B.Math 2013 entrance paper-problem-3-solution

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Problem 3.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $|f(x+y) - f(x-y) - y| \leq y^2$ for all $x, y \in \mathbb{R}$.Show that,$f(x) = \frac{x}{2} + a$,where a is a constant.

Solution:

Let's define a function $g : \mathbb{R} \to \mathbb{R}$,by $g(x) = f(x) - (\frac{x}{2})$.

The given condition reads as,$|f(x+y) - f(x-y) - y| \leq y^2$

$\implies |f(x+y) - (\frac{x+y}{2}) - f(x-y) + (\frac{x-y}{2})| \leq y^2$

$\implies |\{f(x+y) - (\frac{x+y}{2})\} - \{f(x-y) - (\frac{x-y}{2})\}| \leq y^2$

$\implies |g(x+y) - g(x-y)| \leq y^2.$

Now.let's calculate the following limit.

$\lim_{x \to c} \frac{g(x)-g(c)}{x-c} = \lim_{x \to c} \frac{g(\frac{x+c}{2} + \frac{x-c}{2}) - g(\frac{x+c}{2} - \frac{x-c}{2})}{x-c}$,

but the above condition says that, $|g(\frac{x+c}{2} + \frac{x-c}{2}) - g(\frac{x+c}{2} - \frac{x-c}{2})| \leq (\frac{x-c}{2})^2$;

Dividing both side of the inequality by $|x-c|$,

we see that $|\frac{g(\frac{x+c}{2} + \frac{x-c}{2}) - g(\frac{x+c}{2} - \frac{x-c}{2})}{x-c}| \leq \frac{|x-c|}{4}$.

now taking limit as $x \to c$,and using sandwich principle we observe that

$\lim_{x \to c} \frac{g(x)-g(c)}{x-c} = 0.$

Hence,g is differentiable on whole of $\mathbb{R}$ and $g'(c) = 0 \ \forall c \in \mathbb{R}$

$\implies g$ is a constant function.
$\implies g(x) = a$ for some constant $a \in \mathbb{R}$.
$\implies f(x) - (\frac{x}{2}) = a$
$\implies f(x) = (\frac{x}{2}) + a$. **QED**

ISI B.Stat. (Hons.) & B.Math. (Hons.) 2013 Subjective Solution Paper

1. S = $\log_a bc + \log_b ac + \log_c ba$

= $\log_a b + \log_a c + \log_b c + \log_b a + \log_c a + \log_c b$

= $\frac{logb}{loga} + \frac{logc}{loga} + \frac{logc}{logb} + \frac{loga}{logb} + \frac{loga}{logc} + \frac{logb}{logc}$

= $\left(\frac{logb}{loga} + \frac{loga}{logb}\right) + \left(\frac{logc}{loga} + \frac{loga}{logc}\right) + \left(\frac{logc}{logb} + \frac{logb}{logc}\right)$

Now applying AM ≥ GM inequalities we get,

≥ 2 + 2 + 2 = 6.

2. f'(x) = $\frac{2sinx-1}{(x+2cosx)^2} = 0 \; gives \; x = \frac{\pi}{6}$.

f(0) = $\frac{1}{2}$; f$\left(\frac{\pi}{6}\right) = \frac{1}{\frac{\pi}{6}+\sqrt{3}} \approx 0.44$

The function is decreasing in the interval $\left[0, \frac{\pi}{6}\right)$

Now, f$\left(\pi - \frac{\pi}{6}\right) = \frac{1}{\frac{5\pi}{6}+\sqrt{3}} \approx 1.13$

For x in $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$, f(x) is positive & then again it becomes decreasing function.

So, the range of the function is $\left(0, \frac{6}{5\pi - 6\sqrt{3}}\right]$.

3. Given that

$$-y^2 \le f(x+y) - f(x-y) - y \le y^2$$

$$=> \; -y^2 + y \le f(x+y) - f(x-y) \le y^2 + y$$

$$=> \; \frac{-y^2 + y}{(x+y)-(x-y)} \le \frac{f(x+y) - f(x-y)}{(x+y)-(x-y)} \le \frac{y^2 + y}{(x+y)-(x-y)}$$

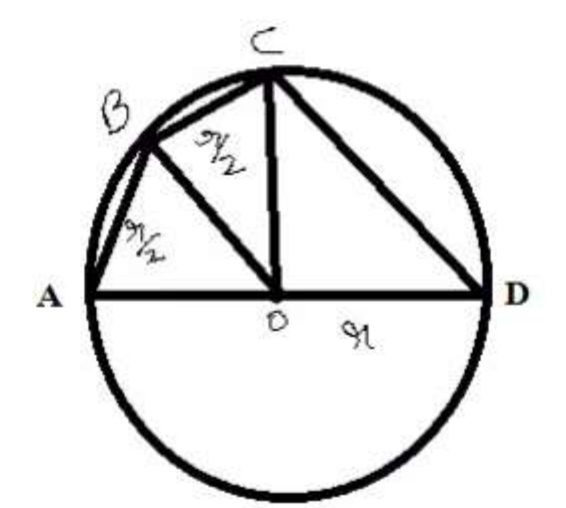
$$=> \; \lim_{2y\to 0}\left(-\frac{y}{2} + \frac{1}{2}\right) \le \lim_{2y\to 0}\frac{f(x+y) - f(x-y)}{(x+y)-(x-y)} \le \lim_{2y\to 0}\left(\frac{y}{2} + \frac{1}{2}\right)$$

$$=> \; \frac{1}{2} \le \lim_{2y\to 0}\frac{f(x+y) - f(x-y)}{(x+y)-(x-y)} \le \frac{1}{2}$$

*S*o, f(x) is a linear function y = ax + c, where a = $\frac{1}{2}$ & c is a constant.

4. Suppose there is no such player. Say X be the player with highest number of names in his list. Let A be the set of players in list of X whom X has directly defeated and B be the set of other players of X's list. Since there is no player with all others names, say X does not have Y's name. This implies Y has beaten X and hence in Y's list, there is X and also each member of set A. Now the members of set B are in X's list because each is beaten by someone of set A. Now if a member of set A beats Y, that would imply that Y is in X's list, which is not so. Hence Y has beaten each member of set A, which further implies he has the names of set B also in his list. Thus Y is a player with more names in his list than that of X, a contradiction.

5.



Let $\angle AOB = \angle BOC = m$

Cosm = $\frac{OA^2+OB^2-AB^2}{2\times OA\times OB} = \frac{r^2+r^2-\frac{r^2}{4}}{2r^2} = \frac{7}{8}$

Then Cos2m = 2Cos^{2}m – 1 = $\frac{34}{64}$

Using Cosine Rule in ΔAOC, we get $AC^2 = OA^2 + OC^2 - 2\times OA\times OC\cos 2m$

So, $AC^2 = \frac{2r^2\times 30}{64}$

Now, $CD^2 = AD^2 - AC^2 = \frac{2r^2\times 98}{64}$

So, CD = $\frac{7r}{4}$

$\Leftrightarrow \frac{CD}{r} = \frac{7}{4}$.

6.

Assume P(x) – Q(x) = F(x). We will show that F(x) has repeated roots of x = 1. To prove this we will show that the derivatives of F(x) also has roots x = 1.

$$(P(x))^3-(Q(x))^3=P(x^3)-Q(x^3)$$.

=> $(P-Q)^3+3PQ(P-Q)=P(x^3)-Q(x^3)$
=> $(F(x))^3+3PQF(x)=F(x^3)$

Since sum of coefficients of P(x) and Q(x) are equal hence P(1) = Q(1). Thus F(1) = P(1) – Q(1) = 0.

Taking the Derivative we have

$$3(F(x))^3F'(x)+3(P'QF+PQ'F+PQF')=3x^2F'(x^3)$$

If we replace x by 1; all terms containing F will vanish. Since P(1) and Q(1) equals S

We have $S^2F'(1)=F'(1)$ implying either F'(1) = 0 or $S^2=1$

If $S^2\neq 1$ we continue the differentiation. Againing ignoring all terms containing F and F' (since F(1) = F'(1) = 0), we have $S^2F^2(1)=3F^2(1)$ implying either $S^2=3\vee F^2(1)=0$.

By induction we can easily show that if we perform a times differentiation we will have $F^a(1)=0$ or $s^2=3^{a-1}$ (one 3 will be generated in each differentiation from the x^3 term; all the derivatives till a-1th derivative will be igonored as at x = 1 they are 0.

(This is a sketch of the solution. The induction should be implemented at this step.)

Hence $F(x)=P(x)-Q(x)=(x-1)^aR(x)$ and $S^2=3^{a-1}$.

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7. For N be a positive integer N(N – 101) be a perfect square.

Since 101 is a prime number, so both N and N - 101 should be perfect squares.

Note that all consecutive perfect squares have a difference of successive odd numbers like

$2^2 - 1^2 = 3$

$3^2 - 2^2 =$

$4^2 - 3^2 = 7$

…………

$51^2 - 50^2 = 101$

So, N = 51^2 or 2601 is our required solution.

8. Since ABCD is a square so AB || CD.

Hence the equation to the line passing through C and D must be y = x + c (where c is a constant) That is the slope is 1 (as the line AB is y = x +8; it has slope 1) and the y-intercept is c.

Let the line y = x + c intersect the parabola $y=x^2$ at α,β (x coordinates).

Thus $x^2=x+c$ has the solutions α,β .

Using Sridhar's formula $x=\frac{1\pm\sqrt{1+4c}}{2}$. These must be the values of α,β . Thus $\alpha-\beta=\sqrt{(1+4c)}$

Distance between the lines y = x+8 and y = x + c is $\frac{|c-8|}{\sqrt{2}}$

Also the coordinates of C and D are $(\alpha,\alpha+c),(\beta,\beta+c)$. Hence the length CD = $\sqrt{2}(\alpha-\beta)$ (using distance formula). Since $\alpha-\beta=\sqrt{(1+4c)}$; the length of CD = $\sqrt{2}\sqrt{(1+4c)}$.

This length CD equals the distance between lines y = x + 8 and y = x +c since ABCD is a square.

Thus $\frac{|c-8|}{\sqrt{2}}$ = $\sqrt{2}\sqrt{(1+4c)}$; squaring both sides we have $(c-8)^2=4(1+4c)$.

From here we solve for c: $c^2-16c+64=4+16c$ or $c^2-32c+60=0$
Thus $(c-30)(c-2)=0$

Therefore c may have two values : 30 and 2

Hence the possible length of sides are $11\sqrt{2}$ or $3\sqrt{2}$