ISI B. Stat B. Math Admission Test 2015 Solved Paper

1. Let $0 < a_1 < a_2 < \ldots < a_n$ be n real numbers. Show that the equation $a_1/(a_1 - x) + a_2/(a_2 - x) + \ldots + a_n/(a_n - x) = 2015$ has exactly n real roots.

Solution:

This is a polynomial of degree n on x. So, there are n number of roots.

We have to prove that n roots are real.

There is no complex root of the equation.

Let, there are two complex roots s + ir and s − ir (As complex roots come in conjugate pair)

Now, s + ir will satisfy the equation.

⇨ $a_1/(a_1 - s - ir) + a_2/(a_2 - s - ir) + \ldots + a_n/(a_n - s - ir) = 2015$

Similarly, $a_1/(a_1 - s + ir) + a_2/(a_2 - s + ir) + \ldots + a_n/(a_n - s + ir) = 2015$

Now, subtracting the equations we get,

$\{a_1/(a_1 - s - ir) - a_1/(a_1 - s + ir)\} + \{a_2/(a_2 - s - ir) - a_2/(a_2 - s + ir)\} + \ldots + \{a_n/(a_n - s - ir) - a_n/(a_n - s + ir)\} = 2015 - 2015$

$a_1(2ir)/\{(a_1 - s)^2 + r^2\} + a_2(2ir)/\{(a_2 - s)^2 + r^2\} + \ldots + a_n(2ir)/\{(a_n - s)^2 + r^2\} = 0$

$(2ir)[a_1/\{(a_1 - s)^2 + r^2\} + a_2/(a_2 - s)^2 + r^2\} + \ldots + a_n/\{(a_n - s)^2 + r^2\}] = 0$

Now, $a_1, a_2, \ldots, a_n$ all are greater than 0.

⇨ The expression inside the square bracket is greater than 0 as denominators are sum of square numbers and greater than 0.

⇨ r has to be 0.

⇨ The imaginary part of the roots are zero.

⇨ The roots are no more complex.

⇨ Our assumption was wrong.

⇨ There is no complex root of the equation.

⇨ All roots are real.

⇨ There are n number of real roots.

2. **Let R denote the set of real numbers. Find all functions f: R→R, satisfying |f(x) – f(y)| = 2|x – y| for all x, y ∈ R. Justify your answer.**

Solution:

We have, $|f(x) - f(y)| = 2|x - y|$

⇨ $|\{f(x) - f(y)\}/(x - y)| = 2$

⇨ $\lim |\{f(x) - f(y)]/(x - y)|$ as $x \to y$ = $\lim (2)$ as $x \to y$

⇨ $|f'(y)| = 2$

⇨ $|f'(x)| = 2$

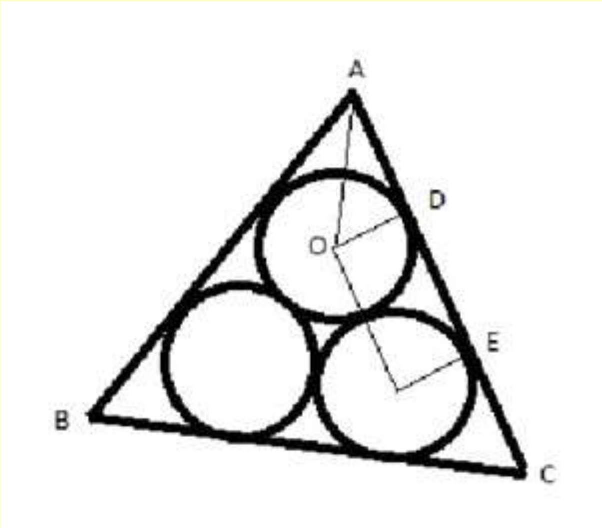
⇨ $f'(x) = 2, -2$

⇨ $\int d\{f(x)\} = 2\int dx, -2\int dx$

⇨ $f(x) = 2x + c_1, -2x + c_2$ (Answer)

3. **Three circles of unit radius tangentially touch each other in the plane. Consider the triangle enclosing them such that each side of the triangle is tangential to two of these three circles. Find the length of each side of the triangle.**

Solution:



Now, ABC must be equilateral.

Angle OAD = 30°

From triangle OAD we have, cot30° = AD/OD

- ⇨ AD = $1\sqrt{3}$ (OD = radius of the circle = 1 given)
- ⇨ AD = $\sqrt{3}$
- ⇨ CE = AD = $\sqrt{3}$

Now, DE = distance between two centres of the circles = 1 + 1 = 2 (As they are tangential)

- ⇨ AC = side of the triangle = AD + DE + CE = $\sqrt{3} + 2 + \sqrt{3} = 2(1 + \sqrt{3})$ (Answer)

4. Let a and b real numbers. Define a function f: R →R, where R denotes the set of real numbers, by the formula $f(x) = x^2 + ax + b$. Assume that the graph of f intersects the co-ordinate axes in three distinct points. Prove that the circle passing through these three points also passes through the point (0, 1).

Solution:

Let the equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$
Now, putting y = 0 in the equation of parabola to get the points on x axis, we get

$x^2 + ax + b = 0$

Now, putting y = 0 in the equation of circle we get, $x^2 + 2gx + c = 0$

Now, both the above two equations have same roots as the circle passes through the points where the parabola cuts the x-axis.

So, comparing the coefficients we get, 2g = a and c = b

So, the equation of the circle becomes, $x^2 + y^2 + ax + 2fy + b = 0$

Now, putting x = 0 in the equation of the parabola we get the point where it cuts the y-axis.

- ⇨ y = b

Now, putting x = 0 in the equation of circle we get, $y^2 + 2fy + b = 0$

Now, y = b is a root of the equation as the circle passes through the point where the parabola cuts the y-axis.

- ⇨ $b^2 + 2fb + b = 0$
- ⇨ $2f = -(b + 1)$

Now, the equation of circle becomes, $x^2 + y^2 + ax - (b + 1)y + b = 0$

Clearly, (0, 1) satisfies the equation of the circle.

⇨ The circle passes through the point (0, 1).

5. Find all positive integer n for which $5^n + 1$ is divisible by 7. Justify your answer.

Solution: Now, $5^n + 1 \equiv 0 \pmod 7$

⇨ $(-2)^n + 1 \equiv 0 \pmod 7$

⇨ $(-1)^n*2^n + 1 \equiv 0 \pmod 7$

We must have $2^n \equiv 1 \pmod 7$ and n must be odd.

Then the above equation is true.

Now, $2^3 \equiv 1 \pmod 7$

And we need to have, $2^n \equiv 1 \pmod 7$

⇨ n must be equal to 3m

And also m is odd as n needs to be odd.

Putting 2m+1 in place of m we get, n = 3(2m + 1) = 6m + 3.

6. Let $P(x) = x^7 + x^6 + b_5x^5 + b_4x^4 + \ldots + b_0$ and $Q(x) = x^5 + c_4x^4 + c_3x^3 + \ldots + c_0$ be polynomials with integer coefficients. Assume that P(i) = Q(i), i = 1, 2, ..., 6. Then, show that there exists a negative integer r such that P(r) = Q(r).

Solution:

Let, F(x) = P(x) – Q(x)

⇨ 1, 2, ..., 6 are roots of F(x).

Now, F(x) is of degree 7.

So, there are 7 roots of F(x).

Let the unknown root be r.

Now, $F(x) = x^7 + x^6 + (b_5 - 1)x^5 + (b_4 - c_4)x^4 + (b_3 - c_3)x^3 + \ldots + (b_0 - c_0)$

Clearly, sum of the roots of F(x) = -1/1

⇨ 1 + 2 + ... + 6 + r = -1

⇨ 6×7/2 + r = -1

⇨ 21 + r = -1

⇨ r = -22 = negative.

7. Let S = {1, 2, 3,….,l}. For every non-empty subset A of S, let m(A) denote the maximum element of A. Then, show that

$\sum m(A) = (l - 1)2^l + 1$

Where the summation in the left hand side of the above question is taken over all non-empty subsets A of S.

Solution:

Let us take any element i of S i.e. i ∈ S.

Now, i will be maximum element of all the subsets if we choose element from the set A = {1, 2, 3, ..., i – 1}.

Now, we can choose 0 element from A in ${}^{(i-1)}C_0$ way.

We can choose 1 element from A in ${}^{(i-1)}C_1$ way.

We can choose 2 element from A in ${}^{(i-1)}C_2$ ways.

...

....

We can choose (i – 1) element from A in ${}^{(i-1)}C_{(i-1)}$ way.
So, total number of subset where i is maximum element is
${}^{(i-1)}C_0 + {}^{(i-1)}C_1 + {}^{(i-1)}C_2 + \ldots + {}^{(i-1)}C_{(i-1)} = 2^{(i-1)}$.

So, sum of the maximum elements in those sets = $i2^{(i-1)}$

We have to prove that, $\sum i2^{(i-1)}$ (summation running over i = 1 to i = l)
$= (l - 1)2^l + 1$

We will prove it by induction.

For l = 1, LHS = $1\times 2^{(1-1)} = 1$ and RHS = $(1 - 1)\times 2^1 + 1 = 1$

So, this is true for l = 1.

Let this is true for l = n i.e. we have, $\sum i2^{(i-1)}$ (summation running over i = 1 to i = n) = $(n - 1)2^n + 1$

Now, for l = n + 1,

LHS = $\sum i\times2^{(i-1)}$ (summation running over i = 1 to i = n + 1)

= $\sum i\times2^{(i-1)}$ (summation running over i = 1 to i = n) + $(n + 1)\times2^{(n+1-1)}$

= $(n - 1)\times2^{n} + 1 + (n + 1)\times2^{n}$

= $2^{n}(n - 1 + n + 1) + 1$

= $2n\times2^{n} + 1$

= $n\times2^{(n+1)} + 1$

For j = n + 1, RHS = $(n + 1 - 1)2^{(n+1)} + 1 = n\times2^{(n+1)} + 1$

By the principle of induction this is true.

8. 1. Let $m_1 < m_2 < \ldots. < m_k$ be positive integers such that $1/ m_1, 1/m_2, \ldots., 1/m_k$ are in arithmetic progression. Then prove that $k < m_1 + 2$.
2. For any integer k > 0, give an example of a sequence of k positive integers whose reciprocals are in arithmetic progression.

Solution: Try Yourself.