B. Stat. (Hons.) Admission test 2011 Solved Paper

Group A: ***Each of the following questions has exactly one correct option and you have to identify it.***

1. **The limit lim [{1 – cos($\sin^2$αx)}/x] as x -> 0**
   - **(a) Equals 1**
   - **(b) Equals α**
   - **(c) Equals 0**
   - **(d) Does not exist**

Solution: (C)
Now, lim [{1 – cos($\sin^2$αx)}/x] as x -> 0
= lim [{sin($\sin^2$αx)×2sin(αx)cos(αx)×α}/1] as x -> 0 (Applying L'Hospital)
= sin($\sin^2$0)2sin(0)cos(0)×α/1 = 0

2. **The set of all x for which the function f(x) = $\log_{1/2}(x^2 - 2x - 3)$ is defined and monotone increasing is**
   - **(a) (-∞, 1)**
   - **(b) (-∞, -1)**
   - **(c) (1, ∞)**
   - **(d) (3, ∞)**

Solution: (B)

Now, $x^2 - 2x - 3 > 0$

⇨ $(x - 1)^2 > 4$

⇨ x – 1 > 2 or x – 1 < -2

⇨ x > 3 or x < -1

f(x) = {log(x^2 – 2x – 3)}/log(1/2) = -{log(x^2 – 2x – 3)}/log(2)

f′(x) = -(1/log2)(2x – 2)/(x^2 – 2x – 3)

f′(x) = -(1/log2)2(x – 1)/{(x + 1)(x – 3)}

To be monotonically increasing f′(x) must be > 0

Let us take x = 4, we get f′(4) < 0

⇨ Options (c) and (d) cannot be true.

Now, let us put x = 0, we get, f’(0) < 0

⇨ Option (a) cannot be true.

3. Let a line with slope of 60° be drawn through the focus F of the parabola $y^2 = 8(x + 2)$. If the two points of intersection of the line with the parabola are A and B and the perpendicular bisector of the chord AB intersects the x-axis at the point P, then the length of the segment PF is

(a) 16/3
(b) 8/3
(c) 16√3/3
(d) 8√3

Solution: (A)

The centre of the parabola is at (-2, 0).

Comparing the equation of parabola with $y^2 = 4a(x - \alpha)$ we get, a = 2.

Hence, coordinate of focus F is (0, 0)

The equation of the line passing through focus and having slope tan60° is,
$y = \sqrt{3}x$

Now, solving the equation of this line with parabola will give A and B.

Now, $(\sqrt{3}x)^2 = 8(x + 2)$

⇨ x = 4, -4/3

Hence coordinate of A and B are $(4, 4\sqrt{3})$ and $(-4/3, -4\sqrt{3}/3)$

Midpoint of AB = $(4/3, 4\sqrt{3}/3)$ and

slope of AB = $\{4\sqrt{3} - (-4\sqrt{3}/3)\}/\{4/3-(-4/3)\} = \sqrt{3}$

⇨ Slope of perpendicular bisector of AB = $-1/\sqrt{3}$

⇨ Equation of perpendicular bisector of AB is, $y - 4\sqrt{3}/3 = (-1/\sqrt{3})(x - 4/3)$

Putting y = 0 we get x-coordinate of P, x = 16/3.

⇨ Coordinate of P is (16/3, 0)

⇨ Now, distance between F (0, 0) and P (16/3, 0) is 16/3.

4. Suppose z is a complex number with |z| < 1.
Let w = (1 + z)/(1 − z). Which of the following is always true?
[Re(w) is the real part of w and Im(w) is the imaginary part of w]
(a) Re(w) > 0
(b) Im(w) ≥ 0
(c) |w| ≤ 1
(d) |w| ≥ 1

Solution: (A)

Let, $z = re^{i\theta}$

Now, $w = (1 + r\cos\theta + ir\sin\theta)/(1 - r\cos\theta - ir\sin\theta)$

$= (1 + r\cos\theta + ir\sin\theta)(1 - r\cos\theta + ir\sin\theta)/\{(1 - \cos\theta)^2 + \sin^2\theta\}$

$= (1 - r^2\cos^2\theta + 2ir\sin\theta - r^2\sin^2\theta)/(2 - 2\cos\theta)$

$= (1 - r^2)/2(1 - \cos\theta) + ir\sin\theta/(1 - \cos\theta)$

$Re(z) = (1 - r^2)/\{2(1 - \cos\theta)\}$

Here, $r = |z| < 1$ and $\cos\theta < 1$

⇨ $Re(z) > 0$

5. Among all the factors of $4^6 6^7 21^8$, the number of factors which are perfect squares is
(a) 240
(b) 360
(c) 400
(d) 640.

Solution: (C)

Now, $4^6 6^7 21^8 = 2^{19} 3^{15} 7^8$

There are 2, 4,...,18 = 9 even numbers we can select from power of 2.

There are 2, 4,...,14 = 7 even numbers we can select from power of 3.

There are 2, 4, 6, 8 = 4 even numbers we can select from power of 7.

We can select 1 number from 9 numbers in ${}^9C_1$ ways. We can select 1 number from 7 numbers in ${}^7C_1$ ways. We can select 1 number from 4 numbers in ${}^4C_1$ ways.

So, number of factors which contain all prime factors (2, 3, 7) and square are ${}^9C_1 \times {}^7C_1 \times {}^4C_1 = 252$.

Similarly number of factors which contain 2 and 3 prime factors and not 7 and square are ${}^9C_1 \times {}^7C_1 = 63$.

Similarly, number of factors which contain 3 and 7 prime factors and not 2 and square are ${}^7C_1 \times {}^4C_1 = 28$.

Number of factors which contain 2 and 7 prime factors and not 3 and square are ${}^9C_1 \times {}^4C_1 = 36$

Number of factors which contain only factor 2 and not 3 and 7 and square are ${}^9C_1 = 9$

Number of factors which contain only factor 3 and not 2 and 7 and square are ${}^7C_1 = 7$

Number of factors which contain only factor 7 and not 2 and 3 and square are ${}^4C_1 = 4$

And 1.

Therefore total number of factors which are perfect square = 252 + 63 + 28 + 36 + 9 + 7 + 4 + 1 = 400.

6. **Let A be the set {1, 2,….,20}. Fix two disjoint subsets S_1 and S_2 of A, each with exactly three elements. How many 3-element subsets of A are there, which have exactly one element common with S_1 and at least one element common with S_2?**
   (a) 51
   (b) 102
   (c) 135
   (d) 153

Solution: (C)

Two cases are there. One, select one element from S_1, select one element from S_2 and select one element from rest 14 numbers. We can do it in ${}^3C_1 \times {}^3C_1 \times {}^{14}C_1 = 126$ ways.

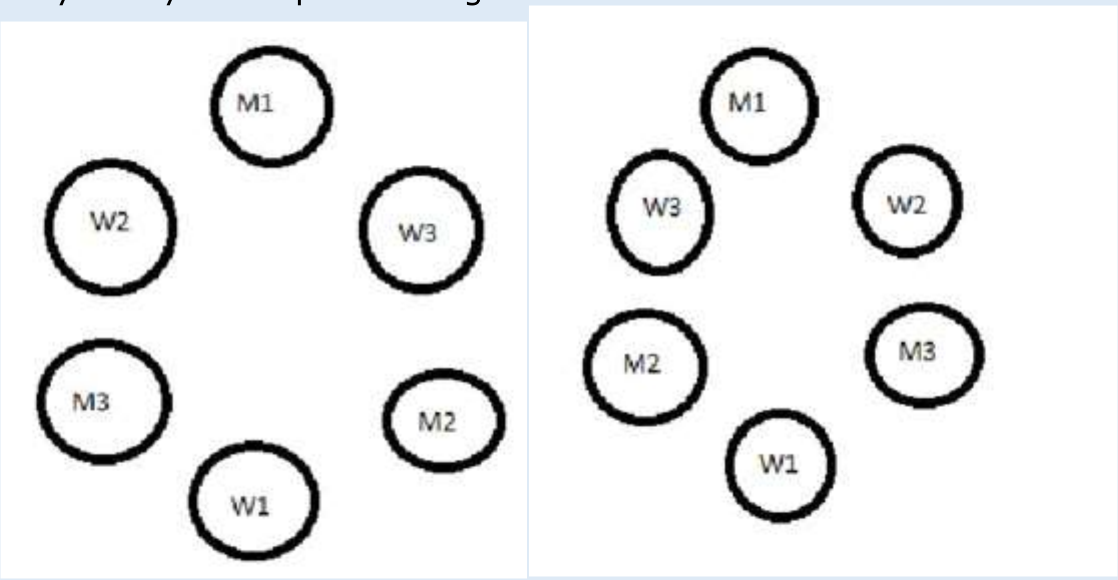
Two, select one element from S_1 and select two elements from S_2. We can do it in ${}^3C_1 \times {}^3C_2 = 9$ ways.

Therefore required number of subsets = 126 + 9 = 135

7. **In how many ways can 3 couples sit around a round table such that men and women alternate and none of the couples sit together?**
   (a) 1
   (b) 2
   (c) 5!/3
   (d) None of these.

Solution: (B)

Only 2 ways as depicted in figure.



8. **The equation $x^3 + y^3 = xy(1 + xy)$ represents**
   (a) Two parabolas intersecting at two points
   (b) Two parabolas touching at one point
   (c) Two non-intersecting hyperbolas
   (d) One parabola passing through the origin.

Solution: (A)

We have, $x^3 + y^3 = xy(1 + xy)$

⇨ $x^3 - x^2y^2 + y^3 - xy = 0$

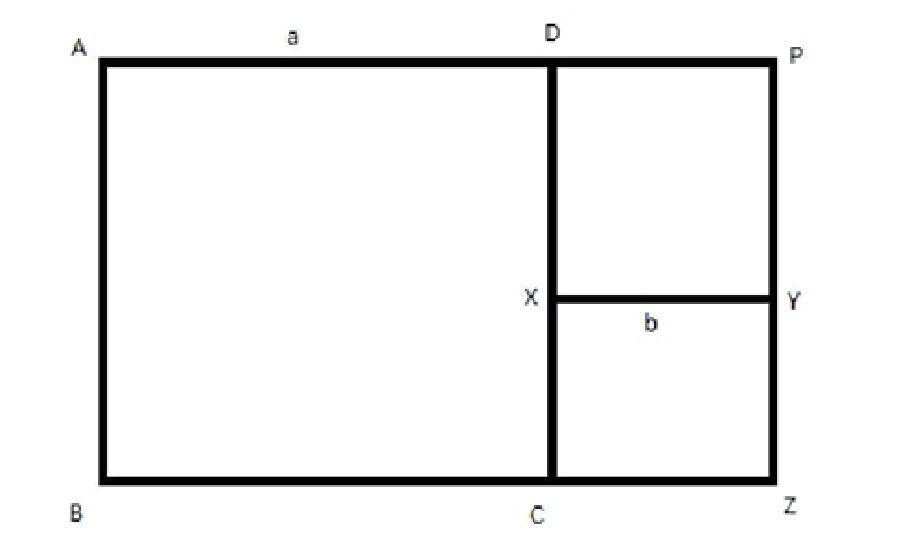
⇨ $-x^2(y^2 - x) + y(y^2 - x) = 0$

⇨ $(y^2 - x)(-x^2 + y) = 0$

⇨ $(y^2 - x)(x^2 - y) = 0$

⇨ This is equation of two parabolas $y^2 = x$ and $x^2 = y$. The two parabolas intersect at (0, 0) and (1, 1).

9. Consider the diagram below where ABZP is a rectangle and ABCD and CXYZ are squares whose areas add up to 1.



The maximum possible area of the rectangle ABZP is
(a) $1 + 1/\sqrt{2}$
(b) $2 - \sqrt{2}$
(c) $1 + \sqrt{2}$
(d) $(1 + \sqrt{2})/2$

Solution: (D)

Area of square ABCD = a^2 and area of square CXYZ = b^2

We have, $a^2 + b^2 = 1$

Now, length of ABZP rectangle = (a + b) and breadth = a

Area of ABZP rectangle = a(a + b) which we have to maximize.

Let, a = cosA, then b = sinA (as $a^2 + b^2 = 1$)

$a(a + b) = \cos A(\cos A + \sin A) = \cos^2 A + \sin A\cos A$

$= (1/2)(2\cos^2 A+2\sin A\cos A)$

$= (1/2)(1 + \cos 2A + \sin 2A)$

$= ½ + (1/\sqrt{2})\{(1/\sqrt{2})\cos 2A + (1/\sqrt{2})\sin 2A\}$

$= ½ + (1/\sqrt{2})(\sin 45°\cos 2A + \cos 45°\sin 2A)$

$= ½ + (1/\sqrt{2})\sin(2A + 45°)$

Now, $\sin(2A + 45°) \leq 1$

⇨ $\frac{1}{2} + (1/\sqrt{2})\sin(2A + 45°) \le \frac{1}{2} + 1/\sqrt{2}$

⇨ Maximum possible area of the rectangle ABZP = $\frac{1}{2} + 1/\sqrt{2} = (1 + \sqrt{2})/2$

10. Let A be the set {1, 2,…,6}. How many functions f from A to A are there such that the range of f has exactly 5 elements?

(a) 240
(b) 720
(c) 1800
(d) 10800

Solution: (D)

Now, there is nothing mentioned for domain of f.

So we will consider the domain of f consists of exactly 6 elements.

Now, 2 numbers will map to 1 number in range set A.

We can choose 2 numbers out of 6 numbers in $^6C_2 = 15$ ways.

Now, take the 2 numbers which are getting mapped into 1 number as unit.

So, there are 5! Cases the 5 numbers (2 unit + 4 distinct) can map into 5 numbers in range set A.

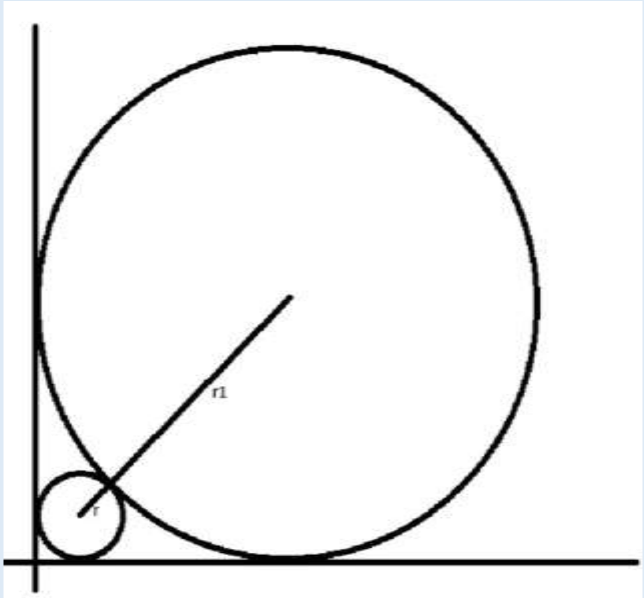
Now, we can choose 5 numbers in range set A from 6 numbers in $^6C_5 = 6$ ways.

So, number of such functions = 15×5!×6 = 10800

11. Let C_1, C_2 and C_3 be three circles lying in the same quadrant, each touching both the axes. Suppose also that C_1 touches C_2 and C_2 touches C_3. If the area of the smallest circle is 1 unit, then area of the largest circle is

(a) $\{(\sqrt{2} + 1)/(\sqrt{2} - 1)\}^4$
(b) $(1 + \sqrt{2})^2$
(c) $(2 + \sqrt{2})^2$
(d) 2^4

Solution: (A)



Let the radius of the smallest circle is r, that of larger is r_1 and that of largest is r_2.

Coordinate of the centre of the smallest circle is (r, r); that of larger circle is (r_1, r_1) and that of largest circle is (r_2, r_2)

We have, $r_1 + r = \sqrt{(r_1 - r)^2 + (r_1 - r)^2}$

⇨ $r_1 + r = \sqrt{2}(r_1 - r)$

⇨ $(r_1 + r)/(r_1 - r) = \sqrt{2}$

⇨ $r_1/r = (\sqrt{2} + 1)/(\sqrt{2} - 1)$

Similarly, $r_2/r_1 = (\sqrt{2} + 1)/(\sqrt{2} - 1)$

Now, multiplying the two equations we get, $r_2/r = \{(\sqrt{2} + 1)/(\sqrt{2} - 1)\}^2$

⇨ $(\pi r_2^2) = (\pi r^2)\{(\sqrt{2} + 1)/(\sqrt{2} - 1)\}^4$

⇨ Area of largest circle = $\{(\sqrt{2} + 1)/(\sqrt{2} - 1)\}^4$ (As $\pi r^2 = 1$)

12. Let [x] denote the largest integer less than or equal to to x. Then $\int[x^k + n]dx$ (integration running from 0 to $n^{1/k}$) equals

(a) **$n^2 + \Sigma i^{1/k}$ summation runs from i = 1 to i = n**

(b) **$2n^{(1+k)/k} - \Sigma i^{1/k}$ summation runs from i=1 to i=n**

(c) **$2n^{(1+k)/k} - \Sigma i^{1/k}$ summation runs from i = 1 to i = n – 1**

(d) **None of these.**

Solution: (B)

Now, $\int[x^k + n]dx$ (integration running from 0 to $n^{1/k}$)

$= \int[x^k + n]dx$ (integration running from 0 to $1^{1/k}$) + $\int[x^k + n]dx$ (integration running from $1^{1/k}$ to $2^{1/k}$)

$= \int[x^k + n]dx$ (integration running from $2^{1/k}$ to $3^{1/k}$) + $\int[x^k + n]dx$ (integration running from $3^{1/k}$ to $4^{1/k}$) + ... + $\int[x^k + n]dx$ (integration running from $(n - 1)^{1/k}$ to $n^{1/k}$)

$= n\int dx$ (integration running from 0 to $1^{1/k}$) + $(n + 1)\int dx$ (integration running from $1^{1/k}$ to $2^{1/k}$) + $(n + 2)\int dx$ (integration running from $2^{1/k}$ to $3^{1/k}$) + $(n + 3)\int dx$ (integration running from $3^{1/k}$ to $4^{1/k}$) + ... + $(2n - 1)\int dx$ (integration running from $(n - 1)^{1/k}$ to $n^{1/k}$)

$= n\times1^{1/k} + (n + 1)(2^{1/k} - 1^{1/k}) + (n + 2)(3^{1/k} - 2^{1/k}) + (n + 3)(4^{1/k} - 3^{1/k}) + ... + (2n - 1)\{n^{1/k} - (n - 1)^{1/k}\}$

$= \{n\times1^{1/k} - (n + 1)\times1^{1/k}\} + \{(n + 1)\times2^{1/k} - (n + 2)\times2^{1/k}\} + \{(n+ 2)\times3^{1/k} - (n + 3)\times3^{1/k}\} + ... + \{(2n - 2)\times(n - 1)^{1/k} - (2n - 1)(n - 1)^{1/k}\} + 2n\times n^{1/k} - n^{1/k}$

$= 2n\times n^{1/k} - 1^{1/k} - 2^{1/k} - 3^{1/k} - ...... - (n - 1)^{1/k} - n^{1/k}$

$= 2n^{(1 + k)/k} - \sum i^{1/k}$ (summation runs from i = 1 o i = n)

13. Consider the function
$f(x)= x(x - 1)e^{2x}$ if $x \le 0$
$f(x) = x(1 - x)e^{-2x}$ if $x > 0$
Then f(x) attains its maximum value at
(a) $1 - 1/\sqrt{2}$
(b) $1 + 1/\sqrt{2}$
(c) $-1/\sqrt{2}$
(d) $1/\sqrt{2}$

Solution: (C)

Clearly, $f(1 - 1/\sqrt{2}) > 0$ and $f(1 + 1/\sqrt{2}) < 0$

⇨ $f(1 - 1/\sqrt{2}) > f(1 + 1/\sqrt{2})$

Option (b) cannot be true.

Now, $f(-1/\sqrt{2}) = (-1/\sqrt{2})((-1/\sqrt{2} - 1)e^{\sqrt{2}} = (1/\sqrt{2})(1 + 1/\sqrt{2})e^{\sqrt{2}}$

$f(1/\sqrt{2}) = (1/\sqrt{2})(1 - 1/\sqrt{2})e^{-\sqrt{2}}$

Now, $f(-1/\sqrt{2})/f(1/\sqrt{2}) = \{(1 + 1/\sqrt{2})/(1 - 1/\sqrt{2})\}e^{2\sqrt{2}} > 1$

⇨ $f(-1/\sqrt{2}) > f(1/\sqrt{2})$

⇨ Option (d) cannot be true.

Now, $f(1 - 1/\sqrt{2})/f(-1/\sqrt{2})$
$= (1 - 1/\sqrt{2})(1/\sqrt{2})e^{-2(1 - 1/\sqrt{2})}/\{(1/\sqrt{2})(1 + 1/\sqrt{2})e^{\sqrt{2}}\}$
$= \{(\sqrt{2} - 1)/(\sqrt{2} + 1)\}e^{-2} < 1$

⇨ $f(-1/\sqrt{2}) > f(1 - /\sqrt{2})$

⇨ Option (a) cannot be true.

14. Consider the function $f(x) = x^n(1 - x)^n/n!$, where $n \geq 1$ is a fixed integer. Let $f^{(k)}$ denote the k-th derivative of f. Which of the following is true for all $k \geq 1$?

(a) $f^{(k)}(0)$ and $f^{(k)}(1)$ are integers.

(b) $f^{(k)}(0)$ is an integer, but not $f^{(k)}(1)$

(c) $f^{(k)}(1)$ is an integer, but not $f^{(k)}(0)$

(d) Neither $f^{(k)}(1)$ nor $f^{(k)}(0)$ is an integer.

Solution: (A)

Let us take n = 3.

Now, $f(x) = x^3(1 - x)^3/3!$

⇨ $f^{(1)}(x) = (1/3!)\{3x^2(1 - x)^3 - 3x^3(1 - x)^2\}$

⇨ $f^{(1)}(0)$ and $f^{(1)}(1) = 0$ and integer.

⇨ $f^{(2)}(x) = (3/3!)\{2x(1 - x)^3 - 3x^2(1 - x)^2 - 3x^2(1 - x)^2 + 2x^3(1 - x)\}$

⇨ $f^{(2)}(0)$ and $f^{(2)}(1) = 0$ and integer

⇨ $f^{(3)}(x) = (3/3!)\{2(1 - x)^3 - 6x(1 - x)^2 - 12x(1 - x)^2 + 12x^2(1 - x) + 6x^2(1 - x) - 2x^3\}$

⇨ $f^{(3)}(x) = (3\times2/3!)\{(1 - x)^3 - 3x(1 - x)^2 - 6x(1 - x)^2 + 6x^2(1 - x) + 3x^2(1 - x) - x^3\}$

⇨ $f^{(3)}(x) = (1 - x)^3 - 3x(1 - x)^2 - 6x(1 - x)^2 + 6x^2(1 - x) + 3x^2(1 - x) - x^3$

⇨ $f^{(3)}(0) = 1$ and $f^{(3)}(1) = -1$, both are integers.

Now, there is nothing in denominator.

⇨ $f^{(k)}(x)$ is always integer.

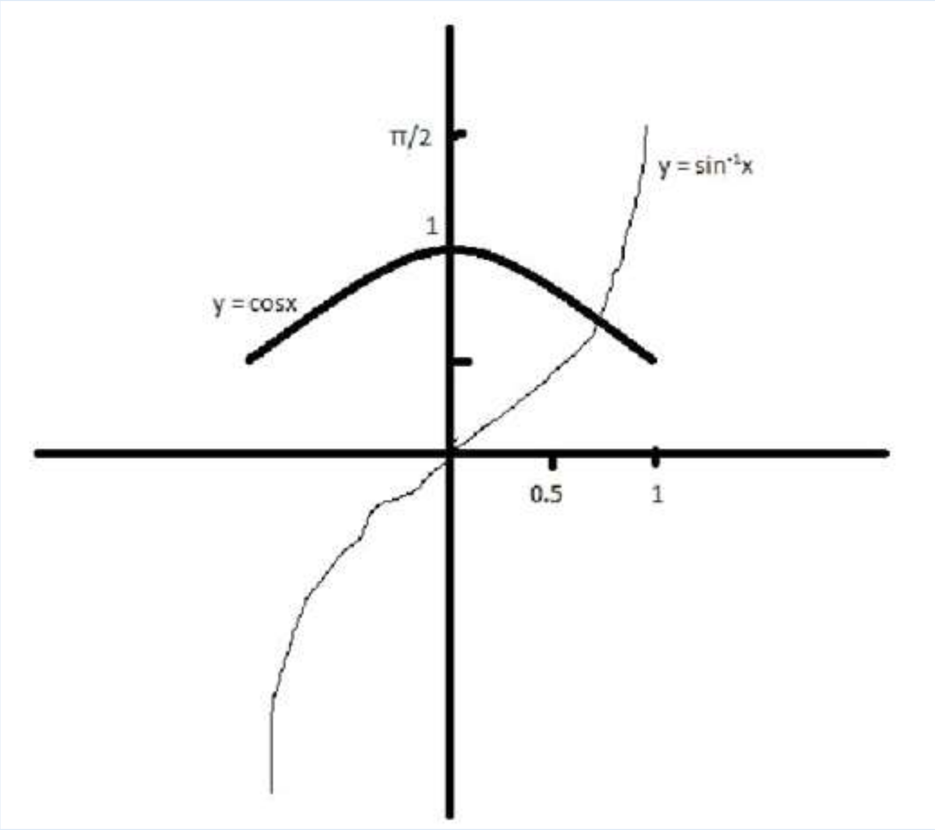
15. The number of solutions of the equation sin(cosΘ) = Θ, $-1 \le \theta \le 1$, is

(a) 0
(b) 1
(c) 2
(d) 3

Solution: (B)

Now, sin(cosΘ) = Θ

cosΘ = $\sin^{-1}$Θ



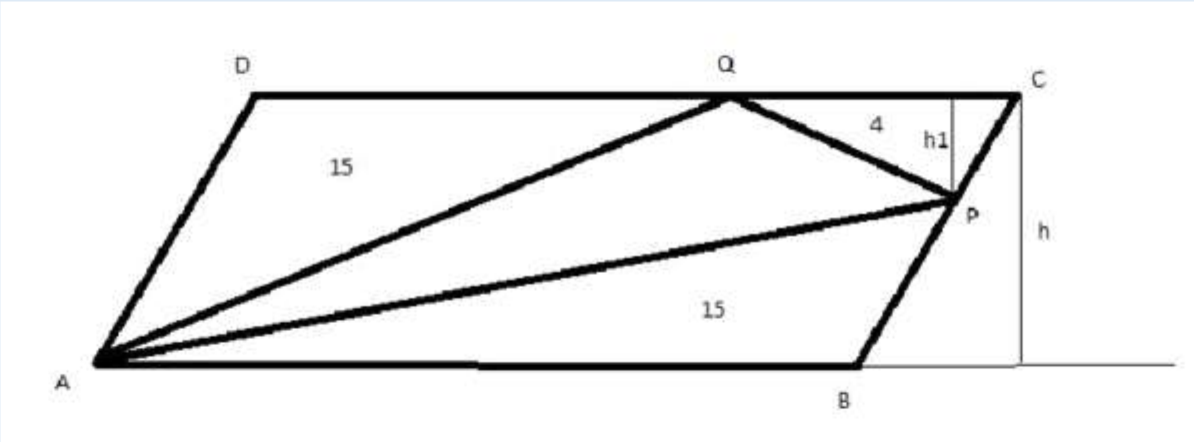
One intersection point.

⇨ One solution

16. Suppose ABCD is a parallelogram and P, Q are points on the sides BC and CD respectively, such that PB = αBC and DQ = βDC. If the area of the triangles ABP, ADQ, PCQ are 15, 15 and 4 respectively, then the area of APQ is

(a) 14
(b) 15
(c) 16
(d) 18.

Solution: (C)



Now, PB = αBC and CQ = βDC

Note that h_1/h = PC/BC as the triangles have equal angles.

⇨ $h_1 = h \times (BC - PB)/BC = h \times (BC - \alpha BC)/BC = h(1 - \alpha)$

Now, area of triangle PCQ = $(1/2) \times CQ \times h_1 = 4$

⇨ $(1/2) \times (1 - \beta) \times DC \times (1 - \alpha) \times h = 4$ (As CQ = DC − DQ = DC − β×DQ = (1- β) × DQ ....(1)

Now, area of triangle ADQ = $(1/2) \times DQ \times h = 15$

⇨ $(1/2) \times \beta \times DC \times h = 15$ ....(2)

Now, area of triangle ABP = $(1/2) \times AB \times \alpha \times h = 15$ ....(3)

Now, dividing (1) by (2) we get, $(1 - \beta)(1 - \alpha)/\beta = 4/15$ ....(4)

Now, dividing (2) by (3) we get, $\beta/\alpha = 1$ (As AB = DC)

$\beta = \alpha$

Putting $\beta = \alpha$ in (4) we get, $(1 - \alpha)^2/\alpha = 4/15$

⇨ $15\alpha^2 - 34\alpha + 15 = 0$

⇨ $\alpha = \{34 \pm \sqrt{(34^2 - 4\times15\times15)}\}/2\times15 = \{17 \pm \sqrt{(17^2 - 15^2)}\}/15$

⇨ $\alpha = \{17 \pm 8\}/15 = 9/15 = 3/5$ ($\alpha < 1$)

⇨ $AB\times h = 15\times2/(3/5)$ (From (3))

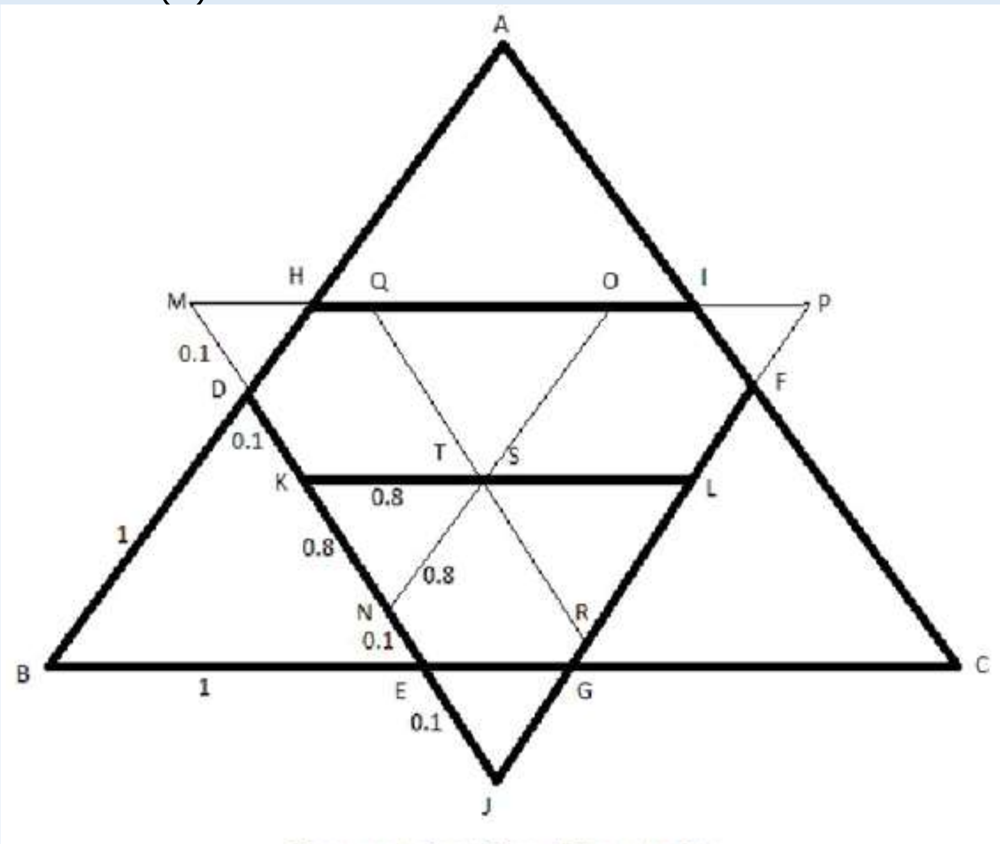
⇨ Area of the parallelogram = 50

⇨ Area of triangle APQ = 50 – (15 + 15 + 4) = 16.

17. Consider an equilateral triangle ABC with side 2.1 cm. You want to place a number of smaller equilateral triangles, each with side 1 cm, over the triangle ABC, so that the triangle ABC is fully covered. What is the minimum number of smaller triangles that you need?

(a) 4
(b) 5
(c) 6
(d) 7.

Solution: (C)



Please note S and T are different points

At first triangles BDE, CFG, AHI are placed.

Now, triangle JKL is placed.

EG = BC − BE − CG = 2.1 − 1 − 1 = 0.1

- ⇨ EJ = 0.1
- ⇨ KE = KJ − EJ = 1 − 0.1 = 0.9
- ⇨ DK = DE − KE = 1 − 0.9 = 0.1

Now, triangle MNO is placed.

Similarly, MD = NE = 0.1

- ⇨ KN = DE − DK − NE = 1 − 0.1 − 0.1 = 0.8
- ⇨ KS = 0.8
- ⇨ SL = KL − KS = 1 − 0.8 = 0.2

Now, if we place triangle PQR then the rest of the region of triangle ABC will be filled up as, TL = 0.8 and SL = 0.2 (Similarly).

So, triangles required = BDE, CFG, AHI, JKL, MNO, PQR.

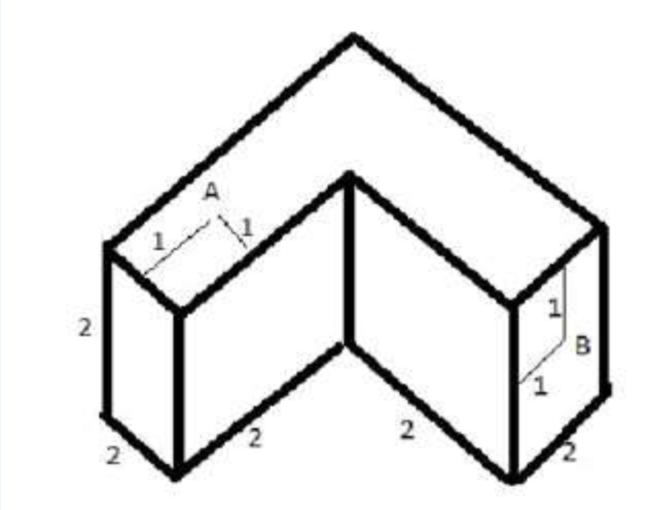
So, number of triangles required = 6.

18. A regular tetrahedron has all its vertices on a sphere of radius R. Then the length of each edge of the tetrahedron is

(a) (√2/√3)R
(b) (√3/2)R
(c) (4/3)R
(d) (2√2/√3)R

Solution: (D)

19. Consider the L-shaped brick in the diagram below.



If an ant starts from A, find the minimum distance it has to travel along the surface to reach B.

(a) $\sqrt{5}$
(b) $2\sqrt{5}$
(c) $(3/2)\sqrt{5}$
(d) $3\sqrt{5}$

Solution: (C)

20. Let $f(x) = (\tan x)^{3/2} - 3\tan x + \sqrt{\tan x}$. Consider the three integrals $I_1 = \int f(x)dx$ (integration running from 0 to 1); $I_2 = \int f(x)dx$ (integration running from 0.3 to 1.3) and $I_3 = \int f(x)dx$ (integration running from 0.5 to 1.5). Then,

(a) $I_1 > I_2 > I_3$
(b) $I_2 > I_1 > I_3$
(c) $I_3 > I_1 > I_2$
(d) $I_1 > I_3 > I_2$

Solution: (D)

Let us first find $\int f(x)dx$ then we will put the different limits and will get to the answer.

Now, $\int f(x)dx$

$= \int\{(\tan x)^{3/2} - 3\tan x + \sqrt{\tan x}\}dx$

$= \int\sqrt{(\tan x)}(1 + \tan x)dx - 3\int \tan x dx$

Let $\tan x = z^2$ in th first integral.

⇨ $x = \tan^{-1}(z^2)$

⇨ $dx = dz/(1+z^4)$

Putting the value we get,

$\int f(x)dx = 2\int\{(z^2 + z^4)/(1 + z^4\}\}dz + 3\ln|\cos x|$

$= 2\int\{(1 + z^4 + z^2 - 1)/(1 + z^4)\}dz + 3\ln|\cos x|$

$= 2\int\{(1 + z^4)/(1 + z^4)\}dz + 2\int\{(z^2 - 1)/(1 + z^4)\}dz + 3\ln|\cos x|$

$= 2\int dz + 2\int\{(1 - 1/z^2)/(z^2 + 1/z^2)\}dz + 3\ln|\cos x|$

$= 2z + 2\int(1 - 1/z^2)/[\{(z + 1/z)^2 - 2\}]dz + 3\ln|\cos x|$

Let, $z + 1/z = t$

$(1 - 1/z^2)dz = dt$

Putting the value we get,

$\int f(x)dx = 2z + 2\int dt/(t^2 - 2) + 3\ln|\cos x|$

$= 2z + (1/\sqrt{2})[\ln|t - \sqrt{2}| - \ln|t + \sqrt{2}|] + 3\ln|\cos x|$

$= 2\sqrt{\tan x} + (1/\sqrt{2})\ln|(z + 1/z - \sqrt{2})/(z + 1/z + \sqrt{2})| + 3\ln|\cos x|$

$= 2\sqrt{\tan x} + (1/\sqrt{2})\ln|(z^2 - \sqrt{2}z + 1)/(z^2 + \sqrt{2}z + 1)| + 3\ln|\cos x|$

$= 2\sqrt{\tan x} + (1/\sqrt{2})\ln|(\tan x - \sqrt{(2\tan x)} + 1)/(\tan x + \sqrt{(2\tan x)} + 1| + 3\ln|\cos x|$

Now, putting the limits we get, $I_1 > I_3 > I_2$

Group B: ***Each of the following questions has either one or two correct options and you have to identify all the correct options.***

21. Let a < b < c be three real numbers and w denote a complex cube root of unity. If $(a + bw\ cw^2)^3 + (a + bw^2 + cw)^3 = 0$, then which of the following must be true?

(a) $a + b + c = 0$
(b) $abc = 0$
(c) $ab + bc + ca = 0$
(d) $b = (c + a)/2$.

Solution: (D)

Now, $(a + bw + cw^2)^3 + (a + bw^2 + cw)^3 = 0$

⇨ $(a + bw + cw^2 + a + bw^2 + cw)\{(a + bw + cw^2)^2 + (a + bw^2 + cw)^2 – (a + bw + cw^2)(a + bw^2 + cw)\} = 0$

⇨ $\{2a + b(w + w^2) + c(w + w^2)\}\{(a + bw + cw^2 – a – bw^2 – cw)^2 + 2(a + bw + cw^2)(a + bw^2 + cw) - (a + bw + cw^2)(a + bw^2 + cw)\} = 0$

⇨ $(2a – b – c)[\{bw(1 – w) – cw(1 – w)\}^2 + (a + bw + cw^2)(a + bw^2 + cw)] = 0$

⇨ $\{w(1 – w)\}^2(b – c)^2 + a^2 + b^2w^3 + c^2w^3 + abw^2 + acw + abw + bcw^2 + caw^2 + bcw^4 = 0$

⇨ $w^2(1 – 2w + w^2)(b – c)^2 + a^2 + b^2 + c^2 – ab – ac – bc = 0$

⇨ $(w^2 – 2 + w)(b – c)^2 + a^2 + b^2 + c^2 – ab – bc – ca = 0$

⇨ $-3(b^2 – 2bc + c^2) + a^2 + b^2 + c^2 – ab – bc – ca = 0$

⇨ $a^2 – 2b^2 – 2c^2 – ab + 5bc – ca = 0$

If we put $b = (c + a)/2$ the equation gets satisfied. No other option is correct.

22. Suppose f is continuously differentiable up to 3rd order ad satisfies $\int\{6f(x) + x^3f'''(x)\}dx = f''(1)$ (integration running from x = 0 to x = 1). Which of the following must be true?

(a) $f(1) = 0$
(b) $f'(1) = 2f(1)$
(c) $f'(1) = f(1)$
(d) $f'(1) = 0$

Solution: (B)

$\int\{6f(x) + x^3f'''(x)\}dx = f''(1)$ (integration running from x = 0 to x = 1)

$6\int f(x)dx + \int x^3f'''(x)dx = f''''(1)$ (integration running from x = 0 to x = 1)

⇨ $6\int f(x)dx + x^3f''(x)$ (limit 0 to 1) - $\int 3x^2f''(x)dx = f''(1)$ (integration running from x = 0 to x = 1)

⇨ $6\int f(x)dx + f''(1) - 3x^2f'(x)$(limit 0 to 1) + $3\int 2xf'(x)dx = f''(1)$ (integration running from x = 0 to x = 1)

⇨ $6\int f(x)dx - 3f'(1) + 6xf(x)$ (limit 0 to 1) - $6\int f(x)dx = 0$ (integration running from x = 0 to x = 1)

⇨ $-3f'(1) + 6f(1) = 0$

⇨ $f''(1) = 2f(1)$

23. Let $f(x) = ax^2 + bx + c$ for some real numbers a, b, and c.
If $f(-5) \geq 10$, $f(-3) < 6$ and $f(2) \geq 7$, then which of the following cannot be true?
(a) $f(3) = 6$
(b) $f(3) \geq 16$
(c) $f(4) = 5$
(d) $f(4) \geq 6.2$

Solution: (A) & (C)

24. Consider the sequence x_n, $n \geq 1$, defined as:
$x_n = \{(1 + 2/n^a)^\wedge(-n^b)\}n^c$
where, a, b and c are real numbers. Which of the following are true?
(a) If $b < a$, $x_n \to 0$ as $n \to \infty$
(b) If $a < b$, $x_n \to 0$ as $n \to \infty$
(c) If $a = b$ and $c > 0$, $x_n \to \infty$ as $n \to \infty$
(d) If $a = b$ and $c < 0$, $x_n \to \infty$ as $n \to \infty$

Solution: (B) & (C)

25. The value of $n^{1/n} - 1$
(a) Tends to 0 as $n \to \infty$
(b) Is greater than (logn)/n for all $n \geq 3$
(c) Is greater than log(n) for all $n \geq 3$
(d) Is greater than $1/\sqrt{n}$ for all $n \geq 3$

Solution: (A) & (B)

Let, $x = n^{1/n}$

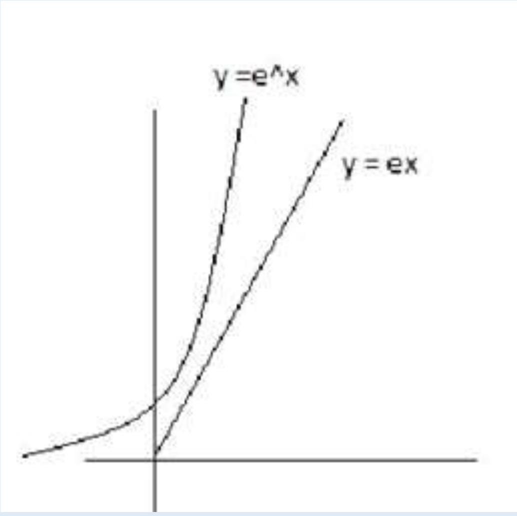
$\log x = (1/n)\log n$
$= \{(n-1)/n\}\{1/(n-1)\}\log\{1 + (n-1)\}$
$= (1 - 1/n)\{\log(1 + (n-1))\}/(n-1) \to (1-0)\times 1$ as $n \to \infty = 1$

⇨ $n^{1/n} - 1 \to 1 - 1 = 0$ as $n \to \infty$

Let us take the inequality, $x - 1 < \log x$

⇨ $e^{x-1} < x$

⇨ $ex > e^x$



From the graph it is clear that the inequality doesn't hold true.

⇨ $x - 1 > \log x$

Putting $x = n^{1/n}$ we get,
$n^{1/n} - 1 > \log n^{1/n}$
$n^{1/n} - 1 > (\log n)/n$

26. If the complex numbers 1 + i and 5 – 3i represent two diagonally opposite vertices of a square, which of the following complex numbers can represent another vertex of the square?
(a) 5 + 2i
(b) 3 + 2√2 – i
(c) 1 – 3i
(d) 4 + 2√2 + 2√2i

Solution: (C)

The coordinates are (1, 1) and (5, -3)

Let us take option (a) i.e. (5, 2)

Now, $\sqrt{\{(5 - 1)^2 + (2 - 1)^2\}} = \sqrt{17}$

Now, $\sqrt{\{(5 - 5)^2 + (2 + 3)^2\}} = 5$

The distances are not equal.

Option (a) cannot be true.

Now, let us take option (b) i.e. $(3 + 2\sqrt{2}, -1)$

Now, $\sqrt{\{(3 + 2\sqrt{2} - 1)^2 + (-1 - 1)^2} = \sqrt{(16 + 8\sqrt{2})}$

Now, $\sqrt{\{(3 + 2\sqrt{2} - 5)^2 + (-1 - 3)^2\}} = \sqrt{(28 - 8\sqrt{2})}$

The distances are not equal.

⇨ Option (b) cannot be true.

Let us take option (c) i.e. (1, -3)

Now, $\sqrt{\{(1 - 1)^2 + (-3 - 1)^2\}} = 4$

Now, $\sqrt{\{(1 - 5)^2 + (-3 + 3)^2\}} = 4$

Distances same but it can be a rhombus.

Now, we will calculate the slope of the two lines and if the product is -1 then it must be a square.

Now, $(-3 - 1)/(1 - 1) = \infty$

⇨ The line is perpendicular to x-axis.

Now, $(-3 + 3)/(5 - 1) = 0$

⇨ The line is parallel to x-axis.

⇨ The angle between the two straight lines is 90°

⇨ Option (c) is correct.

Now, let us take option (d) i.e. $(4 + 2\sqrt{2}, 2\sqrt{2})$

Now, $\sqrt{\{(4 + 2\sqrt{2} - 1)^2 + (2\sqrt{2} - 1)^2} = \sqrt{(26 + 8\sqrt{2})}$

Now, $\sqrt{\{4 + 2\sqrt{2} - 5)^2 + (2\sqrt{2} + 3)^2\}} = \sqrt{(26 + 8\sqrt{2})}$

Distances are same but it can be a rhombus.

So, $(2\sqrt{2} - 1)/(4 + 2\sqrt{2} - 1) = (2\sqrt{2} - 1)/(3 + 2\sqrt{2})$

And, $(2\sqrt{2} + 3)/(4 + 2\sqrt{2} - 5) = (2\sqrt{2} + 3)/(2\sqrt{2} - 1)$

So, the multiplication of the slopes is 1 and not -1.

⇨ They are not perpendicular to each other.

⇨ It cannot be a coordinate of a square.

⇨ Option (d) cannot be true.

27. Suppose x and y are two positive numbers satisfying the equation $x^y = y^x$. Which of the following are true?

(a) For all x > 1, there always exist a y > x such that the above equation holds.

(b) For all x > e there is always a y > x such that the above equation holds.

(c) For all 1 < x < e there is always a y > x such that the above equation holds.

(d) If x < 1, the y must be equal to x.

Solution: (C) & (D)

28. Consider 6 points on the plane no three of which are collinear. An edge is a straight line joining one point to another. Two points are called connected if one can go from one point to another through edges. Suppose you are only told how many edges are there in total, but not where they are. Which of the following are true?

(a) If you are told there are 7 edges, you cannot be sure that all pairs of points are connected.

(b) If you are told that there are 9 edges, you can always ensure that all pairs of points are connected.

(c) If you are told that there are 12 edges, you cannot be sure that all pairs of points are connected.

(d) If you are told that there are 13 edges, you can always ensure that all pairs of points are connected.

Solution: (A) & (D)

There are 6 points. If we leave a point isolated i.e. not connected then we can join rest of the 5 points with ${}^{5}C_2 = 10$ edges.

So, maximum number of edges required to be sure that all the points are connected is 11.

29. Suppose f : R -> R is differentiable everywhere. Which of the following conditions imply that |f(x)| is also differentiable?
(a) f(x) = 0 whenever f'(x) = 0.
(b) f'(x) = 0 whenever f(x) = 0.
(c) f'(x) never takes the value 0.
(d) f(x) never takes the value 0.

Solution: (B) & (D)

Let us take an example.

Let, f(x) = |x – a|
⇨ f(x) = x – a if x > a
⇨ f(x) = a – x if x < a

And f(a) = 0

Now. lim [{f(x) – f(a)}/(x – a)] as x -> a+
= lim {(x – a – 0)/(x – a)} as x -> a+
= lim (1/1) as x -> a+ (Applying L'Hospital rule)
= 1

Now, lim [{f(x) – f(a)}/(x – a)] as x -> a-
= lim {(a – x – 0)/(x – a)} as x -> a-
= lim (-1/1) as x -> a- (Applying L'Hospital rule)
= -1

So, f(x) is not differentiable at x = a.

Now, lim [{f(x) – f(a)}/(x – a)] as x -> a+ or a-
= lim f'(x)/1 as x -> a+ or a-

If f'(a) = 0 then it is differentiable.

We will demonstrate this with an example.

Let f(x) = $|(x – a)^3|$
⇨ f(x) = $(x – a)^3$ if x > a
⇨ f(x) = $(a – x)^3$ if x < a

And f(a) = 0

Now, lim [{f(x) – f(a)}/(x – a)] as x -> a+
= lim $\{(x – a)^3 – 0\}/(x – a)$ as x -> a+
= lim $(x – a)^2$ as x-> a+
= 0

Now, lim [{f(x) − f(a)}/(x − a)] as x -> a-
= lim $\{(a - x)^3 - 0\}/(x - a)$ as x -> a-
= lim $-(x - a)^2$ as x -> a- = 0
⇨ f(x) is differentiable as f''(x) = 0 when f(x) = 0 at x = a.

30. Let the coordinates of the centre of a circle be (-7/10, 2√2). Then the number of points (x, y) on the circle such that both x and y are rational

(a) Cannot be 3 or more.
(b) At least 1, but at most 2.
(c) At least 2, but infinitely many.
(d) Infinitely many.

Solution: (A)

The equation of the circle is $(x + 7/10)^2 + (y - 2\sqrt{2})^2 = r^2$ (where r is radius)

$x^2 + y^2 + (7/5)x - 4\sqrt{2}y + (7/5)^2 + 8 - r^2 = 0$.
Let $r^2 = 4\sqrt{2}t$ where $t > 0$
Let, y = -t.

Then the equation becomes, $x^2 + (7/5)x + (7/5)^2 + 8 + t^2 = 0$
Now, the discriminant of the equation is, $(7/5)^2 - 4\{(7/5)^2 + 8 + t^2\}$.

Clearly it is < 0.
⇨ No real solution for x.

Now, let y = 0, the equation becomes, $x^2 + (7/5)x + (7/5)^2 + 8 - r^2 = 0$
The disciminant of the equation is,

$(7/5)^2 - 4\{(7/5)^2 + 8 - r^2\} = 4r^2 - 3(7/5)^2 - 32$.

Depending on the value of r it can be > 0 also it is possible that it is < 0.

Also it can be a square number of some rational number, so x is rational. Also if $4r^2 - 3(7/5)^2 - 32 = 0$ then the roots are equal i.e. only one solution for x.

So, it can have 0 solution, 1 solution or 2 solutions but not more than that.