B.Math 2013 entrance paper-problem-3-solution

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Problem 3.
Let \( f: \mathbb{R} \to \mathbb{R} \) be a function satisfying \( |f(x + y) - f(x - y) - y| \leq y^2 \) for all \( x, y \in \mathbb{R} \). Show that, \( f(x) = \frac{x}{2} + a \), where \( a \) is a constant.

Solution:
Let’s define a function \( g: \mathbb{R} \to \mathbb{R} \), by \( g(x) = f(x) - \left( \frac{x}{2} \right) \).

The given condition reads as, \( |f(x + y) - f(x - y) - y| \leq y^2 \)

\[ \Rightarrow |f(x + y) - \left( \frac{x + y}{2} \right) - f(x - y) + \left( \frac{x - y}{2} \right)| \leq y^2 \]

\[ \Rightarrow |\{f(x + y) - \left( \frac{x + y}{2} \right)\} - \{f(x - y) - \left( \frac{x - y}{2} \right)\}| \leq y^2 \]

\[ \Rightarrow |g(x + y) - g(x - y)| \leq y^2. \]

Now, let’s calculate the following limit.

\[ \lim_{x \to c} \frac{g(x) - g(c)}{x - c} = \lim_{x \to c} \frac{g\left( \frac{x + c}{2} + \frac{c - x}{2} \right) - g\left( \frac{x + c}{2} - \frac{c - x}{2} \right)}{x - c}, \]

but the above condition says that, \( |g\left( \frac{x + c}{2} + \frac{c - x}{2} \right) - g\left( \frac{x + c}{2} - \frac{c - x}{2} \right)| \leq \left( \frac{x - c}{2} \right)^2; \)

Dividing both side of the inequality by \( |x - c| \),

we see that \( \left| \frac{g\left( \frac{x + c}{2} + \frac{c - x}{2} \right) - g\left( \frac{x + c}{2} - \frac{c - x}{2} \right)}{x - c} \right| \leq \left( \frac{x - c}{2} \right)^2 \)

now taking limit as \( x \to c \), and using sandwich principle we observe that

\[ \lim_{x \to c} \frac{g(x) - g(c)}{x - c} = 0. \]

Hence, \( g \) is differentiable on whole of \( \mathbb{R} \) and \( g'(c) = 0 \) \( \forall c \in \mathbb{R} \)

\[ \Rightarrow g \text{ is a constant function.} \]

\[ \Rightarrow g(x) = a \text{ for some constant } a \in \mathbb{R}. \]

\[ \Rightarrow f(x) - \left( \frac{x}{2} \right) = a \]

\[ \Rightarrow f(x) = \left( \frac{x}{2} \right) + a. \text{ QED} \]
1. \( S = \log_a bc + \log_b ac + \log_c ba \)
   \[= \log_a b + \log_a c + \log_b a + \log_b c + \log_c a + \log_c b \]
   \[= \frac{\log b}{\log a} + \frac{\log c}{\log a} + \frac{\log c}{\log b} + \frac{\log a}{\log b} + \frac{\log a}{\log c} + \frac{\log b}{\log c} \]
   \[= \left( \frac{\log b}{\log a} + \frac{\log a}{\log b} \right) + \left( \frac{\log c}{\log a} + \frac{\log a}{\log c} \right) + \left( \frac{\log c}{\log b} + \frac{\log b}{\log c} \right) \]

   Now applying AM \( \geq \) GM inequalities we get,
   \[\geq 2 + 2 + 2 = 6.\]

2. \( f'(x) = \frac{2 \sin x - 1}{(x + 2 \cos x)^2} = 0 \) gives \( x = \frac{\pi}{6}. \)

   \[ f(0) = \frac{1}{2}, \quad f\left(\frac{\pi}{6}\right) = \frac{1}{\frac{\pi}{6} + \sqrt{3}} \approx 0.44 \]

   The function is decreasing in the interval \( \left[0, \frac{\pi}{6}\right) \)

   Now, \( f\left(\pi - \frac{\pi}{6}\right) = \frac{1}{\frac{\pi}{6} + \sqrt{3}} \approx 1.13 \)

   For \( x \) in \( \left[\frac{\pi}{6}, \frac{5\pi}{6}\right), \) \( f(x) \) is positive & then again it becomes decreasing function.

   So, the range of the function is \( \left(0, \frac{6}{5\pi - 6\sqrt{3}}\right). \)

3. Given that

   \[-y^2 \leq f(x + y) - f(x - y) - y \leq y^2\]
   \[\Rightarrow -y^2 + y \leq f(x + y) - f(x - y) \leq y^2 + y\]
   \[\Rightarrow \frac{-y^2 + y}{(x + y) - (x - y)} \leq \frac{f(x + y) - f(x - y)}{(x + y) - (x - y)} \leq \frac{y^2 + y}{(x + y) - (x - y)}\]
   \[\Rightarrow \lim_{2y \to 0} \left( -\frac{y}{2} + \frac{1}{2} \right) \leq \lim_{2y \to 0} \frac{f(x + y) - f(x - y)}{(x + y) - (x - y)} \leq \lim_{2y \to 0} \left( \frac{y}{2} + \frac{1}{2} \right)\]
   \[\Rightarrow \frac{1}{2} \leq \lim_{2y \to 0} \frac{f(x + y) - f(x - y)}{(x + y) - (x - y)} \leq \frac{1}{2}\]

   So, \( f(x) \) is a linear function \( y = ax + c, \) where \( a = \frac{1}{2} \) & \( c \) is a constant.
4. Suppose there is no such player. Say X be the player with highest number of names in his list. Let A be the set of players in list of X whom X has directly defeated and B be the set of other players of X's list. Since there is no player with all others names, say X does not have Y's name. This implies Y has beaten X and hence in Y's list, there is X and also each member of set A. Now the members of set B are in X's list because each is beaten by someone of set A. Now if a member of set A beats Y, that would imply that Y is in X's list, which is not so. Hence Y has beaten each member of set A, which further implies he has the names of set B also in his list. Thus Y is a player with more names in his list than that of X, a contradiction.

5.

Let \( \angle AOB = \angle BOC = m \)

\[
\cos m = \frac{OA^2 + OB^2 - AB^2}{2 \times OA \times OB} = \frac{r^2 + r^2 - \frac{r^2}{4}}{2r^2} = \frac{7}{8}
\]

Then \( \cos 2m = 2\cos^2 m - 1 = \frac{34}{64} \)

Using Cosine Rule in \( \triangle AOC \), we get \( AC^2 = OA^2 + OC^2 - 2 \times OA \times OC \cos 2m \)

So, \( AC^2 = \frac{2r^2 \times 30}{64} \)

Now, \( CD^2 = AD^2 - AC^2 = \frac{2r^2 \times 98}{64} \)

So, \( CD = \frac{7r}{4} \)

\[
\iff \frac{CD}{r} = \frac{7}{4}
\]
6.

Assume \( P(x) - Q(x) = F(x) \). We will show that \( F(x) \) has repeated roots of \( x = 1 \). To prove this we will show that the derivatives of \( F(x) \) also has roots \( x = 1 \).

\[
(F(x))^2 - (Q(x))^2 = P(x^2) - Q(x^2)
\]

\[
\Rightarrow (F(x))^3 + 3PQ(P(x) - Q(x)) = P(x^2) - Q(x^2)
\]

\[
\Rightarrow (F(x))^3 + 3PQF(x) = F(x^2)
\]

Since sum of coefficients of \( P(x) \) and \( Q(x) \) are equal hence \( P(1) = Q(1) \). Thus \( F(1) = P(1) - Q(1) = 0 \).

Taking the Derivative we have

\[
3(F(x))F'(x) + 3(P'QF + PQ'F + PQF') = 3x^2F'(x)
\]

If we replace \( x \) by \( 1 \), all terms containing \( F \) will vanish. Since \( P(1) \) and \( Q(1) \) equals \( S \)

We have \( S^2 F'(1) = F'(1) \) implying either \( F'(1) = 0 \) or \( S^2 = 1 \).

If \( S^2 \neq 1 \), we continue the differentiation. Again ignoring all terms containing \( F \) and \( F' \) (since \( F(1) = F'(1) = 0 \)), we have \( S^2 F''(1) = 3 F''(1) \) implying either \( S^2 = 3 \) or \( F''(1) = 0 \).

By induction we can easily show that if we perform a times differentiation we will have \( F^n(1) = 0 \) or \( S^2 = 3^{n-1} \). (one 3 will be generated in each differentiation from the \( x^3 \) term; all the derivatives till a-1th derivative will be ignored as at \( x = 1 \) they are 0.

(This is a sketch of the solution. The induction should be implemented at this step.)

Hence \( F(x) = P(x) - Q(x) = (x - 1)^n R(x) \) and \( S^2 = 3^{n-1} \).

7. For \( N \) be a positive integer \( N(N - 101) \) be a perfect square.

Since 101 is a prime number, so both \( N \) and \( N - 101 \) should be perfect squares.

Note that all consecutive perfect squares have a difference of successive odd numbers like

\[
2^2 - 1^2 = 3
\]

\[
3^2 - 2^2 =
\]

\[
4^2 - 3^2 = 7
\]

\[
................
\]

\[
51^2 - 50^2 = 101
\]

So, \( N = 51^2 \) or 2601 is our required solution.
8. Since ABCD is a square so AB \parallel CD.

Hence the equation to the line passing through C and D must be \( y = x + c \) (where \( c \) is a constant). That is the slope is 1 (as the line AB is \( y = x + 8 \); it has slope 1) and the y-intercept is \( c \).

Let the line \( y = x + c \) intersect the parabola \( y = x^2 \) at \( \alpha, \beta \) (x coordinates).

Thus \( x^2 = x + c \) has the solutions \( \alpha, \beta \).

Using Sridhar’s formula \( x = \frac{1 \pm \sqrt{1+4c}}{2} \). These must be the values of \( \alpha, \beta \). Thus \( \alpha - \beta = \sqrt{1+4c} \).

Distance between the lines \( y = x + 8 \) and \( y = x + c \) is \( \frac{|c-8|}{\sqrt{2}} \).

Also the coordinates of C and D are \( (\alpha, \alpha+c), (\beta, \beta+c) \). Hence the length \( CD = \sqrt{2}(\alpha - \beta) \) (using distance formula). Since \( \alpha - \beta = \sqrt{1+4c} \); the length of \( CD = \sqrt{2\sqrt{1+4c}} \).

This length \( CD \) equals the distance between lines \( y = x + 8 \) and \( y = x + c \) since ABCD is a square.

Thus \( \frac{|c-8|}{\sqrt{2}} = \sqrt{2\sqrt{1+4c}} \); squaring both sides we have \( (c-8)^2 = 4(1+4c) \).

From here we solve for \( c \): \( c^2 - 16c + 64 = 4 + 16c \) or \( c^2 - 32c + 60 = 0 \)
Thus \( (c-30)(c-2) = 0 \).

Therefore \( c \) may have two values: 30 and 2.

Hence the possible length of sides are \( 11\sqrt{2} \) or \( 3\sqrt{2} \).