B.Stat. (Hons.) & B.Math. (Hons.) Admission Test: 2013

Multiple-Choice Test
Time: 2 hours

1. Let $i = \sqrt{-1}$ and $S = \{i + i^2 + \cdots + i^n : n \geq 1\}$. The number of distinct *real* numbers in the set S is

(A) 1 (B) 2 (C) 3 (D) infinite.

2. From a square of unit length, pieces from the corners are removed to form a regular octagon. Then, the value of the area removed is

(A) $1/2$ (B) $1/\sqrt{2}$ (C) $\sqrt{2} - 1$ (D) $(\sqrt{2} - 1)^2$.

3. We define the *dual* of a line $y = mx + c$ to be the point $(m, -c)$. Consider a set of n non-vertical lines, $n > 3$, passing through the point $(1, 1)$. Then the duals of these lines will always

(A) be the same (B) lie on a circle (C) lie on a line

(D) form the vertices of a polygon with positive area.

4. Suppose α, β and γ are three real numbers satisfying $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$. Then the value of $\cos(\alpha - \beta)$ is

(A) $-\frac{1}{2}$ (B) $-\frac{1}{4}$ (C) $\frac{1}{4}$ (D) $\frac{1}{2}$.

5. The value of $\lim_{x\to\infty} (3^x + 7^x)^{\frac{1}{x}}$ is

(A) 7 (B) 10 (C) e^7 (D) ∞.

6. The distance between the two foci of the rectangular hyperbola defined by $xy = 2$ is

(A) 2 (B) $2\sqrt{2}$ (C) 4 (D) $4\sqrt{2}$.

7. Suppose f is a differentiable and increasing function on $[0, 1]$ such that $f(0) < 0 < f(1)$. Let $F(t) = \int_0^t f(x)dx$. Then

(A) F is an increasing function on $[0, 1]$

(B) F is a decreasing function on $[0, 1]$

(C) F has a unique maximum in the open interval $(0, 1)$

(D) F has a unique minimum in the open interval $(0, 1)$.

8. In an isosceles triangle ΔABC, the angle $\angle ABC = 120^o$. Then the ratio of two sides $AC : AB$ is

(A) 2:1 (B) 3: 1 (C) $\sqrt{2} : 1$ (D) $\sqrt{3} : 1$.

9. Let x, y, z be positive real numbers. If the equation

$$x^2 + y^2 + z^2 = (xy + yz + zx)\sin\theta$$

has a solution for θ, then x, y and z must satisfy

(A) $x = y = z$ (B) $x^2 + y^2 + z^2 \leq 1$

(C) $xy + yz + zx = 1$ (D) $0 < x,\ y,\ z \leq 1$.

10. Suppose $\sin\theta = \frac{4}{5}$ and $\sec\alpha = \frac{7}{4}$ where $0 \leq \theta \leq \frac{\pi}{2}$ and $-\frac{\pi}{2} \leq \alpha \leq 0$. Then $\sin(\theta + \alpha)$ is

(A) $\dfrac{3\sqrt{33}}{35}$ (B) $-\dfrac{3\sqrt{33}}{35}$ (C) $\dfrac{16 + 3\sqrt{33}}{35}$ (D) $\dfrac{16 - 3\sqrt{33}}{35}$.

11. Let $i = \sqrt{-1}$ and $z_1, z_2, \ldots$ be a sequence of complex numbers defined by $z_1 = i$ and $z_{n+1} = z_n^2 + i$ for $n \geq 1$. Then $|z_{2013} - z_1|$ is

(A) 0 (B) 1 (C) 2 (D) $\sqrt{5}$.

12. The last digit of the number $2^{100} + 5^{100} + 8^{100}$ is

(A) 1 (B) 3 (C) 5 (D) 7.

13. The maximum value of $|x - 1|$ subject to the condition $|x^2 - 4| \leq 5$ is

(A) 2 (B) 3 (C) 4 (D) 5.

14. Which of the following is correct?

(A) $ex \leq e^x$ for all x. (B) $ex < e^x$ for $x < 1$ and $ex \geq e^x$ for $x \geq 1$.

(C) $ex \geq e^x$ for all x. (D) $ex < e^x$ for $x > 1$ and $ex \geq e^x$ for $x \leq 1$.

15. The area of a regular polygon of 12 sides that can be inscribed in the circle $x^2 + y^2 - 6x + 5 = 0$ is

(A) 6 units (B) 9 units (C) 12 units (D) 15 units.

16. Let $f(x) = \sqrt{\log_2 x - 1} + \frac{1}{2}\log_{\frac{1}{2}} x^3 + 2$. The set of all real values of x for which the function $f(x)$ is defined and $f(x) < 0$ is

(A) $x > 2$ (B) $x > 3$ (C) $x > e$ (D) $x > 4$.

17. Let a be the largest integer *strictly* smaller than $\frac{7}{8}b$ where b is also an integer. Consider the following inequalities:

(1) $\frac{7}{8}b - a \leq 1$ (2) $\frac{7}{8}b - a \geq \frac{1}{8}$

and find which of the following is correct.

(A) Only (1) is correct. (B) Only (2) is correct.

(C) Both (1) and (2) are correct. (D) None of them is correct.

18. The value of $\lim_{x \to -\infty} \sum_{k=1}^{1000} \frac{x^k}{k!}$ is

(A) $-\infty$ (B) ∞ (C) 0 (D) e^{-1}.

19. For integers m and n, let $f_{m,n}$ denote the function from the set of integers to itself, defined by

$$f_{m,n}(x) = mx + n.$$

Let $\mathcal{F}$ be the set of all such functions,

$$\mathcal{F} = \{f_{m,n} : m, n \text{ integers } \}.$$

Call an element $f \in \mathcal{F}$ *invertible* if there exists an element $g \in \mathcal{F}$ such that $g(f(x)) = f(g(x)) = x$ for all integers x. Then which of the following is true?

(A) Every element of $\mathcal{F}$ is invertible.

(B) $\mathcal{F}$ has infinitely many invertible and infinitely many non-invertible elements.

(C) $\mathcal{F}$ has finitely many invertible elements.

(D) No element of $\mathcal{F}$ is invertible.

20. Consider six players P_1, P_2, P_3, P_4, P_5 and P_6. A team consists of two players. (Thus, there are 15 distinct teams.) Two teams play a match exactly once if there is no common player. For example, team $\{P_1, P_2\}$ can not play with $\{P_2, P_3\}$ but will play with $\{P_4, P_5\}$. Then the total number of possible matches is

(A) 36 (B) 40 (C) 45 (D) 54.

21. The minimum value of $f(\theta) = 9\cos^2\theta + 16\sec^2\theta$ is

(A) 25 (B) 24 (C) 20 (D) 16.

22. The number of 0's at the end of the integer

$$100! - 101! + \cdots - 109! + 110!$$

is

(A) 24 (B) 25 (C) 26 (D) 27.

23. We denote the largest integer less than or equal to z by $[z]$. Consider the identity

$$(1+x)(10+x)(10^2+x)\cdots(10^{10}+x) \;=\; 10^a + 10^b x + a_2 x^2 + \cdots + a_{11} x^{11}.$$

Then

(A) $[a] > [b]$ (B) $[a] = [b]$ and $a > b$

(C) $[a] < [b]$ (D) $[a] = [b]$ and $a < b$.

24. The number of four tuples (a, b, c, d) of *positive integers* satisfying all three equations

$$\begin{aligned} a^3 &= b^2 \\ c^3 &= d^2 \\ c - a &= 64 \end{aligned}$$

is

(A) 0 (B) 1 (C) 2 (D) 4.

25. The number of real roots of $e^x = x^2$ is

(A) 0 (B) 1 (C) 2 (D) 3.

26. Suppose $\alpha_1, \alpha_2, \alpha_3$ and α_4 are the roots of the equation $x^4 + x^2 + 1 = 0$. Then the value of $\alpha_1^4 + \alpha_2^4 + \alpha_3^4 + \alpha_4^4$ is

(A) -2 (B) 0 (C) 2 (D) 4.

27. *Among the four time instances* given in the options below, when is the angle between the minute hand and the hour hand the smallest?

(A) 5:25 p.m. (B) 5:26 p.m. (C) 5:29 p.m. (D) 5:30 p.m.

28. Suppose all roots of the polynomial $P(x) = a_{10}x^{10} + a_9 x^9 + \cdots + a_1 x + a_0$ are real and smaller than 1. Then, for any such polynomial, the function

$$f(x) = a_{10}\frac{e^{10x}}{10} + a_9\frac{e^{9x}}{9} + \cdots + a_1 e^x + a_0 x, \;\; x > 0$$

(A) is increasing (B) is either increasing or decreasing

(C) is decreasing (D) is neither increasing nor decreasing.

29. Consider a quadrilateral $ABCD$ in the XY-plane with all of its angles less than 180^o. Let P be an arbitrary point in the plane and consider the six triangles each of which is formed by the point P and two of the points A, B, C, D. Then the total area of these six triangles is minimum when the point P is

(A) outside the quadrilateral

(B) one of the vertices of the quadrilateral

(C) intersection of the diagonals of the quadrilateral

(D) none of the points given in (A), (B) or (C).

30. The graph of the equation $x^3 + 3x^2y + 3xy^2 + y^3 - x^2 + y^2 = 0$ comprises

(A) one point

(B) union of a line and a parabola

(C) one line

(D) union of a line and a hyperbola.

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Short-Answer Type Test

Time: 2 hours

1. Let a, b, c be real numbers greater than 1. Let S denote the sum

$$S = \log_a bc + \log_b ca + \log_c ab.$$

Find the smallest possible value of S.

2. For $x \geq 0$ define

$$f(x) = \frac{1}{x + 2\cos(x)}.$$

Determine the set $\{y \in \mathbb{R} : y = f(x), x \geq 0\}$.

3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function satisfying

$$|f(x+y) - f(x-y) - y| \leq y^2$$

for all $x, y \in \mathbb{R}$. Show that $f(x) = \frac{x}{2} + c$, where c is a constant.

4. In a badminton singles tournament, each player played against all the others exactly once and each game had a winner. After all the games, each player listed the names of all the players she defeated as well as the names of all the players defeated by the players defeated by her. For instance, if A defeats B and B defeats C, then in the list of A both B and C are included. Prove that at least one player listed the names of all other players.

5. Let AD be a diameter of a circle of radius r. Let B, C be points on the semi-circle (with C distinct from A) so that $AB = BC = \frac{r}{2}$. Determine the ratio of the length of the chord CD to the radius.

6. Let $p(x), q(x)$ be distinct polynomials with real coefficients such that the sum of the coefficients of each of the polynomials equals s. If $(p(x))^3 - (q(x))^3 = p(x^3) - q(x^3)$, then prove the following:

1. $p(x) - q(x) = (x-1)^a r(x)$ for some integer $a \geq 1$ and a polynomial $r(x)$ with $r(1) \neq 0$.
2. $s^2 = 3^{a-1}$ where a is as given in (a).

7. Let N be a positive integer such that $N(N-101)$ is the square of a positive integer. Then determine all possible values of N. (Note that 101 is a prime number).

8. Let $ABCD$ be a square with the side AB lying on the line $y = x + 8$. Suppose C, D lie on the parabola $x^2 = y$. Find the possible values of the length of the side of the square.