TRANSFORMATIONS OF RANDOM VARIABLES

BY

TANUJIT CHAKRABORTY

Indian Statistical Institute

Mail : tanujitisi@gmail.com

Distributions of Functions of Random Variables

A. The Transformation Technique :—

- **Univariate Case :** — Let the distribution of X is given. To find the distribution of $Y = g(X)$ where $g(\cdot)$ is a function.

(i) Discrete Case : If X is a discrete R.V. with mass points $x_1, x_2, \ldots$, then the distribution of $Y = g(X)$ is determined directly by Probability Laws. It may be noted that the several values of X give the same value of $Y = g(X)$.

Then, $$P[Y=y] = \sum_{\{i : y = g(x_i)\}} P[X = x_i]$$

☆ **Ex. 1.** Let X be RV with PMF $P[X=-2] = \frac{1}{5}$, $P[X=-1] = \frac{1}{6}$, $P[X=0] = \frac{1}{5}$, $P[X=1] = \frac{1}{15}$, $P[X=2] = \frac{11}{30}$. Find the PMF of $Y = X^2$.

Soln. → The set of mass points of X is $A = \{-2, -1, 0, 1, 2\}$ and the set of mass points of $Y = X^2$ is $B = \{0, 1, 4\}$.

Hence, the PMF of $Y = X^2$ is given by

$$P[Y=y] = \begin{cases} P[Y=0] = P[X=0] = \frac{1}{5}, & \text{if } y=0 \\ P[Y=1] = P[X=\pm 1] = \frac{7}{30}, & \text{if } y=1 \\ P[Y=4] = P[X=\pm 2] = \frac{17}{30}, & \text{if } y=4 \\ 0 & , \text{OW} \end{cases}$$

☆ **Ex. 2.** If $f_X(x) = \begin{cases} \frac{1}{6} & , x = -2, -1, 0, 1, 2, 3. \\ 0 & , \text{OW} \end{cases}$

is the PMF of a RV X, find the PMF of $Y = X^2$.

Soln. → The set of mass points of X is $A = \{-2, -1, 0, 1, 2, 3\}$ and the set of mass points of $Y = X^2$ is $B = \{0, 1, 4, 9\}$.

Hence, the PMF of $Y = X^2$ is given by,

$$P[Y=y] = \begin{cases} P[Y=0] = P[X=0] = \frac{1}{6}, & \text{if } y=0 \\ P[Y=1] = P[X=\pm 1] = \frac{2}{6}, & \text{if } y=1 \\ P[Y=4] = P[X=\pm 2] = \frac{2}{6}, & \text{if } y=4 \\ P[Y=9] = P[X=+3] = \frac{1}{6}, & \text{if } y=9 \\ 0 & , \text{OW} \end{cases}$$

$$= \begin{cases} 1/6 & , \text{if } y=0 \\ 1/3 & , \text{if } y=1 \\ 1/3 & , \text{if } y=4 \\ 1/6 & , \text{if } y=9 \\ 0 & , \text{OW} \end{cases}$$

☆ Ex.3. If $X \sim \text{Poisson}(\lambda)$. Find the distn. of $Y = e^X$.

Soln. → The set of mass points of X is $A = \{0, 1, 2, \ldots\}$ and that of $Y = e^X$ is $B = \{1, e, e^2, e^3, \ldots\}$

For $y \in B$,

$$P[Y=y] = P[e^X = y] = P[X = \ln y]$$

$$= \frac{e^{-\lambda} \cdot \lambda^{\ln y}}{(\ln y)!}$$

Hence, the PMF of Y is, $f_Y(y) = \begin{cases} \dfrac{e^{-\lambda} \cdot \lambda^{\ln y}}{\lfloor \ln y}, & y \in B \\ 0, & \text{OW} \end{cases}$

- **Theorem1.** Let X be a r.v. defined on $(\Omega, \mathcal{A}, P)$, also let g be a borel measurable function on $\mathbb{R}$, then $g(X)$ is also be a r.v.

- **Theorem 2.** Given a r.v. X with known D.F., then the D.F. of the R.V. $g(X)$, where g is a borel-measurable function can be determined.

Example:— Let X be a r.v. with d.f. F, then the following are also R.V. s, → $|X|$, $aX+b$, X^k (k is a positive integer), X^+ (where, $X^+ = X$, if $x \geq 0$; $= 0$, if $x < 0$), X^-.

Let us find the DF of the above R.V. s:—

i) The DF of $|X|$:— $G(y) = P(|X| \leq y)$

$$= P(-y \leq X \leq y)$$

$$= P(X \leq y) - P(X < -y)$$

$$= F(y) - F(-y-0).$$

ii) The DF of $aX+b$:—

$$H(y) = P(aX+b \leq y)$$

$$= \begin{cases} P\left(X \leq \frac{y-b}{a}\right), & \text{if } a > 0 \\ P\left(X \geq \frac{y-b}{a}\right), & \text{if } a < 0 \end{cases}$$

$$= \begin{cases} F\left(\frac{y-b}{a}\right), & \text{if } a > 0 \\ 1 - F\left(\frac{y-b}{a} - 0\right), & \text{if } a < 0 \end{cases}$$

iii) The DF of X^K : —— $H(y) = P(X^K \le y)$

$$= \begin{cases} P(X \le y^{1/K}) & \text{if } K \text{ is odd.} \\ P(-y^{1/K} \le X \le y^{1/K}) & \text{if } K \text{ is even.} \end{cases}$$

$$= \begin{cases} F(y^{1/K}) & \text{if } K \text{ is odd} \\ F(y^{1/K}) - F(-y^{1/K} - 0) & \text{if } K \text{ is even} \end{cases}$$

iv) The DF of X^+ is : ——

$$P(X^+ \le x) = \begin{cases} 0 & \text{if } x < 0 \\ P(X^+ = 0) = P(X \le 0) = F(0) = F(x) & \text{if } x = 0 \\ P(0 < X \le x) + P(X < 0) & \text{if } x > 0 \end{cases}$$

v) The DF of X^- is : ——

$$P(X^- \le x) = \begin{cases} P(X^- \le 0) = 1 & \text{if } x > 0 \\ P(X \le x) & \text{if } x \le 0 \end{cases}$$

☆ Ex.4. Let X be a Poisson RV with p.m.f. $f(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$, $x = 0, 1, 2..$
Find the distn. of $Y = P(Y=y)$, where $Y = X^2 + 3$.

Soln. → $Y = X^2 + 3$ maps $A = \{0, 1, 2, \ldots\}$ on to $B = \{3, 4, 7, 12, 19, \ldots\}$. The inverse mapping is $X = \sqrt{Y-3}$. and since there is no negative values in A, we take the positive square root.

$$\therefore P(Y=y) = P[X = \sqrt{y-3}]$$

$$= \frac{e^{-\lambda} \cdot \lambda^{\sqrt{y-3}}}{(\sqrt{y-3})!}, \quad y \in B.$$

☆ Ex.5. $X \sim bin(n, p)$, $x = 0, 1, \ldots, n$.
Find the PMF's of i) $Y = aX + b$, ii) $Z = X^2$, iii) $W = \sqrt{X}$.

Soln. → i) $Y = aX + b$ maps $A = \{0, 1, 2, \ldots\}$ onto $B = \{b, a+b, 2a+b, \ldots\}$
The inverse mapping is $X = \frac{Y-b}{a}$, and since there is no negative values in A, we take the positive values.

$$P[Y=y] = P[X = \tfrac{y-b}{a}] = \binom{n}{\frac{y-b}{a}} p^{\frac{y-b}{a}} q^{n - \frac{y-b}{a}}, \quad y \in B$$

ii) $Z = X^2$ maps $A = \{0, 1, 2, \ldots\}$ onto $B = \{0, 1, 4, 9, \ldots\}$

$$P(Z = z) = P(X^2 = z) = P(X = \sqrt{z}) = \binom{n}{\sqrt{z}} p^{\sqrt{z}} q^{n - \sqrt{z}}, \quad z \in B$$

iii) $W = \sqrt{X}$ maps $A = \{0, 1, 2, \ldots\}$ onto $B = \{0, 1, \sqrt{2}, \ldots\}$
$\Rightarrow \sqrt{X} = W \Rightarrow X = W^2$

$$P(W = \omega) = P(X = \omega^2) = \binom{n}{\omega^2} p^{\omega^2} q^{n - \omega^2}; \quad \omega \in B.$$

(ii) Continuous Case :—

• **Theorem:**— Let X be a R.V. of continuous type with PDF f. Let $y = g(x)$ be differentiable $\forall x$ and either $g'(x) > 0 \ \forall x$; or $g'(x) < 0 \ \forall x$, then $Y = g(X)$ is also a r.v. of continuous type with p.d.f. given by,

$$h(y) = \begin{cases} f(g^{-1}(y)) \cdot \left| \frac{d g^{-1}(y)}{dy} \right| & , \alpha < y < \beta \\ 0 & , \text{ow} \end{cases}$$

where, $\alpha = \min\{g(-\infty), g(\infty)\}$, $\beta = \max\{g(-\infty), g(+\infty)\}$

Proof:→

Case I:— If g is differentiable for all x and $g'(x) > 0 \ \forall x$ then g is continuous and strictly increasing, the limits α, β exist (may be infinite) and the inverse function $x = g^{-1}(y)$ exists and it is strictly increasing and differentiable. The d.f. of Y for $\alpha < y < \beta$ is given by,

$$P[Y \le y] = P[g(X) \le y]$$
$$= P[X \le g^{-1}(y)]$$
$$= F(g^{-1}(y)).$$

The p.d.f. of Y is $h(y) = \frac{d}{dy} P(Y \le y)$

$$= \frac{d}{dy} F(g^{-1}(y))$$
$$= f(g^{-1}(y)) \cdot \frac{d}{dy}(g^{-1}(y))$$

Case II:— Similarly, if $g'(x) < 0 \ \forall x$, then g is strictly decreasing and we have

$$P[Y \le y] = P[g(X) \le y]$$
$$= P[X \ge g^{-1}(y)]$$
$$= 1 - F(g^{-1}(y))$$

so that, $h(y) = -f(g^{-1}(y)) \cdot \frac{d}{dy}(g^{-1}(y))$

but in this case, $\frac{d}{dy}(g^{-1}(y)) < 0$, since both g and g^{-1} are strictly decreasing.

Combining both the cases, we have the PDF of $Y = g(X)$ as

$$h(y) = f(g^{-1}(y)) \left| \frac{d}{dy}(g^{-1}(y)) \right| \quad , \alpha < y < \beta$$
$$= 0 \qquad , \text{ow}$$

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Remark :-

i) If the conditions of this theorem are violated then we should return to the previous method of finding the distribution.

ii) If the PDF f vanishes outside an interval $[a,b]$ of finite length, we need only to assume that g is differentiable in (a,b) and either $g'(x)>0$ or $g'(x)<0$ throughout the interval. Then we take $\alpha = \min\{g(a), g(b)\}$, $\beta = \max\{g(a), g(b)\}$

Ex. 1. If $X \sim U(0,1)$. Find the PDF of i) $Y = e^X$, ii) $Y = -2\ln X$.

Soln. → The PDF of X is $f(x) = \begin{cases} 1 & \text{if } 0<x<1 \\ 0 & , \text{ow} \end{cases}$

i) $Y = e^X$

Here, $g(x) = e^x$

$\therefore g'(x) = e^x > 0$

$\therefore g$ is monotonically increasing.

The inverse function is, $x = g^{-1}(y) = \ln y$.

$$\therefore \frac{d}{dy}\left(g^{-1}(y)\right) = \frac{1}{y}.$$

$$\therefore h(y) = f(\ln y)\left|\frac{1}{y}\right| \quad , \quad 0<\ln y<1$$
$$= 0 \quad , \text{ow}$$

or, $h(y) = \begin{cases} \frac{1}{y} & \text{if } 1<y<e \\ 0 & , \text{ow} \end{cases}$

ii) $Y = -2\ln X$.

Here, $g(x) = -2\ln x$

$\therefore g'(x) = -\frac{2}{x}$

$\therefore g$ is monotonically decreasing.

$$x = g^{-1}(y) = e^{-\frac{1}{2}y}.$$

$$\therefore \frac{d}{dy}\left(g^{-1}(y)\right) = -\frac{1}{2}\cdot e^{-y/2}$$

$$\therefore h(y) = f(e^{-y/2})\cdot\left|-\frac{1}{2}e^{-y/2}\right|$$

$$= \begin{cases} \frac{1}{2}e^{-y/2} & , \text{if } 0<e^{-y/2}<1 \\ 0 & \text{ow} \end{cases}$$

$$= \begin{cases} \frac{1}{2}e^{-y/2} & , \text{if } 0<y<\infty \\ 0 & , \text{ow}. \end{cases}$$

$\therefore Y \sim Exp$ (with mean 2).

☆ **Ex. 2.** $f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{1}{2} & \text{if } 0 < x \le 1 \\ \frac{1}{2x^2} & \text{if } 1 < x < \infty \end{cases}$

Find the PDF of $Y = \frac{1}{X}$.

Soln. →

$g(x) = \frac{1}{x}$, $x \in (0, \infty)$, $y \in (0, \infty)$

$g'(x) = -\frac{1}{x^2} < 0$,

$x = \frac{1}{y} = g^{-1}(y)$.

$\therefore \frac{d}{dy}\left(g^{-1}(y)\right) = -\frac{1}{y^2}$

$\therefore h(y) = f\left(-\frac{1}{y}\right)\left|-\frac{1}{y^2}\right|$

$= \frac{1}{y^2} f\left(\frac{1}{y}\right)$.

$\therefore h(y) = \begin{cases} 0 & , \text{if } \frac{1}{y} \le 0 \\ \frac{1}{2y^2} & , \text{if } 0 < \frac{1}{y} \le 1 \\ \frac{y^2}{2y^2} = \frac{1}{2} & \text{if } 1 < \frac{1}{y} < \infty \end{cases}$

$= \begin{cases} \frac{1}{2y^2} & \text{if } 1 \le y < \infty \\ \frac{1}{2} & \text{if } 0 < y < 1 \end{cases}$.

☆ **Ex. 3.** $f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{OW}. \end{cases}$

Find the PDF of $U = \frac{X}{1+X}$.

Soln. →

$g(x) = \frac{x}{1+x}$, $U + UX = X$

$\Rightarrow X = \frac{U}{1-U}$.

$g'(x) = \frac{(1+x) - x}{(1+x)^2}$

$= \frac{1}{(1+x)^2}$.

$x = g^{-1}(u) = \frac{u}{1-u}$. $\therefore \frac{d}{du}(g^{-1}(u)) = \frac{1}{(1-u)^2}$.

$\therefore h(u) = f\left(\frac{u}{1-u}\right)\left|\frac{1}{(1-u)^2}\right|$, $0 < \frac{u}{1-u} < 1$

$= \begin{cases} \frac{1}{(1-u)^2} & \text{if } 0 < u < \frac{1}{2} \\ 0 & , \text{OW} \end{cases}$

☆ Ex. 4. Let X has Pareto distn. with PDF

$$f_X(x) = \begin{cases} \dfrac{\theta}{x^{\theta+1}}, & \text{if } x>0 \\ 0, & \text{ow} \end{cases}$$

Find the distribution of $Y = \log_e X$.

Soln.→ Note that $Y = \log_e X$ is strictly increasing function from $\mathcal{X} = \{x : x > 1\}$ onto $D = \{y : y > 0\}$

Also note that, $Y = \log_e x$

$$\Rightarrow x = e^y.$$

∴ The PDF of Y in

$$f_Y(y) = \begin{cases} f_X(e^y) \cdot \left|\dfrac{de^y}{dy}\right|, & \text{if } y \in D \\ 0, & \text{if } y \notin D \end{cases}$$

$$= \begin{cases} \dfrac{\theta}{(e^y)^{\theta+1}} \cdot e^y & \text{if } y > 0 \\ 0, & \text{ow} \end{cases}$$

$$= \begin{cases} \theta e^{-\theta y}, & \text{if } y > 0 \\ 0, & \text{ow} \end{cases}$$

Hence $Y = \log_e X$ follows an exponential distribution with mean $\frac{1}{\theta}$.

Probability Integral Transformation :—

Let X be a continuous R.V. with PDF $f(x)$. Then

$$Y = F(X) = \int_{-\infty}^{X} f(t)\,dt \text{ follows } U(0,1) \text{ distribution.}$$

Soln.→ $y = F(x) = \int_{-\infty}^{x} f(t)\,dt$ is a strictly monotonic function from $\{x : f(x) > 0\}$ onto $D = \{y : 0 \le y \le 1\}$

For $y \in D$, the DF of $Y = F(x)$ is

$$G(y) = P[Y \le y] = P[F(x) \le y] = P[X \le F^{-1}(y)]$$

As $y = F(x)$ is strictly increasing,

$$\Rightarrow F(F^{-1}(y)) = y.$$

Hence the PDF of $Y = F(x)$ is

$$g(y) = \begin{cases} 1 & \text{if } y \in D \\ 0 & \text{if } y \notin D \end{cases}$$

$$= \begin{cases} 1 & \text{if } 0 \le y \le 1 \\ 0 & \text{ow} \end{cases}$$

Hence, $Y = \int_{-\infty}^{x} f(t)\,dt$ follows uniform distn. over $(0,1)$.

☆ **Ex.5.** Let X has the PDF $f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{o/w} \end{cases}$

Find the distribution of $Y = 1 - e^{-\lambda x}$.

Soln. → Here $\mathscr{X} = \{x : f_X(x) > 0\} = \{x : x > 0\}$

For $x \in \mathscr{X}$, the CDF of X is,

$$F(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_{0}^{x} \lambda e^{-\lambda t}\,dt$$
$$= 1 - e^{-\lambda x}.$$

Hence, $Y = 1 - e^{-\lambda x} = F(x)$ is a probability integral transformation. Hence, $Y = 1 - e^{-\lambda x}$ follows uniform distribution over the interval $(0, 1)$.

☆ **Ex.6.** Let u be an observed value of $U \sim R(0, 1)$. Then obtain an observed value of $X \sim$ Exponential with mean $\frac{1}{\lambda}$.

Soln. →

The PDF of X is $f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{o/w} \end{cases}$

For $x > 0$, The DF, $F(x) = \int_0^x \lambda \cdot e^{-\lambda t}\,dt$
$$= 1 - e^{-\lambda x}.$$

Hence, $F(x) = 1 - e^{-\lambda x}$ follows $R(0, 1)$ by Probability integral transformation.

As $U \sim R(0, 1)$ and $F(x) \sim R(0, 1)$.

Taking $u = F(x) \Rightarrow u = 1 - e^{-\lambda x} \Rightarrow x = -\frac{1}{\lambda}\log(1 - u)$ is an observed value of $X \sim$ Exp. with mean $\frac{1}{\lambda}$.

• **Theorem:** — If the transformation $y=g(x)$ is not one-to-one transformation from $\mathscr{X}$ onto D, i.e. for a point in D, there exists more than one points in $\mathscr{X}$, then $\mathscr{X}$ can be decomposed into a finite (or, even countable) number of disjoint sets, $\mathscr{X}_1, \mathscr{X}_2, \dots, \mathscr{X}_m$, say, so that $y=g(x)$ is one to one from each $\mathscr{X}_i$ onto D, $i=1(1)m$.

Let $x=g_i^{-1}(y)$ be the inverse of $y=g(x)$ on $\mathscr{X}_i$, $i=1(1)m$. Then the PDF of $Y=g(X)$ is

$$f_Y(y)=\begin{cases}\sum_{i=1}^{m} f_X(g_i^{-1}(y))\left|\frac{d}{dy}g_i^{-1}(y)\right|, & \text{if } y\in D\\ 0, & \text{ow}\end{cases}$$

Ex.1. Let $X\sim N(0,1)$. Find the PDF of $Y=X^2$.

soln. → Here, $f(x)=\frac{1}{\sqrt{2\pi}}\cdot e^{-x^2/2}$, $x\in\mathbb{R}$, $\mathscr{X}=\mathbb{R}$

$y=g(x)=x^2$

$g'(x)=2x \begin{cases} >0 & \text{if } x>0\\ <0 & \text{if } x<0.\end{cases}$

Clearly, $y=x^2$ is not a one-to-one transformation from $\mathscr{X}$ onto $D=\{y: y\geq 0\}$

Note that, $y=x^2 \Rightarrow x=\pm\sqrt{y}$.

Decompose $\mathscr{X}$ into two parts; $\mathscr{X}_1=\{x, x<0\}$

and $\mathscr{X}_2=\{x: x\geq 0\}$

Then $y=x^2$ is one-to-one transformation from each $\mathscr{X}_i$ onto D, $i=1,2$.

Note that, $x=-\sqrt{y}$, if $x\in\mathscr{X}_1$

$=g_1^{-1}(y)$

$x=+\sqrt{y}$, if $x\in\mathscr{X}_2$

$=g_2^{-1}(y)$.

The PDF of $Y=X^2$ is,

$$f_Y(y)=\begin{cases} f_X(g_1^{-1}(y))\left|\frac{d}{dy}g_1^{-1}(y)\right|+f_X(g_2^{-1}(y))\left|\frac{d}{dy}g_2^{-1}(y)\right|, & \text{if } y\in D\\ 0, & \text{ow}\end{cases}$$

$$=\begin{cases}\frac{1}{\sqrt{2\pi}}\cdot e^{-y/2}\cdot\left|-\frac{1}{2\sqrt{y}}\right|+\frac{1}{\sqrt{2\pi}}\cdot e^{-y/2}\cdot\left|\frac{1}{2\sqrt{y}}\right|, & \text{if } y>0\\ 0, & \text{ow}\end{cases}$$

$\therefore f_Y(y) = \begin{cases} \dfrac{e^{-y/2} \cdot y^{1/2-1}}{\Gamma(\frac{1}{2}) \cdot 2^{1/2}} & , \text{if } y>0 \\ 0 & , \text{elsewhere} \end{cases}$

Hence, $Y \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$.

☆ Ex. 2. Let, $X \sim$ Standard Laplace distn.
Find the distribution of $Y = |X|$.

Soln. → Here, $f_X(x) = \frac{1}{2} e^{-|x|}$, $x \in \mathbb{R}$ and $\mathscr{X} = \mathbb{R}$

Here, $y = |X|$ is not one-to-one transformation from $\mathscr{X} = \mathbb{R}$ on $\mathcal{D} = \{y : y \geq 0\}$.

Note that, $y = |x|$
$\Rightarrow x = \pm y$

Decomposed $\mathscr{X}$ into two parts:
$\mathscr{X}_1 = \{x : x < 0\}$ and $\mathscr{X}_2 = \{x ; x \geq 0\}$

Then, $x = -y$, $x \in \mathscr{X}_1$
$x = +y$, $x \in \mathscr{X}_2$

The PDF of $Y = |X|$ is

$$f_Y(y) = \begin{cases} f_X(-y)\left|\frac{d}{dy}(-y)\right| + f_X(y)\left|\frac{d}{dy}(y)\right| & , \text{if } y \in \mathcal{D} \\ 0 & , \text{ow} \end{cases}$$

$$= \begin{cases} \frac{1}{2} \cdot e^{-|y|} \cdot |-1| + \frac{1}{2} e^{-|y|} \cdot |1| & , \text{if } y>0 \\ 0 & , \text{ow} \end{cases}$$

$$= \begin{cases} e^{-y} , & \text{if } y>0 \\ 0 & , \text{ow} \end{cases}$$

$\Rightarrow Y = |X|$ follows exponential with mean 1;

Ex. 3. Let X be a R.V. with PDF

$$f_X(x) = \begin{cases} \frac{2x}{\pi^2}, & 0 < x < \pi \\ 0, & \text{ow} \end{cases}$$

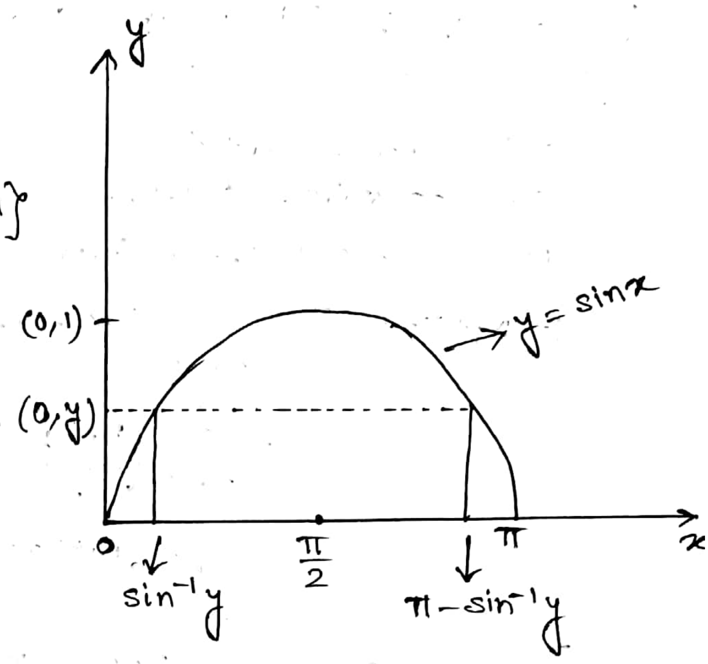
Find the PDF of $Y = \sin X$.

Soln. →

Here
$\mathcal{X} = \{x : 0 < x < \pi\}$ and $D = \{y : 0 < y < 1\}$

Note that $y = \sin x$ is not one to one transformation from $\mathcal{X}$ onto D. Decomposed $\mathcal{X}$ as $\mathcal{X}_1 = \{x : 0 < x < \frac{\pi}{2}\}$

and, $\mathcal{X}_2 = \{x : \frac{\pi}{2} \le x < \pi\}$



Then $x = \sin^{-1}y$, if $x \in \mathcal{X}_1$

$= \pi - \sin^{-1}y$, if $x \in \mathcal{X}_2$

The PDF of $Y = \sin X$ is

$$f_Y(y) = \begin{cases} f_X(\sin^{-1}y)\left|\frac{d(\sin^{-1}y)}{dy}\right| + f_X(\pi - \sin^{-1}y)\left|\frac{d(\pi - \sin^{-1}y)}{dy}\right|, & \text{if } y \in D \\ 0, & \text{ow} \end{cases}$$

$$= \begin{cases} \frac{2\sin^{-1}y}{\pi^2} \cdot \frac{1}{\sqrt{1-y^2}} + \frac{2(\pi - \sin^{-1}y)}{\pi^2} \cdot \frac{1}{\sqrt{1-y^2}}, & \text{if } 0 < y < 1 \\ 0, & \text{ow} \end{cases}$$

$$= \begin{cases} \frac{2}{\pi} \frac{1}{\sqrt{1-y^2}}, & 0 < y < 1 \\ 0, & \text{ow}. \end{cases}$$

- Several Variables :—

(i) Discrete Case :—

☆ Ex. 1. If $X_1 \sim Bin(n_1, p)$ and $X_2 \sim Bin(n_2, p)$ independently, find $P[X_1+X_2=s]$ and identify the distn. of X_1+X_2.

Soln. → Let $S = X_1 + X_2$

S takes values $0, 1, 2, \ldots, n_1+n_2$.

for, $s = \overline{0, 1, 2, \ldots, n_1+n_2}$,

$$P[X_1+X_2=s] = \sum_{x_1=0}^{m} P[X_1 = x_1, X_2 = s-x_1] \text{, where } m = \min\{n_1, s\}$$

$$= \sum_{x_1=0}^{m} \binom{n_1}{x_1} p^{x_1} q^{n_1-x_1} \binom{n_2}{s-x_1} p^{s-x_1} q^{n-s+x_1}$$

$$= p^s q^{n_1+n_2-s} \left\{ \sum_{x_1=0}^{m} \binom{n_1}{x_1}\binom{n_2}{s-x_1} \right\}$$

[Note that, $(1+t)^{n_1}(1+t)^{n_2} = (1+t)^{n_1+n_2}$

$$\Rightarrow \left\{ \sum_{i=0}^{n_1} \binom{n_1}{i} t^i \right\} \left\{ \sum_{j=0}^{n_2} \binom{n_2}{j} t^j \right\} = \left\{ \sum_{k=0}^{n_1+n_2} \binom{n_1+n_2}{k} t^k \right\}$$

Equating the coefficient of t^s, we get,—

$$\sum_{x_1=0}^{m} \binom{n}{x_1}\binom{n_2}{s-x_1} = \binom{n_1+n_2}{s} \quad]$$

$$\therefore P[X_1+X_2=s] = \begin{cases} \binom{n_1+n_2}{s} p^s q^{n_1+n_2-s} & , s = 0(1)\overline{n_1+n_2} \\ 0 & , \text{ow} \end{cases}$$

and so $S = X_1+X_2 \sim Bin(n_1+n_2, p)$.

[This property is known as Reproductive Property of Binomial distribution.]

Remark:— If $X \sim Bin(n_i, p)$, $i = 1(1)k$, independently, then $\sum_{i=1}^{k} X_i \sim Bin\left(\sum_{i=1}^{k} n_i, p\right)$, by induction.

Ex. 2. If $X_1 \sim Bin(n_1, p)$ and $X_2 \sim Bin(n_2, q)$, independently, then find the distribution of $(X_1 - X_2 + n_2)$ and hence the distn. of $(X_1 - X_2)$.

[C. U. 2011]

Soln. → $X_1 \sim Bin(n_1, p)$

$n_2 - X_2 \sim Bin(n_2, p)$, independently.

By reproductive property,

$X_1 + (n_2 - X_2) \sim Bin(n_1 + n_2, p)$.

Now, let $D = X_1 - X_2$; the the PMF of D is,

$$P[D = d] = P[X_1 - X_2 = d] = P[X_1 - X_2 + n_2 = d + n_2]$$

$$= \begin{cases} \binom{n_1+n_2}{d+n_2} p^{d+n_2} q^{n_1-d}, & \text{if } d = -n_2(1)n_1. \\ 0 & \text{ow} \end{cases}$$

Ex. 3. If $X_1, X_2 \overset{i.i.d.}{\sim} Bin(n, \frac{1}{2})$; Find the distn. of $(X_1 - X_2)$ and show that it is symmetric about '0'.

Soln. → $X_1 \sim Bin(n, \frac{1}{2})$

and, $n - X_2 \sim Bin(n, \frac{1}{2})$, independently.

Then, $X_1 + n - X_2 \sim Bin(2n, \frac{1}{2})$.

Let, $D = X_1 - X_2$

The PMF of D is

$$P[D = d] = P[X_1 - X_2 = d]$$

$$= P[X_1 - X_2 + n = n + d]$$

$$= \begin{cases} \binom{2n}{n+d} \left(\frac{1}{2}\right)^{2n}, & d = -n(1)n \\ 0 & , \text{ow} \end{cases}$$

Note that,

$$P[D = -d] = \binom{2n}{n-d} \frac{1}{2^{2n}} = \binom{2n}{n+d} \cdot \frac{1}{2^{2n}} = P[D = d]$$

$\forall\, d = -n(1)n.$

Hence, $D = (X_1 - X_2)$ is symmetrically distributed about '0'.

☆ Ex. 4. If $X_1 \sim Bin(n_1, p)$ and $X_2 \sim Bin(n_2, p)$ independently, then find the conditional distribution of X_1 given $X_1+X_2=s$.

Soln. → By reproductive property,

$X_1+X_2 \sim Bin(n_1+n_2, p)$; for $s = 0, 1, \ldots, n_1+n_2$,

$$P[X_1=x_1 / X_1+X_2=s] = \frac{P[X_1=x_1 ; X_1+X_2=s]}{P[X_1+X_2=s]}$$

$$= \frac{P[X_1=x_1 ; X_2=s-x_1]}{P[X_1+X_2=s]}$$

$$= \begin{cases} \dfrac{\binom{n_1}{x_1} p^{x_1} q^{n_1-x_1} \binom{n_2}{s-x_1} p^{s-x_1} q^{n-s+x_1}}{\binom{n_1+n_2}{s} p^s q^{n_1+n_2-s}} & \text{if } x=0(1)\min(n_1, s) \\ 0 & \text{, ow.} \end{cases}$$

$$= \frac{\binom{n_1}{x_1}\binom{n_2}{s-x_1}}{\binom{n_1+n_2}{s}} \text{, if } x=0(1)\min(n_1, s).$$

Here, $X_1 / X_1+X_2=s$ follows a Hypergeometric distn. with parameters (n_1, n_2, s).

Remark:-

1) It is important to note that the conditional distn. is free from the parameter 'p'.

2) $E(X_1 / X_1+X_2=s) = s \cdot \dfrac{n_1}{n_1+n_2}$.

⇒ The regression of X_1 on (X_1+X_2) is linear and

$$\beta_{X_1, X_1+X_2} = \frac{n_1}{n_1+n_2}.$$

☆ **Ex.5.** If $X_1 \sim P(\lambda_1)$ and $X_2 \sim P(\lambda_2)$, independently, then show that $(X_1 + X_2) \sim P(\lambda_1 + \lambda_2)$.

[Reproductive property of Poisson distribution]

Soln. →

The joint PMF of X_1, X_2 is

$$f(x_1, x_2) = \frac{e^{-\lambda_1} \lambda_1^{x_1}}{x_1!} \cdot \frac{e^{-\lambda_2} \lambda_2^{x_2}}{x_2!}, \quad x_i = 0, 1, 2, \ldots \infty$$

Let $Y = X_1 + X_2$ be a function of X_1 and X_2 then the PMF of Y is,

$$P[Y = y] = f_X(y) = \sum_{x_1=0}^{\infty} \sum_{x_2=0}^{\infty} e^{-(\lambda_1+\lambda_2)} \cdot \frac{\lambda_1^{x_1} \lambda_2^{x_2}}{x_1!\, x_2!}$$
$$\{(x_1, x_2) : x_1 + x_2 = y\}$$

$$= e^{-(\lambda_1+\lambda_2)} \sum_{x_1=0}^{\infty} \frac{\lambda_1^{x_1} \lambda_2^{y-x_1}}{x_1!\,(y-x_1)!}$$

$$= e^{-(\lambda_1+\lambda_2)} \cdot \frac{1}{y!} \sum_{x_1=0} \frac{y!}{x_1!\,(y-x_1)!} \lambda_1^{x_1} \lambda_2^{y-x_1}$$

$$= e^{-(\lambda_1+\lambda_2)} \frac{1}{y!} (\lambda_1 + \lambda_2)^y, \quad y = 0, 1, 2, \ldots$$

$\therefore Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

☆ **EX. 6.** If $X_1 \sim P(\lambda_1)$ and $X_2 \sim P(\lambda_2)$, independently, then show that $X_1 / X_1 + X_2 = s \sim Bin\left(s, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$.

Soln. → By reproductive property,

$X_1 + X_2 \sim P(\lambda_1+\lambda_2)$.

For $s = 0, 1, 2, 3, \ldots\ldots$

$$P[X_1 = x_1 / X_1 + X_2 = s] = \frac{P[X_1 = x_1, X_2 = s - x_1]}{P[X_1 + X_2 = s]}$$

$$= \begin{cases} \dfrac{e^{-\lambda_1} \cdot \dfrac{\lambda_1^{x_1}}{x_1!} \cdot e^{-\lambda_2} \cdot \dfrac{\lambda_2^{s-x_1}}{(s-x_1)!}}{e^{-(\lambda_1+\lambda_2)} \cdot \dfrac{(\lambda_1+\lambda_2)^s}{s!}} & , \text{if } x_1 = 0(1)s \\ 0 & , \text{ow} \end{cases}$$

$$= \begin{cases} \dfrac{s!}{x_1!(s-x_1)!} \left(\dfrac{\lambda_1}{\lambda_1+\lambda_2}\right)^{x_1} \left(\dfrac{\lambda_2}{\lambda_1+\lambda_2}\right)^{s-x_1}, & x_1 = 0(1)s \\ 0 & , \text{ow} \end{cases}$$

$$= \begin{cases} \binom{s}{x_1} p^{x_1} q^{s-x_1} & , x_1 = 0, 1, \ldots, s \\ 0 & , \text{ow} \end{cases}$$

, where $p = \dfrac{\lambda_1}{\lambda_1+\lambda_2}$.

Hence, $X_1 / X_1 + X_2 = s \sim Bin\left(s, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$.

Remark:— $E(X_1 / X_1 + X_2 = s) = s \cdot \dfrac{\lambda_1}{\lambda_1+\lambda_2}$.

⇒ The regression of X_1 on X_1+X_2 is linear and

$$\beta_{X_1, X_1+X_2} = \frac{\lambda_1}{\lambda_1+\lambda_2}.$$

☆ Ex.7. If $X_i \sim P(\lambda_i)$, $i=1,2,\ldots$, independently, find the conditional distn. of (X_1, X_2) given $X_1+X_2=s$.

Soln. → For $s=0,1,2,3,\ldots$,

$$P[(X_1,X_2)/X_1+X_2=s] = \frac{P[X_1=x_1, X_2=x_2; X_1+X_2=s]}{P[X_1+X_2=s]}$$

$$= \begin{cases} 0 & , \text{if } x_1+x_2 \neq s \\ \dfrac{P[X_1=x_1, X_2=x_2]}{P[X_1+X_2=s]} & , \text{if } x_1+x_2=s. \end{cases}$$

[For, $x_1+x_2=s$, $\{X_1=x_1, X_2=x_2\} \subseteq \{X_1+X_2=s\}$]

$$= \begin{cases} 0 & , \text{if } x_1+x_2 \neq s \\ \dfrac{e^{-\lambda}\cdot\dfrac{\lambda_1^{x_1}}{x_1!}\cdot e^{-\lambda_2}\cdot\dfrac{\lambda_2^{x_2}}{x_2!}}{e^{-(\lambda_1+\lambda_2)}\cdot\dfrac{(\lambda_1+\lambda_2)^s}{s!}} & , \text{if } x_1+x_2=s. \end{cases}$$

$$= \begin{cases} \binom{s}{x_1}\left(\dfrac{\lambda_1}{\lambda_1+\lambda_2}\right)^{x_1}\left(\dfrac{\lambda_2}{\lambda_1+\lambda_2}\right)^{x_2} & , \text{if } x_1 \text{ and } x_2 \text{ are non-negative integers such that, } x_1+x_2=s. \\ 0 & , \text{ow} \end{cases}$$

$\therefore (X_1, X_2)/X_1+X_2=s \sim B\left(s, \dfrac{\lambda_1}{\lambda_1+\lambda_2}\right)$.

☆ Ex.8. If $X_1 \sim Bin(5, \frac{1}{4})$, $X_2 \sim Bin(7, \frac{7}{4})$, find the distn. of $(X_1 - X_2)$.

Ans:- Same as example 2.

☆ Ex. 9. If $X_i \sim P(\lambda_i)$, $i=1(1)k$, independently, find the conditional distn. of $\left(X_1, X_2, \ldots, X_k / \sum_{i=1}^{k} X_i = s\right)$. (C.U.)

Soln:→ For $s = 0, 1, 2, \ldots$

$$P\left[X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k / \sum_{i=1}^{k} X_i = s\right]$$

$$= \frac{P\left[X_1 = x_1, \ldots, X_k = x_k ; \sum_{i=1}^{k} X_i = s\right]}{P\left[\sum_{i=1}^{k} X_i = s\right]}$$

$$= \begin{cases} 0 & , \text{if } \sum_{i=1}^{k} x_i \neq s. \\ \dfrac{P[X_1 = x_1, \ldots, X_k = x_k]}{P\left[\sum_{i=1}^{k} X_i = s\right]} & , \text{if } \sum_{i=1}^{k} x_i = s. \end{cases}$$

$\left[\text{For } \sum_{i=1}^{k} x_i = s,\ \{X_1 = x_1, \ldots, X_k = x_k\} \subseteq \left\{\sum_{i=1}^{k} X_i = s\right\}\right]$

$$= \begin{cases} \dfrac{\prod_{i=1}^{k} \left\{ e^{-\lambda_i} \cdot \dfrac{\lambda_i^{x_i}}{x_i!} \right\}}{e^{-\sum_{i=1}^{k}\lambda_i} \cdot \dfrac{\left(\sum_{i=1}^{k} \lambda_i\right)^s}{s!}} & , \text{if } x_1, \ldots, x_k \text{ are non-negative integers such that } \sum_{i=1}^{k} x_i = s. \\ 0 & , \text{ow} \end{cases}$$

$$= \begin{cases} \dfrac{s!}{x_1!\, x_2! \cdots x_k!} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k} & , \text{if } x_1, x_2, \ldots, x_k \text{ are non-negative integers such that } \sum_{i=1}^{k} x_i = s. \\ 0 & , \text{ow} \end{cases}$$

where $p_i = \dfrac{\lambda_i}{\sum_{i=1}^{k} \lambda_i}$, $\sum_{i=1}^{k} p_i = 1$.

Remark:– Here, we are considering the distn. of k R.V.'s, i.e. the conditional distn. of $(X_1, \ldots, X_k)$ given $\left(\sum_{i=1}^{k} X_i = s\right)$. Clearly, the R.V.'s are linearly related. This type of distn. is known as Singular distribution in k-dimension. But if we consider the distn of $(X_1, \ldots, X_{k-1})$ given $\sum_{i=1}^{k} X_i = s$, then $\sum_{i=1}^{k-1} X_i \leq s$, then $(X_1, \ldots, X_{k-1})$ are not linearly related and the distn. becomes a non-singular distn. in $(k-1)$ dimension.

Ex.10. If X and Y are i.i.d. geometric RV's. find the distn. of X given $X+Y=s$. [C.U. 2007]

Soln. → Let, $f(x) = \begin{cases} pq^x, & x = 0,1,2,\ldots \\ 0, & \text{ow} \end{cases}$

be the common PMF of X and Y.

For $s = 0,1,2,\ldots$

$$P[X+Y=s] = \sum_{x=0}^{s} P[X=x, Y=s-x]$$

$$= \sum_{x=0}^{s} pq^x \cdot pq^{s-x}$$

$$= \sum_{x=0}^{s} p^2 q^s$$

$$= (s+1)p^2q^s.$$

Hence, $P[X+Y=s] = \begin{cases} \binom{s+2-1}{2-1} p^2 q^s, & s = 0,1,2,\ldots \\ 0, & \text{ow} \end{cases}$

$\Rightarrow X+Y \sim NB(2,p)$.

For $s = 0,1,2,\ldots$

$$P[X=x / X+Y=s] = \frac{P[X=x; X+Y=s]}{P[X+Y=s]}$$

$$= \frac{P[X=x, Y=s-x]}{P[X+Y=s]}$$

$$= \begin{cases} \dfrac{(pq)^x \cdot (pq)^{s-x}}{(s+1)p^2q^s}, & x = 0,1,2,\ldots,s \\ 0, & \text{ow} \end{cases}$$

$$= \begin{cases} \dfrac{1}{s+1}, & x = 0,1,\ldots,s \\ 0, & \text{ow} \end{cases}$$

Hence $X / X+Y=s$ has a uniform distribution over $A = \{0,1,2,\ldots,s\}$.

(ii) Continuous Case :

Theorem: Let X_1 and X_2 be two RV's with joint PDF $f_{X_1X_2}(x_1,x_2)$. Let

$$\mathcal{X} = \{(x_1,x_2) : f_{X_1,X_2}(x_1,x_2) > 0\}$$

Assume that

i) $y_1 = g_1(x_1,x_2)$ and $y_2 = g_2(x_1,x_2)$ define a one-to-one tramformation from $\mathcal{X}$ onto D.

ii) the first order partial derivative of $x_1 = h_1(y_1,y_2)$ and $x_2 = h_2(y_1,y_2)$ w.r.t. y_1 and y_2 are continuous on D.

iii) The jacobian $J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$ is non-zero on D.

Then the PDF of (Y_1, Y_2) is

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} f_{X_1,X_2}(h_1(y_1,y_2), h_2(y_1,y_2)) \cdot |J|, & \text{if } (y_1,y_2) \in D \\ 0, & \text{OW} \end{cases}$$

- **Remark:** The jacobian J is also known as the Local magnification factor.

☆ **Example 1.** Let $X \sim \text{Gamma}(\alpha)$ and $Y \sim \text{Gamma}(\beta)$, independently. Show that $U = X+Y$ and $V = \frac{X}{X+Y}$ are also independently distributed. Also, identify their distributions. [2010]

Soln. → The PDF of (X,Y) is

$$f_{X,Y}(x,y) = \begin{cases} \frac{e^{-x-y} \cdot x^{\alpha-1} \cdot y^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}, & \text{if } x,y > 0 \\ 0, & \text{OW} \end{cases}$$

Here, $u = x+y$, $v = \frac{x}{x+y}$

$\Rightarrow x = uv$, $y = u(1-v)$

Clearly, $0 < u < \infty$, $0 < v < 1$

Range:-
$0 < x < \infty$
$0 < y < \infty$
$\Rightarrow 0 < x+y < \infty$,
$\Rightarrow x < x+y$
$\Rightarrow \frac{x}{x+y} < 1$,
$\therefore 0 < \frac{x}{x+y} < 1$

The jacobian of the transformation is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u$$

The PDF of (U, V) is —

$$f_{U,V}(u,v) = \begin{cases} \dfrac{e^{-u} \cdot (uv)^{\alpha-1} \{u(1-v)\}^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \cdot |-u| & , \text{if } 0<u<\infty,\ 0<v<1 \\ 0 & , \text{ow} \end{cases}$$

$$= \begin{cases} \dfrac{e^{-u} \cdot u^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \cdot \dfrac{v^{\alpha-1}(1-v)^{\beta-1}}{\beta(\alpha,\beta)} & \text{if } 0<u<\infty, 0<v<1 \\ 0 & , \text{ow} \end{cases}$$

$$= f_U(u) \cdot f_V(v) \quad \forall (u,v)$$

where,

$$f_U(u) = \begin{cases} \dfrac{e^{-u} \cdot u^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} & , 0<u<\infty \\ 0 & , \text{o.w.} \end{cases}$$

and,

$$f_V(v) = \begin{cases} \dfrac{v^{\alpha-1} \cdot (1-v)^{\beta-1}}{\beta(\alpha,\beta)} & , 0<v<1 \\ 0 & , \text{ow} \end{cases}$$

are the marginal PDFs.

Hence, U and V are independently distributed.

Clearly, $U = X+Y \sim \text{Gamma}(\alpha+\beta)$, and

$V = \dfrac{X}{X+Y} \sim$ first kind Beta (α, β).

☆ Ex. 2. (Continuation) Let $X \sim \text{Gamma}(\alpha)$ and $Y \sim \text{Gamma}(\beta)$ independently. Then show that $E\left(\dfrac{X}{X+Y}\right) = \dfrac{E(X)}{E(X+Y)}$.

Soln. → $E(X) = E\left\{(X+Y) \cdot \dfrac{X}{(X+Y)}\right\}$

$= E(U \cdot V)$

$= E(U)E(V)$, due to independence.

$= E(X+Y) \cdot E\left(\dfrac{X}{X+Y}\right)$.

$$\Rightarrow E\left(\frac{X}{X+Y}\right) = \frac{E(X)}{E(X+Y)}$$

☆**Ex. 3.** Let $X_1, X_2 \overset{iid}{\sim} N(0,1)$

i) Find the distn. of $\frac{X_1+X_2}{\sqrt{2}}$ and $\frac{X_1-X_2}{\sqrt{2}}$.

ii) Argue that $2X_1X_2$ and $(X_1^2 - X_2^2)$ have the same distribution.

Soln. → The PDF of (X_1, X_2) is

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi} \cdot e^{-\frac{1}{2}(x_1^2+x_2^2)} \;, \quad (x_1,x_2) \in \mathbb{R}^2$$

i) Let, $Y_1 = \frac{X_1+X_2}{\sqrt{2}}$ and $Y_2 = \frac{X_1-X_2}{\sqrt{2}}$.

$$\therefore y_1 = \frac{x_1+x_2}{\sqrt{2}} \text{ and } y_2 = \frac{x_1-x_2}{\sqrt{2}}$$

$$\Rightarrow x_1 = \frac{y_1+y_2}{\sqrt{2}}, \quad x_2 = \frac{y_1-y_2}{\sqrt{2}}.$$

Clearly, $(y_1, y_2) \in \mathbb{R}^2$.

Jacobian is $J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{vmatrix} = -1$.

The PDF of (Y_1, Y_2) is

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{2\pi} \cdot e^{-\frac{1}{2}\left\{\left(\frac{y_1+y_2}{\sqrt{2}}\right)^2 + \left(\frac{y_1-y_2}{\sqrt{2}}\right)^2\right\}} \cdot |J| \quad \text{if } y_1, y_2 \in \mathbb{R}$$

$$= \frac{1}{2\pi} \cdot e^{-\frac{1}{2}(y_1^2+y_2^2)} \;, \quad y_1, y_2 \in \mathbb{R}$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}y_1^2}}_{\downarrow} \cdot \underbrace{\frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}y_2^2}}_{\downarrow} \;, \quad y_1, y_2 \in \mathbb{R}$$

$$= f_{Y_1}(y_1) \cdot f_{Y_2}(y_2) \;, \quad y_1, y_2 \in \mathbb{R}$$

Hence, $Y_1, Y_2 \overset{iid}{\sim} N(0,1)$.

ii) Let $U = 2X_1X_2$ where $X_1, X_2 \overset{iid}{\sim} N(0,1)$

and Let $V = X_1^2 - X_2^2 = 2 \cdot \frac{X_1+X_2}{\sqrt{2}} \cdot \frac{X_1-X_2}{\sqrt{2}}$

$= 2Y_1Y_2$, where $Y_1, Y_2 \overset{iid}{\sim} N(0,1)$

Note that U and V both are twice the product of two iid $N(0,1)$ variables and they must have the same distribution.

☆ **Ex.4.** Let X and Y be independent RV's with common PDF, $f(x) = \begin{cases} e^{-x} & , \text{if } x>0 \\ 0 & , \text{ow} \end{cases}$

Find the distribution of $U = X - Y$.

Soln. → Here

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y} & , \text{if } x>0, y>0 \\ 0 & , \text{ow}. \end{cases}$$

Let, $U = X - Y$ and $V = X + Y$.

$\therefore u = x - y$, $v = x + y$

$\Rightarrow x = \dfrac{u+v}{2}$ and $y = \dfrac{v-u}{2}$

Note that, $0 < x < \infty$, $0 < y < \infty$

$\Rightarrow 0 < u + v < \infty$, $0 < v - u < \infty$

$\Rightarrow -u < v < \infty$, $u < v < \infty$.

$\Rightarrow 0 < |u| < v < \infty$.

Jacobian is $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$.

The PDF of (U,V) is

$$f_{UV}(u,v) = \begin{cases} e^{-v} \cdot \left|\frac{1}{2}\right| & , \text{if } 0<|u|<v<\infty. \\ 0 & , \text{ow} \end{cases}$$

$$= \begin{cases} \frac{1}{2} e^{-v} & , \text{if } 0<|u|<v<\infty. \\ 0 & , \text{ow} \end{cases}$$

The PDF of U is,

$$f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u,v)\, dv$$

$$= \int_{|u|}^{\infty} e^{-v} \cdot \frac{1}{2}\, dv,$$

$$= \frac{1}{2} e^{-|u|}, \quad u \in \mathbb{R}$$

which is the PDF of standard Laplace distribution.

Hence, $U = X - Y$ follows standard Laplace distn.

☆ **Ex. 5.** Let $X, Y \overset{iid}{\sim} N(0,1)$. Show that $U = \frac{X}{Y}$ has a Standard Cauchy distribution. What would be the distn. of $\frac{X}{|Y|}$?

[C.U. 2011]

Soln. → Here, $f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$, $(x,y) \in \mathbb{R}^2$

Let, $U = \frac{X}{Y}$ and $V = Y$.

$\therefore u = \frac{x}{y}$, $v = y$ $\left[\begin{array}{l} -\infty < u < \infty \\ -\infty < v < \infty \end{array}\right]$

$\Rightarrow x = uv$, $y = v$

$$\therefore J = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Clearly, $(u,v) \in \mathbb{R}^2$

The PDF of (U,V) is —

$$f_{U,V}(u,v) = \frac{1}{2\pi} e^{-(1+u^2)\frac{v^2}{2}} |v|, \quad (u,v) \in \mathbb{R}^2$$

The PDF of U is

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,dv = \int_{-\infty}^{\infty} \frac{1}{2\pi} \cdot e^{-(1+u^2)\frac{v^2}{2}} |v|\,dv$$

$$= \frac{2}{2\pi} \int_0^{\infty} e^{-(1+u^2)\frac{v^2}{2}} v\,dv$$

$$= \frac{1}{\pi} \int_0^{\infty} e^{-(1+u^2)z}\,dz, \quad \text{where } z = \frac{v^2}{2} \Rightarrow dz = v\,dv$$

$$= \frac{1}{\pi} \cdot \frac{\Gamma(1)}{(1+u^2)}, \quad u \in \mathbb{R}$$

$$= \frac{1}{\pi(1+u^2)}, \quad u \in \mathbb{R}$$

or,
$$= 2\int_0^{\infty} \frac{v}{2\pi} e^{-\frac{1}{2}v^2(1+u^2)}\,dv$$

$$= \int_0^{\infty} \frac{1}{\pi} \cdot \frac{1}{(1+u^2)} \cdot e^{-z}\,dz \quad \begin{array}{l} \frac{1}{2}v^2(1+u^2) = z \\ \therefore v\,(u^2+1)\,dv = dz \end{array}$$

$$= \frac{1}{\pi(1+u^2)} \left[-e^{-z}\right]_0^{\infty}$$

$$= \frac{1}{\pi(1+u^2)}.$$

Hence, $U = \frac{X}{Y} \sim C(0,1)$ distn.

Let, $W = \frac{X}{|Y|}$, The DF of W is $F_W(w) = P[W \le w]$

$$F_W(w) = P[W \le w / Y < 0]\, P[Y < 0] + P[W \le w / Y > 0]\, P[Y > 0]$$

$$= \frac{1}{2}\left\{P\left[\frac{X}{-Y} \le w\right] + P\left[\frac{X}{Y} \le w\right]\right\}$$

$$= \frac{1}{2}\left\{P[-U \le w] + P[U \le w]\right\}$$

$$= \frac{1}{2} \cdot 2 \cdot P[U \le w] \quad [\because U \sim C(0,1) \text{ is symmetrical about '0'.}]$$

$[\Rightarrow f_U(-u) = f_U(u)$

$\Rightarrow$ U and $-U$ have identical distribution$]$

$\therefore F_W(w) = F_U(w) \quad \forall\, w$

$\Rightarrow W = \frac{X}{|Y|} \sim C(0,1)$.

Ex. 6. [Box-Muller Transformation]

Let $X_1, X_2 \overset{iid}{\sim} R(0,1)$. Show that —

$U_1 = \sqrt{-2\ln X_1}\cos(2\pi X_2)$

$U_2 = \sqrt{-2\ln X_1}\sin(2\pi X_2)$ [C.U. 2003]

are standard normal variables.

Soln. → The PDF of (X_1, X_2) is $f_{X_1,X_2}(x_1,x_2) = \begin{cases} 1 & , 0<x_1,x_2<1 \\ 0 & , \text{ow} \end{cases}$

Here, $u_1 = \sqrt{-2\ln x_1}\cos(2\pi x_2)$

$u_2 = \sqrt{-2\ln x_1}\sin(2\pi x_2)$

$\therefore u_1^2 + u_2^2 = -2\ln x_1$

$\Rightarrow x_1 = e^{-\frac{1}{2}(u_1^2+u_2^2)}$

and $\tan(2\pi x_2) = \frac{u_2}{u_1}$

$\Rightarrow x_2 = \frac{1}{2\pi}\tan^{-1}\left(\frac{u_2}{u_1}\right)$.

Note that, $0<x_1<1$, $0<x_2<1$

$\Rightarrow -2\ln x_1 > 0$, $0<2\pi x_2<2\pi$

$\Rightarrow \sqrt{-2\ln x_1} > 0$, $-1 \le \cos(2\pi x_2), \sin(2\pi x_2) \le 1$.

$\Rightarrow u_1, u_2 \in \mathbb{R}$.

The Jacobian is $J = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} \end{vmatrix}$

$$= \begin{vmatrix} e^{-\frac{1}{2}(u_1^2+u_2^2)}\cdot(-u_1) & e^{-\frac{1}{2}(u_1^2+u_2^2)}\cdot(-u_2) \\ \frac{1}{2\pi\left\{1+\left(\frac{u_2}{u_1}\right)^2\right\}}\cdot\left(-\frac{u_2}{u_1^2}\right) & \frac{1}{2\pi\left\{1+\left(\frac{u_2}{u_1}\right)^2\right\}}\cdot\frac{1}{u_1} \end{vmatrix}$$

$$= \frac{e^{-\frac{1}{2}(u_1^2+u_2^2)}}{2\pi\left\{1+\left(\frac{u_2}{u_1}\right)^2\right\}} \begin{vmatrix} -u_1 & -u_2 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1} \end{vmatrix}$$

$$= -\frac{1}{2\pi}\cdot e^{-\frac{1}{2}(u_1^2+u_2^2)}.$$

The PDF of (U_1, U_2) is

$$f_{U_1,U_2}(u_1,u_2) = 1.\left|-\frac{1}{2\pi}\cdot e^{-\frac{1}{2}(u_1^2+u_2^2)}\right| , (u_1,u_2)\in\mathbb{R}^2$$

$$= \frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{1}{2}u_1^2}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u_2^2} , (u_1,u_2)\in\mathbb{R}^2$$

$$= f_{U_1}(u_1)\cdot f_{U_2}(u_2) , u_1, u_2 \in \mathbb{R}.$$

Hence, U_1 and $U_2 \overset{iid}{\sim} N(0,1)$.

☆ **Ex.7.** If $X \sim Beta(a,b)$ and $Y \sim Beta(c,d)$, independently, where $a = c+d$, then S.T.

$$XY \sim Beta(c, b+d).$$

Soln. →

Here
$$f_{XY}(x,y) = \begin{cases} \dfrac{x^{a-1}(1-x)^{b-1}}{\beta(a,b)} \cdot \dfrac{y^{c-1}(1-y)^{d-1}}{\beta(c,d)} & , \text{if } 0<x,y<1 \\ 0 & , \text{OW} \end{cases}$$

Let, $U = XY$, $V = X$

$\therefore u = xy$, $v = x$

$\Rightarrow x = v$, $y = \frac{u}{v}$.

Note that, $0<x<1$, $0<y<1$

$\Rightarrow 0<v<1$, $0<u<v$

$\Rightarrow 0<u<v<1$.

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{1}{v}.$$

The PDF of (U,V) is

$$f_{U,V}(u,v) = \begin{cases} \dfrac{v^{a-1}(1-v)^{b-1}}{\beta(a,b)\,\beta(c,d)} \left(\frac{u}{v}\right)^{c-1}\left(1-\frac{u}{v}\right)^{d-1} \cdot \left|-\frac{1}{v}\right| & , 0<u<v<1 \\ 0 & , \text{OW} \end{cases}$$

$$= \begin{cases} \dfrac{(1-v)^{b-1}\, u^{c-1}\,(v-u)^{d-1}}{\beta(a,b)\,\beta(c,d)} & , 0<u<v<1 \\ 0 & , \text{OW} \end{cases}$$

Here,
$$f_U(u) = \begin{cases} \dfrac{\int_u^1 (1-v)^{b-1}\, u^{c-1}\,(v-u)^{d-1}\,dv}{\beta(a,b)\,\beta(c,d)} & , 0<u<1 \\ 0 & , \text{OW} \end{cases}$$

$$= \begin{cases} \dfrac{u^{c-1}}{\beta(a,b)\,\beta(c,d)} \int_u^1 (1-v)^{b-1}(v-u)^{d-1}\,dv & , 0<u<1 \\ 0 & , \text{OW} \end{cases}$$

[Let, $\frac{v-u}{1-u} = z$, $dv = (1-u)\,dz$

and $1 - \frac{v-u}{1-u} = 1-z \Rightarrow \frac{1-v}{1-u} = 1-z$

$\Rightarrow (1-v) = (1-z)(1-u)$]

$$= \begin{cases} \dfrac{u^{c-1}(1-u)^{b+d-1}}{\beta(a,b)\beta(c,d)} \displaystyle\int_0^1 z^{d-1}(1-z)^{b-1}\,dz, & 0<u<1 \\ 0 & , OW \end{cases}$$

$$= \begin{cases} \dfrac{\beta(d,b)}{\beta(a,b)\beta(c,d)} \cdot u^{c-1}(1-u)^{b+d-1} & ; 0<u<1 \\ 0 & ; OW \end{cases}$$

$$= \begin{cases} \dfrac{u^{c-1}(1-u)^{b+d-1}}{\beta(c,b+d)} & , 0<u<1 \\ 0 & , OW \end{cases}$$

$$\left[\because \frac{\beta(d,b)}{\beta(a,b)\beta(c,d)} = \frac{\Gamma(a+b)}{\Gamma(c)\Gamma(d+b)} = \frac{\Gamma(c+d+b)}{\Gamma(c)\Gamma(b+d)} = \beta(c,b+d), \text{ as } a=c+d \right]$$

Ex.8. Let $X_1, X_2 \overset{iid}{\sim} R(0,1)$. Find the PDF of i) X_1+X_2, ii) X_1-X_2, iii) X_1X_2, iv) $\dfrac{X_1}{X_2}$.

Soln. → Here, $f_{X_1X_2}(x_1,x_2) = \begin{cases} 1 & , 0<x_1, x_2<1 \\ 0 & , OW \end{cases}$

i) and ii)

Let $U = X_1+X_2$, $V = X_1-X_2$

$\therefore u = x_1+x_2$, $v = x_1-x_2$

$\therefore x_1 = \dfrac{u+v}{2}$, $x_2 = \dfrac{u-v}{2}$.

Note that, $0<x_1<1$, $0<x_2<1$

$\Rightarrow 0 < \dfrac{u+v}{2} < 1$, $0 < \dfrac{u-v}{2} < 1$

$\Rightarrow 0 < u+v < 2$, $0 < u-v < 2$

Jacobian is, $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$.

The PDF of (U,V) is

$$f_{U,V}(u,v) = \begin{cases} 1 \cdot \left|-\frac{1}{2}\right| & , 0<v+u<2, \ 0<u-v<2 \\ 0 & , OW \end{cases}$$

i) To find the PDF of U: —

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,dv$$

Now, $0 < u+v < 2$, $0 < u-v < 2$

$\Rightarrow -u < v < 2-u$, $u-2 < v < u$

$\Rightarrow \max\{-u, u-2\} < v \leq \min\{2-u, u\}$

and $0 < u < 2$.

when, $0 < u < 1$, $-u < v < u$

and, $1 \leq u < 2$, $u-2 < v < 2-u$

$$\therefore f_U(u) = \begin{cases} \int_{-u}^{u} \frac{1}{2}\,dv, & 0 < u < 1 \\ \int_{u-2}^{2-u} \frac{1}{2}\,dv, & 1 \leq u < 2 \\ 0, & ow \end{cases}$$

$$= \begin{cases} u, & 0 < u < 1 \\ 2-u, & 1 \leq u < 2 \\ 0, & ow \end{cases}$$

ii) To find the PDF of V: —

Note that, $0 < u+v < 2$ and $0 < u-v < 2$

$\Rightarrow -v < u < 2-v$ and $v < u < 2+v$

$\Rightarrow \max\{-v, v\} < u < \min\{2-v, 2+v\}$

and $-1 < v < 1$

when, $-1 < v < 0$, then $-v < u < 2+v$

and $0 < v < 1$, then $v < u < 2-v$

The PDF of V is, —

$$f_V(v) = \begin{cases} \int_{-v}^{2+v} \frac{1}{2}\,du, & \text{if } -1 < v < 0 \\ \int_{v}^{2-v} \frac{1}{2}\,du, & \text{if } 0 \leq v < 1 \\ 0, & ow \end{cases}$$

$$= \begin{cases} 1+v, & \text{if } -1 < v < 0 \\ 1-v, & \text{if } 0 \leq v < 1 \\ 0, & ow \end{cases}$$

$$= \begin{cases} 1-|v|, & \text{if } -1 < v < 1 \\ 0, & ow. \end{cases}$$

iii) Let, $U = X_1 X_2$ and $V = X_2$

$\therefore u = x_1 x_2$ and $v = x_2$

$\Rightarrow x_1 = \frac{u}{v}$, $x_2 = v$.

Note that, $0 < x_1 < 1$, $0 < x_2 < 1$

$\Rightarrow 0 < \frac{u}{v} < 1$, $0 < v < 1$

$\Rightarrow 0 < u < v$, $0 < v < 1$

$\Rightarrow 0 < u < v < 1$.

Jacobian is $J = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$

The PDF of (U,V) is

$$f_{U,V}(u,v) = \begin{cases} 1 \cdot \left|\frac{1}{v}\right| & \text{, if } 0 < u < v < 1 \\ 0 & \text{, ow} \end{cases}$$

$$= \begin{cases} \frac{1}{v} & , 0 < u < v < 1 \\ 0 & , \text{ow} \end{cases}$$

The PDF of U is,

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, dv$$

$$= \begin{cases} \int_u^1 \frac{1}{v}\, dv & , 0 < u < 1 \\ 0 & , \text{ow} \end{cases}$$

$$= \begin{cases} -\ln u & , 0 < u < 1 \\ 0 & , \text{ow} \end{cases}$$

iv) Let, $U = \frac{X_1}{X_2}$ and $V = X_2$.

$\therefore u = \frac{x_1}{x_2}$, and $v = x_2$

$\Rightarrow x_1 = uv$, $x_2 = v$.

Note that, $0 < x_1 < 1$, $0 < x_2 < 1$

$\Rightarrow 0 < uv < 1$, $0 < v < 1$

$\Rightarrow 0 < v < \frac{1}{u}$, $0 < v < 1$

$\Rightarrow 0 < v < \min\{\frac{1}{u}, 1\}$ and $0 < u < \infty$.

If $0 < u < 1$, then $0 < v < 1$

and if $u \geq 1$, then $0 < v < \frac{1}{u}$,

Jacobian is, $J = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$

The PDF of (U, V) is

$$f_{U,V}(u,v) = \begin{cases} 1 \cdot |v|, & \text{if } 0 < v < \min\{\frac{1}{u}, 1\} \text{ and } 0 < u < \infty \\ 0, & \text{ow} \end{cases}$$

The PDF of U is —

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, dv$$

$$= \begin{cases} \int_0^1 v\, dv, & \text{if } 0 < u < 1 \\ \int_0^{1/u} v\, dv, & \text{if } 1 \leq u < \infty \\ 0, & \text{ow} \end{cases}$$

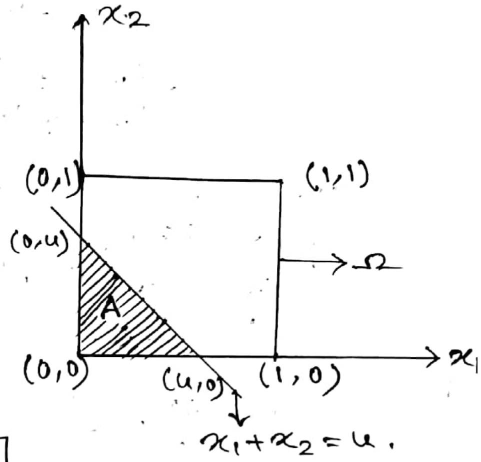
$$= \begin{cases} \frac{1}{2}, & \text{if } 0 < u < 1 \\ \frac{1}{2u^2}, & \text{if } 1 \leq u < \infty \\ 0 & \text{ow} \end{cases}$$

☆ Ex.9. Let $X_1, X_2 \overset{iid}{\sim} R(0,1)$. Find out CDF and hence the PDF of X_1+X_2. How should the above result be modified in case X_1 and $X_2 \overset{iid}{\sim} R(a,b)$?

Soln. →

$$F_U(u) = P[U \le u]$$
$$= P[X_1+X_2 \le u]$$
$$= \iint_{x_1+x_2 \le u} f_{X_1X_2}(x_1,x_2)\,dx_1\,dx_2$$

Here, $U = X_1+X_2$ takes values between 0 and 2.
Note that for $0<u<1$,

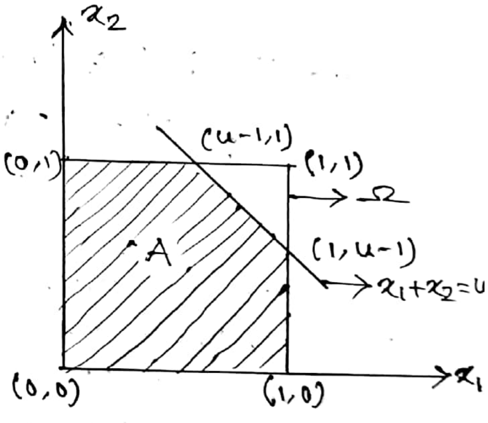


$$P[U \le u] = P[X_1+X_2 \le u]$$
$$= \frac{\text{Area of the region A}}{\text{Area of the sample space } (\Omega)}.$$

[Using the concept of Geometric probability, as (X_1,X_2) is uniformly distributed over Ω. Here,
$\Omega = \{(x_1,x_2) : 0<x_1,x_2<1\}$
and $A = \{(x_1,x_2) : x_1+x_2 \le u\} \subseteq \Omega$]

$$\therefore P[U \le u] = \frac{\frac{1}{2}u^2}{1^2} = \frac{1}{2}u^2, \text{ for } 0<u<1.$$

For $1 \le u < 2$, ——



$$P[U \le u] = P[X_1+X_2 \le u]$$
$$= \frac{\text{Area of the Region A}}{\text{Area of the sample space } (\Omega)}$$
$$= \frac{1^2 - \frac{1}{2}(2-u)^2}{1^2}$$
$$= 1-\frac{1}{2}(2-u)^2$$

Hence the CDF of U is —
$$F_U(u) = \begin{cases} 0, & u \le 0 \\ \frac{1}{2}u^2, & 0<u<1 \\ 1-\frac{1}{2}(2-u)^2, & 1 \le u < 2 \\ 1, & u \ge 2 \end{cases}$$

and the PDF of U is ——
$$f_U(u) = \begin{cases} u, & 0<u<1 \\ 2-u, & 1 \le u < 2 \\ 0, & \text{ow} \end{cases}$$

☒ Modification:— $X_i \overset{iid}{\sim} R(a,b)$, $i=1,2$.
$$\Rightarrow U_i = \frac{X_i-a}{b-a} \overset{iid}{\sim} R(0,1), \; i=1,2.$$

☆ **Ex.10.** Let $X_1, X_2 \overset{iid}{\sim} R(0,1)$

Find the CDF and then PDF of

i) $|X_1 - X_2|$, ii) $X_1 X_2$, iii) $X_1 - X_2$, iv) $\frac{X_1}{X_2}$.

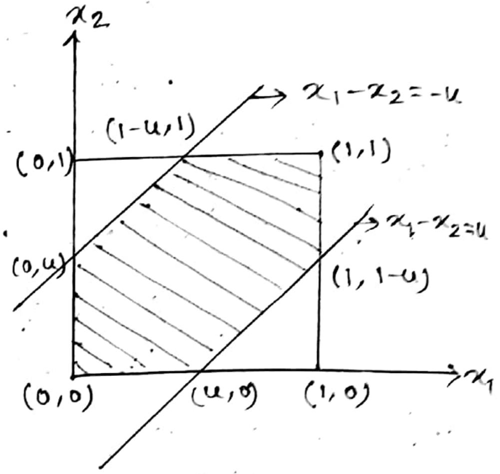
Soln. →

i) Let $U = |X_1 - X_2|$

Note that U takes values between 0 and 1.

For, $0 < u < 1$,

$$P[U \le u] = P[|X_1 - X_2| \le u]$$
$$= P[-u \le (X_1 - X_2) \le u]$$
$$= \frac{\text{Area of the shaded region}}{\text{Area of the sample space } (\Omega)}$$



[Using the concept of Geometric probability, as, (X_1, X_2) is uniformly distributed over Ω]

$$P[U \le u] = \frac{1^2 - 2 \cdot \frac{1}{2}(1-u)^2}{1^2}$$
$$= 1 - (1-u)^2$$

Hence, the CDF of U is

$$F_U(u) = \begin{cases} 0 & , u \le 0 \\ 1-(1-u)^2, & 0 < u < 1 \\ 1 & , u \ge 1 \end{cases}$$

and the PDF of U is

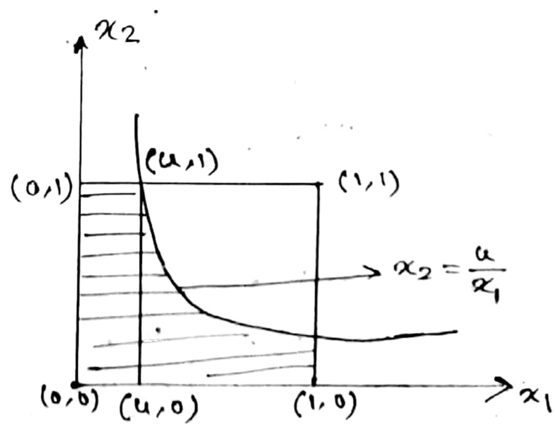
$$f_U(u) = \begin{cases} 2(1-u) & , 0 < u < 1 \\ 0 & , \text{ow} \end{cases}$$

ii) Let, $U = X_1 X_2$

Then U takes values between 0 and 1.

For, $0 < u < 1$;

$$P[U \le u] = P[X_1 X_2 \le u]$$
$$= \frac{\text{Area of the shaded region}}{\text{Area of the sample space } (\Omega)}$$



$$= \frac{u x_1 + \int_u^1 x_2 \, dx_1}{1^2}$$
$$= u + \int_u^1 \frac{u}{x_1} dx_1$$
$$= u + u[\ln x_1]_u^1 = u(1 - \ln u)$$

The CDF of U is

$$F_U(u) = \begin{cases} 0 & , u \leq 0 \\ u(1 - \ln u) & , 0 < u < 1 \\ 1 & , u \geq 1 \end{cases}$$

The PDF of U is,

$$f_U(u) = \begin{cases} -\ln u & , 0 < u < 1 \\ 0 & , \text{o/w}. \end{cases}$$

iii)

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Ex.12. Let X and Y are independently distributed with densities,

$$f_X(x) = \begin{cases} \frac{1}{\pi\sqrt{1-x^2}} & , |x|<1 \\ 0 & , \text{OW} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{y}{\sigma^2} \cdot e^{-\frac{1}{2}\cdot\frac{y^2}{\sigma^2}} & , y>0 \\ 0 & , \text{OW} \end{cases}$$

Show that $XY \sim N(0, \sigma^2)$. [2001]

Soln. →

$$f_{X,Y}(x,y) = \begin{cases} \frac{y e^{-\frac{y^2}{2\sigma^2}}}{\pi\sigma^2\sqrt{1-x^2}} & , \text{if } |x|<1, y>0 \\ 0 & , \text{OW} \end{cases}$$

Let, $U = XY$, and $V = Y$

$\therefore$ $u = xy$ and $v = y$

$\Rightarrow x = \frac{u}{v}$, $y = v$.

Note that, $|x|<1$, $0<y<\infty$

$\Rightarrow \left|\frac{u}{v}\right| < 1$, $0<v<\infty$

$\Rightarrow |u| < v$, $0<v<\infty$

$\Rightarrow 0 < |u| < v < \infty$.

Note that,

$$J = \frac{1}{v}.$$

The PDF of (U,V) is

$$f_{UV}(u,v) = \begin{cases} \frac{v}{\pi\sigma^2\sqrt{1-\frac{u^2}{v^2}}} \cdot e^{-\frac{1}{2}\frac{v^2}{\sigma^2}} \cdot \left|\frac{1}{v}\right| & \text{if } 0<|u|<v<\infty \\ 0 & , \text{OW} \end{cases}$$

$$= \begin{cases} \frac{v e^{-\frac{v^2}{2\sigma^2}}}{\pi\sigma^2\sqrt{v^2-u^2}} & ; 0<|u|<v<\infty \\ 0 & ; \text{OW} \end{cases}$$

The PDF of U is

$$f_U(u) = \int_{|u|}^{\infty} \frac{v e^{-\frac{v^2}{2\sigma^2}}}{\pi\sigma^2\sqrt{v^2-u^2}}\, dv \quad , u \in \mathbb{R}$$

$$= \int_{0}^{\infty} \frac{e^{-\frac{u^2+z^2}{2\sigma^2}} \cancel{z}\, dz}{\pi\sigma^2\sqrt{\cancel{z^2}}} \quad ; \text{ let, } v^2-u^2=z^2 \Rightarrow v\,dv = z\,dz$$

$$= \frac{e^{-\frac{u^2}{2\sigma^2}}}{\pi\sigma^2} \int_0^{\infty} e^{-\frac{z^2}{2\sigma^2}}\, dz$$

$$= \frac{e^{-u^2/2\sigma^2}}{\pi\sigma^2} \left(\frac{1}{2}\sigma\sqrt{2\pi}\right) , \quad u \in \mathbb{R}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-u^2/2\sigma^2} \quad ; u \in \mathbb{R}$$

Hence, $U = XY \sim N(0, \sigma^2)$.

☆ __Ex. 13.__ Mention and justify the suitable transformation from the $R(0,1)$ distribution to get

i) Exponential distribution ii) Normal distribution.

__Soln__:- i) Let $X \sim R(0,1)$

$\therefore$ PDF of X is given by, $f_X(x) = 1 \quad , 0<x<1$

Let us consider the following transformation

$X \to Y$ such that, $Y = -2\ln X \quad , y>0$

$$\Rightarrow X = e^{-y/2}$$

$\therefore$ Jacobian of the transformation is given by,

$$|J| = \left|\frac{\partial x}{\partial y}\right| = \frac{1}{2} e^{-y/2}$$

$\therefore$ PDF of Y is given by,

$$f_Y(y) = \frac{1}{2} e^{-y/2} \quad , y>0$$

$\therefore Y \sim \text{Exp}(2)$.

ii) See Box-muller transformation.

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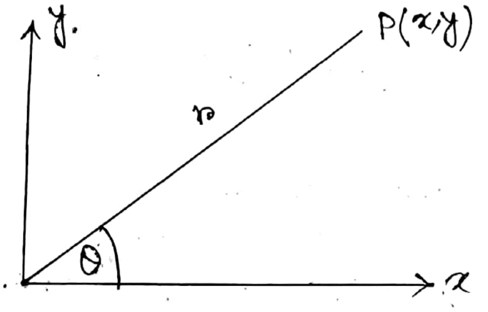
Polar Transformation :— Let (x,y) be a point in $\mathbb{R}^2$. Then the polar co-ordinate is given by (r,θ) where

$$x = r\cos\theta$$
$$y = r\sin\theta$$

where $0 < r < \infty$, $0 < \theta < 2\pi$



$$\text{Jacobian} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r$$

In particular, if $0 < x, y < \infty$

Then, $x = r\cos\theta$

$y = r\sin\theta$, where $0 < r < \infty$, $0 < \theta < \frac{\pi}{2}$

In particular, if $-\infty < x < \infty$, $0 < y < \infty$

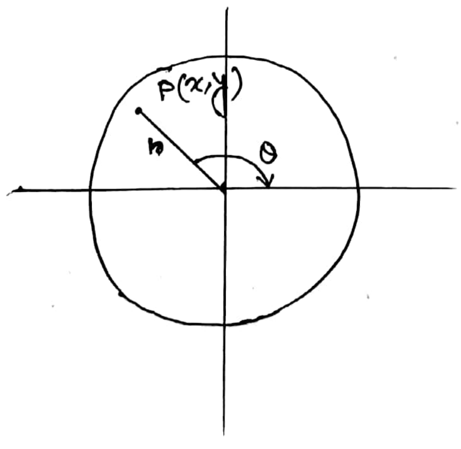
then, $x = r\cos\theta$,

$y = r\sin\theta$, where $0 < r < \infty$, and $0 < \theta < \pi$

In particular, if $(x,y) \in \{(x,y) : x^2 + y^2 \leq a^2\}$

then, $x = r\cos\theta$

$y = r\sin\theta$, where $0 < r < a$ and $0 < \theta < 2\pi$



☆ Ex.1. If (X, Y) is uniformly distributed over a region bounded by a circle of radius 'a', find the PDF of $\sqrt{X^2+Y^2}$.

Soln. → $f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi a^2} &, x^2+y^2 \le a^2 \\ 0 &, OW \end{cases}$

Let, $x = r\cos\theta$,
$y = r\sin\theta$,

As $x^2+y^2 \le a^2$, $0 < r < a$ and $0 < \theta < 2\pi$

Here, $J = r$.

The PDF of (r, θ) is

$$g(r,\theta) = \begin{cases} \frac{1}{\pi a^2} \cdot |r| & \text{if } 0 < r < a \text{ and } 0 < \theta < 2\pi. \\ 0 &, OW \end{cases}$$

Note that, $r = \sqrt{x^2+y^2}$,

The PDF of r is

$$g_1(r) = \begin{cases} \int_0^{2\pi} \frac{r}{\pi a^2} d\theta, & \text{if } 0 < r < a \\ 0 &, OW \end{cases}$$

$$= \begin{cases} \frac{2r}{a^2} &, 0 < r < a \\ 0 \end{cases}$$

☆ Ex.2. Let, $f_{X,Y}(x,y) = \begin{cases} 4xy e^{-(x^2+y^2)} &, \text{if } x>0, y>0, \\ 0 &, OW \end{cases}$

be the PDF of (X, Y). Find the PDF of (X^2+Y^2).

Soln. → Let $x = r\cos\theta$,
$y = r\sin\theta$.

As $x>0$, $y>0$, $0 < r < \infty$
and $0 < \theta < \frac{\pi}{2}$.

Here, $J = r$,

The PDF of (r, θ) is

$$g(r,\theta) = \begin{cases} 4r^2 \sin\theta\cos\theta . e^{-r^2} |r|. &, \text{if } 0 < r < \infty \text{ and } 0 < \theta < \pi/2. \\ 0 &, OW \end{cases}$$

Note that $r = \sqrt{x^2+y^2}$;

The PDF of r is

$$g_1(r) = \begin{cases} 4r^3 e^{-r^2} \int_0^{\pi/2} \sin\theta\cos\theta \, d\theta, & 0<r<\infty \\ 0 & , ow \end{cases}$$

$$= \begin{cases} 2r^3 e^{-r^2}, & 0<r<\infty \\ 0 & , ow \end{cases}$$

To find the distn. of $(X^2+Y^2) = R^2$

Let $U = R^2$, $r = \pm\sqrt{u}$, as $0<r<\infty$,

Hence, the PDF of $u = r^2$ is —

$$f_U(u) = \begin{cases} g_1(\sqrt{u}) \left| \frac{d\sqrt{u}}{du} \right| & , 0<u<\infty \\ 0 & , ow \end{cases}$$

$$= \begin{cases} \frac{e^{-u} \cdot u^{2-1}}{\Gamma(2)}, & 0<u<\infty \\ 0 & , ow \end{cases}$$

Hence, $r^2 = X^2+Y^2 \sim \text{Gamma}(2)$

☆ Ex. 3. If $X, Y \overset{iid}{\sim} N(0,1)$ and $x = r\cos\theta$, $y = r\sin\theta$, then show that $r^2 \sim$ Exp. with mean 2 and, $\theta \sim U(0, 2\pi)$, independently.

Soln. $f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$, $(x,y) \in \mathbb{R}^2$

Here, $x = r\cos\theta$, $y = r\sin\theta$.

As $(x,y) \in \mathbb{R}^2$, $0<r<\infty$, and $0<\theta<2\pi$

$J = r$

The PDF of (r,θ) is $\rightarrow$

$$g(r,\theta) = \begin{cases} \frac{1}{2\pi} e^{-r^2/2} \cdot |r| & , \text{if } 0<r<\infty \text{ and } 0<\theta<2\pi \\ 0 & , ow \end{cases}$$

$$= \begin{cases} \left(r e^{-r^2/2}\right)\left(\frac{1}{2\pi}\right) & , \text{if } 0<r<\infty \text{ and } 0<\theta<2\pi \\ 0 & , ow \end{cases}$$

$$= g_1(r) \cdot g_2(\theta) \quad \forall (r,\theta)$$

Hence, r with PDF

$$g_1(r) = \begin{cases} r \cdot e^{-r^2/2} & , 0 < r < \infty \\ 0 & , ow \end{cases}$$

and $\Theta \sim U(0, 2\pi)$, independently.

Let $u = r^2$,

$\therefore r = \sqrt{u}$, as $0 < r < \infty$.

Hence, $$f_U(u) = \begin{cases} g_1(\sqrt{u}) \left| \frac{d\sqrt{u}}{du} \right| & , 0 < u < \infty \\ 0 & , ow. \end{cases}$$

$$= \begin{cases} \frac{1}{2} e^{-u/2} & , 0 < u < \infty \\ 0 & , ow \end{cases}$$

Hence, $U = r^2 \sim$ Exp. with mean 2,

and, $\Theta \sim U(0, 2\pi)$, independently.

× —— ×

Result:–

If $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$ is not one-to-one transformation from $\mathcal{X}$ in to D.

Then $X_1 = h_{1i}(y_1, y_2)$

$X_2 = h_{2i}(y_1, y_2)$, $i = 1(1)k$.

and the PDF of (Y_1, Y_2) is

$$f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} \sum_{i=1}^{k} f_{X_1 X_2}(h_{1i}(y_1, y_2)), (h_{2i}(y_1, y_2)) |J_i| & , \text{if } (y_1, y_2) \in D \\ 0 & , ow \end{cases}$$

☆ **Ex. 1.** If $X, Y \overset{iid}{\sim} N(0,1)$. Find the distn.s of $U = \sqrt{X^2+Y^2}$ and $V = \frac{X}{Y}$.

Soln. → $f_{X,Y}(x,y) = \frac{1}{2\pi} \cdot e^{-\frac{1}{2}(x^2+y^2)}$, $(x,y) \in \mathbb{R}^2$

Note that, $u = \sqrt{x^2+y^2}$, $v = \frac{x}{y}$

$\Rightarrow u = |y| \cdot \sqrt{1+v^2}$, $x = vy$

$\Rightarrow x = \pm \frac{uv}{\sqrt{1+v^2}}$, $y = \pm \frac{u}{\sqrt{1+v^2}}$

Let, $x_1 = \frac{uv}{\sqrt{1+v^2}}$, $y_1 = \frac{u}{\sqrt{1+v^2}}$

Then for a pair (U,V), there are two points of (x,y):

(x_1, y_1) , $(-x_1, -y_1)$

The transformation is not one-to-one.

Clearly, $0 < u < \infty$, $v \in \mathbb{R}$

Now, $J_1 = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial y_1}{\partial u} & \frac{\partial y_1}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{v}{\sqrt{v^2+1}} & \frac{u}{(v^2+1)^{3/2}} \\ \frac{1}{\sqrt{v^2+1}} & -\frac{uv}{(v^2+1)^{3/2}} \end{vmatrix}$

$= -\frac{u}{1+v^2} = J_2$

Hence, the PDF of (U,V) is

$$f_{U,V}(u,v) = \begin{cases} f_{X,Y}(x_1, y_1)\,|J_1| + f_{X,Y}(-x_1,-y_1)\,|J_2| & \text{, if } 0<u<\infty,\ -\infty<v<\infty \\ 0 & \text{, ow} \end{cases}$$

$$= \begin{cases} \frac{2}{2\pi} \cdot e^{-\frac{u^2}{2}} \cdot \left| -\frac{u}{1+v^2} \right| & \text{, if } 0<u<\infty, \text{ and } -\infty<v<\infty. \\ 0 & \text{, ow} \end{cases}$$

$$= \begin{cases} \left(u e^{-u^2/2}\right) \cdot \frac{1}{\pi(1+v^2)} & \text{, if } 0<u<\infty \text{ and } v \in \mathbb{R} \\ 0 & \text{, ow} \end{cases}$$

Hence, $U = \sqrt{X^2+Y^2}$ has the PDF

$$f_U(u) = \begin{cases} u e^{-u^2/2} & , 0<u<\infty \\ 0 & , \text{ow} \end{cases}$$

and $V \sim$ Cauchy $(0,1)$, independently.

Ex.2. If $X, Y \overset{iid}{\sim} N(0,1)$, find the distn. of

$$U = \frac{XY}{\sqrt{X^2+Y^2}}, \text{ and } V = \frac{X^2-Y^2}{\sqrt{X^2+Y^2}}.$$ [WBSU/11]

Soln. → $f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}, (x,y) \in \mathbb{R}^2$

Let, $x = r\cos\theta$, $y = r\sin\theta$,

Here, $0 < r < \infty$, $0 < \theta < 2\pi$,

$\therefore J = r$,

The PDF of (r, θ) is —

$$g(r,\theta) = \begin{cases} r e^{-r^2/2} \cdot \frac{1}{2\pi}, & 0 < r < \infty \text{ and } 0 < \theta < 2\pi \\ 0, & \text{ow} \end{cases}$$

Here, $u = r\sin\theta\cos\theta = \frac{r}{2}\sin 2\theta$

and $v = r\cos 2\theta$

Clearly, $(U; V) \in \mathbb{R}^2$

$$J_1 = \frac{\partial(r,\theta)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(r,\theta)}} = \frac{1}{\begin{vmatrix} \frac{1}{2}\sin 2\theta & r\cos 2\theta \\ \cos 2\theta & -2\sin 2\theta \end{vmatrix}} = -\frac{1}{r} = J_2$$

Here, $(2u)^2 + v^2 = r^2$ [a pair (u,v) is obtained, for two pairs: (r,θ), $(r, \theta+2\pi)$. The transformation is not one-to-one]

$\Rightarrow r = \sqrt{4u^2+v^2}$

The PDF of (U,V) is

$$f_{U,V}(u,v) = \frac{2 \cdot e^{-\frac{4u^2+v^2}{2}}}{2\pi} \cdot \left(\sqrt{4u^2+v^2}\right) \left| -\frac{1}{\sqrt{4u^2+v^2}} \right|, (u,v) \in \mathbb{R}^2$$

$$= \frac{1}{\frac{1}{2} \cdot \sqrt{2\pi}} \cdot e^{-\frac{u^2}{2 \cdot \frac{1}{4}}} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-v^2/2} \; ; (u,v) \in \mathbb{R}^2$$

$$= f_U(u) \cdot f_V(v), \; u, v \in \mathbb{R}$$

Hence, $U \sim N(0, \frac{1}{4})$ and $V \sim N(0,1)$, independently.

▨ EXAMPLES ON THREE VARIABLES :—

☆ **Ex.1.** Let X_1, X_2, X_3 be iid RV's with PDF

$$f(x) = \begin{cases} e^{-x}, & x>0 \\ 0, & \text{OW} \end{cases}$$

Show that →

$$Y_1 = X_1+X_2+X_3$$

$$Y_2 = \frac{X_1+X_2}{X_1+X_2+X_3}$$

$$Y_3 = \frac{X_1}{X_1+X_2}$$

, are independently distributed. [C.U. 2001]

Identify their distribution.

Soln. →

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = \begin{cases} e^{-(x_1+x_2+x_3)}, & \text{if } x_i>0 \ \forall\ i=1,2,3. \\ 0, & \text{OW} \end{cases}$$

Here, $x_1+x_2+x_3 = y_1$

$x_1+x_2 = y_1 y_2$

$x_3 = y_1(1-y_2)$

$\because x_1+x_2 < x_1+x_2+x_3$

$\Rightarrow \frac{x_1+x_2}{y_1} < 1, \ \frac{x_1}{x_1+x_2} < 1.$

Clearly, $0<y_1<\infty$ and $0<y_2, y_3<1$.

The Jacobian is

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{vmatrix}$$

$$= \begin{vmatrix} y_2y_3 & y_1y_3 & y_1y_2 \\ y_2(1-y_3) & y_1(1-y_3) & -y_1y_2 \\ 1-y_2 & -y_1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ y_2(1-y_3) & y_1(1-y_3) & -y_1y_2 \\ 1-y_2 & -y_1 & 0 \end{vmatrix} \quad R_1' \to R_1+R_2+R_3$$

$$= -y_1^2 y_2$$

The PDF of (Y_1, Y_2, Y_3) is

$$f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3) = \begin{cases} e^{-y_1} \, |y_1^2 y_2|, & \text{if } 0<y_1<\infty \text{ and } 0<y_2, y_3<1. \\ 0, & \text{OW} \end{cases}$$

$$= \begin{cases} \frac{e^{-y_1} \cdot y_1^{3-1}}{\Gamma(3)} \cdot 2y_2 \cdot 1, & \text{if } 0<y_1<\infty, \ 0<y_2, y_3<1 \\ 0, & \text{OW} \end{cases}$$

$$= f_{Y_1}(y_1) f_{Y_2}(y_2) f_{Y_3}(y_3) \quad \forall\ (y_1, y_2, y_3)$$

where,

$$f_{Y_1}(y_1) = \begin{cases} \frac{e^{-y_1} \cdot y_1^{3-1}}{\Gamma(3)}, & 0<y_1<\infty \\ 0, & \text{OW} \end{cases} \quad \therefore Y_1 \sim \text{Gamma}(3)$$

$$f_{Y_2}(y_2) = \begin{cases} \frac{y_2^{2-1}(1-y_2)^{1-1}}{B(2,1)}, & 0<y_2<1 \\ 0, & \text{OW} \end{cases} \quad \therefore Y_2 \sim \beta_1(2,1)$$

and,

$$f_{Y_3}(y_3) = \begin{cases} 1, & 0<y_3<1 \\ 0, & \text{OW} \end{cases} \quad , \therefore Y_3 \sim U(0,1).$$

[Due to independence]

☆ Ex.2. Let $f_{X,Y,Z}(x,y,z) = \begin{cases} \dfrac{6}{(1+x+y+z)^4}, & \text{if } x,y,z>0 \\ 0, & \text{ow} \end{cases}$ be the PDF of (X,Y,Z). Find the distn. of $U = X+Y+Z$.

Soln. →

$$U = X+Y+Z$$
$$V = \frac{X+Y}{X+Y+Z}$$
$$W = \frac{X}{X+Y},$$

Here, $0 < u < \infty$
$0 < v, w < 1$.

$$J = -u^2 v.$$

$$f_{U,V,W}(u,v,w) = \begin{cases} \dfrac{6}{(1+u)^4} \cdot |-u^2 v|, & 0 < u < \infty \text{ and } 0 < v, w < 1 \\ 0, & \text{ow} \end{cases}$$

The PDF of U is,

$$f_U(u) = \begin{cases} \int_0^1 \left(\int_0^1 \dfrac{6u^2 v}{(1+u)^4} dv \right) dw, & \text{if } 0 < u < \infty \\ 0, & \text{ow} \end{cases}$$

$=$

$$= \begin{cases} \dfrac{1}{\beta(3,1)} \cdot \dfrac{u^{3-1}}{(1+u)^{3+1}}, & 0 < u < \infty \\ 0, & \text{ow} \end{cases}$$

Hence, $U \sim$ Second Kind Beta $(3,1)$

☆ Ex.3. If $X_1, X_2, X_3 \overset{iid}{\sim} N(0,1)$. find the distn.s of

$$Y_1 = \frac{X_1+X_2+X_3}{\sqrt{3}}$$
$$Y_2 = \frac{X_1-X_2}{\sqrt{2}}$$
$$Y_3 = \frac{X_1+X_2-2X_3}{\sqrt{6}}$$

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Soln. →

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = \left(\frac{1}{2\pi}\right)^{3/2} \cdot e^{-\frac{1}{2}(x_1^2+x_2^2+x_3^2)} \quad ; x_i \in \mathbb{R}$$

Note that, $\underset{\sim}{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$= A\underset{\sim}{x}$, where A is orthogonal, i.e. $AA^T = I_3$.

$\therefore \underset{\sim}{x} = A^{-1}\underset{\sim}{y} = A^T\underset{\sim}{y}$ and

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\Rightarrow x_1 = \frac{y_1}{\sqrt{3}} + \frac{y_2}{\sqrt{2}} + \frac{y_3}{\sqrt{6}}$$

$$x_2 = \frac{y_1}{\sqrt{3}} - \frac{y_2}{\sqrt{2}} + \frac{y_3}{\sqrt{6}}$$

$$x_3 = \frac{y_1}{\sqrt{3}} + 0 \cdot y - \frac{2y_3}{\sqrt{6}}$$

$$\text{Jacobian} = \left|\frac{\partial(\text{old variable})}{\partial(\text{new variable})}\right| = \left|\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{y}}\right| \text{ or } \left|\frac{\partial(x_1,x_2,x_3)}{\partial(y_1,y_2,y_3)}\right|$$

$$= \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{vmatrix} = |A^T| = \pm 1$$

Note that, $\underset{\sim}{y}'\underset{\sim}{y} = \underset{\sim}{x}'A'A\underset{\sim}{x} = \underset{\sim}{x}'\underset{\sim}{x} \Rightarrow \sum_{i=1}^{3} y_i^2 = \sum_{i=1}^{3} x_i^2$.

Clearly, $y_i \in \mathbb{R}$, $i = 1,2,3$.

The PDF of (Y_1, Y_2, Y_3) is

$$f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3) = \frac{1}{(2\pi)^{3/2}} \cdot e^{-\frac{1}{2}\sum_{i=1}^{3} y_i^2} \cdot |\pm 1| \quad , y_i \in \mathbb{R}$$

$$= \prod_{i=1}^{3} \left\{ \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}y_i^2} \right\} = \prod_{i=1}^{3} f_{Y_i}(y_i).$$

Hence, $Y_i \overset{iid}{\sim} N(0,1)$, $i = 1,2,3$.

B. MGF Technique : — If the joint distribution of $X_1, \ldots, X_n$ is known and its joint MGF exists, then we can determine the MGF of $Y_1, \ldots, Y_K$, where $Y_i = g_i(x_1, x_2, \ldots, x_n)$.

Here, $M_{Y_1, \ldots, Y_K}(t_1, \ldots, t_K) = E\left[e^{t_1 Y_1 + \cdots + t_K Y_K}\right]$

$$= E\left[e^{\sum_{i=1}^{k} t_i \cdot g_i(x_1, \ldots, x_n)}\right]$$

if the resulting function of $t_1, \ldots, t_K$ can be recognized as the MGF of some known distribution, it follows, by uniqueness of MGF, that $Y_1, Y_2, \ldots, Y_K$ has that joint distn..

☆ Ex.1. Let $X \sim N(0,1)$. Find the distribution of $Y = X^2$ using MGF. (2011)

Soln.→ $M_Y(t) = E[e^{tY}] = E[e^{tX^2}]$

$$= \int_{-\infty}^{\infty} e^{tx^2} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2} \, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-(1-2t)x^2/2} \, dx$$

$$= \frac{1}{\sqrt{1-2t}} \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{2 \cdot \frac{1}{(1-2t)}}}}{\sqrt{\frac{1}{1-2t}} \cdot \sqrt{2\pi}} \, dx \quad , \text{if } (1-2t) > 0$$

$$= (1-2t)^{-1/2} \int_{-\infty}^{\infty} n\left(x/0, \frac{1}{1-2t}\right) dx$$

$$= (1-2t)^{-1/2} \quad , t < \frac{1}{2}$$

which is the MGF of Gamma$(\frac{1}{2}, \frac{1}{2})$.

Hence, $Y = X^2 \sim$ Gamma $(\frac{1}{2}, \frac{1}{2})$.

☆ Ex.2. If $X_i \sim N(\mu_i, \sigma_i^2)$, $i = 1(1)n$, independently, then show that, $\sum a_i X_i \sim N\left(\sum a_i \mu_i, \sum a_i^2 \sigma_i^2\right)$. [C.U. 2010]

Soln.→ Let $Y = \sum_{i=1}^{n} a_i X_i$

$M_Y(t) = E[e^{tY}] = E\left[e^{t \sum_{i=1}^{n} a_i X_i}\right]$

$$= \prod_{i=1}^{n} E\left[e^{t a_i X_i}\right] \quad , \text{due to independence of } X_i\text{'s.}$$

$$= \prod_{i=1}^{n} M_{X_i}(t a_i)$$

$$= \prod_{i=1}^{n} \left\{ e^{(t a_i)\mu_i + \frac{1}{2} \cdot (t a_i)^2 \sigma_i^2} \right\}$$

$$= e^{t \sum_{i=1}^{n} a_i \mu_i + \frac{1}{2} t^2 \sum_{i=1}^{n} a_i \sigma_i^2}$$

which is the MGF of $N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i \sigma_i^2\right)$.

Remark:→ If we put, $a_i = \frac{1}{n}$ and $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$ then $\overline{X} \sim N(\mu, \frac{\sigma^2}{n})$.

☆ Ex. 3. If $X_1 \sim \Lambda(\mu_1, \sigma_1^2)$ and $X_2 \sim \Lambda(\mu_2, \sigma_2^2)$, independently.
Find the distribution of
i) $a.X_1^b.X_2^c$, ii) X_1X_2, iii) $\frac{X_1}{X_2}$, iv) $\sqrt{X_1X_2}$.

Soln →

i) Let $Y = a.X_1^b.X_2^c$, $a>0$

$\ln Y = \ln a + b\ln X_1 + c\ln X_2$

$$M_{\ln Y}(t) = E\left[e^{t.\ln Y}\right] = E\left[e^{t\ln a + tb\ln X_1 + tc\ln X_2}\right]$$

$$= e^{t\ln a} . E\left[e^{tb.\ln X_1}\right].E\left[e^{tc\ln X_2}\right]$$

$$= e^{t\ln a}. M_{\ln X_1}(tb) . M_{\ln X_2}(tc).$$

$$= e^{t\ln a} . e^{tb\mu_1 + \frac{1}{2}t^2b^2\sigma_1^2} . e^{tc\mu_2 + \frac{1}{2}t^2c^2\sigma_2^2}$$

[$\because \ln X_i \sim N(\mu_i, \sigma_i^2)$, $i=1,2$, independently]

$$= e^{t(\ln a + b\mu_1 + c\mu_2) + \frac{1}{2}b^2(b^2\sigma_1^2 + c^2\sigma_2^2)}$$

which is the MGF of $N(\ln a + b\mu_1 + c\mu_2, b^2\sigma_1^2 + c^2\sigma_2^2)$.

By uniqueness of MGF,

$\ln Y \sim N(\ln a + b\mu_1 + c\mu_2, b^2\sigma_1^2 + c^2\sigma_2^2)$

$\Rightarrow Y \sim \Lambda(\ln a + b\mu_1 + c\mu_2, b^2\sigma_1^2 + c^2\sigma_2^2)$.

ii)

☆ Ex. 4. $\boxed{\text{Outline 10.15}}$

If $X_i \overset{iid}{\sim} N(0,1)$, $i = 1(1)4$, show that $U = X_1X_2 - X_3X_4$ has the PDF $f_U(u) = \frac{1}{2}e^{-|u|}$, $u \in \mathbb{R}$

Soln → $M_U(t) = E\left[e^{t(X_1X_2 - X_3X_4)}\right]$

$= E\left[e^{tX_1X_2}\right] E\left[e^{-tX_3X_4}\right]$, as X_i's are independent.

Now, $$E\left[e^{tX_1X_2}\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{tx_1x_2} \cdot \frac{1}{2\pi} \cdot e^{-\frac{x_1^2 + x_2^2}{2}} dx_1 dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} \cdot e^{-\frac{x_1^2 - 2tx_1x_2 + x_2^2}{2}} dx_1 dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} \cdot e^{-\frac{(x_1 - tx_2)^2 + (1-t^2)x_2^2}{2}} dx_1 dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(x_1 - tx_2)^2}{2} - \frac{(1-t^2)x_2^2}{2}} dx_1 dx_2$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(1-t^2)x_2^2}{2}} \left\{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(x_1 - tx_2)^2}{2}} dx_1\right\} dx_2$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x_2^2}{2\left(\frac{1}{1-t^2}\right)}} \left\{\int_{-\infty}^{\infty} n(x_1 / tx_2, 1) dx_1\right\} dx_2$$

$$= \frac{1}{\sqrt{1-t^2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{x_2^2}{2\left(\frac{1}{1-t^2}\right)}}}{\sqrt{2\pi} \cdot \sqrt{\frac{1}{1-t^2}}} dx_2, \text{ if } 1-t^2 > 0$$

$$= \frac{1}{\sqrt{1-t^2}} \int_{-\infty}^{\infty} n\left(x_2 / 0, \frac{1}{1-t^2}\right) dx_2$$

$$= \frac{1}{\sqrt{1-t^2}}, \quad |t| < 1.$$

Now, $M_U(t) = E\left[e^{tX_1X_2}\right] \cdot E\left[e^{-tX_3X_4}\right]$

$= E\left[e^{tX_1X_2}\right] \cdot E\left[e^{-tX_1X_2}\right]$ as X_i's are i.i.d.

$= \frac{1}{\sqrt{1-t^2}} \cdot \frac{1}{\sqrt{1-(-t)^2}}$, $|t| < 1$

$= \frac{1}{1-t^2}$, $|t| < 1$, which is the MGF of standard Laplace distribution. By uniqueness of MGF, $U = X_1X_2 - X_3X_4$ follows Standard Laplace distribution.

☆ **Ex. 5.** $\boxed{\text{Outline 10.7}}$ If $X_1, \ldots, X_n \overset{iid}{\sim} R(0,1)$, find the distn. of $G = \left(\prod_{i=1}^{n} X_i\right)^{1/n}$. Indicate how the result can be modified in case of distn. in $R(a,b)$. (2001)

Soln. →
$$G = \left(\prod_{i=1}^{n} X_i\right)^{1/n}$$
$$\therefore \ln G = \frac{1}{n}\sum_{i=1}^{n} \ln X_i$$
$$\Rightarrow -2n\ln G = \sum_{i=1}^{n}(-2\ln X_i)$$

Let, $Y = -2n\ln G$,
$$M_Y(t) = E(e^{tY}) = E\left[e^{t\left(\sum_{i=1}^{n} -2\ln X_i\right)}\right]$$
$$= \left\{E\left[e^{t(-2\ln X_1)}\right]\right\}^n, \text{ as } X_i\text{'s are i.i.d.}$$
$$= \left\{E\left[X_1^{-2t}\right]\right\}^n \qquad \left[\begin{array}{l} e^{-2t\ln X_1} \\ = e^{\ln(X_1^{-2t})} \\ = X_1^{-2t} \text{ as } a^{\log_a M} = M \end{array}\right]$$
$$= \left\{\int_0^1 (x_1^{-2t} \cdot 1)\, dx_1\right\}^n$$
$$= \left(\frac{1}{1-2t}\right)^n \text{ if } 1-2t>0$$

$$\left[\because \int_0^1 x_1^{-2t}\,dx_1 = \lim_{a\to 0^+}\int_a^1 x_1^{-2t}\,dx_1 = \lim_{a\to 0^+}\left[\frac{x_1^{1-2t}}{1-2t}\right]_a^1 = \frac{1-\lim_{a\to 0^+} a^{1-2t}}{1-2t} = \frac{1}{1-2t}, \text{ if } 1-2t>0\right]$$

$$\therefore M_Y(t) = (1-2t)^{-n}, \quad t<\frac{1}{2}$$
which is the MGF of Gamma$\left(\frac{1}{2}, n\right)$.

By uniqueness of MGF, $Y \sim \text{Gamma}\left(\frac{1}{2}, n\right)$
$$\therefore f_Y(y) = \begin{cases} \dfrac{e^{-y/2}\cdot y^{n-1}}{2^n\cdot\Gamma(n)} & ; 0<y<\infty \\ 0 & , \text{ow} \end{cases}$$

Here, $Y = -2n\ln G$
$$\Rightarrow G = e^{-y/2n}$$
As $0<y<\infty$, $0<g<1$,
The PDF of G is
$$f_G(g) = \begin{cases} \dfrac{g^n(-2n\ln g)^{n-1}}{2^n\Gamma(n)}\left|\dfrac{d(-n2\ln g)}{dg}\right| & , 0<g<1 \\ 0 & \text{ow} \end{cases}$$
$$= \begin{cases} \dfrac{n^n \cdot g^{n-1}\cdot(-\ln g)^{n-1}}{\Gamma(n)} & , 0<g<1 \\ 0 & , \text{ow} \end{cases} \longrightarrow (*)$$

Modification: ~

$X_i \overset{iid}{\sim} R(a,b)$, $i = 1(1)n$

$U_i = F_i(X_i) = \dfrac{X_i - a}{b-a} \overset{iid}{\sim} R(0,1)$, $i = 1(1)n$.

Define, $G^* = \left\{\prod_{i=1}^{n}\dfrac{X_i-a}{b-a}\right\}^{1/n} = \left\{\prod_{i=1}^{n} U_i\right\}^{1/n}$

will have the same distn. as given by $(*)$.

[C]. CDF Technique : Let X be a RV with DF $F_X(x)$, then the DF of $Y=g(X)$ is

$$F_Y(y)=P[Y\le y]=P[g(X)\le y]=P[X\in A], \text{ where, } A=\{x: g(x)\le y\}$$

which can be evaluated in terms of $F_X(x)$.

☆ **Ex.1.** Let $X\sim R(0,1)$. Find the distribution of $Y=-2\ln X$.

Soln → $f_X(x)=\begin{cases} 1 & , 0<x<1 \\ 0 & , \text{ow} \end{cases}$

where $0<x<1$, $0<y<\infty$,

For $0<y<\infty$,

$$\begin{aligned} F_Y(y) &= P[Y\le y] = P[-2\ln X\le y] \\ &= P[X\ge e^{-y/2}] \\ &= 1-P[X\le e^{-y/2}] \\ &= 1-F_X(e^{-y/2}). \end{aligned}$$

The PDF of Y is,

$$f_Y(y)=\begin{cases} -f_X(e^{-y/2})\cdot \dfrac{d(e^{-y/2})}{dy} & , \text{if } 0<y<\infty \\ 0 & , \text{ow} \end{cases}$$

$$=\begin{cases} \frac{1}{2}e^{-y/2} & , 0<y<\infty \\ 0 & , \text{ow} \end{cases}$$

☆ **Ex.2.** Let X be a RV with PDF $f_X(x)$. Find the PDF of $Y=X^2$.

Soln → The CDF of $Y=X^2$ is

$$\begin{aligned} F_Y(y) &= P[Y\le y] \\ &= P[X^2\le y] \\ &= \begin{cases} 0 & , y\le 0 \\ P[-\sqrt{y}\le X\le \sqrt{y}] & , y>0 \end{cases} \\ &= \begin{cases} 0 & , \text{if } y\le 0 \\ F_X(+\sqrt{y})-F_X(-\sqrt{y}) & , \text{if } y>0 \end{cases} \end{aligned}$$

The PDF of $Y=X^2$ is

$$f_Y(y)=\begin{cases} \{f_X(\sqrt{y})+f_X(-\sqrt{y})\}\dfrac{1}{2\sqrt{y}} & , \text{if } y>0 \\ 0 & , \text{ow} \end{cases}$$

Ex. 3. If $X \sim N(0,1)$, find the distn. of $Y = X^2$.

Soln →

$$P(Y \le y)$$
$$= P(X^2 \le y)$$
$$= P[-\sqrt{y} \le X \le \sqrt{y}]$$
$$= F(\sqrt{y}) - F(-\sqrt{y}).$$

$$\therefore h(y) = \frac{d}{dy} P(Y \le y)$$
$$= \frac{1}{2} y^{1/2 - 1} f(\sqrt{y}) + \frac{1}{2} y^{1/2-1} f(-\sqrt{y})$$
$$= \frac{1}{2} y^{-1/2} \left[f(\sqrt{y}) + f(-\sqrt{y}) \right]$$
$$= \frac{1}{2} y^{-1/2} \left[\frac{2}{\sqrt{2\pi}} e^{-y/2} \right], \quad 0 < y < \infty$$
$$= \frac{y^{-1/2}}{\sqrt{2\pi}} \cdot e^{-y/2}, \quad 0 < y < \infty.$$

☆ **Ex. 4.** If $X \sim R(-1, 2)$, find the distn. of $Y = X^2$.

Soln →

$$f_X(x) = \begin{cases} \frac{1}{3}, & -1 < x < 2 \\ 0, & \text{o.w.} \end{cases}$$

[$1 < y < 4 \Rightarrow -2 \le \sqrt{y} \le 2$
$\Rightarrow -2 \le -\sqrt{y} \le 0, \; 0 \le \sqrt{y} \le 2.$
$\Rightarrow -2 \le x \le 2$ but $-1 < x < 2$
$\therefore -1 \le x \le y.$]

Here, $y = x^2$

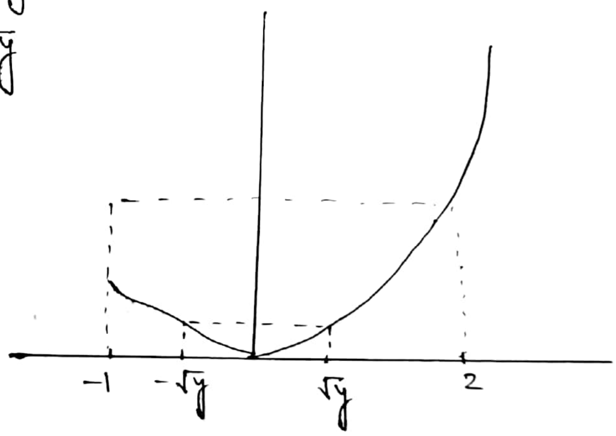
as $-1 < x < 2$, $0 < y < 4$

For $0 < y \le 1$, $x^2 \le y \Rightarrow -\sqrt{y} \le x \le \sqrt{y}$

For $1 < y < 4$, $x^2 \le y \Rightarrow -1 \le x \le \sqrt{y}$

The DF of $Y = X^2$ is

$$F_Y(y) = P[Y \le y] = P[X^2 \le y]$$



If $0 < y < 1$,

$$F_Y(y) = P[-\sqrt{y} \le X \le \sqrt{y}]$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

If $1 < y < 4$,

$$F_Y(y) = P[-1 \le X \le \sqrt{y}]$$
$$= F_X(\sqrt{y}) - F_X(-1)$$

The PDF of Y is
$$f_Y(y) = \begin{cases} f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} + f(-\sqrt{y}) \cdot \frac{1}{2\sqrt{y}}, & 0 < y \le 1 \\ f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}}, & 1 < y < 4 \\ 0, & \text{OW} \end{cases}$$

$$= \begin{cases} \frac{1}{3\sqrt{y}}, & 0 < y \le 1 \\ \frac{1}{6\sqrt{y}}, & 1 < y < 4 \\ 0, & \text{OW} \end{cases}$$

☆ **Ex. 5.** If $X \sim R(0, 2\pi)$, find the distn. of $Y = \sin x$.

Soln. →

☆ **Ex. 6.** Let X be a RV with DF $F_X(x)$, Define $Y = \begin{cases} X, & X<a \\ a, & X \geq a \end{cases}$

$= \min\{X, a\}$

Find the DF of Y.

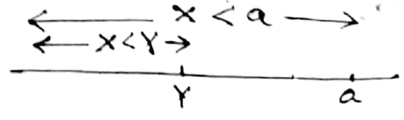
Soln. →

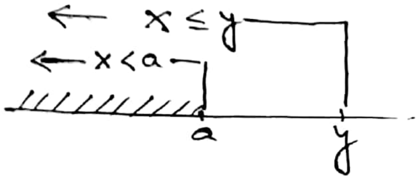
$F_Y(y) = P[Y \leq y] = P[Y \leq y \,;\, X<a] + P[Y \leq y \,;\, X \geq a]$

$= P[X \leq y \,;\, X<a] + P[a \leq y \,;\, X \geq a]$

If $Y<a$, $F_Y(y) = P[X \leq y \,;\, X<a] + 0$

If $Y \geq a$, $F_Y(y) = P[X<a] + P[X \geq a] = 1$





☆ **Ex. 7.**

Let X be a RV with D.F. $F_X(x)$. Find D.F. of

i) $Y = \begin{cases} a, & X < a \\ X, & X \geq a \end{cases}$

$= \max\{X, a\}$

ii) $Y = \begin{cases} a, & X \leq a \\ X, & a < X < b \\ b, & X \geq b \end{cases}$

Soln. →

☆ **Ex. 8.**

Let $X_1, \ldots, X_n$ be i.i.d. RV's with common D.F. $F(x)$. Find D.F.'s of

$$X_{(n)} = \max_{i=1(1)n} \{X_i\}, \quad X_{(1)} = \min_{i=1(1)n} \{X_i\}$$

Soln. →

$$F_{X_{(n)}}(x) = P[X_{(n)} \leq x]$$

$$= P\left[\max_{i=1(1)n} \{X_i\} \leq x\right]$$

$$= P[X_1 \leq x, \ldots, X_n \leq x]$$

$$= \left\{P[X_1 \leq x]\right\}^n, \text{ as } X_i\text{'s are i.i.d.}$$

$$= \left\{F(x)\right\}^n.$$

$$F_{X_{(1)}}(x) = P[X_{(1)} \leq x]$$

$$= P\left[\min_{i=1(1)n} \{X_i\} \leq x\right]$$

$$= 1 - P\left[\min_{i=1(1)n} \{X_i\} > x\right]$$

$$= 1 - P[X_1 > x, \ldots, X_n > x]$$

$$= 1 - \left\{P[X_1 > x]\right\}^n, \text{ as } X_i\text{'s are i.i.d.}$$

$$= 1 - \left\{1 - F(x)\right\}^n.$$

Remark:– In particular, if X is an absolutely continuous R.V.'s with PDF $f(x)$, $i = 1(1)n$, then the PDF of $X_{(n)}$ is

$$f_{X_{(n)}}(x) = n\left\{F(x)\right\}^{n-1} \cdot f(x)$$

and the PDF of $X_{(1)}$ is

$$f_{X_{(1)}}(x) = n\left\{1 - F(x)\right\}^{n-1} f(x).$$

☆ Ex. 9. If $x_1, \ldots, x_n$ be iid R.V.'s from $R(0, \theta)$ distn, find the distn of $X_{(n)}$ and $X_{(1)}$. Also find $E[X_{(n)}]$ and $E[X_{(1)}]$.

Soln. → Here,
$$F(x) = \begin{cases} 0 & , x \le 0 \\ \frac{x}{\theta} & , 0 < x < \theta \\ 1 & , x \geq \theta \end{cases}$$

Now, $F_{X_{(n)}}(x) = \{F(x)\}^n$
$$= \begin{cases} 0 & , x \le 0 \\ \left(\frac{x}{\theta}\right)^n & , 0 < x < \theta \\ 1 & , x \geq \theta \end{cases}$$

and $F_{X_{(1)}}(x) = 1 - \{1 - F(x)\}^n$
$$= \begin{cases} 0 & , x \le 0 \\ 1 - \left(1 - \frac{x}{\theta}\right)^n & , 0 < x < \theta \\ 1 & , x \geq \theta. \end{cases}$$

For a non-negative R.V. Y,
$$E(Y) = \int_0^\infty \{1 - F_Y(y)\}\, dy, \text{ provided } E(Y) \text{ exists.}$$

Here,
$$E(X_{(n)}) = \int_0^\infty \{1 - F_{X_{(n)}}(x)\}\, dx$$
$$= \int_0^\theta \left(1 - \frac{x^n}{\theta^n}\right) dx$$
$$= \theta - \frac{1}{\theta^n} \cdot \frac{\theta^{n+1}}{n+1}$$
$$= \frac{n}{n+1} \cdot \theta$$

∴ θ is unbiasedly estimated by $\frac{n+1}{n} X_{(n)}$ as $E\left(\frac{n+1}{n} X_{(n)}\right) = \theta$

and,
$$E(X_{(1)}) = \int_0^\infty \{1 - F_{X_{(1)}}(x)\}\, dx$$
$$= \int_0^\theta \left(1 - \frac{x}{\theta}\right)^n dx$$
$$= \frac{1}{\theta^n} \int_0^\theta (\theta - x)^n\, dx$$
$$= \frac{1}{\theta^n} \left[\frac{(\theta - x)^{n+1}}{-(n+1)}\right]_0^\theta$$
$$= \frac{1}{\theta^n} \left\{0 + \frac{\theta^{n+1}}{n+1}\right\}$$
$$= \frac{\theta}{n+1}.$$

∴ θ is unbiasedly estimated by $(n+1) X_{(1)}$.

☆ Ex. 10. Let $X \sim N(0,1)$.
Find the PDF of $Y = \begin{cases} -X, & \text{if } X < 0 \\ X^2, & \text{if } X \geq 0 \end{cases}$

Soln. →

$$F_Y(y) = P[Y \leq y]$$
$$= P[Y \leq y ; X < 0] + P[Y \leq y ; X \geq 0]$$
$$= P[-X \leq y ; X < 0] + P[X^2 \leq y ; X \geq 0]$$
$$= P[X \geq -y, X < 0] + P[X^2 \leq y, X \geq 0]$$
$$= \begin{cases} 0, & \text{if } y \leq 0 \\ P[-y \leq X < 0] + P[-\sqrt{y} \leq X \leq \sqrt{y}, X \geq 0], & \text{if } y > 0 \end{cases}$$
$$= \begin{cases} 0, & \text{if } y \leq 0 \\ F_X(\sqrt{y}) - F_X(-\sqrt{y}), & \text{if } y > 0 \end{cases}$$

The pdf of Y is, —

$$f_Y(y) = \begin{cases} f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} + f_X(-\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} \end{cases}$$

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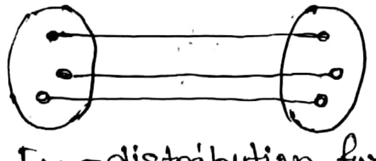
ALTERNATIVE APPROACH

Case-1.

Let X be a continuous random variable having PDF $f_X(\cdot)$ and $g(\cdot)$ be a bijection. Here the object is to find the distribution and hence the PDF of $g(X)$.



Let $Y = g(X)$

$F_Y(y) = P[Y \leq y]$ $\quad$ F_Y = distribution function of Y.

$= P[g(X) \leq y]$

$= P[X \leq g^{-1}(y)] = F_X[g^{-1}(y)]$

$\therefore f_Y(y) = \frac{d}{dy} F_X(g^{-1}(y)) = \frac{d}{dy} F_Y(y)$.

* $P[Y \leq y_0]$

$= P[g(X) \leq y_0] = P[X \leq g^{-1}(y_0)] = \int_{-\infty}^{g^{-1}(y_0)} f_X(x)dx$

Consider the transformation,

$y = g(x)$

$\therefore x = g^{-1}(y)$

$\therefore dx = \frac{d}{dy}(g^{-1}(y)) \cdot dy$

$= J \cdot dy$

$\therefore P[Y \leq y_0]$

$= \int_{-\infty}^{y_0} f_X(g^{-1}(y)) |J| dy$

J is termed as Jacobian of transformation.

$\therefore P[Y \leq y_0]$

$= \int_{-\infty}^{y_0} f_X(y) dy$, by the definition of distribution function.

$\therefore f_Y(y) = f_X(g^{-1}(y)) |J|$.

Example: →

1) $X \sim R(0,1)$
$-2\ln X \sim ?$

$\Rightarrow$ $f_X(x) = I_x(0,1)$

Consider the transformation $x \to y = -2\ln x$.

$\therefore x = e^{-y/2}$

Clearly,
$x \in (0,1)$

$|J| = \left|\frac{dx}{dy}\right| = \frac{1}{2} e^{-y/2}$

$\therefore f_Y(y) = |J| I_y(0,\infty)$

$= \frac{1}{2} e^{-y/2}$

$\therefore -2\ln X \sim$ Exponential with mean 2 $(= \chi_2^2)$.

2) $\boxed{\begin{array}{l} X \sim \beta_1(\theta,1) \\ Y = -2\theta \ln X \sim ? \end{array}}$

$\Rightarrow$ $f_X(x) = \dfrac{x^{\theta-1}}{\beta(\theta,1)} I_x(0,1)$, $\theta > 0$

$= \theta x^{\theta-1} I_x(0,1)$, $\theta > 0$

Consider the tramformation, $x \to y \ni$

$y = -2\theta \ln x$

$\therefore x = e^{-y/2\theta}$

$\Rightarrow |J| = \left|\dfrac{dx}{dy}\right| = \dfrac{1}{2\theta} \cdot e^{-y/2\theta}$.

$\therefore f_Y(y) = \theta \left(e^{-y/2\theta}\right)^{\theta-1} \cdot \dfrac{1}{2\theta} \cdot e^{-y/2\theta} I_y(0,\infty)$

$= \dfrac{1}{2} e^{-y/2} I_y(0,\infty)$

$\Rightarrow Y = -2\theta \ln X \sim \chi^2_2$.

3) $\boxed{\begin{array}{l} X \sim \beta_2(m,n). \\ Y = \dfrac{X}{1+X} \sim ? \end{array}}$

$\Rightarrow$ $f_X(x) = \dfrac{x^{m-1}}{(1+x)^{m+n}} \cdot \dfrac{1}{\beta(m,n)} I_x(0,\infty)$; when $m>0, n>0$

Consider the transformation $x \to y \ni$ $y = \dfrac{x}{x+1}$

$|J| = \left|\dfrac{dx}{dy}\right| = \dfrac{(1-y)+y}{(1-y)^2} = \dfrac{1}{(1-y)^2}$. $\Rightarrow x = \dfrac{y}{1-y}$

$\therefore f_Y(y) = \dfrac{\left(\frac{y}{1-y}\right)^{m-1}}{\left(1+\frac{y}{1-y}\right)^{m+n}} \cdot \dfrac{1}{(1-y)^2} \cdot \dfrac{1}{\beta(m,n)} \cdot I_y(0,1)$.

$= y^{m-1}(1-y)^{n+1} \cdot \dfrac{1}{(1-y)^2} \cdot \dfrac{1}{\beta(m,n)} I_y(0,1)$

$= y^{m-1} \cdot (1-y)^{n-1} \cdot \dfrac{1}{\beta(m,n)} I_y(0,1)$

4) $\boxed{\begin{array}{l} X \sim N(\mu,\sigma^2) \\ \alpha+\beta X \sim ? \end{array}}$

$\Rightarrow$ let, $Y = \alpha + \beta X$

$F_Y(y) =$ distribution function of Y

$= P[\alpha + \beta X \le y]$

$= P\left[\dfrac{X-\mu}{\sigma} \le \dfrac{\frac{y-\alpha}{\beta} - \mu}{\sigma}\right]$

$= \Phi\left(\dfrac{y-(\alpha+\beta\mu)}{\beta\sigma}\right)$

$f_Y(y) = \dfrac{1}{\beta\sigma} \phi\left(\dfrac{y-(\alpha+\beta\mu)}{\beta\sigma}\right)$, $y \in \mathbb{R}$

$\therefore Y \sim N(\alpha+\mu\beta, \beta^2\sigma^2)$.

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Case-II Let X be a continuous RV having PDF $f_X(\cdot)$ and g be a function that is not 'one-to-one'.

Objective $g(X) \sim ?$

$y = g(x) = |x|$

$\therefore g(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Define, $g_1(x) = x$,

$g_1 : \mathbb{R}^+ \cup \{0\} \longrightarrow \mathbb{R}^+ \cup \{0\}$

$g_2(x) = -x$,

$g_2 : \mathbb{R}^- \longrightarrow \mathbb{R}^+$.

Partition the domain of g, i.e. $\mathbb{R}$ into $\mathbb{R}^+ \cup \{0\}$ and $\mathbb{R}^-$ and define, g_1 and g_2 so that both became bijections.

$F_Y(y_0) = P[Y \leq y_0]$, F_Y :- distribution function of Y.

$= P[g(X) \leq y_0]$

$= P[X \in A]$, $A = \{x : g(x) \leq y_0\}$

$= \int_A f_X(x)dx$

Consider the transformation $x \to y \ni y = g(x)$

$$F_Y(y_0) = \int_{A \cap S} f_X(x)\,dx$$

$$= \int_{A \cap (\bigcup_{j=1}^{k} S_j)} f_X(x)dx = \int_{\bigcup_{j=1}^{k}(A \cap S_j)} f_X(x)dx$$

$$= \sum_{j=1}^{k} \delta_j \int_{A \cap S_j} f_X(x)dx \quad ; \quad \delta_j = 1 \text{ or } 0$$

according as $x \in S_j$ or, not.

Let, $y = g(x)$

$g : S \longrightarrow S'$

Partition S into $S_1, S_2, \ldots, S_k \ni$ the transformation

$g : S_j \longrightarrow S_j'$ became bijections, i.e.

$g^{-1}(y)$ exists when $y \in S_j'$.

$\Rightarrow g^{-1}(y) \in S_j$, $j = 1, 2, \ldots, k$.

Let, $g_j : S_j \longrightarrow S_j'$, where $g_j(x) = g(x) \ \forall \ x \in S_j$

it is to be noted that $S_1', S_2', \ldots, S_k'$ may not be a partition of S'.

$$\therefore F_Y(y_0) = \sum_{j=1}^{k} \delta_j \int_{-\infty}^{y_0} f_X(g_j^{-1}(y)) \left| \frac{d g_j^{-1}(y)}{dy} \right| dy \quad ; \quad \delta_j = 1 \text{ or } 0$$

according as $y \in S_j$ or, not

$$= \int_{-\infty}^{y_0} \sum_{j=1}^{k} \delta_j f_X(g_j^{-1}(y)) \left| \frac{d}{dy} g_j^{-1}(y) \right| dy$$

$$\therefore f_Y(y) = \sum_{j=1}^{k} \delta_j f_X(g_j^{-1}(y)) \left| \frac{d}{dy} g_j^{-1}(y) \right|$$

Example 5. $\boxed{\begin{array}{l} X \sim N(0,\sigma^2) \\ |X| \sim ? \end{array}}$

$\Rightarrow$ $X \to Y = g(X)$

$g \neq$ bijection.

$g(X) = |X| : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$

Define, $g_1(x) = x : \mathbb{R}^+ \to \mathbb{R}^+$

$g_2(x) = -x : \mathbb{R}^- \to \mathbb{R}^+$

$\Rightarrow g_i^{-1} = h_i$

$\therefore x = h_1(y) = y$ and $x = h_2(y) = -y$

$\therefore f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, x \in \mathbb{R}$

when, $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}$.

Here, $\mu = 0$

J_1 = Jacobian of transformation $[x \to y : y = g_1(x)]$

$= \frac{d}{dy} h_1(y) = 1$

J_2 = Jacobian of transformation $[x \to y : y = g_2(x)]$

$= \frac{d}{dy} h_2(y) = -1$

$\therefore |J_1| = |J_2| = 1$.

Note that, $x \in \mathbb{R} \Rightarrow y \in \mathbb{R}^+$.

PDF of Y is

$f_Y(y) = \Big[f_X(h_1(y))|J_1| + f_X(h_2(y))|J_2| \Big] I_y(\mathbb{R}^+)$

$= \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2\sigma^2} \cdot y^2} + \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2} \cdot y^2}$

In particular, if $\mu = 0$

$f_Y(y) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sigma} \cdot e^{-y^2/2\sigma^2} \; I_y\{\mathbb{R}^+\}$

62. $\boxed{\begin{array}{l} X \sim U(-1,2) \\ Y = |X| \sim ? \end{array}}$ $\qquad f_X(x) = \begin{cases} \frac{1}{3} & \text{if } -1 < x < 2 \\ 0 & \text{, ow} \end{cases}$

$\Rightarrow$ Clearly, $y = |x|$ is not a bijection.

Partition $(-1,2)$ into $(-1,0)$, $[0,2)$ and define

$y = g_1(x) = |x| \quad \therefore (-1,0) \to (0,1)$

$\Rightarrow y = -x$

$\therefore x = g_1^{-1}(y) = -y$

$\therefore |J_1| = 1$

and $g_2(x) = |x| \quad \therefore [0,2) \to [0,2)$

$\Rightarrow y = g(x)$

$\therefore x = y$

$\therefore |J_2| = 1$

PDF of Y is, $\longrightarrow$ $f_Y(y) = f_X(-y)|J_1|\delta_1 + f_X(y)|J_2| \cdot \delta_2$

[$\delta_1 = 1$ or, 0 according as $y \in (0,1)$

$\delta_2 = 1$ or, 0 according as $y \in (0,2)$

let us ignore the case $y = 0$ as y is continuous R.V.]

$f_Y(y) = \begin{cases} \frac{1}{3} + \frac{1}{3} & \text{if } y \in (0,1). \\ \frac{1}{3} & \text{if } y \in [1,2). \\ 0 & \text{, ow} \end{cases}$

Case-III Suppose $X_1, X_2, \ldots, X_n$ are jointly distributed continuous random variables having joint PDF $f_{X_1 X_2 \cdots X_n}(x_1, x_2, \ldots, x_n)$; Consider the transformation

$$(X_1, X_2, \ldots, X_n) \longrightarrow (Y_1, Y_2, \ldots, Y_n)$$

where, $Y_1 = g_1(X_1, X_2, \ldots, X_n)$
$Y_2 = g_2(X_1, X_2, \ldots, X_n)$
$\vdots$
$Y_n = g_n(X_1, X_2, \ldots, X_n)$

Further assume that the transformation is 'one-to-one'.
Hence, $\exists\ h_1, h_2, \ldots, h_n \ni$
$X_1 = h_1(Y_1, Y_2, \ldots, Y_n)$
$X_2 = h_2(Y_1, Y_2, \ldots, Y_n)$
$\vdots$
$X_n = h_n(Y_1, Y_2, \ldots, Y_n)$

Hence, the jacobian of the transformation is —

$$J\left(\frac{x_1, x_2, \ldots, x_n}{y_1, y_2, \ldots, y_n}\right) = \det\left(\left(\frac{\partial x_i}{\partial y_j}\right)\right)$$

Here the joint PDF of $Y_1, Y_2, \ldots, Y_n$ will be, —

$$f_{Y_1, Y_2, \ldots Y_n}(y_1, y_2, \ldots, y_n) = f_{X_1, X_2, \ldots, X_n}(h_1(\underset{\sim}{y}), \ldots, h_n(\underset{\sim}{y}))\,|J|$$

Case-IV Suppose $X_1, X_2, \ldots, X_n$ are jointly distributed continuous RV having the joint PDF $f_{X_1 X_2 \cdots X_n}(x_1, x_2, \ldots, x_n)$; consider the transformation

$$(X_1, X_2, \ldots, X_n) \longrightarrow (Y_1, Y_2, \ldots, Y_n).$$

where, $Y_1 = g_1(X_1, \ldots, X_n)$
$Y_2 = g_2(X_1, \ldots, X_n)$
$\vdots$
$Y_n = g_n(X_1, \ldots, X_n)$, $\underset{\sim}{Y}: S \rightarrow S'$

Let us assume that the transformation is not one-to-one.
Partition S into $S_1, S_2, \ldots, S_k$ so that the transformation

$\underset{\sim}{Y}: S_r \rightarrow S_r'$; $r = 1(1)k$,

became one-to-one.

Hence there exists $h_1^{(r)}, h_2^{(r)}, h_3^{(r)}, \ldots, h_n^{(r)}$
for $\underset{\sim}{X} \in S_r$
$X_1 = h_1^{(r)}(Y_1, Y_2, \ldots, Y_n)$
$\vdots$
$X_n = h_n^{(r)}(Y_1, Y_2, \ldots, Y_n)$

$\therefore$ The jacobian of the transformation $S_r \rightarrow S_r'$ is

$$J_r = \det\left(\left(\frac{\partial h_i^{(r)}(\underset{\sim}{y})}{\partial y_j}\right)\right)$$

Hence, the PDF of Y is given by.

$$f_Y(y) = \sum_{r=1}^{k} \delta_r f_X(h_1^{(r)}(\underset{\sim}{y}), h_2^{(r)}(\underset{\sim}{y}) \ldots, h_n^{(r)}(\underset{\sim}{y}))\,|J|$$

where, $\delta_r = 1$ or, 0 according as $\underset{\sim}{y} \in S_r'$ or, not.

Example :- 7.

$$\boxed{\begin{array}{c} X \sim N(0,1) \\ Y \sim N(0,1) \\ U = \frac{X}{|Y|} \sim ? \\ X \,\&\, Y \text{ are independent.} \end{array}}$$

(C.U. 2011)

$\Rightarrow$ Let, $V = |Y|$

$$f_{XY}(x,y) = \frac{1}{2\pi} \cdot e^{-\frac{1}{2}(x^2+y^2)} I_x(-\infty,\infty) I_y(-\infty,\infty).$$

Let us take the transformation

$$U = \frac{X}{|Y|} \text{ and } V = |Y|$$

$$-\infty < u < \infty, \quad 0 < v < \infty$$

$$\Rightarrow u = \frac{x}{|y|}, \text{ and } v = |y| \quad \text{and} \quad y = v, \; y \geq 0$$
$$\qquad\qquad\qquad = -v, \; y < 0$$

$\therefore x = uv$

$\therefore |J|$ = Jacobian of the transformation [when $y \geq 0$]

$$= \left| \begin{matrix} v & u \\ 0 & 1 \end{matrix} \right| = |v| = v$$

$$|J| = [\text{when } y < 0] = \left| \begin{matrix} v & u \\ 0 & -1 \end{matrix} \right| = |-v| = v.$$

$$\therefore f_{UV}(u,v) = \frac{v}{\cancel{2}\pi} \cdot e^{-\frac{1}{2}(u^2+1)v^2} \cdot \cancel{2} \; I_u(-\infty,\infty) I_v(0,\infty).$$

$$\therefore f_U(u) = \int_0^\infty \frac{v}{\pi} e^{-\frac{1}{2}v^2(u^2+1)} dv$$

$$\frac{1}{2}v^2(u^2+1) = z$$

$$v(u^2+1)dv = dz$$

$$= \frac{1}{\pi(1+u^2)} \int_0^\infty e^{-z} dz$$

$$= \frac{1}{\pi(1+u^2)} I_u(-\infty,\infty)$$

$$\therefore \frac{X}{Y} \stackrel{D}{=} \frac{X}{|Y|} \sim C(0,1).$$

Ex. 8.

$$\boxed{\begin{array}{c} X \sim \gamma(\alpha, m) \\ Y \sim \gamma(\alpha, n) \\ U = X+Y \sim ? \\ X \,\&\, Y \text{ are independent.} \end{array}}$$

$\Rightarrow$ Joint PDF of X & Y is,

$f_{XY}(x,y) = f_X(x) f_Y(y)$ [Product of marginal PDFs]

$$= \frac{\alpha^{m+n} e^{-\alpha(x+y)} x^{m-1} y^{n-1}}{\Gamma m \; \Gamma n} I_x(0,\infty) I_y(0,\infty).$$

Let us take the transformation

$$(X, Y) \longrightarrow (U, Y) \quad \text{. where}$$

Let, $U = X+Y$, $V = Y$

$\therefore$ $x = u - v$, $y = v$ $\Big\}$ $x>0, y>0 \Rightarrow u>v>0$

$$J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$$\therefore f_{UV}(u,v) = \frac{\alpha^{m+n} e^{-\alpha u}}{\Gamma m\, \Gamma n} (u-v)^{m-1} v^{n-1} I_{u>v>0}$$

The PDF of U is, —

$$f_U(u) = \int_0^u \frac{\alpha^{m+n} e^{-\alpha u} (u-v)^{m-1} (v)^{n-1}}{\Gamma m\, \Gamma n} dv$$

$$= \frac{e^{-\alpha u} \alpha^{m+n} u^{m+n-1}}{\Gamma m+n} I_u(0,\infty).$$

$\therefore X+Y \sim \gamma(\alpha, m+n)$.

Ex. 9.

$X \sim \gamma(\alpha, m)$, $Y \sim \gamma(\alpha, n)$,
X & Y are independent.
$U = \frac{X}{Y} \sim ?$

$$\Rightarrow f_{XY}(x,y) = \frac{\alpha^{m+n}}{\Gamma m\, \Gamma n} \cdot e^{-\alpha(x+y)} x^{m-1} y^{n-1} I_x(0,\infty) I_y(0,\infty)$$

$(X,Y) \longrightarrow (U,Y)$, where $U = \frac{X}{Y}$.

$x>0, y>0 \Rightarrow u>0$, $x = uy$

The Jacobian of the transformation is

$$J = \left|\frac{\partial x}{\partial u}\right| = |y| = y$$

The joint density (U,Y) is

$$f_{UY}(u,y) = \frac{\alpha^{m+n} e^{-\alpha y(1+u)} (uy)^{m-1} y^{n-1}}{\Gamma m\, \Gamma n} I_u(0,\infty) I_y(0,\infty)$$

$$\therefore f_U(u) = \int_0^\infty f_{UY}(u,y)\, dy$$

$$= \frac{\alpha^{m+n} u^{m-1}}{\Gamma m\, \Gamma n} \int_0^\infty e^{-\alpha y(1+u)} y^{m+n-1} dy$$

$$= \frac{\cancel{\alpha^{m+n}} \cdot u^{m-1}}{\Gamma m\, \Gamma n} \cdot \frac{\Gamma m+n}{\cancel{\alpha^{m+n}} (1+u)^{m+n}} I_u(0,\infty)$$

$$= \frac{1}{\beta(m,n)} \cdot \frac{u^{m-1}}{(1+u)^{m+n}} I_u(0,\infty)$$

$\Rightarrow U = \frac{X}{Y} \sim \beta_2(m,n)$.

Ex. 10.

$X \sim \gamma(m)$, $Y \sim \gamma(n)$, $Z \sim \gamma(p)$
X, Y, Z are independent.

i) $\left(\frac{X}{X+Y}, \frac{X+Y}{X+Y+Z}\right) \sim ?$

ii) $\left(\frac{Y}{X+Y}, \frac{Z}{X+Y+Z}\right) \sim ?$

iii) $\left(\frac{X}{X+Y+Z}, \frac{Y}{Y+Z}\right) \sim ?$

iv) $\left(\frac{X}{X+Y+Z}, \frac{Y}{X+Y+Z}\right) \sim ?$

i) Soln. → The joint PDF of X, Y, Z is

$$f_{XYZ}(x,y,z) = \frac{e^{-x} \cdot e^{-y} \cdot e^{-z} \cdot x^{m-1} \cdot y^{n-1} \cdot z^{p-1}}{\lceil m \lceil n \lceil p} \quad I_{x,y,z}(0,\infty)$$

$$= \frac{e^{-(x+y+z)} \cdot x^{m-1} y^{n-1} z^{p-1}}{\lceil m \lceil n \lceil p} \quad I_{x,y,z}(0,\infty)$$

Let us make the transformation,

$(X, Y, Z) \longrightarrow (U, V, W)$.

where, $U = \frac{X}{X+Y}$, $V = \frac{X+Y}{X+Y+Z}$, $W = X+Y+Z$

$X+Y = VW$

$X = UVW$

$Y = WV(1-U)$

$Z = W(1-V)$

$$J\left(\frac{x,y,z}{u,v,w}\right) = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} & \frac{dx}{dw} \\ \frac{dy}{du} & \frac{dy}{dv} & \frac{dy}{dw} \\ \frac{dz}{du} & \frac{dz}{dv} & \frac{dz}{dw} \end{vmatrix}$$

$$= \begin{vmatrix} vw & uw & uv \\ -vw & w(1-u) & v(1-u) \\ 0 & -w & 1-v \end{vmatrix}$$

$$= \begin{vmatrix} vw & uw & uv \\ 0 & w & v \\ 0 & -w & 1-v \end{vmatrix} \quad R_2' \to R_2 + R_1$$

$= vw^2$.

$\therefore |J| = vw^2 \quad [\because 0 < v < 1,\ w^2 > 0]$

$$\left.\begin{array}{l} x = uvw \\ y = vw(1-u) \\ z = w(1-v) \end{array}\right\} \quad \begin{array}{l} x>0,\ y>0,\ z>0 \\ \Rightarrow 0<u<1,\ 0<v<1,\ w>0 \end{array}$$

If the joint PDF of (U,V,W) is —

$$f_{U,V,W}(u,v,w) = \frac{e^{-w}(uvw)^{m-1}\{vw(1-u)\}^{n-1}\{w(1-v)\}^{p-1}}{\Gamma m\ \Gamma n\ \Gamma p}|J|$$

where, $0<u<1$
$0<v<1$
$w>0$

$$= \frac{e^{-w}u^{m-1}v^{m+n-1}w^{p+n+m-1}(1-u)^{n-1}(1-v)^{p-1}}{\Gamma m\ \Gamma n\ \Gamma p}$$

where, $0<u<1$
$0<v<1$
$w>0$

$$= \frac{e^{-w}w^{m+n+p-1}}{\Gamma m+n+p} \cdot \frac{u^{m-1}(1-u)^{n-1}}{\beta(m,n)} \cdot \frac{v^{m+n-1}(1-v)^{p-1}}{\beta(m+n,p)}$$

where, $0<u<1$
$0<v<1$
$w>0$

$$= f_W(w) \cdot f_U(u) f_V(v)$$

where, $f_W(w) = \frac{e^{-w} \cdot w^{m+n+p-1}}{\Gamma m+n+p} I_w(0,\infty)$; $\therefore W \sim \gamma(m+n+p)$

$f_U(u) = \frac{u^{m-1} \cdot (1-u)^{n-1}}{\beta(m,n)} I_u(0,1)$; $\therefore U \sim \beta_1(m,n)$.

$f_V(v) = \frac{v^{m+n-1}(1-v)^{p-1}}{\beta(m+n,p)} I_v(0,1)$; $\therefore V \sim \beta_1(m+n,p)$.

where, W, U, V are independently distributed.

ii) The joint PDF of (X,Y,Z) is, —

$$f_{XYZ}(x,y,z) = \frac{e^{-(x+y+z)}x^{m-1}y^{n-1}z^{p-1}}{\Gamma m\ \Gamma n\ \Gamma p} I_x(0,\infty) I_y(0,\infty) I_z(0,\infty)$$

Let us consider the transformation

$$(X,Y,Z) \longrightarrow (U,V,W)$$

where, $U = \frac{Y}{X+Y}$,

$V = \frac{Z}{X+Y+Z}$,

$W = X+Y+Z$.

$\therefore Z = VW$

$\therefore Y = UW(1-V)$

$\therefore X = W - VW - UW(1-V)$
$= W(1-U)(1-V)$

let us take the tramformations

$$\begin{array}{l} z = vw \\ y = uw(1-v) \\ x = w(1-u)(1-v) \end{array} \quad \begin{array}{l} x,y,z>0 \\ \Rightarrow 0<u<1 \\ 0<v<1,\ 0<w<\infty. \end{array}$$

The jacobian of the transformation is

$$J\left(\frac{x,y,z}{u,v,w}\right) = \begin{vmatrix} -w(1-v) & -w(1-u) & (1-u)(1-v) \\ w(1-v) & -uw & u(1-v) \\ 0 & w & v \end{vmatrix}$$

$$= w^2(1-v).$$

$\therefore |J| = w^2(1-v).$

$\therefore$ the joint PDF of (U,V,W) is

$$f_{U,V,W}(u,v,w) = \frac{e^{-w}\{w(1-u)(1-v)\}^{m-1}\{uw(1-v)\}^{n-1}\{vw\}^{p-1}\cdot w^2(1-v)}{\Gamma m\,\Gamma n\,\Gamma p} \times I_u(0,1)I_v(0,1)I_w(0,\infty)$$

$$= \frac{w^{m+n+p-1}\cdot e^{-w}}{\Gamma m+n+p}\cdot\frac{\frac{u^{n-1}(1-u)^{m-1}}{\Gamma m\,\Gamma n}}{\Gamma m+n}\cdot\frac{\frac{v^{p-1}(1-v)^{m+n-1}}{\Gamma p\,\cdot\,\Gamma m+n}}{\Gamma m+n+p} I_u(0,1)I_v(0,1)I_w(0,\infty)$$

$$= \frac{e^{-w}\,w^{m+n+p-1}}{\Gamma m+n+p}\cdot\frac{u^{n-1}(1-u)^{m-1}}{\beta(m,n)}\cdot\frac{v^{p-1}(1-v)^{m+n-1}}{\beta(m+n,p)} I_u(0,1)I_v(0,1)I_w(0,\infty)$$

$$= f_W(w)\cdot f_U(u)\cdot f_V(v).$$

where U, V, W are independently distributed random variables.

where, $W \sim \gamma(m+n+p)$

$U \sim \beta_1(m,n)$

$V \sim \beta_1(m+n,p)$

iii) The joint PDF of (X,Y,Z) is —

$$f_{XYZ}(x,y,z) = \frac{e^{-(x+y+z)}\cdot x^{m-1}\cdot y^{n-1} z^{p-1}}{\Gamma m\,\Gamma n\,\Gamma p} I_x(0,\infty)I_y(0,\infty)I_z(0,\infty).$$

$(X,Y,Z) \longrightarrow (U,V,W).$

$U = \frac{X}{X+Y+Z}$ $\qquad \therefore X = UW$ $\qquad \therefore 0<u<1,$

$V = \frac{Y}{Y+Z}.$ $\qquad \therefore Y = V(W-UW)$ $\qquad 0<v<1$

$\Rightarrow Y = VW(1-U)$ $\qquad \therefore 0\leq w<\infty.$

$W = X+Y+Z.$ $\qquad \therefore Z = W-UW-VW(1-U)$

$= W(1-U)(1-V).$

let us take the transformations, $x = wu$, $y = vw(1-u)$, $z = (1-u)(1-v)w.$

$$\therefore J\left(\frac{x,y,z}{u,v,w}\right) = \begin{vmatrix} w & 0 & u \\ -vw & w(1-u) & v(1-u) \\ -w(1-v) & -w(1-u) & (1-u)(1-v) \end{vmatrix} = w^2 \begin{vmatrix} 1 & 0 & u \\ -v & (1-u) & (1-u)v \\ -(1-v) & -(1-u) & (1-u)(1-v) \end{vmatrix}$$

$= w^2(1-u)$

is the jacobian of the transformation.

$\therefore$ The joint PDF of (U,V,W) is

$$f_{UVW}(u,v,\omega) = \frac{e^{-\omega}\{u\omega\}^{m-1}\{v\omega(1-u)\}^{n-1}\{\omega(1-u)(1-v)\}^{p-1}(1-u)\omega^2}{\Gamma m\,\Gamma n\,\Gamma p} \times I_u(0,1)\,I_v(0,1)\,I_\omega(0,\infty)$$

$$= \frac{e^{-\omega}\,.\,\omega^{m+n+p-1}}{\Gamma m+n+p} \cdot \frac{u^{m-1}.(1-u)^{n+p-1}}{\beta(m,n+p)} \cdot \frac{v^{n-1}(1-v)^{p-1}}{\beta(n,p)} \cdot I_u(0,1) \times I_v(0,1) \times I_\omega(0,\infty).$$

$\therefore$ U, V, W are independently distributed random variables.

where,

$U \sim \beta_1(m, n+p)$

$V \sim \beta_1(n,p)$

$W \sim \gamma(m+n+p)$.

iv) The joint PDF of (X,Y,Z) is

$$f_{XYZ}(x,y,z) = \frac{e^{-(x+y+z)}.x^{m-1}.y^{n-1}z^{p-1}}{\Gamma m\,\Gamma n\,\Gamma p} I_x(0,\infty) I_y(0,\infty) I_z(0,\infty)$$

$$(X,Y,Z) \longrightarrow (U,V,W)$$

$U = \frac{X}{X+Y+Z}$ $\qquad\therefore X = UW$

$V = \frac{Y}{X+Y+Z}$ $\qquad\therefore Y = VW$

$\qquad\therefore Z = W - UW - VW = W(1-U-V)$.

$W = X+Y+Z$

Let us ~~take~~ the transformation,

$x = u\omega$ $\qquad x, y, z > 0$

$y = v\omega$ $\qquad \Rightarrow 0<u<1,\ 0<v<1$

$z = \omega(1-u-v)$ $\qquad \therefore 0<\omega<\infty$.

Jacobian of the transformation is

$$J\left(\frac{x,y,z}{u,v,\omega}\right) = \begin{vmatrix} \omega & 0 & u \\ 0 & \omega & v \\ -\omega & -\omega & 1-u-v \end{vmatrix} = \omega^2$$

$\therefore |J| = \omega^2$.

The joint distribution of U,V,W is

$$f_{UVW}(u,v,\omega) = \frac{e^{-\omega}\{u\omega\}^{m-1}\{v\omega\}^{n-1}\{\omega(1-u-v)\}^{p-1}.\omega^2}{\Gamma m\,\Gamma n\,\Gamma p} \times I_u(0,1)\,.\,I_v(0,1)\,I_\omega(0,\infty)$$

$$= e^{-\omega}.\omega^{m+n+p-1}.\frac{u^{m-1}v^{n-1}(1-u-v)^{p-1}}{\Gamma m\,\Gamma n\,\Gamma p} I_u(0,1)\,I_v(0,1)\,I_\omega(0,\infty).$$

$$\therefore f_{UV}(u,v) = \int_0^\infty f_{UVW}(u,v,\omega)\,d\omega$$

$$= \frac{u^{m-1}v^{n-1}(1-u-v)^{p-1}}{\Gamma m\,\Gamma n\,\Gamma p}\int_0^\infty e^{-\omega}.\omega^{m+n+p-1}d\omega$$

$$= \frac{u^{m-1}.v^{n-1}(1-u-v)^{p-1}}{\Gamma m\,\Gamma n\,\Gamma p}.\Gamma m+n+p$$

Bivariate dirichlet distribution.

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Ex. 11.

X and $Y \sim R(0,1)$

X & Y are independent.

i) $X+Y \sim ?$

ii) $X-Y \sim ?$

iii) $XY \sim ?$

iv) $\frac{X}{Y} \sim ?$

v) $|X-Y| \sim ?$

ANS:-

i) $Z = X+Y$

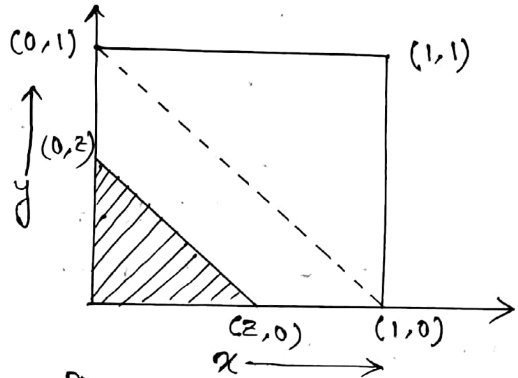
$0 < X, Y < 1$

$\Rightarrow 0 < Z < 2.$

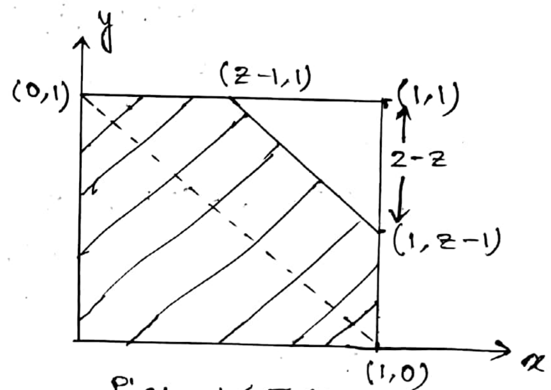
Distribution function of Z is

$$F_Z(z) = P[Z \le z]$$
$$= P[Y \le z - X]$$

$$= \begin{cases} 0, & z \le 0 \\ \frac{1}{2}z^2, & \text{if } 0<z<1 \\ 1-\frac{1}{2}(2-z)^2 & \text{if } 1<z<2 \\ 1, & \text{if } z \ge 2 \end{cases}$$



Pic: $0 < Z < 1$



Pic:- $1 < Z < 2$

PDF of Z is,—

$$\therefore f_Z(z) = \begin{cases} z, & \text{if } 0<z<1 \\ 2-z, & \text{if } 1<z<2 \\ 0, & \text{ow} \end{cases}$$

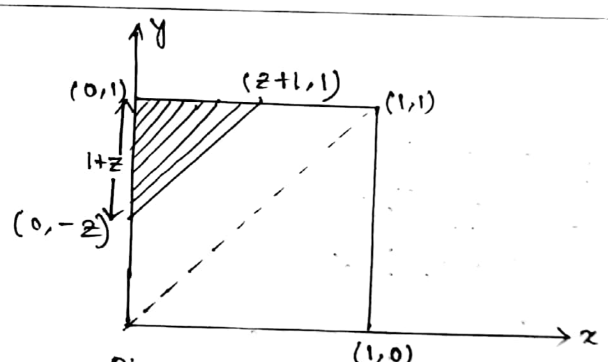
ii) $Z = X - Y$

$0 < X, Y < 1$

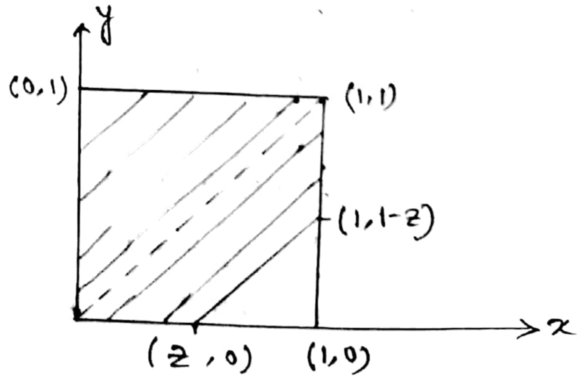
$\Rightarrow -1 < Z < 1$

$$F_Z(z) = P[X - Y \le z]$$
$$= P[Y \ge X - z]$$

$$= \begin{cases} 0, & \text{if } z \le -1 \\ \frac{1}{2}(z+1)^2, & \text{if } -1<z<0 \\ 1-\frac{1}{2}(1-z)^2, & \text{if } 0<z<1 \\ 1, & z \ge 1. \end{cases}$$



Pic:- $-1 < z < 0$



$0 < z < 1$

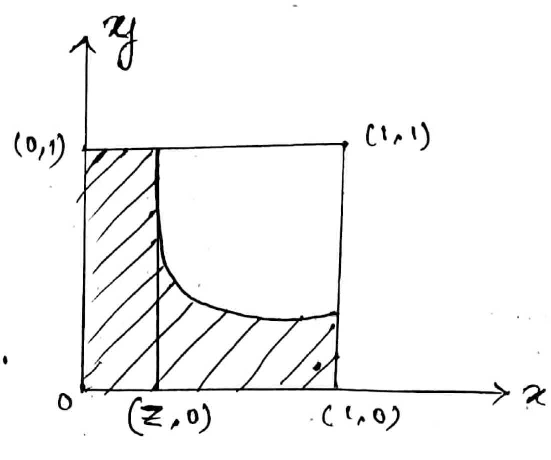
PDF of Z is,—

$$f_Z(z) = \begin{cases} z+1, & \text{if } -1<z<0 \\ 1-z, & \text{if } 0<z<1 \\ 0, & \text{ow} \end{cases}$$

iii) $Z = XY$

$0 < X, Y < 1 \Rightarrow 0 < XY < 1.$

$\Rightarrow 0 < Z < 1$



Pic:- $0 < z < 1$

$F_Z(z) = P[XY \le z]$

$$= \begin{cases} 0 & , \text{if } z \le 0 \\ \int_0^1 \int_0^z dx\,dy + \int_z^1 \int_0^{z/x} dx\,dy & , \text{if } 0 < z < 1. \\ 1 & , \text{if } z \ge 1 \end{cases}$$

$$= z + \int_z^1 \frac{z}{x}\,dx \quad , \text{if } 0 < z < 1$$

$$= z + z[\ln 1 - \ln z]$$

$$= z - z\ln z = z(1 - \ln z). \quad , \text{if } 0 < z < 1$$

$\therefore$ DF of Z is, :-

$$F_Z(z) = \begin{cases} 0 & \text{if } z \le 0 \\ z(1 - \ln z), & \text{if } 0 < z < 1 \\ 1 & , \text{if } z \ge 1 \end{cases}$$

PDF of Z is, —

$$f_Z(z) = \begin{cases} -\ln z, & \text{if } 0 < z < 1 \\ 0 & , \text{ow}. \end{cases}$$

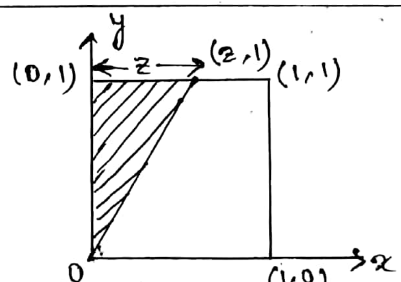
---

iv) $Z = \frac{X}{Y}$

$0 < X, Y < 1$

$\Rightarrow 0 < \frac{X}{Y} < \infty$

$\therefore 0 < Z < \infty.$



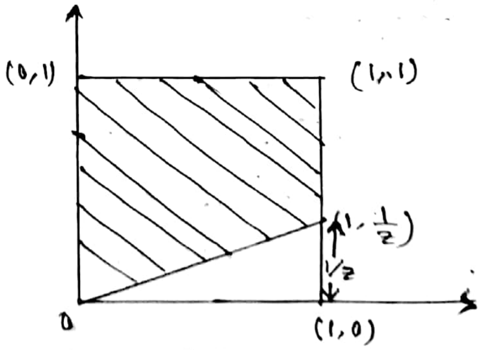
Pic:- $0 < z < 1$

$P[Z \le z]$

$= P[\frac{X}{Y} \le z]$

$= P[Y \ge \frac{1}{z} X]$

$$= \begin{cases} 0 & \text{if } z \le 0 \\ \frac{1}{2} z, & \text{if } 0 < z < 1 \\ 1 - \frac{1}{2z}, & \text{if } z \ge 1 \end{cases}$$



Pic:- $z > 1$

$$\therefore f_Z(z) = \begin{cases} \frac{1}{2} & , \text{if } 0 < z < 1 \\ \frac{1}{2z^2} & , \text{if } z \ge 1 \\ 0 & , \text{ow} \end{cases}$$

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v) $Z = |X-Y|$

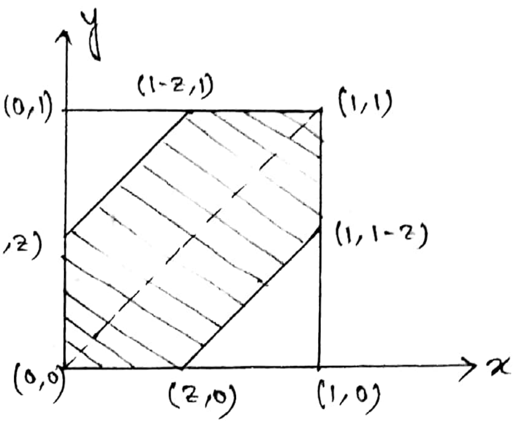
$0 < X, Y < 1$

$\Rightarrow 0 < |X-Y| < 1$

$P[Z \leq z]$

$= P[X - Z \leq Y \leq X + Z]$

$$= \begin{cases} 0 & \text{if } z \leq 0 \\ 1-(1-z)^2 & \text{if } 0<z<1 \\ 1 & \text{if } z \geq 1 \end{cases}$$



Pic:- $0<z<1$

$$\therefore f_Z(z) = \begin{cases} 2(1-z), & \text{if } 0<z<1 \\ 0, & \text{OW.} \end{cases}$$

Ex. 12.

> $X \sim R(0,a)$
> $Y \sim R(0,b)$
> X & Y are independent.
> & $a > b$.
> i) $X+Y \sim ?$
> ii) $X-Y \sim ?$
> iii) $XY \sim ?$
> iv) $\frac{X}{Y} \sim ?$
> v) $|X-Y| \sim ?$

ANS:-

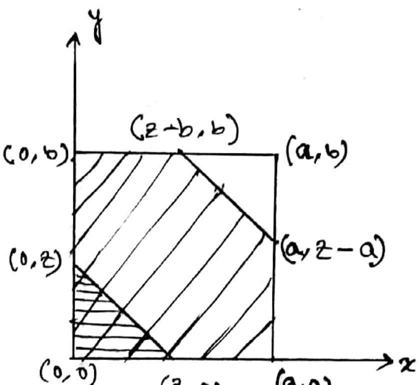
i) $X+Y = Z$,

$0 < x < a,\ 0 < y < b,\ a > b$

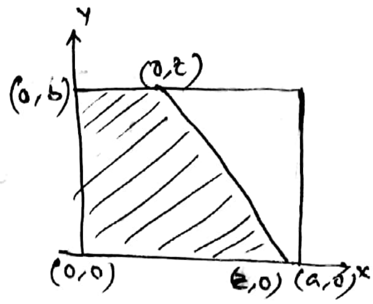
$\Rightarrow 0 < x+y < a+b$.

$\therefore P[Z \leq z] = P[Y \leq z - \cancel{X}]$

$$= \begin{cases} 0, & z \leq 0 \\ \frac{1}{ab} \times \frac{1}{2} z^2, & 0 < z \leq b \\ \frac{1}{ab} \times \frac{b}{2} \times (2z-b), & b < z \leq a \\ \frac{ab - (a+b-z)^2/2}{ab}, & a < z < a+b \\ 1 & \text{if } z \geq a+b. \end{cases}$$



$$\therefore f_Z(z) = \begin{cases} \frac{z}{ab}, & 0 < z \leq b \\ \frac{1}{a}, & b < z \leq a \\ \frac{a+b-z}{ab}, & a < z < a+b \\ 0, & \text{OW} \end{cases}$$



Case: $b < z \leq a$

Area $= \frac{1}{2} \times b \times (z + z - b)$

ii) $X - Y = Z$

$P[Z \le z]$

$= P[X - Y \le z]$

$= P[Y \ge X - z]$

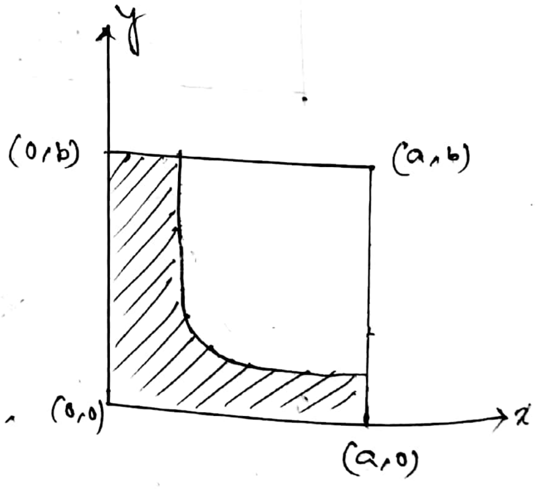
$$= \begin{cases} 0 & , z \le -b \\ \frac{1}{2ab}(b+z)^2 & , -b < z \le 0 \\ \frac{1}{2ab}(b+2z)\,b & , 0 < z \le a-b \\ 1 - \frac{1}{2ab}(a-z)^2 & , a-b < z < a \\ 1 & , z \ge a \end{cases}$$

$$\therefore f_Z(z) = \begin{cases} \frac{1}{ab}(b+z) & , -b < z \le 0 \\ \frac{1}{a} & , 0 < z \le a-b \\ \frac{a-z}{ab} & , a-b < z < a \\ 0 & , \text{ow} \end{cases}$$

iii) $Z = XY$, $0 < z < ab$

$P[Z \le z]$

$= P[Y \le \frac{z}{X}]$



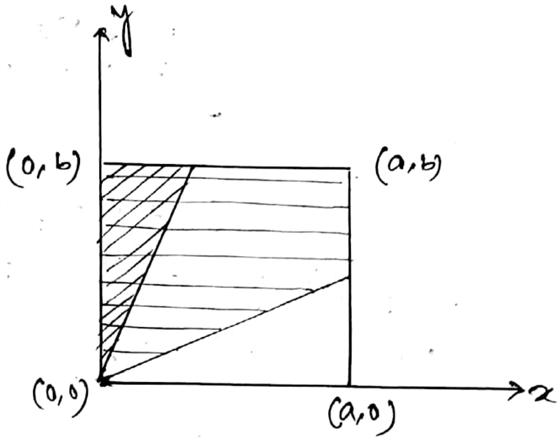
$$= \begin{cases} 0 & , z \le 0 \\ \frac{1}{ab}\left[z + \int_{z/b}^{a} \frac{z}{x}\,dx\right] & , 0 < z < ab \\ 1 & , z \ge ab \end{cases}$$

$$\therefore f_Z(z) = \begin{cases} [1 - b + \ln ba - \ln z]\frac{1}{ab} & , 0 < z < ab \\ 0 & , \text{ow} \end{cases}$$

iv) $Z = \frac{X}{Y}$, $0 < Z$

$P\left[\frac{X}{Y} \le z\right]$

$= P\left[Y \geq \frac{X}{z}\right]$

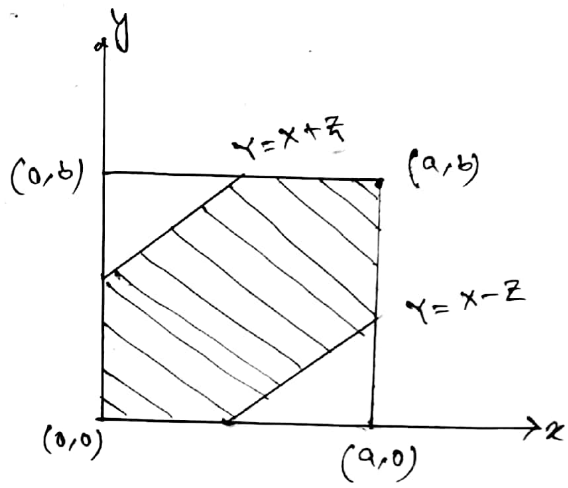


$$= \begin{cases} 0 & , z \le 0 \\ \frac{1}{2ab} \times bz \times b & , 0 < z < 1 \\ 1 - a \cdot \frac{a}{z} \cdot \frac{1}{2ab} & , 1 \le z \end{cases}$$

$$\therefore f_Z(z) = \begin{cases} \frac{b}{2a} , & 0 < z < 1 \\ \frac{a}{2b}\left(\frac{1}{z^2}\right) , & z \geq 1 \\ 0 & , \text{OW} \end{cases}$$

v) $Z = |X - Y|$, $0 < z < a$

$P\left[X - z \le Y \le X + z\right]$



$$= \begin{cases} 0 & , z \le 0 \\ 1 - \frac{1}{2}(b-z)^2 - \frac{1}{2}(a-z)^2 , & 0 < z < a \\ 1 & , z \geq a \end{cases}$$

$$\therefore f_Z(z) = \begin{cases} a + b - 2z , & 0 < z < a \\ 0 & , \text{OW} . \end{cases}$$

Ex.13.

> $X \sim$ Exp. with mean unity,
> $Y \sim$ " " " ",
> X & Y are independent.
> Find the distribution of
> i) $X+Y \sim ?$ ii) $X-Y \sim ?$

[WBSU'11]

ANS:-

(i) & (ii)

Joint PDF of X, Y is

$$f_{XY}(x,y) = e^{-(x+y)} I_x(0,\infty) I_y(0,\infty)$$

Consider the transformation,

$$U = X+Y$$
$$V = X-Y$$

$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}$$

$x>0, y>0 \Rightarrow u>-v, u>v$, i.e. $u>|v|, v \in \mathbb{R}$

or, $|v|<u$, i.e. $-u<v<u$, $u>0$

Jacobian of the transformation is,

$$J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

The PDF of U & V is,—

$$f_{UV}(u,v) = \frac{1}{2}e^{-u} I_u(|v|,\infty) I_v(-\infty,\infty)$$
$$= \frac{1}{2}e^{-u} I_v(-u,u) I_u(0,\infty)$$

$$\therefore f_U(u) = \left(\frac{1}{2}e^{u}\int_{-u}^{u} dv\right) I_u(0,\infty)$$

$$= \frac{1}{2} \cdot e^{-u} \cdot 2u \, I_u(0,\infty)$$
$$= ue^{-u} I_u(0,\infty)$$

$\therefore U \sim \gamma(0,2)$.

$$\therefore f_V(v) = \left(\frac{1}{2}\int_{|v|}^{\infty} e^{-u} du\right) I_v(-\infty,\infty)$$

$$= \frac{1}{2}e^{-|v|} I_v(-\infty,\infty)$$

ii)

X ~ exponential with mean unity

Y ~ exponential with mean unity

Joint PDF of X, Y is, —

$$f_{XY}(x,y) = e^{-(x+y)} I_x(0,\infty) I_y(0,\infty)$$

Consider the transformation, $U = X - Y$
$V = Y$.

$\therefore u + v = x$
$v = y$.

$x>0,\ y>0$

$\Rightarrow u > -v,\ v > 0$

and $v > -u,\ u \in \mathbb{R}$

$|J| = 1$, $\max(-u,0) < v < \infty,\ u \in \mathbb{R}$

The PDF of U, V is, —

$$f_{UV}(u,v) = e^{-(u+2v)} I_v\{\max(-u,0),\infty\} I_u(-\infty,\infty).$$

Case-I → When, $u<0$, $-u<v<\infty$;

Case-II → When, $u>0$, $0<v<\infty$.

$$\therefore f_U(u) = \left(\int_{-u}^{\infty} e^{-(u+2v)}dv\right) I_u(-\infty,0) + \left(\int_0^{\infty} e^{-(u+2v)}dv\right) I_u(0,\infty)$$

$$= \frac{1}{2}.e^{u} I_u(-\infty,0) + \frac{1}{2}e^{-u} I_u(0,\infty)$$

$$= \begin{cases} \frac{1}{2}e^{-|u|} &, u \in \mathbb{R} \\ 0 &, \text{ow} \end{cases}$$

Ex.14.

> X ~ Exp. with mean $\frac{1}{\alpha}$,
> Y ~ Exp. with mean $\frac{1}{\beta}$,
> $\alpha > \beta$, Y, X are indep.
> X + Y ~ ?

ANS :-

$$f_X(x) = \alpha e^{-\alpha x} I_x(0,\infty)$$

$$f_Y(y) = \beta.e^{-\beta y} I_y(0,\infty)$$

$$\therefore f_{XY}(x,y) = \alpha\beta.e^{-(\alpha x+\beta y)} I_x(0,\infty) I_y(0,\infty)$$

$u = x+y$
$v = y$

$\therefore x = u-v$
$y = v$

$\therefore x>0,\ y>0$
$\Rightarrow u>v,\ v>0$
$\therefore u>v>0$

$\therefore |J| = 1.$

Joint PDF of U and V is, —

$$f_{UV}(u,v) = \alpha\beta e^{-\{\alpha(u-v)+\beta v\}} I_u(0,\infty)\, I_v(0,u)$$

$$\therefore f_U(u) = \int_0^u \alpha\beta . e^{-\{\alpha(u-v)+\beta v\}} dv\ I_u(0,\infty)$$

$$= \frac{\alpha\beta}{\alpha-\beta}\left[e^{-\beta u} - e^{-\alpha u}\right] I_u(0,\infty).$$

Ex. 15.

$\boxed{\begin{array}{l} X \sim R(0,1),\ Y \sim R(0,1) \\ X \,\&\, Y \text{ are independent.} \\ X+Y \sim\ ? \end{array}}$

ANS:- $f_{XY}(x,y) = 1 \quad , 0<x<1,\ 0<y<1$

$U = X+Y,\ 0<U<2.$
$V = Y$

$(X,Y) \longrightarrow (U,V) \ni$

$x+y = u \Rightarrow x = -y+u$
$y = v.$

$$J\left(\frac{x,y}{u,v}\right) = 1$$

Case-I $\Rightarrow$ When $0<u<1$, $0<y<1 \Rightarrow 0<v<1$
$0<x<1$,
$\Rightarrow 0<u-v<1 \Rightarrow -u<-v<1-u$
$\Rightarrow u-1<v<u.$

$\max(0, u-1) < v < \min(1,u).$

$f_{UV}(u,v) = 1.|J| = 1,$

$$\therefore f_U(u) = \int_0^u dv = u \text{ when } 0<u<1$$

Case-II → When $1<u<2$, $0<v<1$

$0<x<1$

$\Rightarrow 0<u-v<1$

$\Rightarrow u-1<v<2$

$\Rightarrow \max(0, u-1) < v < \min(1,2)$

$\therefore f_U(u) = \int_{u-1}^{1} dv = 2-u$ when $1<u<2$.

$\therefore f_U(u) = \begin{cases} u & , 0 \le u < 1 \\ 2-u & , 1 \le u < 2 \end{cases}$

This distribution is called triangular distribution.

Ex.16.

$\boxed{\begin{array}{l} Y, X \sim \text{Exp. with mean } 2. \\ X, Y \text{ are independent.} \\ \dfrac{aX+bY}{X+Y} \sim ? \end{array}}$

ANS:-

$$f_{XY}(x,y) = \frac{1}{4} e^{-\frac{1}{2}(x+y)} I_x(0,\infty) I_y(0,\infty).$$

$$u = \frac{ax+by}{x+y}, \quad v = x+y$$

$$u = \frac{ax+by}{x+y} = \frac{b(x+y)+x(a-b)}{x+y} = b + \frac{x}{x+y}(a-b).$$

$$x, y > 0 \Rightarrow 0 < \frac{x}{x+y} < 1$$

$\therefore b < u < a$.

$ax+by = uv$

$$\therefore x = \frac{(u-b)v}{a-b}.$$

$$\therefore y = v - \frac{(u-b)v}{a-b} = \frac{(a-u)v}{a-b}.$$

$x, y > 0 \Rightarrow v > 0$

$$J = \begin{vmatrix} \frac{v}{a-b} & \frac{u-b}{a-b} \\ -\frac{v}{a-b} & \frac{a-u}{a-b} \end{vmatrix} = \frac{1}{(a-b)^2}\left[av - \cancel{uv} + \cancel{uv} - bv\right]$$

$$= \frac{v}{(a-b)}.$$

$$\therefore f_{UV}(u,v) = \frac{1}{4} e^{-v/2} \cdot \frac{v}{(a-b)} I_v(0,\infty) I_u(b,a)$$

$$\therefore f_U(u) = \frac{1}{4(a-b)} \left(\int_0^\infty v e^{-v/2} dv \right) I_u(b,a)$$

$$= \frac{I_u(b,a)}{4(a-b)} \cdot \frac{\Gamma 2}{\left(\frac{1}{2}\right)^2} = \frac{1}{a-b} I_u(b,a)$$

Ex. 17.

> X & Y are independent,
> $X \sim R(0,1)$
> $Y \sim R(0,1)$
>
> $X+Y \sim ?$
> $X-Y \sim ?$

Soln. → $f_{XY}(x,y) = \begin{cases} 1, & \text{if } 0<x,y<1 \\ 0, & \text{ow} \end{cases}$

Consider the transformation,

$U = X+Y$
$V = X-Y$

$\Rightarrow X = \frac{U+V}{2}$,

$Y = \frac{U-V}{2}$

Conditional range of u,—

$x>0, \quad y>0$

$\Rightarrow u>-v \Rightarrow u>v$

$\therefore u > \max(-v, v)$

$x<1, \quad y<1$

$\Rightarrow u<2-v, \Rightarrow u<2+v$

$\therefore u < \min(2-v, 2+v)$

$\therefore \max(-v,v) < u < \min(2-v, 2+v)$

where, v varies from $(-1,1)$.

Conditional range of v,—

$x>0, \quad y>0$

$\Rightarrow v>-u, \quad v<u$

$x<1, \quad y<1$

$\Rightarrow v<2-u, \quad v>u-2$

$\therefore \max(-u, u-2) < v < \min(u, 2-u)$

where, u varies from $(0,2)$.

$\therefore J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \frac{1}{2}$

Joint PDF of U & V is, —

$$f_{UV}(u,v) = \frac{1}{2} I_v\left(\max(-u, u-2), \min(u, 2-u)\right) I_u(0,2).$$
$$= \frac{1}{2} I_u\left\{\max(-v, v), \min(2-v, 2+v)\right\} I_v(-1,1).$$

Probability density of U : —

$\max(-u, u-2) < v < \min(u, 2-u)$, $0 < u < 2$.

Case-I : →

$0 < u < 1 \Rightarrow -u < v < u$

Case-II : →

$1 < u < 2 \Rightarrow u-2 < v < 2-u.$

$$\therefore f_U(u) = \left(\frac{1}{2}\int_{-u}^{u} dv\right) I_u(0,1) + \left(\frac{1}{2}\int_{u-2}^{2-u} dv\right) I_u(1,2)$$

$$= u\, I_u(0,1) + (2-u) I_u(1,2).$$

$$= \begin{cases} u, & 0 < u < 1 \\ 2-u, & 1 < u < 2 \\ 0, & \text{ow} \end{cases}$$

Probability density of V : —

$\max(-v, v) < u < \min(2-v, 2+v)$, $-1 < v < 1$

Case-I : →

$-1 < v < 0 \Rightarrow -v < u < 2+v$

Case-II : →

$0 < v < 1 \Rightarrow v < u < 2-v$

$$\therefore f_V(v) = \left(\frac{1}{2}\int_{-v}^{2+v} du\right) I_v(-1,0) + \left(\frac{1}{2}\int_{v}^{2-v} du\right) I_v(0,1)$$

$$= (1+v) I_v(-1,0) + (1-v) I_v(0,1)$$

$$= \begin{cases} 1+v, & -1 < v < 0 \\ 1-v, & 0 < v < 1 \\ 0, & \text{ow} \end{cases}$$