NUMERICAL ANALYSIS

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Write a short note on error due to rounding off numbers.

Rounding Off:- In numerical computations, we come across numbers which have large numbers of digits, and it is necessary to cut them to a useable number of figures. This process of cutting off superfluous digits and retaining as many as desired is called rounding off. Thus, to round off π to three, four, five figures, we have 3·14, 3·142, 3·1416, respectively.

To round off a number to 'n' significant figures, discard all digits to right of the n^{th} place, if the discarded number is

(i) less than half a unit in the n^{th} place leave the n^{th} digit unchanged.

(ii) greater than half a unit in the n^{th} place, add 1 to the n^{th} digit.

(iii) exactly half a unit in the n^{th} place, leave the n^{th} digit unchanged if it is an even number, but increase it by 1 if it is an odd number; in other words, round off so as to leave the n^{th} digit an even number in such case.

Then the rounded number is said to be correct to 'n' significant figures (The digits or figures that are used to express a number are called significant digits or significant figures).

The following numbers are rounded off correctly to four significant figures:

29·63243 becomes 29·63
81·9773 becomes 81·98
67·495 becomes 67·50

Absolute, Relative and Percentage Errors:— If 'A' is the approximate value of a quantity whose true value (not necessarily known) is T, then the absolute error modulas of 'A' is defined as

$$|e| = |A-T|.$$

The absolute error has the dimension of the quantity, and a better measure of error is given by the dimensionless quantity 'r',

$$r = \frac{|A-T|}{|T|}, \text{ or, } \frac{|A-T|}{|A|}.$$

The percentage error is $P = 100r$.

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OPERATORS: — Operator has no magnitude or numerical value but it has some function or meaning with a function.

(a) **Finite Difference Operator:** — Suppose a function $y=f(x)$ is given in a table for the values $x_0, x_0+h, \dots, x_0+nh$, of its argument x, where h is the interval of differencing.
Let us define a quantity $\Delta f(x_0) = f(x_0+h) - f(x_0)$ is called the first difference of $f(x_0)$.

$$\Delta f(x_0) = f(x_0+h) - f(x_0)$$
$$\Delta f(x_0+h) = f(x_0+2h) - f(x_0+h)$$
$$\vdots$$
$$\Delta f(x_0+\overline{n-1}\,h) = f(x_0+nh) - f(x_0+\overline{n-1}\,h),$$

where Δ is called the forward difference operator.

Difference Table: — It is convenient to arrange the tabular values and their differences for increasing values of the argument in what is called a difference table, as follows:

Argument	Entry	Δ	Δ^2	Δ^3	Δ^4
x_0	$f(x_0)$				
		$\Delta f(x_0)$			
x_1	$f(x_1)$		$\Delta^2 f(x_0)$		
		$\Delta f(x_1)$		$\Delta^3 f(x_0)$	
x_2	$f(x_2)$		$\Delta^2 f(x_1)$		$\Delta^4 f(x_0)$
		$\Delta f(x_2)$		$\Delta^3 f(x_1)$	
x_3	$f(x_3)$		$\Delta^2 f(x_2)$		
		$\Delta f(x_3)$			
x_4	$f(x_4)$				

where $x_i - x_{i-1} = h \ \forall\ i = 1(1)4$.

Here, $f(x_0)$ is the leading term, and $\Delta f(x_0), \Delta^2 f(x_0), \dots$ are leading differences.

(b) **Shifting Operator:** — Define the shifting operator 'E' as the operation of increasing the argument by h, so that $Ef(x_0) = f(x_0+h)$; in general, $E^n f(x_0) = f(x_0+nh)$, where n is positive integer.
Also, $E^u f(x) = f(x+uh)$.

Identity Between Δ and E: —

$$\Delta f(x) = f(x+h) - f(x)$$
$$= Ef(x) - f(x)$$
$$= (E-1)f(x).$$

$$\therefore \boxed{\Delta = E-1}$$

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Properties of Δ and E :-

i) $\Delta c f(x) = c f(x+h) - c f(x)$
$= c\Delta f(x)$.

$E\{c f(x)\} = c f(x+h) = c E f(x)$

ii) $\Delta(u_x + v_x + w_x + \cdots\cdots) = (u_{x+h} + v_{x+h} + \cdots\cdots) - (u_x + v_x + \cdots\cdots)$
$= \Delta u_x + \Delta v_x + \Delta w_x + \cdots\cdots$

$E(u_x + v_x + w_x + \cdots\cdots) = E u_x + E v_x + E w_x + \cdots\cdots$

iii) If m be a positive integer, then $\Delta^m(\Delta^n f(x)) = \Delta^{m+n} f(x)$
and $E^m(E^n f(x)) = E^{m+n} f(x)$,
for any positive integer n.

iv) For any positive integer n,

$f(x+nh) = E^n f(x) = (1+\Delta)^n f(x)$

$= \left\{1 + \binom{n}{1}\Delta + \binom{n}{2}\Delta^2 + \cdots\cdots + \binom{n}{n}\Delta^n\right\} f(x)$

$= f(x) + \binom{n}{1}\Delta f(x) + \binom{n}{2}\Delta^2 f(x) + \cdots\cdots + \binom{n}{n}\Delta^n f(x)$ —— (*)

and $\Delta^n f(x) = (E-1)^n f(x)$

$= \left\{E^n - \binom{n}{1}E^{n-1} + \binom{n}{2}E^{n-2} \cdots\cdots + (-1)^n\binom{n}{n}\right\} f(x)$

$= f(x+nh) - \binom{n}{1} f(x+\overline{n-1}h) + \binom{n}{2} f(x+\overline{n-2}h)$
$\cdots\cdots + (-1)^n f(x)$

The above result (*) is true whatever the form of the function so long as n is a positive integer. If n be other than a positive integer we can't adopt the binomial expansion without further investigation.

It is important to note that we can expand $f(x+nh)$ in terms $\Delta f(x)$, $\Delta^2 f(x)$, ... for all forms of the function if n be a positive integer, but for the other values of n only if $f(x+nh)$ is polynomial.

Application of Δ & E operators :—

1) Find u_6, given $u_0 = -3$, $u_1 = 6$, $u_2 = 8$, $u_3 = 12$.

Solution:-

Difference table :

Argument	Entry	Δ	Δ^2	Δ^3
0	-3			
		$\Delta u_0 = 9$		
1	6		$\Delta^2 u_0 = -7$	
		$\Delta u_1 = 2$		$\Delta^3 u_0 = 9$
2	8		$\Delta^2 u_1 = 2$	
		$\Delta u_2 = 4$		
3	12			

Now, $u_6 = E^6 u_0 = (1+\Delta)^6 u_0$

$= [(1 + 6\Delta + 15\Delta^2 + 20\Delta^3) u_0]$

$= u_0 + 6\Delta u_0 + 15\Delta^2 u_0 + 20\Delta^3 u_0$

$= 126.$

2) Find the missing term in the table

x	0	1	2	3	4
y	1	3	9	-	81

Solution:- As four terms are available, we may assume that third differences are constant, and fourth differences are zero. If the function is $y = f(x)$, we assume that

$\Delta^4 f(x) = 0$, we have $\Delta^4 f(0) = 0$.

i.e. $(E-1)^4 f(0) = 0$

$\Rightarrow (E^4 - 4E^3 + 6E^2 - 4E + 1) f(0) = 0$

$\Rightarrow f(4) - 4f(3) + 6f(2) - 4f(1) + f(0) = 0$

$\Rightarrow f(3) = 31.$

EXAMPLES:-

1. Show that $x^n - \binom{n}{1}(x-y)^n + \binom{n}{2}(x-2y)^n + \dots\dots + (-1)^n (x-ny)^n = y^n . n!$

Solution:- Taking the interval of differencing $= y$

LHS $= \left[1 - \binom{n}{1}E^{-1} + \binom{n}{2}E^{-2} + \dots\dots + (-1)^n \binom{n}{n} E^{-n}\right] x^n$

$= (1 - \frac{1}{E})^n x^n$

$= \Delta^n \frac{x^n}{E^n}$

$= \Delta^n (x - ny)^n$

$= ny(\overline{n-1}y)(\overline{n-2}y) \dots\dots (1.y)$

$= n!\, y^n = \text{RHS}$

2. Prove the identity, —

$$f(a)+f(a+h)\,x/1! + f(a+2h)\,x^2/2! + f(a+3h)\,x^3/3! + \cdots$$

$$=\exp[x]\left[f(a)+ x\Delta f(a) + x^2\Delta^2 f(a)/2! + x^3\Delta^3 f(a)/3! + \cdots\right]$$

ANS:-

$$\text{LHS} = f(a) + Ef(a)\cdot x/1! + E^2 f(a)x^2/2! + E^3 f(a)x^3/3! + \cdots$$

$$= \exp[Ex]\, f(a)$$

$$= \exp[x+\Delta x]\, f(a)$$

$$= \exp[x]\, \exp[\Delta x]\, f(a)$$

$$= \exp[x]\left[1+\Delta x/1! + \Delta^2 x^2/2! + \cdots\right] f(a)$$

$$= \exp[x]\left[f(a) + x\Delta f(a) + x^2\Delta^2 f(a)/2! + \cdots\right]$$

$$= \text{RHS}.$$

3. Find $\Delta^n u_x$, where $u_x = e^{ax+b}$.

ANS:- $\Delta u_x = u_{x+h} - u_x$, where h is the interval of differencing.

$$= e^{a(x+h)+b} - e^{ax+b}$$

$$= e^{ax+b}\left(e^{ah}-1\right)$$

$$= \left(e^{ah}-1\right)u_x$$

Again, $\Delta^2 u_x = \Delta(\Delta u_x)$

$$= \Delta\left[\left(e^{ah}-1\right)u_x\right]$$

$$= \left(e^{ah}-1\right)\Delta u_x$$

$$= \left(e^{ah}-1\right)^2 u_x$$

Similarly, $\Delta^n u_x = \left(e^{ah}-1\right)^n u_x$.

4. S.T. $\Delta^n u_{x-n} = u_x - n u_{x-1} + \frac{n(n-1)}{2!}u_{x-2} - \cdots + (-1)^n u_{x-n}$.

ANS:- $\Delta^n u_{x-n} = (E-1)^n u_{x-n} = (E-1)^n E^{-n} u_x$

$$= \left(1-E^{-1}\right)^n u_x$$

$$= \left\{1 - nE^{-1} + \frac{n(n-1)}{2!}E^{-2} + \cdots + (-1)^n E^{-n}\right\} u_x$$

$$= u_x - nE^{-1}u_x + \frac{n(n-1)}{2!}E^{-2}u_x - \cdots + (-1)^n E^{-n}u_x$$

$$= u_x - n u_{x-1} + \frac{n(n-1)}{2!}u_{x-2} - \cdots + (-1)^n u_{x-n}.$$

5.(a) S.T. $u_0+u_1+\cdots+u_n = \binom{n+1}{1}u_0 + \binom{n+1}{2}\Delta u_0 + \binom{n+1}{3}\Delta^2 u_0 + \cdots + \Delta^n u_0$

ANS:- $LHS = u_0 + Eu_0 + \cdots + E^n u_0$

$= (1+E+E^2+\cdots+E^n)u_0$

$= \dfrac{(E^{n+1}-1)}{E-1}u_0$

$= \dfrac{(1+\Delta)^{n+1}-1}{\Delta}u_0$

$= \dfrac{1}{\Delta}\left[1+\binom{n+1}{1}\Delta+\binom{n+1}{2}\Delta^2+\cdots+\Delta^{n+1}-1\right]u_0$

$= \binom{n+1}{1}u_0 + \binom{n+1}{2}\Delta u_0 + \cdots + \Delta^n u_0$.

(b) Find the sum of the series: $x^3+(x+1)^3+\cdots+(x+m-1)^3$.

ANS:- The series $= \{1+E+E^2+\cdots+E^{m-1}\}x^3$

$= \dfrac{E^m-1}{E-1}x^3 = \dfrac{((1+\Delta)^m-1)}{\Delta}x^3$

$= \left\{\binom{m}{1}+\binom{m}{2}\Delta+\binom{m}{3}\Delta^2+\binom{m}{4}\Delta^3\right\}x^3$

since higher order difference of x^3 will be zero.

6. (a) S.T. $u_x = u_{x-1}+\Delta u_{x-2}+\Delta^2 u_{x-3}+\cdots+\Delta^{n-1}u_{x-n}+\Delta^n u_{x-n}$

ANS:- $RHS = (E^{-1}+\Delta E^{-2}+\cdots+\Delta^{n-1}E^{-n})u_x + \Delta^n E^{-n}u_x$

$= \dfrac{E^{-1}\{1-(\Delta E^{-1})^n\}}{1-\Delta E^{-1}}u_x + \Delta^n E^{-n}u_x$

$= \dfrac{E^n-\Delta^n}{E^n(E-\Delta)}u_x + \Delta^n E^{-n}u_x$

$= (1-\Delta^n E^{-n})u_x + \Delta^n E^{-n}u_x$

$= u_x = LHS$.

(b) $u_1x+u_2x^2+u_3x^3+\cdots = \dfrac{x}{1-x}u_1 + \left(\dfrac{x}{1-x}\right)^2\Delta u_1 + \left(\dfrac{x}{1-x}\right)^3\Delta^2 u_1+\cdots$

ANS:- $RHS = \dfrac{x}{1-x}\left\{1+\dfrac{x}{1-x}\Delta+\left(\dfrac{x\Delta}{1-x}\right)^2+\cdots\right\}u_1$

$= \dfrac{x}{1-x}\left\{\dfrac{1}{\left\{1-\dfrac{x\Delta}{1-x}\right\}}\right\}u_1$

$= \dfrac{x}{1-x}\cdot\dfrac{(1-x)}{1-x(1+\Delta)}u_1 = \dfrac{x}{1-xE}u_1$

$= x\{1+xE+x^2E^2+\cdots\}u_1 = xu_1+x^2u_2+\cdots$

$= RHS$.

7. Show that, $u_{2n} - \binom{n}{1} 2 u_{2n-1} + \binom{n}{2} 2^2 u_{2n-2} - \cdots + (-2)^n u_n = (-1)^n (c - 2an)$
where $u_x = ax^2 + bx + c$.

ANS:- $LHS = \left\{ E^n - \binom{n}{1} E^{n-1} \cdot 2 + \binom{n}{2} E^{n-2} \cdot 2^2 - \cdots + (-2)^n \right\} u_n$

$$= (E-2)^n u_n$$
$$= (-1)^n (1-\Delta)^n u_n$$
$$= (-1)^n (1-\Delta)^n E^n u_0$$
$$= (-1)^n (1-\Delta^2)^n u_0$$
$$= (-1)^n \{ u_0 - n\Delta^2 u_0 \}$$
$$= (-1)^n \{ u_0 - n (u_2 - 2u_1 + u_0) \}$$
$$= (-1)^n (c - 2an)$$

8. S.T. $U_x - U_{x+1} + U_{x+2} - U_{x+3} + \cdots = \frac{1}{2} \left[U_{x-1/2} - \frac{1}{8} \Delta^2 U_{x-3/2} + \frac{1.3}{2!} \left(\frac{1}{8}\right)^2 \Delta^4 U_{x-5/2} - \cdots \right]$

ANS:- $RHS = \frac{1}{2} \left[1 - \frac{1}{8} \Delta^2 E^{-1} + \frac{1.3.}{2!} \left(\frac{1}{8}\right)^2 \Delta^4 E^{-2} - \frac{1.3.5}{3!} \left(\frac{1}{8}\right)^3 \Delta^6 E^{-4} + \cdots \right] U_{x-1/2}$

$$= \frac{1}{2} \left[1 + \frac{1}{2}\left(-\frac{1}{4} \Delta^2 E^{-1}\right) + \frac{1/2\,(1/2+1)}{2!} \left(-\frac{1}{4}\Delta^2 E^{-1}\right)^2 + \cdots \right] U_{x-1/2}$$
$$= \frac{1}{4} \left(1 + \frac{1}{4} \Delta^2 E^{-1}\right)^{-1/2} U_{x-1/2}$$
$$= \frac{1}{2} \left(1 + \frac{\Delta^2}{4E}\right)^{-1/2} E^{-1/2} U_x$$
$$= (4E + \Delta^2)^{-1/2} U_x$$
$$= (2 + \Delta)^{-1} U_x$$
$$= (1+E)^{-1} U_x$$
$$= (1 - E + E^2 - \cdots) U_x$$
$$= L.H.S.$$

9. Find the value of $\Delta x^m - \frac{1}{2}\Delta^2 x^m + \frac{1.3}{2.4}\Delta^3 x^m - \frac{1.3.5}{2.4.6}\Delta^4 x^m + \ldots\ldots$ to m terms.

Solution:- Since $\Delta^r x^m = 0$ for $r > m$, the sum of the series to m terms is the same as the sum to infinity. The expression $= \Delta\left[1 - \frac{1}{2}\Delta + \frac{\frac{1}{2}(\frac{1}{2}+1)}{2!}\Delta^2 - \frac{\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+2)}{3!}\Delta^3 + \ldots\right]x^m$

$$= \Delta(1+\Delta)^{-1/2} x^m$$

$$= \Delta E^{-1/2} x^m$$

$$= \Delta\left(x - \frac{1}{2}\right)^m$$

$= \left(x+\frac{1}{2}\right)^m - \left(x-\frac{1}{2}\right)^m$, if the interval of differencing be taken as unity.

10. Given $f(0)=1$, $f(1)+f(2)=10$, $f(3)+f(4)+f(5)=65$. find $f(4)=?$

Solution:- Assuming $f(x)$ to be a polynomial of degree 2 in x.

Then $f(x) = a + bx + cx^2$

$f(0) = a = 1$

$f(1) + f(2) = a+b+c+a+2b+4c = 10$

$\Rightarrow 3b + 5c = 8$ ——— (*)

$f(3)+f(4)+f(5) = 65$ gives

$a + 3b + 9c + a + 4b + 16c + a + 5b + 25c = 65$

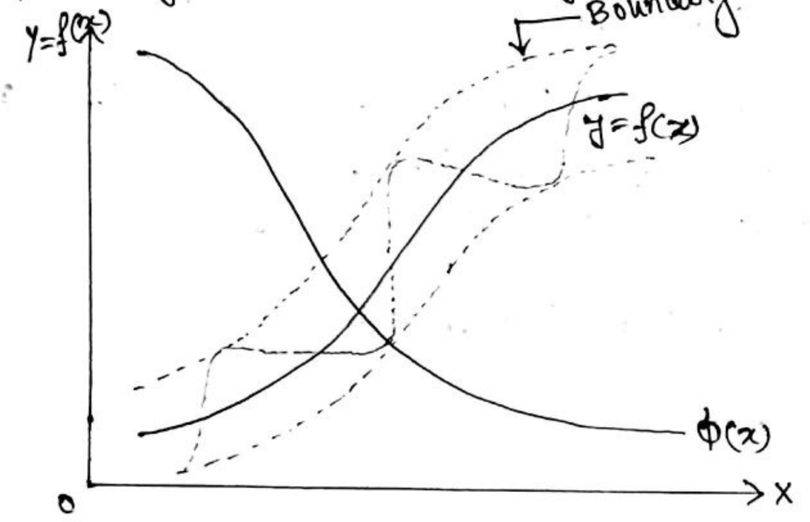
$\Rightarrow 12b + 50c = 62$.

$\Rightarrow 6b + 25c = 62$ ——— (**)

solving (*) & (**), we get, $b =$

Weierstress Theorem :— (Polynomial function to be fitted)

Statement :— Every function which is continuous in an open interval (a,b) can be represented in that interval, to any desired degree of accuracy, by a polynomial. In other words, it is possible to find a polynomial $\phi(x)$ such that $|f(x)-\phi(x)| < \epsilon$, for every values of x in the interval (a,b), where ϵ is any preassigned positive quantity.



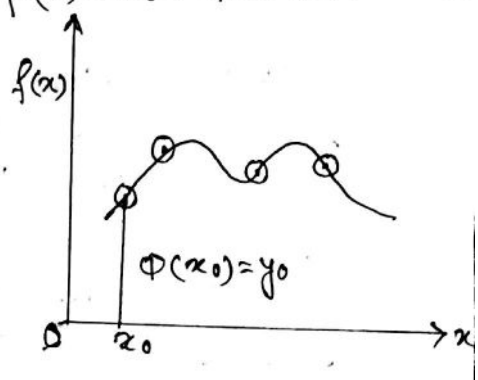
Geometrically, this theorem means that it is possible to find a polynomial whose graph remains with in the region bounded by $y=f(x)+\epsilon$ and $y=f(x)-\epsilon$ for all values $x\in(a,b)$, however small ϵ may be.

Unique :- Data : $(x_i, y_i)\ \forall\ i=0(1)n$, $y_i=f(x_i)$

$\phi(x)$ being a polynomial of degree n, $\phi(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$

$$\underbrace{\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}}_{Y} = \underbrace{\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}}_{\underset{\sim}{a}}$$

$$\Rightarrow \underset{\sim}{a} = A^{-1}Y$$



Result :- For given $(n+1)$ data points (x_i, y_i), $i=0(1)n$, there exists a unique polynomial of degree at most n passing through the $(n+1)$ data points, which implies interpolating polynomial $\phi(x)$ of degree at most n exists and is unique.

Equidistant :— $\phi(x) = a_0+a_1x+a_2x^2+\cdots+a_nx^n$

$$\Delta f(x) = f(x+h) - f(x).$$

Result:- If $f(x)$ is a polynomial of degree n. Then

$$\Delta^n f(x) = \text{constant and}$$
$$\Delta^{n+1} f(x) = 0.$$

Proof:-

Let $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$, $a_n \neq 0$, be a polynomial of degree n.

$$\Delta f(x) = f(x+h) - f(x)$$
$$= \{a_0 + a_1(x+h) + a_2(x+h)^2 + \cdots + a_n(x+h)^n\}$$
$$- \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\}$$
$$= a_1 h + a_2\{(x+h)^2 - x^2\} + \cdots + a_n\{(x+h)^n - x^n\}$$

is a polynomial of degree $(n-1)$.

It gives that an application of Δ on a polynomial reduces its degree by 1.

So, $\Delta^2 f(x)$ is a polynomial of degree $(n-2)$ and so on.

$\Delta^n f(x)$ is a constant, and

$\Delta^{n+1} f(x) = 0$.

Relationship Between Interpolation and Regression :-

Interpolation and regression are founded on different data structure and interrelationship. In interpolation, we assume the functional form $y = f(x)$, where $f(x)$ is a continuous function of x. In regression, we consider the functional form $y = f(x) + \epsilon$, where ϵ is the error term describing the relationship between y and x is an approximate one.

Both in regression and interpolation, we estimate the dependent variable y for a given value of x via fitted function (i.e. the polynomial interpolation formula and regression line).

Explain briefly what is meant by Interpolationship?

ANS:- Interpolation has been defined as the art of reading between the lines of a table, and in elementary mathematics, the term usually denotes the process of computing intermediate values of a function from a set of given or tabular values of that function. But in higher mathematics, we frequently deal with functions whose analytical form is either completely unknown or is of a nature (complicated or otherwise) that the such function can't be easily subjected to such operations as may we required. In either case, it is desirable to replace the given function by another which can readily be handled. this operation of replacing or representing a given function by a simpler one constitutes intepolation in the broad sense of term.

Let $y=f(x)$ be a continuous function given by the set of values $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, where the explicit form of $y=f(x)$ is not known. Let $\phi(x)$ denotes a simpler function so constructed that $f(x)$ and $\phi(x)$ agree at the set of tabular points, i.e. $f(x_i)=\phi(x_i)$ $\forall\ i=1(1)n$. Then if $f(x)$ is replaced by $\phi(x)$ over a given interval, the process constitutes interpolation, and the function $\phi(x)$ is a formula of interpolation.

The function $\phi(x)$ can take a variety of forms. When $\phi(x)$ is a polynomial; the process of representing $f(x)$ by $\phi(x)$ is called polynomial interpolation; and when $\phi(x)$ is finite trigonometric series, the process is called trigonometric interpolation.

As justification, for the approximation of an unknown function by means of polynomial, we state a theorem due to Weirestraess

" If $f(x)$ is continuous in $x_0 \le x \le x_n$, then given any $\epsilon\ (>0)$, there exists a polynomial $\phi(x)$ such that,

$$|f(x) - \phi(x)| < \epsilon \quad \forall\ x \in (x_0, x_n)."$$

(A) Interpolation formulae with equal intervals :-

1) Newton's forward Interpolation formula:—

Data :- $(n+1)$ data points (x_i, y_i), $i = 0(1)n$ and

$x_{i+1} - x_i = h \ \forall \ i = 0(1)n$.

Objective :- Interpolating the unknown function $y = f(x)$

Now, Let us take the nth degree polynomial in the form

$$\phi(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots\cdots + a_n(x-x_0)\cdots(x-x_{n-1})$$

We shall now determine the constants $a_0, a_1, \ldots, a_n$ so as to make $\phi(x_i) = y_i$, $i = 0(1)n$.

Now, $\phi(x_0) = y_0$ or $y_0 = a_0$

Again, $\phi(x_1) = y_1 \Rightarrow a_0 + a_1(x_1 - x_0) = y_1 \Rightarrow a_1 = \dfrac{(y_1 - y_0)}{(x_1 - x_0)} = \dfrac{\Delta y_0}{h}$.

and, $\phi(x_2) = y_2 \Rightarrow a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = y_2$

$$\Rightarrow a_2 = \frac{y_2 - 2y_1 + y_0}{2!\,h^2} = \frac{\Delta^2 y_0}{2!\,h^2} \quad \text{and so on.}$$

Lastly, $\phi(x_n) = y_n$ gives $a_n = \dfrac{\Delta^n y_0}{n!\,h^n}$.

Hence the polynomial interpolation formula becomes

$$\phi(x) = y_0 + (x-x_0)\frac{\Delta y_0}{h} + (x-x_0)(x-x_1)\frac{\Delta^2 y_0}{2!\,h^2} + \cdots\cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})\cdot\frac{\Delta^n y_0}{n!\,h^n}$$

Now, putting $\dfrac{x-x_0}{h} = u$,

$$\phi(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots\cdots + \frac{u(u-1)(u-2)\cdots(u-n+1)}{n!}\Delta^n y_0.$$

This is Newton's forward interpolation formula.
Since the formula contains values of the tabulated function beginning from y_0 forward to the right and none backward; it is called a forward interpolation formula.

2) Newton's backward interpolation formula:

Data:- $(n+1)$ data points (x_i, y_i), $i = 0(1)n$ and

$x_{i+1} - x_i = h \quad \forall\, i = 0(1)n.$

Objective:- Interpolating the unknown function $y = f(x)$.

Here, the approximating polynomial $\phi(x)$ of degree 'n' is taken as

$$\phi(x) = b_0 + b_1(x - x_n) + b_2(x - x_n)(x - x_{n-1}) + \cdots\cdots + b_n(x - x_n)\cdots\cdots(x - x_1)$$

The constants b_i, $i = 0(1)n$ are determined by solving the equations

$$y_i = \phi(x_i), \quad i = 0(1)n.$$

Now, $\phi(x_n) = y_n$ gives $b_0 = y_n$

Again, $\phi(x_{n-1}) = y_{n-1}$ gives $b_0 + b_1(x_{n-1} - x_n) = y_{n-1}$

$$\Rightarrow y_n - b_1 h = y_{n-1}$$

$$\Rightarrow b_1 = \frac{\Delta y_{n-1}}{h}.$$

Again, $\phi(x_{n-2}) = b_0 + b_1(x_{n-2} - x_n) + b_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1}) = y_{n-2}$

$$\Rightarrow y_{n-2} = y_n - \frac{\Delta y_{n-1}}{h} \cdot 2h + b_2 \cdot 2!\, h^2$$

$$\Rightarrow b_2 = \frac{\Delta^2 y_{n-2}}{2!\, h^2}.$$

Lastly, equating $\phi(x_0)$ to y_0, we find

$$b_n = \frac{\Delta^n y_0}{n!\, h^n}.$$

Substituting these values for $b_0, b_1, \ldots\ldots, b_n$, one gets

$$\phi(x) = y_n + (x - x_n)\frac{\Delta y_{n-1}}{h} + (x - x_n)(x - x_{n-1})\frac{\Delta^2 y_{n-2}}{2!\, h^2} + \cdots\cdots$$

$$+ (x - x_n)(x - x_{n-1})\cdots\cdots(x - x_1) \cdot \frac{\Delta^n y_0}{n!\, h^n}.$$

If we put $\frac{x - x_n}{h} = u$, the formula reduces to a simpler form:

$$\phi(x) = y_n + u\Delta y_{n-1} + \frac{u(u+1)}{2!}\Delta^2 y_{n-2} + \frac{u(u+1)(u+2)}{3!}\Delta^3 y_{n-3} + \cdots\cdots$$

$$+ \frac{u(u+1)\cdots\cdots(u+n-1)}{n!}\Delta^n y_0.$$

This is Newton's backward interpolation formula. This contains values of the function beginning from y_n all backward (to the left) and none forward.

Merits:- 1. The newton's formulae can be used only when the given values of the argument are equidistant.

2. When interpolation is required at the end (or beginning) of the table, we use Newton's backward (forward) interpolation formula and it is simple to compute.

3. It is more convenient for hand computation than the Lagrange's formula.

Demerits:- 1. It is only applicable for equal intervals.

2. But sometimes it may be difficult to obtain tabulated values of a function for equidistant values of the argument, to deal with such cases, we require a third formula which can be used even when the values of x are **not** equidistant.

Remark:-

It is known from algebra that the nth degree polynomial which passes through $(n+1)$ points is <u>unique</u>. Hence the various interpolation formula derived here are actually only different forms of the same polynomial. If therefore follows that all the interpolation formulae should give the same functional value.

If $(n+1)$ terms are retained in Newton's forward formula, the polynomial agreeing with $f(x)$ at $x_0, x_1, \ldots, x_n$ is obtained; the retention of $(n+1)$ terms in Newton's backward yields the polynomial agreeing with $f(x)$ at $x_n, x_{n-1}, \ldots, x_{n-n}$. If $(n+1)$ terms are retained in each formula, the two formulae could involve the same ordinates and could yield the same polynomial approximation.

Since the forward formula agrees with $f(x)$ at $x_0, \ldots, x_n$ if we retain $(n+1)$ terms i.e. if we use up to $\Delta^n y_0$ and ignore the higher differences, the formula gives more importance to the beginning of the tabulated values and should be used near the beginning of a table. Similarly, the backward formula should be used near the end of a table.

(B) Interpolation formula with unequal intervals :

Lagrange's Interpolation formula : —

Data:- $(n+1)$ data points (x_i, y_i), $i = 0(1)n$.

Given the function $y = f(x)$, where the values of x need not necessarily be equally spaced.

Objective:- To find an interpolation polynomial formula of degree n, say $\phi(x)$, such that $f(x_i) = \phi(x_i)$ $\forall$ $i = 0(1)n$.
$= y_i$.

Now, let us take the nth degree polynomial in the form

$$\phi(x) = c_0(x-x_1)(x-x_2)\cdots\cdots(x-x_n) + c_1(x-x_0)(x-x_2)\cdots\cdots(x-x_n) + \cdots\cdots + c_n(x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1}) \quad \text{——} (1)$$

$y_0 = \phi(x_0)$ gives, $c_0 = \dfrac{y_0}{(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_n)}$.

In general,

$y_i = \phi(x_i)$ for $i = 1(1)n$.

$$c_i = \frac{y_i}{(x_i-x_0)(x_i-x_1)\cdots\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots\cdots(x_i-x_n)}$$

Putting this values in (1), we get,

$$\phi(x) = \frac{(x-x_1)(x-x_2)\cdots\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\cdots\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}y_1 + \cdots\cdots\cdots + \frac{(x-x_0)(x-x_1)\cdots\cdots\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots\cdots(x_n-x_{n-1})}y_n$$

This is Lagrange's Interpolation formula.

Hence, $$\phi(x) = \sum_{i=0}^{n} \frac{(x-x_0)\cdots\cdots(x-x_{i-1})(x-x_{i+1})\cdots\cdots(x-x_n)}{(x_i-x_0)\cdots\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots\cdots(x_i-x_n)}y_i$$

Merit :- This formula is of more general in nature than Newton's forward and backward interpolation formula because this formula is applicable for any argument values x_i, equidistant or not. It is applicable for any part of the data table as it passes through all the given data points.

Demerit :- On the whole, it is of more labourious. Naturally, where the application of Newton's formulae are possible, it is better not to use this formula.

Problem:-

1. Suppose $(x_0, y_0), \ldots, (x_n, y_n)$ are $(n+1)$ data points of the argument x and entry $y = f(x)$. Then show that the Lagrange's interpolation formula can be represented in the form,

$$\sum_{i=0}^{n} \frac{G(x)}{(x-x_i)G'(x_i)}, \text{ where } G(x) = \prod_{i=0}^{n}(x-x_i) \text{ and } G'(x_i) \text{ is}$$

the value of $G'(x)$ at $x = x_i$.

ANS:- Let the function $y = f(x)$ is given for $(n+1)$ data points (x_i, y_i), $i = 0(1)n$.

Let, $\phi(x)$ be the approximation of $f(x)$ such that $\phi(x_i) = y_i$ $\forall\, i = 0(1)n$, and $\phi(x)$ be the Lagrange's interpolation formula then

$$\phi(x) = \sum_{i=0}^{n} \frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}\, y_i \quad \text{——— (1)}$$

letting, $G(x) = \prod_{i=0}^{n}(x-x_i)$

Then, $\ln G(x) = \sum_{i=0}^{n} \ln(x-x_i)$, then differentiating w.r.t. x,

we get,

$$\frac{G'(x)}{G(x)} = \sum_{i=0}^{n}\frac{1}{(x-x_i)} \Rightarrow G'(x) = \sum_{i=0}^{n}\frac{G(x)}{(x-x_i)}$$

$$\Rightarrow G'(x) = (x-x_1)(x-x_2)\cdots(x-x_n) + (x-x_0)(x-x_2)\cdots(x-x_n) + \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

$$\Rightarrow G'(x_i) = (x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)$$

From (1), the coefficients of y_i is

$$l_i(x) = \frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

$$= \frac{G(x)}{(x-x_i)G'(x_i)}.$$

Hence the Lagrange's formula can be expressed as

$$\phi(x) = \sum_{i=0}^{n}\frac{G(x)}{(x-x_i)G'(x_i)}\, y_i \quad \text{——— } (*)$$

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2. The sum of weights in Lagrange's interpolation formula is unity.

ANS:- We are to consider the following table:

x	$y = f(x) = 1$
x_0	$y_0 = 1$
x_1	$y_1 = 1$
$\vdots$	$\vdots$
x_n	$y_n = 1$

Then the Lagrange's formula gives:-

$$\phi(x) = 1 = \sum_{i=0}^{n} \frac{(x-x_0)\cdots\cdots(x-x_{i-1})(x-x_{i+1})\cdots\cdots(x-x_n)}{(x_i-x_0)\cdots\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

Another Proof:-

Let $\dfrac{1}{G(x)} = \dfrac{A_0}{x-x_0} + \dfrac{A_1}{x-x_1} + \cdots\cdots + \dfrac{A_n}{x-x_n}$, ——— (*)

where $A_0, A_1, \ldots, A_n$ are constants.

On simplification, we get

$$1 = A_0(x-x_1)(x-x_2)\cdots(x-x_n) + A_1(x-x_0)(x-x_2)\cdots(x-x_n) + \cdots\cdots + A_n(x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1}).$$

Putting $x = x_0, x_1, \ldots, x_n$, successively, we get

$$A_0 = \frac{1}{(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_n)} = \frac{1}{G'(x_0)},$$

$$A_1 = \frac{1}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} = \frac{1}{G'(x_1)},$$

$$\vdots$$

$$A_n = \frac{1}{G'(x_n)}.$$

Now, (*) reduces to, $\dfrac{1}{G(x)} = \displaystyle\sum_{i=0}^{n} \frac{A_i}{(x-x_i)} = \sum_{i=0}^{n} \frac{1}{(x-x_i)G'(x_i)}$

$$\Leftrightarrow \sum_{i=0}^{n} \frac{G(x)}{(x-x_i)G'(x_i)} = 1.$$

Inverse Interpolation & Use of Lagrange's formula in Inverse Interpolation :—

It frequently happens that a variable y is given in tabular form as a single valued function of x, say.

$y = f(x)$, and that a value of the independent variable x is required for which the dependent variable y takes on a prescribed value. The process of finding the value of x for a certain value of y is called the inverse interpolation.

Since Lagrange's formula is merely a relation between two variables, either of which may taken as the independent variable, it is evident that considering y as the independent variable, we can write a formula giving x as a function of y. Hence on interchanging x and y in (*), we get

$$\phi(y) = \sum_{i=0}^{n} \frac{G(y)}{(y-y_i)G'(y_i)} x_i$$, which is useful for inverse interpolation.

Uses of Lagrange's Interpolation formula :—

The chief uses of Lagrange's formula are:

1. To find any value of a function when the given values of the independent variable are not equidistant, and
2. to find the value of the independent variable corresponding to a given value of the function;

It is important to note that the Lagrange's formula is applicable to any point of the data table.

Example :- 1. By means of Lagrange's formula, find the value of U_1 given

$$U_0 = 0.4$$
$$U_2 = 0.128$$
$$U_3 = 0.224$$
$$U_4 = 0.376$$

ANS :-

Let us assume that $f(x)$ be a polynomial of degree 3 in x. We now write,

$$u_1 = f(x) = \frac{(1-2)(1-3)(1-4)}{(0-2)(0-3)(0-4)} u_0 + \frac{(1-0)(1-3)(1-4)}{(2-0)(2-3)(2-4)} u_2$$
$$+ \frac{(1-0)(1-2)(1-4)}{(3-0)(3-2)(3-4)} u_3 + \frac{(1-0)(1-2)(1-3)}{(4-0)(4-2)(4-3)} u_4$$

$$= 0.162.$$

2. By means of Lagrange's formula prove that, approximately,
$$u_0 = u_2 - 0.3(u_4 - u_{-4}) + 0.2(u_{-4} - u_{-6}).$$

ANS :- Let us consider u_x to be a polynomial of degree 3.

$$u_0 = \frac{(0+4)(0-2)(0-4)}{(-6+4)(-6-2)(-6-4)} u_{-6} + \frac{(0+6)(0-2)(0-4)}{(-4+6)(-4-2)(-4-4)} u_{-4}$$
$$+ \frac{(0+4)(0+6)(0-4)}{(2+4)(2+6)(2-4)} u_2 + \frac{(0+4)(0+6)(0-2)}{(4+4)(4+6)(4-2)} u_4$$

$$= -0.2u_{-6} + 0.5u_{-4} + u_2 - 0.3u_4$$
$$= u_2 - 0.3(u_4 - u_{-4}) + 0.2(u_4 - u_{-6}).$$

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Errors in Polynomial Interpolation Formula: In the problem of polynomial interpolation with $(n+1)$ data points $(x_0, f(x_0))$, $(x_1, f(x_1)), \ldots, (x_n, f(x_n))$, we approximate a function $f(x)$ by a polynomial $\phi(x)$ of degree at most n in the interval $(x_0, x_n) \ni f(x_i) = \phi(x_i)\ \forall\ i = 0(1)n$. At all other points in this interval (x_0, x_n), $f(x)$ may not be equal to $\phi(x)$. We can write $f(x) = \phi(x) + R(x)$, where $R(x)$ is the error or remainder term.

Now we shall assume that the unknown function $f(x)$ is continuous and posseses continuous derivatives upto order $(n+1)$ within the closed interval $[x_0, x_n]$. To derive remainder term in Newton's forward, backward and Lagrange's formulae (keeping x fixed), we may define an arbitrary function $F(z)$ as

$$F(z) = \left[f(z) - \phi(z)\right] - \left[f(x) - \phi(x)\right]\frac{(z-x_0)(z-x_1)\cdots\cdots(z-x_n)}{(x-x_0)(x-x_1)\cdots\cdots(x-x_n)}.$$

As $F(z)$ satisfies all the conditions of Rolle's theorem in each of the $(n+1)$ sub-intervals, given $x_0, \ldots, x_n$. Hence, $F'(z)$ vanishes at least one time in each of the $(n+1)$ subintervals. Therefore, within the entire interval $F'(z)$ vanishes at least $(n+1)$ times and by repeated application of Rolle's theorem, we can find $F''(z)$ vanishes at least n times and so on. $F^{(n+1)}(z)$ vanishes at least ones in (x_0, x_n), say at ξ, $x_0 < \xi < x_n$.

We may note that the expression $(z-x_0)(z-x_1)\cdots\cdots(z-x_n)$ is a polynomial of degree $(n+1)$ in z with coefficient of $z^{n+1} = 1$. Therefore $(n+1)$th derivative of this expression will be $(n+1)!$

$$\therefore\quad 0 = F^{(n+1)}(\xi)$$

$$= \left[f^{(n+1)}(\xi) - 0\right] - \left(f(x) - \phi(x)\right)\frac{(n+1)!}{(x-x_0)\cdots\cdots(x-x_n)}$$

or, Remainder term, $R_n(x) = f(x) - \phi(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\cdots(x-x_n)$,

where $x_0 < \xi < x_n$.

This is an analytical form of the required remainder term related to Newton's forward, backward and Lagrange's formula.

(a) Newton's forward formula :—

The error term is $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\cdots\cdots(x-x_n)$

In general, the form of $f(x)$ is unknown. When $f^{(n+1)}(\xi)$ does not vary too rapidly in the interval, a useful estimate of the derivative can be obtained as follows:

$$f'(x) \simeq \frac{f(x+h)-f(x)}{h} = \frac{1}{h}\Delta f(x) \Rightarrow D = \frac{1}{h}\Delta \text{ and}$$

$$D^{n+1} = \frac{1}{h^{n+1}}\cdot\Delta^{n+1} \text{ (approximately)}$$

Thus, $f^{(n+1)}(x) \simeq \dfrac{1}{h^{n+1}}\cdot\Delta^{n+1} f(x)$

$$\Rightarrow f^{(n+1)}(\xi) \simeq \frac{1}{h^{n+1}}\Delta^{n+1} f(x_0)$$

Hence the error term in Newton's forward formula to

$$R_n(x) \simeq \frac{u(u-1)(u-2)\cdots\cdots(u-n)}{(n+1)!}\Delta^{n+1} f(x_0), \text{ where } u = \frac{x-x_0}{h}.$$

(b) Newton's Backward formula :— Putting $u = \dfrac{x-x_n}{h}$ in the error term $R_n(x)$, it reduces to

$$R_n(x) = \frac{u(u+1)\cdots\cdots(u+n)}{(n+1)!}\cdot h^{(n+1)} f^{(n+1)}(\xi).$$

Approximating $f^{(n+1)}(\xi)$ by $\dfrac{1}{h^{n+1}}\Delta^{n+1} f(x_0) = \dfrac{1}{h^{n+1}}\nabla^{n+1} f(x_n)$,

the error term is backward formula is,

$$R_n(x) \simeq \frac{u(u+1)\cdots\cdots(u+n)}{(n+1)!}\nabla^{n+1} f(x_n).$$

[$\nabla \equiv 1 - E^{-1}$. $\nabla f(x_n) = f(x_n) - f(x_{n-1}) = \Delta f(x_{n-1})$,

$$\begin{aligned}\nabla^2 f(x_n) = \nabla\{\nabla f(x_n)\} &= \nabla\{f(x_n) - f(x_{n-1})\} \\ &= \Delta f(x_{n-1}) - \Delta f(x_{n-2}) \\ &= \Delta^2 f(x_{n-2}), \text{ etc.}\end{aligned}$$]

Problem 1. Prove that, if 3rd differences are constant,
$$u_x = x u_1 + \frac{x(x^2-1)}{6}\Delta^2 u_0 + y u_0 + \frac{y(y^2-1)}{6}\Delta^2 u_{-1},$$
where $x+y=1$.

Solution:- Since 3rd differences of u_x are constant, u_x is a polynomial of degree 3 and it can be obtained by Lagrange's formula:

$$u_x = \frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)}u_{-1} + \frac{(x+1)(x-1)(x-2)}{(0+1)(0-1)(0-2)}u_0$$
$$+ \frac{(x+1)(x-0)(x-2)}{(1+0)(1-0)(1-2)}u_1 + \frac{(x+1)(x-0)(x-1)}{(2+1)(2-0)(2-1)}u_2$$
$$= -\frac{x(x-1)(x-2)}{6}u_{-1} + \frac{x^3-2x^2-x+2}{2}u_0 + \frac{(2x+x^2-x^3)}{2}u_1 + \frac{x(x^2-1)}{6}u_2 \quad \text{———} (*)$$

Again;
$$x u_1 + \frac{x(x^2-1)}{6}\Delta^2 u_0 + y u_0 + \frac{y(y^2-1)}{6}\Delta^2 u_{-1}$$
$$= x u_1 + \frac{x(x^2-1)}{6}(u_2 - 2u_1 + u_0) + (1-x)u_0 + \frac{(1-x)\{(1-x)^2-1\}}{6} \times \{u_1 - 2u_0 + u_{-1}\}$$
$$= -\frac{x(x-1)(x-2)}{6}u_{-1} + \frac{x^3-2x^2-x+2}{2}u_0 + \frac{(2x+x^2-x^3)}{2}u_1 + \frac{x(x^2-1)}{6}u_2$$
$$= u_x, \text{ from } (*).$$

Problem 2. i) Show that the linear interpolation formula can be expressed in the form
$$\phi(x) = \frac{(x_2-x)f(x_1) + (x-x_1)f(x_2)}{(x_2-x_1)}$$
and the corresponding remainder term $R(x) = f(x) - \phi(x)$ has the following bound:
$$|R(x)| \le \frac{M(x_2-x_1)^2}{8}, \text{ where } |f''(x)| \le M \text{ and } x_1 < x < x_2.$$

ii) A table of natural sines is given with entries for every degree. What is the maximum error of linear interpolation in such a table?

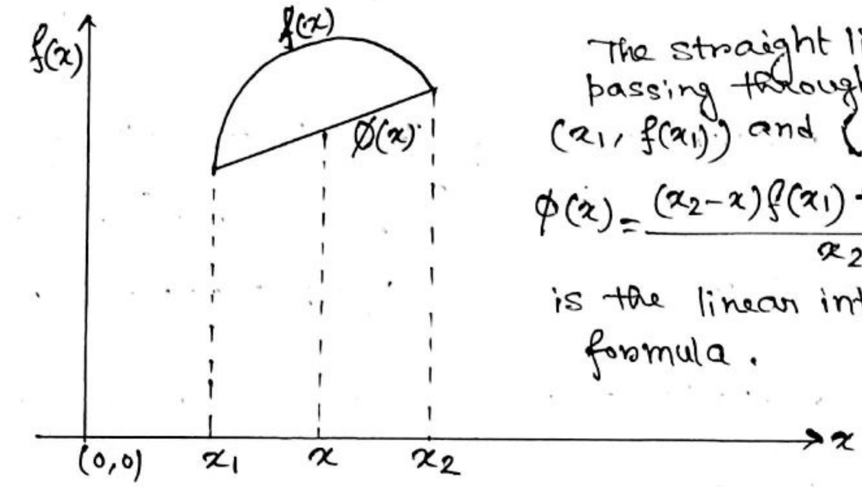
Solution: i) The Lagrange's formula based on the pairs $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is

$$\phi(x) = \frac{(x-x_2)}{(x_1-x_2)} f(x_1) + \frac{(x-x_1)}{(x_2-x_1)} f(x_2)$$

$$= \frac{(x_2-x) f(x_1) + (x-x_1) f(x_2)}{(x_2-x_1)}$$

is the linear interpolation formula.

[Here by linear interpolation formula, we simply mean that the graph of the tabulated function $f(x)$ may be approximated by a straight line $\phi(x)$ with in the interval $[x_1, x_2]$ as shown in figure:



The straight line $\phi(x)$ passing through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is

$$\phi(x) = \frac{(x_2-x) f(x_1) + (x-x_1) f(x_2)}{x_2-x_1}$$

is the linear interpolation formula.]

We know that in polynomial interpolation with $(n+1)$ data points, the remainder term in a formula of degree atmost n is,

$$R(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)(x-x_1)\cdots\cdots(x-x_n) \; ; \; x_0 < \xi < x_n$$

By taking $n=1$, we have the remainder term in the linear interpolation formula

$$R(x) = \frac{f''(\xi)}{2!} (x-x_1)(x-x_2) \; ; \text{ where } x_1 < \xi < x_2$$

$$= \frac{f''(\xi)}{2!} h^2 u(u-1) \qquad \text{———— } (*)$$

where $u = \frac{x-x_1}{h}$, and $h = (x_2-x_1)$.

Now when x lies between x_1 and x_2 then we have $0 < u < 1$, and the maximum value of $|u(u-1)|$ in this interval is $\frac{1}{4}$.

Therefore (*) gives, ———

$$|R(x)| = |f(x) - \phi(x)| \le \frac{f''(\xi)}{2!} \cdot \frac{h^2}{4} = \frac{f''(\xi) h^2}{8}$$

Now, the point ξ is an unknown point in (x_1, x_2), here we are also given $|f''(x)| \le M$, i.e. the second derivative of $f(x)$ is bounded in (x_1, x_2), then the remainder term becomes:

$$\frac{f''(\xi) h^2}{8} \le \frac{M (x_2 - x_1)^2}{8}.$$

$$\Rightarrow |R(x)| \le \frac{M (x_2 - x_1)^2}{8}$$; where $|f''(x)| \le M$ and $x_1 < x < x_2$.

ii) Maximum error of linear interpolation formula is

$\frac{M (x_2 - x_1)^2}{8}$, where $h = (x_2 - x_1)$ is the interval of differencing and $|f''(x)| \le M$ and $x_1 < x < x_2$.

In a table of natural sine $h = x_2 - x_1 = 1^\circ = \left(\frac{\pi}{180}\right)^c$

and $f(x) = \sin x$, $f'(x) = +\cos x$, $f''(x) = -\sin x$.

since $M = \max\{|-\sin x|\} = 1$.

Hence, the maximum error in the linear interpolation is

$$\frac{\left(\frac{\pi}{180}\right)^2}{8} \cdot 1 = \frac{\pi^2}{8 \cdot (180)^2} \quad (\underline{\underline{\text{Ans}}})$$

5) Write a shortnote on Numerical Differentiation.

Numerical Differentiation: — Suppose a function $y = f(x)$, whose analytical form is either unknown or is of such a nature that the mathematical handelling is difficult, is tabulated by a set of data points. On the basis of the data, we are required to calculate the derivative or successive derivative of the function $y = f(x)$ at some value(s) of argument 'x'. The technique that helps us to do this job is known as numerical differentiation.

Numerical differentiation is the process of calculating the derivatives of a function by means of a set of given values of that functions. The problem is solved by representing the function by an interpolation formula and then differentiating this formula as many times as desired.

If the function is given by a table of values for equidistant values of the argument, it should be represented by an interpolation formula employing differences. But if the given values of the function are not for equidistant values of the argument, we must represent the function by Lagrange's formula.

If we desire the derivative at a point near the beginning of a set of tabular values, we use Newton's forward formula; whereas, if we desire the derivative at a point near the end of the table, we use. Newton's backward formula.

Relationship between Δ and D operators :

By Taylor's theorem,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f^{(3)}(x) + \cdots\cdots$$

$$= \left\{1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots\cdots\right\} f(x), \text{ where } D \equiv \frac{d}{dx}.$$

$$= e^{hD} f(x)$$

i.e. $E f(x) = e^{hD} f(x)$

Hence, $E = e^{hD}$

$$\Leftrightarrow hD = \log(1+\Delta)$$

Therefore, $D = \frac{1}{h}\left\{\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots\cdots\right\}$ and also,

$$\boxed{Df(x) = \frac{d}{dx}f(x) = \frac{1}{h}\left\{\Delta f(x) - \frac{\Delta^2}{2}f(x) + \frac{\Delta^3 f(x)}{3} - \frac{\Delta^4 f(x)}{4} + \cdots\cdots\right\}},$$

for tabular values of x.

Since, $D \equiv \frac{1}{h}\left\{\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots\cdots\right\}$, we have

$$D^2 \equiv \frac{1}{h^2}\left\{\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \cdots\cdots\right\}$$

and,

$$D^3 \equiv \frac{1}{h^3}\left\{\Delta^3 - \frac{3}{2}\Delta^4 + \frac{7}{4}\Delta^5 - \cdots\cdots\right\}.$$

Therefore, we have a convenient method for expressing the differential coefficients of a function of x at some tabular value of x in terms of the differences of the function.

▨ The result $\frac{df(x)}{dx} = \frac{1}{h}\left[\Delta f(x) - \frac{\Delta^2 f(x)}{2} + \frac{\Delta^3 f(x)}{3} - \frac{\Delta^4 f(x)}{4} + \cdots\cdots\right]$ can be obtained by the differentiation of the Newton's forward formula:

Consider the Newton's forward formula:

$$f(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \cdots\cdots;$$

where $x = x_0 + uh$.

Then,

$$\frac{d}{dx}f(x) = \frac{df(x)}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{h}\left\{\Delta f(x_0) + \frac{(2u-1)}{2}\Delta^2 f(x_0) + \frac{3u^2-6u+2}{6}\Delta^3 f(x_0) + \frac{4u^3-18u^2+22u+(-6)}{24}\Delta^4 f(x_0) + \cdots\cdots\right\} \quad (*)$$

This formula can be used for computing the value of $\frac{df(x)}{dx}$ for non-tabular values of x.

For tabular values of x, the formula takes a simpler form, for $x = x_0$, we have $u=0$ and hence (*) gives

$$\left.\frac{df(x)}{dx}\right|_{x=x_0} = \frac{1}{h}\left\{\Delta f(x_0) - \frac{\Delta^2 f(x_0)}{2} + \frac{\Delta^3 f(x_0)}{3} - \frac{\Delta^4 f(x_0)}{4} + \cdots\cdots\right\}$$

Differentiating (*) once again, we obtain

$$\frac{d^2 f(x)}{dx^2} = \frac{1}{h^2}\left\{\Delta^2 f(x_0) + \frac{6u-6}{6}\Delta^3 f(x_0) + \frac{12u^2-36u+22}{24}\Delta^4 f(x_0) + \cdots\cdots\cdots\right\}$$

This formula can be used for computing $\frac{d^2 f(x)}{dx^2}$ for non-tabular values of x. Also,

$$\left.\frac{d^2 f(x)}{dx^2}\right|_{x=x_0} = \frac{1}{h^2}\left\{\Delta^2 f(x_0) - \Delta^3 f(x_0) + \frac{11}{12}\Delta^4 f(x_0) - \cdots\cdots\right\}$$

On changing the origin to x, some tabular value of the argument x, we get,

$$\frac{df(x)}{dx} = \frac{1}{h}\left\{\Delta f(x) - \frac{\Delta^2 f(x)}{2} + \frac{\Delta^3 f(x)}{3} - \frac{\Delta^4 f(x)}{4} + \cdots\cdots\right\}$$

and

$$\frac{d^2 f(x)}{dx^2} = \frac{1}{h^2}\left\{\Delta^2 f(x) - \Delta^3 f(x) + \frac{11}{12}\Delta^4 f(x) \cdots\cdots\cdots\right\}.$$

☒ We can find the derivatives of a tabulated function in exactly the same way by differentiating Lagrange's formula:-

Lagrange's formula:

$$f(x) = \sum_{i=0}^{n} \frac{G(x)}{(x-x_i)G'(x_i)} f(x_i), \text{ where } G(x) = \prod_{i=0}^{n}(x-x_i)$$

Differentiating $f(x)$ w.r.t. x, we get

$$\frac{df(x)}{dx} = \sum_{i=0}^{n} \frac{(x-x_i)G'(x) - G(x)}{(x-x_i)^2} \cdot \frac{f(x_i)}{G'(x_i)}$$

Note that, $\ln G(x) = \sum_{j=0}^{n} \ln(x-x_j)$ and $\frac{G'(x)}{G(x)} = \sum_{j=0}^{n} \frac{1}{(x-x_j)}$

$$\Leftrightarrow G'(x) = G(x)\left\{\sum_{j=0}^{n} \frac{1}{x-x_j}\right\}.$$

Hence,

$$\frac{df(x)}{dx} = \sum_{i=0}^{n} \frac{G(x)\left[(x-x_i)\left\{\sum_{j=0}^{n}\frac{1}{x-x_j}\right\} - 1\right]}{(x-x_i)^2 G'(x_i)} f(x_i)$$

$$= \sum_{i=0}^{n} \frac{G(x)\left\{\sum_{j\neq i}\frac{1}{x-x_j}\right\}}{(x-x_i)G'(x_i)} \cdot f(x_i)$$

$$= \frac{(x-x_1)(x-x_2)\cdots\cdots(x-x_n)\left\{\frac{1}{x-x_1}+\frac{1}{x-x_2}+\cdots+\frac{1}{x-x_n}\right\}}{\{(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_n)\}} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)\cdots\cdots(x-x_n)\left\{\frac{1}{x-x_0}+\cdots+\frac{1}{x-x_n}\right\}}{\{(x_1-x_0)\cdots\cdots(x_1-x_n)\}} f(x_1) + \cdots$$

$$\cdots + \frac{(x-x_0)\cdots\cdots(x-x_{n-1})\left\{\frac{1}{x-x_0}+\cdots\cdots+\frac{1}{x-x_{n-1}}\right\}}{\{(x_n-x_0)\cdots\cdots(x_n-x_{n-1})\}} f(x_n),$$

is a polynomial of degree $(n-1)$.

Putting $x = x_j$, we get

$$\left.\frac{df(x)}{dx}\right|_{x=x_i} = \left\{\frac{1}{x_i-x_0}+\cdots\cdots+\frac{1}{x_i-x_{i-1}}+\frac{1}{x_i-x_{i+1}}+\cdots+\frac{1}{x_i-x_n}\right\} \times f(x_i)$$

$$= f(x_i)\left\{\sum_{j\neq i}\frac{1}{x_i-x_j}\right\}$$

Example:- 1. Prove that $\frac{du_x}{dx} = \frac{1}{h}\left(u_{x+h} - u_{x-h}\right) - \frac{1}{2h}\left(u_{x+2h} - u_{x-2h}\right) + \frac{1}{3h}\left(u_{x+3h} - u_{x-3h}\right) - \cdots\cdots$

Solution:-

$$\text{RHS} = \frac{1}{h}\left[(E - E^{-1}) - \frac{1}{2}(E^2 - E^{-2}) + \frac{1}{3}(E^3 - E^{-3}) - \cdots\cdots\right]u_x$$

$$= \frac{1}{h}\left[\left(E - \frac{1}{2}E^2 + \frac{1}{3}E^3 - \cdots\right) - \left(E^{-1} - \frac{E^{-2}}{2} + \frac{E^{-3}}{3} - \cdots\right)\right]u_x$$

$$= \frac{1}{h}\left[\log(1+E) - \log(1+E^{-1})\right]u_x$$

$$= \left\{\frac{1}{h}\log E\right\}u_x$$

$$= \frac{1}{h}\left\{\log\left(e^{hD}\right)\right\}u_x$$

$$= Du_x$$

$$= \frac{d}{dx}(u_x) \text{ , since } E \equiv e^{hD}.$$

2. Prove that $\frac{du_x}{dx} = \frac{2}{3}\left(u_{x+1} - u_{x-1}\right) - \frac{1}{12}\left(u_{x+2} - u_{x-2}\right)$, approximately.

Solution:- Since the formula for $\frac{du_x}{dx}$ is based on four values $u_{x-2}, u_{x-1}, u_{x+1}, u_{x+2}$, so u_x can be approximated by a third degree polynomial. Let, $u_x = a + bx + cx^2 + dx^3 + ex^4$

Note that, $\left.\frac{du_x}{dx}\right|_{x=0} = b$ and

$$\frac{2}{3}(u_1 - u_{-1}) - \frac{1}{12}(u_2 - u_{-2}) = \frac{2}{3}(2b + 2d) - \frac{1}{12}(4b + 16d)$$
$$= b.$$

Hence, $\frac{2}{3}(u_1 - u_{-1}) - \frac{1}{12}(u_2 - u_{-2}) = \left.\frac{d}{dx}(u_x)\right|_{x=0}$

Changing the origin, we get $\frac{du_x}{dx} = \frac{2}{3}(u_{x+1} - u_{x-1}) - \frac{1}{12}(u_{x+2} - u_{x-2})$ holds approximately. When u_x is a polynomial of degree 4, then this formula gives exact result.

3. Prove that $\frac{du_x}{dx} = \frac{u_{x+h} - u_{x-h}}{2h}$, approximately.

Solution:- Let $u_x = a + bx + cx^2$

Then $\frac{du_x}{dx} = b + 2cx$ and $\frac{u_{x+h} - u_{x-h}}{2h} = b + 2cx$

So, $\frac{du_x}{dx} = \frac{u_{x+h} - u_{x-h}}{2h}$, this formula holds exactly for quadratic functions u_x, for other functions, this gives approximate values of $\frac{du_x}{dx}$.

4. The first differences of the first differential coefficient of $\log u_x$ are in G.P. Determine the form of u_x.

Solution:- Since $\Delta \frac{d}{dx} \log u_x$'s are in GP, hence we take

$$\Delta \frac{d}{dx} \log u_x = a.r^x$$

Note that, $\Delta r^x = r^{x+1} - r^x = r^x(r-1)$

$$\Rightarrow \Delta\left\{\frac{r^x}{r-1}\right\} = r^x.$$

$$\therefore \Delta \frac{d}{dx} \log u_x = \Delta\left\{\frac{a}{r-1} \cdot r^x\right\} \Rightarrow \frac{d}{dx}\log u_x = \frac{ar^x}{r-1} + k$$

Integrating, $\log u_x = \frac{a}{r-1} \cdot \frac{r^x}{\log_e r} + kx + k_1$

$$\therefore u_x = e^{k^* \cdot r^x + kx + k_1}, \text{ where } k^* = \frac{a}{r-1} \cdot \frac{1}{\log_e r}$$

$$= \alpha e^{kx + k^* r^x}, \quad \alpha = e^{k_1}.$$

Maximum & Minimum value of a tabulated function :—

Let $y = f(x)$ be a tabulated function given by the values (x_0, y_0), (x_1, y_1), ..., with interval of differencing 'h', then, we can approximate the function $y = f(x)$ by the Newton's forward interpolation formula:

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots \quad \text{———} (*)$$

The points of maxima or minima of the tabulated function may be found by equating the first derivative of $y = f(x)$ to zero and solving for x.

Differentiating $(*)$ w.r.t. x, we obtain

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2} \cdot \Delta^2 y_0 + \frac{3u^2 - 6u + 2}{6}\Delta^3 y_0 + \cdots\right]$$

Hence, for points of maxima or minima, $\frac{dy}{dx} = 0$; i.e.

$$\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \left(\frac{3u^2 - 6u + 2}{6}\right)\Delta^3 y_0 + \cdots = 0 \quad \text{———} (**)$$

Now, **determinating** the LHS of $(**)$, for simplicity, after the third difference; i.e., assuming $y = f(x)$ as a cubic in x, and we obtain the second degree equation in u.

$$c_0 + c_1 u + c_2 u^2 = 0, \text{ where } c_0 = \Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0,$$

$$c_1 = \Delta^2 y_0 - \Delta^3 y_0, \quad c_2 = \frac{1}{2}\Delta^3 y_0.$$

The values of x are then found by the relation $x = x_0 + uh$.

Example:– Assume that $y = f(x)$ has an extremum in the interval $[-1, 1]$. Show that Newton's forward interpolation formula upto 2^{nd} difference would provide an approximate value of the extremum value equal to $f(0) - \frac{1}{8}\left[\frac{f(1) - f(-1)}{f(1) - 2f(0) + f(-1)}\right]^2$

In what way would you be sure of this value to be a maximum?

Solution:– Assuming $f(x) = f(-1) + x\,\Delta f(-1) + \frac{x(x-1)}{2}\Delta^2 f(-1)$

For extremum, $\frac{d\,f(x)}{dx} = 0$, i.e. $\Delta f(-1) + \frac{2x-1}{2}\Delta^2 f(-1) = 0$

$$\Rightarrow x = \frac{\Delta^2 f(-1) - 2\Delta f(-1)}{2\Delta^2 f(-1)} = x^*, \text{say}$$

The extremum value of $f(x)$ is

$$f(x^*) = f(-1) + x^*\left(f(0) - f(-1)\right) + \frac{x^*(x^*-1)}{2}\Delta^2 f(-1)$$

$$= f(0) + (x^*-1)\left(f(0) - f(1)\right) + \frac{x^*(x^*-1)}{2}\Delta^2 f(-1)$$

$$= f(0) + (x^*-1)\left\{\Delta f(-1) + \frac{x^*}{2}\Delta^2 f(-1)\right\}$$

$$= f(0) - \left\{\frac{\Delta^2 f(-1) + 2\Delta f(-1)}{2\Delta^2 f(-1)}\right\}\left\{\Delta f(-1) + \frac{\Delta^2 f(-1) - 2\Delta f(-1)}{4}\right\}$$

$$= f(0) - \frac{\left\{\Delta^2 f(-1) + 2\Delta f(-1)\right\}^2}{8\Delta^2 f(-1)}$$

$$= f(0) - \frac{\left\{f(1) - f(-1)\right\}^2}{8\left\{f(1) - 2f(0) + f(-1)\right\}}$$

Note that, $\frac{d^2 f(x)}{dx^2} = \Delta^2 f(-1)$ and if $\Delta^2 f(-1)$ is negative, then this value is a maximum.

5) <u>Write a short note on Numerical integration.</u>

Numerical integration is the process of computing the value of a definite integral from a set of numerical values of the integrand, when applied to the integration of a function of a single variable, the process is sometimes called mechanical quadrature.

In calculating the definite integral $\int_a^b f(x)dx$, sometimes it so happens that the analytical form of the integral $y = f(x)$ is either totally unknown or is such of a complicated nature that the calculation of the definite integral by analytical methods of integration is either difficult or impossible. In such cases, the process of numerical integration helps us to approximate the value of the definite integral.

The problem of numerical integration may be stated as follows:
Given a set of data points $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_n, y_n)$ of a function $y = f(x)$, it is required to compute the value of the definite integral $\int_a^b f(x)dx$. The problem of numerical integration is solved by representing the integrand by a suitable interpolation formula $\phi(x)$ and then integrate $\phi(x)$ between the desired limits a and b. This gives

$\int_a^b f(x)dx \simeq \int_a^b \phi(x)dx$. The assumption under which we work here is,

"both the limits a and b are finite and $f(x)$ is continuous over the closed interval $[a, b]$".

<u>A General Integration formula:</u>~ Different integration formulae can be obtained depending upon the types of interpolation formula used to replace the integral. We therefore derive a general formula for numerical integration using Newton's forward interpolation formula.

<u>Quadrature Problem:</u>— When the process of numerical integration is applied to the integration of a function of a single variable, then we called the process quadrature problem.

Derivation of General Quadrature formula :-

Suppose $y=f(x)$ is known for $(n+1)$ data points: $(x_0,y_0),(x_1,y_1),\ldots\ldots,(x_n,y_n)$; where $x_i - x_{i-1} = h \ \forall\ i=0(1)n$, $x_0=a$, $x_n=b$.

To derive general quadrature formula for $\int_a^b f(x)dx$, we replace $f(x)$ by $\phi(x)$, where,

$$\phi(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots\cdots + \frac{u(u-1)(u-2)\cdots(u-n+1)}{n!}\Delta^n y_0$$

with $u=\dfrac{x-x_0}{h}$; by Newton's forward interpolation formula.

Now, $$I = \int_a^b f(x)dx = \int_{x_0}^{x_0+nh} f(x)\,dx \simeq \int_{x_0}^{x_0+nh} \phi(x)\,dx$$

$$= h\int_0^n \phi(x_0+hu)\,du$$

$$= h\left[y_0 u + \frac{u^2}{2}\Delta y_0 + \frac{\Delta^2 y_0}{2!}\left(\frac{u^3}{3}-\frac{u^2}{2}\right) + \cdots\cdots\right]_0^n$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{\Delta^2 y_0}{2!}\left(\frac{n^3}{3}-\frac{n^2}{2}\right) + \cdots\cdots\right]$$

This formula is called "<u>General Gauss Legendre quadrature formula</u>" for equidistant ordinants.

From this general formula, we can obtain a variety of quadrature formulae by putting $n=1,2,3,\ldots\ldots$, etc.

☒ Now, we shall discuss two useful formulae of Numerical integration;

1) Trapezoidal Rule.

2) Simpson's 1/3rd Rule.

(1) **TRAPEZOIDAL RULE :-** Here we assume that the integrand $y = f(x)$ is such that it can be well represented by a straight line in any interval of width 'h'. This means $f(x)$ can be represented or replaced by a first-degree polynomial.

Then, from Newton's forward interpolation formula,

$y = f(x) = y_0 + u\Delta y_0$, as far as first differences, where $u = \frac{x - x_0}{h}$.

Then,
$$\int_{x_0}^{x_0+h} f(x)dx = h\int_0^1 [y_0 + u\Delta y_0]\, du$$

[since, $x = x_0 + uh$
$dx = h\,du$

x	x_0	x_0+h
u	0	1

]

$$= h\left[uy_0 + \frac{u^2}{2}\Delta y_0\right]_0^1$$
$$= h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$
$$= \frac{h}{2}[y_0 + y_1]$$

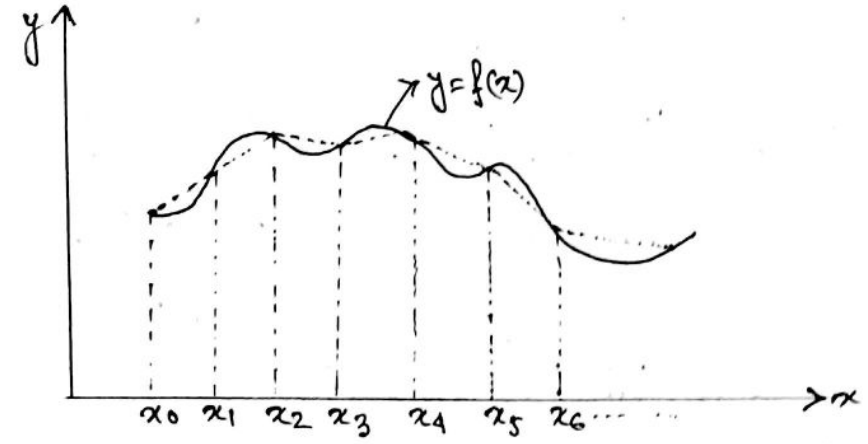
Similarly,
$$\int_{x_0+h}^{x_0+2h} y\,dx = \frac{h}{2}[y_1 + y_2], \ldots\ldots, \int_{x_0+\overline{n-1}h}^{x_0+nh} y\,dx = \frac{h}{2}[y_{n-1} + y_n]$$

Adding all these, we get finally,

$$I = \int_{x_0}^{x_0+nh} y\,dx = \frac{h}{2}[y_0 + y_1] + \frac{h}{2}[y_1 + y_2] + \cdots\cdots + \frac{h}{2}[y_{n-1} + y_n]$$
$$= \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots\cdots + y_{n-1})\right]$$

This is known as Trapezoidal Rule.

- It is useful where h is small, because for any small segment of a smooth curve can be well approximated by a straight line.
- Geometric Significance :- The geometrical significance of this rule is that the curve $y = f(x)$ is replaced by n straight lines segments, joining the pairs of points : (x_0, y_0) & (x_1, y_1) ; (x_1, y_1) & (x_2, y_2); ……; (x_{n-1}, y_{n-1}) & (x_n, y_n). Then the area under the curve $y = f(x)$ between the ordinates $x = x_0$ and $x = x_n$, and the x-axis is approximately equal to the sum of the areas of the n trapeziums obtained.



:- Geometry of Trapezoidal Rule :-

(2) SIMPSON'S ONE-THIRD RULE :- Here we assume that the integrand $y = f(x)$ is such that it can be replaced by a second-degree polynomial over any interval of width '2h'; equivalently, $\Delta^2 f(x)$ can be regarded as a constant.

Therefore, from Newton's forward interpolation formula, we have
$y = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0$, as far as second differences, $u = \frac{x - x_0}{h}$.

Then,
$$\int_{x_0}^{x_0+2h} y\,dx = h\int_0^2 \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0\right] du$$
[since $x = x_0 + uh$
$dx = h\,du$

x	x_0	x_0+2h
u	0	2

$$= h\left[uy_0 + \frac{u^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)\Delta^2 y_0\right]_0^2$$
$$= h\left[2y_0 + 2\Delta y_0 + \frac{1}{2}\left(\frac{8}{3} - 2\right)\Delta^2 y_0\right]$$
$$= h\left[2y_0 + 2(y_1 - y_0) + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$
$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

Similarly,
$$\int_{x_0+2h}^{x_0+4h} y\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$$

and so on. finally we have (assuming n is even)
$$\int_{x_0+\overline{n-2}h}^{x_0+nh} y\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$$

Summing up we get (assuming n is even),

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+\cdots+y_{n-1}) + 2(y_2+y_4+\cdots+y_{n-2})\right]$$

This formula is known as <u>Simpson's 1/3rd Rule.</u>

- This rule is simple with high accuracy and the most useful of all quadrature formulae. In this case, we assume that the entire interval of integration is divided into an even number of subintervals.

- <u>Geometric Significance:</u> — The geometric significance of Simpson's $\frac{1}{3}$rd rule is that we replace the graph of $y=f(x)$ by $\frac{n}{2}$ arcs of second degree polynomials or parabolas with vertical axes passing (or joining) the points: $\{(x_0,y_0),(x_1,y_1),(x_2,y_2)\}$; $\{(x_2,y_2),(x_3,y_3),(x_4,y_4)\}$; ..... ; $\{(x_{n-2},y_{n-2}),(x_{n-1},y_{n-1}),(x_n,y_n)\}$
Then the area under the curve $y=f(x)$ between the ordinates $x=x_0$ and $x=x_n$, and the x-axis is equal to the sum of the areas under the $\frac{n}{2}$ parabolas.

⇨ Derive Simpson's 1/3rd rule by the use of the Lagrange's formula.

<u>Solution:-</u> Given u_0, u_1, u_2; u_x can be expressed in terms of u_0, u_1, u_2 as

$$u_x = \frac{(x-1)(x-2)}{(0-1)(0-2)}u_0 + \frac{x(x-2)}{1(1-2)}u_1 + \frac{x(x-1)}{2\cdot(2-1)}u_2$$

$$= \frac{1}{2}(x^2-3x+2)u_0 - (x^2-2x)u_1 + \frac{1}{2}(x^2-x)u_2$$

Therefore,

$$\int_0^2 u_x\,dx = \left[u_0\left(\frac{x^3}{6}-\frac{3x^2}{4}+x\right) + u_1\left(x^2-\frac{x^3}{3}\right) + u_2\left(\frac{x^3}{6}-\frac{x^2}{4}\right)\right]_0^2$$

$$= \frac{1}{3}\left[u_0 + 4u_1 + u_2\right]$$, <u>which is Simpson's rule.</u>

"When the integrand is a polynomial of degree one, two or three; Simpson's $\frac{1}{3}$rd rule gives the exact value of the definite integral" – comment with justification.

OR

"Simpson's $\frac{1}{3}$rd rule is true to one more order of differences than it was originally assumed" – Comment with justification.

Solution:– Let $u_x = a+bx+cx^2+dx^3$, then $\int_{-1}^{1} u_x dx = \int_{-1}^{1}(a+bx+cx^2+dx^3)dx$

$$= 2a + \frac{2c}{3}.$$

Also, $\frac{1}{3}\left[4u_0 + (u_{-1}+u_1)\right]$

$$= 2a + \frac{2c}{3}.$$

Hence, $\int_{-1}^{1} u_x dx = \frac{1}{3}\left[u_{-1}+u_1+4u_0\right]$

$\Leftrightarrow \int_{0}^{2} u_x dx = \frac{1}{3}\left[u_0+u_2+4u_1\right]$ gives the exact result

if $u_x = a+bx+cx^2+dx^3$.

Note that, $u_x = a+bx+cx^2+dx^3$ is a polynomial of degree

(i) three if $d \neq 0$

(ii) two if $c \neq 0, d=0$

(iii) one if $b \neq 0, c=d=0$.

Hence, Simpson's rule is true for one more order of differences than it was originally assumed, i.e. to third differences.

In other words, Simpson's rule give exact result for polynomials of degree less than or equal to three.

Example:– Obtain the approximate formula $\int_{-1}^{1} u_x dx = \frac{13(u_{-1}+u_1)-(u_3+u_{-3})}{12}$ showing up to what order of differences it holds.

Solution:– since the four values of u_x are given, u_{-1}, u_1, u_{-3}, u_3; then u_x can be taken as a polynomial of degree 3.

Let, $u_x = a+bx+cx^2+dx^3$

Now, $\int_{-1}^{1} u_x dx = \int_{-1}^{1}(a+bx+cx^2+dx^3)dx = \left[ax+\frac{bx^2}{2}+\frac{cx^3}{3}+\frac{dx^4}{4}\right]_{-1}^{1}$

$$= 2a + \frac{2c}{3}$$, and

$$\frac{13(u_{-1}+u_1)-(u_{-3}+u_3)}{12} = \frac{13(2a+2c)-(2a+18c)}{12}$$

$$= 2a + \frac{2c}{3}.$$

Hence, $$\int_{-1}^{1} u_x\,dx = \frac{13(u_{-1}+u_1)-(u_{-3}+u_3)}{12}$$

The formula holds exactly if u_x is a polynomial of degree ≤ 3 and hence for the other function, the formula gives an approximate value of $\int_{-1}^{1} u_x\,dx$.

PROBLEM. 1. If u_x is of the form $(a+bx+cx^2)$, find a formula for $\int_0^1 u_x\,dx$ in terms of u_0, u_1, u_2.

Solution:- The interpolation formula which involves the terms u_0, u_1, u_2 is

$$u_x = u_0 + x\Delta u_0 + \frac{x^2-x}{2}\Delta^2 u_0$$

$$\therefore \int_0^1 u_x\,dx = \left[x u_0 + \frac{x^2}{2}\Delta u_0 + \left(\frac{x^3}{6}-\frac{x^2}{4}\right)\Delta^2 u_0\right]_0^1$$

$$= u_0 + \frac{1}{2}\Delta u_0 - \frac{1}{12}\Delta^2 u_0$$

$$= u_0 + \frac{1}{2}(u_1-u_0) - \frac{1}{12}(u_2-2u_1+u_0)$$

$$= \frac{1}{12}\left[5u_0+8u_1-u_2\right]$$, which is the required formula.

Problem 2. If $u_x = a+bx+cx^2$, derive a formula for $\int_0^1 u_x\,dx$ in terms of u_{-1}, u_1, u_2 using Lagrange's formula.

Solution:- Since u_x is a quadratic, it can be expressed by Lagrange's formula as

$$u_x = \frac{(x-1)(x-2)}{(-1-1)(-1-2)}u_{-1} + \frac{(x+1)(x-2)}{(1+1)(1-2)}u_1 + \frac{(x+1)(x-1)}{(2+1)(2-1)}u_2$$

$$= \frac{1}{6}(x^2-3x+2)u_{-1} - \frac{1}{2}(x^2-x-2)u_1 + \frac{1}{3}(x^2-1)u_2.$$

$$\therefore \int_0^1 u_x\,dx = \left[\frac{1}{6}\left(\frac{x^3}{3}-\frac{3x^2}{2}+2x\right)u_{-1} - \frac{1}{2}\left(\frac{x^3}{3}-\frac{x^2}{2}-2x\right)u_1 + \frac{1}{3}\left(\frac{x^3}{3}-x\right)u_2\right]_0^1$$

$$= \frac{1}{36}\left[5u_{-1}+39u_1-8u_2\right]$$

Problem 3. If $\int_0^m f(x)\,dx = a_1 f(x_1) + a_2 f(x_2) + \cdots + a_n f(x_n)$, where $f(x)$ is a polynomial in x of the $(n-1)^{th}$ degree $(n>2)$, prove that

$$3\sum_{r=1}^{n} a_r x_r^2 = m^3.$$

Solution:- Consider an ordinary polynomial in x of degree $(n-1)$:

$$f(x) = \sum_{i=0}^{n-1} l_i x^i \; ; \; i = 0(1)\overline{n-1}, \text{ where } l_i \in R$$

Then
$$\int_0^m \left(\sum_{i=0}^{n-1} l_i x^i \right) dx = \sum_{r=1}^{n} a_r f(x_r)$$

$$\Rightarrow \sum_{i=0}^{n-1} l_i \left(\int_0^m x^i dx \right) = \sum_{r=1}^{n} a_r \left(\sum_{i=0}^{n-1} l_i x_r^i \right)$$

$$\Rightarrow \sum_{i=0}^{n-1} l_i \left[\frac{x^{i+1}}{i+1} \right]_0^m = \sum_{i=0}^{n-1} l_i \left(\sum_{r=1}^{n} a_r x_r^i \right)$$

$$\Rightarrow \sum_{i=0}^{n-1} l_i \frac{m^{i+1}}{i+1} = \sum_{i=0}^{n-1} l_i \left(\sum_{r=1}^{n} a_r x_r^i \right)$$

Equating the coefficient of l_2, we get

$$\frac{m^3}{3} = \sum_{r=1}^{n} a_r x_r^2 \Rightarrow 3\sum_{r=1}^{n} a_r x_r^2 = m^3$$

In general, equating the coefficient of l_K,

$$(K+1)\sum_{r=1}^{n} a_r x_r^K = m^{K+1}, \; K = 0(1)\overline{n-1}.$$

Error terms of Quadrature formulae :

Let $y=f(x)$ be continuous and possess continuous derivatives in $[x_0, x_n]$. Expanding $y=f(x)$ in a Taylor's series around $x=x_0$, we obtain

$$y=f(x)=y_0+(x-x_0)y_0'+\frac{(x-x_0)^2}{2!}y_0''+\frac{(x-x_0)^3}{3!}y_0'''+\cdots\cdots \longrightarrow (*)$$

(a) Trapezoidal rule : Using (*), we have $\int_{x_0}^{x_1} y\,dx$, where

$$\int_{x_0}^{x_1} y\,dx=\int_{x_0}^{x_1}\left[y_0+(x-x_0)y_0'+\frac{(x-x_0)^2}{2!}y_0''+\cdots\cdots\right]dx$$

$$=\left[xy_0+\frac{(x-x_0)^2}{2}y_0'+\frac{(x-x_0)^3}{3.2!}y_0''+\cdots\cdots\right]_{x_0}^{x_1}$$

$$=hy_0+\frac{h^2}{2}y_0'+\frac{h^3}{6}y_0''+\cdots\cdots\cdots,$$ which is the true value of the integral.

(The value of the integral $\int_{x_0}^{x_1} y\,dx$) $-$ (the value of the integral given by the Trapezoidal rule to evaluate $\int_{x_0}^{x_1} y\,dx$) is $E(x_0)=\int_{x_0}^{x_1} y\,dx-\frac{h}{2}[y_0+y_1]$

$$=-\frac{h^3}{12}y_0''+\cdots\cdots\cdots$$

Similarly, the error due to the use of Trapezoidal rule to evaluate $\int_{x_i}^{x_{i+1}} y\,dx$ is $E(x_i)=-\frac{h^3}{12}y_i''+\cdots\cdots\cdots\,;\ i=1(1)\overline{n-1}$.

Hence, the total error to evaluate $\int_{x_0}^{x} y\,dx$ by Trapezoidal rule is

$$E=-\frac{h^3}{12}\left\{y_0''+y_1''+\cdots\cdots+y_{n-1}''\right\}+\cdots\cdots$$

$$\simeq -\frac{h^3}{12}\left\{y_0''+y_1''+\cdots\cdots+y_{n-1}''\right\}$$

Therefore, $|E|\le \frac{nh^3}{12}y''(\xi)$, where $y''(\xi)=\max_{0\le i\le \overline{n-1}}\left\{|y_i''|\right\}$.

(b) Simpson's 1/3rd rule: Let $y=f(x)$ be continuous and posseses continuous derivatives in $[x_0, x_n]$, n is even.
Then Taylor's series expansion gives around $x=x_0$

$$y=f(x)=y_0+(x-x_0)y_0'+\frac{(x-x_0)^2}{2!}y_0''+\cdots \longrightarrow (*)$$

From (*), we get,

$$\int_{x_0}^{x_2} y\,dx = \left[xy_0+\frac{(x-x_0)^2}{2}y_0'+\frac{(x-x_0)^3}{3!}y_0''+\cdots\right]_{x_0}^{x_2}$$

$$=2hy_0+\frac{(2h)^2}{2}y_0'+\frac{(2h)^3}{6}y_0''+\frac{(2h)^4}{24}y_0'''+\cdots\infty$$

$$=2hy_0+2h^2y_0'+\frac{4}{3}h^3y_0''+\frac{2}{3}h^4y_0'''+\cdots\infty$$, which is the true value of the integral.

By Simpson's 1/3rd rule, we get the value of the integral as follows:

$$\int_{x_0}^{x_2} y\,dx = \frac{h}{3}[y_0+4y_1+y_2] = \frac{h}{3}[f(x_0)+4f(x_1)+f(x_2)]$$

$$=\frac{h}{3}\Big[y_0+4\Big\{y_0+hy_0'+\frac{h^2}{2!}y_0''+\frac{h^3}{3!}y_0'''+\frac{h^4}{4!}y_0^{iv}+\cdots\Big\}$$

$$+\Big\{y_0+(2h)y_0'+\frac{(2h)^2}{2!}y_0''+\frac{(2h)^3}{3!}y_0'''+\frac{(2h)^4}{4!}y_0^{iv}+\cdots\Big\}\Big]$$

$$=2hy_0+2h^2y_0'+\frac{4}{3}h^3y_0''+\frac{2}{3}h^4y_0'''+\frac{5}{18}h^5y_0^{iv}+\cdots \quad [\text{Using } (*)]$$

Hence, the error due to the use of Simpson's Rule to evaluate $\int_{x_0}^{x_2} y\,dx$ is

$$E(x_0)=\int_{x_0}^{x_2} y\,dx - h[y_0+4y_1+y_2]$$

$$=-\frac{h^5}{90}y_0^{iv}+\cdots$$

Hence the total error due to the use of Simpson's rule to evaluate $\int_{x_0}^{x_n} y\,dx$ is

$$E \simeq -\frac{h^5}{90}\left[y_0^{iv}+y_2^{iv}+y_4^{iv}+\cdots+y_{n-2}^{iv}\right]$$

Therefore,

$$|E| \le \frac{nh^5}{180}\max_i\{|y_i^{iv}|\}, \quad i=0,2,\ldots,\overline{n-2}$$

Remark:- Now, if the integrand is a polynomial of degree one, two or three then the y_i^{iv} vanishes for all $i=0,2,\ldots,\overline{n-2}$,

i.e.,

$$E \simeq -\frac{h^5}{90}\times 0 = 0$$

i.e. There will be no error if the integrand is a polynomial of degree one, two or three,

i.e. Simpson's 1/3rd rule gives the exact value of the integration in this case.

Examples:- 1. By a double application of Simpson's rule, derive the formula $\int_{x_0}^{x_2}\int_{y_0}^{y_2} f(x,y)\,dx\,dy = \frac{hk}{9}\left[(f_{00}+f_{02}+f_{20}+f_{22})+4(f_{01}+f_{10}+f_{12}+f_{21}) + 16 f_{11}\right]$

Solution:-

$$\int_{x_0}^{x_2}\left[\int_{y_0}^{y_2} f(x,y)\,dy\right]dx = \int_{x_0}^{x_2} \frac{k}{3}\left[f(x,y_0)+4f(x,y_1)+f(x,y_2)\right]dx$$

$$= \frac{k}{3}\left[\int_{x_0}^{x_2} f(x,y_0)\,dx + 4\int_{x_0}^{x_2} f(x,y_1)\,dx + \int_{x_0}^{x_2} f(x,y_2)\,dx\right]$$

$$= \frac{k}{3}\left[\frac{h}{3}\{f_{00}+4f_{10}+f_{20}\} + 4\cdot\frac{h}{3}\{f_{01}+4f_{11}+f_{21}\} + \frac{h}{3}\{f_{02}+4f_{12}+f_{22}\}\right]$$

$$= \frac{hk}{9}\left[(f_{00}+f_{02}+f_{20}+f_{22}) + 4(f_{01}+f_{10}+f_{12}+f_{21}) + 16 f_{11}\right].$$

2. Obtain an upper bound to the error in evaluating
i) $\int_1^2 \log x\,dx$ by Simpson's Rule, (ii) $\int_0^1 \frac{1}{1+x}dx$ by Trapezoidal Rule.

Solution:-

(i) Here, $y = \log_e x$. The error due to Simpson's $1/3$rd Rule is
$E = -\frac{h^5}{90} y_0^{iv} + \ldots\ldots$. Here, we get $x_0 = 1, x_1 = 1.5, x_2 = 2$; hence,
$h = 0.5 = \frac{1}{2}$. Now, $y^{iv} = -\frac{6}{x^3}$ and $y_0^{iv} = -\frac{6}{x_0^3} = -6$.
Therefore, $E = -\frac{\left(\frac{1}{2}\right)^5}{90}\cdot(-6) = \frac{1}{480}$.

(ii) $y = \frac{1}{1+x}$. The error due to Trapezoidal rule is
$E = -\frac{h^3}{12} y_0'' + \ldots\ldots$. Here, $x_0 = 0$, $x_1 = 1$
$h = 1$, now $y'' = \frac{2}{(1+x)^3}$, $y_0'' = \frac{2}{(1+x_0)^3} = 2$

Therefore, $E = -\frac{(1)^3}{12}(2) = \frac{1}{6}$.

THE NUMERICAL SOLUTION OF EQUATIONS

In scientific studies, a frequently occuring problem is to find the roots of equations of the form $f(x)=0$. If $f(x)$ is a polynomial equation of degree less than or equal to four, then algebraic formulae are available for expressing the roots interms of the coefficients. Algebra is silent, however, on the solution of such types of equation as (i) $4x+5\log x=c$, (ii) $2e^{-x}+3\tan x=5$, etc. These are transcendental equation (a non-algebraic equation), and no general method exists for finding their roots in terms of their coefficients. When the coefficients of such equations are pure numbers, called numerical equations; however, it is always impossible to compute the roots to any desired degree of accuracy.

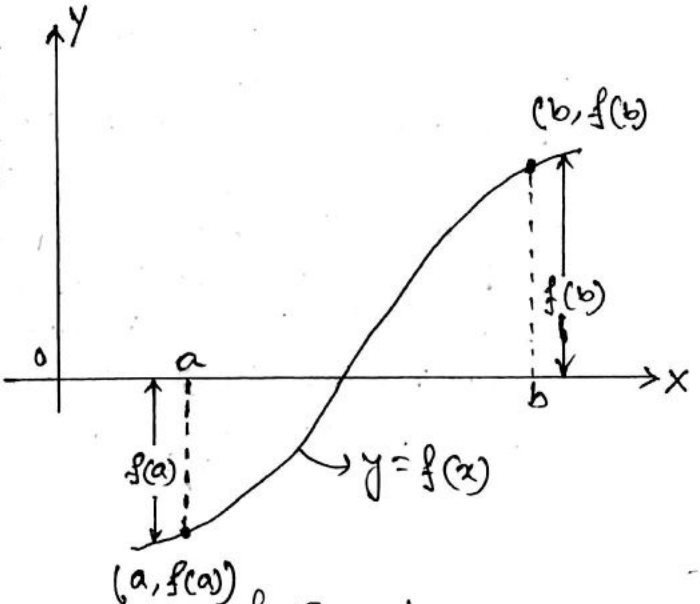
The object of the present chapter is to describe the most useful methods for finding the roots of any numerical equations.

Finding approximate values of the Roots :- In finding the real roots of a numerical equation by any method, it is necessary first to find an approximate value of the root from a graph or otherwise.

Let $f(x)=0$ denotes the equation whose roots are to be found. Then if we take a set of rectangular co-ordinate axes and plot the graph $y=f(x)$, it is evident that abscissae of the points where the graph crosses the x-axis are the real roots of the given equation, for at these points y is zero.

Even more useful and important than a graph is the following fundamental theorem :

" If $f(x)$ is continuous on $[a,b]$ and if $f(a)$ and $f(b)$ have opposite signs, then there is at least one real root between 'a' and 'b'."



In most cases the approximate values of the real root of $f(x)=0$ are most easily found by writing the equation in the form

$$f_1(x)=f_2(x),$$

and then plotting on the same axes the two equations

$$y=f_1(x) \text{ and } y=f_2(x).$$

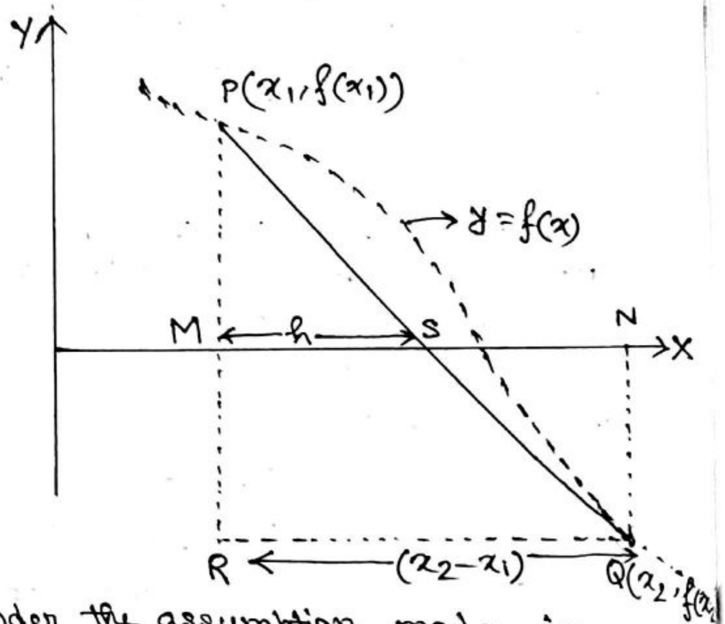
The abscissae of the points of intersection of these two curves are the real roots of the given equation, for at these points $f_1(x)=f_2(x)$ and consequently $f(x)=0$.

A The method of False Position (Regula Falsi):

The oldest method for computing the real roots of a numerical equation $f(x)=0$ is the method of false position or regula falsi. In this method we find two numbers x_1 and x_2 such that $f(x_1)$ and $f(x_2)$ have opposite signs. Hence, a real root of $f(x)=0$ must lie between x_1 and x_2.

Since any portion of a smooth curve is practically straight for a short distance, the method of false position is based on the principle that the graph of $y=f(x)$ is a straight line between the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

To derive a formula for computing the roots, let figure represents a magnified view of that part. Then from the similar triangles PMS and PRQ we have



$$\frac{MS}{MP} = \frac{RQ}{RP}$$

$$\Rightarrow \frac{h}{|f(x_1)|} = \frac{(x_2 - x_1)}{|f(x_1)| + |f(x_2)|}$$

$$\Rightarrow h = \frac{(x_2 - x_1)|f(x_1)|}{|f(x_1)| + |f(x_2)|}$$

The value of the desired root, under the assumption made, is

$x' = x_1 + MS = x_1 + h$. Hence, $x' = x_1 + \dfrac{(x_2 - x_1)|f(x_1)|}{|f(x_1)| + |f(x_2)|}$ —— (*)

This value of x is not, however, the true value of the root, because the graph of $y=f(x)$ is not a perfectly straight line between the points P and Q. It is merely a close approximation to the true. If now $f(x')$ and $f(x_2)$ are of opposite signs, then the root lies between x' and x_2, and we replace x_1 by x' in (*) and obtain the next approximation, otherwise x_2 by x' in (*) and obtain the next approximation. We shall repeat this process a number of times till we get the root correct up to the desired number of decimal places.

[B]. The method of Iteration :

<u>Ques</u>:- (a) Describe the method of fixed point iteration.

(b) Derive the condition for convergence of the iteration process.

(c) Give a geometrical interpretation of the procedure.

<u>Solution</u>:-

(a) <u>Fixed point Iteration process</u>: When a numerical equation $f(x)=0$ can be expressed in the form,

$$x = \phi(x) \quad \text{------} \langle i \rangle$$

The real roots can be found by the process of iteration.

The process is finite: We find from a graph or otherwise an approximate value x_0 of the desired root. We then substitute this in the RHS of $\langle i \rangle$ and get a better approximation x_1, given by the equation

$$x_1 = \phi(x_0)$$

the succeeding approximations are

$$x_2 = \phi(x_1)$$
$$x_3 = \phi(x_2)$$
$$\vdots$$
$$x_n = \phi(x_{n-1})$$

We shall repeat this process until we get root correct to desired number of places.

(b) <u>A sufficient condition for convergence of the Iteration Process</u>:

We shall now determine the condition underwhich the iteration process converges. The true value of the root satisfies the equation

$x = \phi(x)$, and the first approximation satisfies,

$x_1 = \phi(x_0)$; Therefore $x - x_1 = \phi(x) - \phi(x_0)$

$\Rightarrow (x - x_1) = (x - x_0)\phi'(\xi_0)$; $x_0 \le \xi_0 \le x$, by themean value theorem $\dfrac{\phi(x)-\phi(x_0)}{x-x_0} = \phi'(\xi_0)$

similarly, for the n^{th} approximation,

$$(x - x_n) = (x - x_{n-1})\phi'(\xi_{n-1}) \;,\; x_{n-1} \le \xi_{n-1} \le x.$$

Multiplying all these equations together, we get

$$(x-x_1)(x-x_2)\cdots\cdots(x-x_n) = (x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1})\phi'(\xi_0)\cdots\cdots\phi'(\xi_{n-1})$$

$$\Rightarrow (x - x_n) = (x-x_0)\phi'(\xi_0)\,\phi'(\xi_1)\cdots\cdots\phi'(\xi_{n-1})$$

If $|\phi'(\xi_i)| \le M < 1$, for all $i = 0(1)\overline{n-1}$, then we get

$$|x - x_n| \le |x - x_0| M^n$$

Since $0 \le M < 1$, as $n \to \infty$, $M^n \to 0$ and x_n converges to the true root 'x'.

Hence, a sufficient condition for convergence of an iteration process $x = \phi(x)$ is that $|\phi'(x)| < 1$ in the neighbourhood of the desired root containing the initial approximation x_0.

Convergence Factor or Rate of Convergence of iteration Process:-

We define the convergence factor ρ_n as the ratio of the error in x_{n+1} to the error in x_n. We have

$$x - x_{n+1} = \phi(x) - \phi(x_n)$$
$$= (x - x_n)\,\phi'(\xi_n), \quad x_n < \xi_n < x.$$

If x_n is near to the true root 'x', then $\phi'(\xi_n) \simeq \phi'(x)$; and $\rho_n = \dfrac{x - x_{n+1}}{x - x_n} \simeq \phi'(x)$. The number $\phi'(x)$ may be called the asymptotic convergence factor. Since

$$(x - x_{n+1}) = (x - x_n)\,\rho_n$$

, the convergence in iteration is linear convergence.

Hence, the speed of convergence of the iteration depends on the value of $\phi'(x)(\rho_n)$, the smaller the value of $|\phi'(x)|$ the more rapid the convergence, provided $|\phi'(x)| < 1$.

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(c) Geometry of Iteration Process: — It is instructive to look at the geometric picture of the iteration process. We denote the successive approximations to the root by $x_0, x_1, x_2, \ldots$ and then the relations

$x_1 = \phi(x_0)$, $x_2 = \phi(x_1)$, $x_3 = \phi(x_2)$, etc. can be pictured as points by the following geometric construction:
Draw the graphs of $y = x$ and $y = \phi(x)$. Since $|\phi'(x)| < 1$ for convergence, the inclination of the curve $y = \phi(x)$ must be less than $45°$ in the neighbourhood of x_0.

Now to trace the convergence of the iteration process, draw the ordinate $\phi(x_0)$. Then from the point $P_0[x_0, \phi(x_0)]$ draw a line parallel to OX until intersects the line $y = x$ at $Q_1[x_1, \phi(x_0)]$. Note that this point Q_1 is the geometric representation of the 1st iteration equation $x_1 = \phi(x_0)$. Then draw Q_1P_1, P_1Q_2, Q_2P_2, etc., as indicated in the figure.

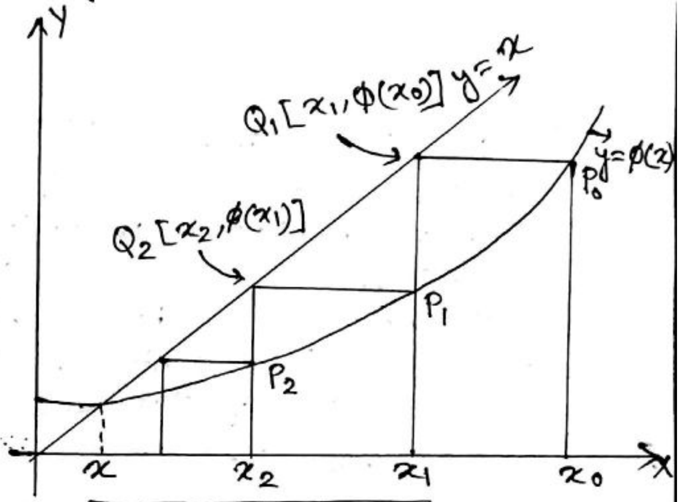


Fig: $0 < \phi'(x) < 1$

Note that the co-ordinates of these Q_i's satisfy the corresponding iteration equations $x_i = \phi(x_{i-1})$. Thus the points $Q_1, Q_2, Q_3, \ldots$ approach the point of intersection of the curves $y = x$ and $y = \phi(x)$, which is the root of $x - \phi(x) = 0 \Leftrightarrow f(x) = 0$.

If $0 < \phi'(x) < 1$, the sequence of iterates will converge to x from one direction but if $-1 < \phi'(x) < 0$, the sequence of iterates oscillate about x with decreasing amplitude and converges to x.

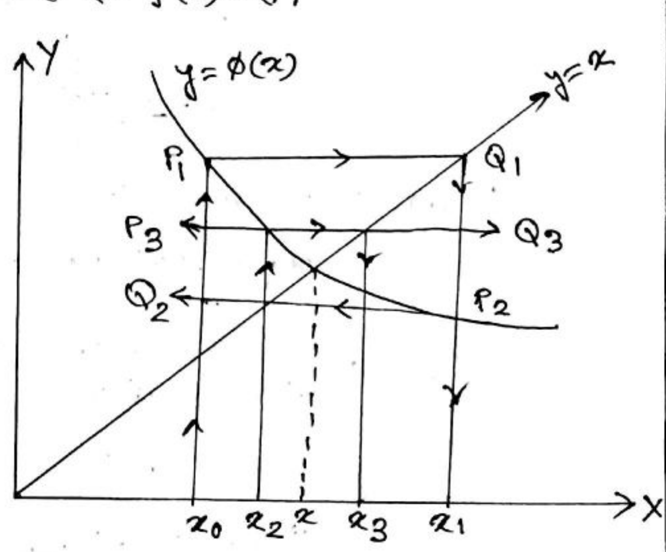


Fig: $-1 < \phi'(x < 0$

C. The Newton Raphson Method :

Ques: (a) Describe the Newton-Raphson method for solution of equations involving one unknown.

(b) Write short notes on,

(i) Geometrical significance of the method.

(ii) Initial approximation of the root.

(iii) Convergence of Newton Raphson method.

(iv) Convergence factor or Rate of convergence.

(v) Disadvantages of this method.

(vi) The inherent error in the Newton-Raphson method.

Solution:-

(a) The Newton Raphson Method: When the derivative of $f(x)$ is a simple expression and easily found, the real roots of $f(x)=0$ can be computed rapidly by a process called the Newton-Raphson method.

Let x_0 be an approximate value of the desired root and h be the correction which must be applied to x_0 in order to get the exact value of the desired root. This gives

$$f(x_0+h)=0$$

$$\Rightarrow f(x_0)+f'(x_0)h+f''(x_0+\theta h)\cdot\frac{h^2}{2!}=0, \quad 0<\theta<1, \text{ expanding by Taylor's theorem.}$$

Now, if h is relatively small, we can neglect the term containing h^2 and we get the simple equation

$$f(x_0)+hf'(x_0)=0$$

$$\Rightarrow h=-\frac{f(x_0)}{f'(x_0)}$$ is an approximation of h as

$$h_1=-\frac{f(x_0)}{f'(x_0)}$$

$\therefore$ The improved value of the root is,

$$x_1=x_0+h_1=x_0-\frac{f(x_0)}{f'(x_0)}.$$

Using x_1 as the value of the desired root we get the second approximation of h as

$$h_2=-\frac{f(x_1)}{f'(x_1)}$$

$$\therefore x_2=x_1+h_2=x_1-\frac{f(x_1)}{f'(x_1)}.$$

On successive approximation, we finally get,

$$x_n=x_{n-1}-\frac{f(x_{n-1})}{f'(x_{n-1})}$$

The process is repeated till we get the root correct upto the desired number of decimal places.

• **Remark:-** The Newton-Raphson formula $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$, can be written in the form $x_n = \phi(x_{n-1})$, where $\phi(x) = \{x - f(x)/f'(x)\}$. Hence the Newton-Raphson method is seen to be a special case of the iteration: $x = \phi(x)$ where $\phi(x) = x - \frac{f(x)}{f'(x)}$.

(b)

(ii) Initial approximation of the root :- Let x_0 be an approximate value of the desired root of an equation $f(x) = 0$ and h be the correction that must be applied to x_0, inorder to get the exact value of the desired root.

$\therefore f(x_0 + h) = 0$

Expanding $f(x_0+h)$ by Taylor's theorem, we get,

$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0 + \theta h) = 0$, where $0 < \theta < 1$.

Since $|h|$ is small, the term containing h^2 is negligible, so, we get,

$$f(x_0) + hf'(x_0) = 0$$

$\Rightarrow h = -\frac{f(x_0)}{f'(x_0)}$, This gives the first approximation of h as $h_1 = -\frac{f(x_0)}{f'(x_0)}$.

The improved value of the root is,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

This procedure is called the "Initial approximation of root".

⟨iii⟩ Convergence of the Newton - Raphson Method :

The Newton Raphson formula $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$ shows that the method is really an iteration method and it can be written as $x_n = \phi(x_{n-1})$, where $\phi(x) = x - \frac{f(x)}{f'(x)}$.

This method of iteration converges if $|\phi'(x)| < 1$ in the neighbourhood of the true root. Note that

$$\frac{d}{dx}\phi(x) = \frac{d}{dx}\left\{x - \frac{f(x)}{f'(x)}\right\} = \frac{f(x)\,f''(x)}{[f'(x)]^2}$$

Therefore, a sufficient condition for convergence of the Newton-Raphson method is $\left|\frac{f(x)\,f''(x)}{[f'(x)]^2}\right| < 1 \Leftrightarrow |f(x)\,f''(x)| < [f'(x)]^2$, in the neighbourhood of the desired true root.

Hence if $f'(x) \neq 0$ and $f''(x)$ is finite on any interval containing the true root of $f(x)=0$, then $\frac{f(x)\,f''(x)}{[f'(x)]^2} \simeq 0$ since $f(x) \simeq 0$ when x is close enough to the true root; that is, the convergence factor $\rho_n = \phi'(x_n) = \frac{f(x_n)\,f''(x_n)}{[f'(x_n)]^2}$ tends to zero when and if x_n tends to the true root.

⟨iv⟩ Convergence factor or rate of convergence :

The convergence factor of the Newton-Raphson method is

$$\rho_n \simeq \phi'(x) = \frac{f(x)\,f''(x)}{[f'(x)]^2}.$$

In order to examine the behaviour of the error $\{x - x_n\}$, we write the equation $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ in the form:

$$x - x_{n+1} = x - x_n - \frac{f(x) - f(x_n)}{f'(x_n)}, \text{ since } f(x) = 0.$$

$$\simeq -\frac{1}{2}(x - x_n)^2 \cdot \frac{f''(x_n)}{f'(x_n)}, \text{ since } f(x) = f(x_n) + (x - x_n)f'(x_n) + \frac{1}{2}(x - x_n)^2 f''(x_n) + \cdots$$

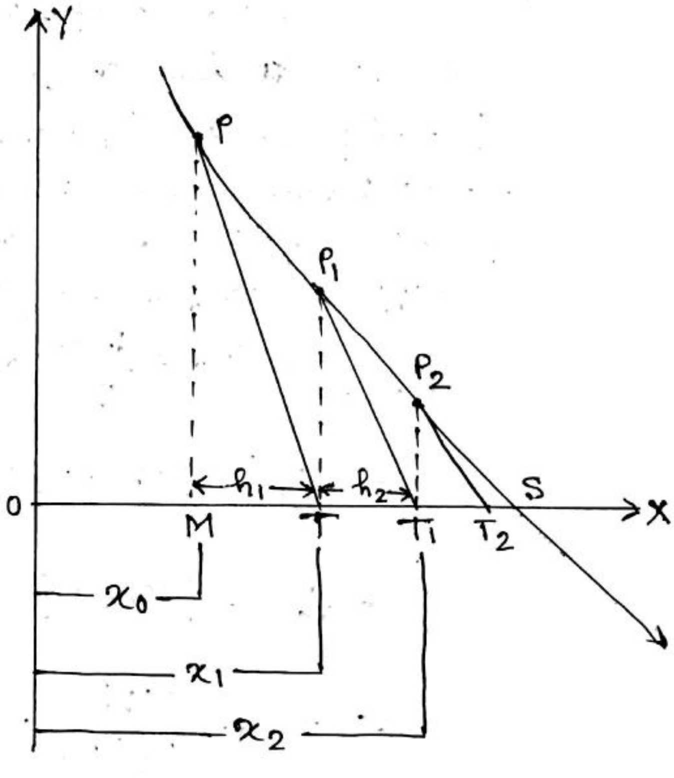
Thus, if the process converges to the true root x, then $x - x_{n+1} \simeq -\frac{f''(x)}{2f'(x)}(x - x_n)^2$; that is, if we let $\epsilon_n = x - x_n$, then $\epsilon_{n+1} \simeq \left\{-\frac{f''(x)}{2f'(x)}\right\}\epsilon_n^2$, for sufficiently large n.

It is important to note that ϵ_{n+1} tends to be proportional to be square of ϵ_n, for large n.

Hence the Newton-Raphson method has a <u>second-order</u> or <u>quadratic convergence</u> rate, whereas the other iteration methods generally has a <u>linear convergence rate</u>.

(i) Geometric Significance of Newton-Raphson Method:

Let the figure represents a magnified view of the graph of $y = f(x)$ where it crosses the x-axis. Suppose we draw a tangent from the point $P(x_0, f(x_0))$. This tangent will intersect the x-axis in some point T. Then we draw another tangent from $P_1(x_1, f(x_1))$, where $x_1 = OT = x_0 + h_1$. This tangent will meet the x-axis in some point T_1 between T and S. Then we may draw a third tangent from $P_2(x_2, f(x_2))$ where $x_2 = x_1 + h_2 = OT_1$, this tangent cutting the x-axis at a point T_2 between T_1 and S, and so on. It is evident that if the curvature of the graph does not change sign between P and S, the points $T, T_1, T_2, \ldots$ will approach the point S as a limit; i.e., the intercepts $OT, OT_1, OT_2, \ldots$ will approach to OS as a limit. But OS represents the true root, hence the lengths $OT_1, OT_2, \ldots$ are successive approximations to the desired root. This is the geometric significance of the Newton-Raphson method.



[From figure, let

$MT = h_1$, $TT_1 = h_2$ and so on.

The slope of the graph at $P(x_0, f(x_0))$ is $f'(x_0)$, but from figure, the slope of the tangent at P

$= \tan(\angle STP)$

$= -\dfrac{f(x_0)}{h_1}$

$\Rightarrow f'(x_0) = -\dfrac{f(x_0)}{h_1}$

$\Rightarrow h_1 = -\dfrac{f(x_0)}{f'(x_0)}$, and,

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$, which is the fundamental formula in Newton-Raphson method.]

Hence, in Newton-Raphson method the graph of the given function is replaced by a tangent at each successive step in the approximation process.

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(V) Disadvantages of this method: In Newton - Raphson method, it is evident that the larger the derivative $f'(x)$, the smaller is the correction which must be applied to get the correct value of the root. This means that when the graph is nearly vertical where it crosses the x-axis the correct root can be found with great rapidly and very little labour. On the other hand, this method should not be used when $f'(x) \simeq 0$ near the desired root.

(vi) The Inherent Error in the Newton-Raphson Method:

If 'x_0' is an approximate value of a root of $f(x)=0$ and h is the necessary correction, so that $f(x_0+h)=0$, then we have

$$0 = f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0+\theta h) \; ; \; 0<\theta<1 \quad (*)$$

In the Newton-Raphson method we neglect the term involving h^2 and got an approximate value 'h_1' from the equation:

$$f(x_0) + h_1 f'(x_0) = 0 \quad (**)$$

subtracting (**) from (*), we have

$$(h-h_1) f'(x_0) + \frac{h^2}{2} f''(x_0+\theta h) = 0$$

Let M denotes the maximum value of $f''(x)$ in the neighbourhood of x_0+h_1. then, we may have

$$Mh^2 + 2f'(x_0)h - 2f'(x_0)h_1 = 0$$

$$\Rightarrow h = \frac{-f'(x_0) + \sqrt{[f'(x_0)]^2 + 2Mf'(x_0)h_1}}{M}$$

$$= \frac{1}{M}\left[-f'(x_0) + f'(x_0)\left(1 + \frac{2Mh_1}{f'(x_0)}\right)^{1/2}\right]$$

$$= \frac{1}{M}\left[-f'(x_0) + f'(x_0)\left\{1 + \frac{Mh_1}{f'(x_0)} - \frac{1}{2}\cdot\frac{M^2h_1^2}{\{f'(x_0)\}^2} + \frac{1}{2}\cdot\frac{M^3h_1^3}{\{f'(x_0)\}^3} - \cdots\right\}\right]$$

$$= h_1 - \frac{Mh_1^2}{2f'(x_0)} + \frac{M^2h_1^3}{2[f'(x_0)]^2} - \cdots$$

Hence, error $= h - h_1 = -\dfrac{Mh_1^2}{2f'(x_0)} + \dfrac{M^2h_1^3}{2[f'(x_0)]^2} - \cdots$

Since h_1 is always a small decimal, it is evident that the principal part of the error is contained in the first term, so that we may neglect the term involving h_1^3.

The formula for the error reduces to $E_1 \le \dfrac{Mh_1^2}{2f'(x_0)}$, and this is an error in $x_1 = x_0 + h_1$

The error in x_n is therefore $E_n \le \dfrac{Mh_n^2}{2f'(x_{n-1})}$.

Simultaneous Equation in Several unknowns :—

Explain briefly how you can extend (i) the Newton-Raphson method, (ii) the method of iteration to the case of two unknowns.

Solution:-

(i) The Newton-Raphson method for simultaneous equations :-

Case of Two Equation in Two Unknowns :-

Let the given equations be $f(x,y)=0$, $g(x,y)=0$. Now if (x_0, y_0) be an initial approximation to the true root and h, k be the corrections, so that

$$x = x_0 + h$$
$$y = y_0 + k$$

Then we have

$$f(x_0+h, y_0+k) = 0$$
$$g(x_0+h, y_0+k) = 0$$

Expanding by Taylor's series, we have

$$f_0 + h\left(\frac{\partial f}{\partial x}\right)_0 + k\left(\frac{\partial f}{\partial y}\right)_0 + \cdots\cdots = 0$$
$$g_0 + h\left(\frac{\partial g}{\partial x}\right)_0 + k\left(\frac{\partial g}{\partial y}\right)_0 + \cdots\cdots = 0$$

; where $\left(\frac{\partial f}{\partial x}\right)_0 = \left(\frac{\partial f}{\partial x}\right)_{(x,y)=(x_0,y_0)}$

and $f_0 = f(x_0, y_0)$, etc. Neglecting the second and higher-order terms of h and k, we obtain

$$h\left(\frac{\partial f}{\partial x}\right)_0 + k\left(\frac{\partial f}{\partial y}\right)_0 = -f_0$$
$$h\left(\frac{\partial g}{\partial x}\right)_0 + k\left(\frac{\partial g}{\partial y}\right)_0 = -g_0$$

Solving these, we find the first correction to be as

$$h_1 = \begin{vmatrix} -f_0 & \left(\frac{\partial f}{\partial y}\right)_0 \\ -g_0 & \left(\frac{\partial g}{\partial y}\right)_0 \end{vmatrix} \Big/ J_0 \quad \text{and} \quad k_1 = \begin{vmatrix} \left(\frac{\partial f}{\partial x}\right)_0 & -f_0 \\ \left(\frac{\partial g}{\partial x}\right)_0 & -g_0 \end{vmatrix} \Big/ J_0$$

where, $J_0 = \begin{vmatrix} \left(\frac{\partial f}{\partial x}\right)_0 & \left(\frac{\partial f}{\partial y}\right)_0 \\ \left(\frac{\partial g}{\partial x}\right)_0 & \left(\frac{\partial g}{\partial y}\right)_0 \end{vmatrix} \neq 0$. Then the first approximations are given by $x_1 = x_0 + h_1$
$y_1 = y_0 + h_2$

Therefore the second corrections to be

$$h_2 = \begin{vmatrix} -f_1 & \left(\frac{\partial f}{\partial y}\right)_1 \\ -g_1 & \left(\frac{\partial g}{\partial y}\right)_1 \end{vmatrix} \Big/ J_1 \quad \text{and} \quad k_2 = \begin{vmatrix} \left(\frac{\partial f}{\partial x}\right)_1 & -f_1 \\ \left(\frac{\partial g}{\partial x}\right)_1 & -g_1 \end{vmatrix} \Big/ J_1$$

where, $J_1 = \begin{vmatrix} \left(\frac{\partial f}{\partial x}\right)_1 & \left(\frac{\partial f}{\partial y}\right)_1 \\ \left(\frac{\partial g}{\partial x}\right)_1 & \left(\frac{\partial g}{\partial y}\right)_1 \end{vmatrix} \neq 0$

The process is to be repeated till we obtain the roots to the desired accuracy.

(ii) The method of Iteration for Simultaneous Equations :

Case of Two Unknowns : Consider two simultaneous equations $f(x,y)=0$, $g(x,y)=0$, we may write these given equations in the equivalent forms,

$$x = F(x,y)$$
$$y = G(x,y)$$

Then, if (x_0, y_0) be the approximations to a pair of roots, improved values are found by the steps indicated below:

1st approx : $x_1 = F(x_0, y_0)$
$y_1 = G(x_0, y_0)$

2nd approx : $x_2 = F(x_1, y_1)$
$y_2 = G(x_1, y_1)$

$\vdots$

nth approx: $x_n = F(x_{n-1}, y_{n-1})$
$y_n = G(x_{n-1}, y_{n-1})$

The process is to be repeated till we obtain the roots to the desired accuracy.

A sufficient condition for convergence of the iteration process is that $\left|\frac{\partial F}{\partial x}\right| + \left|\frac{\partial F}{\partial y}\right| < 1$, $\left|\frac{\partial G}{\partial x}\right| + \left|\frac{\partial G}{\partial y}\right| < 1$ in the neighbourhood of (x_0, y_0), provided the iteration is started with (x_0, y_0) sufficiently near the true root.

The method can obviously be generalised to any number of equations.

Examples:- Show that $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$ is the iteration process of finding the square root of N. Also, find the iteration process of finding the cubic root of an integer.

Solution:- Let $f(x) = x^2 - N$. Note that $f(x)=0 \Rightarrow x = \sqrt{N}$; i.e. the square root of N is a solution of $f(x)=0$.

By the formula of Newton-Raphson method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\{x_n\}^2 - N}{2x_n} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right).$$

Since the Newton-Raphson process is a particular case of iteration process, therefore the iteration formula for the square root of N is

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

Similarly, let $f(x) = x^3 - N$. Then the formula of iteration is

$$x_{n+1} = x_n - \frac{\{x_n\}^3 - N}{3\{x_n\}^2} = \frac{2}{3}x_n + \frac{N}{3(x_n)^2}.$$

Runge-kutta Method :- Useful to find numerical solution of ordinary differential equations.

Let us consider the equation $y = f(x, y)$ with initial values (x_0, y_0).

Let h be the length of the interval between equidistant values of x.

The first increment in y is computed by the following set of formulae

$$k_1 = f(x_0, y_0)\, h$$
$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) h$$
$$k_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) h$$
$$k_4 = f(x_0 + h, y_0 + k_3)\, h$$

Then $\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

and so on for the succeeding intervals.

- **Example:-** Use Runge-kutta method to approximate y, when $x = 0.1$, given that $y = 1$, when $x = 0$ and $\frac{dy}{dx} = x + y$.

Solution:- Here $x_0 = 0$, $y_0 = 1$, $f(x, y) = x + y$; $h = 0.1$

$$k_1 = f(x_0, y_0)\, h = 1 \times 0.1 = 0.1$$
$$k_2 = f(x_0 + 0.05, y_0 + 0.05)\, h = 0.11$$
$$k_3 = f(x_0 + 0.05, y_0 + 0.055)\, h = 0.1(0 + 0.05 + 1.055) = 0.1105$$
$$k_4 = f(x_0 + 0.1, y_0 + 0.1105)\, h = 0.1\{0 + 0.1 + 1 + 0.1105\} = 0.12105$$

$\therefore \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.11034$

$\therefore x_1 = x_0 + h = 0.1$

$\therefore y_1 = y_0 + k = 1.1103$.

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■ State Stirling's approximation to the factorial of a large integer n. Indicate its use with example. Prove it.

Statement:- **Stirling's Approximation**

For large n, $n! \simeq \sqrt{2\pi}\, n^{n+1/2} e^{-n}$, where '$\simeq$' indicates that the ratio of two sides tend to unity when $n \to \infty$.

Uses:- This formula can be used to obtain a good approximation of $n!$ for large values of n. It may be noted that the difference between the two sides of the formula increases if the value of n increases, but it is the percentage that decreases steadily and stirling's approximation is accurate even for small n.

Example:- For Poisson distn., $MD_\mu(X)$ is given by

$$MD_\mu(X) = \frac{2e^{-\lambda}.\lambda^{\lambda+1}}{\lambda!}$$

Using stirling's approximation for large λ,

$$MD_\mu(X) \doteq \frac{2e^{-\lambda}.\lambda^{\lambda+1}}{\sqrt{2\pi}\,\lambda^{\lambda+1/2}.e^{-\lambda}}$$

$$= \sqrt{\frac{2\lambda}{\pi}}$$

So, Stirling's formula provides here a good approximation of $\lambda!$.

Proof:- Consider the sequence $\{d_n\}$ defined by

$$d_n = \log n! - (n+1/2)\log n + n \quad \text{-------- } (*)$$

Now, $d_n - d_{n+1} = -\log(n+1) - (n+1/2)\log n + (n+3/2)\log(n+1) - 1$

$$= (n+\tfrac{1}{2})\log\left(\frac{n+1}{n}\right) - 1$$

$$= \frac{2n+1}{2}\log\left\{\frac{1+(2n+1)^{-1}}{1-(2n+1)^{-1}}\right\} - 1$$

$$= \frac{1}{3(2n+1)^2} + \frac{1}{5(2n+1)^4} + \cdots\cdots \quad \langle 1\rangle$$

Thus $d_n - d_{n+1} \geq 0$ $(n = 1, 2, \ldots)$, therefore $\{d_n\}$ decreases. By comparison of the RHS of (1) with a geometric series having ratios $(2n+1)^{-2}$, one can note that,

$$0 < d_n - d_{n+1} < \frac{1}{3\{(2n+1)^2 - 1\}} = \frac{1}{12n} - \frac{1}{12(n+1)}$$

$$\Rightarrow d_n - \frac{1}{12n} < d_{n+1} - \frac{1}{12(n+1)} \quad \text{———} \langle 2\rangle$$

$\langle 2\rangle$ shows that $\{d_n - \frac{1}{12n}\}$ is increasing. It follows that a finite limit $C = \lim_{n\to\infty} d_n$ exists. But in view of $\langle 2\rangle$, the relation $d_n \to C$ is equivalent to $n! \simeq e^C n^{n+1/2} e^{-n}$ ———$\langle 3\rangle$

Let $e^c = A$, now to complete the proof, we have to evaluate A.
From ⟨3⟩ we get,

$$U_n = \frac{n!}{n^n e^{-n}\sqrt{n}} \longrightarrow A \text{ as } n \to \infty$$

So, the subsequence $\{U_{2n}\}$, where,

$$U_{2n} = \frac{(2n)!}{2n^{2n} e^{-2n}\sqrt{2n}} \text{ converges to } A.$$

Now, $$\left(\frac{U_n^2}{U_{2n}}\right)^2 = \frac{(n!)^4 2^{4n} 2n}{\{2n!\}^2 n^2} \longrightarrow \left(\frac{A^2}{A}\right)^2 = A^2$$

So, $$A^2 = \frac{(n!)^4 2^{4n}}{\{(2n!)\}^2 (2n+1)} \cdot \frac{2n(2n+1)}{n^2}$$

$$= \frac{(n!)^4 2^{4n}}{\{2n!\}^2 (2n+1)} \cdot 2\left\{2 + \frac{1}{n}\right\}$$

$$\to \frac{\pi}{2} \cdot 2 \cdot 2$$ [By Walli's formula as $n \to \infty$
i.e. $\frac{\pi}{2} = \lim_{n\to\infty}\left[\frac{2^{4n}(n!)^4}{\{2n!\}^2(2n+1)}\right]$]

$$= 2\pi$$

$$\therefore A = \sqrt{2\pi}$$

from ⟨3⟩, we have

$$n! \simeq \sqrt{2\pi}\, n^{n+1/2} e^{-n}$$ [Proved]

— This is the required Stirling's formula.

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PRACTICALS ON NUMERICAL ANALYSIS

1. (a) Given that $\sum_{1}^{10} f(x) = 500426$, $\sum_{4}^{10} f(x) = 329240$, $\sum_{7}^{10} f(x) = 175212$, $f(10) = 40365$. Find $f(1)$.

(b) Using the following table, compute the value of the definite integral
$$\int_{0.125}^{0.185} e^x \, dx.$$

x	0.12	0.13	0.14	0.15	0.17	0.18	0.19
e^x	1.127497	1.138828	1.150274	1.161834	1.185305	1.197217	1.20925[illegible]

Solution:- (a) Define,
$$u_1 = \sum_{1}^{10} f(x) = 500426$$
$$u_4 = \sum_{4}^{10} f(x) = 329240$$
$$u_7 = \sum_{7}^{10} f(x) = 175212$$
$$u_{10} = \sum_{10}^{10} f(x) = 40365$$

Argument x	U_x	ΔU_x	$\Delta^2 U_x$	$\Delta^3 U_x$
1				
4				
7				
10				

$$\therefore f(1) = \sum_{1}^{10} f(x) - \sum_{2}^{10} f(x) = u_1 - u_2.$$

To find u_2, we shall apply Newton's forward formula:
$$u_2 = u_1 + k\Delta u_1 + \frac{k(k-1)}{2!}\Delta^2 u_1 + \frac{k(k-1)(k-2)}{3!}\Delta^3 u_1 \text{; where } k = \frac{2-1}{3} = \frac{1}{3}.$$

(b) We know that $\int_{0.125}^{0.185} e^{x}dx = e^{0.185} - e^{0.125}$

Now, $e^{0.125}$ and $e^{0.185}$ will be obtained by applying Newton's forward and backward formular respectively.

Observe that $\rightarrow$ $\Delta^3 f(x) \simeq$ constant $\forall x$.

Then, $f(0.125) = f(0.12) + u\Delta f(0.12) + \frac{u^2-u}{2!}\Delta^2 f(0.12) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(0.1$

where, $u = \frac{0.125-0.12}{0.01}$

2. Let $f(x)$ be the PDF of a R.V. The functional form of $f(x)$ is unknown except for its value for some values of x as shown below:

x:	0	1	1.5
$f(x)$:	0.4179	0.25360	0.1357

You may also assume that the domain of $f(x)$ to be $[-2, 2]$. Find an approximating polynomial for $f(x)$. Suppose further that you are given the information that $f(x)$ is symmetric about '0'. Then make necessary modification and get the approximating polynomial.

Solution:- Given that $f(0) =$

$f(1) =$

$f(1.5) =$

$$\int_{-2}^{2} f(x)dx = 1.$$

From these 4 information, it is possible to find 4 unknowns and with the help of these 4 unknowns, we assume that,

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 ;$$

Now, $f(0) = 0.4179$ gives $a_0 = 0.4179$

$f(1) = a_0 + a_1 + a_2 + a_3 =$ ___

$f(1.5) =$

Now, $\int_{-2}^{2} f(x)dx = \int_{-2}^{2} (a_0 + a_1 x + a_2 x^2 + a_3 x^3) dx = 1$

$\Rightarrow 4a_0 + 16 \cdot \frac{a^2}{3} = 1.$

$\therefore f(x) =$ ________.

For $f(x)$ to be symmetric about '0'.

We have, $f(x) = f(-x) \ \forall x$.

$\Rightarrow f(x)$ contains only even power of x.

$\therefore$ With the help of 4 unknowns,

$f(x) = b_0 + b_1 x^2 + b_2 x^4 + b_3 x^6$

Now, $b_0 = f(0) = 0.4179$

$f(1) =$.

Now, $\int_0^2 f(x)dx = \frac{1}{2}$.

$\Rightarrow$

3) (a) The following table gives values of a function $g(x)$ for a few values of x:

x	1·1	1·2	1·3	1·4	1·5
$g(x)$	0·769	0·472	0·103	−0·344	−0·875

Using an appropriate interpolation formula, obtain the value of x in the interval $[1\cdot1, 1\cdot5]$ for which $g(x) = 0$.

(b) The following table gives the values of the following incomplete normal probability integral $F(x)$ for different values of x. $F(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$

Using this table, find t_0 and t_1 $\ni$ $P[X > t_0] = 0\cdot33$, where $X \sim N(50, 5^2)$ and $P[Y \le t_1] = 0\cdot33$, where $Y \sim \chi_1^2$.

x	0·00	0·05	0·10	......	0·40	0·50	0·60
$F(x)$	0·50000	0·48006	0·46017	......	0·34458	0·30854	0·27425

Solution:- (a) To find x for which $y = g(x) = 0$. This is a problem of inverse interpolation formula.
Here, we shall use Lagrange's inverse interpolation formula:

$$x = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)}x_0 + \frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)}x_1 + \cdots\cdots$$

Here, $y = 0$ is given.

(b) Here to find t_0,

$$P[X > t_0] = 0.33$$

$$\Rightarrow P\left[\frac{X-50}{5} > \frac{t_0-50}{5}\right] = 0.33$$

$$\Rightarrow F\left(\frac{t_0-50}{5}\right) = 0.33$$

Note that $F(x) = 0.33$ is in the last half of the table. To find, t_0 for which $F\left(\frac{t_0-50}{5}\right) = 0.33$, we shall use Lagrange's formula for inverse interpolation based on the last 6 values.

$$x = \frac{(y-y_1)(y-y_2)\cdots\cdots(y-y_5)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_5)} x_0 + \cdots \cdots\cdots$$

Here, $y = 0.33$

$$\therefore \frac{t_0-50}{5} = x = \_\_\_\_\_\_$$

$$\Rightarrow t_0 \doteq \_\_\_\_\_\_$$

To find $t_1 \ni$

$$P[Y \leq t_1] = 0.33, \quad Y \sim \chi_1^2$$

$$P[Z^2 \leq t_1] = 0.33, \quad Z \sim N(0,1)$$

$$P[Z^2 > t_1] = 0.67$$

$$\Rightarrow P[|Z| > t_1] \doteq 0.67$$

$$\Rightarrow 2P[Z > t_1] = 0.67$$

$$\Rightarrow F(t_1) = 0.335$$

Now, find t_1 as done in above.

DIFFERENTIATION

4. (a) The following table gives the values of the probability generating function $P(s)$ of a certain RV X for different values of s.

s	$P(s)$
0.5	0.3679
1.5	2.7183
2.0	7.3891
2.5	20.0855
3.0	54.5981

Find approximate values of $E(X)$ and $V(X)$.

<u>Solution:</u>- We know that $E(X) = P'(1)$ and $E(X(X-1)) = P''(1)$

Then the relationship between D and Δ is

$$D \equiv \frac{1}{h}\log_e(1+\Delta) = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \cdots\cdots\right]$$

Hence $P'(1) = \left.\frac{d}{ds}P(s)\right|_{s=1}$

$$= \frac{1}{h}\left\{\Delta P(1) - \frac{\Delta^2 P(1)}{2} + \frac{\Delta^2 P(1)}{3} \cdots\cdots\right\}$$

Use $P(1) = 1$.

s	$P(s)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5

Here, $h = 0.5$

and $P'(1) =$ ________.

Now, $D^2 \equiv \frac{1}{h^2}\left\{\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \cdots\cdots\right\}^2$

$$\equiv \frac{1}{h^2}\left\{\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \cdots\cdots\right\}$$

Hence, $P''(1) = \frac{1}{h^2}\left\{\Delta^2 P(1) - \Delta^3 P(1) + \frac{11}{12}\Delta^4 P(1)\right\}$

$=$ ________.

$\therefore E(X) =$ ________

$\therefore E(X(X-1)) =$ ________

and $Var(X) = E\{X(X-1)\} + E(X) - E^2(X)$.

$=$ ________.

(b) The following table gives the values of the MGF $M(t)$ of a certain RV X for different values of t.

t	$M(t)$
-0.01	0.98028
0.01	1.0203
0.02	1.04123
0.03	1.03279

Obtain the approximate value of $E(X)$ and $Var(X)$.

Solution:– We know that $M(0) = 1$

$E(X) = M'(0)$, $E(X^2) = M''(0)$.

t	$M(t)$	$\Delta M(t)$	$\Delta^2 M(t)$	$\Delta^3 M(t)$	$\Delta^4 M(t)$
-0.01					
0.00	1				
0.01					$\Delta^4 M(-0.01)$
0.02					$\Delta^4 M(0.00)$ equal
0.03					

Assuming $\Delta^4 M(t) = \text{constant}$,

$$\therefore M'(0) = \frac{1}{h}\left[\Delta M(0) - \frac{\Delta^2}{2} M(0) + \frac{\Delta^3}{3} M(0) \cdots\right]$$

$$\& \; M''(0) = \frac{1}{h^2}\left[\Delta^2 M(0) - \Delta^3 M(0) + \frac{11}{12}\Delta^4 M(0) \cdots\right]$$

Based on 5 values, $M(t)$ can be approximated by a 4^{th} degree polynomial.

INTEGRATION

5. (a) Evaluate $I = \int_0^1 \frac{dx}{1+x}$, correct to three decimal places, using both the trapezoidal and Simpson's rule with $h = 0.25$. Find the exact value of I. Which formula is accurate?

(b) Find from the following table the area bounded by the curve $y = f(x)$ and the x-axis from $x = 7.47$ to $x = 7.52$.

x	7.47	7.48	7.49	7.50	7.51	7.52
$f(x)$	1.93	1.95	1.98	2.01	2.03	2.06

(c) Evaluate $\int_{\sqrt{3}}^{\sqrt{5}} x.e^{-x^2} dx$, by numerical integration.

(d) Using Numerical integration, find an approximate value of π.

OR

Evaluate $\int_0^1 \frac{1}{1+x^2} dx$ by numerical integration. Hence find the value of π.

<u>Solution:-</u> (a) $I_T = \int_0^1 \frac{dx}{1+x} = \int_0^1 f(x)\,dx$

$$= \frac{0.25}{2}\left\{f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + f(1)\right\}$$, by Trapezoidal Rule.

Again, $I_S = \int_0^1 f(x)dx$

$$= \frac{0.25}{3}\left\{f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + f(1)\right\}$$, by Simpson's $1/3$rd rule.

$$I = \int_0^1 \frac{dx}{1+x} = \log 2 = \underline{\hspace{3cm}}.$$

Now, compute $E_1 = I - I_T$, $E_2 = I - I_S$.

Smaller the error, better the accuracy.

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(b) The area bounded by the curve $y = f(x)$ and x-axis from $x = 7.47$ to $x = 7.52$ is $\int_{7.47}^{7.52} f(x)\,dx = I$.

By Trapezoidal Rule, $I = \frac{h}{2}\left[y_0 + y_5 + 2(y_1 + y_2 + y_3 + y_4)\right]$, $h = 0.01$.

(c) $$I = \int_{\sqrt{3}}^{\sqrt{5}} x e^{-x^2}\,dx = \frac{1}{2}\int_3^5 e^{-u}\,du \;,\; u = x^2$$

$$= \frac{1}{2}\int_3^5 e^{-x}\,dx = \frac{1}{2}\left(e^{-3} - e^{-5}\right).$$

$$= \frac{1}{2}\int_3^5 f(x)\,dx$$

$$= \frac{1}{2} \times I.$$

(d) $$I = \int_0^1 f(x)\,dx \;,\; f(x) = \frac{1}{1+x^2}$$

x	0	1/6	2/6	3/6	4/6	5/6	1
$f(x)$							

Using Simpson's 1/3rd rule,

$$I = \frac{h}{3}\left\{y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right\}, \text{ where } h = 1/6.$$

$$= \_\_\_\_\_\_\_\_ .$$

Now, $I = \int_0^1 \frac{dx}{1+x^2} = \tan^{-1}(1) = \frac{\pi}{4}$.

$\Rightarrow 4I = \pi = \_\_\_\_\_\_ .$

6. (a) Find the value of the integral by numerical integration and then the exact value of the integral. Then compute the error.

$$\int_1^{4.4}\int_2^{2.6} \frac{dy\,dx}{xy}.$$

(b) By using a suitable quadrature formula find the value of the following integral for $n=1$,

$$n^{-1/2}\left\{B\left(\tfrac{1}{2},\tfrac{n}{2}\right)\right\}^{-1}\int_0^{1.96}\left(1+n^{-1}t^2\right)^{-\frac{n+1}{2}}dt$$

Evaluate the exact value by integral calculus and determine the error.

Solution:- (a) $I=\left(\int_1^{4.4}\frac{dx}{x}\right)\left(\int_2^{2.6}\frac{dy}{y}\right)=I_1\times I_2$.

Evaluate I_1 and I_2 by Simpson's $1/3$rd rule.

$I_S=$ ________.

Exact value: $I=[\log x]_1^{4.4}[\log y]_2^{2.6}$

$=[\log 4.4][\log 1.3]=$ ________.

Error $=|I-I_S|=$ ________.

(b) For $n=1$,

$$I=\frac{1}{\pi}\int_0^{1.96}\frac{dt}{1+t^2}.$$

Evaluate I by Simpson's $1/3$rd rule.

$I_S=$ ________.

$I=\frac{1}{\pi}\tan^{-1}(1.96)=$ ________.

Percentage error $=\frac{|I-I_S|}{I}\times 100\%$

$=$ ________.

NUMERICAL SOLUTION OF EQUATIONS

7. Find the real root of $2x - \log_{10}x - 7 = 0$. First verify whether the condition for convergence of the iteration method is satisfied or not, if it is satisfied use the iteration method, otherwise use any other suitable method, correct upto five decimal places.

Solution:- The equation is $2x - \log_{10}x - 7 = 0$ can be expressed in the form $x = \phi(x)$ in two ways:

(i) $x = \dfrac{\log_{10}x + 7}{2} = \phi_1(x)$

(ii) $x = 10^{2x-7} = \phi_2(x)$

choosing the form $\phi_1(x) = \dfrac{\log_{10}x + 7}{2}$

Trial root:- Let $f(x) = 2x - \log_{10}x - 7$

Note that $f(3) = -1 - \log_{10}3 < 0$

and $f(4) = 1 - \log_{10}4 > 0$

Hence, a (true) root of $f(x) = 0$ lies between 3 and 4.

Take, $x_0 = 3.5$ as the trial root.

Condition for Convergence :-

$$|\phi_1'(x)|_{x=x_0} = \frac{1}{2} \cdot \frac{\log_{10}e}{3.5} < 1$$

Hence, the iteration process, $x_{n+1} = \phi(x_n)$, $n = 0, 1, 2, \ldots$ converges.

Now, the successive improvements are given by

$$x_1 = \phi_1(x_0) = \frac{\log_{10}3.5 + 7}{2} = 3.772034 = x_1$$

$$x_2 = \phi_1(x_1) = \frac{\log_{10}x_1 + 7}{2} = 3.788288$$

$$x_3 = \phi_1(x_2) = 3.789221$$

$$x_4 = \phi_1(x_3) = 3.789275$$

$$x_5 = \phi_1(x_4) = 3.789278$$

$\therefore$ Trial root $= 3.789278 \cong 3.78928$.

calculation should be carried out by taking at least 6 digits after decimal places.

8. Find by Newton-Raphson method a real root of the equation $x^3 - x - 4 = 0$ correct to 4 places of decimals.

Solution:- $f(x) = x^3 - x - 4$

Trial root: $f(1) = -4 < 0$

$f(2) = 2 > 0$

Taking $x_0 = 1.5$ as the trial root.

Apply Newton-Raphson method :—

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - x_n - 4}{3x_n^2 - 1}$$

$$= \frac{2x_n^3 + 4}{3x_n^2 - 1}.$$

The successive improvements are;

$$x_1 = \frac{2x_0^3 + 4}{3x_0^2 - 1} = 1.86956$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{2x_1^3 + 4}{3x_1^2 - 1} = 1.79945$$

$x_3 =$

9. (a) Find by iteration method the value of $\sqrt{26}$ correct upto five decimal places.

(b). An art collection stands at a distance x feet from the wall where a picture of height four feet is hung. If the bottom of the picture is 15 inches above his eye level, then find using any numerical methods, the value of x for which the angle θ subtended at the eye by the picture is maximized. Note that maximizing θ is same as maximizing $\tan\theta$.

(c) The equation $x^4 - 8.2x^3 - 39.41x^2 - 62.26x + 30.25 = 0$ has a double root in the neighbourhood of $x=1$, Determine it correct upto 5 significant figures by any numerical method known to you.

(d) Determine correct upto 4 decimal places, a non-negative value of x for which the following function of x is minimized.

$$f(x) = \frac{1}{4}x^5 + 4\sin x - 5x.$$

Solution:- (a) Let $x = \sqrt{26}$;

$$x^2 - 26 = 0$$

To find a positive solution of the equation $f(x)=0$, where $f(x) = x^2 - 26$.

By N.R. Method, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{x_n^2 - 26}{2x_n}$$

$$= \frac{1}{2}\left(x_n + \frac{26}{x_n}\right) = \phi(x_n)$$

Trial root, $x_0 = 5$

Here, we shall use iteration process, $x_{n+1} = \phi(x_n)$.

[It is important to note that NR method is a special case of iteration]

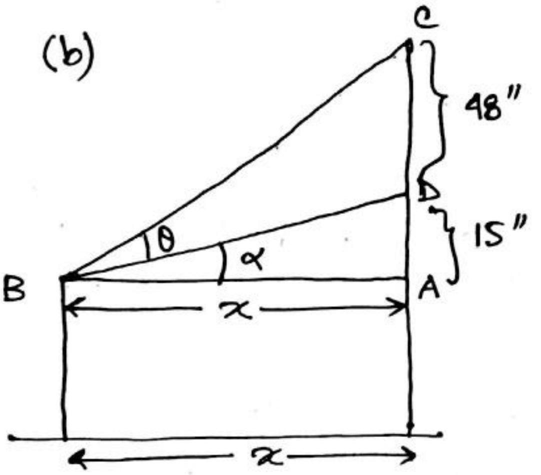
$$x_1 = \phi(x_0) = \frac{1}{2}\left(x_0 + \frac{26}{x_0}\right) = 5.1$$

$$x_2 = \phi(x_1) = 5.0990196$$

$$x_3 = \phi(x_2) = 5.0990195$$

$$\therefore \sqrt{26} = 5.09902.$$

(b)



Here $\tan\theta = \tan(\theta+\alpha-\alpha)$

$$= \frac{\tan(\theta+\alpha) - \tan\alpha}{1+\tan(\theta+\alpha)\tan\alpha}$$

$$= \frac{\frac{63}{x} - \frac{15}{x}}{1+\frac{63}{x}\cdot\frac{15}{x}}$$

$$= f(x)$$

To maximize $f(x)$, we are to solve,

$$f'(x) = 0 \text{ and } f''(x) < 0.$$

Use iteration method or N-R Method.

(c) Let $f(x) = x^4 - 8.2x^3 - 39.41x^2 - 62.26x + 30.25$.
The equation $f(x)=0$ has a root near 1 with multiplicity 2.

[If $x=\alpha$ is a double root of $f(x)=0$, then

$$f(x) = (x-\alpha)^2 g(x)$$

$$\therefore f'(x) = (x-\alpha)^2 g'(x) + 2(x-\alpha) g(x)$$

$$= (x-\alpha)\{(x-\alpha)g'(x) + 2g(x)\}$$

$$= (x-\alpha) h(x), \text{ say}$$

$\Rightarrow x=\alpha$ is a root of $f'(x)=0$]

To get a double root, we shall solve the equation

$$f'(x) = 0 = 4x^3 - 24.6x^2 - 78.82x - 62.26$$

Use NR or Iteration method, with $x_0 = 1$ as a trial root.

For iteration, $x = \sqrt[3]{\frac{24.6x^2 + 78.82x + 62.26}{4}}$

$$x_1 = \sqrt[3]{\frac{24.6x_0^2 + bx_0 + c}{4}}$$

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(d) To find the value of x for which $f(x)$ is minimum.

i.e. $f'(x)=0$ and $f''(x)>0$

Solving, $f'(x)=0=\frac{5}{4}x^4+4\cos x-5=g(x)$, say.

<u>Trial root:-</u>

$$g(0)=-1,\ g(2)=13.35>0$$

$$g(1)=-1.58.$$

$\Rightarrow x_0=1.5$ is a trial root.

<u>Iteration method:-</u>

$$x=\left\{\frac{4}{5}(5-4\cos x)\right\}^{1/4}=\phi(x),\text{ say}$$

<u>Condition of Convergence:-</u>

$$\phi'(x)=\left(\frac{4}{5}\right)^{1/4}\left(\frac{1}{4}\right)(5-4\cos x)^{-3/4}(4\sin x)$$

$$\phi'(1.5)=\left(\frac{4}{5}\right)^{1/4}\cdot\frac{\sin x}{(5-4\cos x)^{3/4}}<1.$$

<u>Successive Approximations:-</u>

$$x_1=\phi(x_0)$$

$$x_2=\phi(x_1)$$

$$\vdots$$

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10. (a) Find a real root of the equation

$$x = 0.3x^2 + 0.8$$
$$y = 0.3xy^2 + 0.7$$

by a suitable method.

Solution:-

Let
$$x = 0.3x^2 + 0.8 = F(x,y)$$
$$y = 0.3xy^2 + 0.7 = G(x,y)$$

Trial root:- $(x_0, y_0) = (1,1)$.

Condition for Convergence:-

$$\left\{\left|\frac{\partial F}{\partial x}\right| + \left|\frac{\partial F}{\partial y}\right|\right\}_{(x_0, y_0)} = [0.6x_0 + 0] = 0.6 < 1$$

$$\left\{\left|\frac{\partial F}{\partial x}\right| + \left|\frac{\partial F}{\partial y}\right|\right\}_{(1,1)} = |0.y^2 + 0.6x_0y_0| = 0.6 < 1$$

Successive Approximation:-

$$\begin{cases} x_1 = F(x_0, y_0) = \\ y_1 = G(x_0, y_0) = \end{cases}$$

$$\begin{cases} x_2 = F(x_1, y_1) = \\ y_2 = G(x_1, y_1) = \end{cases}$$

(b) Obtain by N-R Method a real solution (correct upto 4 decimal places) of the equations

$$\begin{cases} x + 3\log_{10} x - y^2 = 0 \\ 2x^2 - xy + 1 - 5x = 0 \end{cases}$$

[Take the trial root as $(x, y) = (3\cdot4, 2\cdot2)$]

<u>Solution</u>:- Let $f(x,y) = x + 3\log_{10}x - y^2$

$g(x,y) = 2x^2 - xy - 5x + 1$

To solve:

$$\left.\begin{array}{l} f(x,y) = 0 \\ g(x,y) = 0 \end{array}\right\}.$$

Trial root: $(x_0, y_0) = (3\cdot4, 2\cdot2)$

<u>NR Method</u>:- $x_1 = x_0 + h_1$, $h_1 = \begin{vmatrix} -f_0 & \left(\frac{\partial f}{\partial y}\right)_0 \\ -g_0 & \left(\frac{\partial g}{\partial y}\right)_0 \end{vmatrix} \Big/ J_0$

$y_1 = y_0 + k_1$,

$$k_1 = \begin{vmatrix} \left(\frac{\partial f}{\partial x}\right)_0 & -f_0 \\ \left(\frac{\partial g}{\partial x}\right)_0 & -g_0 \end{vmatrix} \Big/ J_0$$

where, $J_0 = \begin{vmatrix} \left(\frac{\partial f}{\partial x}\right)_0 & \left(\frac{\partial f}{\partial y}\right)_0 \\ \left(\frac{\partial g}{\partial x}\right)_0 & \left(\frac{\partial g}{\partial y}\right)_0 \end{vmatrix}$

Now, $\frac{\partial f}{\partial x} = 1 + \frac{3}{x}\log_{10}e \Rightarrow \left(\frac{\partial f}{\partial x}\right)_0 = 1 + \frac{3}{x_0}\log_{10}e =$

$\frac{\partial f}{\partial y} = -2y \Rightarrow \left(\frac{\partial f}{\partial y}\right)_0 =$

$\frac{\partial g}{\partial x} = 4x - y - 5 \Rightarrow \left(\frac{\partial g}{\partial x}\right)_0 = 4x_0 - y_0 - 5 =$

$\frac{\partial g}{\partial y} = -x \Rightarrow \left(\frac{\partial g}{\partial y}\right)_0 = -x_0 =$

$f_0 = f(x_0, y_0)$, $g_0 = g(x_0, y_0)$

Now, $J_0 =$

$h_1 =$

$k_1 =$

$\Rightarrow$ $x_1 =$

$y_1 =$