\{1 + m(m+1)\}]$
$= \tan^{-1}(m + 1) - \tan^{-1}(m)$

$\tan^{-1}\{1/(1 + 1 + 1^2)\} = \tan^{-1}(2) - \tan^{-1}(1)$

$\tan^{-1}\{1/(1 + 2 + 2^2)\} = \tan^{-1}(3) - \tan^{-1}(2)$

$\tan^{-1}\{1/(1 + 3 + 3^2)\} = \tan^{-1}(4) - \tan^{-1}(3)$

.....

....

$\tan^{-1}\{1/(1 + n + n^2)\} = \tan^{-1}(n + 1) - \tan^{-1}(n)$

$\sum\tan^{-1}\{1/(1 + m + m^2)\} = \tan^{-1}(n + 1) - \tan^{-1}(1)$ where m runs from 1 to n.

The given expression becomes,
$\lim\{\tan^{-1}(n + 1) - \tan^{-1}(1)\}$ as $n \to \infty$

$= \tan^{-1}(\infty) - \tan^{-1}(1)$

$= (\pi/2) - (\pi/4)$

$= \pi/4$

16. Let n be an integer. The number of primes which divide both $n^2 - 1$ and $(n + 1)^2 - 1$ is

(a) At most one.
(b) Exactly one.
(c) Exactly two.
(d) None of the above.

Solution: (A)

Now, $n^2 - 1 = (n + 1)(n - 1)$ and $(n + 1)^2 - 1 = n(n + 2)$

Now, n + 1 is relatively prime with both n and n + 2 as they are consecutive integers.

So, only possible case is a prime may divide n – 1 and n + 2 as n – 1 and n are also relatively prime.

We see that6 if 3 divides n – 1 then 3 divides n + 2.

We also see that if 3 divides n or n +1 then 3 doesn't divide the rest three.

So, at most one prime (3) can divide both $n^2 - 1$ and $(n + 1)^2 -1$.

17. The value of $\lim[\sum\{6n/(9n^2 - r^2)\}]$ as $n \to \infty$ is

(a) 0
(b) log(3/2)
(c) log(2/3)
(d) log(2)

Solution: (D)

18. A person X standing at a point P on a flat plane starts walking. At each step, he walks exactly 1 foot in one of the directions North, South, East or West. Suppose that after 6 steps X comes to the original position P. Then the number of distinct paths that X can take is

- **(a) 196**
- **(b) 256**
- **(c) 344**
- **(d) 400**

Solution: (D)

Total number of cases = 100×4 = 400.

19. Consider the branch of the rectangular hyperbola xy = 1 in the first quadrant. Let P be a fixed point on this curve. The locus of the mid-point of the line segment joining P and an arbitrary point Q on the curve is part of

- **(a) A hyperbola**
- **(b) A parabola**
- **(c) An ellipse**
- **(d) None of the above.**

Solution: (A)

Let co-ordinate of point P is (a, b)

Let co-ordinate of point Q is (h, k)

Co-ordinate of mid-point of P and Q is {(h + a)/2, (k + b)/2}

Now, P and Q lies on the hyperbola xy = 1

So, we have, ab = 1 and hk = 1.

Putting (h + a)/2 = x and (k + b)/2 = y i.e. h = 2x − a and k = 2y − b we get,

(2x − a)(2y − b) = 1

⇨ 4xy – 2ay – 2bx + ab = 1

⇨ 4xy – 2ay – 2bx = 0 (As ab = 1)

⇨ xy – bx/2 – ay/2 = 0

Which is the equation of a hyperbola.

20. The digit at the unit place of

$(1! - 2! + 3! - ..... + 25!)^{(1! - 2! + 3! - .... + 25!)}$ is

(a) 0
(b) 1
(c) 5
(d) 9

Solution: (B)

Now, $1! - 2! + 3! - 4! + 5! = 1 - 2 + 6 - 24 + 120 = 101$

Now, last digit of $- 6! + 7! - .... + 25!$ Is zero.

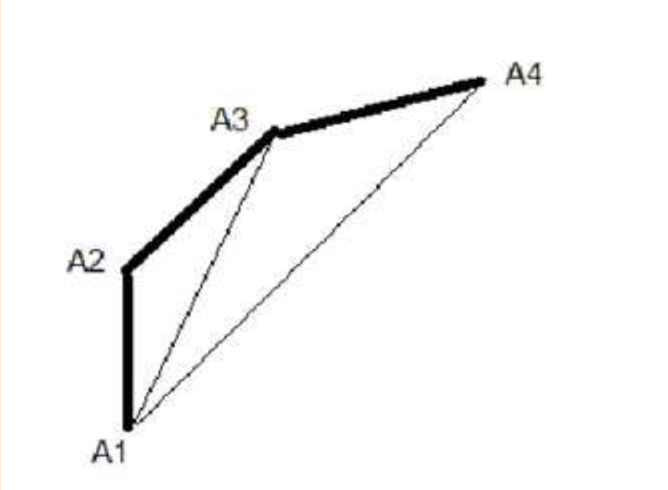
⇨ $1! - 2! + 3! - .... + 25!$ Has the last digit as 1.

⇨ The last digit of $(1! - 2! + 3! - .... + 25!)^{(1! - 2! + 3! - ..... + 25!)}$ is 1.

21. Let $A_1, A_2, ..., A_n$ be the vertices of a regular polygon and A_1A_2, $A_2A_3, ..., A_{n-1}A_n$, A_nA_1 be its n sides. If $(1/A_1A_2) - (1/A_1A_4) = (1/A_1A_3)$, then the value of n is

(a) 5
(b) 6
(c) 7
(d) 8

Solution: (C)



Let, the side of the regular polygon is a.

Now, angle $A_1A_2A_3 = (n - 2)\pi/n$

In triangle $A_1A_2A_3$, $A_1A_2 = A_2A_3$

⇨ Angle $A_2A_1A_3$ = Angle $A_2A_3A_1$

⇨ Angle $A_2A_3A_1 = (1/2)[\pi - (n - 2)\pi/n] = \pi/n$

In triangle $A_1A_2A_3$, $A_1A_3/\sin\{(n - 2)\pi/n\} = a/\sin(\pi/n)$

⇨ $A_1A_3 = a\sin(\pi - 2\pi/n)/\sin(\pi/n) = a\sin(2\pi/n)/\sin(\pi/n) = 2a\cos(\pi/n)$

Let, Angle $A_3A_1A_4 = A$

Now, angle $A_1A_3A_4 = (n - 2)\pi/n - \pi/n = \pi - 3\pi/n$

Angle $A_1A_4A_3 = \pi - A - \pi + 3\pi/n = 3\pi/n - A$

Now, in triangle $A_1A_3A_4$ we have,

$A_1A_4/\sin(\pi - 3\pi/n) = a/\sin A = A_1A_3/\sin(3\pi/n - A)$

From, $a/\sin A = A_1A_3/\sin(3\pi/n - A)$ we get,

$\cot A = \{2\cos(\pi/n) + \cos(3\pi/n)\}/\sin(3\pi/n)$ (using $\sin(3\pi/n - A)$

$= \sin(3\pi/n)\cos A - \cos(3\pi/n)\sin A$)

⇨ $\sin A = \sin(3\pi/n)/\sqrt{[1 + 4\cos(\pi/n)\cos(3\pi/n) + 4\cos^2(\pi/n)]}$

Now, from the relation, $A_1A_4/\sin(\pi - 3\pi/n) = a/\sin A$ we get,

$A_1A_4/\sin(3\pi/n) = a\sqrt{[1 + 4\cos(\pi/n)\cos(3\pi/n) + 4\cos^2(\pi/n)]}/\sin(3\pi/n)$

⇨ $A_1A_4 = a\sqrt{[1 + 4\cos(\pi/n)\{4\cos^3(\pi/n) - 3\cos(\pi/n)\} + 4\cos^2(\pi/n)]}$

⇨ $A_1A_4 = a(4\cos^2(\pi/n) - 1)$

Now, given $1/A_1A_2 - 1/A_1A_3 = 1/A_1A_4$

⇨ $(1/a) - 1/2a\cos(\pi/n) = 1/a(4\cos^2(\pi/n) - 1)$

⇨ $1 - 1/2\cos(\pi/n) = 1/(4\cos^2(\pi/n) - 1)$

This equation is satisfied by n = 7.

22. Suppose that α and β are two distinct numbers in the interval (0, π). If sinα + sinβ = √3(cosα – cosβ) then the value of sin3α + sin3β is

(a) 0

(b) 2sin{3(α + β)/2}

(c) 2cos{3(α – β)/2}

(d) Cos{3(α – β)/2}

Solution: (A)

Now, sinα + sinβ = √3(cosα – cosβ)

⇨ 2sin{(α + β)/2}cos{(α – β)/2} = √3*2sin{(α + β)/2}sin{(β – α)/2}

⇨ cos{(α – β)/2} = -√3sin{(α – β)/2}

⇨ tan{(α – β)/2} = -1/√3

⇨ tan{(α – β)/2} = tan(π – π/6)

⇨ (α – β)/2 = π – π/6

Now, sin3α + sin3β

= 2sin{3(α + β)/2}cos{3(α – β)/2}

= 2sin{3(α + β)/2}cos{3(π – π/6)}

= 2sin{3(α + β)/2}cos(3π – π/2)

= 2sin{3(α + β)/2}{-cos(π/2)}

= 0 (As cos(π/2) = 0)

23. Consider the function h(x) = x^2 – 2x + 2 + 4/(x^2 – 2x + 2), x Є R. Then h(x) – 5 = 0 has

(a) No solution

(b) Only one solution

(c) Exactly two solutions

(d) Exactly three solutions.

Solution: (D)

Now, h(x) – 5 = 0

⇨ x^2 – 2x + 2 + 4/(x^2 – 2x + 2) – 5 = 0

⇨ $(x^2 - 2x + 2)^2 - 5(x^2 - 2x + 2) + 4 = 0$ (As $x^2 - 2x + 2 \neq 0$ because roots are imaginary)

⇨ $(x^2 - 2x + 2)^2 - 4(x^2 - 2x + 2) - (x^2 - 2x + 2) + 4 = 0$

⇨ $(x^2 - 2x + 2)(x^2 - 2x + 2 - 4) - (x^2 - 2x + 2 - 4) = 0$

⇨ $(x^2 - 2x + 2)(x^2 - 2x - 2) - (x^2 - 2x - 2) = 0$

⇨ $(x^2 - 2x - 2)(x^2 - 2x + 2 - 1) = 0$

⇨ $(x^2 - 2x - 2)(x^2 - 2x + 1) = 0$

⇨ $(x^2 - 2x - 2)(x - 1)^2 = 0$

So, $x = 1$ is a solution and $x^2 - 2x - 2 = 0$

⇨ $x = \{2 \pm \sqrt{(4 + 8)}\}/2$

⇨ $x = 1 \pm \sqrt{3}$

So, the equation has 3 solutions viz. $x = 1, 1 + \sqrt{3}$ and $1 - \sqrt{3}$

24. Consider the quadratic equation $x^2 + bx + c = 0$. The number of pairs (b, c) which the equation has solutions of the form $\cos\alpha$ and $\sin\alpha$ for some α is

(a) 0

(b) 1

(c) 2

(d) Infinite.

Solution: (D)

25. Let $\theta_1 = 2\pi/3$, $\theta_2 = 4\pi/7$, $\theta_3 = 7\pi/12$. Then

(a) $\sin\theta_1^{\sin\theta_1} < \sin\theta_2^{\sin\theta_2} < \sin\theta_3^{\sin\theta_3}$

(b) $\sin\theta_2^{\sin\theta_2} < \sin\theta_1^{\sin\theta_1} < \sin\theta_3^{\sin\theta_3}$

(c) $\sin\theta_3^{\sin\theta_3} < \sin\theta_1^{\sin\theta_1} < \sin\theta_2^{\sin\theta_2}$

(d) $\sin\theta_1^{\sin\theta_1} < \sin\theta_3^{\sin\theta_3} < \sin\theta_2^{\sin\theta_2}$

Solution: (D)

26. Consider the following two curves on the interval (0, 1) :
$C_1 : y = 1 - x^4$ and $C_2 : y = \sqrt{(1 - x^2)}$. Then on (0, 1)
(a) C_1 lies above C_2
(b) C_2 lies above C_1
(c) C_1 and C_2 intersect at exactly one point
(d) None of the above.

Solution: (C)

If we look for the point of intersection of C_1 and C_2 we get,

$$1 - x^4 = \sqrt{(1 - x^2)}$$

⇨ $(1 + x^2)(1 - x^2) = \sqrt{(1 - x^2)}$
⇨ $(1 + x^2)\sqrt{(1 - x^2)} = 0$
⇨ $1 - x^2 = 0$ (As $1 + x^2$ cannot be equal to 0)
⇨ $x = 1$ (As -1 doesn't fall in the interval (0, 1))
⇨ C_1 and C_2 intersect at exactly one point.

27. Let f be a real valued function on R such that f is twice differentiable. Suppose that f' vanishes only at 0 and f'' is everywhere negative. Define a function h by $h(x) = (x - a)^2 - f(x)$, where $a > 0$. Then

(a) h has a local minima in (0, a)
(b) h has a local maxima in (0, a)
(c) h is monotonically increasing in (0, a)
(d) h is monotonically decreasing in (0, a).

Solution: (A)

Now, $h(x) = (x - a)^2 - f(x)$
⇨ $h'(x) = 2(x - a) - f'(x)$
⇨ $h''(x) = 2 - f''(x)$

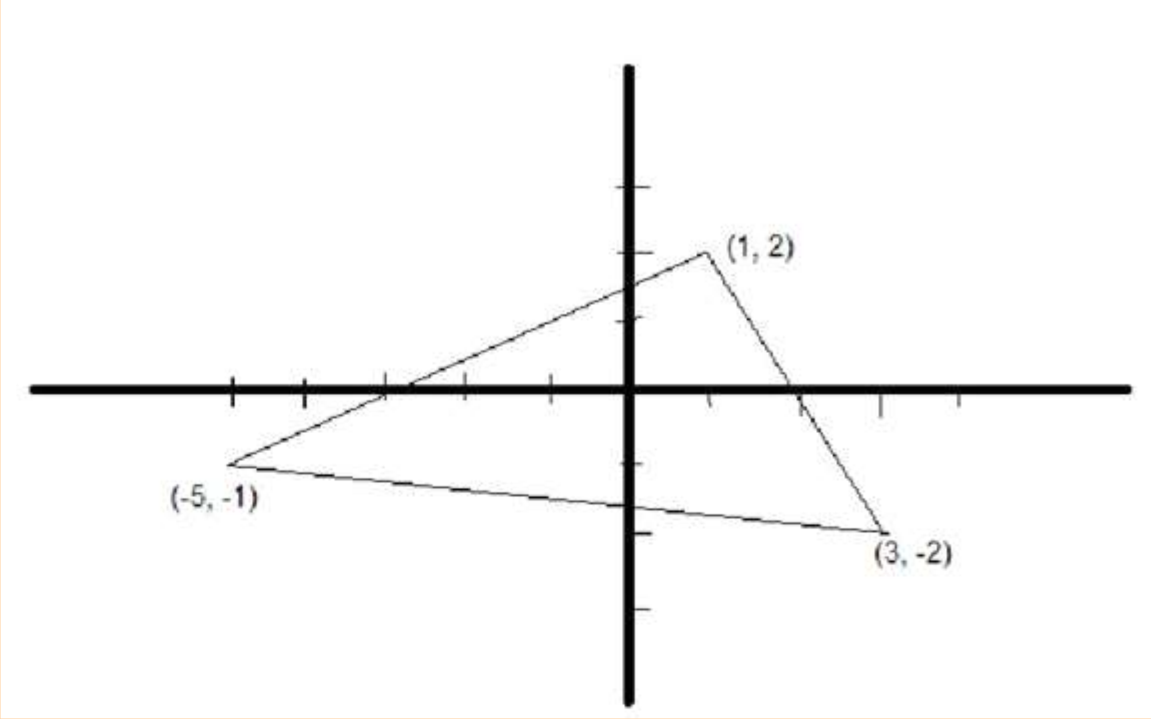
Now, $f''(x)$ is negative everywhere.
⇨ $h''(x) > 0$ everywhere.
⇨ h(x) has a local minima.

Now, $h'(x) = 0$ gives, $x = a + f'(x)/2$

28. Consider the triangle with vertices (1, 2), (-5, -1) and (3, -2). Let Δ denote the region enclosed by the above triangle. Consider the function f : Δ -> R defined by f(x, y) = |10x – 3y|. Then the range of f is the interval

(a) [0, 36]
(b) [0, 47]
(c) [4, 47]
(d) [36, 47]

Solution: (B)



Clearly (0, 0) is inside the triangle.

Hence, f(0, 0) = 0 in R. So, 0 is in the range of f.

Option (c) and (d) cannot be true.

Now, (-5, -1) is in Δ

f(-5, -1) = |10×(-5) – 3×(-1)| = 47

So, 47 is in range of f.

Option (a) cannot be true.

29. For every positive integer n, let <n> denote the integer closest to √n. Let A_k = {n > 0 : <n> = k}. The number of elements in A_{49} is

- **(a) 97**
- **(b) 98**
- **(c) 99**
- **(d) 100**

Solution: (B)

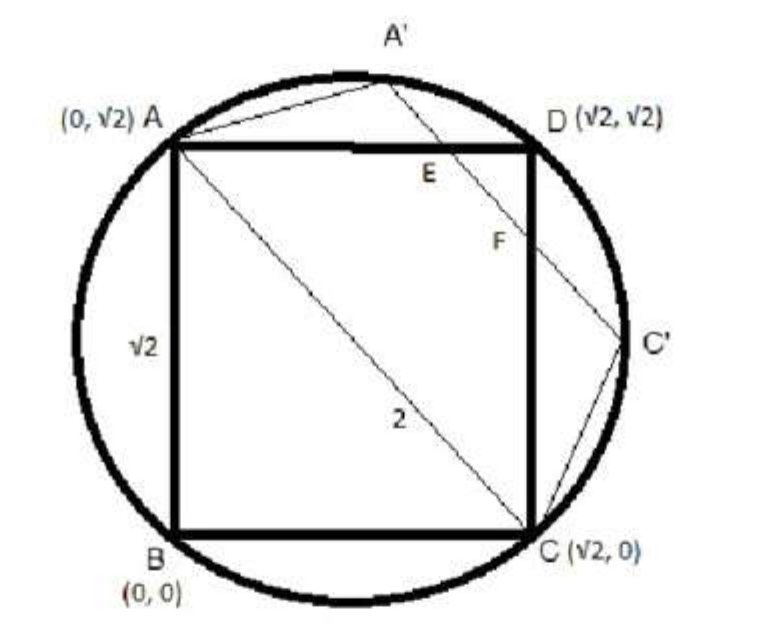
$49.5^2 = 2450.25$ and $48.5^2 = 2352.25$

Number of elements in $A_{49} = 2450 - 2353 + 1 = 98$.

30. Consider a square ABCD inscribed in a circle of radius 1. Let A′ and C′ be two points on the (smaller) arcs AD and CD respectively, such that A′ABCC′ is a pentagon in which AA′ = CC′. If P denotes the area of the pentagon A′ABCC′ then

- **(a) P cannot be equal to 2.**
- **(b) P lies in the interval (1, 2].**
- **(c) P is greater than or equal to 2.**
- **(d) None of the above.**

Solution: (D)



Clearly from the picture none of option (a), (b), (c) always correct because A′ and C′ can move from A to D and C to D respectively. When A′ and C′ very close to D i.e. A′D = C′D -> 0 then area of the trapezium A′ACC′ is near to 1 and when A′ and C′ are close to A and C respectively i.e. AA′ and CC′ both -> 0 then area of trapezium A′ACC′ is near to 0.

Also, clearly, area of triangle AA′E = area of triangle CC′F

If area of triangle AA′E > (area of triangle DEF)/2 then area of the pentagon A′ABCC′ greater than 2 as area of triangle ADC = 1

When A′ is at position such that E is the mid-point of AD.

Then area of triangle DEF = $(1/2)\times(1/\sqrt{2})\times(1/\sqrt{2}) = ¼$

Then, AE = $1/\sqrt{2}$ and perpendicular from A′ to AE = $1 - 1/\sqrt{2}$

So, area of triangle A′AE = $(1/2)\times(1/\sqrt{2})\times(1 - 1/\sqrt{2}) = (\sqrt{2} - 1)/4$

⇨ Area of triangle A′AE + area of triangle C′CF = $(\sqrt{2} - 1)/2$

Now, $(\sqrt{2} - 1)/2 - ¼ = (2\sqrt{2} - 3)/4 > 0$

So, Area of the pentagon can be > 2 and also from above discussion we have area of pentagon can be < 2 and so it can be equal to 2.