PRACTICALS ON STATISTICS

BY

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PROBLEMS ON LINEAR ALGEBRA

1) (a) Determine k so that the set S is linearly independent in E^3
(i) $S = \{(1,2,1), (k,3,1), (2,k,0)\}$
(ii) $S = \{(k,1,1), (1,k,1), (1,1,k)\}$

(b) For what values of α will the vectors $(1,5,7), (4,0,\alpha), (1,0,0)$ form a basis for E^3? [C.U. 2009]

ANS :-
(a) (i) Construct a matrix with the given vectors :

$$A = \begin{pmatrix} 1 & 2 & 1 \\ k & 3 & 1 \\ 2 & k & 0 \end{pmatrix}$$

As this vectors are linearly independent, so, rank(A) = 3.

$\therefore |A| \neq 0$

$$\Rightarrow \begin{vmatrix} 1 & 2 & 1 \\ k & 3 & 1 \\ 2 & k & 0 \end{vmatrix} \neq 0$$

$\Rightarrow (k^2 - 6) - (k-4) \neq 0$

$\Rightarrow (k^2 - k - 2) \neq 0$

$\Rightarrow (k-2)(k+1) \neq 0 \Rightarrow k \neq 2, -1.$

(ii) Construct a matrix with the given vectors :

$$A = \begin{pmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{pmatrix}$$

As this vectors are Linearly independent, so, rank(A) = 3

$\therefore |A| \neq 0$

$\Rightarrow (k-1)^2 (k+2) \neq 0$

$\Rightarrow k \neq -2, 1.$

(b) As the vectors $(1,5,7), (4,0,\alpha), (1,0,0)$ are linearly independent, so we construct a matrix A with rank(A) = 3.

$$A = \begin{pmatrix} 1 & 5 & 7 \\ 4 & 0 & \alpha \\ 1 & 0 & 0 \end{pmatrix}$$

$\Rightarrow |A| \neq 0$

$\Rightarrow 5\alpha \neq 0$

$\Rightarrow \alpha \neq 0.$

2) Extend the set $\{(2,3,-1),(1,-2,-9)\}$ of vectors to a basis for E3. Also find an orthonormal basis for E3.

ANS:-

$(0,0,1) \in E^3$ and the set $\{(2,3,-1),(1,-2,-9),(0,0,1)\}$ of vectors form a basis of E3, as the vectors $(0,0,1),(2,3,-1),(1,-2,-9)$ are LIN and they span in E^3.

$$\underset{\sim}{v}_1 = (0,0,1)$$

$$\therefore \underset{\sim}{u}_1 = \frac{\underset{\sim}{v}_1}{\|\underset{\sim}{v}_1\|} = \frac{(0,0,1)}{\sqrt{0^2+0^2+1^2}} = (0,0,1)$$

$$\therefore \underset{\sim}{v}_2 = (2,3,-1) - \{(2,3,-1)'(0,0,1)\}(0,0,1)$$
$$= (2,3,-1) + (0,0,1)$$
$$= (2,3,0)$$

$$\therefore \underset{\sim}{u}_2 = \frac{\underset{\sim}{v}_2}{|\underset{\sim}{v}_2|} = \frac{(2,3,0)}{\sqrt{4+9}} = \frac{1}{\sqrt{13}}(2,3,0)$$

$$\therefore \underset{\sim}{v}_3 = (1,-2,-9) - (\underset{\sim}{u}_1'\underset{\sim}{a}_3)\underset{\sim}{u}_1 - (\underset{\sim}{u}_2'\underset{\sim}{a}_3)\underset{\sim}{u}_2$$
$$= (1,-2,-9) + 9(0,0,1) + \frac{4}{\sqrt{13}}\left(\frac{1}{\sqrt{13}}(2,3,0)\right)$$
$$= (1,-2,-9) + (0,0,9) + \left(\frac{8}{13}, \frac{12}{13}, 0\right)$$
$$= \left(\frac{21}{13}, -\frac{14}{13}, 0\right)$$
$$= \frac{7}{13}(3,-2,0)$$

$$\therefore \underset{\sim}{u}_3 = \frac{\underset{\sim}{v}_3}{\|\underset{\sim}{v}_3\|} = \frac{7/13\,(3,-2,0)}{\sqrt{\left(\frac{21}{13}\right)^2 + \left(\frac{14}{13}\right)^2}} = \frac{\cancel{7}}{\cancel{13}}(3,-2,0) \times \frac{\cancel{13}}{\cancel{7}\sqrt{13}}$$
$$= \frac{1}{\sqrt{13}}(3,-2,0)$$

We know, $\{\underset{\sim}{u}_1, \underset{\sim}{u}_2, \underset{\sim}{u}_3\}$ forms orthonormal basis for E3.
$\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3\}$ forms orthogonal basis for E3.

In general, $\underset{\sim}{v}_k = \underset{\sim}{a}_k - \sum_{i=1}^{k-1} (\underset{\sim}{u}_i'\underset{\sim}{a}_k)\, u_i$ & $\underset{\sim}{u}_k = \frac{\underset{\sim}{v}_k}{\|\underset{\sim}{v}_k\|}$.

$\therefore$ the orthonormal basis for E^3 is

$$\left\{(0,0,1), \frac{1}{\sqrt{3}}(2,3,0), \frac{1}{\sqrt{13}}(3,-2,0)\right\}$$

3) Find the dimension of the vector space generated by the vectors:

$$\underset{\sim}{\alpha}_1' = (0 \ 1 \ 2 \ 3)$$
$$\underset{\sim}{\alpha}_2' = (2 \ -1 \ 5 \ 4)$$
$$\underset{\sim}{\alpha}_3' = (4 \ 0 \ 6 \ 1)$$
$$\underset{\sim}{\alpha}_4' = (0 \ -2 \ 4 \ 7)$$

Find a vector in the space orthogonal to the vector space spanned by $\underset{\sim}{\alpha}_1', \underset{\sim}{\alpha}_2', \underset{\sim}{\alpha}_3', \underset{\sim}{\alpha}_4'$. [C.U. 2001]

ANS:-

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & -1 & 5 & 4 \\ 4 & 0 & 6 & 1 \\ 0 & -2 & 4 & 7 \end{pmatrix}$$

Now, to find Rank (A), we will reduce this matrix into its echelon form (E), and the solution of $E\underset{\sim}{x} = \underset{\sim}{0}$ is a vector which is orthogonal to the vector space spanned by $\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \underset{\sim}{\alpha}_3, \underset{\sim}{\alpha}_4$.

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & -1 & 5 & 4 \\ 4 & 0 & 6 & 1 \\ 0 & -2 & 4 & 7 \end{pmatrix} \underset{R_2 \leftrightarrow R_1}{\sim} \begin{pmatrix} 2 & -1 & 5 & 4 \\ 0 & 1 & 2 & 3 \\ 4 & 0 & 6 & 1 \\ 0 & -2 & 4 & 7 \end{pmatrix}$$

$$\underset{R_3' = R_3 - 2R_1}{\sim} \begin{pmatrix} 2 & -1 & 5 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & -4 & -7 \\ 0 & -2 & 4 & 7 \end{pmatrix} \underset{R_4' = R_3 + R_4}{\sim} \begin{pmatrix} 2 & -1 & 5 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & -4 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underset{\substack{R_1' = R_1 + R_2 \\ R_3' = R_3 - 2R_2}}{\sim} \begin{pmatrix} 2 & 0 & 7 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -8 & -13 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{\substack{R_1' = R_1/2 \\ R_3' = \frac{R_3}{-8}}}{\sim} \begin{pmatrix} 1 & 0 & 7/2 & 7/2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & \frac{13}{8} \\ 0 & 0 & 0 & 0 \end{pmatrix} = E$$

$\therefore$ Rank (A) = 3.

$\therefore$ dim (S4) = 3.

Now, $E\begin{pmatrix} \underset{\sim}{x}_1 \\ \underset{\sim}{x}_2 \\ \underset{\sim}{x}_3 \\ \underset{\sim}{x}_4 \end{pmatrix} = \underset{\sim}{0}$ gives —

$$\Rightarrow \underset{\sim}{x}_1 + \frac{7}{2}\underset{\sim}{x}_3 + \frac{7}{2}\underset{\sim}{x}_4 = \underset{\sim}{0}$$
$$\underset{\sim}{x}_2 + 2\underset{\sim}{x}_3 + 3\underset{\sim}{x}_4 = \underset{\sim}{0}$$
$$\underset{\sim}{x}_3 + \frac{13}{8}\underset{\sim}{x}_4 = \underset{\sim}{0}$$

Let, $x_4 = t$, then $x_3 = -\frac{13}{8}t$, $x_2 = \frac{1}{4}t$, $x_1 = \frac{35}{18}t$.

$$\therefore \underset{\sim}{x} = t\left(\frac{35}{18}, \frac{1}{4}, -\frac{13}{8}, 1\right), \ t \in \mathbb{R}.$$

4) The following vectors from a spanning set for a vector space S_4:

$$\alpha_1' = (1.000 \quad 0.313 \quad 0.280 \quad 0.156)$$
$$\alpha_2' = (0.333 \quad 1.313 \quad 0.628 \quad 0.315)$$
$$\alpha_3' = (0.309 \quad 0.553 \quad 0.480 \quad 0.165)$$
$$\alpha_4' = (1.024 \quad 1.073 \quad 0.428 \quad 0.306)$$

i) Find the $\dim(S_4)$?

ii) Find a basis for S_4?

iii) If $\dim(S_4) < 4$, find a basis for $O(S_4)$? [C.U.1996]

ANS:- Dimension of the vector space (S) spaned by α_i's, $i=1(1)4$, = the rank of $A = \begin{pmatrix} \alpha_1' \\ \alpha_2' \\ \alpha_3' \\ \alpha_4' \end{pmatrix}$

Now,

$$A = \begin{pmatrix} 1.000 & 0.313 & 0.280 & 0.156 \\ 0.333 & 1.313 & 0.628 & 0.315 \\ 0.309 & 0.553 & 0.480 & 0.165 \\ 1.024 & 1.073 & 0.428 & 0.306 \end{pmatrix}$$

$R_4' = R_3 + R_4 - R_2 - R_1$
$$\begin{pmatrix} 1.000 & 0.313 & 0.280 & 0.156 \\ 0.333 & 1.313 & 0.628 & 0.315 \\ 0.309 & 0.553 & 0.480 & 0.165 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$R_2' \to R_2 - 0.333R_1$

$R_3' = R_3 - 0.309R_1$
$$\begin{pmatrix} 1.000 & 0.313 & 0.280 & 0.156 \\ 0 & 1.2088 & 0.5348 & 0.2630 \\ 0 & 0.4563 & 0.3935 & 0.1168 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$R_2' = R_2/1.2088$

$R_3' = R_3 - .4563R_2$
$$\begin{pmatrix} 1.000 & 0.313 & 0.280 & 0.156 \\ 0 & 1.000 & 0.4424 & 0.2176 \\ 0 & 0 & 0.1918 & 0.0175 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$R_3' = R_3/.1916$
$$\begin{pmatrix} 1.000 & 0.313 & 0.280 & 0.156 \\ 0 & 1.000 & 0.4424 & 0.2176 \\ 0 & 0 & 1.000 & 0.0913 \\ 0 & 0 & 0 & 0 \end{pmatrix} = E$$

$\therefore$ Rank (A) = No. of non-null rows in it's echelon form
= 3

$\therefore \dim(S) = 3$

ii) The basis of S is $\{(1.000 \quad 0.313 \quad 0.280 \quad 0.156),$
$(0 \quad 1.000 \quad 0.4424 \quad 0.2176), (0 \quad 0 \quad 1.000 \quad 0.0913)\}$

iii) $E\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \underset{\sim}{0}$

$$\Rightarrow \begin{aligned} 1.000x_1 + 0.313x_2 + 0.280x_3 + 0.156x_4 &= 0 \\ 1.000x_2 + 0.4424x_3 + 0.2176x_4 &= 0 \\ 1.000x_3 + 0.0913x_4 &= 0 \end{aligned}$$

Let, $x_4 = t, t \in \mathbb{R}$.

$$\begin{cases} \therefore x_3 = -0.0913t \\ \therefore x_2 = -0.1772t \\ \therefore x_1 = -0.0749t \end{cases}$$

$\therefore O(S) = t(-0.0749, -0.1772, -0.0913, 1), t \in \mathbb{R}$.

5) S and T are subspaces of V_4 given by
$S = \{(x_1, x_2, x_3, x_4) : 2x_1 + x_2 + 3x_3 + x_4 = 0\}$
$T = \{(x_1, x_4, x_3, x_4) : x_1 + 2x_2 + x_3 + 3x_4 = 0\}$
Find a basis and the dimension of (i) $S \cap T$, (ii) $S+T$.

ANS:- Here, $S \cap T = \left\{(x_1, x_2, x_3, x_4) : \begin{matrix} 2x_1 + x_2 + 3x_3 + x_4 = 0 \\ x_1 + 2x_2 + x_3 + 3x_4 = 0 \end{matrix}\right\}$

Let, $\underset{\sim}{x} \in S \cap T$, then

$2x_1 + x_2 + 3x_3 + x_4 = 0 \Rightarrow 2x_1 + x_2 = -3x_3 - x_4$ —— ①

$x_1 + 2x_2 + x_3 + 3x_4 = 0 \Rightarrow x_1 + 2x_2 = -x_3 - 3x_4$ —— ②

Solving ① & ②, we get, $x_1 = \dfrac{-5x_3 + x_4}{3}$

$x_2 = \dfrac{x_3 - 5x_4}{3}$

Here, $\underset{\sim}{x} = \left(\dfrac{-5x_3 + x_4}{3}, \dfrac{x_3 - 5x_4}{3}, x_3, x_4\right)$

$= x_3\left(-\frac{5}{3}, \frac{1}{3}, 1, 0\right) + x_4\left(\frac{1}{3}, -\frac{5}{3}, 0, 1\right)$

Here, $\{(-\frac{5}{3}, \frac{1}{3}, 1, 0), (\frac{1}{3}, -\frac{5}{3}, 0, 1)\}$ forms a basis.

$\therefore \dim(S \cap T) = 2$.

Now, $x \in S$

$\therefore \underset{\sim}{x} = (x_1, x_2, x_3, x_4)$, $2x_1 + x_2 + 3x_3 + x_4 = 0$

$= (x_1, x_2, x_3, -2x_1 - x_2 - 3x_3)$

$= x_1(1, 0, 0, -2) + x_2(0, 1, 0, -1) + x_3(0, 0, 1, -3)$

Clearly, $\{(1,0,0,-2), (0,1,0,-1), (0,0,1,-3)\}$ forms a basis for S.

$\therefore \dim(S) = 3$.

Now, $\underset{\sim}{x} \in T$

$\therefore \underset{\sim}{x} = (x_1, x_2, x_3, x_4)$; $x_1 + 2x_2 + x_3 + 3x_4 = 0$

$= (-2x_2 - x_3 - 3x_4, x_2, x_3, x_4)$

$= x_2(-2, 1, 0, 0) + x_3(-1, 0, 1, 0) + x_4(-3, 0, 0, 1)$

Clearly, $\{(-2,1,0,0), (-1,0,1,0), (-3,0,0,1)\}$ forms a basis for T.

$\therefore \dim(T) = 3$.

$\therefore \dim(S+T) = \dim(S) + \dim(T) - \dim(S \cap T)$

$= 3 + 3 - 2 = 4$

Clearly, $S+T \subseteq V_4$

And $\dim(S+T) = \dim(V_4)$

$\Rightarrow S+T = V_4$.

$\Rightarrow \{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3, \underset{\sim}{e}_4\}$ is a basis of $(S+T)$.

6) If $U = \text{span}\{(1,2,1),(2,1,3)\}$, $W = \text{span}\{(1,0,0),(0,1,0)\}$. Show that U and W are subspaces of $\mathbb{R}^3$. Determine a basis and dimension of $U \cap W$ and $U+W$.

ANS:- Let $\underset{\sim}{x} \in U \cap W$

$\therefore \underset{\sim}{x} = l_1(1,2,1) + l_2(2,1,3)$
$= l_1(1,0,0) + l_2(0,1,0)$

$\therefore l_1 + 2l_2 = l_1$
$2l_1 + l_2 = l_2$
$l_1 + 3l_2 = 0$

$\therefore l_1 = -3l_2$

$\therefore \underset{\sim}{x} = l_2\{(-3)(1,2,1) + (2,1,3)\} = l_2(-1,-5,0)$
$= t(1,5,0)$, $t \in \mathbb{R}$

$\therefore U \cap W = \{t(1,5,0) : t \in \mathbb{R}\}$

$\therefore$ Basis of $U \cap W = \{(1,5,0)\}$

$\therefore \dim(U \cap W) = 1$.

$(1,2,1),(2,1,3) \in U$ and they are linearly independent, so therefore they form a basis of U.

$\therefore \dim(U) = 2$

Similarly, $\dim(W) = 2$

$\therefore \dim(U+W) = \dim(U) + \dim(W) - \dim(U+W)$
$= 2+2-1 = 3$.

Clearly, $U+W \subseteq \mathbb{R}^3$

$\therefore \dim(U+W) = \dim(\mathbb{R}^3)$

$\& U+W = \mathbb{R}^3$

$\therefore \{\underset{\sim}{e_1}, \underset{\sim}{e_2}, \underset{\sim}{e_3}\}$ is a basis of $(U+W)$.

7) Given $S_1 = \{(1,2,3),(0,1,2),(3,2,1)\}$ and $S_2 = \{(1,-2,3),(-1,1,-1),(1,-3,4)\}$. Determine the dimension and a basis for

i) $[S_1] \cap [S_2]$, ii) $[S_1] + [S_2]$,

where $[S]$ denotes the span of S. [C.U. 2004] i)

$$S_1 \cap S_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & -2 & 3 \\ -1 & 1 & -2 \\ 1 & -3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 5 & -5 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & -5 & 1 \end{bmatrix} \quad \begin{aligned} R_3' &= R_3 + 3R_5 \\ R_4' &= R_4 + R_5 \\ R_5' &= R_5 + R_6 \\ R_6' &= R_6 - R_1 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & -5 & 1 \end{bmatrix} \quad \begin{aligned} R_3' &= R_3/5 \\ R_4' &= R_4 - 2R_3 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -5 & 1 \end{bmatrix} \quad \begin{aligned} R_4' &= R_4 + R_3 \\ R_3' &= R_3 - R_2 \end{aligned} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & -5 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} R_3' &= R_3/3 \\ R_4 &\leftrightarrow R_6 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} R_2' &= R_2 - 2R_3 \\ R_4' &= R_4 + 5R_2' - R_3 \end{aligned}$$

Here, ~~[illegible]~~ $\dim(S_1 \cap S_2) = 3$

Now, $\text{dis}(S) = \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2)$

$$S_1 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -4 & -8 \end{pmatrix}$$

$$\xrightarrow{R_3' = R_3 + 4R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore \dim(S_1) = 2$

$$S_2 = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 1 & -3 & 4 \end{pmatrix} \xrightarrow[R_3' \to R_3 - R_1]{R_2' \to R_1 + R_2} \begin{pmatrix} 1 & -2 & 3 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow[R_2' \to R_2/4]{R_3' \to R_3 - R_2} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\dim(S_2) = 2$

$$\begin{aligned} \therefore \dim(S_1 + S_2) &= \text{dis}(S_1) + \dim(S_2) - \dim(S_1 \cap S_2) \\ &= 2 + 2 - 3 \\ &= 1. \end{aligned}$$

8) Construct an orthogonal matrix $A^{4\times 4}$ whose first row is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. find A^{-1}. The matrix B is obtained by replacing the third column of A by ($2\times$ 3rd column of A). Using A^{-1}, find B^{-1}. [C.U. 2002]

ANS:- Note that $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3, \underset{\sim}{e}_4\}$ form a basis for E^4.

Now, $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}\underset{\sim}{e}_1 + \frac{1}{2}\underset{\sim}{e}_2 + \frac{1}{2}\underset{\sim}{e}_3 + \frac{1}{2}\underset{\sim}{e}_4$

$\therefore \left\{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \underset{\sim}{e}_2, \underset{\sim}{e}_3, \underset{\sim}{e}_4\right\}$ forms a basis for E^4.

Taking, $\underset{\sim}{v}_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$$\therefore \underset{\sim}{u}_1 = \frac{\underset{\sim}{v}_1}{\|\underset{\sim}{v}_1\|} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$\underset{\sim}{v}_2 = (0,1,0,0) - \left\{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)(0,1,0,0)\right\}\frac{1}{2}(1,1,1,1)$$

$$= (0,1,0,0) - \frac{1}{4}(1,1,1,1)$$

$$= \left(-\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$$

$$\therefore \underset{\sim}{u}_2 = \frac{\underset{\sim}{v}_2}{\|\underset{\sim}{v}_2\|} = \frac{\left(-\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}\right)}{\sqrt{\frac{1}{16}+\frac{9}{16}+\frac{1}{16}+\frac{1}{16}}} = -\frac{1}{\sqrt{12}}(1,-3,1,1)$$

$$\underset{\sim}{v}_3 = \underset{\sim}{e}_3 - \{\underset{\sim}{e}_3\cdot\underset{\sim}{u}_1\}u_1 - \{\underset{\sim}{e}_3\cdot\underset{\sim}{u}_2\}\underset{\sim}{u}_2$$

$$= (0,0,1,0) - \left\{(0,0,1,0)\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right\}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$- \left\{(0,0,1,0)\left(-\frac{1}{\sqrt{12}}, \frac{3}{\sqrt{12}}, -\frac{1}{\sqrt{12}}, -\frac{1}{\sqrt{12}}\right)\right\}\left(-\frac{1}{\sqrt{12}}, \frac{3}{\sqrt{12}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{12}}\right)$$

$$= \left(-\frac{1}{3}, 0, \frac{2}{3}, -\frac{1}{3}\right)$$

$$\therefore \underset{\sim}{u}_3 = \frac{\underset{\sim}{v}_3}{\|\underset{\sim}{v}_3\|} = \frac{\left(-\frac{1}{3}, 0, \frac{2}{3}, -\frac{1}{3}\right)}{\sqrt{\frac{1}{9}+0+\frac{4}{9}+\frac{1}{9}}} = \frac{1}{\sqrt{6}}(-1,0,2,-1).$$

$$\underset{\sim}{v}_4 = (0,0,0,1) - \left\{(0,0,0,1)\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right\}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$- \left\{(0,0,0,1)\left(-\frac{1}{\sqrt{12}}\right)(1,-3,1,1)\right\}\left(-\frac{1}{\sqrt{12}}\right)(1,-3,1,1)$$

$$- \left\{(0,0,0,1)\left(\frac{1}{\sqrt{6}}\right)(-1,0,2,-1)\right\}\left(\frac{1}{\sqrt{6}}\right)(-1,0,2,-1)$$

$$= \left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right)$$

$$\therefore \underset{\sim}{u}_4 = \frac{\underset{\sim}{v}_4}{\|\underset{\sim}{v}_4\|} = \frac{\left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right)}{\sqrt{\frac{1}{4}+0+0+\frac{1}{4}}} = \frac{1}{\sqrt{2}}(-1,0,0,1)$$

Hence, $\{\underset{\sim}{u}_1, \underset{\sim}{u}_2, \underset{\sim}{u}_3, \underset{\sim}{u}_4\}$ is a set of 4 orthonormal vectors.

Hence, $A = \begin{pmatrix} \underset{\sim}{u}_1 \\ \underset{\sim}{u}_2 \\ \underset{\sim}{u}_3 \\ \underset{\sim}{u}_4 \end{pmatrix}$ is an orthonormal matrix with first

now, $\underset{\sim}{u}_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{\sqrt{12}} & \frac{3}{\sqrt{12}} & -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{6}} & 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

As A is an orthogonal matrix, so $A' = A^{-1}$.

$$\therefore A^{-1} = A' = \begin{bmatrix} \frac{1}{2} & -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{3}{\sqrt{12}} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{\sqrt{12}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{2} & -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Here, $B = AE_3(2)$, where $E_3(2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\text{Now, } B^{-1} = [A E_3(2)]^{-1}$$
$$= E_3^{-1}(2) A^{-1}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{3}{\sqrt{12}} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{\sqrt{12}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{2} & -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{3}{\sqrt{12}} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2\sqrt{12}} & \frac{1}{\sqrt{6}} & 0 \\ \frac{1}{2} & -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{6}} & \frac{1}{2} \end{pmatrix}$$

9) Let A and B be defined by

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & 1 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & -2 & -1 \end{pmatrix}$$

(i) Show that the column space of B is a subspace of A.
(ii) Find a matrix C such that AC = B. [C.U. 2000]

ANS:— (i) Column space of A $= C(A) = \left\{ l_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + l_2 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + l_3 \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} ; \; l_1, l_2, l_3 \in \mathbb{R} \right\}$

Column space of B $= C(B) = \left\{ l \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} ; l \in \mathbb{R} \right\}$

Now, for any vector $x \in C(B)$

we have, $\underset{\sim}{x} = l \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$= l_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + l_2 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + l_3 \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$, where $l = l_1$ & $l_2 = l_3 = 0$.

which belongs to $C(A)$.

$\therefore x \in C(A) \Rightarrow x \in C(B)$.

$\therefore C(B) \subseteq C(A)$

(ii) Here, note that, $r(A) = 2$, i.e. $r(A) < 3$,
Since 1st & 2nd rows are LD.

$\Rightarrow$ A is singular.

$\Rightarrow A^{-1}$ does not exist.

[If A^{-1} exists, then $AC = B \Rightarrow C = A^{-1}B$ and it is unique]

Hence, choice of C is not unique.

Note that,

$$\underset{(A)}{\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & 1 & 1 \end{pmatrix}} \underset{(C)}{\begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} = \underset{(B)}{\begin{pmatrix} 0 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & -2 & -1 \end{pmatrix}}$$

Alternatice choice of $C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

10) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \\ 9 & 16 & 23 \end{pmatrix}$. Suppose $r(A) = r$.

i) Find r. ii) Write A as the sum of r matrices each of rank unity. iii) Find an orthonormal basis of the row-space of A.

[C.U. 2008]

ANS:-

Note that, $(9, 16, 23)$
$= 3(1,2,3) + 2(3,5,7)$

i) $$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \\ 3\times1+2\times1 & 3\times2+2\times5 & 3\times3+2\times7 \end{pmatrix}$$

clearly, Rank$(A) = 2$, as there have two LIN rows.

ii) $$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 3 & 6 & 9 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 3 & 5 & 7 \\ 6 & 10 & 14 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 3\times1 & 3\times2 & 3\times3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 3 & 5 & 7 \\ 3\times2 & 5\times2 & 7\times2 \end{pmatrix}$$

$= A_1 + A_2$, where Rank$(A_i) = 1$, $\forall i = 1, 2$.

iii) $\{(1,2,3), (3,5,7)\}$ form a basis of the row space of A.

$\therefore \underset{\sim}{v}_1 = (1,2,3)$

$$\therefore \underset{\sim}{u}_1 = \frac{\underset{\sim}{v}_1}{||\underset{\sim}{v}_1||} = \frac{(1,2,3)}{\sqrt{1+4+9}} = \frac{1}{\sqrt{14}}(1,2,3)$$

$$\therefore \underset{\sim}{v}_2 = (3,5,7) - \left\{(3,5,7)\frac{1}{\sqrt{14}}(1,2,3)\right\}\frac{1}{\sqrt{14}}(1,2,3)$$

$$= \left(\frac{4}{7}, \frac{1}{7}, -\frac{2}{7}\right)$$

$$\therefore \underset{\sim}{u}_2 = \frac{\underset{\sim}{v}_2}{||\underset{\sim}{v}_2||} = \frac{7}{\sqrt{21}}\left(\frac{4}{7}, \frac{1}{7}, -\frac{2}{7}\right)$$

$$= \frac{1}{\sqrt{21}}(4, 1, -2)$$

11) Suppose $A = \begin{pmatrix} 7 & -12 & 4 \\ 3 & -2 & 5 \\ -6 & 11 & 8 \end{pmatrix}$.

(a) Find A^{-1}.

(b) Find a matrix B such that $AB = \begin{pmatrix} 8 & -12 & 4 \\ 3 & -1 & 5 \\ -6 & 11 & 9 \end{pmatrix}$.

(c) Write A as a sum of symmetric and a skew-symmetric matrices.

(d) Let V be a vector space generated by the first two column vectors of A. Then write the third column vector of A as the sum of the two non-zero vectors such that one is a member of V and the other is orthogonal to V. [C.U. 1998]

ANS:-

(d) $V = \left\{ l_1 \begin{pmatrix} 7 \\ 3 \\ -6 \end{pmatrix} + l_2 \begin{pmatrix} -12 \\ -2 \\ 11 \end{pmatrix} : l_1, l_2 \in R \right\}$.

Let, $\begin{pmatrix} 4 \\ 5 \\ 8 \end{pmatrix} = \underset{\sim}{x} + \underset{\sim}{y}$ where $\underset{\sim}{x} \in V$ and $\underset{\sim}{y}$ is $\perp$ to V.

Now, $\underset{\sim}{x} \in V$, $\underset{\sim}{x} = l_1 \begin{pmatrix} 7 \\ 3 \\ -6 \end{pmatrix} + l_2 \begin{pmatrix} -12 \\ -2 \\ 11 \end{pmatrix}$

Here, $\underset{\sim}{y}$ is $\perp$ to V $\Rightarrow$ $7y_1 + 3y_2 - 6y_3 = 0$

$-12y_1 - 2y_2 + 11y_3 = 0$

Let, $y_3 = t$, now, $y_1 = \frac{21t}{22}$, $y_2 = -\frac{5t}{22}$; $y = t \begin{pmatrix} 21/22 \\ -5/22 \\ 1 \end{pmatrix}$

$\therefore \underset{\sim}{x} = l_1 \begin{pmatrix} 7 \\ 3 \\ -6 \end{pmatrix} + l_2 \begin{pmatrix} -12 \\ -2 \\ 11 \end{pmatrix}$; $l_1, l_2 \in R$ $= 22t \begin{pmatrix} 21 \\ -5 \\ 22 \end{pmatrix}$ $= l_3 \begin{pmatrix} 21 \\ -5 \\ 22 \end{pmatrix}$

$\therefore \begin{pmatrix} 4 \\ 5 \\ 8 \end{pmatrix} = l_1 \begin{pmatrix} 7 \\ 3 \\ -6 \end{pmatrix} + l_2 \begin{pmatrix} -12 \\ -2 \\ 11 \end{pmatrix} + l_3 \begin{pmatrix} 21 \\ -5 \\ 22 \end{pmatrix}$; $l_1, l_2, l_3 \in R$.

12) (a) Suppose $A = \begin{pmatrix} 4 & 3 & 6 \\ 5 & 9 & 8 \\ 2 & 7 & 1 \end{pmatrix}$. Find two non-singular matrices P and Q such that $PAQ = I_3$. [C.U. 1999]

(b) Find inverse of $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & -1 \\ -1 & 2 & 0 \end{pmatrix}$, by Pivotal Condensation or sweeping-out method.

(c) Obtain the fully reduced normal form of $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix}$

(d) Express $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ as the product of elementary matrices.

ANS:-

(i) By problem, $PAQ = I_3$

$\therefore$ Rank $(PAQ) = $ Rank (I_3)

$\therefore$ rank$(A) = 3$, as P and Q are non-singular matrices and rank is not altered by non-singular matrix multiplication.

$\Rightarrow$ A is non-singular.

$\Rightarrow A^{-1}$ exists.

$\therefore A^{-1} A I_3 = I_3$

$\Rightarrow PAQ = I_3$, where $P = A^{-1}$, $Q = I_3$.

(ii) $(A \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 & 0 & 1 \end{array}\right)$

R [Row operations]

$\sim$

$\sim$

$\sim$

$\sim$

$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 7 & -1 & 5 \end{array}\right)$

$\therefore A^{-1} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \\ 7 & -1 & 5 \end{pmatrix}$

(c) $[I_3 \mid A^{3\times 3} \mid I_3]$

$$= \left[\begin{array}{ccc|ccc|c} 1 & 0 & 0 & 1 & 0 & 1 & \\ 0 & 1 & 0 & 2 & 1 & 3 & I_3 \\ 0 & 0 & 1 & 1 & 1 & 2 & \end{array}\right]$$

$R_2' \to R_2 - 2R_1$

$R_3' \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|ccc|c} 1 & 0 & 0 & 1 & 0 & 1 & \\ -2 & 1 & 0 & 0 & 1 & 1 & I_3 \\ -1 & 0 & 1 & 0 & 1 & 1 & \end{array}\right]$$

$R_3' \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$C_3' \to C_3 - C_1 - C_2$

$$\sim \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ -2 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|cc|c} P & I_2 & 0 & Q \\ & 0 & 0 & \end{array}\right]$$

Hence, $P = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

(d) $[I_3 \mid A^{3\times 3} \mid I_3]$

$$= \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim$$

$$\sim$$

$$\sim \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] ; P = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}, Q = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$\therefore A = P^{-1} I_3 Q^{-1} = P^{-1} Q^{-1}$.

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} P \quad [R_3' \to R_3 + R_2]$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} P \quad [R_2' \to R_1 + R_2]$$

$$\therefore P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} Q$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} Q$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} Q$$

$$\Rightarrow Q^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

12) (a) Solve the following system of linear equations using a numerical method:

$$x_1 - 2x_2 + 3x_3 + 4x_4 = 4.5$$
$$3x_1 - x_2 + 2x_3 + 5x_4 = 9.5$$
$$2x_1 + 4x_2 + 5x_3 + x_4 = 15$$
$$4x_1 + 2x_2 - x_3 + 3x_4 = 12$$

[C.U. 2007]

ANS:-

$$\left(\begin{array}{cccc|c} 1 & -2 & 3 & 4 & 4.5 \\ 3 & -1 & 2 & 5 & 9.5 \\ 2 & 4 & +5 & 1 & 15 \\ 4 & 2 & -1 & 8 & 12 \end{array}\right)$$

$R_2' = R_2 - 3R_1$

$R_3' = R_3 - 2R_1$

$R_4' = R_4 - 4R_1$

$$\left(\begin{array}{cccc|c} 1 & -2 & 3 & 4 & 4.5 \\ 0 & 5 & -7 & -7 & \\ 0 & & & & \\ 0 & & & & \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \end{array}\right)$$

$$\therefore \left(\quad\right)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \left(\quad\right)$$

$\therefore x_1$

x_2

x_3

x_4

(b) For what value of k the planes

$x-4y+5z=k$

$x-y+2z=3$ (i) intersect in a line ?,

$2x+y+z=0$, (ii) intersect in a point ?

ANS:-

Augmented matrix $= [A : b]$

$$= \left[\begin{array}{ccc|c} 1 & -4 & 5 & k \\ 1 & -1 & 2 & 3 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

$R_2' \to R_2 - R_1$, $R_3' \to R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -4 & 5 & k \\ 0 & 3 & -3 & 3-k \\ 0 & 9 & -9 & -2k \end{array}\right]$$

$R_3' \to R_3 - 3R_2$, $R_2' \to R_2/3$

$$\sim \left[\begin{array}{ccc|c} 1 & -4 & 5 & k \\ 0 & 1 & -1 & 3-k/3 \\ 0 & 0 & 0 & k-9 \end{array}\right]$$

(i) If $k=9$, then $\text{Rank}(A:b) = 2 = \text{Rank}(A)$
$\Rightarrow$ The system is consistent and has infinitely many solutions and intersect in a line.

(ii) $R(A:b) = R(A) = 3$ is not satisfied by no value of k.

13) (i) Identify the definiteness of the following quadratic forms:

$x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 4x_2x_3 + 2x_1x_3$.

(ii) $2x_1^2 + 2x_2^2 + 5x_3^2 - 4x_1x_2 - 2x_1x_3 + 2x_2x_3$.

ANS:- (i) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$; $a_{11} = 1 > 0$, $\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 > 0$,

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = 1 > 0$

$\therefore$ A is a p.d. matrix.

(ii) $A = \begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & 1 \\ -1 & 1 & 5 \end{pmatrix}$

$a_{11} = 2 > 0$, $\begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix} = 0$, $\begin{vmatrix} 2 & -2 & -1 \\ -2 & 2 & 1 \\ -1 & 1 & 5 \end{vmatrix} = 0$

$\therefore$ A is a p.s.d. matrix.

14) S.T. $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 11 \end{pmatrix}$ is p.d. and find the matrix
(i)

B such that $A = BB^T$.

ANS:-

$a_{11} = 1 > 0$, $\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 1 > 0$, $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 11 \end{vmatrix} = 1 > 0$

$\therefore$ All the principal minors of A is positive.

$\therefore$ A is p.d. matrix.

$$\left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 5 & 7 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 7 & 11 & 0 & 0 & 1 \end{array}\right)$$

$R_2' \to R_2 - 2R_1$
$C_2' \to C_2 - 2C_1$
$$\left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 3 & 1 & -2 & 0 \\ -2 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 & 11 & 0 & 0 & 1 \end{array}\right)$$

$R_3' \to R_3 - 3R_1$
$C_3' \to C_3 - 3C_1$
$$\left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & -2 & -3 \\ -2 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ -3 & 0 & 1 & 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$R_3' \to R_3 - R_2$
$C_3' \to C_3 - C_2$
$$\left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & -2 & -1 \\ -2 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

& $CAC' = I_3 \Rightarrow A = C^{-1} I_3 (C^{-1})'$

$= (C^{-1})(C^{-1})'$

$= BB^T$

where $B = C^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}^{-1}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}$

(ii) Solve the equations:

$$5x_1 - 2x_2 - x_3 = 4z$$
$$5x_2 - 2x_3 - x_1 + 3z = 0$$
$$5x_3 - 2x_1 - x_2 = 3z$$

to express x_i's in terms of z. For what value of z, do x_1, x_2, x_3 further satisfy $x_1 + x_2 + x_3 = 1$?

[C.U. 1991]

ANS:- Augmented matrix is $[A \mid b]$

$$= \left(\begin{array}{ccc|c} 5 & -2 & -1 & 4z \\ -1 & 5 & -2 & -3z \\ -2 & -1 & 5 & 3z \end{array}\right)$$

$$\underset{R_2 \leftrightarrow R_1}{\sim} \left(\begin{array}{ccc|c} -1 & 5 & -2 & -3z \\ 5 & -2 & -1 & 4z \\ -2 & -1 & 5 & 3z \end{array}\right)$$

$$\underset{\substack{R_2' \leftrightarrow R_2 + 5R_1 \\ R_3' \to R_3 - 2R_1}}{\sim} \left(\begin{array}{ccc|c} -1 & 5 & -2 & -3z \\ 0 & 23 & -11 & -11z \\ 0 & -11 & 9 & 9z \end{array}\right)$$

$$\underset{\substack{R_1' \to R_1(-1) \\ R_2' \to R_2/23 \\ R_3' \to R_3/11}}{\sim} \left(\begin{array}{ccc|c} 1 & -5 & 2 & 3z \\ 0 & 1 & -11/23 & -11z/23 \\ 0 & 1 & -9/11 & -\frac{9z}{11} \end{array}\right)$$

$$\underset{R_3' \to R_3 - R_2}{\sim} \left(\begin{array}{ccc|c} 1 & -5 & 2 & 3z \\ 0 & 1 & -11/23 & -11/23z \\ 0 & 0 & -86/253 & -\frac{86}{253}z \end{array}\right)$$

$$\underset{R_3' = R_3/-\frac{86}{253}}{\sim} \left(\begin{array}{ccc|c} 1 & -5 & 2 & 3z \\ 0 & 1 & -11/23 & -11/23z \\ 0 & 0 & 1 & z \end{array}\right)$$

$\therefore x_3 = t,\ x_2 = -\frac{11}{23}t,$

$\Rightarrow x_2 = 0$

$\Rightarrow x_1 = t$

$\therefore (x_1, x_2, x_3) = t(1, 0, 1)$ is a solution.

Again $x_1 + x_2 + x_3 = 1 \Rightarrow t + 0 + t = 1 \Rightarrow t = 1/2$

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15) i) Determine the null space of $A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & -3 & 1 \\ -1 & -1 & 0 & -5 \end{bmatrix}$.

Also find dim (N(A)) and rank (A).

ANS:- To find dim(N(A)), we use,

$$\dim(N(A)) = 4 - \text{rank}(A).$$

We know, $\{\underset{\sim}{x} : A\underset{\sim}{x} = 0\} = [N(A)]$

Here,

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & -3 & 1 \\ -1 & -1 & 0 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & -1 & -3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = H$$

Rank (A) = 2.

$$H\underset{\sim}{x} = \underset{\sim}{0}$$

$\Rightarrow x_1 + x_2 + 5x_4 = 0$

$\Rightarrow x_3 + 3x_4 = 0 \Rightarrow x_3 = -3x_4$

$\therefore x_4 = t,\ x_3 = -3t,\ x_2 = -3t,\ x_1 = -2t$

$$[N(A)] = t\begin{pmatrix} 1 \\ -3 \\ -3 \\ -2 \end{pmatrix}$$

$\therefore \dim(N(A)) = 4 - 2 = 2.$

(ii) Find an orthogonal matrix which diagonalizes $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$. Also, find A^8.

ANS:- The characteristic equation of A is

$$0 = |A - \lambda I_3| = \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 6-\lambda & -2 & 2 \\ 0 & 2-\lambda & 2-\lambda \\ 2 & -1 & 3-\lambda \end{vmatrix} \quad R_2' \to R_2 + R_3$$

$$= (2-\lambda) \begin{vmatrix} 6-\lambda & -2 & 2 \\ 0 & 1 & 1 \\ 2 & -1 & 3-\lambda \end{vmatrix}$$

$$= (2-\lambda)\{(6-\lambda)(3-\lambda+1) + 2(-2-2)\}$$

$$= (2-\lambda)(\lambda-2)(\lambda-8)$$

$\therefore \lambda = 2, 2, 8$ are the eigen values of A.

For $\lambda = 8$,

$(A - 8I_3)\underset{\sim}{x} = \underset{\sim}{0}$

$$\Rightarrow \begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1 + x_2 - x_3 = 0 \\ 2x_1 + 5x_2 + x_3 = 0 \\ 2x_1 - x_2 - 5x_3 = 0 \end{cases}$$

Let, $x_3 = t$, $\Rightarrow x_1 + x_2 = t$.

$\Rightarrow x_2 = t - x_1$

$\therefore x_2 = -t$, & $x_1 = 2t$.

$$\therefore x = t\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad \underset{\sim}{a_3} = \frac{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}}{\sqrt{4+1+1}} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

For $\lambda = 2$, $(A - 2I_3)\underset{\sim}{x} = 0$

$$\Rightarrow \begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow 2x_1 - x_2 + x_3 = 0$

$x_2 = t_1$, $x_3 = t_2$, $x_1 = \frac{t_1 - t_2}{2}$. $\quad \underset{\sim}{x} = t_1\begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + t_2\begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$

$\underset{\sim}{x_1} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\underset{\sim}{x_2} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ are LIN eigen vectors when $\lambda = 2$.

$$\underset{\sim}{v}_1 = \underset{\sim}{a}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$\underset{\sim}{u}_1 = \frac{\underset{\sim}{a}_1}{\|\underset{\sim}{a}_1\|} = \frac{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}{\sqrt{5}} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$\underset{\sim}{v}_2 = \underset{\sim}{a}_2 - (\underset{\sim}{u}_1 \cdot \underset{\sim}{a}_2)\underset{\sim}{u}_1$$

$$= \begin{pmatrix} -4/5 \\ 2/5 \\ 2 \end{pmatrix}$$

$$\underset{\sim}{u}_2 = \frac{\underset{\sim}{v}_2}{\|\underset{\sim}{v}_2\|} = \frac{1}{\sqrt{30}}\begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$$

$$\underset{\sim}{v}_3 = \underset{\sim}{a}_3 - (\underset{\sim}{a}_3 \underset{\sim}{u}_1)u_1 - (\underset{\sim}{a}_3 \underset{\sim}{u}_2)\underset{\sim}{u}_2$$

$$= \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\therefore \underset{\sim}{u}_3 = \frac{\underset{\sim}{v}_3}{\|\underset{\sim}{v}_3\|} = \frac{\frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}}{1} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\therefore Q = (\underset{\sim}{u}_1 \quad \underset{\sim}{u}_2 \quad \underset{\sim}{u}_3)$$

$$= \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{30}} & \frac{2}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{pmatrix} \leftarrow \text{orthogonal matrix.}$$

$$\therefore Q'AQ' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

λ_i^m is the eigen value of the matrix A^m.

So, $$Q'A^8Q = \begin{pmatrix} 2^8 & 0 & 0 \\ 0 & 2^8 & 0 \\ 0 & 0 & 8^8 \end{pmatrix}$$

$$\Rightarrow A^8 = Q'\begin{pmatrix} 256 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 16777216 \end{pmatrix} Q$$

as $Q^{-1} = Q'$

PRACTICAL PROBLEMS ON DESCRIPTIVE STATISTICS

Q.NO.1. [CU'03] The sound intensity levels measured in decibels at 50 construction sites are

68, 63, 59, 77, 60, 57, 63, 62, 64, 73, 63, 70, 71, 65, 68, 67,
67, 62, 56, 61, 69, 64, 58, 73, 68, 66, 65, 64, 68, 67, 69, 68,
67, 70, 69, 61, 65, 62, 68, 62, 69, 64, 82, 86, 65, 69, 68, 70, 66.

(i) Prepare a frequency table for considering 6 classes of equal width, (ii) Calculate the exact mean and variance of sample from the raw data. (iii) Estimate the mean and variance from the grouped frequency distribution by assuming each value is equal to the mid-point of the class to which it belongs.
(iv) Compare the answers in (ii) and (iii) and in this connection mention about the role of Sheppard's correction.

Q.NO.2. [CU'96] A chemical compound contains 12.5% of iron was given to two technicians A and B for chemical analysis. A made of 15 determinations, and B 10 determinations, of the percentage of iron. Their results are given in the following table

Determinations by A			Determinations by B	
12.46	12.11	12.47	12.23	12.11
11.80	12.44	12.11	11.91	12.45
12.40	11.85	12.56	12.45	12.39
11.95	12.12	12.65	12.22	12.37
12.77	12.43	12.72	12.05	12.65

(i) Find seperately for A and B various measures of central tendency and dispersion. Also, find their respective coefficients of variation.
(ii) Based on the above measures prepare a report comparing the accuracy and consistency of the two technicians.

Q.NO.3. The following tables relate to the weights of new born babies recorded at the different clinics. One of the clinics is located in a locality where the average family income is also homogeneous throughout the locality where as the other clinic caters to the same category but also has a considerable number of citizens from a significant lower income group. Examine the locations, dispersions and the shapes of the two distributions and interpret your findings.

[CU'1995]

Weights in (kgs.)	Clinic I	Clinic II
0 – 1	0	0
2 – 3	1	43
3 – 4	60	230
4 – 5	304	372
5 – 6	318	320
6 – 7	56	57
>7	3	2
	0	0

Q.No.4. [CU'2001] The scores in English of 250 candidates appearing in an Examination have

$$m_1' = 39.7213 \quad , \quad s = 9.8894$$

$$\frac{m_3}{m_2^{3/2}} = -0.1182 \quad , \quad b_2 = 2.9719.$$

It is later found on scrutiny that score 61 has been wrongly recorded as 51. Obtain the correct values of b_1 and b_2.

Q.No.5. Particular relating to the monthly wage distributions of two manufacturing firms are given below:

Measures	Firm A	Firm B
Mean wage	Rs. 1477	1495
Median wage	Rs. 1389	1354
Modal wage	Rs. 1350	1312
Quartiles	Rs. 1278, 1422	1262, 1435
S.D.	Rs. 87	Rs. 99

Compare the two distributions w.r.t. the characteristic central tendency, dispersion and skewness, kurtosis.

Q.No.6. The weights (in grams) of 25 indicator housing used on gauges are as follows:

102.0	106.3	106.6	108.8	107.7
106.1	105.9	106.7	106.8	110.2
101.7	106.6	106.3	110.2	109.9
102.0	105.8	106.1	106.7	1007.3
102.0	106.8	110.0	107.9	109.3

(a) Construct an ordered stem-leaf display using integers as the stems and tenths as the leaves.
(b) Find the five-number summary of the data and draw a box-plot.
(c) Are there any suspected outliers?

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Q.No.7. The following table gives the yearly corn yield (X) in bushels per acre, in six Corn Belt states (Iowa, Illinois, Nebraska, Missouri, Indiana, Ohio) and rain fall measurements in inches (Y) in six states from 1915 to 1927.

Year:	1915	1916	1917	1918	1919	1920	1921	1922	1923	1924	1925	1926	1927
X :	33.3	29.7	35.0	29.9	35.2	38.3	35.2	35.5	36.7	26.8	38.0	31.7	32.6
Y :	16.5	9.3	9.4	9.0	9.5	11.6	12.1	8.0	10.7	13.9	11.3	11.6	10.4

Fit a linear regression model : $X = \alpha + \beta Y + \epsilon$ to the data by the method of least squares, making the usual assumptions.

[CU'1997]

Q.No.8. In an experiment on certain fertilizers were applied at various levels (in appropriate units) with resulting yields (in appropriate unit) as follows:

Fertilizer level (x) :	0	5	10	15	20	25	30
Yield (y) :	27.1	32.1	35.0	36.2	36.9	36.1	35.2

(i) Fit an appropriate polynomial to the given data.
(ii) Obtain a suitable measure of association between x and y and comment
(iii) Obtain the optimum fertilizer level which maximizes yield.

Q.No.9. [CU'2001]
The following data relate to the height (x) and weight (y) of 15 students

x	5'6"	5'8"	5'3"	5'9"	5'6"	5'6"	5'9"	5'3"	5'6"	5'6"	5'9"	5'6"	5'3"	5'9"	5'3"
y	55	52	60	61	57	54	60	51	58	50	59	55	44	65	47

Compute r_{yx} and e_{yx}. Is the regression linear? If not, compute a measure of deviation from linearity.

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Q.No.10. Two supervisors ranked 12 workers working under them in order of efficiency as follows. Compute Spearman's rank correlation coefficient between the two rankings. Also compute Kendall's τ.

Worker :	A	B	C	D	E	F	G	H	I	J	K	L
Supervisor I :	5	6	1	2	3	8	8	4	7	11	10	12
Supervisor II :	5	5	1	1	1	9	7	4	8	10	12	10

Q.No.11. [CU'2008]

The following table gives data on income in thousand dollars (x), the number of families (N) at income x and the number of families owing a house (n).

x :	10	13	15	20	25	30	35	40
N :	60	80	100	70	65	50	40	25
n :	18	28	45	36	39	33	30	20

Suggest an appropriate regression equation to explain the effect of income on owning a house. Also estimate the parameters of this equation using the above data and predict there from the proportion of families of income 32 thousands dollars who own a house.

Solutions:→ PROBLEMS ON DESCRIPTIVE STATS.

1. Minimum value = 56 , Maximum value = 86

Range = 30.

(i) Considering 6 classes of width 5:

Class-interval limit	Class-boundary	Frequency (f_i)	Tally marks	Cls. mark (x_i')
53-58	52.5 - 58.5	2	//	55.5
59-64	58.5 - 64.5	16	卌 卌 卌 /	61.5
65-70	64.5 - 70.5	25	卌 卌 卌 卌 卌	67.5
71-76	70.5 - 76.5	4	////	73.5
77-82	76.5 - 82.5	2	//	79.5
83-88	82.5 - 88.5	1	/	85.5
TOTAL		50		

(i) From the raw data,

$$\text{mean}(\bar{x}) = \frac{1}{50}\sum_{i=1}^{50} x_i$$

$$\text{Variance}(s^2) = \frac{1}{50}\sum_{i=1}^{50} x_i^2 - \bar{x}^2$$

(ii) From the grouped data,

$$\text{mean}(\bar{x}') = \frac{1}{\sum_{i=1}^{6} f_i}\sum_{i=1}^{6} x_i' f_i$$

$$\text{Variance}(s'^2) = \frac{1}{50}\sum_{i=1}^{6} x_i'^2 f_i - \bar{x}'^2$$

(iv) The mean and variance computed from the raw data and from grouped data are different.

In general, frequency distribution, we assume that all the values in a class are equal to the mid-point which is true if values are uniformly distributed over the class. so mean and variance computed from the raw data and grouped data are different due to the error due to grouping. The correction for the error due to grouping is given by Sheppard's correction.

$$m_1'(\text{corrected}) = m_1'$$

$$m_2(\text{corrected}) = m_2 - \frac{c^2}{12}, \quad c = \text{class width}$$

2. (i) Exact mean of A $= \frac{1}{15}\sum_{i=1}^{15} x_{iA} =$

Exact mean of B $= \frac{1}{10}\sum_{i=1}^{10} x_{iB} =$

Measure of Dispersion, $SD(A) = \sqrt{\frac{1}{15}\sum_{i=1}^{15} x_{iA}^2 - \bar{x}_A^2}$

$=$

$$SD(B) = \sqrt{\frac{1}{10}\sum_{i=1}^{10} x_{iB}^2 - \bar{x}_B^2} =$$

$$RMSD_A(12.5) = \sqrt{\frac{1}{15}\sum_{i=1}^{15} (x_{iA} - 12.5)^2}$$

$$RMSD_B(12.5) = \sqrt{\frac{1}{10}\sum_{i=1}^{10} (x_{iB} - 12.5)^2}$$

Co-efficient of variation, $CV(A) = \frac{SD(A)}{\bar{x}_A} \times 100\% =$

$$CV(B) = \frac{SD(B)}{\bar{x}_B} \times 100\% =$$

(ii) As a measure of accuracy, we use RMSD about 12·5. The smaller the RMSD, is more accuracy.

As a measure of consistency, we use C.V. The smaller C.V. is more consistency in Data set.

3) The frequency distribution of clinic-I is highly positively skewed and that of clinic-II is near about symmetric. [To get the measure of distribution we may draw histogram of the frequency distribution].

(i) As a measure of location, we may use median.

∴ Median of Clinic I is

∴ Median of Clinic II is

(ii) For Dispersion, we use $Q.D. = \frac{Q_3 - Q_1}{2}$.

∴ Q.D. of Clinic I is

& Q.D. of clinic II is

(iii) For shape, we use, $S_K = \frac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1}$

∴ Skewness of clinic I is

∴ Skewness of clinic II is

iv) For peackedness, we use, $K_p = \frac{Q_3 - Q_1}{2(P_{90} - P_{10})}$

∴ kurtosis of Clinic I is

∴ kurtosis of Clinic II is

The smaller the value of K_p, the higher the kurtosis.

Weights in (kgs)	Clinic I	Clinic II	≤ I	≤ II
0-1	0	0	0	0
1-2	1	43	1	43
2-3	60	230	61	273
3-4	304	372	365	645
4-5	318	320	683	965
5-6	56	54	739	1019
6-7	3	2	742	1021

$P_{90}^{I} = 4 + \frac{667.8 - 365}{318} \times 1$
$= 4.95$

$P_{10}^{I} = 3 + \frac{74.2 - 61}{304}$
$= 3.04$

$Q_1^{I} = 3 + \frac{185.5 - 61}{304} \times 1$
$= 3.41$

$Q_2^{I} = 4 + \frac{371 - 365}{318} \times 1$
$= 4.02$

$Q_3^{I} = 4 + \frac{556.5 - 365}{318} \times 1$
$= 4.60$

4)

$n = 250$

$m_1'(\text{incorrected}) = 39.7213$

$\Rightarrow \frac{1}{250}\sum_{i=1}^{50} x_i(\text{inconnected}) = 39.7213$

$\Rightarrow \sum_{i=1}^{50} x_i(\text{inconnected}) = 9930.325$

$\therefore \sum_i x_i(\text{connected}) = (9930.325 - 51 + 61)$
$= 9940.325$

$\therefore m_1'(\text{connected}) = 39.7613.$

$s(\text{inconnected}) = \sqrt{\frac{1}{250}\sum x_i^2(\text{inconnect}) - \bar{x}^2(\text{inconnect})}$

$= 9.8894$

$\therefore \sum x_i^2(\text{inconnected}) = 418895.4765$

$\therefore \sum x_i^2(\text{connected}) = 418895.4765 - 51^2 + 61^2$
$= 420015.4765$

$\therefore s(\text{connected}) = \sqrt{\frac{1}{250}\sum x_i^2(\text{connected}) - \bar{x}^2(\text{connected})}$

$= 9.9549$

$b_1(\text{inconnected}) = \frac{m_3(\text{inconnected})}{m_2^{3/2}(\text{inconnected})} = -0.1182$

$\Rightarrow m_3(\text{inconnected}) = -114.32134.$

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5) For Firm A: Mean > Median > Mode and the distribution is +vely skewed.

For Firm B: Mean > Median > Mode & the distn. is +vely skewed.

(For skewed distn. measures should be based on quantiles)

Measures:-

(i) Location : Median (Q_2)

(a) Median of the Firm A is 1389

(b) " " " " B is 1354

(ii) Dispersion: $Q.D. = \frac{Q_3 - Q_1}{2}$.

(a) Q.D. of Firm A is $\frac{1422-1278}{2} = 72$

(b) Q.D of Firm B is $\frac{1435-1262}{2} = 86.5$

(iii) skewness: $S_K = \frac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1}$

(a) S_K of firm A is $(1422+1278-2\times1350)/144 = 0$

(b) S_K of firm B is $(1262+1435-2\times1312)/173 = 0.42$

(iv) kurtosis: $K_p = \frac{Q_3 - Q_1}{2\times SD}$

(a) K_p of firm A is $(1422-1278)/2\times87 = 0.8276$

(b) K_p of firm B is $(1435-1262)/2\times99 = 0.8739$

[$(Q_3 - Q_1)$ represents the length within which we have central 50% value. The smaller the length $(Q_3-Q_1)/2$, the higher the kurtosis. To set a unit free measure, we divide $\frac{Q_3-Q_1}{2}$ by S.D.]

6) (a) Consider the integer as stems and decimals as leaves.

Stem	Leaf
101	7
102	0 0 0
105	8 9
106	1 3 3 6 6 7 7 8 8
107	3 7 9
108	8
109	1 3 9
110	0 2 2

Comment: The frequency distribution is located at 106 (median or mode) & is +vely skewed.

(b) Minimum value = 101.7, maximum value = 110.2

$Q_1 = \frac{25}{4}$ th ordered value = 6 th ordered value + $\frac{1}{4}$(7 th − 6th) ordered value

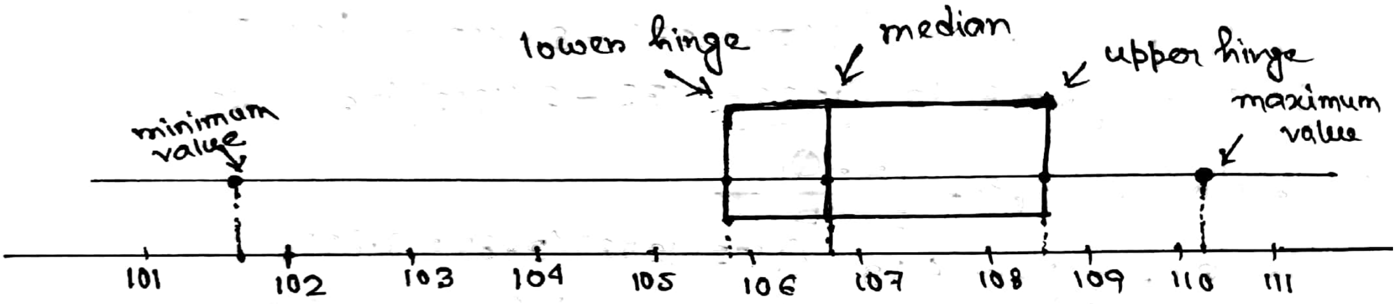
$= 105.9 + \frac{1}{4}(.2) = 105.95$ gram

$Q_2 = \frac{25}{2}$ th ordered value = 12th ordered value + $\frac{1}{2}$(13th − 12th) ordered value.

$= 106.7 + \frac{1}{2} \times 0 = 106.7$ gm.

$Q_3 = \frac{25 \times 3}{4}$ th ordered value = 18th ordered value + $\frac{3}{4}$(19th − 18th) ordered value.

$= 107.9 + \frac{3}{4}(.9) = 108.575$ gm.



Comment:— The frequency distribution is located at (near about) 107. The H-~~shaped~~ spread is small. Hence, the dispersion of the data is not high. The distance between upper hinge and median is greater than that of lower hinge and median. The distribution is +vely skewed. The length of H-spread is small w.r.t. the range, the kurtosis is high.

(c) The data values that are smaller than $L_H - \frac{3}{2}H$ or greater than $L_U + \frac{3}{2}H$ are called outliers.

$L_H - \frac{3}{2}H = 105.95 - \frac{3}{2}(108.575 - 105.95)$

$= 102.0125$

$L_U + \frac{3}{2}H = 108.575 + \frac{3}{2}(108.575 - 105.95)$

$= 112.5125$.

In the data, 101.7, 102.0, 102.0, 102.0 are the outliers of the data set.

7) Linear Regression model:

$$X = \alpha + \beta Y + \epsilon$$

The constant α, β are determined by minimising error sum of square or residual sum of square.

$$S = \sum_{i=1}^{n} (x_i - \alpha - \beta y_i)^2$$

The normal equations are:

$$\sum_{i=1}^{n} x_i = n\alpha + \beta \sum_{i=1}^{n} y_i$$

$$\sum_{i=1}^{n} x_i y_i = \alpha \sum_{i=1}^{n} y_i + \beta \sum_{i=1}^{n} y_i^2$$

From data, $n = 13$, $\sum_i x_i = 437.9$, $\sum_i y_i = 143.3$,

$\sum x_i y_i = 4419.35$, $\sum y_i^2 = 1543.67$.

The normal equations are:

$$437.9 = 13\alpha + (\beta \times 143.3) \quad \text{--- } ①$$

$$4419.35 = 143.3\alpha + 1543.67\beta \quad \text{--- } ②$$

① $\times 143.3$ − ② $\times 13$, we get

$$\beta = 11.34.$$

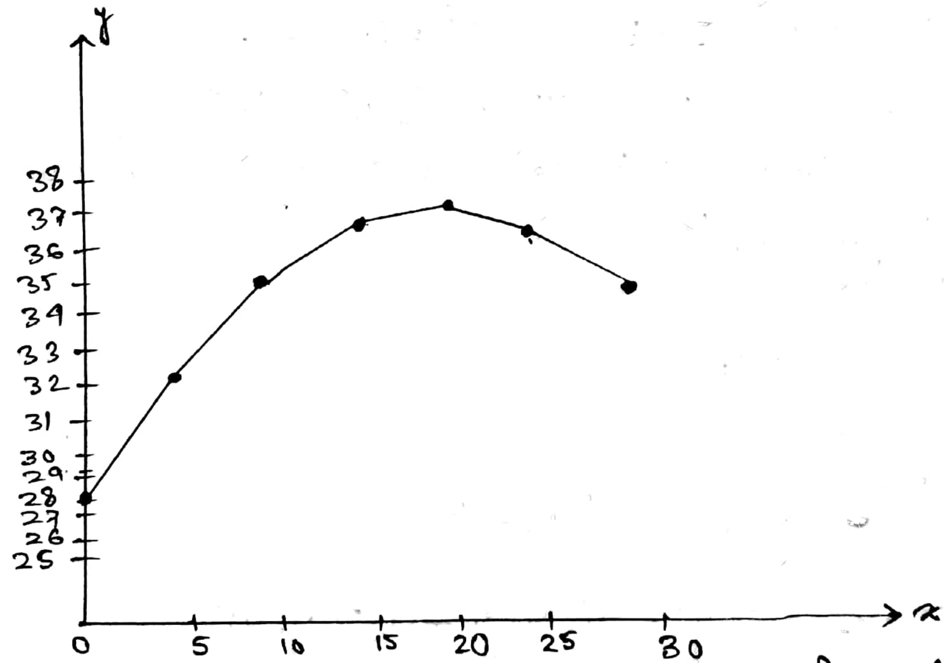
$$\therefore \alpha = -91.36$$

$$\therefore \hat{\alpha} = -91.36, \quad \hat{\beta} = 11.34$$

Hence, the predicting formula is —

$$X = -91.36 + 11.34Y.$$

8)



From scatter plot, it is clear that the relationship between x & y is approximately a quadratic.

(ai) A measure of association between x and y is the measure of usefulness of the 2nd degree polynomial (least square) regression as a predicting formula, i.e.

$$r_2^2 = \frac{\sum(Y_{pi}-\bar{Y}_p)^2}{\sum(y_i-\bar{y})^2} = 1 - \frac{\sum(y_i-Y_{pi})^2}{\sum(Y_i-\bar{y})^2}$$

x	y	$u_i=\frac{x_i-15}{5}$	u_i^2	u_i^3	u_i^4	u_iy_i	$u_i^2y_i$	y_i^2
0	27·1	−3						
5	32·1	−2						
10	35·0	−1						
15	36·2	0						
20	36·9	1						
25	36·1	2						
30	35·2	3						

Let, $Y = a_0 + a_1u + a_2u^2$

the constants a_0, a_1, a_2 are determined by minimizing

$$\sum_{i=1}^{n}(y_i - a_0 - a_1u_i - a_2u_i^2)$$

∴ Normal eqn are:-

$$\sum y_i = na_0 + a_1\sum u_i + a_2\sum u_i^2$$

$$\sum u_iy_i = a_0\sum u_i + a_1\sum u_i^2 + a_2\sum u_i^3$$

$$\sum u_i^2y_i = a_0\sum u_i^2 + a_1\sum u_i^3 + a_2\sum u_i^4$$

Here, $n=7$, $\sum u_i = 0$, $\sum u_i^2 = 28$, etc.

Now, the total variability $= \sum y_i^2 - n\bar{y}^2$

& unexplained variability or Residual sum of squares (RSS)

$$= \sum_{i=1}^{n}(y_i - Y_{pi})^2$$

$$= \sum_i (y_i - Y_{pi})e_i$$

$$= \sum_i y_ie_i$$

$$= \sum_i y_i(y_i - \hat{a}_0 - \hat{a}_1u_i - \hat{a}_2u_i^2)$$

$$= \sum_i y_i^2 - \hat{a}_0\sum_i y_i - \hat{a}_1\sum_i u_iy_i - \hat{a}_2\sum_i y_iu_i^2$$

$\therefore r_2^2 = 1 - \dfrac{\text{Unexplained variability}}{\text{Total variability}}$,

(iii) $y = \hat{a}_0 + \hat{a}_1 u + \hat{a}_2 u^2$, $\quad u = \dfrac{x-15}{5}$

$\dfrac{dy}{du} = \hat{a}_1 + 2\hat{a}_2 u$

$\Rightarrow u = -\dfrac{\hat{a}_1}{2\hat{a}_2}$, since $\dfrac{dy}{dx} = 0$

& $\dfrac{d^2y}{du^2} = 2\hat{a}_2 < 0$

Hence, y is maximum when $u_0 = -\dfrac{\hat{a}_1}{2\hat{a}_2}$

$\Rightarrow x = 15 + 5u_0$

9)

x_i	y_{ij}	$\bar{y}_i$	n_i	$n_i\bar{y}_i^2$
$x_1 = 5'3'' = 63''$	52, 60, 51, 44, 47	50.8	5	12903
$x_2 = 5'6'' = 66''$	55, 57, 54, 58, 50, 55	54.8	6	18018
$x_3 = 5'9'' = 69''$	61, 60, 58, 65	61	4	14884

Now, $e_{yx}^2 = \dfrac{\sum_{i=1}^{k} n_i(\bar{y}_i - \bar{y})^2}{\sum_i\sum_j (y_{ij} - \bar{y})^2} = \dfrac{\sum n_i\bar{y}_i^2 - n\bar{y}^2}{\sum_i\sum_j y_{ij}^2 - n\bar{y}^2}$, where

$\bar{y} = \dfrac{\sum_{i=1}^{k} n_i\bar{y}_i}{\sum_{i=1}^{k} n_i}$, $\quad \bar{y}_i = \dfrac{1}{n_i}\sum_{j=1}^{n_i} y_{ij}$, $i = 1(1)k$

$\bar{x} = \dfrac{1}{\sum_i n_i}\sum_i n_i x_i$, $\quad \bar{x} = 65.8$, $\quad \bar{y} = \dfrac{\sum_i\sum_j y_{ij}}{n} = 55.13$

$$\therefore e^2_{yx} = \frac{45805 - 45590}{46039 - 45590} = 0.4788$$

$$\therefore e_{yx} = 0.692$$

$$r^2_{yx} = \frac{\sum_{i=1}^{k} n_i x_i \bar{y}_i - n\bar{x}\bar{y}}{\sqrt{\sum_i n_i x_i^2 - n\bar{x}^2}\sqrt{\sum_i \sum_j y_{ij}^2 - n\bar{y}^2}}$$

$$= \frac{54538.8 - 54413.31}{8.97 \times 21.20}$$

$$= 0.1658$$

$$\therefore r_{yx} = 0.2585$$

the regression equation is linear iff $e^2_{yx} = r^2_{yx}$. A measure of deviation of the (true) regression equation from linearity is $d^2_{yx} = e^2_{yx} - r^2_{yx} = 0.4788 - 0.1658$
$= 0.413$.

10) Both series of efficiency have some tie. So, we replace the tied rank by their average of those ...

Workers	A	B	C	D	E	F	G	H	I	J	K	L
I	5	6	1	2	3	8.5	8.5	4	7	11	10	12
II	5.5	5.5	2	2	2	9	7	4	8	10.5	12	10.5

$$r_R = \frac{\frac{n^2-1}{12} - \frac{T_u + T_v}{2} - \frac{1}{2n}\sum d_i^2}{\sqrt{\frac{n^2-1}{12} - T_u}\sqrt{\frac{n^2-1}{12} - T_v}}$$

11)

Explanatory variable (x): income.

Response Variable (y): owning a house.

Hence y is Binary.

Hence, a logistic regression may be appropriate.

In logistic regression,

$$\bar{y}_x = \frac{e^{\alpha+\beta x}}{1+e^{\alpha+\beta x}}$$

$$\Rightarrow \ln\left(\frac{\bar{y}_n}{1-\bar{y}_n}\right) = \alpha+\beta x.$$

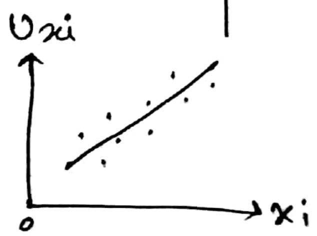
$$\Rightarrow u_x = \alpha+\beta x$$

$\Rightarrow x$ & u_x are linearly related.

x	y - values	Array total $n_i = \sum_{j=1}^{N_i} y_{ij}$	$\bar{y}_{x_i} = \frac{n_i}{N_i}$	U_{x_i}
$x_1 = 10$	y_{11} . . . . . y_{1N_1}			
$x_2 = 13$	y_{21} . . . . . y_{2N_2}			
$x_3 = 15$	. - - - -			
⋮				
$x_8 = 40$	y_{81} . . . . y_{8N_8}			

Plot the points (x_i, U_{x_i}), $i = 1(1)8$.

From graph the points (x_i, U_{x_i}) lie near a straight line.



Hence logistic regression is appropriate.

Hence, α & β can be determined by minimising $\sum w_i (U_{x_i} - \alpha - \beta x_i)^2$

where $w_i = N_i \bar{y}_{x_i} (1 - \bar{y}_{x_i})$

Normal eqn.,
$$\sum w_i U_{x_i} = \alpha \sum w_i + \beta \sum x_i w_i$$

$$\sum w_i x_i U_{x_i} = \alpha \sum x_i w_i + \beta \sum x_i^2 w_i$$

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U_{x_i}	w_i	$x_i w_i$	$x_i^2 w_i$	$w_i U_{x_i}$	$w_i x_i U_{x_i}$

Fitted logistic regression is

$$Y = \frac{e^{\hat{\alpha} + \hat{\beta} x}}{1 + e^{\hat{\alpha} + \hat{\beta} x}}$$

PROBLEMS ON CATEGORICAL DATA ANALYSIS

1. In one phase of a study regarding the effectiveness of several drugs on Post-operating nausea 167 patients were assigned at random, 30 to Drug-P, 67 to Drug-C and the remaining 70 to Placebo. The no. of patients suffering from severe, moderate, slight or no nausea is shown below:

	SEVERE	MODERATE	SLIGHT	NO NAUSEA	TOTAL
PLACEBO	8	8	19	35	70
DRUG-P	2	3	5	20	30
DRUG-C	3	4	15	45	67
TOTAL	13	15	39	100	167

Calculate a suitable measure of association.

2. A drug, supposed to have some effect in curing diabetis was treated on 100 patients in a certain hospital and their records were compared with 100 other patients not treated with drug. Study the efficacy of the drug in curing diabetis.

	Cured	Not cured
Treated	54	46
Untreated	24	76

3. The following table gives the data on the results of a visual test and a Balance test performed on 413 male college students.

	Left eyed	Ambiocular	Right-eyed	TOTAL
Left handed	48	25	52	125
Ambidestrous	32	13	25	70
Right handed	94	33	91	218
TOTAL	174	71	168	413

Compute at least two measures of association and comment.

4. A study was conducted to find out what people of different educational level feel about the role of the caste in life.

Role / Edu. level	No Role	Little Role	Large Role	Complete Role
Upto cls-8 (Gr-I)	2	8	25	36
Cls-9 - H.S. (Gr-II)	10	15	14	12
Graduate (Gr-III)	29	24	2	1

(a) Compute a suitable measure to find if there is any association between educational level and the individual's perception towards the role of region or caste in social life. Interpret your finding.

(b) Merge the columns 'No role' and 'little role' and rename it as 'minor role', similarly merge, the last two columns 'Large role' and 'complete Role' and rename it as 'Major Role'. Find the odds ratio of

i) Group I with Group II

ii) Group III w.r.t Group I, w.r.t. the Minor and Major roles and interpret your findings.

PROBLEMS ON CATEGORICAL DATA ANALYSIS

Solutions:-

1. We now calculate $\sum_i \sum_j \frac{(n_{ij})^2}{n_i n_j}$

$$= \frac{8^2}{13\times70} + \frac{2^2}{13\times30} + \frac{3^2}{13\times67} + \frac{8^2}{70\times15} + \frac{3^2}{15\times30} + \frac{4^2}{15\times67} + \frac{19^2}{39\times70} + \frac{5^2}{39\times30} + \frac{15^2}{39\times67} + \frac{35^2}{100\times70} + \frac{20^2}{100\times30} + \frac{45^2}{100\times67}$$

$= 1.0381$.

Mean-square contingency:-

$$\chi^2 = N\left\{\sum_{i=1}^{3}\sum_{j=1}^{4} \frac{(n_{ij})^2}{n_i n_j} - 1\right\}$$

$= 167(1.0381-1)$

$= 6.3827$.

Karl Pearson's coefficient of mean-square contingency:-

$$C = \sqrt{\frac{\chi^2}{N+\chi^2}} = 0.1916$$

Tschuprow's coefficient:-

$$T^2 = \frac{\chi^2}{N\sqrt{(s-1)(t-1)}}, \quad s=4, t=3$$

$= 0.1247$.

2.

	Cured (B)	Not cured (β)	Total
Treated (A)	(AB) = 54	($A\beta$) = 46	100
Untreated (α)	(αB) = 24	($\alpha\beta$) = 76	100

A measure of association between A and B in a 2X2 table is:

(i) Yule's coefficient of association:-

$$Q = \frac{(AB)(\alpha\beta) - (A\beta)(\alpha B)}{(AB)(\alpha\beta) + (A\beta)(\alpha B)}$$

$= 0.5760$

(ii) Yule's coefficient of colligation:-

$$Y = \frac{\sqrt{(AB)(\alpha\beta)} - \sqrt{(A\beta)(\alpha B)}}{\sqrt{(AB)(\alpha\beta)} + \sqrt{(A\beta)(\alpha B)}}$$

$= 0.3169$.

Hence, A and B have moderate positive association.
i.e. Drug has moderate effect on Diabetis.

3. Here both the characters 'Balance Test' (A) and 'Visual test' (B) are in normal scale.

A \ B	B_1	B_2	B_3	TOTAL
A_1	$f_{11} = 48$	$f_{12} = 25$	$f_{13} = 52$	$f_{10} = 125$
A_2	$f_{21} = 32$	$f_{22} = 13$	$f_{23} = 25$	$f_{20} = 70$
A_3	$f_{31} = 94$	$f_{32} = 33$	$f_{33} = 91$	$f_{30} = 218$
TOTAL	$f_{01} = 174$	$f_{02} = 71$	$f_{03} = 168$	413

Here, we shall use measures based on χ^2.

$$\chi^2 = N \left\{ \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{f_{ij}^2}{f_{0j} f_{i0}} - 1 \right\}$$

$= 2.3738$

(i) Karl Pearson's coefficient :- $C_{AB} = \sqrt{\dfrac{\chi^2}{N + \chi^2}}$

$= \sqrt{\dfrac{2.3738}{413 + 2.3738}}$

$= 0.0756$.

(ii) Tschuprow's coefficient :- $T_{AB} = \sqrt{\dfrac{\chi^2}{N\sqrt{(r-1)(s-1)}}}$

$= 0.05361$.

4. (a) Here both the characters → educational level (X) and Role of religion (Y) are in ordinal scale.
We shall use measures based on the no. of concordant pairs (C) and no. of discordant pairs (D).

A pair (i,j) of individuals with scores (X_i, Y_i) and (X_j, Y_j) is in concordance if

$$\{X_i > X_j, Y_i > Y_j\} \text{ or } \{X_i < X_j, Y_i < Y_j\}$$

and discordant if

$$\{X_i > X_j, Y_i < Y_j\} \text{ or } \{X_i < X_j, Y_i > Y_j\}.$$

a pair (i,j) is on tie w.r.t. X if $X_i = X_j$.

Here, C = No. of concordant pairs

$$= 2(15+14+12+24+2+1) + 8(14+12+2+1) + 25(2+1) + 36(0) + 10(24+12+1) + 15(2+1) + 14\times1$$

$$= 1022.$$

D = No. of discordant pairs

$$= 8(10+29) + 25(10+15+29+24) + 36(10+15+14+29+24+2) + 15\times29 + 14(29+24) + 12(29+24+2)$$

$$= 7483.$$

Goodman-Kruskal (γ) measures of association:-

$$\gamma = \frac{C-D}{C+D} = -.7597.$$

So, we can say educational level and the individual's perception towards the role of religion is negatively associated.

(b)

	Minor Role	Major Role
Gr I	$f_{11} = 10$	$f_{12} = 61$
Gr II	$f_{21} = 25$	$f_{22} = 26$
Gr III	$f_{31} = 53$	$f_{32} = 3$

(i) Sample odds ratio of Gr. I with Gr. II $= \dfrac{f_{11} f_{22}}{f_{21} f_{12}}$

$= 0.17$.

Hence the success in 'minor role' in GrI is less likely than GrII, i.e., Hence, more people feel that religion has 'Major Role' from GrI compare to GrII.

(ii) Sample odds ratio of GrIII w.r.t. GrI is

$$\hat{\theta} = \frac{f_{11} f_{22}}{f_{21} f_{22}} = 107.76$$

	Minor	Major
Gr III	$f_{11} = 53$	$f_{12} = 3$
Gr I	$f_{21} = 10$	$f_{22} = 61$

The success 'minor role' in GrIII is more likely than GrI.

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PRACTICAL PROBLEMS FROM VITAL STATISTICS

Q.1. [CU'92, 09] Consider the following data set for two countries

Country I

Age Group	Population size (in'000) P_x	No. of Males (in '000) mP_x	No. of Deaths D_x	Number of death in males mD_x
< 1	100	50	750	400
1 – 4	400	200	3000	1600
5 – 20	1500	750	12000	6500
21 – 100	4000	1500	31000	12000

Country II

Age Group	Population size (in '000)	No. of Males (in'000)	No. of Deaths	Number of deaths in males
< 1	15	7.5	12	6
1 – 4	60	30	45	24
5 – 20	200	100	160	85
21 – 100	440	170	350	200

(i) Compute the CDR for both the countries.

(ii) Compute the ASDR for both the countries seperately for male, female and total population.

(iii) In order to compare the mortality situation of the two countries propose a good measure and calculate its value for both the countries.

Q.2. [CU'05]

Fill in the blanks of the following table which are marked with question marks:

Age(x)	l_x	d_x	q_x	p_x	L_x	T_x	$\mathring{e}_x$
20	6,93,435	?	?	?	?	35081126	?
21	6,90,673	–	–	–	–	–	?

Q.3. The number of persons dying at age 75 is 476 and the complete expectation of life at 75 and 76 years are 3.92 and 3.66 years. Find the numbers living at ages 75 and 76.

Q.4. [CU'08,95] From the following data relating to a particular community compute the GRR and the NRR. Interpret your results.

Age of mother	No. of Women	No. of Births	Survival factor
15-19	9000	140	0.920
20-24	9200	1312	0.914
25-29	8900	1067	0.908
30-34	8600	771	0.891
35-39	8400	468	0.879
40-44	8500	160	0.869

Assume that 48.7 % of the total are female births. Survival factor gives the Rate of survival from Birth to the mid point of the corresponding age-group.

[CU'07]

Q.5. The following table provides data of female population and the number of live births in different age groups in the USA in 2004.

Age Group (years)	Female Population ('000)	Live Births ('000)
14-19	20,724	422
20-24	20,973	1034
25-29	19,555	1104
30-34	20,467	966
35-39	21,050	476
40-44	23,055	104
45-49	22,121	6

Given that in 2004, the total population of the USA was 293,657,00 and the sex-ratio at birth was 105 male births per 100 female births, Compare, for the year 2004,
(a) the CBR , (b) the GFR (c) the ASFRs (d) the TFR
(e) the GRR.

Q.6. [CU'06] In 1951, the total number of live births in W.B. was estimated as 399680. The table below records the number of females in the child bearing age intervals and the survival rates for WB and also the number of female live births in India during the year mentioned. Can you gather any idea about the GRR and NRR for WB from the given data?

Age	Female Population in WB (100)	Survival rate (100000)	Total Number of female live-births in India (100)
15-19	12652	59753	4632
20-24	11403	56924	14443
25-29	10001	54032	14058
30-34	8346	50352	8329
35-39	6897	45921	4036
40-44	5696	40356	2158
45-49	4728	36205	689

Q.7. [CU'1997] The following informations are obtained from a sample survey

Age-group	Number of women	Proportion of Marriage	No. of births	Proportion surviving from birth to mid point of age group among married women
15–19	16592	0·82	2642	0·902
20–24	19137	0·83	4272	0·891
25–29	10860	0·84	2179	0·878
30–34	4990	0·85	790	0·865
35–39	2463	0·74	203	0·849
40–44	928	0·64	47	0·830

(a) Calculate the GRR and NRR, assuming that there was no illegimate birth and sex ratio at birth is 1000 : 942 in favour of male.

(b) Determine the probability of a female children dying (after marriage at 32 age.

Q.8. The following table gives the female ASFR for 1993 and the life table for females of India with cohort $l_0 = 1{,}000$.

Age of Years	ASFR	Female life-table stationary popln.
15–19	0·0696	4180
20–24	0·2346	4123
25–29	0·1897	4063
30–34	0·1143	4001
35–39	0·0611	3934
40–44	0·0285	3860
45–49	0·0101	3763

Compute TFR, GRR and NRR for India assuming sex ratio at birth is 1·05 : 1.

Q.10. Consider a population on July 1, 1985 equal to 10,00,000 and growing at 2% per year as a continuous instantaneous rate, the crude rate of natural increase for 1995 was 3% and the crude rate was 1%. Determine the number of live births in 1995.

Q.11. [CU'07] For a stationary population with radix $l_0 = 1,00,000$, out of the children born in 1980, number of deceased was 20,000 and the number of deceased in 1981 was 5000. Given the ASDRs of this population for the following ages (l.b.d)

x	0	1	2	3	4
m_x	·4073	·021	·0023	·0017	·0012

(a) Compute complete expectation of life at age 4, given the same at birth is 65.89 years.

(b) What is the chance that two new born babies will survive 4 years after their birth?

Q.9. [CU'03] The following table gives the ten decennial census popln. of two countries, say, A and B.

Year :	1901	1911	1921	1931	1941	1951	1961	1971
Popln. of Country A (in million) :	283·3	252·0	251·2	278·9	318·5	361·0	439·1	547·0
Popln. of Country B (in million) :	5·3	7·2	9·6	12·9	17·1	23·2	31·4	38·5

Year :	1981	1991
Popln. of C-A (in million) :	653·8	823·8
Popln. of C-B (in million) :	50·2	62·9

Justify by some graphical procedure which of these countries population growth follows approximately Logistic model and comment on it. Fit a logistic model to the population that fits as well.

PROBLEMS ON VITAL STATS.

1) (iii) To compare the mortality situation of country I and country II, we consider STDR based on age & sex specific death rates,

$$STDR_I = \frac{\sum {}_n^m m_x \, {}_n^m P_x^S + \sum {}_n^f m_x \, {}_n^f P_x^S}{\sum {}_n^m P_x^S + \sum {}_n^f P_x^S}.$$

Here, we select ${}_n^m P_x^S = \frac{{}_n^m P_x^{I} + {}_n^m P_x^{II}}{2}$, ${}_n^f P_x^S = \frac{{}_n^f P_x^{I} + {}_n^f P_x^{II}}{2}$

2.

2) $d_x = l_x - l_{x+1}$

$\therefore d_{20} = 6,93,435 - 6,90,673 = 2762$

$\therefore q_x = \frac{d_x}{l_x} \Rightarrow q_{20} = \frac{d_{20}}{l_{20}} = 3.98 \times 10^{-3}$

$\therefore p_x = 1 - q_x \Rightarrow p_{20} = 1 - q_{20} = 0.996$

$L_x = \frac{l_x + l_{x+1}}{2}$, assuming that deaths are uniformly distributed.

$\therefore L_{20} = \frac{1}{2}(l_{20} + l_{21}) = 692054$.

$e_x^0 = \frac{T_x}{l_x} = \frac{T_{20}}{l_{20}} = e_{20}^0 = 50.59$

$\therefore T_x = L_x + T_{x+1}$ $\qquad e_{21}^0 = \frac{T_{21}}{l_{21}}$

$\Rightarrow T_{x+1} = T_x - L_x$

$\Rightarrow T_{21} = T_{20} - L_{20} = 34389072$,

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3)

$$\mathring{e}_{75} = 3.92, \quad \mathring{e}_{76} = 3.66$$

$$d_{75} = 476$$

$$\Rightarrow l_{75} - l_{76} = 476$$

$$\therefore \mathring{e}_{75} \simeq e_{75} + \frac{1}{2}$$

$$\Rightarrow e_{75} = \mathring{e}_{75} - \frac{1}{2} = 3.92 - .5 = 3.42$$

$$\therefore e_{76} = 3.66 - .5 = 3.16$$

$$\therefore \frac{e_{76}}{e_{75}} = \frac{\left(\sum_{t=1}^{\infty} l_{76+t}\right) / l_{76}}{\left(\sum_{t=1}^{\infty} l_{75+t}\right) / l_{75}}$$

$$= \frac{l_{75}}{l_{76}} \times \frac{\sum_{t=1}^{\infty} l_{76+t}}{l_{76} + \sum_{t=1}^{\infty} l_{76+t}}$$

$$= \frac{l_{75}}{l_{76}} \times \frac{1}{\frac{l_{76}}{\sum_{t=1}^{\infty} l_{76+t}} + 1}$$

$$= \frac{l_{75}}{l_{76}} \times \frac{1}{1 + \frac{1}{e_{76}}}$$

$$\Rightarrow \frac{l_{76}}{l_{75}} = \frac{e_{75}}{1 + e_{76}}$$

$$\Rightarrow l_{76} = l_{75} \times \left(\frac{e_{75}}{1 + e_{76}}\right)$$

$$= (476 + l_{76}) \times \frac{e_{75}}{1 + e_{76}}$$

$$\Rightarrow l_{76}\left(1 - \frac{e_{75}}{1 + e_{76}}\right) = 476 \times \frac{e_{75}}{1 + e_{76}}$$

$$\Rightarrow l_{76} = \frac{476 \times e_{75}}{1 + e_{76} - e_{75}} = 2200$$

$$\Rightarrow l_{75} = 476 + 2200 = 2676.$$

4) Age-specific fertility rate of age group $[x, x+5)$ is

$$ {}_5^f i_x = \frac{{}_5^f B_x}{{}_5^f P_x} $$

$$ \therefore GRR = 5 \times \sum_x \frac{{}_5^f B_x}{{}_5^f P_x} = 5 \times \sum_x {}_5^f i_x $$

$$ NRR = 5 \sum_x {}_5^f i_x \times {}_5^f R_x $$

where ${}_5^f R_x$ is the survivors factor per year in the age-group $[x, x+5)$.

Here, $\frac{{}_5^f B_x}{{}_5 B_x} = 48.7\% = \frac{48.7}{100} \;\forall x$.

Age Group	No. of women ${}_5^f P_x$	No. of births ${}_5 B_x$	${}_5^f B_x = \frac{48.7}{100} \times {}_5 B_x$	${}_5^f i_x = \frac{{}_5^f B_x}{{}_5^f P_x}$	survival Rate ${}_5^f R_x$	${}_5^f i_x \times {}_5^f R_x$
15-19	9000	140			0.920	
					⋮	
				.2153679		0.19414

$$ GRR = 5 \times \sum_x {}_5^f i_x = 5 \times .2153679 = 1.0768 $$

$$ \therefore NRR = 5 \times \sum_x {}_5^f i_x \times {}_5^f R_x = 5 \times .19414 = 0.9707 $$

"GRR = 1.0768" indicates the no. 1.0768 of daughters would be born on the average, to each of a group of females beginning life together, supposing none of them died before reaching the end of the reproductive period and all of them experience, throughout the reproductive period.

"NRR .9707" implies a group of 10000 females is expected to be replaced by 9707 females in the next generation under the given rates of fertility and mortality and the population will shows a tendency to decrease. Hence the population will ultimately decrease and will ultimately die out, unless the fertility and mortality change.

5) 1. $P = 293,857,000$

$$\frac{m_B}{f_B} = \frac{105}{100}$$

(a) $CBR = \frac{B}{P} \times 1000$

$$= \frac{4112,000}{293,657,000} \simeq 14$$

(b) $GFR = \frac{B}{\sum_x {}_5^fP_x} \times 1000$

(c) $ASFR = \frac{{}_5B_x}{{}_5^fP_x} \times 1000 = {}_5i_x$

(d) $TFR = 5 \times \sum_x {}_5i_x = 5 \times$ sum of ASFR's.

(e) $GRR = \left(TFR \times \frac{f_B}{B}\right)$; $GRR = \frac{100}{205} \times \frac{TFR}{1000}$.

8. 6) $B = 399680$

To compute GRR & NRR of W.B., we should have information regarding ${}^{f}_{5}B_x$.

Assuming, $\dfrac{{}^{f}_{5}B_x}{{}^{f}B} = \dfrac{{}^{f}_{5}B_x^{I}}{{}^{f}B^{I}} \quad \forall x$

$\Rightarrow {}^{f}_{5}B_x = \dfrac{{}^{f}_{5}B_x^{I}}{{}^{f}B^{I}} \times {}^{f}B$

Assume, $\dfrac{{}^{f}B}{B} = \dfrac{100}{205} \Rightarrow {}^{f}B = \dfrac{100}{205} \times B$

$= \dfrac{100}{205} \times 399680$

$= 194966$

Age	Female Popln. ${}^{f}_{5}P_x$	Surviyal Rate (${}^{f}_{5}R_x$)	Total no. of female live births in india ${}^{f}_{5}B_x^{I}$	${}^{f}_{5}B_x$	${}^{f}_{5}i_x$	${}^{f}_{5}i_x \times {}^{f}_{5}R_x$
15-19	1265200	·59753	463200	18680	·01476	·00882
20-24	1140300	·56924	1444300	58246	·05108	·02908
25-29	1000100	·54032	1405800	56693	·05669	·03063
30-34	886600	·50352	832900	33589	·04015	·02022
35-39	684700	·45921	403600	16276	·02377	·01091
40-44	569600	·40356	215800	8703	·01528	·00617
45-49	472800	·36205	68900	2779	·00588	·00213
TOTAL	5969300		4834500	194966	·20761	·10797

${}^{f}_{5}B_x = \dfrac{{}^{f}_{5}B_x^{I}}{{}^{f}B^{I}} \times 194966, \quad {}^{f}_{5}i_x = \dfrac{{}^{f}_{5}B_x}{{}^{f}_{5}P_x}$

$\therefore GRR = 5 \times \sum_x {}^{f}_{5}i_x$

$= 5 \times 0{\cdot}20761 = 1{\cdot}03805$

$\therefore NRR = 5 \times \sum_x {}^{f}_{5}i_x \times {}^{f}_{5}R_x$

$= 5 \times 0{\cdot}10797$

$= 0{\cdot}53985$.

Q. 7)
(a) there was no illegimate birth. We consider ${}^f_5P_x$ = No. of women in age group $[x, x+5) \times$ proportion of marriage.

$$\frac{f_B}{m_B} = \frac{942}{1000} \Rightarrow \frac{f_B}{B} = \frac{942}{1942}$$

$$\therefore GRR = 5 \times \frac{f_B}{B} \times \sum_x \frac{{}_5B_x}{{}^f_5P_x} = 5 \times \frac{f_B}{B} \times \sum_x {}_5i_x$$

Age Group	No. of women	Proportion of marriage	${}^f_5P_x$	${}_5B_x$	${}_5i_x$	${}^f_5R_x$	${}^f_5R_x \times {}_5i_x$
15-19	16592	0.82	13605	2642	0.19419	0.902	0.17516
20-24	19137	0.83	15884	4272	0.26895	0.891	0.28963
25-29	10860	0.84	9122	2179	0.23887	0.878	0.20973
30-34	4990	0.85	4241	790	0.18628	0.865	0.16113
35-39	2463	0.74	1823	203	0.11138	0.849	0.09456
40-44	925	0.64	592	47	0.07939	0.830	0.06589
TOTAL	—	—	—	—	1.07906	—	0.57328

$$\therefore GRR = 5 \times \frac{942}{1942} \times 1.07906 = 2.61708$$

$$\therefore NRR = 5 \times \frac{942}{1942} \times 0.57328 = 1.3904$$

(b) Probability of the female children dying after marriage at 32 age $= \{1 - \text{Probability of surviving at age } 32\} \times \{\text{Prob. of married}\}$

$= (1 - 0.865) \times 0.85$

$= 0.11475$

Q. 8) $NRR = \sum_x {}^f_5i_x \times \frac{{}^f_5L_x}{{}^fl_0}$, where ${}^f_5L_x = {}^fL_x + {}^fL_{x+1} + \cdots + {}^fL_{x+4}$

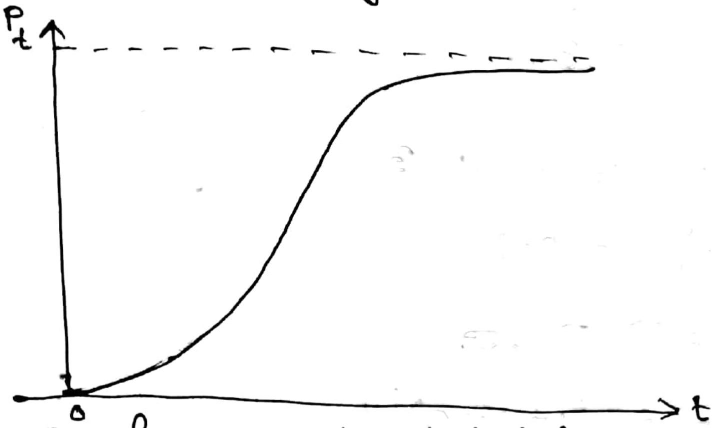
Age Group	ASFR (${}_5i_x$)	${}^f_5L_x$	${}^fR_0 = \frac{{}^f_5L_x}{{}^fl_0[=1000]}$	${}_5i_x \times {}^fR_0$
15-19	0.0696	4180	4.180	.29023
20-24	0.2346	4123	4.123	.96726
25-29	0.1897	4063	4.063	.77075
30-34	0.1143	4001	4.001	.45731
35-39	0.0611	8934	3.934	.24037
40-44	0.0289	3860	3.860	.11001
45-49	0.0101	3763	3.763	.03801
TOTAL	0.7079	27924	27.924	2.87394

$$\therefore TFR = 5 \sum_x {}_5i_x = 5 \times 0.7079 = 3.5395$$

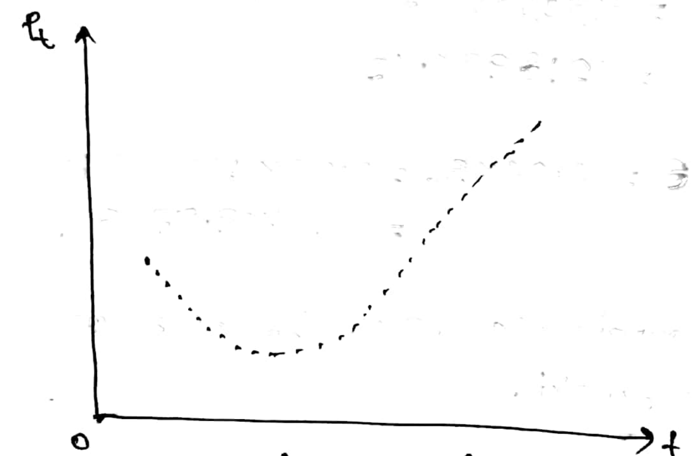
$$\therefore GRR = \frac{f_B}{B} \times TFR = \frac{100}{205} \times 3.5395 = 1.7266$$

$$\therefore NRR = \frac{f_B}{B} \times \sum_x {}_5i_x \times {}^fR_0 = \frac{100}{205} \times 2.87394 = 1.40192$$

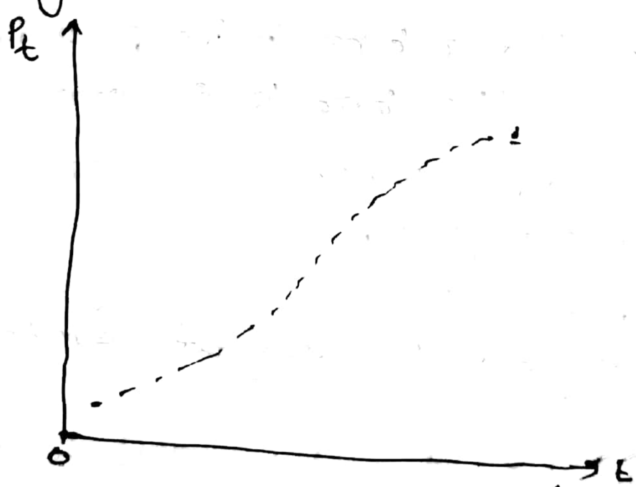
9) ⇒ The shape of the logistic curve is given by :



Plot (t, P_t) for both the population on graph paper.



Clearly the population figures for the country I is not following the logistic law.



Clearly, II population figures are following logistic law approximately.

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10)

CBR − CDR = Crude rate of natural increase

$= \left(\frac{B}{P} - \frac{D}{P}\right) 100\%$

$= 3\%$

$\Rightarrow \frac{B}{P} - \frac{D}{P} = 0.03$ $\left[\frac{D}{P} \times 100 = 1\right]$

$\Rightarrow \frac{B}{P} = 0.04$

$P_0 = 10,000,00$

$P_{10} = P_0 (1+0.02)^{10}$

$= 1000000 (1+0.02)^{10}$

$= 1218994.42$

$\therefore B = 0.04 \times P_{10} = 0.04 \times 1218994.42$

$= 48759.7768$

11) (a) To calculate the value q_0, we use the formula given by kuczynski.

$$\hat{q}_0 = 1 - \left(1 - \frac{D'''}{B_0}\right)\left(1 - \frac{D''}{B_{-1} - D'}\right)$$

B_{-1} = No. of children born in the preceding calender year.

B_0 = No. of children born in the current calender year.

D' = No. of children born and deceased in the preceding calender year.

D'' = No. of children born in the preceding calender year & deceased in the current calender year before reaching age 1 and

D''' = No. of children born and deceased in the current calender year.

$B_{-1} = 1,00,000$

$B_0 = 1,00,000$

$D' = 20,000$

$D'' = 5,000$

$D''' = 20,000$

$\hat{q}_0 = 1 - \left(1 - \frac{20000}{100000}\right)\left(1 - \frac{5000}{80000}\right)$

$= 0.25$

$$m_x = \frac{d_x}{L_x}$$

$$\Rightarrow m_x = \frac{d_x}{l_x - (1-a_x)d_x}, \quad a_1 = 0.43$$

$$a_2 = 0.45$$

$$= \frac{q_x}{1-(1-a_x)q_x}, \quad a_3 = 0.47$$

$$a_4 = 0.49$$

(b) Required probability $= \left(\frac{l_4}{l_0}\right)^2$.

* ——————— *

PROBLEMS ON INDEX NUMBER — FOR PRAC. EXAM

1) The following table shows the quantities consumed and the values (price × quantity) of 5 commodities for 3 successive years.

Commodities	1990 Quantity	1990 Value	1991 Quantity	1991 Value	1992 Quantity	1992 Value
I	50	350	60	420	70	490
II	120	600	140	700	100	800
III	30	330	20	200	15	225
IV	20	360	15	300	10	220
V	5	40	5	50	5	60

Calculate the price index number for 1992 taking 1990 as the base period adopting chain base formula and using Paasches formula at each stage. Also verify whether the circular test is satisfied by the Paasches formula or not on the basis of the above data. [C.U. 1996, 2010]

Soln. → The chain index number for 1992 w.r.t. base 1990 is

$$P'_{90,92} = P_{90,91} \times P_{91,92}$$

$$= \frac{\Sigma p_{91} q_{91}}{\Sigma p_{90} q_{91}} \times \frac{\Sigma p_{92} q_{92}}{\Sigma p_{91} q_{92}} \text{, by paasches formula.}$$

$=$

Note that, Price $(p) = \frac{\text{Value}}{\text{Quantity}}$.

For circular test, $P_{90,91} \cdot P_{91,92} \cdot P_{92,90} = 1$.

$$\Leftrightarrow P_{90,91} \cdot P_{91,92} = P_{90,92}$$

$$\Leftrightarrow P'_{90,92} = P_{90,92}$$

i.e. Chain index = Fixed base index.

Hence, $P_{90,92} = \frac{\Sigma p_{92} \cdot q_{92}}{\Sigma p_{90} q_{92}} =$

Hence, $P'_{90,92} \neq P_{90,92}$

$\Rightarrow$ The circular test is not satisfied by the Paasches formula.

2. The following table shows the average per capita consumption of cereals and prices of cereals in rural India for four different periods T_1, T_2, T_3, T_4.

Commodities	Consumptions (in kg) per month				Price (Rs. per kg)			
	T_1	T_2	T_3	T_4	T_1	T_2	T_3	T_4
Rice	4.69	3.94	3.58	4.28	5.37	5.75	6.15	6.30
Wheat	5.51	6.93	6.43	6.78	4.94	5.15	5.02	5.10
Others	7.66	8.19	7.75	7.71	3.32	2.94	3.25	3.52

Calculate the price index number of cereals for period T_4 taking T_1 as the base adopting chain-base formula and using any uniformly suitable formula at each stage. If after calculations it is found that consumption figures in T_4 are in error by 5%, do you think that your index is going to be affected by this? Give reason for your answer. [C.U. 1992]

Soln. → If we use Laspeyre's formula, then

$$P'_{14} = P_{12} \cdot P_{23} \cdot P_{34}$$

$$= \frac{\sum p_2 q_1}{\sum p_1 q_1} \times \frac{\sum p_3 q_2}{\sum p_2 q_2} \times \frac{\sum p_4 q_3}{\sum p_3 q_3} = \_\_\_\_$$

is independent of q_4 (the consumption figure for T_4). Hence, it will not affect the calculation of chain index.

If we use Paasche's formula, then

$$P'_{14} = P_{12} \cdot P_{23} \cdot P_{34}$$

$$= \frac{\sum p_2 q_2}{\sum p_1 q_2} \times \frac{\sum p_3 q_3}{\sum p_2 q_3} \times \frac{\sum p_4 q_4}{\sum p_3 q_4} = \cdots$$

which depends on q_4.

Hence, all the consumption figures in T_4 are in error by 5%, then the corrected figures are

$$q'_{4i} = q_{4i} \times \frac{105}{100} \text{ or } q_{4i} \times \frac{95}{100}$$

and hence,

$$\frac{\sum p_{4i} q'_{4i}}{\sum p_{3i} q'_{4i}} = \frac{\sum p_{4i} \frac{105}{100} q_{4i}}{\sum p_{3i} \frac{105}{100} q_{4i}} = \frac{\sum p_{4i} q_{4i}}{\sum p_{3i} q_{4i}}$$

Ultimately, it will not affect the calculation of the index.

3. The data below show the percentage increases in price of a few tabulated food items and weights attached to each of them. Calculate the index number for the food group.

Food items :	Rice	Wheat	Dal	Ghee	Oil	Spices	Milk	Fish	Others
Weights :	30	11	8	5	5	3	7	9	19
Percentage increase in price :	180	202	115	212	175	517	260	426	332

Using the above Food Index and information given below. Calculate CLI.

Group :	Food	Clothing	Fuel and Light	Rents and Rates	Miscellaneous
Index :	—	310	220	150	300
Weight :	60	5	8	9	18

[C.U. 2000]

Soln. :→ The percentage increase in price for the ith item

$$(r) = \frac{p_{1i} - p_{0i}}{p_{0i}} \times 100$$

$$\Rightarrow \frac{p_{1i}}{p_{0i}} \times 100 = (100 + r)$$

The food index is computed by

$$I_F = \frac{\sum \frac{p_{1i}}{p_{0i}} \times w_i}{\sum w_i} \times 100 = \frac{\sum \left(\frac{p_{1i}}{p_{0i}} \times 100\right) \times w_i}{\sum w_i}$$

Food item	Weights (w_i)	$\frac{p_{1i}}{p_{0i}} \times 100$	$\left(\frac{p_{1i}}{p_{0i}} \times 100\right) w_i$
	$\sum w_i =$		$\sum \left(\frac{p_{1i}}{p_{0i}} \times 100\right) w_i =$

Hence, $I_F = \frac{\text{Column 4 total}}{\text{Column 2 total}} = \ldots$

The $CLI = \frac{\sum I_i w_i}{\sum w_i}$

4. The following table shows the group indices and the corresponding weights for the year 1995 with 1981 as the base year of a given community.

Group	Group index	Weight
Food	212·45	65·3
Clothing	328·06	4·8
fuel and light	345·89	8·5
House rent	173·41	7·6
Miscellaneous	201·35	13·8

(a) Find the CLI for the year 1995.
(b) What is the purchasing power in 1995 as compared to 1981.
(c) If Mr. Dasgupta's salary increased from 2400 Rs in 1981 to Rs. 4950 in 1995, how has his economic status change?
(d) The weights are proportional to the consumption expenditure of each group. Suppose Mr. Dasgupta has to maintain the same status for each of the 1st four groups and can only adjust his spending on Miscellaneous items to come to terms with his income changes.
Find his spendings for each of the groups in 1995.
(e) What would be Mr. Dasgupta's weights for each of the groups in 1995? [C.U. 2001, 2009]

Soln. :→

(a) $$C.L.I = \frac{\sum I_i w_i}{\sum w_i} = 224.842$$

(b) The purchasing power of money in 1995 w.r.t. 1981 is $$\frac{1}{\left(C.L.I/100\right)} = \frac{100}{CLI} = 0.44756.$$

(c) CLI for 1995 w.r.t. 1981 as base year is 224·842 means that if some individual spends Rs. 100 in 1981, in 1995 he/she has to spend Rs. 224·842 to maintain the same standard of living as she/he has in 1981.
For Mr. Dasgupta's expenditure on 1981 was Rs. 2400. In 1995 he has to spend Rs. 2400 × 2·24842 = 5396·2 to maintain the same standard of living as he has in 1981. But his salary in 1995 was 4950. He has to adjust Rs. (5396·2 − 4950) = 446·2 Rs. i.e. his economic status falls in 1995 as compared to 1981.

(d)

Group	Index	Weights	In 1981 Expenditure	In 1995 ~~to~~ ~~expenditure~~ expenditure to maintain the same standard of living
Food	212.45	65.3	2400 X 65.3/100 = 1567.8	1567.8X 212.45/100 = 3329.51
Clothing	328.06	4.8	115.2	377.925
Fuel & light	345.89	8.5	204	705.615
House rent	173.41	7.6	182.4	316.299
Miscellaneous	201.35	13.8	326.4	666.206
TOTAL =		100	2400	5396.2

In 1981, expenditure on a group = Income $\times \frac{w_i}{\Sigma w_i}$

Hence his spendings on the first four groups remain same and the deficit of Rs. 446.27 will be adjusted on Miscelleneous group and the spending on Miscellaneous is Rs. (666.86 – 446.27)

= 220.59 Rs. in 1995.

(e) In 1995, the weight of a group = $\frac{\text{Expenditure on the Group}}{\text{salary (4950)}} \times 100$

Group	Expenditure	Weights
Food	3329.51	3329.51X100/4950 = 67.2
Clothing	377.925	377.925X100/4950 = ...
⋮	⋮	⋮
Miscelleneous	220.59	220.59X100/4950 = ...
TOTAL =	4950	100

— x —

Q. 5. The following data relate to the urban middle-class people of a particular region in the years 2003 and 2005.

Group	% of total expenditure	Group Index (Base: 2000) 2003	2005
Food	35	118	122
Clothing	15	112	118
Housing	20	113	115
Transport	10	112	117
Durable Goods	8	105	110
Others	12	120	125

(a) Compute the CLI for 2003 and 2005 with Base year 2000.
(b) If a family saved 15% of its monthly income in 2003, find the relative change in its average savings over the years 2003-2005, assuming that it maintained the same standard of living as in 2003. (C.U.-2008)

Soln.
(a)

$$CLI_{2003} = \frac{\sum I_{1i} w_i}{\sum w_i} = \frac{11470}{100} = 114.70$$

$$CLI_{2005} = \frac{\sum I_{2i} w_i}{\sum w_i} = \frac{11890}{100} = 118.90$$

∵ Salary is constant, i.e. Rs. 100,
so, weights of the group = Expenditure of the group.

(b) Let Rs. 100 be the salary in 2003. Saving = Rs. = 15.
Expenditure = Rs. 85

CL.I.	2003	2005
	114.70	118.90

If some individual spent Rs. 114.70 in 2003, in 2005 he has spent Rs. $\frac{118.90}{114.70} \times 85$ = ~~[illegible]~~
= 88.11 Rs.

In 2005, his salary is 100 Rs.
Exp. ~~Expenditure~~ = 88.11 Rs.
Saving = 11.89 Rs. ∴ saving % = $\frac{11.89}{100} \times 100$
= 11.89%

———X———

6. The following table gives (with two missing values) the overall and groupwise CLI (with 2000 as the base year) with six different expenditure groups and their respective weights, for the urban middle class people of a particular city, in 2004 and 2005.

Group	Weight	Group Index 2004	Group Index 2005
Food	350	117	120
Clothing	156	113	118
Housing	187	118	-
Transport	108	112	117
Dourable Goods	76	102	1171
Others	123	121	125
CLI	TOTAL = 1000	= 115.4	= 119.5

(a) If Mr. X saved 20% of his salary in 2004, Determine the relative change in his average savings, relative to 2004, in 2005 if his salary increased by 10% and he maintained the same standard of living as in 2004. [C.U. 2007]

Soln. (a) Let Rs. 100 be the salary in 2004. Saving = Rs. 20

Exp. = Rs. 80.

CLI.	2004	2005
	115.4	119.5

If X spends Rs. 115.4 in 2004, in 2005, he has spent.

$$Rs.\ \frac{119.5}{115.4} \times 80 = 82.84\ Rs.$$

In 2005, his salary is 110.

Exp. = Rs. 82.84.

Saving = Rs. 27.16

$$Saving\ \% = \frac{27.16}{110} \times 100 = 24.69\%$$

— x —

PRACTICALS ON PROBABILITY DISTRIBUTION

1. (a) Given that $\sum^{10} f(x) = 500426$, $\sum_{4}^{10} f(x) = 329240$, $\sum_{7}^{10} f(x) = 175212$. $f(10) = 40365$. Find $f(1)$.

(b)

1. (a) The no. of females in each of 100 queues of length 10 at a metro railway station in kolkata. The data are shown below:

Count	0	1	2	3	4	5	6	7	8	9	10
Freq	1	3	4	23	25	19	18	5	1	1	0

Propose an appropriate theoretical distn and fit it to the above data. Also comment on the fitting.

Solution:- Let X : "No of females in a queue of length 10"

Consider "getting a female in queue" as a success.

Assuming probability of successes in each queue is p (constant).

Then, Binomial model is appropriate to the given RV X.

Then PMF of X is

$$p(x) = \begin{cases} \binom{10}{x} p^{x} \cdot (1-p)^{10-x} & ; x = 0(1)10 \\ 0 & ; \text{ow} \end{cases}$$

By method of moments, $\mu_1' = m_1'$.

$$\Rightarrow 10p = \bar{x} = \sum_{i=1}^{10} x_i f_i \Big/ \sum_{i=1}^{10} f_i = 435/100 = 4.35.$$

$$\Rightarrow \hat{p} = \frac{\bar{x}}{10} = 0.435.$$

Hence the fitted distn is given by,

$$p(x) = \begin{cases} \binom{10}{x} (\hat{p})^{x} (1-\hat{p})^{10-x} & ; x = 0(1)10 \\ 0 & ; \text{ow} \end{cases}$$

Comment on Fitting :- Expected frequency of the value 'x' is

$$N \times P[X=x]$$
$$= 100 \times p(x).$$

[Consider getting a value 'x' as a success. Then
f_x = the frequency of 'x' in N observation
= the no. of successes in N Bernoulli trials
$\sim Bin(N, P[X=x])$
$\therefore E(f_x) = N \times P[X=x]$]

x	$p(x)$	Exp. freq = $N.p(x)$	Observed freq.

(b) When the first proof of 200 pages of an encyclopaedia of 500 pages were read, the distn. of printing mistakes were found to be as shown in the table:

No. of misprints on Page:	0	1	2	3	4	5
Frequency :	112	63	20	3	1	1

Fit a suitable probability distribution to the data.

Estabish the total cost of correcting the first proof of the whole encyclopaedia by using the information given below:

No. of misprints on page:	0	1	2	3	4	5 or more
Cost of detection and correction (dollars) per page:	0.10	0.16	0.23	0.29	0.34	0.36

Solution:- Let X denotes the number of misprints on a page. Getting a misprint on a page is a rare event. Hence X is expected to follow Poisson distribution.

Assume that $X \sim P(\lambda)$.

The PMF of X is

$$p(x) = \begin{cases} e^{-\lambda} \cdot \frac{\lambda^x}{x!} & , x = 0,1,2,3,\ldots \\ 0 & , ow \end{cases}$$

where $\lambda > 0$.

By method of moments, $\mu_1' = m_1'$

$$\Rightarrow \hat{\lambda} = \bar{x} = \sum_{i=0}^{5} x_i f_i / \Sigma f_i = \_\_\_\_\_\_$$

Hence the fitted distn. is

$$p(x) = \begin{cases} e^{-\hat{\lambda}} \cdot \frac{(\hat{\lambda})^x}{x!} & , x = 0,1,2,\ldots \\ 0 & , ow \end{cases}$$

The expected frequency of the value 'x' is

$$N.P[X=x] = 5000 \times p(x)$$

Table showing expected freq. and cost of correction:-

x	$p(x)$	Exp. freq	Cost per page	Cost of correction = Exp. freq × Cost per page
0			0.10	
1			0.16	
2			0.23	
3			0.29	
4			0.34	
≥ 5			.36	
	1	N		

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2. (a) A farmer sells bean seeds in packets of 100 and agrees to refund the price if the number of seeds germinating from a packet is less than 75. He knows from past experience than on an average 80% of the seeds germinate and it costs him Rs. 200 for a packet of seeds. How should he fix the price of a packet so as to ensure an average profit of 25% ?

Solution:- X : The no. of seeds germinated in packet.

Here, $X \sim Bin(100, p = 0.8)$.

Let x be the selling price.

Define, $Z = \begin{cases} (x-2) & \text{if } x \geq 75 \\ -2 & \text{if } x < 75 \end{cases}$

Now, $E(Z) = \frac{x}{4} = (x-2)P[X \geq 75] + (-2)P[X < 75]$

$= (x-2)[1 - P[X<75]] - 2P[X<75]$

By CLT, $\frac{x-80}{\sqrt{16}} \overset{a}{\sim} N(0,1)$

$P[X \geq 75] = 1 - \Phi\left(\frac{5}{4}\right)$

$\Rightarrow P[X<75] = \Phi\left(\frac{5}{4}\right)$

(b) Let $X \sim Bin(2, p)$, $Y \sim Pois(\lambda = 1)$ and X and Y are independently distributed. It is known that $P[Y > X] = 1 - 2e^{-1}$. What is the probability that X is strictly positive?

Solution:- $1 - 2e^{-1} = P[Y > X] = 1 - P[Y \leq X]$

$= 1 - \sum_{x=0}^{2} P[Y \leq x, X = x]$

$= 1 - \{P[Y \leq 0]P[X=0] + P[Y \leq 1]P[X=1] + P[Y \leq 2]P[X=2]\}$

$= 1 - \left[e^{-1}\binom{2}{0}p^0(1-p)^0 + \left\{e^{-1} + \frac{e^{-1}}{1!}\right\}\binom{2}{1} p^0(1-p)^2 + \left\{e^{-1} + \frac{e^{-1}}{1!} + \frac{e^{-1}}{2!}\right\}\binom{2}{2}p^2\right]$

$\Rightarrow 1 + 2p - \frac{p^2}{2} = 2$

$\Rightarrow p^2 - 4p + 2 = 0$

$\Rightarrow p = \frac{4 \pm \sqrt{4^2 - 4.1.2}}{2}$

$\Rightarrow p = 2 \pm \sqrt{2}$.

* $P[Y \leq 0] = P[Y=0]$

$P[Y \leq 1] = P[Y=0] + P[Y=1]$.

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Now, required probability is $= P[X > 0]$

$$= 1 - P[X = 0]$$

$$= 1 - (1-p)^2$$

$$= \left[1 - (\sqrt{2} - 1)^2\right]$$

$$= 2(\sqrt{2} - 1).$$

(c) Suppose that the number of accidents per week at an industrial plant is a poisson random variable with mean four. Suppose also that the number of workers injured in different accidents are independent Poisson RVs with a common mean 2. Assume that the number of workers injured in each accident is independent of the number of accidents that occur. What are the mean and variance of the number of injured during a week?

Solution:- Let N denotes the no. of accidents per week and X denotes the no. of workers injured in the ith accident.

Here $N \sim P(\lambda = 4)$ and $X \sim P(\lambda = 2)$, independently.

Hence, the number of injured during a week is

$$S_N = X_1 + X_2 + \cdots + X_N$$

$$E(S_N) = E(N)\,E(X_1) = 2 \times 4 = 8.$$

(d) A sample of size 10 is drawn from a normal population with mean and variance both equal to $\theta\,(>0)$ and the first quartile equal to 2.65. Find the probability that

(i) the first four observations are negative

(ii) four of the observations are negative.

Solution:- Let $X_1, X_2, \ldots, X_n$ be a r.v. from $N(\theta, \theta)$; $\theta > 0$.

By definition of 1st quartile,

$$P[X_1 \le Q_1] = 0.25$$

$$\Rightarrow P\left[\frac{X_1 - \theta}{\sqrt{\theta}} \le \frac{Q - \theta}{\sqrt{\theta}}\right] = 0.25$$

$$\Rightarrow \Phi\left(\frac{Q - \theta}{\sqrt{\theta}}\right) = 0.25$$

$$\Rightarrow \Phi\left(-\frac{Q_1 - \theta}{\sqrt{\theta}}\right) = 0.75$$

From Biometrica,

x	$\Phi(x)$
0.67	0.74857
0.68	0.75175

$$\therefore \; -\frac{Q_1-\theta}{\sqrt{\theta}} = 0.6745$$

$$\Rightarrow \frac{2.65-\theta}{\sqrt{\theta}} = 0.6745$$

$$\Rightarrow \sqrt{\theta} = \frac{0.6745 + \sqrt{(0.6745)^2 + 4\times 2.65}}{2}$$

as $\sqrt{\theta} > 0$.

$\therefore \sqrt{\theta} = 1.99 \simeq 2.$

$\therefore \theta = 4.$

Hence, $X \sim N(4, 2^2)$.

(i) The probability that first four observations are negative,

$$P[X_1 < 0, X_2 < 0, X_3 < 0, X_4 < 0]$$

$$= \{P[X_1 < 0]\}^4, \text{ as } X_i\text{'s are i.i.d.}$$

$$= \left\{\Phi\left[\frac{0-\theta}{\sqrt{\theta}}\right]\right\}^4$$

$$= \{\Phi(-\sqrt{\theta})\}^4$$

$$= \Phi^4(-2) = \_\_\_\_\_\_\_\_.$$

(ii) Let Y denotes the no. of negative values in $X_1, X_2, \ldots, X_{10}$.

Then $Y \sim Bin(10, p)$, where, $p = P[X_1 < 0] = \Phi(-2)$

$\therefore$ Required Probability $= P[Y=4] = \_\_\_\_\_\_\_\_.$

$$= \binom{10}{4} p^4 (1-p)^6 = \_\_\_\_\_\_\_\_.$$

3. (i) Twenty five leaves were selected at random from each of six similar apple trees. The number of adult female European red mites of each leaf was counted, the resulting information is summarized in the table below:

No. of Mites per leaf :	0	1	2	3	4	5	6	7
Frequency :	70	38	17	10	9	3	2	1

(a) Fit a negative binomial distribution to the data.
(b) Plot the observed and fitted values on the same graph paper, and comment on the goodness of fit after visual inspection.

Solution:- (i) Let X denotes the no. of adult female European red mites on a leaf.

(a) Assume that $X \sim NB(r, p)$

The PMF of X is $f(x) = \begin{cases} \binom{x+r-1}{x} p^r q^x ; & x = 0, 1, 2, \ldots \\ 0 & ; \text{ow} \end{cases}$

where $0<p<1$ and $p+q=1$ and $r>0$.

By method of moments,

$$\mu_1' = \bar{x}, \mu_2 = s^2$$

$$\Rightarrow \frac{rq}{p} = \bar{x}, \quad \frac{rq}{p^2} = s^2$$

$$\Rightarrow \bar{p} = \frac{\bar{x}}{s^2} \text{ and } \hat{r} = \bar{x} \cdot \frac{\hat{p}}{1-\hat{p}}.$$

From data, $\bar{x} = \dfrac{\sum_{i=0}^{7} i f_i}{\sum f_i} = \_\_\_\_\_\_.$

$$s^2 = \left(\sum_{i=0}^{7} i^2 f_i \Big/ \sum f_i\right) - \bar{x}^2 = \_\_\_\_\_\_.$$

$$\hat{p} = \_\_\_\_\_\_$$

$$\hat{r} = \_\_\_\_\_\_.$$

The fitted NB distribution is

$$p(x) = \begin{cases} \binom{x+\hat{r}-1}{x} (\hat{p})^r (\hat{q})^x, & x = 0, 1, 2, \ldots \\ 0 & , \text{ow} \end{cases}$$

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(b)

x	$p(x)$	Exp. freq. $N \times p(x)$	Obs. freq.
0			
1			
2			
3			
4			
5			
6			
≥ 7	(by subtraction)		

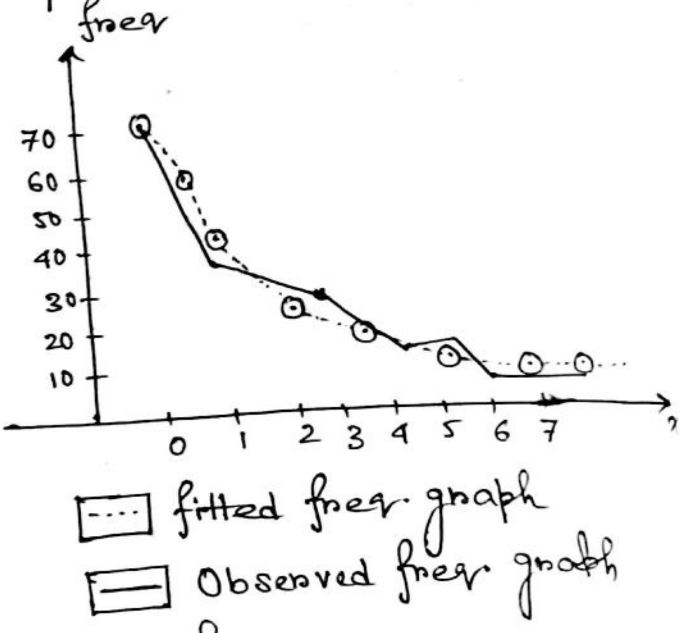
Here $p(x+1) = \left(\frac{x+r}{x+1}\right)\hat{q} \cdot p(x)$.

$p(1) = r.\hat{q} \cdot p(0)$

$\therefore p(0) = \hat{p}^{r} = \_\_\_\_\_\_\_\_$

$p(1) = \left(\frac{r}{1} . \hat{q}\right) p(0) = \_\_\_\_\_\_$

$p(2) = \left(\frac{r+1}{2} \hat{q}\right) p(1) = \_\_\_\_\_\_$



If the both frequency graphs are very near to each other throughout range, then the fitting is satisfactory.

Remark:- It is observed that the red mites on a leaf form a cluster. Therefore it seems like that NB model is appropriate for X.

$\boxed{\text{Alt:-}}$ For a poisson distribution, we should have $\bar{x} \simeq s^2$. For a negative binomial distn., we should have $\bar{x} << s^2$.

(ii) The following table gives the freq. distn. of the number of albino children in families of five children including at least one albino child (Pearson's data)

No. of albino in family	1	2	3	4	5	TOTAL
No. of families	25	23	10	1	1	60

Fit an appropriate probability distribution to the data. Also compare the observed and expected frequencies.

Solution:- Let X denotes the no. of albino children in families of five children. Then, considering "getting an albino children" as a success,

$$X \sim Bin(n=5, p).$$

The PMF of X is $p(x) = \begin{cases} \binom{5}{x} p^x q^{5-x}, & x=0(1)5 \\ 0 & , ow \end{cases}$

Let Y denotes the no. of albino children in a family of five having at least one albino. The data is given on the R.V. Y. Note that Y is the RV "X is truncated at $x=0$".

The PDF of Y is $p(y) = \begin{cases} \dfrac{\binom{5}{y} p^y (1-p)^{5-y}}{1-(1-p)^5} & , y=1(1)5 \\ 0 & , ow \end{cases}$

By method of moments, $E(Y) = \bar{y}$,

$$\Rightarrow \frac{5p}{1-(1-p)^5} = \bar{y} \text{ ———— } (*)$$

$$\bar{y} = \sum_{i=1}^{5} i f_i \Big/ \sum_{i=1}^{5} f_i = 1.833.$$

Solving (*) by Iteration method:-

$$p = \frac{\bar{y}}{5}\left\{1-(1-p)^5\right\} = \phi(p), \text{say}$$

Trial root:- $p_0 = \frac{\bar{y}}{5} =$ ______

Condition of Convergence:- $|\phi'(p_0)| =$ ______ $< 1.$

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Successive approximation, $p_1 = \phi(p_0)$

$p_2 = \phi(p_1)$

$\vdots$

$\hat{p} = \_\_\_\_$

$\therefore$ The fitted distribution is $p'(y) = \begin{cases} \dfrac{\binom{5}{y} \hat{p}^y (1-\hat{p})^{5-y}}{1-(1-\hat{p})^5}, & y = 1(1)5 \\ 0, & \text{ow} \end{cases}$

(iii) The table gives the frequency distribution of the number of dust nuclei in a small volume of air that fell onto a stage in a chamber containing moisture and filtered air. It is suspected that a number of zero counts were wrongly rejected on the ground that the apparatus was not working.

No. of dust nuclei	0	1	2	3	4	5	6	7	8	Total
Frequency	23	56	88	95	73	40	17	5	3	400

Fit an appropriate distribution to the frequency distribution, omitting the zero counts.

Solution:- Let X denotes the number of dust nuclei in a small volume of air.

Here, the poisson distribution is appropriate for X. Let,

$X \sim P(\lambda)$.

Let, Y be the RV of X truncated at $x=0$.

Hence, the PMF of Y is,

$$p(y) = \begin{cases} \dfrac{e^{-\lambda} \cdot \frac{\lambda^y}{y!}}{1-e^{-\lambda}}, & \text{if } y = 1, 2, 3, \ldots \\ 0, & \text{ow} \end{cases}$$

By method of moments,

$$E(Y) = \bar{y}$$

$$\Rightarrow \frac{\lambda}{1-e^{-\lambda}} = \bar{y} = \frac{\sum_{i=1}^{8} i.f_i}{\sum_{i=1}^{8} f_i} = \underline{\qquad\qquad}.$$

$$\Rightarrow \lambda = \bar{y}\left(1-e^{-\lambda}\right) = \phi(\lambda).$$

<u>Trial root :-</u> $\lambda_0 = \bar{y} = \underline{\qquad\qquad}.$

<u>Condition of Convergence :-</u> $|\phi'(\lambda_0)| = \underline{\qquad\qquad} < 1.$

Successive improvements are :

$$\lambda_1 = \phi(\lambda_0)$$

$$\lambda_2 = \phi(\lambda_1)$$

$$\vdots$$

$\hat{\lambda} = \underline{\qquad\qquad}$, correct to two decimal places.

Hence, fitted PMF of Y is,

$$p(y) = \begin{cases} e^{-\hat{\lambda}} \cdot \frac{(\hat{\lambda})^y}{y!} & , y = 1, 2, 3, \ldots \\ 0 & , \text{ow} \end{cases}$$

4. The Life time (T) in hours of an electron tube manufactured in a company is a r.v. with the d.f.

$$F(t) = 1 - e^{-t/\theta}, \quad t > 0.$$

A sample of 337 electron tubes give the following frequency distribution of T.

Life (in hours) :	0-50	50-100	100-150	150-200	200-300	300-400	400-6
Frequency :	100	68	48	31	42	21	27

(i) Estimate θ from the data and compare the observed and expected frequencies.

(ii) Estimate the probability that a supply of 20 tubes will not last more than 1900 hours, if they used one at a time successively.

(iii) Find the number of electronic tubes likely to burn away within 80 hours of their life and also the average life of these tubes.

Solution:-

The DF of T is

$$F(t) = 1 - e^{-t/\theta}, \quad t > 0$$

PDF of T is

$$f(t) = \frac{1}{\theta} e^{-t/\theta}, \quad \text{if } t > 0$$

(i) By method of moments,

$$\mu_1'(T) = \bar{t}$$

$$\Rightarrow \theta = \bar{t} = \frac{\sum_{i=1}^{7} t_i \cdot f_i}{\sum_{i=1}^{7} f_i} = \hat{\theta}$$

where, t_i, f_i are mid point and frequency of the i^{th} class.

The fitted distribution is given by,

$$F(t) = 1 - e^{-t/\hat{\theta}}, \quad t > 0.$$

$\therefore$ Expected frequency of the class interval (a, b) is

$$N.P[a < T < b] = N\{F(b) - F(a)\}$$

$$= N\{e^{-a/\hat{\theta}} - e^{-b/\hat{\theta}}\}$$

Computation of Expected Frequencies :-

Class interval (a,b)	$P[a < T < b]$	Exp. Freq. $N.P[a<T<b]$	Observed Freq.
0 - 50			
50 - 100			
100 - 150			
⋮			

(ii) Let T_i denotes the life-time of the i^{th} tube, $i = 1(1)20$.
Here, $T_i \overset{iid}{\sim} Exp(\theta)$, $i = 1(1)20$.

Required Probability $= P\left[\sum_{i=1}^{20} T_i < 1900\right]$

$$= P[T < 1900], \quad T \sim Gamma(20, \theta).$$

$$= \int_0^{1900} \frac{e^{-t/\theta} . t^{20-1}}{\theta^{20} . \Gamma 20} dt$$

$$= \int_0^{1900/\theta} \frac{e^{-x} . x^{20-1}}{\Gamma 20} dx \quad \left[\because \frac{2T_i}{\theta} \overset{iid}{\sim} \chi^2_2, \Rightarrow \frac{2T}{\theta} = \sum_{i=1}^{20} \frac{2T_i}{\theta} \Rightarrow \frac{2T}{\theta} \sim \chi^2_{40}\right]$$

$$= \Gamma_{\frac{1900}{\theta}}(20)$$

So, estimated probability $= \Gamma_{\frac{1900}{\hat{\theta}}}(20)$.

[Use Pearson Table for Incomplete Gamma]

(iii) The no. of tubes which burns away within 80 hours.

$$= N.P[T < 80]$$

$$= 337.\left\{1 - e^{-\frac{80}{\theta}}\right\}$$

$$= \_\_\_\_\_\_ .$$

The distribution of lifetime of a tube which burns away with 80 hours is

$$f'(t) = \begin{cases} \dfrac{\frac{1}{\theta} e^{-t/\theta}}{1 - e^{-80/\theta}} & , 0 < t < 80 \\ 0 & , \text{ow} \end{cases}$$

$$E(T') = \int_0^{80} \frac{t.\frac{1}{\theta} e^{-t/\theta}}{1 - e^{-80/\theta}} dt = \theta - 80\left(\frac{e^{-80/\theta}}{1 - e^{-80/\theta}}\right).$$

$$\hat{E}(T') = \hat{\theta} - 80.\frac{e^{-80/\hat{\theta}}}{1 - e^{-80/\hat{\theta}}}.$$

5. (a) The following table gives the frequency distribution of the length of the left middle finger of 3000 criminals, obtained in the course of a study in clinical anthropometry.

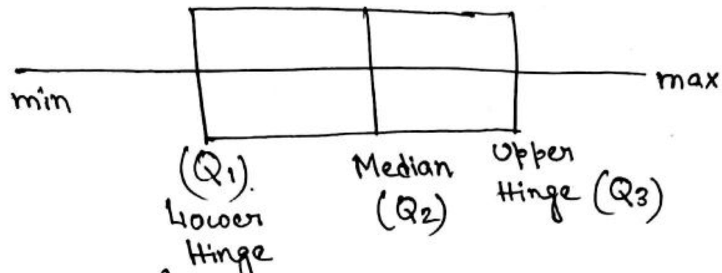
Length (cm): (Mid. point of class interval)	9.5	9.8	10.1	10.4	10.7	10.0	11.3	11.6	11.9	12.2	12.5
Frequency:	1	4	24	67	193	417	575	691	509	306	131

Length (cm):	12.8	13.1	13.4
Frequency:	63	16	3

(i) Display the data graphically be means of box-plot.
(ii) Fit an appropriate probability distribution to the data.
(iii) Plot observed and estimated frequencies on a graph paper and comment on the fit.

Solution:-

(i)



(ii) Here the freq. distn. is more or less symmetric and the variable given continuous. Let X denotes the length of the left middle finger of a criminal.

Assume that $X \sim N(\mu, \sigma^2)$.

The PDF of X is $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$; $x \in \mathbb{R}$.

By method of moments,

$$\mu_1' = m_1', \ \mu_2 = m_2.$$

$$\Rightarrow \begin{cases} \hat{\mu} = \bar{x} = \dfrac{\sum x_i f_i}{\sum f_i} = \_\_\_\_\_\_ \\ \hat{\sigma}^2 = s^2 = \dfrac{\sum x_i^2 f_i}{\sum f_i} = \_\_\_\_\_\_ = \bar{x}^2. \end{cases}$$

Hence, the fitted PDF of X is

$$f(x) = \frac{1}{\hat{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\hat{\mu}}{\hat{\sigma}}\right)^2}; \ x \in \mathbb{R}.$$

(iii)

Cls Mark	Class Boundaries
9.5	9.35 – 9.65
9.8	9.65 – 9.95
⋮	⋮

Exp. freq. of the class (a, b) is

$$= N.P[a < X < b] = N\left[\Phi\left(\frac{b-\hat{\mu}}{\hat{\sigma}}\right) - \Phi\left(\frac{a-\hat{\mu}}{\hat{\sigma}}\right)\right]$$

(x) Class boundaries	$\Phi\left(\frac{x-\hat{\mu}}{\hat{\sigma}}\right)$	$\Phi\left(\frac{x-\hat{\mu}}{\hat{\sigma}}\right)$	Exp. Freq $N.\Delta\Phi$	Observed Freq.
$-\infty$	0	–		
9.35	–			
9.65	–			
⋮	⋮			
13.65	–			
∞	1	...		

Comment:- If the graph of observed and expected frequencies are very close to each other then the fitting is good.

(b) The following table gives the frequency distribution of annual salaries of individuals in rupees obtained from Indian Income Tax Returns (1995)

Statistics of Individual salaries Assessed, 1955

Annual Salary (rupees)	Frequency
below 5000	27,000
5001 – 10000	60,000
10001 – 25000	90,000
25001 – 40000	26,032
40001 – 80000	13,000
80001 – 100000	60000
100001 – 200000	1178
200001 and above	312

Plot the data, using a suitable scale, and find out whether Pareto's law will give an adequate fit over the entire range of the observed distn., if not, try to fit the data in a suitable range.

<u>Solution:-</u> Let X denotes the salary of an individual.

Assume that the distribution of X is given. Then the DF of X is

$$F(x) = 1 - \left(\frac{x_0}{x}\right)^{\gamma} \quad \text{if } x > x_0$$

$$\Rightarrow 1 - F(x) = \left(\frac{x_0}{x}\right)^{\gamma}$$

$$\Rightarrow P[X \leq x] = \left(\frac{x_0}{x}\right)^{\gamma}$$

$$\Rightarrow \ln p = \ln x_0^{\gamma} - \gamma \ln x$$

$$\Rightarrow \ln p = \alpha - \gamma \ln x$$, where, p is the proportion of individuals whose income is above x.

Note that, $-\ln p = -\alpha + \gamma \ln x$.

Class	Freq.	$p = \frac{\text{c.f. of} \geq \text{type}}{N}$	$\ln p$	$\ln x$

x = upper bound of the class.

From graph, we get —— "Except the first two points, all the points are near about a straight line".
Hence, the Pareto law is appropriate after including 3rd class.

Fitting of Pareto Distribution :— Here $\quad -\ln p = (-\alpha) + \ln x$

$$\Rightarrow p' = -\alpha + \gamma x' \quad (p' = \ln p,\ x' = \ln x)$$

Hence (p', x') are linearly related.
The parameters α, γ are estimated by method of least squares based on the values (p_i', x_i'), from 3rd class and above.

Normal Equations are ——

$$\sum p_i' = N'(-\alpha) + \gamma \sum x_i'$$

$$\sum x_i' p_i' = -\alpha \sum x_i' + \gamma \sum x_i'^2$$

$$\Rightarrow \hat{\alpha} = \_\_\_\_\_\_, \quad \hat{\gamma} = \_\_\_\_\_\_.$$

Also $\hat{x}_0 = \_\_\_\_\_\_, \quad \hat{\gamma} = \_\_\_\_\_\_.$

Hence, the fitted Pareto law is given by the D.F.

$$F(x) = 1 - \left(\frac{\hat{x}_0}{x}\right)^{\hat{\gamma}}, \text{ if } x > x_0.$$

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(c) The differences (in mins) of the actual arrival times and the scheduled arrival are tabulated below:

Difference (in mins)	No. of days
< 4	12
$4-7$	23
$7-55$	141
$55-90$	14
>90	11

(i) Fit a log-normal distribution to the data.

(ii) Use the fitted distribution to estimate the probability that the train will arrive at 1hr. 15mins late on a typical day.

<u>Solution:-</u> Let X denotes the differences (in mins) of the actual arrival times and the scheduled arrival time.

(i) Here $X \sim \Lambda(\mu, \sigma^2)$.

Note that —
$$p = P[X \le \xi_p] = P[\ln X \le \ln \xi_p]$$
$$= P\left[\frac{\ln X - \mu}{\sigma} \le \frac{\ln \xi_p - \mu}{\sigma}\right]$$
$$= \Phi\left(\frac{\ln \xi_p - \mu}{\sigma}\right)$$
$$= \Phi(z_p) \quad [\because \ln X \sim N(\mu, \sigma^2)]$$
$$\Rightarrow \xi_p = e^{\mu + \sigma z_p} \text{ ; where } \Phi(z_p) = p.$$

To product the tails and to cover up the most of the values, we shall use P_{10} and P_{90}.

Now, equating the sample and population percentile, we get —
$$P_{10} = \hat{P}_{10} = 5.5 \Rightarrow e^{\mu + \sigma(z_{0.1})} = \hat{P}_{10}$$
$$P_{90} = \hat{P}_{90} = 72.5 \Rightarrow e^{\mu + \sigma(z_{0.9})} = \hat{P}_{90}$$
$$\Rightarrow \ln \hat{P}_{10} = \mu + \sigma(z_{0.1}) \;\&\; \ln \hat{P}_{90} = \mu + \sigma(z_{0.9})$$
$$\Rightarrow \hat{\mu} = \underline{\qquad\qquad}, \quad \hat{\sigma} = \underline{\qquad\qquad}.$$

(ii) Required Probability
$$= P[74.5 < X < 75.5]$$
$$= \Phi\left(\frac{\ln 74.5 - \hat{\mu}}{\hat{\sigma}}\right) - \Phi\left(\frac{\ln 74.5 - \hat{\mu}}{\hat{\sigma}}\right).$$

PRACTICALS ON STATISTICAL INFERENCE I

ESTIMATION

1. The length of life recorded in hours for 10 electron tubes were:
980, 1020, 995, 1015, 990, 1030, 975, 950, 1050, 870
Assume that life-times are distributed in the form:

$$f(t) = \frac{1}{\theta} e^{-t/\theta}, \text{ if } t > 0, \text{ where } \theta > 0$$

(i) Obtain an estimate of θ and the estimated S.E. of this estimate.
(ii) Estimate also the probability that an electron tube will survive at least 100 hours.
(iii) Determine the lower confidence limit with confidence coefficient 0.95 to θ and to the probability of survival for 100 hours or more.

__Solution:-__ (i) By method of moments :-

$$E(T) = \bar{t}$$

$$\Rightarrow \hat{\theta} = \bar{t} = \frac{1}{10}\sum_{i=1}^{10} t_i = \_\_\_\_\_\_ .$$

$$\text{S.E. of } \hat{\theta} = SE(\hat{\theta}) = \sqrt{Var(T)} = \sqrt{\frac{\theta^2}{n}} = \frac{\theta}{\sqrt{n}} .$$

$$\therefore \text{Estimated S.E. is } \widehat{SE}(\hat{\theta}) = \frac{\hat{\theta}}{\sqrt{n}} = \frac{\bar{t}}{\sqrt{10}} = \_\_\_\_\_\_ .$$

(ii) Probability that an electron tube will survive at least 100 hours

$$= P[T_i > 100]$$

$$= e^{-100/\theta}$$

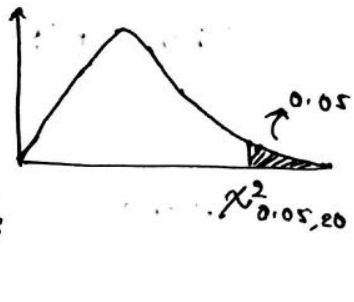
$$\therefore \hat{p} = e^{-100/\hat{\theta}} = e^{-100/\bar{t}} = \_\_\_\_\_\_ .$$

(iii) Note that $\frac{2T_i}{\theta} \sim \chi^2_2$, $i = 1(1)10$

$$\Rightarrow 2\sum_{i=1}^{10} \frac{T_i}{\theta} \sim \chi^2_{20}$$

Now, $$P\left[2\frac{\sum_{i=1}^{10} T_i}{\theta} \leq \chi^2_{0.05,20}\right] = 0.95$$

$$\Rightarrow P\left[\frac{20.\bar{T}}{\chi^2_{0.05,20}} < \theta < \infty\right] = 0.95$$



$\chi^2_{0.05,20}$

Hence, the observed lower confidence limit is

$$\left(\frac{20\bar{t}}{\chi^2_{0.05,20}}, \infty\right) = (\_\_\_ , \infty)$$

2. Let $X_1, X_2, \ldots, X_{10}$ be ten independent and identically distributed random variables where each X_i is normally distributed with mean 2·08. Further suppose 9·68% of its value is negative. If $X_{(1)} < X_{(2)} < \cdots < X_{(10)}$ be the ordered arrangement of X_i's, then calculate $P[X_{(3)} \geq 5 \cdot 28]$.

Solution:- The pdf of $X_{(r)}$ is

$$f_{X_{(r)}}(x) = \frac{\lfloor n}{\lfloor r-1 \; \lfloor n-r} \{F(x)\}^{r-1} \cdot f(x) \{1-F(x)\}^{n-r}$$

$$f_{X_{(3)}}(x) = \frac{\lfloor 10}{\lfloor 2 \; \lfloor 7} \{F(x)\}^2 \{1-F(x)\}^7 f(x)$$

and $$P[X_{(3)} \geq 5 \cdot 28] = \int_{5 \cdot 28}^{\infty} \frac{1}{\beta(3,8)} \{F(x)\}^2 \cdot \{1-F(x)\}^7 f(x)\,dx$$

$$= \int_{F(5 \cdot 28)}^{1} \frac{1}{\beta(3,8)} z^2 (1-z)^7 dz$$

Now, $X_i \overset{iid}{\sim} N(2 \cdot 08, \sigma^2)$

and $$P[X_i < 0] = \frac{9 \cdot 68}{100}$$

$$\Rightarrow P\left[\frac{X_i - 2 \cdot 08}{\sigma} < \frac{0 - 2 \cdot 08}{\sigma}\right] = 0 \cdot 0968$$

$$\Rightarrow \Phi\left(-\frac{2 \cdot 08}{\sigma}\right) = 0 \cdot 0968 \;;\; X_i \sim N(2 \cdot 08, \sigma^2)$$

$$\Rightarrow \Phi\left(\frac{2 \cdot 08}{\sigma}\right) = 0 \cdot 9132$$

$$\Rightarrow \sigma = \underline{\qquad\qquad}.$$

Then $$F(5 \cdot 28) = P[X_i < 5 \cdot 28] = P\left[\frac{X_i - 2 \cdot 08}{\sigma} < \frac{5 \cdot 28 - 2 \cdot 08}{\sigma}\right]$$

$$= \Phi\left(\frac{5 \cdot 28 - 2 \cdot 08}{\sigma}\right) = \underline{\qquad\qquad} = x_0, \text{ say}.$$

Hence, $$P[X_{(3)} \geq 5 \cdot 28] = 1 - P[X_{(3)} \leq 5 \cdot 28]$$

$$= 1 - \int_0^{x_0} \frac{z^{3-1}(1-z)^{8-1}}{\beta(3,8)} dz$$

$$= 1 - I_{x_0}(3,8)$$ [Use Pearson table for incomplete beta]

* TESTING OF HYPOTHESIS *

1. Let X be equal to the thickness of spearment gum manufactured for bending machine. Assume that the distn. of X is normal. The target thickness is 7.5 hundred of an age. The following 10 thickness of hundred of an age for pieces of gum that were selected randomly from the production line are

7.65, 7.60, 7.65, 7.70, 7.55, 7.40, 7.40, 7.50, 7.50, 7.55.

i) At $\alpha = 0.05$ significance level, was the company successful to meet the target thickness?

ii) What is the appropriate p- value of your test?

iii) Is $\mu = 7.50$ contained in a 95% confidence interval for μ?

__Solution:-__ Let $X \sim N(\mu, \sigma^2)$

From the sample, $\bar{x} = \frac{1}{10}\sum_{i=1}^{10} x_i = 7.55$

and $s^2 = \frac{1}{10}\sum_{i=1}^{10} x_i^2 - \bar{x}^2 = 0.097$

(i)

To test $H_0: \mu = 7.5$ Vs. $H_1: \mu \neq 7.5$

Test statistic is $T = \frac{(\bar{X} - \mu_0)\sqrt{n}}{S} \sim t_{n-1}$, under H_0.

Here $T = \frac{(\bar{X} - 7.5)\sqrt{10}}{S} \sim t_9$, under H_0.

__Critical region:-__ If the observed value of $|T| > t_{\alpha/2, 9}$; we shall reject H_0 at α level of significance.

i.e. $\left| \left(\frac{\bar{x} - 7.5}{s}\right)\sqrt{10} \right| > t_{0.025, 9}$; at $\alpha = 0.05$

From table, $t_{0.025, 9} = 2.262$.

Observed value, $\frac{(\bar{x} - 7.5)\sqrt{10}}{s} = \left(\frac{7.55 - 7.5}{0.097}\right)\sqrt{10}$

$= 1.63 < t_{0.025, 9} = 2.262$

Hence, there is no reason to reject H_0 at $\alpha = 0.05$ level of significance, i.e., the company was successful to meet the target value.

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(ii) $p\text{-value} = P_{H_0}\left[|T| \geq |t_0|\right]$

$= 2P_{H_0}\left[T \geq |t_0|\right]$, t_0 is the observed value of T.

$= 2 \cdot P_{H_0}\left[T \geq 1.63\right]$ [From Biometrica table]

$= 2 \times 0.1$

$= 0.2.$

As p-value is quite large, the observed value is a likely value under H_0.

(iii) 95% confidence interval for μ is

$\left(\bar{x} - t_{0.025,9} \cdot \frac{s}{\sqrt{n}}, \bar{x} + t_{0.025,9} \cdot \frac{s}{\sqrt{n}}\right)$

$= \left(7.55 - 2.62 \times \frac{0.097}{\sqrt{10}}, \_\_\_\_\right) = (7.48, 7.62)$ when $\mu = 7.5$.

2. The heart weights in groups of 12 females and 15 male cats are given below:

Male: 12·7, 15·6, 9·1, 12·1, 8·3, 11·2, 9·4, 8·0, 14·9, 10·7, 13·6, 9·6, 11·7, 9·3, 7·6

Female: 7·4, 7·3, 7·1, 9·0, 7·6, 9·5, 10·1, 10·2, 10·1, 9·5, 8·7, 7·2

Does the heart of a male cat on an average weight for male that of a female cat. State clearly any assumption you use.

Solutions:– Let X and Y denote the heart weights of a male and a female.

We assume that, $\left.\begin{matrix} X \sim N(\mu, \sigma_1^2) \\ Y \sim N(\mu, \sigma_2^2) \end{matrix}\right\}$ independently

To test $H_0: \mu_1 = \mu_2$ Vs. $H_1: \mu_1 > \mu_2$

We also assume that $\sigma_1 = \sigma_2 = \sigma$ (unknown)

Let $x_{11}, x_{12}, \ldots, x_{1n}$ be a r.s. from $N(\mu_1, \sigma^2)$

Let $x_{21}, x_{22}, \ldots, x_{2n}$ be a r.s. from $N(\mu_2, \sigma^2)$.

Here $n_1 = 15$, $n_2 = 12$

$$\bar{x}_1 = \frac{1}{15}\sum_{i=1}^{15} x_{1i} = \_\_\_\_$$

$$\bar{x}_2 = \frac{1}{12}\sum_{i=1}^{12} x_{2i} = \_\_\_\_$$

$$s_1^2 = \frac{1}{n_1 - 1}\left\{\sum x_{1i}^2 - n_1\bar{x}_1^2\right\} = \_\_\_\_$$

$$s_2^2 = \frac{1}{n_2 - 1}\left\{\sum x_{2i}^2 - n_2\bar{x}_2^2\right\} = \_\_\_\_.$$

We know, $s^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2} = \_\_\_\_\_\_$ is an UE of σ^2.

Test statistic:-

$$T = \frac{\bar{X}_1 - \bar{X}_2}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1+n_2-2} \text{, under } H_0.$$

Critical Region:-

$$\frac{(\bar{x}_1 - \bar{x}_2)}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} > t_{\alpha; n_1+n_2-2}$$

So, observed value of T is $\frac{\bar{x}_1 - \bar{x}_2}{s\sqrt{\frac{1}{15} + \frac{1}{12}}} = \_\_\_\_\_\_$.

From table, $t_{0.05,\, 15+12-2} = t_{0.05, 25} = 1.708$.

Conclusion:-

3) In an experiment to investigate, the effect of light on root growth in musters sidlings, to growth of siblings, these grown in identical condition, except that one was kept in the dark, find the other was explodes in sunlight during the day. After a certain period of time, the root lengths in mm of all the sidlings are measured. The following table gives the date obtained performed appropriate statistical list of significance to assist whether

i) light effects, root growth, & to the variances of the root length in the two population to be equal but unknown.

ii) Light effects in both growths in root lengths is the same.

The table makes two rows first for light and second for darkness.

L : 21, 39, 31, 30, 52, 39, 55, 50, 29, 17

D : 22, 16, 20, 14, 32, 28, 36, 41, 17, 22

Solution:-

i) To test :- $H_0 : \mu_1 = \mu_2$ against $H_1 : \mu_1 > \mu_2$.
If H_1 is accepted, then light affects the root growth.
(same as previous problem)

ii) To test $H_0 : \sigma_1 = \sigma_2$ vs. $H_1 : \sigma_1 \neq \sigma_2$

Test statistic :- $F = \dfrac{S_1^2}{S_2^2} \sim F_{n_1-1, n_2-1}$

Critical Region :- $\dfrac{s_1^2}{s_2^2} > F_{\alpha/2; n_1-1, n_2-1}$ or $< F_{1-\alpha/2; n_1-1, n_2-1}$.

where, $n_1 = 10$, $n_2 = 10$

Observed value of F is $\dfrac{s_1^2}{s_2^2} =$ ______ .

$\alpha = 0.1$.

From table, $F_{0.05, 9, 9} = 3.18$.

$$F_{0.95, 9, 9} = \frac{1}{F_{0.05, 9, 9}} = \frac{1}{3.18}$$

Conclusion:-

4) [CU'2006]

The following table contains observations on the supline systolic and diastolic blood pressures (in mm of Hg) for 15 patients with moderate hypertension, immediately before and two hours after taking a drug, captopril.

(i) Perform an appropriate test for the hypothesis that the drug is successful in reducing the diastolic blood pressure.

(ii) Obtain a 95% confidence interval for the mean difference in the systolic blood pressure before and after treatment.

Patient No.	Systolic BP		Diastolic BP	
	before	After	Before	After
1	210	201	130	125
2	169	165	122	121
3	187	166	124	121
4	160	157	104	106
5	167	147	112	101
6	176	145	101	85
7	185	168	121	98
8	206	180	124	105
9	173	147	115	103
10	146	136	102	98
11	174	151	98	90
12	201	168	119	98
13	198	179	106	110
14	148	129	107	103
15	154	131	100	82

Solution:- (i) Let X and Y denote the diastolic BP of a patient before and after taking the drug.

Assume that, $(X, Y) \sim BN(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$.

To test whether the drug is successful in reducing the diastolic BP, i.e. to test:

$H_0: \mu_x = \mu_y$ Vs. $H_1: \mu_x > \mu_y$.

Let (x_i, y_i), $i=1(1)15$, be 15 paired samples on (X, Y).

Define, $D_i = X_i - Y_i$, $i=1(1)n$.

$\Rightarrow D_i \overset{iid}{\sim} N(\mu_D, \sigma_D^2)$, where $\mu_D = \mu_x - \mu_y$.

Here to test, $H_0: \mu_D = 0$ Vs. $H_1: \mu_D > 0$,

<u>Test statistic :-</u> $\frac{\sqrt{n}(\bar{D}-0)}{s_D} \sim t_{n-1}$

<u>Critical region:-</u> $\frac{\sqrt{n}\cdot\bar{d}}{s_d} > t_{\alpha;n-1}$.

From the data:- for $n=15$,

Patient No.	1	2	3	...	15
d :					

$\bar{d} = \frac{1}{15}\sum d_i =$ ______

$s_d^2 = \frac{1}{n-1}\sum_{i=1}^{n}(d_i-\bar{d})^2 = \frac{1}{14}\left\{\sum d_i^2 - 15\bar{d}^2\right\} =$ ______ .

Observed value of the test statistic is $\frac{\sqrt{n}\cdot\bar{d}}{s_d} =$ ______ .

Tabulated value: $t_{0.05,14} =$ ______ .

<u>Conclusion:-</u>

(ii) Let U and V denote the systolic BP of a patient before and after taking drug.

Assume that, $(U,V) \sim BN(\mu_U, \mu_V, \sigma_U^2, \sigma_V^2, \rho)$

Let (U_i, V_i), $i=1(1)15$ be the 15 given paired sample on (U,V).

Define, $D_i = U_i - V_i \overset{iid}{\sim} N(\mu_D^2, \sigma_D^2)$, $\mu_D = \mu_U - \mu_V$.

$$\Rightarrow \frac{\sqrt{n}(\bar{D}-\mu_0)}{S_D} \sim t_{n-1}.$$

Now, 95% C.I. for μ_D is $\left(\bar{d} - t_{0.05,14}\cdot\frac{s_d}{\sqrt{15}}, \bar{d} + t_{0.05,14}\cdot\frac{s_d}{\sqrt{15}}\right)$

Here,

Patient No.	1	2	...	15
d =				

$\bar{d} =$ ______

$s_d =$ ______

Hence the observed 95% C.I. for μ_D is

$$(\quad , \quad).$$

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5) [CU'2005]

The members of a team of nine men are asked to load and fire naval guns by one of two methods M_1 and M_2, attempting to get off as many rounds per minute as possible. Two sets of such firing were made per method. The following table gives the outcome of this experiment.

Participate No.	Number of rounds fired per minute in			
	first set		second set	
	by M_1	by M_2	by M_1	by M_2
1	20.2	14.2	24.1	16.2
2	22.0	14.1	23.5	16.1
3	23.1	14.1	22.9	16.1
4	26.2	18.0	26.9	19.1
5	22.6	14.0	24.6	18.1
6	22.9	12.2	23.7	13.8
7	23.8	12.5	24.9	15.4
8	22.9	13.7	25.0	16.0
9	21.8	12.7	23.5	15.1

In the light of this data, are we justified in inferring that method M_1 is significantly better than M_2? Obtain a 95% C.I. for the difference in the mean performance by the two methods.

Solution:- Let (X_i, Y_i) be the paired samples in the 1st set by M_1 and M_2, $i=1(1)9$.

Let (X_i', Y_i') be the paired samples in the 2nd set by M_1 and M_2 for $i=1(1)9$.

We assume that, $\left.\begin{array}{l}(X_i, Y_i) \overset{iid}{\sim} BN, \ i=1(1)9 \\ (X_i', Y_i') \overset{iid}{\sim} BN, \ i=1(1)9\end{array}\right\}$ independent.

As (X_i, Y_i), (X_i', Y_i') are observed on the i^{th} gun man.

Note that, $\left.\begin{array}{l}X_i \overset{iid}{\sim} \text{Normal} \\ X_i' \overset{iid}{\sim} \text{Normal}\end{array}\right.$, $i=1(1)9$.

$$\Rightarrow \frac{X_i + X_i'}{2} \overset{iid}{\sim} N\left(\mu_{M_1}, \sigma_{M_1}^2\right)$$

Similarly, $\dfrac{Y_i + Y_i'}{2} \overset{iid}{\sim} N\left(\mu_{M_2}, \sigma_{M_2}^2\right)$.

But $X_i + X_i'$, $Y_i + Y_i'$ are observed on the same i^{th} gun man, $i = 1(1)9$.

So, $\left(\frac{X_i + X_i'}{2}, \frac{Y_i + Y_i'}{2}\right) \overset{iid}{\sim} BN\left(\mu_{M_1}, \mu_{M_2}, \sigma_{M_1}^2, \sigma_{M_2}^2, \rho\right)$

To test $H_0: \mu_{M_1} = \mu_{M_2}$ Vs. $H_1: \mu_{M_1} > \mu_{M_2}$.

Define, $D_i = \frac{(X_i + X_i') - (Y_i + Y_i')}{2} \overset{iid}{\sim} N(\mu_D, \sigma_D^2)$, $i = 1(1)9$.

where, $\mu_D = \mu_{M_1} - \mu_{M_2}$.

Now, To test, $H_0: \mu_D = 0$ Vs. $H_1: \mu_D > 0$

(Use Paired t-test).

6. (a) The correlation coefficient between head-length and stature for a sample of 36 members of an Indian tribe has been found to be 0·4339. Is it reasonable to assume that in the population of the characters are uncorrelated?

(b) Examine on the basis of the data given in the table below whether the variances of muscle weights of right and left legs of rabbits are equal.

Sample no. of rabit	Weight (gms) of anterior muscle of	
	left leg	Right leg
1	5·0	4·9
2	4·8	5·0
3	4·3	4·3
4	5·1	5·3
5	4·1	4·1
6	4·0	4·0
7	7·1	6·9
8	5·9	6·3
9	5·3	5·2
10	5·3	5·5
11	5·3	5·5
12	5·9	5·9
13	6·5	6·8
14	6·3	6·3
15	6·6	6·6
16	5·2	6·3

(c) Given in the table are the means, the s.d.s and the correlation coefficient of scores x, y on two halves of a psychological test on 20 days.

(i) Examine whether the scores on the two halves are uncorrelated.

(ii) Examine whether the scores on the two halves are equally variable.

Score	Mean	S.d.	Correlation
x	45·5	9·51	0·76
y	50·2	5·25	

(a) Let (X_i, Y_i), $i = 1(1)36$, be the paired values on head-length and stature for 36 indian tribe. Assume that $(X_i, Y_i) \sim BN$ with correlation coefficient ρ.

To test $H_0: \rho = 0$ Vs. $H_1: \rho \neq 0$

Test statistic:-

$$\frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \sim t_{n-2} \text{, under } H_0.$$

Critical region:-

$$\left| \frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \right| > t_{\alpha/2,\, n-2}$$

Here $n = 36$, $r = 0.4339$

Observed value of the test statistic is $\frac{r\sqrt{n-2}}{\sqrt{1-r^2}} = \frac{0.4339 \times \sqrt{34}}{\sqrt{1-(0.4339)^2}}$

$= 2.808$.

From Biometrica, Vol-I, for $\alpha = 0.05$

$t_{0.025,30} = 2.042$

$t_{0.025,40} = 2.021$

Here, $\left| \frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \right| = 2.808 > t_{0.025,34}$

$\Rightarrow$ H_0 is rejected at 5% level based on the given information.

$\Rightarrow$ The population characters are not uncorrelated.

(b) Let (X_i, Y_i) denote the weights of right and left legs of the i^{th} rabit, $i = 1(1)6$.

Assume that $(X_i, Y_i) \overset{iid}{\sim} BN(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$, $i = 1(1)n$, $n = 16$.

To test $H_0: \sigma_x = \sigma_y$ Vs. $H_1: \sigma_x \neq \sigma_y$

Define, $U_i = X_i + Y_i$
$V_i = X_i - Y_i$, $i = 1(1)16$.

Note that $\rho_{UV} = 0$, under H_0.

To test, $H_0: \sigma_x = \sigma_y$

$\Leftrightarrow H_0': \rho_{UV} = 0$

To test $H_0: \rho_{UV} = 0$, based on the paired sample (U_i, V_i), $i = 1(1)16$; compute

$$r_{UV} = \frac{\frac{1}{n}\sum u_i v_i - \bar{u}\bar{v}}{\sqrt{\left\{\frac{1}{n}\sum u_i^2 - \bar{u}^2\right\}}\sqrt{\left\{\frac{1}{n}\sum v_i^2 - \bar{v}^2\right\}}}.$$

Serial no	1	2	. . . . . .	16
U_i				
V_i				

Test statistic: $\frac{r_{UV}\sqrt{n-2}}{\sqrt{1-r_{UV}^2}} \sim t_{n-2}$, under H_0'.

(c) (i) Here we are to test whether $\rho = 0$ or not.

(ii) To test $\sigma_1 = \sigma_2$

If $\rho = 0$, then it becomes a testing of equality of variances in uncorrelated test.

If $\rho \neq 0$, it becomes a testing of equality of variances in correlated test.

To test $\mu_1 = \mu_2$

If $\rho = 0$, & $\sigma_1 = \sigma_2$ are accepted, then it becomes a Fisher's t-test.

If $\rho \neq 0$ then it becomes a paired t-test.

7. (a) In an excavation 10 fossils were discovered, of which 6 could be definitely classified as male and 3 as a female. The sex of the last fossil could not be precisely determined. Are these findings compatible with a sex ratio?

Solution:- Let X denotes the no. of male fossils out of 9 fossils obtained.

Assuming probability of getting a male fossils $= p$,

$X \sim Bin(9, p)$.

To test $H_0: p = 1/2$ against $H_1: p \neq 1/2$.

Here the observed value of X is $x_0 = 6$.

$$\text{The } p\text{-value} = 2 \min \{ P_{H_0}[X \geq x_0], P_{H_0}[X \leq x_0] \}$$

$$= 2 \times \min \left\{ \sum_{x=6}^{9} \binom{9}{x} \left(\frac{1}{2}\right)^9, \sum_{x=0}^{6} \binom{9}{x} \left(\frac{1}{2}\right)^9 \right\}$$

$$= 2 \times \frac{\binom{9}{6} + \binom{9}{7} + \binom{9}{8} + 1}{2^9}$$

$$= 0.50$$

Let $\alpha = 0.05$ be the chosen level of significance.

Clearly, $p\text{-value} > 0.05 = \alpha$.

$\Rightarrow H_0$ is accepted.

$\Rightarrow$ The sex-ratio is 1:1 is supported by the data.

(c) The following table gives the result of an experiment to compare the effect of newly discovered medicine on a certain disease with that of the prevailing treatment (control). Do the data confirm the superiority of the new drug?

Treatment	Cured	Not cured	Total
Control	3	5	8
New medicine	5	3	8
Total	8	8	8

Solution:- Let X_1 denotes the no. of cured out of 8 patients treated under control.

Let X_2 " .. " .. .. ..
under new medicine.

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Let $X_1 \sim Bin(8, p_1)$ $\Big\rangle$ independently ; $n_1 = 8, n_2 = 8$
$X_2 \sim Bin(8, p_2)$

To test whether new medicine is superior.

$\Rightarrow$ To test $H_0: p_1 = p_2$ Vs. $H_1: p_1 < p_2$.

From data, the observed value of X_1 and X_2 are $x_{10} = 3, x_{20} = 5$

Then $X_1 + X_2 = X$ has the observed value $x_0 = 8$.

In the testing problem,

$$p\text{-value is} = P_{H_0}\left[X_1 \le x_{10} \mid X = x_0\right]$$

$$= \sum_{x_1=0}^{x_{10}} \frac{\binom{n_1}{x_1}\binom{n_2}{x_0 - x_1}}{\binom{n_1+n_2}{x_0}} = \sum_{x_1=0}^{3} \frac{\binom{8}{x_1}\binom{8}{8-x_1}}{\binom{16}{8}}$$

$$= \_\_\_\_\_\_\_\_\_\_.$$

<u>Conclusion:-</u>

8) (a) The number of deaths from drowning in a certain river in two consecutive months were 8 and 5. Are these fluctuations due to chance?

(b) A newspaper in a certain city observed that driving conditions have much improved because the number of fatal automobile accidents in last year was 9 where as the average number per year over the past several year was 15. Is the statement justified? If further, the number of fatal automobile accidents in the first six month of the current year is given to be 3, could you modify your earlier conclusion?

(c) A 5 cm specimen of a new type of fibre is found to have 13 defects while the manufacturer claims that there are no more than 150 defects per 100 cm. Do the above data support this claim?

Solution:- (a) Let X_1 and X_2 denote the number of deaths of drowning in a certain river in two consecutive months.

Let $\left.\begin{array}{l} X_1 \sim P(\lambda_1) \\ X_2 \sim P(\lambda_2) \end{array}\right\}$ independently

To test whether the observed values are the values of the same population or not.

$\Leftrightarrow$ To test $H_0: \lambda_1 = \lambda_2$ vs. $H_1: \lambda_1 \neq \lambda_2$.

Define, $X = X_1 + X_2 \sim P(\lambda)$, under $H_0: \lambda_1 = \lambda_2 = \lambda$.

The observed values of X_1 and X are $x_{10} = 8$, $x_0 = 8+5 = 13$.

$\therefore$ p-value $= 2 \times \min \left\{ P_{H_0}[X_1 \geq x_{10} \mid X = x_0],\ P_{H_0}[X_1 \leq x_{10} \mid X = x_0] \right\}$

$$= 2 \times \min \left\{ \sum_{x=x_{10}}^{x_0} \binom{x_0}{x} \frac{1}{2^{x_0}},\ \sum_{x=0}^{x_{10}} \binom{x_0}{x} \frac{1}{2^{x_0}} \right\}$$

$$= 2 \times \min \left\{ \sum_{x=8}^{13} \binom{13}{x} \frac{1}{2^{13}},\ \sum_{x=0}^{8} \binom{13}{x} \frac{1}{2^{13}} \right\}$$

$=$ ________

Conclusion:-

(b) Let X denotes the number of accidents per year.

We assume that $X \sim P(\lambda)$

To test $H_0: \lambda = 15$ Vs. $H_1: \lambda < 15$

Observed value of X is $x_0 = 9$

p-value $= P_{H_0}[X \le x_0]$

$$= \sum_{x=0}^{9} e^{-15} \cdot \frac{15^x}{x!} = 1 - e^{-15} \sum_{x=0}^{15} \frac{15^x}{x!} = \_\_\_\_\_\_ .$$

<u>Conclusion</u>:- Let $\alpha = 0.05$, If p-value $< \alpha = 0.05$ then H_0 is rejected.

Let X' denotes the no. of accidents per six month. Then

$X' \sim Poi(\lambda/2)$.

The observed value of X' is $x_0' = 3$.

$$\therefore\ p\text{-value} = P_{H_0}[X' \le x_0'] = \sum_{x=0}^{3} e^{-7.5} \frac{(7.5)^x}{x!} = \_\_\_\_\_\_ .$$

<u>Conclusion</u>:-

(c) Let X denotes the no. of defects in a 5 cm specimen. We assume that

$X \sim P(\lambda)$.

Manufacturer claims:

There are not more than 150 defects in 100 cm.

$\Rightarrow$ " " " " " 7.5 " " 5 cm.

To test $H_0: \lambda_0 = 7.5$ Vs. $H_1: \lambda > 7.5$

The observed value of X is $x_0 = 13$

$$\therefore\ p\text{-value} = P_{H_0}[X \ge 13]$$

$$= \sum_{x=13}^{\infty} e^{-7.5} \frac{(7.5)^x}{x!}$$

$$= 1 - \sum_{x=0}^{13} e^{-7.5} \cdot \frac{(7.5)^x}{x!}$$

$$= \_\_\_\_\_\_ .$$

9. For 20 pairs of fathers and sons, the regression equation of height of son (y) on height of father (x), both measured in cm, was found to be

$$y = 9.29 + 0.932x$$

For 20 pairs, $\bar{x} = 168.17$, $\Sigma(x_i - \bar{x})^2 = 777.80$ and $\Sigma(y_i - \bar{y})^2 = 939.42$. Test whether the regression coefficient differs significantly from unity. Find the 95% confidence limits to the conditional mean of y given $x = 177$. Also, the prediction limits of the height of a son when the height of the father is known to be 177 cm.

Solution:- Let $\eta_x = \alpha + \beta x$ be the regression equation of y on x.

The least square linear regression equation of y on x is

$$y = a + bx$$

i.e. $y = 9.29 + 0.932x$.

To test $H_0: \beta = 1$ Vs. $H_1: \beta \neq 1$

Test statistic:- $\dfrac{(b-1)\sqrt{S_{xx}}}{s_{y.x}} \sim t_{n-2}$, under H_0.

From data; $b = 0.932$; $n = 20$

$$S_{xx} = \Sigma(x_i - \bar{x})^2 =$$

$$s^2_{y.x} = \frac{1}{n-2}\sum_{i=1}^{n} \{y_i - \bar{y} - b(x_i - \bar{x})\}^2$$

$$= \frac{\Sigma(y_i - \bar{y})^2 - b^2\,\Sigma(x_i - \bar{x})^2}{n-2}.$$

$$= \frac{S_{yy} - b^2 S_{xx}}{n-2} = \underline{\qquad}.$$

If the observed $\left|\dfrac{(b-1)\sqrt{S_{xx}}}{s_{y.x}}\right| > t_{\alpha/2,\, n-2}$.

We shall reject H_0 at α-level.

Now, observed $\left| \dfrac{(b-1)\sqrt{S_{xx}}}{s_{y.x}} \right| = $ ______ .

and from table, for $\alpha = 0.05$, $t_{0.025,18} =$ ______ .

Conclusion :-

■ To find 95% C.I. for $\eta_x = \alpha + \beta x$, when $x = 177$.

Let $Y_x = a + bx$ be the predictor value when x is given.

$$\frac{Y_x - \eta_x}{s_{y.x}\sqrt{\frac{1}{n} + \frac{(x-\bar{x})^2}{S_{xx}}}} \sim t_{n-2}. \qquad \left[\begin{array}{l} Y_x = a + bx = \text{——} \\ \& \ \bar{x} = 168.17 \end{array} \right]$$

The 95% C.I. for η_x is $\left(Y_x - t_{0.025,18} \cdot s_{yx}\sqrt{\frac{1}{n} + \frac{(x-\bar{x})^2}{S_{xx}}}, \ Y_x + t_{0.025,18} \cdot s_{y.x}\sqrt{\frac{1}{n} + \frac{(x-\bar{x})^2}{S_{xx}}} \right)$

where, $n = 20$, $x = 177$,

■ To find 95% prediction limit for y when $x = 177$

$$\frac{y - Y_x}{s_{y.x}\sqrt{1 + \frac{1}{n} + \frac{(x-\bar{x})^2}{S_{xx}}}} \sim t_{n-2}.$$

The 95% prediction limit for y when $x = 177$ is

$$\left(Y_{177} \mp t_{0.025,18} \cdot s_{yx}\sqrt{1 + \frac{1}{20} + \frac{(177-\bar{x})^2}{S_{xx}}} \right).$$

Problems on Large Sample [Only CU problems]

1. In a sample of size 100 from a bivariate popln, the correlation coefficient is found to be 0.25. Test whether it is
(i) significant (ii) significantly less than 0.5,

Solution:- Here $n = 100$
$r = 0.25$

(i) To test $H_0: \rho = 0$ Vs $H_1: \rho > 0$.

[exact test: $\frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \sim t_{n-2}$]

Using Fisher's Z-transformation,

$$\sqrt{n-3}\,(Z - \xi) \overset{a}{\sim} N(0,1)$$

Here $Z = \frac{1}{2}\log\left(\frac{1+r}{1-r}\right) = $ ______

$$\xi_0 = \frac{1}{2}\log\left(\frac{1+\rho}{1-\rho}\right) = 0.$$

$\therefore$ Under H_0,

$$\tau = \sqrt{n-3}\,(Z - 0) \overset{a}{\sim} N(0,1).$$

Critical region:- observed $\tau > \tau_\alpha$.

(ii) To test $H_0: \rho = 0.5$ Vs $H_1: \rho < 0.5$

Here $\xi_0 = \frac{1}{2}\log\left(\frac{1+0.5}{1-0.5}\right) =$

Under H_0,

$$\tau = \sqrt{n-3}\,(T - \xi_0) \overset{a}{\sim} N(0,1).$$

Critical region:- observed $T < -\tau_\alpha$.

2. To examine a manufacturer claim that not more than 1.9% of the products are defectives, then 630 items are put to impection and 25 found defected can be consider the claim to be justified.

Sol. To $H_0: p = 0.019$ vs. $H_1: p > 0.019$

Here $n = 830$

$$\hat{p} = \frac{25}{830}.$$

Using $\sin^{-1}$ transformation.

Under H_0,

$$\tau = \sqrt{4n}\left(\sin^{-1}\sqrt{\hat{p}} - \sin^{-1}\sqrt{0.019}\right) \overset{a}{\sim} N(0,1)$$

Critical region: observed $\tau > \tau_\alpha$.

3. A proponent of inovative teaching methods wishes to compare the effectiveness of teaching english by the traditional classroom lecture system and by the extensive use of audio-visual aid. To do so, 100 students are selected at random from a class of 250 and assign to audio-visual instruction. The remaining 150 students are taught english in classroom lecture. At the end of the term, all 250 students are given a test; the no. of students from each group to pass the test is recored in the following table

Medium	Pass	Fail
Audio-visual instruction	63	37
classroom lecture	107	43

(a) Find the 95% C.I. for the difference between success rate for the two methods of instructions.

(b) Do the data strongly support that a better passing rate is achieved using the classroom lecture that is acheived using A.V. method.

(c) Explain whether your inferential procedure coucial depends on the term 'random'.

Sol. Let p_1 and p_2 be the passing rates in Audio-visual and classroom lecture.

Here $\hat{p}_1 = \frac{63}{100}$, $\hat{p}_2 = \frac{107}{150}$.

(a) Here $P\left[\sin^2\left(\sin^{-1}\sqrt{\hat{p}_i} - \frac{\tau_{\alpha/4}}{\sqrt{4n_i}}\right) \le p_i \le \sin^2\left(\sin^{-1}\sqrt{\hat{p}_i} + \frac{\tau_{\alpha/4}}{\sqrt{4n_i}}\right)\right]$

$= 1-\alpha$, $i = 1,2$.

$\Rightarrow P[L_i \le p_i \le U_i] = 1-\alpha$, $i = 1,2$.

Then $P[L_1 \le p_1 \le U_1, L_2 \le p_2 \le U_2]$

$= P[A_1 \cap A_2] \ge P[A_1] + P[A_2] - 1 = 1-\alpha/2 + 1-\alpha/2 - 1$

$= 1-\alpha.$

$\Rightarrow P[L_1 - U_2 \le p_1 - p_2 \le U_1 - L_2] \ge 1-\alpha.$

Here $L_1 =$

$L_2 =$.

$U_1 =$

$U_2 =$

and $\alpha = 0.05$.

Hence $(L_1 - U_2, U_1 - L_2) = (\quad , \quad)$

is an observed C.I for $(p_1 - p_2)$ with confidence leve 0.95.

(b) To test $H_0: p_1 = p_2$ against $H_1: p_1 < p_2$

[Here $\sqrt{4n_i}\left(\sin^{-1}\sqrt{\hat{p}_i} - \sin^{-1}\sqrt{p_i}\right) \overset{a}{\sim} N(0,1)$, $i = 1,2$, independently as the groups are selected randomly.]

Here, $\sin^{-1}\sqrt{\hat{p}_i} \overset{a}{\sim} N\left(\sin^{-1}\sqrt{p_i}, \frac{1}{4n_i}\right)$, $i = 1,2$ independently.

Now, $\sin^{-1}\sqrt{\hat{p}_1} - \sin^{-1}\sqrt{\hat{p}_2} \overset{a}{\sim} N\left(\sin^{-1}\sqrt{p_1} - \sin^{-1}\sqrt{p_2}, \frac{1}{4n_1} + \frac{1}{4n_2}\right)$

Under H_0, $$\tau = \frac{\sin^{-1}\sqrt{\hat{p}_1} - \sin^{-1}\sqrt{\hat{p}_2}}{\sqrt{\frac{1}{4n_1} + \frac{1}{4n_2}}} \overset{a}{\sim} N(0,1)..$$

Critical region:- Observed $\tau < -\tau_\alpha$.

(c) As the groups are selected randomly, the data obtained from the two groups then constitute two independent random samples. Therefore the estimates $\hat{p}_1$ and $\hat{p}_2$ are independently distributed and accordingly we obtain the test statistic.

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4. For 600 beans of particular variety, the frequency distn. of breath (in m.m) has $\gamma_1 = -0.093$, $\gamma_2 = -0.125$, where γ_1 and γ_2 are sample measures of skewness and kurtosis. Examine if the popln. can be supposed to be normal.

Sol. Here to test H_0 : the data is a r.s. from normal popln.,
As the measures skewness g_1 and kurtosis g_2 are given, then H_0 reduces to $H_{01}: \gamma_1 = 0$ and $H_{02}: \gamma_2 = 0$ as far as given information is concerned.

If both H_{01} and H_{02} are accepted, then we accept H_0.

Under H_0, $g_1 \overset{a}{\sim} N\left(0, \frac{6}{n}\right)$, $g_2 \overset{a}{\sim} N\left(0, \frac{24}{n}\right)$.

Test statistic: $\tau_1 = \sqrt{\frac{n}{6}}\, g_1 \overset{a}{\sim} N(0,1)$

$$\tau_2 = \sqrt{\frac{n}{24}}\, g_2 \overset{a}{\sim} N(0,1).$$

If observed $|\tau_1| > \tau_{\alpha/2}$, we reject H_{01}.
If observed $|\tau_2| > \tau_{\alpha/2}$, we reject H_{02}.

[Prob. of accepting $H_0 = P\left[\, |\tau_1| < \tau_{\alpha/2},\ |\tau_2| < \tau_{\alpha/2} \right]$

$$\geq (1-\alpha) + (1-\alpha) - 1 = 1 - 2\alpha.$$

$\Rightarrow$ prob. of rejecting $H_0 \leq 2\alpha$.]

[Calculation & Conclusion: suggesion: use $\alpha = 1\%$.]

5. The corr. coeff. between streature (cm) and nasal height (cm) of a group of 106 males is 0.672 and that for a female group of 117 females is 0.725. Can you treat the corr. coefficients to differ significantly.

Sol. Let ρ_1, ρ_2 be the correlation coefficient between stature and nasal height of male and female, respectively.
Assuming the popln.s are bivariate.
To test $H_0: \rho_1 = \rho_2$ vs. $H_1: \rho_1 \neq \rho_2$.

Here $Z_i = \frac{1}{2}\log\left(\frac{1+r_i}{1-r_i}\right)$

$$\xi_i = \frac{1}{2}\log\left(\frac{1+\rho_i}{1-\rho_i}\right)$$

Here, $Z_i \overset{a}{\sim} N\left(\xi_i + \frac{\rho_i}{2(n-1)}, \frac{1}{n_i - 3}\right)$, $i = 1, 2$, independently.

Under $H_0: \rho_1 = \rho_2 = \rho$

$$Z_1 - Z_2 \overset{a}{\sim} N\left(0, \frac{1}{n_1 - 3} + \frac{1}{n_2 - 3}\right).$$

where $E(Z_1 - Z_2) = \frac{\rho}{2}\left\{\frac{1}{n_1 - 1} - \frac{1}{n_2 - 1}\right\} = \frac{\rho}{2} \cdot \frac{n_1 - n_2}{(n_1 - 1)(n_2 - 1)} \simeq 0.$

Test statistic:-

$$\tau = \frac{(Z_1 - Z_2)}{\sqrt{\frac{1}{n_1 - 3} + \frac{1}{n_2 - 3}}} \overset{a}{\sim} N(0,1), \text{ under } H_0.$$

6. Consider the following set $A_1 = \{x : -\infty < x \le 0\}$

$A_i = \{x : i-2 \le x \le i-1\}$, $i = 2(1)7$.

$A_8 = \{x : 6 \le x < \alpha\}$

A certain Hypothesis H_0 assign probabilities p_{i0} to this sets A_i is accordence with

$$p_{i0} = \int_{A_i} \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-3}{2}\right)^2} dx,$$

the hypothesis can be tested on the basis of the observed frequencies of the sets A_i, $i = 2(1)8$, which are respectively 60, 96, 140, 210, 172, 160, 88, 74.

Would H_0 is accepted at the 5% level of significance?

Sol. To test $H_0 : p_{i0} = \int_{A_1} \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{8}(x-3)^2} dx$, $i = 1(1)6$.

$$\Leftrightarrow H_0 : X \sim N(3, 2^2).$$

Here $p_{10} = P[X \le 0] = \Phi\left(-\frac{3}{2}\right)$

$p_{20} = P[0 < X \le 1] = \Phi(-1) - \Phi\left(-\frac{3}{2}\right)$

$\vdots$

$p_{80} = P[6 < X < \alpha] = 1 - \Phi\left(\frac{3}{2}\right)$

Class	Observed freq. (O_i)	Exp. Freq. $e_i = np_{i0}$
A_1		
A_2		
$\vdots$		
A_8		

This is a test of Goodness of fit, $n = \sum_{i=1}^{8} f_i$.

Test statistic

$$\chi^2 = \sum \frac{O_i^2}{e_i} - n \sim \chi^2_{8-1}, \text{ under } H_0.$$

Critical region:- Observed $\chi^2 > \chi^2_{\alpha, 7}$.

7. A survey of drivers was taken to see if they have been is an accident during the previous year, and if so was it the minor and major accident. The results are tabulated by age group

Age	None	minor	major
		Accident type	
Under 18	67	10	5
18 - 25	42	6	5
26 - 40	75	8	4
40 - 65	56	4	6
over 65	57	15	1

Use an appropriate test to check if the data suggest that the nature of the accidents depends on age.

Sol. Let A: Nature of Accident

B: Age

To test A and are independent.

The test statistic is : $\chi^2 = n\left\{\sum_{i=1}^{r}\sum_{j=1}^{s} \frac{f_{ij}^2}{f_{io}f_{oj}} - 1\right\} \sim \chi^2_{(r-1)(s-1)}$

Critical region:-

observed $\chi^2 > \chi^2_{\alpha,(r-1)(s-1)}$.

8. To compare the results of college 1 and 2 the following data on the final examination results have been obtained.

College	Excellent	Good	Mediocare	Bad
		Results		
1	62	173	28	8
2	51	126	70	17

State the null hypothesis and test it to determine if the results vary between the colleges.

Sol.

$$\sum_{i=1}^{k} \frac{(f_{i1} - n_1 p_{i1})^2}{n_1 p_{i1}} + \sum_i \frac{(f_{i2} - n_2 p_{i2})^2}{n_2 p_{i2}} + \cdots\cdots$$

$$= \sum_{j=1}^{l}\sum_{i=1}^{k} \frac{(f_{ij} - n_i p_{ij})^2}{n_i p_{ij}} \sim \chi^2_{l(k-1)}.$$

Under Ho: $\sum\sum \frac{\left(f_{ij} - n_j \frac{f_{io}}{n}\right)^2}{n_j \cdot \frac{f_{io}}{n}}$; $\hat{p}_{io} = \frac{f_{io}}{n}$.

$$\chi^2 = n\left\{\sum\sum \frac{f_{ij}^2}{n_j f_{io}} - 1\right\} \sim \chi^2_{(l-1)(k-1)}.$$

Let A denotes the result of a college.

	A			
	A_1	A_2	A_3	A_4
College I (sample I)	$f_{11} = 62$	$f_{21} = 173$	$f_{31} = 28$	$f_{41} = 8$
College II sample II	f_{12}	f_{22}	f_{32}	f_{42}
	f_{10}	f_{20}	f_{30}	f_{40}

$n_1 =$ ______

$n_2 =$ ______

f_{ij}, $i = 1(1)4$, $j = 1(1)2$.

To test Ho: the homogeneity of the result in the colleges.

Test statistic, under Ho,

$$\chi^2 = n\left\{\sum_{j=1}^{2}\sum_{i=1}^{4}\frac{f_{ij}^2}{n_j f_{i0}} - 1\right\} \sim \chi^2_{(2-1)(4-1)}$$

Critical region:- observed $\chi^2 > \chi^2_{\alpha,3}$.

9. The no. of occurences of a word in 48 essays per thousand essays examined, written by Hamilton and 50 essays written by Maddisson. The following table is obtained.

Author	No. of essays according to rate of occurence of 'to' words	
	Upto 45	above 45
Hamilton	37	11
Madison	48	2

Test whether the data occurence of 'to' vary significantly between the two authors by applying the χ^2 test, and also by applying the formulae for exact probability.

Sol. 1→ Let A denotes the rate of occurence of 'to'.

Let B_1, B_2 denote the Authors : Hamilton and Madison.

	A_1	A_2	
B_1	$a = 37$	$c = 11$	48
B_2	$b = 48$	$d = 2$	50
	85	13	$N = 98$

To test Ho: A and B are independent.

Applying Yates continuity correction to Pearsonian χ^2, we get

$$\chi^2 = \frac{\left\{|ad - bc| - \frac{N}{2}\right\}^2}{(a+d)(c+d)(a+c)(b+d)} \sim \chi^2_1.$$

Critical region:- Observed $\chi^2 > \chi^2_{\alpha,1}$.

Exact Prob. Test:- Under H_0, given marginals, the prob. of obtaining the cell frequency $\boxed{\begin{matrix} a & c \\ b & d \end{matrix}}$ is

$$p_d = \frac{\binom{a+c}{a}\binom{b+d}{b}}{\binom{N}{a+b}} = \frac{\lfloor a+c \; \lfloor b+d \; \lfloor a+b \; \lfloor c+d}{\lfloor N \; \lfloor a \; \lfloor b \; \lfloor c \; \lfloor d}.$$

p-value $= p_2 + p_1 + p_0$ [since here $d = 2$]

p value $< \alpha \Rightarrow$ reject H_0.

$p_2 \rightarrow$

37	11	48
48	2	50
85	13	98

$p_1 \rightarrow$

36	12	48
49	1	50
85	13	98

$p_0 \rightarrow$

35	13	48
50	0	50
85	13	98

10. The variance of the stature in cm for males in 8 different capital through out the world are given below. Examine whether the variance differs from capital to capital. If not, find the pooled estimates of the variance.

Capital	Sample size	Variance of the stature (in cm^2)
1	299	38.8306
2	77	24.8924
3	131	39.5068
4	59	24.4685
5	124	24.3490
6	170	32.4325
7	337	39.2262
8	139	33.4436

Sol. Let σ_i be the popl^n s.d. of the i^{th} capital.
Assuming the popl^n distributions are normal,

$$\ln s_i \overset{a}{\sim} N\left(\ln \sigma_i, \frac{1}{2n_i}\right), i = 1(1)8.$$

$$\Rightarrow \sum_{i=1}^{8} \sqrt{2n_i}(\ln s_i - \ln \hat{\sigma})^2 \sim \chi^2_7, \text{ under } H_0: \sigma_1 = \sigma_2 = \cdots = \sigma_8 = \sigma.$$

where, $\ln \hat{\sigma} = \frac{\sum 2n_i \ln s_i}{\sum 2n_i}$ = pooled estimate

Critical region: observed $\chi^2 > \chi^2_{\alpha,7}$.

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11. The correlation between height and weight was formed to be 0.7, 0.8, 0.95 for samples of thousands object each for three different ethenic groups of children aged between 1 to 3 years. Test whether there is significant evidence for the dependence of the correlation on ethencity. If not, find the pool estimate of the common correlation.

Sol. Let ρ_i be the correlation coefficient between height and weight of the i^{th} ethnic group, $i=1,2,3$.

Assume, the popln. distn.s are BN.

To test $H_0: \rho_1=\rho_2=\rho_3$ ag. H_1: not H_0.

Here, $\sqrt{n_i-3}\,(Z_i-\xi_i) \overset{a}{\sim} N(0,1)$, $i=1,2,3$, independently.

where, $Z_i = \frac{1}{2}\log\left(\frac{1+r_i}{1-r_i}\right) =$

$\xi_i = \frac{1}{2}\log\left(\frac{1+\rho_i}{1-\rho_i}\right) =$

Under $H_0: \rho_1=\rho_2=\rho_3=\rho$,

$$\sum_{i=1}^{3}(n_i-3)(Z-\hat{\xi})^2 \sim \chi^2_{3-1}, \text{ where } \hat{\xi} = \frac{\sum(n_i-3)Z_i}{\sum(n_i-3)} = \bar{Z}.$$

Critical region:- Observed $\chi^2 > \chi^2_{\alpha;2}$

□ If $H_0: \rho_1=\rho_2=\rho_3=\rho$, the pooled estimate of the common correlation ρ is given by

$$\hat{\xi} = \bar{Z}$$

$$\Rightarrow \frac{1}{2}\log\left(\frac{1+\hat{\rho}}{1-\hat{\rho}}\right) = \bar{Z}$$

$$\Rightarrow \hat{\rho} = \frac{e^{2\bar{Z}}-1}{e^{2\bar{Z}}+1} = \underline{\qquad\qquad}.$$

12. Sample of sizes 75, 125, 100, 150, 130 from 5 poisson popln. give the mean values as 115/75, 197/125, 141/100, 237/150, 185/130, respectively. Do the popln have the same mean value?

Solution:-

To test $H_0: \lambda_1=\lambda_2=\lambda_3=\lambda_4=\lambda_5$

Under H_0: $\lambda_i=\lambda$, $i=1(1)5$.

$$4n_i(\sqrt{x_i}-\sqrt{\lambda})^2 \sim \chi^2_{5-1}.$$

where $\sqrt{\hat{\lambda}} = \sum 4n_i\sqrt{\bar{x}_i} \big/ \sum 4n_i$

Problems on Statistical Inference

1. The following observations (Y) are drawn from $N(\theta, \theta^k)$. Obtain the MLE of θ in each case $k = 0, 1,$ and 2. Are they unique?

20·2, 22·9, 23·3, 20·0, 19·4, 22·0, 22·1, 22·0, 21·9, 21·5, 19·7, 21·5, 20·9.

Hint:- Let $y_1, y_2, \ldots, y_n$ be a r.s. from $N(\theta, \theta^k)$, $k = 0, 1, 2$.

$\boxed{k=0}$ The MLE of θ is $\hat{\theta} = \bar{y} =$ ______ .

$\boxed{k=1}$ $N(\theta, \theta)$

$$\hat{\theta} = -1 + \frac{1 + \sqrt{\frac{4}{n}\sum y_i^2}}{2} = \text{______}$$

$\boxed{k=2}$ $N(\theta, \theta^2)$, $\theta \neq 0$

$$L(\theta \mid y_1, \ldots, y_n) = \left(\frac{1}{\sqrt{2\theta^2\pi}}\right)^n \cdot e^{-\frac{1}{2\theta^2}\sum_{i=1}^{n}(y_i-\theta)^2}, \quad \theta \neq 0$$

$$0 = \frac{\partial \ln L}{\partial \theta}$$

$$0 = -\frac{n}{\theta} + \frac{\sum y_i^2}{\theta^3} - \frac{\sum y_i}{\theta^2}.$$

$$\Rightarrow \theta^2 + \theta \cdot \bar{y} - \frac{1}{n}\sum y_i^2 = 0$$

$$\Rightarrow \hat{\theta} = \frac{-\bar{y} \pm \sqrt{\bar{y}^2 + \frac{4}{n}\sum y_i^2}}{2} = \text{______}, \text{______}$$

[MLE is not unique]

2. The general experiment on linsead, the observed frequencies for petal and stigma colour are given in the following table along with probabilities of the cell in terms of a parameter θ.

Stigma Colour	Petal colour: Lilac	Petal colour: deep lilac
White	$357\left(\frac{2+\theta}{4}\right)$	$33\left(\frac{1-\theta}{4}\right)$
Purple	$37\left(\frac{1-\theta}{4}\right)$	$94\left(\frac{\theta}{4}\right)$

Estimate θ by the method of MLE, [( ) → Probability]

Also estimate the variance of this estimate of θ.

Sol.

Likelihood function,

$$L(\theta \mid n_1, n_2, n_3, n_4) = \frac{\lfloor n}{\prod_{i=1}^{4} \lfloor n_i} \left(\frac{2+\theta}{4}\right)^{n_1} \left(\frac{1-\theta}{4}\right)^{n_2} \left(\frac{1-\theta}{4}\right)^{n_3} \left(\frac{\theta}{4}\right)^{n_4}$$

Likelihood equation:

$$0 = \frac{\partial \ln L}{\partial \theta} = \frac{n_1}{2+\theta} - \frac{n_2+n_3}{1-\theta} + \frac{n_4}{\theta}.$$

$$\Rightarrow n\theta^2 - \{n_1 - 2(n_2+n_3) - n_4\}\theta - 2n_4 = 0, \quad n = \sum_{i=1}^{4} n_i$$

$$\Rightarrow \hat{\theta} = \text{______}.$$

For large n,

$$\hat{\theta} \overset{a}{\sim} N\left(\theta, \frac{1}{I_n(\theta)}\right), \text{ where } \quad I_n(\theta) = E\left(-\frac{\partial^2}{\partial\theta^2}\ln L\right)$$

$$\therefore I_n(\theta) = E\left\{\frac{n_1}{(2+\theta)^2} + \frac{n_2+n_3}{(1-\theta)^2} + \frac{n_4}{\theta^2}\right\}$$

$$= n\left\{\frac{\frac{2+\theta}{4}}{(2+\theta)^2} + \frac{2\left(\frac{1-\theta}{4}\right)}{(1-\theta)^2} + \frac{\theta/4}{\theta^2}\right\}$$

$$= \frac{n}{4}\left\{\frac{1}{2+\theta} + \frac{2}{1-\theta} + \frac{1}{\theta}\right\}$$

Asymptotic variance is $Var(\hat{\theta}) = \frac{1}{I_n(\theta)}$.

$$\hat{var}(\hat{\theta}) \simeq \frac{1}{I_n(\hat{\theta})} = \frac{n}{4}\left\{\frac{1}{2+\hat{\theta}} + \frac{2}{1-\hat{\theta}} + \frac{1}{\hat{\theta}}\right\}$$

3) The length of life recorded in hours for 10 electron tubes were

980, 1020, 995, 1015, 990, 1030, 975, 950, 1050, 870.

Assume that life times are distributed in the form:

$$f(t,\theta) = \frac{1}{\theta}e^{-(t-\theta)}, \theta>0, 0<t<\infty.$$

Obtain the MLE of θ and the estimated standard error of this estimate. Estimate also the probability that an electron tube will survive at least 100 hours. Given estimate of the large sample standard error of this estimated probability. Determine the lower confidence limit. Confidence coefficient = 0.05, to the true prob. of survivor for 100 hours or more.

(i) By using the exact distn. of MLE of θ,

(ii) Assuming some approximate distn. for the estimated prob. of survivor.

Sol. The likelihood function is

$$L(\theta | t_1, t_2, \ldots, t_n) = \frac{1}{\theta^n} e^{-\sum_{i=1}^{n} t_i/\theta}, \theta>0.$$

Here $n=10$.

Likelihood equation is $\quad 0 = \frac{\partial}{\partial\theta}\ln L = -\frac{n}{\theta} + \frac{\sum t_i}{\theta^2}$

$$\Rightarrow \hat{\theta} = \bar{t} = \_\_\_\_\_\_ \text{ is the MLE of } \theta.$$

Now, $S.E.(\hat{\theta}) = \sqrt{V(\bar{t})} = \sqrt{\frac{\theta^2}{n}} = \frac{\theta}{\sqrt{n}}$.

and $\hat{S.E.}(\hat{\theta}) = \frac{\hat{\theta}}{\sqrt{n}} = \frac{\bar{t}}{\sqrt{n}} = \_\_\_\_$

To estimate $p = P[T \geq 100] = e^{-100/\theta}$

MLE of p is $\hat{p} = e^{-100/\hat{\theta}} = \_\_\_\_\_$.

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For large sample,

$$\hat{\theta} \overset{a}{\sim} N\left(\theta, \frac{1}{I_n(\theta)}\right), \text{ where } I_n(\theta) = \frac{n}{\theta^2}.$$

$$\Rightarrow \hat{p} = e^{-100/\theta} \overset{a}{\sim} N\left(p, \frac{\left\{\frac{\partial}{\partial\theta} p\right\}^2}{I_n(\theta)}\right)$$

For large n,

$$Var(\hat{p}) = \frac{\left\{ p \cdot \frac{100}{\theta^2} \right\}^2}{\frac{n}{\theta^2}}.$$

$$\Rightarrow S.\hat{E}.(\hat{p}) = \frac{p \cdot \frac{100}{\theta^2}}{\sqrt{\frac{n}{\theta^2}}} = \frac{100p}{\sqrt{n}\,\theta}.$$

$$\text{Hence, } \hat{SE}(\hat{p}) \simeq \frac{100\hat{p}}{\hat{\theta}\sqrt{n}} = \_\_\_\_\_\_.$$

Confidence limit of p:-

(i) Exact C.I.:- $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} t_i$

Here $y_i = \frac{2t_i}{\theta} \overset{iid}{\sim} \chi^2_2$, $i = 1(1)n$.

$$\Rightarrow 2\frac{\Sigma t_i}{\theta} \sim \chi^2_{2n}$$

$$\Rightarrow \frac{2n\hat{\theta}}{\theta} \sim \chi^2_{2n}$$

$$\therefore P\left[\frac{2n\hat{\theta}}{\theta} < \chi^2_{\alpha, 2n}\right] = 1-\alpha$$

$$\Rightarrow P\left[\theta > \frac{2n\hat{\theta}}{\chi^2_{\alpha, 2n}}\right] = 1-\alpha.$$

$$\Rightarrow P\left[e^{-100/\theta} > e^{-\frac{\chi^2_{\alpha,2n}}{2n\hat{\theta}} \cdot 100}\right] = 1-\alpha.$$

$$\Rightarrow P\left[p > e^{-100 \cdot \frac{\chi^2_{\alpha,2n}}{2n\hat{\theta}}}\right] = 1-\alpha.$$

Lower confidence limit with confidence coefficien 0.95 is

$$p_L = e^{-\frac{100 \cdot \chi^2_{0.05, 20}}{20 \times 2 \times \hat{\theta}}} = \_\_\_\_\_\_.$$

(ii) Approximate confidence limit:-

$$\hat{p} \overset{a}{\sim} N\left(p, SE^2(\hat{p})\right)$$

$$\Rightarrow \frac{\hat{p}-p}{\hat{SE}(\hat{p})} \overset{a}{\sim} N(0,1).$$

Hence for large n,

$$P\left[\left|\frac{\hat{p}-p}{S.\hat{E}(\hat{p})}\right| < \tau_{\alpha/2}\right] = 1-\alpha.$$

$$\Rightarrow P\left[\hat{p} - \tau_{\alpha/2} \cdot S.\hat{E}(\hat{p}) < p < \hat{p} + \tau_{\alpha/2} \cdot S.\hat{E}.(\hat{p})\right] = 1-\alpha.$$

4) At time $t=0$, 20 identical components are put on test. The life distn. of each is exponential with mean θ, after 24 hours. We were found that 15 of the 20 components are still working. Derive the MLE of θ. Also give an estimate of the S.E. of the estimator.

Hint:- Here $T \sim$ Exp with mean θ

Let Y = the no. of bulbs survived upto 24 hours out of 20 bulbs.

Clearly, $Y \sim Bin(n=20, p)$; where $p = P[T>24] = e^{-24/\theta}$,

MLE of p is $\hat{p} = \frac{y}{n} = \frac{15}{20} = 0.75$

$$\Rightarrow e^{-24/\hat{\theta}} = \hat{p}$$

$$\Rightarrow \hat{\theta} = \frac{-24}{\ln(.75)} = \_\_\_\_\_ .$$

5) In a life testing experiment, 10 electric lamp are put to test. The lamps are burnt at a streech for 20 hours. 2 of this survive time to termination (in hours) of life. for the remaining lamps, are observed and given below

9.8, 15.8, 17.2, 11.2, 13.8, 18.9, 14.6, 19.6

find an estimate of the mean life θ assuming the life distn. to be exponential. If all the bulbs survive, what could have been your estimate of θ.

Sol. $T \sim$ Exp. with mean θ.

$$p = P[T>20] = e^{-20/\theta}$$

Let $x_1, \ldots, x_8$ be the lifetime of 8 lamps.

Likelihood function is $L(\theta) = \prod_{i=1}^{8} \left\{ \frac{1}{\theta} \cdot e^{-x_i/\theta} \right\} \cdot p^2$

$$= \frac{1}{\theta^8} \cdot e^{-\sum_{i=1}^{8} x_i/\theta} \cdot e^{-40/\theta}$$

$$= \frac{1}{\theta^8} \cdot e^{-\frac{8(\bar{x}+5)}{\theta}}$$

Likelihood equation is :- $0 = \frac{\partial}{\partial\theta} \ln L(\theta) = -\frac{8}{\theta} + \frac{8(\bar{x}+5)}{\theta^2}$

$$\Rightarrow \hat{\theta} = \bar{x} + 5 = \_\_\_\_\_ .$$

If all 10 bulbs survived, the likelihood equation is

$$L(\theta) = \left\{ e^{-20/\theta} \right\}^{10} = e^{-200/\theta}$$

$L(\theta)$ is max. iff $200/\theta$ is minimum.

iff θ is max.

Hence, MLE does not exist.

6) The following in the freq. distn. of 154 obsn. drawn at random from a multinomial popln. with 6 classes

Class no	1	2	3	4	5	6
No. of observation in the sample	79	32	5	6	17	15

denoting the probabilities of 6 classes by $\pi_1, \pi_2, \ldots, \pi_6$, respectively. Find the MLE of $\pi_1 - 2\pi_2 + 6\pi_5$.

Sol. → Let f_i, $i=1(1)6$ be the frequency of the i^{th} class in a r.s. of size $n=154$.

Likelihood function:-

$$L(\pi_i | f_i) = \frac{\lfloor n}{\prod_{i=1}^{n} \lfloor f_i} \prod_{i=1}^{6} \pi_i^{f_i}, \text{ where } \sum_{i=1}^{6} \pi_i = 1.$$

and $0 < \pi_i < 1$, $i=1(1)6$.

To maximine $\ln L$ subject to $\sum_{i=1}^{6} \pi_i = 1$

let, $F = \ln L + \lambda \left(\sum_{i=1}^{n} \pi_i - 1 \right)$

$$= \text{constant} + \sum_{i=1}^{6} f_i \cdot \ln \pi_i + \lambda \left(\sum_{i=1}^{n} \pi_i - 1 \right)$$

Solve:- $0 = \frac{\partial F}{\partial \pi_i} = \frac{f_i}{\pi_i} + \lambda$

$$\Rightarrow \pi_i = -\frac{f_i}{\lambda}.$$

and $1 = \sum \pi_i = \frac{-\sum f_i}{\lambda} = -\frac{n}{\lambda}$.

$$\Rightarrow -\frac{1}{\lambda} = \frac{1}{n}$$

$$\Rightarrow \hat{\pi}_i = \frac{f_i}{n}, \text{ are the MLE's.}$$

$$\Rightarrow \hat{\underset{\sim}{\pi}} = \left(\frac{f_1}{n}, \frac{f_2}{n}, \ldots, \frac{f_6}{n} \right).$$

The MLE of $(\pi_1 - 2\pi_2 + 6\pi_5)$

$$= (1, -2, 0, 0, 6, 0) \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \\ \pi_6 \end{pmatrix}$$

$$= \underset{\sim}{l}' \underset{\sim}{\pi}$$

is $\underset{\sim}{l}' \hat{\underset{\sim}{\pi}} = \hat{\pi}_1 - 2\hat{\pi}_2 + 6\hat{\pi}_5$.

7) The IQ's of 10 tenagers are
98, 114, 105, 101, 123, 117, 106, 92, 110, 108, where as those of 6 tenagers belonging to other ethnic group are
122, 105, 95, 126, 114, 108.
Assuming that the data can be looked upon as a independent r.s. from Normal popln. with mean μ_1 and μ_2 and common variance σ^2. Estimate the parameters by the method of MLE.

Sol. Let $X_1, X_2, \dots, X_{10} \overset{iid}{\sim} N(\mu_1, \sigma^2)$
$Y_1, Y_2, \dots, Y_6 \overset{iid}{\sim} N(\mu_2, \sigma^2)$ independent.

Likelihood Equation:-

$$L(\mu_1, \mu_2, \sigma \mid x_i, y_j) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^{16} \cdot \exp\left[-\frac{\Sigma(x_i-\mu_1)^2 + \Sigma(y_j-\mu_2)^2}{2\sigma^2}\right]$$

Likelihood Equation:-

$$① \rightarrow 0 = \frac{\partial \ln L}{\partial \mu_1} = -\frac{2\Sigma(x_i-\mu_1)(-1)}{2\sigma^2}.$$

$$\Rightarrow \mu_1 = \bar{x}.$$

$$② \rightarrow 0 = \frac{\partial \ln L}{\partial \mu_2} \Rightarrow \mu_2 = \bar{y}.$$

$$③ \rightarrow 0 = \frac{\partial \ln L}{\partial \sigma} = \frac{-16}{\sigma} + \frac{\Sigma(x_i-\mu_1)^2 + \Sigma(y_j-\mu_2)^2}{\sigma^3}$$

$$\Rightarrow \hat{\sigma}^2 = \frac{\Sigma(x_i-\hat{\mu}_1)^2 + \Sigma(y_j-\hat{\mu}_2)^2}{16} = \_\_\_\_\_.$$

8) Score obtained from 10 students in two parallel forms of a test of a test are recorded below:
Assuming that score in the two sets arise from a bivariate normal popln. with the common mean μ, common s.d. σ and the corrln. coeff. ρ, find the MLE of μ, σ, ρ.

1st form	2nd form
41	47
52	54
49	50
44	48
32	40
62	65
40	38
45	42
39	45
48	45

Sol. Let (X_i, Y_i), $i=1(1)10$, be a r.s. from $BN(\mu, \mu, \sigma^2, \sigma^2, \rho)$

Let $U_i = X_i + Y_i \overset{iid}{\sim} N\left(2\mu, 2\sigma^2(1+\rho)\right) = N\left(\mu_U, \sigma_U^2\right)$

$V_i = X_i - Y_i \overset{iid}{\sim} N\left(0, 2\sigma^2(1-\rho)\right)$, $i=1(1)n$, indep.

$\equiv N\left(0, \sigma_V^2\right)$

MLEs:- (i) $\begin{cases} \hat{\mu}_U = \bar{u} \\ 2\hat{\mu} = \bar{u} = \bar{X} + \bar{Y} \\ \Rightarrow \hat{\mu} = \dfrac{\bar{X}+\bar{Y}}{2}. \end{cases}$

(ii) $\hat{\sigma}_U^2 = s_u^2 = \frac{1}{n}\sum(u_i - \bar{u})^2 = \frac{1}{n}\sum\left(x_i - \bar{x} + y_i - \bar{y}\right)^2$

$= \frac{1}{n}\sum_i\left[(x_i-\bar{x})^2 + (y_i-\bar{y})^2 + 2(x_i-\bar{x})(y_i-\bar{y})\right]$

$\Rightarrow 2\hat{\sigma}^2(1+\hat{\rho}) = s_x^2 + s_y^2 + 2s_{xy}$.

(iii) $\hat{\sigma}_V = s_V^2 = \frac{1}{n}\sum(v_i - \bar{v})^2 = \frac{1}{n}\sum_i\left([x_i - \bar{x}]^2 + [y_i - \bar{y}]^2 - 2(x_i-\bar{x})(y_i-\bar{y})\right)$

$\Rightarrow 2\hat{\sigma}^2(1-\rho) = s_x^2 + s_y^2 - 2s_{xy}$

(ii) + (iii) gives

$$4\hat{\sigma}^2 = 2\left(s_x^2 + s_y^2\right)$$

$$\Rightarrow \hat{\sigma}^2 = \frac{s_x^2 + s_y^2}{2} = \_\_\_\_\_\_.$$

(ii) - (iii) gives

$$4\hat{\sigma}^2\hat{\rho} = 4s_{xy}$$

$$\Rightarrow \hat{\rho} = \frac{2s_{xy}}{s_x^2 + s_y^2} = \_\_\_\_\_\_$$

9. Suppose heights of 6 pairs of identical adult male bengali films, there observed that

5′6″; 5′5″; 6′1″; 5′11″; 5′7″; 5′8″; 6′2″; 6′; 5′4″; 5′3″; 5′5″; 5′3″;

Suppose it is known that average height of adult bengali is 5′5″ and s.d. is 3′3″. Find the MLE of ρ.

Assuming that pairs of height follow a BN distn., with unknown corr. coeff. ρ.

Sol.→ Assume that, $(X, Y) \sim BN(\mu, \mu, \sigma, \sigma, \rho)$

$\mu = 5'5''$

$\sigma = 3'3''$

and ρ is unknown.

$$\Rightarrow (U, V) = \left(\frac{x-\mu}{\sigma}, \frac{Y-\mu}{\sigma}\right) \sim BN(0, 0, 1, 1, \rho)$$

Likelihood function is :—

$$L(\rho \mid (u_i, v_i)) = \left\{\frac{1}{2\pi\sqrt{1-\rho^2}}\right\}^n \cdot e^{-\frac{1}{2(1-\rho^2)}\left\{\Sigma u_i^2 + \Sigma v_i^2 - 2\rho \Sigma u_i v_i\right\}}$$

Likelihood equation:-

$$0 = \frac{\partial}{\partial \rho}\ln L = -\frac{n(-2\rho)}{2(1-\rho^2)} - \frac{1}{2(1-\rho^2)}\left\{-2\Sigma u_i v_i\right\} + \frac{\Sigma u_i^2 + \Sigma v_i^2 - 2\rho \Sigma u_i v_i}{2(1-\rho^2)^2}(-2\rho)$$

$$\Rightarrow \frac{\rho}{(1-\rho^2)}\left\{n + \frac{1}{\rho}\Sigma u_i v_i - \frac{1}{1-\rho^2}\left(\Sigma u_i^2 + \Sigma v_i^2 - 2\rho \Sigma u_i v_i\right)\right\} = 0$$

$$\Rightarrow n\rho^3 - \rho^2 \Sigma u_i v_i + \rho\left(\Sigma u_i^2 + \Sigma v_i^2 - n\right) - \Sigma u_i v_i = 0$$

$$\Rightarrow \rho^3 + a\rho^2 + b\rho + c = 0, \text{ say.}$$

Use some numerical method of solution.

10. Estimate the value of θ by the ML method and find out 95% C.I. on the basis of this sample

3.58, 2.86, 3.16, 2.46, 6.33, 8.31, 15.17, 9.59

drawn from the cauchy population —

$$PDF(x) = \frac{1}{\pi\left\{1+(x_i-\theta)^2\right\}}, \ \theta \in \mathbb{R}, \ -\infty < x_i < \infty.$$

<u>Sol.</u> $$L(\theta \mid x_1, \ldots, x_n) = \frac{1}{\pi^n \cdot \prod_{i=1}^{n}\left\{1+(x_i-\theta)^2\right\}}, \ \theta \in \mathbb{R}.$$

Likelihood equation:-

$$0 = \frac{\partial}{\partial \theta}\ln L = \sum_{i=1}^{n}\frac{2(x_i-\theta)}{1+(x_i-\theta)^2} = f(\theta), \text{ say.}$$

By NR method,

$$\theta_{n+1} = \theta_n - \left\{\frac{f(\theta)}{f'(\theta)}\right\}_{\theta=\theta_n}$$

$$= \theta_n + \left\{\frac{\frac{\partial \ln L}{\partial \theta}}{-\frac{\partial^2 \ln L}{\partial \theta^2}}\right\}_{\theta=\theta_n}$$

$$\simeq \theta_n + \left\{\frac{\frac{\partial \ln L}{\partial \theta}}{I_n(\theta)}\right\}_{\theta=\theta_n}$$, replacing $-\frac{\partial^2 \ln L}{\partial \theta^2}$ by its expectation $I_n(\theta)$.

Here $I_n(\theta) = \frac{n}{2}$.

$$\theta_{n+1} = \theta_n + \frac{4}{n}\sum_{i=1}^{4}\frac{(x_i-\theta_n)}{1+(x_i-\theta_n)^2}$$

Trial root $\theta_1 = \tilde{x}$ = the sample median = ________.

$\theta_2 =$ ________

$\theta_3 =$ ________

etc. $\hat{\theta} =$ ________.

For large sample,

$$\hat{\theta} \overset{a}{\sim} N\left(\theta, \frac{1}{I_n(\theta)}\right)$$

$$\Rightarrow \hat{\theta} \overset{a}{\sim} N\left(\theta, \frac{2}{n}\right)$$

$$\Rightarrow 1-\alpha = P\left[\left|\frac{\hat{\theta}-\theta}{\sqrt{\frac{2}{n}}}\right| < \tau_{\alpha/2}\right]$$

$$= P\left[\hat{\theta} - \sqrt{\frac{2}{n}}\tau_{\alpha/2} < \theta < \hat{\theta} + \sqrt{\frac{2}{n}}\tau_{\alpha/2}\right]$$

11. To test a null hypothesis $H_0: p = 0.6$ of a popln. $\sim B(m=5, p)$ vs. $H_1: p > 0.6$. Suggest an UMP test with exact size α. the choice of α is left to you. If the observed no. of success is 4, construct a randomised test of exact size 0.1 and conclude.

Sol. To test $H_0: p = 0.6$ vs. $H_1: p > 0.6$

Let x be an observation from $B(m=5, p)$

MP test of its size of testing $H_0: p = p_0$ Vs. $H_1: p = p_1$, $p_1 > p_0 = 0.6$ is

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \frac{f(x, p_1)}{f(x, p_0)} > c \\ 0 & \text{ow} \end{cases}$$

$$= \begin{cases} 1 & , x > k \\ 0 & , \text{ow} \end{cases}$$

where, $P_{H_0}[\phi(\underset{\sim}{X})] = \alpha$ (say)

and the test $\phi(x)$ is independent of $p_1 (> p_0)$.

Hence, the test $\phi(x)$ is UMP of $H_0: p = 0.6$ vs. $H_1: p > 0.6$ of its size.

Let $k = 4$,

Then $$\phi(x) = \begin{cases} 1 & , x > 4 \\ 0 & , \text{ow} \end{cases}$$

is a UMP test of size $= E_{H_0}[\phi(X)] = P_{H_0}[X > 4]$

$$= \binom{5}{5}(0.6)^5(0.4)^0$$

$$= 0.07776$$

If $\phi(x) = \begin{cases} 1 & , x > 3 \\ 0 & , \text{ow} \end{cases}$

then size $= P_{H_0}[X > 3]$

$$= \binom{5}{4}(0.6)^4(0.4)^1 + (0.6)^5$$

$$= 0.2592 + 0.07776 > 0.1$$

To get the exact size 0.1, it is required to randomize at $x=4$.

Let $\phi(x) = \begin{cases} 1 & , x > 4 \\ \gamma & , x = 4 \\ 0 & , \text{ow} \end{cases}$

where γ is such that $0.1 = E_{H_0}[\phi(X)]$

$$= 1 \cdot P_{H_0}[X > 4] + \gamma \cdot P_{H_0}[X = 4]$$

$$= 0.07776 + \gamma(0.2592)$$

$$\Rightarrow \gamma = 0.086$$

$\therefore \quad \phi(x) = \begin{cases} 1 & , x > 4 \\ 0.086 & , x = 4 \\ 0 & , x < 4 \end{cases}$

is the UMP test at level $\alpha = 0.1$.

Here $x = 4$ is the observed value.

Hence, we reject H_0 with probability $\gamma = 0.086$ and accept H_0 with prob. $1-\gamma$.

Draw a 3-digited random number, then the prob. of $R = \{$the selected no. $\leq 0.85\}$ is $P(R) = \frac{86}{1000}$.

Let the selected no. be 126.

Then we accept $H_0: p = 0.6$ at level 0.1.

12. Suppose the waiting time denoted by x in minutes for a bus is uniformly distributed over $U[0,\theta]$. To test the hypothesis $H_0: \theta = 10$ vs. $H_1: \theta > 10$ on the basis of a sample of size 6 to decision rules are proposed. Reject H_0 if i) $\max\{x_1, \ldots, x_6\} > c_1$ (ii) (no. of $\{x_1, \ldots, x_6\} > 8) > c_2$

Find the values of c_1 and c_2 taking level of significance is 0.05. Also, draw power curve for both these procedure and comment on the relative performance.

Sol. Let $X_1, \ldots, X_6 \overset{iid}{\sim} R(0,\theta)$

To test $H_0: \theta = 10$ vs. $H_1: \theta \neq 10$

(a) $\phi_1(x) = \begin{cases} 1 & \text{if } x_{(6)} > c_1 \\ 0 & \text{ow} \end{cases}$

(b) $\phi_2(x) = \begin{cases} 1 & \text{if } y > c_2 \\ 0 & \text{ow} \end{cases}$

where Y = the no. of x_i's which are greater than 8.

$\therefore Y \sim Bin(6, p)$.

$\therefore p = \int_8^\theta \frac{1}{\theta} d\theta = \left(1 - \frac{8}{\theta}\right) = P[X_i > 8]$

Power function:-

(a) $\beta_1(\theta) = P_\theta[X_{(6)} > c_1]$

$= 1 - P_\theta[X_{(6)} \leq c_1]$

$= 1 - \left(\frac{c_1}{\theta}\right)^6$.

(b) $\beta_2(\theta) = P_\theta[Y > c_2]$

$= \sum_{y=c_2+1}^{6} \binom{6}{y} \left(1 - \frac{8}{\theta}\right)^y \left(\frac{8}{\theta}\right)^{6-y}$

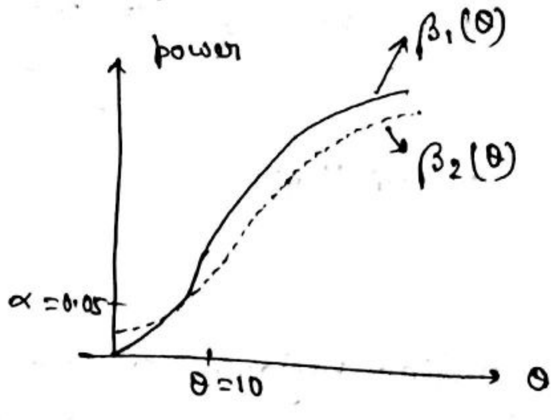
Here $\alpha = 0.05$,

& $0.05 = \beta_1(\theta)\Big|_{\theta=10} = 1 - \left(\frac{c_1}{10}\right)^6$

$\Rightarrow c_1 = 10 \cdot (0.95)^{1/6}$.

and $0.05 = \beta_2(\theta)\Big|_{\theta=10}$

$= \sum_{y=c_2+1}^{6} \binom{6}{y} \left(\frac{1}{5}\right)^y \left(\frac{4}{5}\right)^{6-y}$



Comment:- the power of the test (a) which is based on exact X_i's is greater than that of the test (b) which is based on Bernoulli trial.

—: Non-Parametric Inference:—

13. The following are 15 measurements of the octane rating of a certain kind of Gasoline

97.5, 95.2, 97.3, 96.0, 96.8, 100.3, 97.4, 95.3, 93.2, 99.1, 96.1, 97.6, 98.2, 98.5, 94.9.

Use an exact non-parametric test and also an approximate test to examine whether or not the average octane rating of the given kind of Gasoline is 98.5.

Sol. X : the measurement of the octane of gasoline.

Let $x_1, x_2, \ldots, x_n$, $n = 15$ be given random samples.

To test $H_0: \xi_{1/2} = 98.5$ Vs. $H_1: \xi_{1/2} \neq 98.5$

(a) Exact Non-Parametric Test:- Count the no. of +ve signs among $x_1 - \xi_0, \ldots, x_n - \xi_0$.

We have two +ve sign and 12 -ve sign and one observation x_{14} is $= 98.5$. We ignore the value x_{14} from the sample.

Now, we have a r.s. of size $n = 14$.

Let Y = the no. of +ve sign among $(X_i - \xi_0)$'s, $i = 1(1)14$.

Under H_0, $Y \sim Bin(n=14, p=\frac{1}{2})$.

The observed value of Y is $y_0 = 2$.

$$\text{The p-value} = 2 \cdot \min\{P_{H_0}[Y \geq y_0],\ P_{H_0}[Y \leq y_0]\}$$
$$= 2 \cdot P_{H_0}[Y \leq y_0]$$
$$= 2 \cdot \sum_{y=0}^{2} \binom{14}{y} \cdot \frac{1}{2^{14}} = \underline{\qquad}.$$

If p-value < 0.05, we reject H_0, at 5% level of significance.

(b) Assume that the distn. of X is $N(\mu, \sigma^2)$, whether the actual distn. is normal or not.

Here, $\xi_{1/2} = \mu$.

To test $H_0: \mu = 98.5$ Vs. $H_1: \mu \neq 98.5$

t-test:- $$t = \frac{\sqrt{n}(\bar{x} - 98.5)}{s} \sim t_{n-1}, \text{ under } H_0.$$

$n = 15$.

Computation:-

Critical region:- Observed $|t| > t_{\alpha/2, 14}$.

The test procedure described is an approximate test procedure, as the actual distn. of x may not be normal.

14. The following table shows Hamilton depression scale factor measurements in 9 patients suffering from depression, taken before (X) and after (Y) a visit to therapist :

X	1.83	0.50	1.62	2.48	1.68	1.88	1.55	3.06	1.3
Y	.878	.647	.598	2.05	1.06	1.29	1.06	3.14	1.29

Perform a suitable (a) Parametric (b) non parametric test to judge whether the therapy can be considered to be effective.

Sol. (a) Parametric:-

Assume that $(X,Y) \sim BN(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$

To judge whether the therapy is considered to be effective, i.e.

$H_0: \mu_x = \mu_y$ alternative $H_1: \mu_x > \mu_y$.

This is a paired-t testing problem.

Define $d_i = x_i - y_i \overset{iid}{\sim} N(\mu_d, \sigma_d^2)$,

where $\mu_d = \mu_x - \mu_y$.

Test statistic $\dfrac{\sqrt{n}(\bar{d} - 0)}{s_d} \sim t_{n-1}$, under H_0.

Computation:-

	1	2	...	9
d_i				

$\bar{d} =$ ______

$s_d^2 = \frac{1}{n-1}\left\{\sum d_i^2 - n\bar{d}^2\right\}$

$=$ ______.

Critical region:- Observed $t > t_{\alpha,8}$.

(b) Non-Parametric:-

Here $d_i = x_i - y_i$, $i = 1(1)9$.

And we are interested in testing H_0: median$(d) = 0$ Vs. H_1: median$(d) > 0$

Test:- $H_0: \xi_{1/2}(d) = 0$ Vs. $H_1: \xi_{1/2}(d) > 0$

We shall use sign test based on the values $d_1, d_2, \ldots, d_9$ as a r.s. from an absolutely continuous univariate distn.

Let Z = the no. of +ve d_i's

$\sim Bin(n=9, p=1/2)$, under H_0.

The observed value of Z is $Z_0 =$ ______.

p-value $= P_{H_0}[Z \geq Z_0] =$ ______.

15. Conduct the parametric and non-parametric test for differences of location for the data given below;

loop I : 9 7 12 16 14

loop II : 5 2 8 4 20

Sol. Let $X_1, \ldots, X_m$, $m=5$ and $Y_1, \ldots, Y_n$, $n=5$ be the given r.s. from gr-I, gr-II, respectively.

Parametric test:- (Fisher's t-test)

Assumption:- $x_1, \ldots, x_m \overset{iid}{\sim} N(\mu_x, \sigma_x^2)$, $y_1, \ldots, y_n \overset{iid}{\sim} N(\mu_y, \sigma_y^2)$ independently distributed

Assume that $\sigma_x = \sigma_y$

To test: $H_0: \mu_x = \mu_y$ vs. $H_1: \mu_x \neq \mu_y$

Fisher's t-statistic:-
$$t = \frac{\bar{x} - \bar{y}}{s\sqrt{\frac{1}{m} + \frac{1}{n}}} \sim t_{m+n-2}.$$

Non-parametric test:- (Median Test)

Assume that the distn. of the two groups are absolutely continuous and independent.

To test $H_0: \xi_{1/2}(x) = \xi_{1/2}(y)$ vs. $H_1: \xi_{1/2}(x) \neq \xi_{1/2}(y)$

Let $\tilde{Z}$ be the median in the combined data set.

Define, V = the no. of x_i's which are $\leq \tilde{Z}$

Under H_0,
$$P[V=v] = \frac{\binom{m}{v}\binom{n}{p-v}}{\binom{m+n}{p}}, \quad v = 0(1)n$$
$m+n = 10 = 2p$
$\therefore p = 5$.

Combine the data in an increasing order, we have

2, 4, 5, 7, 9, 12, 14, 16, 20.

$$\tilde{Z} = \frac{8+9}{2} = 8.5.$$

The observed of $V = 1 = v$

$$\therefore \text{p-value} = 2 \min\{P_{H_0}[V \geq v_0], P_{H_0}[V \leq v_0]\}$$
$$= 2 P_{H_0}[V \leq v_0 = 1]$$
$$= 2 \cdot \frac{\binom{5}{v}\binom{n}{5-v}}{\binom{10}{5}}$$

ANOVA

1. The following measurements refer to the no. of hours in which 9 patients are free from pain after taking placebo, a new drug and aspirins:

Group	Observations			
Placebo	0.0	1.0		
New drug	2.3	3.5	2.8	2.5
Aspirin	3.1	2.7	3.8	

(i) Test whether the new drug is more effective than the others

(ii) Test if the effects of the new drug is the same as the average effect of the other two.

Sol. The factor considered here is drug (A) with levels Placebo (A_1), New drug (A_2) and Aspirin (A_3). The data given is one-way classified data. Let y_{ij} denotes the j^{th} observations in the i^{th} level or group, $i=1,2,3$, $j=1(1)n_i$, $n_1=2$, $n_2=4$, $n_3=3$.

Model [Fixed Effects]:-

$$y_{ij} = \mu + \alpha_i + e_{ij} \quad \text{with } \sum n_i\alpha_i = 0.$$

where $e_{ij} \overset{iid}{\sim} N(0, \sigma_e^2)$, $i=1(1)3$, $j=1(1)n_i$.

Here $\mu_i = \mu + \alpha_i$ is mean-effect (fixed) of the i^{th} level.

To test $H_0: \alpha_1 = \alpha_2 = \alpha_3 = 0$.

Computation:-

$$G = \sum_{i=1}^{3}\sum_{j=1}^{n_i} y_{ij} = \_\_\_\_\_\_.$$

$$C.F. = \frac{G^2}{n} = \_\_\_\_\_\_ \quad \text{with } n=9.$$

$$SS(\text{Total}) = \sum\sum y_{ij}^2 - CF = \_\_\_\_\_\_.$$

$$SS(\text{between}) = \sum_{i=1}^{3} \frac{T_{i0}^2}{n_i} - CF = \_\_\_\_\_\_$$

$$= \frac{T_{10}^2}{n_1} + \frac{T_{20}^2}{n_2} + \frac{T_{30}^2}{n_3} - CF = \_\_\_\_.$$

where $T_{i0} = \sum_{j=1}^{n_i} y_{ij}$

ANOVA Table:-

Source of Variation	d.f.	SS	MS	F	
Between groups.	3-1=2	SS (between) =	MSB	$F = \frac{MSB}{MSE}$	$F_{0.05;\,2,6}$ = ___
Within groups [Error]	9-3=6	SSE = ___	MSE		
Total =	9-1=8	SS(total) =			

If observed $F > F_{0.05;2,6}$, reject H_0.

(i) To test $H_{01}: \mu_2 = \mu_1$, Vs. $H_{01}': \mu_2 > \mu_1$

and $H_{02}: \mu_2 = \mu_3$ Vs $H_{02}': \mu_2 > \mu_3$.

To test H_{01}, the test statistic

$$\frac{\bar{y}_{20} - \bar{y}_{10}}{\sqrt{MSE\left(\frac{1}{n_2} + \frac{1}{n_1}\right)}} \sim t_6 \text{, under } H_{01}.$$

Reject H_{01}, if observed $t > t_{\alpha;6}$.

To test H_{02},

the test statistic is $t_1 = \dfrac{\bar{y}_{20} - \bar{y}_{30}}{\sqrt{MSE\left(\frac{1}{n_2} + \frac{1}{n_3}\right)}}$

(ii) To test $H_{03}: \mu_2 = \dfrac{\mu_1 + \mu_3}{2}$.

$\Leftrightarrow H_{03}: \mu_1 - 2\mu_2 + \mu_3 = 0$

Vs. $H_{03}': \mu_1 - 2\mu_2 + \mu_3 \neq 0$.

Test statistic $(t) = \dfrac{\bar{y}_{10} - 2\bar{y}_{20} + \bar{y}_{30}}{\sqrt{MSE\left(\frac{1}{n_1} + \frac{4}{n_2} + \frac{1}{n_3}\right)}} \sim t_6$, under H_{03}.

Critical region: observed $|t| > t_{\alpha/2, 6}$.

2. 4 randomly selected typist work on each of 4 given type writers of different make and speeds were recorded wise of typiest and typewriter.

Type - writer	Typiest			
1	30, 32	40, 30	25, 40	20, 30
2	40, 35	32, 25	20, 30	31, 21
3	20, 31	25, 25	26, 30	25, 21
4	27, 28	31, 24	30,	19, 25

Analyse the data and give estimate of the parameters.

Sol. Type writer [factor A] : A_1, A_2, A_3, A_4

Typists [factor B] : B_1, B_2, B_3, B_4

The effects of different levels of A are fixed. As 4 typists are selected randomly, there effects are random, the data given is a two-way classification data with 2 obs.n. per cell. Let y_{ijk} be the k^{th} obsn. corresponding to the i^{th} level of A and j^{th} level of B.

$k = 1(1)m = 2$, $i = 1(1)p = 4$; $j = 1(1)q = 4$.

Model:- [Two way classified data with mixed effects]

$$y_{ijk} = \mu + \underset{\text{(fixed)}}{\alpha_i} + \underset{\text{(random)}}{b_j} + c_{ij} + e_{ijk}$$

where $\sum_{i=1}^{p} \alpha_i = 0$, $\sum_{i=1}^{p} c_{ij} = 0$, $j = 1(1)q$

and $\{b_j\}$, $\{c_{ij}\}$ and $\{e_{ijk}\}$ are jointly normal,

$$e_{ijk} \overset{iid}{\sim} N(0, \sigma_e^2).$$

Define, $\sigma_A^2 = \frac{1}{p-1} \sum_{i=1}^{p} \alpha_i^2$

$$\sigma_B^2 = Var(b_j)$$

$$\sigma_{AB}^2 = \frac{1}{p-1} \sum_{i=1}^{p} V(c_{ij})$$

To test $H_0: \sigma_A^2 = 0$ and to estimate the parameters σ_e^2, σ_B^2, σ_{AB}^2.

Computation:- $G = \sum_i \sum_j \sum_k y_{ijk} =$ ______

C.F. $G^2/n =$ ______

$n = pqm = 32$.

$$SS(\text{Total}) = \sum_i \sum_j \sum_k y_{ijk}^2 - C.F.$$

Defined, $T_{i00} = \sum_j \sum_k y_{ijk}$

$$T_{0j0} = \sum_i \sum_k y_{ijk}$$

$$T_{ij0} = \sum_k y_{ijk}.$$

A \ B	B_1	B_2	B_3	B_4	Total
A_1	$T_{110}=$ ___	$T_{120}=$ ___	$T_{130}=$ ___	$T_{140}=$ ___	$T_{100}=$ ______
A_2	$T_{210}=$ ___	$T_{220}=$ ___			$T_{200}=$ ______
A_3					
A_4					
	$T_{010}=$	$T_{020}=$			$G=$ ______

$$SSA = \sum_{i=1}^{p} \frac{T_{i00}^2}{qm} - CF = \_\_\_\_\_\_$$

$$SSB = \sum_{j} \frac{T_{0j0}^2}{pm} - CF = \_\_\_\_\_\_$$

$$SS(AB) = \sum_i \sum_j \frac{T_{ij0}^2}{m} - \sum_i \frac{T_{i00}^2}{qm} - \sum_j \frac{T_{0j0}^2}{pm} + C.F.$$

$$= \left(\sum_i \sum_j \frac{T_{ij0}^2}{m} - CF \right) - SSA - SSB.$$

ANOVA Table

Source of Variation	d.f.	SS	MS	E(MS)
Due to A (fixed)	$p-1=3$	SSA	$MSA = \frac{SSA}{3}$	$\sigma_e^2 + m\sigma_{AB}^2 + qm\sigma_A^2$
Due to B (random	$q-1 = 3$	SSB	$MSB = \frac{SSB}{3}$	$\sigma_e^2 + pm\sigma_B^2$
Due to AXB	$(p-1)(q-1) = 9$	SS(AB)	$MS(AB) = \frac{SS(AB)}{9}$	$\sigma_e^2 + m\sigma_{AB}^2$
Error	$pq(m-1) = 16$	SSE	$MSE = \frac{SSE}{16}$	$\sigma_e^2 = \_\_\_$
Total	$pqm-1 = 31$	SST		

$$F_A = \frac{MSA}{MS(AXB)}.$$

If observed $F > F_{\alpha;3,9}$, reject $H_A: \sigma_A^2 = 0$.

[If H_A is rejected, give $\hat{\alpha}_i = \bar{y}_{i00} - \bar{y}_{000} = \_\_\_\_\_\_$]

The estimates of the parameters $\hat{\sigma}_e^2 = MSE = \_\_\_\_\_\_$

$$\hat{\sigma}_{AB}^2 = \frac{MS(AB) - MSE}{m} = \_\_\_\_\_\_$$

$$\hat{\sigma}_B^2 = \frac{MSB - MSE}{pm} = \_\_\_\_\_\_.$$

8. In an experiment on yield of sugar beets (tons/acre) there were two levels of irrigation treatment and three of fertilizer treatment and each combination of treatments was carried out in 5 replicants. The following values of sum of squares (SS) were obtained : SS (Irrigation) = 120.0 ; SS (Fertilizer) = 221.7 ; SS (Interaction) = 35.0 ; SS (Error) = 108.0 ;

Assuming that the irrigation & fertilizer effects are random, estimate the components of variance. Also, test whether the variance of the irrigation component is zero.

Sol. Here two effects are : Irrigation (A) : A_1 A_2

Fertilizer (B) : B_1 B_2 B_3

Both the effects are random.

It is two way classified data with m=5 obsn. per cell.

$p=2, q=3, m=5.$

Model:- $y_{ijk} = \mu + a_i + b_j + c_{ij} + e_{ijk}$, where

$a_i \overset{iid}{\sim} N(0, \sigma_A^2)$

$b_j \overset{iid}{\sim} N(0, \sigma_B^2)$

$c_{ij} \overset{iid}{\sim} N(0, \sigma_{AB}^2)$ are independent.

$e_{ijk} \overset{iid}{\sim} N(0, \sigma_e^2)$

Note that, $\sigma^2 = V(y_{ijk}) = \sigma_A^2 + \sigma_B^2 + \sigma_{AB}^2 + \sigma_e^2$.

ANOVA MODEL:-

Source of variation	Df	SS	MS	E(MS)
due to A	2−1=1	120	120	$\sigma_e^2 + m\sigma_{AB}^2 + qm\sigma_A^2$
due to B	3−1=2	221.7	110.85	$\sigma_e^2 + m\sigma_{AB}^2 + pm\sigma_B^2$
due to AB	2	35	17.5	$\sigma_e^2 + m\sigma_{AB}^2$
Error	24 = pq(m−1)	108	4.5	$\sigma_e^2 = 4.5$
Total	29 = pqm − 1			

Here $\hat{\sigma}_e^2 = MSE = 4.5$

$$\hat{\sigma}_{AB} = \frac{MS(AB) - MSE}{m} = \frac{17.5 - 4.5}{5} = 2.6$$

$$\hat{\sigma}_A^2 = \frac{MSA - MS(AB)}{qm} = \frac{120 - 17.5}{18} = \_\_\_$$

$$\sigma_B^2 = \frac{MSB - MS(AB)}{pm} = \frac{110.85 - 17.5}{10} = \_\_\_ .$$

Reject $H_0: \sigma_A^2 = 0$ if observed $F = \frac{MSA}{MS(AB)} > F_{\alpha;1,2}$.

DESIGN

1. In the experiment described below 4 materials were tested in each of 4 runs on a machine with 4 different position. The letters A to D refer to the 4 materials. The layout of the expt. is given below. Here the figures denote the loss in weight in a run of standard length.

	Position in machine			
Run	4	2	1	3
2	A (251)	B (241)	D (227)	C (229)
3	D (234)	C (237)	A (274)	B (226)
1	C (235)	D (236)	B (218)	A (268)
4	B (195)	A (270)	C (230)	D (225)

(a) Analyse the data and comment.

(b) If the variation due to the different position of the machine is ignored, will you modify your conclusion?

Sol. Factor A (Row): Factor runs are the four levels A

Factor B (Column): Four positions of the machine are the four levels of B.

Treatment: A, B, C, D.

(a) Experiment is conducted according to the LSD with four treatments A, B, C, D.

Let y_{ijk} be the obsn. of the k^{th} treatment in the $(i,j)^{th}$ cell.

Model:-

$$y_{ijk} = \mu + \alpha_i + \beta_j + \tau_k + e_{ijk}$$

with $\sum \alpha_i = \sum \beta_j = \sum \tau_k = 0$,

$$e_{ijk} \overset{iid}{\sim} N(0, \sigma_e^2)$$

To test: $H_0: \tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$.

Computation:-

$$G = \sum_i \sum_j \sum_k y_{ijk} = \_\_\_\_\_$$

$$C.F. = G^2/n, \quad m = 4.$$

$T_{i00} \to$ row totals

$T_{0j0} \to$ column totals

$T_{00k} \to$ treatment totals

$$SS(row) = \sum_{i=1}^{m} \frac{T_{i00}^2}{m} - CF = \_\_\_\_\_.$$

$$SS(column) = \sum_{j=1}^{m} \frac{T_{0j0}^2}{m} - CF$$

$$SS(treatment) = \sum_k \frac{T_{00k}^2}{m} - CF$$

ANOVA Table

Source of Variation	d.f.	SS	MS
row	$m-1=3$	SSR	
column	$=3$	SSC	
Treatment	$=3$	SS(tr)	MS(tr)
Error	6	—	MSE
Total =	$m^2-1=15$	SS(Total)	

$F = \frac{MS(tr)}{MSE}$

If observed $F > F_{\alpha;3,6}$, we reject H_0.

If H_0 is rejected, then, the different treatment has different effects in general.

(b) If the variation due to different position of the machine is ignored that is columns are ignored, then corresponding 4 rows as 4 blocks, the design of experiment reduced to RBD.

Let y_{ik} be the observation on the k^{th} treatment in the i^{th} block (row).

Model:- $y_{ik} = \mu + \alpha_i + \tau_k + e'_{ik}$

where, $e'_{ik} \overset{iid}{\sim} N(0, \sigma_e'^2)$.

$$\sum_i \alpha_i = 0 = \sum_k \tau_k$$

To test $H_0: \tau_k = 0$, $k = 1(1)4$.

Computation:- SS(Block) = SS(row)

SS(treatment) in RBD is same as the SS(tr) in LSD.

But SS(column) is added to SSE of LSD to get the SSE* in RBD.

ANOVA table

Source of Variation	d.f.	SS.
Row	$m-1=3$	SSR
Treatment	3	SS(tr)
Error	$3+6=9$	$SSE^* = SSE + SS(column)$
Total	$m^2-1=15$	SS(total)

$F^* = \frac{MS(tr)}{MSE^*}$

If observed $F^* > F_{0.05;\,3,9}$, reject H_0.

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2) In order to compare the hardness of alloy three furnaces (F) and three levels of moulds (M) were tried. The lay out as well as the hardness (in suitable unit) are shown below:

Rep-I

F_1	M_3 156	M_2 118	M_1 140
F_2	M_1 130	M_3 174	M_2 157
F_3	M_1 114	M_2 161	M_3 141

Rep II

M_2 104	M_1 89	M_3 117
M_3 112	M_1 89	M_2 81
M_1 103	M_2 132	M_3 133

Rep III

M_1 103	M_2 126	M_3 149
M_3 144	M_1 124	M_2 129
M_3 100	M_1 91	M_2 97

Analyse the data.

Sol. **Split Plot Design:-**

factor A : $(A_1, A_2, \ldots, A_p)$

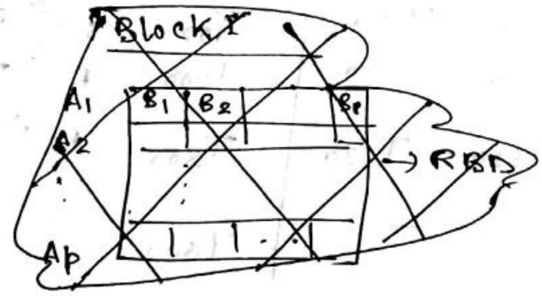
↳ they exclude use of small plots

i.e. they are applied on large plots.

→ their effects are known to be different / effect of A_i's are to be tested with less precision.

Another factor:- $B_1, B_2, \ldots, B_q$.

→ they are applicable small plots and we want test B and A×B more accurately than A.



F denoted by F_1, F_2, F_3, M denoted by M_1, M_2, M_3.
Here $p=3$, levels of F arranged is a R.B.D. using $r=3$ blocks or replicable and the factor M at $q=3$ levels, are applied to the plots of a block after subdividing each plot into $q=3$ subplots. This is a split plot design.

Model:- $$y_{ijk} = \underbrace{\mu + b_i + \tau_j + e_{ij}}_{\text{model for whole plot treatment F by RBD}} + \underbrace{\gamma_k + \delta_{jk} + e_{ijk}'}_{\text{model for subplot treatment in whole plot.}}$$

$i=1(1)r=3$, $j=1(1)p=3$, $k=1(1)q=3$.

$$\sum \tau_j = \sum \gamma_k = \sum_k \delta_{jk} = \sum_j \delta_{jk} = 0$$

	I		
F_3	M_3	M_2	M_1
F_1	M_1	M_3	M_2
F_2			

where, b_i, e_{ij}, e_{ijk} are independently normal $(0, \sigma_b^2; \sigma_e^2; \sigma_{e'}^2)$.

To test $H_{01}: \tau_j = 0$

$H_{02}: \gamma_k = 0$

$H_{03}: \delta_{jk} = 0$

Computation:—

$$G_1 = \sum_i \sum_j \sum_k y_{ijk} = \_\_\_\_\_\_$$

$$C.F. = G^2/rpq$$

$$SS(total) = \sum\sum\sum y_{ijk} - C.F = \_\_\_\_$$

Whole Plot Analysis:—

$$SS(Block) = \sum_{i=1}^{r} \frac{T_{i00}^2}{pq} - C.F.$$

$$SS(F) = \sum \frac{T_{0j0}^2}{rq} - C.F.$$

$$SSE_I = \left\{ \sum_i \sum_j \frac{T_{ij0}^2}{pq} - C.F. \right\} - SSB - SSF$$

Whole Plot analysis Table:—

Replicate \ F	F_1	F_2	F_3	Total	
I	$T_{110} = 461$	T_{120}	T_{130}	T_{100}	R-I) $F_1(M_1+M_2+M_3) = T_{110}$
II	T_{210}	T_{220}	T_{230}	T_{200}	R-II) $F_1(M_1+M_2+M_3) = T_{210}$
III	T_{310}	T_{320}	T_{330}	T_{300}	R-I) $F_2(M_1+M_2+M_3) = T_{120}$
				T_{000}	

Split plot Analysis:—

$$SS(M) = \sum \frac{T_{00k}^2}{rp} - CF$$

$$SS(F \times M) = \left(\sum \frac{T_{0jk}^2}{r} - CF \right) - SSF - SSM$$

Table for SS (FXM)

F \ M	M_1	M_2	M_3	
F_1	T_{011}	T_{012}	T_{013}	T_{010}
F_2	T_{021}	T_{022}	T_{023}	T_{020}
F_3	T_{031}	T_{032}	T_{033}	T_{030}
	T_{001}	T_{002}	T_{003}	T_{000}

$F_1(M_1+M_1+M_1) = T_{011}$ under R_1-I, II, III

$F_1(M_2+M_2+M_2) = T_{012}$ under R-I, II, III.

ANOVA Table

Source of Variation	df	SS	MS	F
Replicates	$r-1=2$		$MS(F)$	$F_1 = \frac{MSF}{MSE_I}$
whole plot treat (F)	$p-1=2$		$MS(E_I)$	
Error (I)	$(r-1)(p-1)=9$		$MS(M)$	$F_2 = \frac{MSM}{MSE_{II}}$
Subplot treat (M)	$q-1=2$		$MS(F\times M)$	$F_3 = \frac{MS(F\times M)}{MSE_{II}}$
Interaction (F×M)	$(p-1)(q-1)=4$		$MS(E_{II})$	
Error (II)	$p(q-1)(r-1)=12$			
total	$df = rpq-1 = 26$			

$F_1 \leftrightarrow F_{\alpha;2,4}$

$F_2 \leftrightarrow F_{\alpha;2,12}$

$F_3 \leftrightarrow F_{\alpha;4,12}$.

SAMPLE SURVEY

(1). (a) Draw a random sample of size 7 from an exp. popln with mean 2·345.

(b) Draw a r.s. of size 5 from a Cauchy popln. with median 0 and scale 2.

(c) Draw a r.s. of size 6 from a univariate normal popln. with mean 17·95 and s.d. 6·23.

(d) Draw a r.s. of size 5 from the distn.

$P[X=0]=\frac{1}{5}$, $P[X=1]=\frac{2}{5}$, $P[X=2]=\frac{2}{5}$.

Solution:- (a) Let $X \sim Exp(\theta = 2\cdot345)$

$$\left[\ f_X(x) = \frac{1}{\theta} e^{-x/\theta} \qquad F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = 1 - e^{-x/\theta}\ \right]$$

D.F. of X is

$F(x) = 1 - e^{-x/\theta}$, $x>0$

By probability integral transformation,

$U = F(X) \sim R(0,1)$.

If u is an observed sample from $R(0,1)$, then $U = F(x)$

$$\Rightarrow u = 1 - e^{-x/\theta}.$$

$$\Rightarrow x = -\ln(1-u)$$

$= -2\cdot345 \ln(1-u)$ is an observed sample from Exp. with mean $\theta = 2\cdot345$.

We take seven 3-digited random number from Fisher-Yates table.

Page No: 125, row = 5, column = 4

260, 573, 375, 204, 056, 930, 001.

We place decimal points before the selected nos.

Then 0·260, 0·573, 0·375, 0·204, 0·056, 0·930, 0·001 are 7 r.s. from R(0,1).

Serial No.	u_i	$x_i = -2\cdot345\ln(1-u_i)$
1		
2		
3		
4		
5		
6		
7		

(b) $\left[f_X(x) = \dfrac{\sigma}{\pi\{\sigma^2 + (x-\mu)^2\}} \right.$

$\Rightarrow F_X(x) = \displaystyle\int_{-\infty}^{x} f_X(t)\,dt$

$= \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x-\mu}{\sigma}\right)$. Here $\mu = 0, \sigma = 2.\left.\right]$

$F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x}{2}\right)$

where $x \sim R(0,1)$

Here $u = F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x}{2}\right)$.

$\Rightarrow x = 2\tan\left\{\pi\left(u - \frac{1}{2}\right)\right\}$

$= 2\tan x\,\{\pi u - \pi/2\}$

$= -2\cot \pi u.$

(c) $X \sim N(17.95, (6.23)^2)$

$\Rightarrow Z = \dfrac{X - 17.95}{6.23} \sim N(0,1)$

Hence, $U = \Phi\left(\dfrac{X - 17.95}{6.23}\right) \sim R(0,1)$

$\Rightarrow u = \Phi\left(\dfrac{x - 17.95}{6.23}\right)$

$\Rightarrow \Phi^{-1}(u) = \dfrac{x - 17.95}{6.23}$ $\rightarrow$ From Biometrica.

(d) Draw a r.s. of size 5 from the distn

$P[X=0] = \frac{1}{5}$, $P[X=1] = \frac{2}{5}$, $P[X=2] = \frac{2}{5}$.

Let us define the r.v. $Y \sim U(0,1)$.

So, $P[0 < Y < .2] = .2 = P[X=0]$

$P[.2 \le Y < .6] = .4 = P[X=1]$

$P[.6 \le Y < 1] = .4 = P[X=2]$

So, we take 5 3-digited random no. If the no. selected is in between (000 – 199) then we consider is eaeivalent to choosing $X=0$, if the no. is in between (200 – 599), we eaeivalently choose $X=1$, and finally, if the selected no. is in between (600–999), then we choose $X=2$.

(2). For the cost function a) $C = C_0 + \sum C_h n_h$
b) $C = C_0 + \sum C_h \sqrt{n_h}$.

where C_0 and C_1 are known constants. Find the optimum values of n_h's by minimizing $Var(\bar{y}_{st})$ for fixed total cost with total sample size being 60, given the following data:

Stratum	1	2	3	4
N_h	30	40	60	70
S_h	1.5	2.0	3.5	4.0
C_h	1	2	3	.4

Also, compute the optimum allocation due to Neyman & compare the efficiency of the optimum allocations with that of proportional allocation to estimate the population mean $\bar{Y}$.

Solution:- (a) Define, $F = \frac{1}{N^2} \sum N_h^2\left(\frac{1}{n_h} - \frac{1}{N_h}\right) S_h^2 + \lambda\left(C_0 + \sum_{h=1}^{L} C_h n_h - C\right)$

Now, $\frac{\partial F}{\partial n_h} = 0$

$$\Rightarrow \frac{1}{N^2} N_h^2 S_h^2 \left(-\frac{1}{n_h^2}\right) + \lambda . C_h = 0$$

$$\Rightarrow n_h = \lambda_1 \frac{N_h . S_h}{\sqrt{C_h}}$$

Given $\sum_h n_h = 60$

$$\Rightarrow \lambda_1 = \frac{60}{\sum_h \frac{N_h S_h}{\sqrt{C_h}}} = \_\_\_\_\_.$$

Stratum	N_h	S_h	C_h	$\frac{N_h S_h}{\sqrt{C_h}}$	$n_h = \lambda_1 \cdot \frac{N_h S_h}{\sqrt{C_h}}$
1					
2					
3					
4					

(b) Define $F = \frac{1}{N} \sum_{h=1}^{L} N_h^2\left(\frac{1}{n_h} - \frac{1}{N_h}\right) \cdot S_h^2 + \lambda\left(C_0 + \sum_{h=1}^{L} C_h n_h^{1/2} - C\right)$

Now, $0 = \frac{\partial F}{\partial n_h}$

$$\Rightarrow \frac{1}{N^2} \cdot N_h^2 . S_h^2 \left(-\frac{1}{n_h^2}\right) + \lambda . \frac{C_h}{2\sqrt{n_h}} = 0$$

$$\Rightarrow n_h = \lambda_1^* \left(\frac{N_h S_h}{\sqrt{C_h}}\right)^{4/3}$$

Given $\sum_h n_h = 60$ $\Rightarrow \lambda_1^* = \frac{60}{\sum_h \left(\frac{N_h S_h}{\sqrt{C_h}}\right)^{4/3}} = \_\_\_\_\_.$

Stratum	N_h	S_h	C_h	$\frac{N_h S_h}{\sqrt{C_h}}$	$n_h = \lambda . \frac{N_h S_h}{\sqrt{C_h}}$	$\left(\frac{N_h S_h}{\sqrt{C_h}}\right)^{4/3}$	$n_h = \lambda_1^* \left(\frac{N_h S_h}{\sqrt{C_h}}\right)^{4/3}$

Neyman's optimum and proportional allocations:-

Here $n_h \propto \frac{N_h S_h}{\sqrt{C_h}}$.

① If C_h = constant, then $n_h = \lambda_2 N_h S_h$ and $60 = \sum n_h$.

$$\Rightarrow \lambda_2 = \frac{60}{\sum N_h S_h}.$$

$$\therefore n_h = \left(\frac{60}{\sum N_h S_h}\right) . N_h S_h .$$

$\longrightarrow$ optimum allocation.

② Here C_h = constant

S_h = constant

$n_h = \lambda_3 . N_h$

and $\lambda_3 = \frac{60}{\sum N_h} = \frac{60}{N}$.

$\therefore n_h = \left(\frac{60}{N}\right) N_h \rightarrow$ proportional allocation.

③. Ques:- Using the information given below, find the relative standard error that one could expect if a sample of 1% of the viallagers group of district the steps be selected by SRSWOR for estimating the total population Y of the region.

District serial No.	No. of Villages (N_h)	Average pop.ln for Village ($\bar{Y}_h$)	Standard deviation (S_h)
1	1953	487	564
2	1864	829	931
3	1381	822	996
4	1174	1083	1167
5	531	1956	1940
6	1391	664	625
7	1996	456	779
8	1951	372	856
9	3369	339	591

Solution:- Relative standard Error $= RSE(\hat{Y}_{st})$

$$= \frac{S.E(\hat{Y}_{st})}{\hat{Y}_{st}} = \frac{N . SE(\hat{\bar{Y}}_{st})}{N . \hat{\bar{Y}}_{st}}$$

$$= \frac{\left\{\frac{1}{N^2}\sum N_h^2 S_h^2\left(\frac{1}{n_h} - \frac{1}{N_h}\right)\right\}^{1/2}}{\frac{\sum N_h . \bar{Y}_h}{N}}.$$

$$\left[\text{Here } \frac{n_h}{N_h} = \frac{1}{100}\right]$$

$$= \frac{\sqrt{\sum N_h \left(N_h - \frac{N_h}{100}\right) \cdot \frac{S_h^2}{\frac{N_h}{100}}}}{\sum N_h \cdot \bar{Y}_h}$$

$$= \frac{\sqrt{99 \sum N_h S_h^2}}{\sum N_h \bar{Y}_h} = \text{———}$$

District	N_h	$\bar{Y}_h$	S_h	$N_h \bar{Y}_h$	$N_h S_h^2$
1					
2					
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9					

(3) All the farms of a country have been stratified according to size and the following data have been obtained.

Farm Size	No. of Farm (N_h)	Mean ($\bar{Y}_h$)	S.D. (S_h)
0 – 40	394	5.4	8.3
41 – 80	464	16.3	13.3
81 – 120	391	24.3	15.1
121 – 160	334	34.5	19.8
161 – 200	169	42.1	24.5
201 – 240	113	50.1	26.0
241 – 280	148	63.8	35.2

If the first 3 stratum combine into a 1 stratum and last 4 into another. Find the relative loss in efficiency as compared to 7 strata situation in estimating the pop'n mean by stratified Random sampling (a) Proportional allocation (b) optimum allocation that the total no. of farms are selected 100.

Sol. In the new stratified pop'n

Farm Size	No. of firms	$\bar{Y}_h'$	$S_h'^2$
0 – 120	$N_1' = \sum_{h=1}^{3} h_n$		
121 – 280	$N_2' = \sum_{h=4}^{7} h$		

where $\bar{Y}_1' = \dfrac{\sum_{h=1}^{3} N_h \bar{Y}_h}{\sum_{h=1}^{3} N_h}$, $\bar{Y}_2' = \dfrac{\sum_{h=4}^{7} N_h \bar{Y}_h}{\sum_{h=4}^{7} N_h}$

and $S_1'^2 = \dfrac{1}{\sum_{h=1}^{3} N_h - 1} \left\{ \sum_{h=1}^{3} (N_h - 1) S_h^2 + \sum_{h=1}^{3} N_h (\bar{Y}_h - \bar{Y}_1')^2 \right\}$

and $S_2'^2 = \dfrac{1}{\sum_{h=4}^{7} N_h - 1} \left\{ \sum_{h=4}^{7} (N_h - 1) S_h^2 + \sum_{h=4}^{7} N_h (\bar{Y}_h - \bar{Y}_2')^2 \right\}$

(a) $V_{prop} = \dfrac{1-f}{n} \sum_{h=1}^{7} \dfrac{N_h}{N} \cdot S_h^2$ (Given stratification)

and $V_{prop}' = \dfrac{1-f}{n} \sum_{h=1}^{2} \dfrac{N_h'}{N} S_h'^2$ (New stratification)

Loss in eff $= \dfrac{V_{prop}' - V_{prop}}{V_{prop}} = \_\_\_\_\_\_$.

(b) $V_{opt} = \dfrac{\left(\sum_{h=1}^{7} \frac{N_h}{N} \cdot S_h \right)^2}{n} - \dfrac{\sum_{h=1}^{7} \frac{N_h}{N} \cdot S_h^2}{N}$

$V_{opt}' = \dfrac{\sum_{h=1}^{2} \frac{N_h'}{N} \cdot S_h'^2}{n} - \dfrac{\sum_{h=1}^{2} \frac{N_h'}{N} \cdot S_h'^2}{N}$.

Loss in eff $= \dfrac{V_{opt}' - V_{opt}}{V_{opt}} = \_\_\_\_\_\_$.

4 (Ratio - Regression Estimator)

An experimenter makes an farmer's eye-estimate of the weight of peaches on each tree in a orchard of 200 trees. He finds a total weight of 11600 lbs and weight for a SRS of 10 trees which yield the following result;

Serial No. of tree :	1	2	3	4	5	6	7	8	9	10
Actual weight :	61	42	50	58	67	45	39	57	71	53
Estimated is :	59	47	52	60	67	48	44	58	76	58

Compute the ratio and regression estimates of the total actual weight of Peaches of all the 200 trees in the orchard and compare the precisions of 2 estimates.

Solution:— Y = total weight, X = an eye estimate

To estimate the "the weight of Peaches (y)" of all the 200 trees.

Here auxiliary information "an eye estimate of weight of Peaches (x)" is given.

$$X = 11600 \text{ lbs.}$$

Data: (x_i, y_i), $i = 1(1)n$.

(a) Ratio estimation:— Ratio estimate $\hat{Y}_R = \frac{\hat{y}}{\hat{x}} \cdot X = \frac{\bar{y}}{\bar{x}} \cdot X$

where $\bar{y} = \frac{1}{n} \sum y_i =$ ____ , $\bar{x} = \frac{1}{n} \sum x_i =$ ____ .

$\Rightarrow \hat{Y}_R =$ ____ .

$$MSE(\hat{Y}_R) \simeq N^2 \left(\frac{1}{n} - \frac{1}{N}\right)\left(s_y^2 + s_x^2 - 2\hat{R} s_{xy}\right)$$

where $s_x^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$, $s_y^2 = \frac{1}{n-1} \sum (y_i - \bar{y})^2$,

$s_{xy} = \frac{1}{n-1} \sum (x_i - \bar{x})(y_i - \bar{y})$, $\hat{R} = \frac{\bar{y}}{\bar{x}} =$ ____ .

(b) Regression Estimation:—

Regression estimate, $\hat{Y}_{lr} = N \cdot \hat{\bar{Y}}_{lr} = N\{\bar{y} + b(\bar{X} - \bar{x})\}$

$$= N\bar{y} + b(X - N\bar{x})$$

$$= \text{____}$$

$b = \frac{s_{xy}}{s_x^2} =$ ____ ,

$$MSE(\hat{Y}_{lr}) \simeq N^2 \left(\frac{1-f}{n}\right) s_e^2; \quad s_e^2 = s_y^2(1-\rho^2) = s_{y.x}^2$$

where $s_{y.x}^2 = \frac{1}{n-2} \sum_i \left(y_i - \bar{y} - b(x_i - \bar{x})\right)^2$

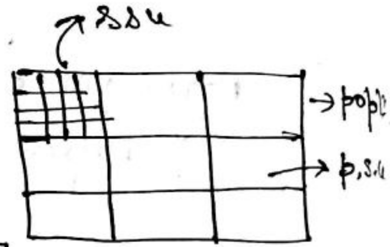
$$= \frac{1}{n-2}\left\{\sum (y_i - \bar{y})^2 - b^2 \sum (x_i - \bar{x})^2\right\}$$

$$= \frac{n-1}{n-2}\left\{s_y^2 - b^2 s_x^2\right\}$$

$$= s_e^2.$$

Comparison:— $\text{Eff}(\text{regression}/\text{ratio}) = \frac{\hat{MSE}(\hat{Y}_R)}{\hat{MSE}(\hat{Y}_{lr})} =$ ____ .

(5). Two - stage - cluster :→ In an experimental investigation, 100 fields each consisting of 16 plots of equal size, were sown with wheat. Out of 100 fields, 10 fields are selected by SRSWOR and out of each field so selected 4 plots are selected by WOR to observe the yield. From the given observation following values are estimated. Sample mean (in kg) for the selected fields are: 4.290, 4.255, 3.795, 4.220, 4.070, 3.636, 4.550, 4.285, 4.375, 3.790. Sample average with variance = .018215 (kg)^2. Estimate the total yield of wheat in the experimental station along with its standard error. Compare the efficiency of this estimate with on that would have been obtained by selecting an SRSWOR of 40 plots out of 1600 plots in the station.



Sol. [First we select n p.s.u from N. p.s.u. From selected n p.s.u., with in each p.s.u. have M units, we select m units.]

$N = 100$ fields (p.s.u.), each consisting of $M = 16$ plots (s.s.u.). Out of N p.s.u., $n = 10$ fields and out of each selected (p.s.u.) fields $m = 4$ plots (s.s.u.) are selected under SRSWOR.

Let y_{ij} be the value obtained for the j^{th} selected s.s.u. in the i^{th} selected p.s.u., $i = 1(1)10 = n$; $j = 1(1)4 = m$.

An unbiased estimate of Y (total yield) is

$$\hat{Y} = N \cdot \bar{\bar{y}} = N\left(\frac{1}{n}\sum_{i=1}^{n}\bar{y}_i\right), \quad \bar{y}_i = \frac{1}{m}\sum_{j=1}^{m} y_{ij}.$$

$$= \underline{\qquad\qquad}.$$

$$V(\hat{Y}) = (MN)^2\left\{\frac{1-f_1}{n}\cdot S_b^2 + \frac{1-f_2}{mn}\cdot S_w^2\right\}$$

and

$$\hat{V}(\hat{\bar{y}}) = (MN)^2\left\{\frac{1-f_1}{n}\cdot s_b^2 + \frac{f_1(1-f_2)}{mn}\cdot s_w^2\right\}$$

where

$$s_b^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\bar{y}_i - \bar{\bar{y}})^2 = \underline{\qquad\qquad}$$

and

$$s_w^2 = \frac{1}{n(m-1)}\sum_{i=1}^{n}\sum_{j=1}^{m}(y_{ij} - \bar{y}_i)^2 = \underline{\qquad}.$$

$$\hat{V}(\hat{Y}) = \underline{\qquad\qquad}.$$

In case of SRSWOR, with sample of size $mn = 40 = n'$. from $MN = 1600 = N'$ plots.

$$V_{SRS}(\hat{Y}) = N'^2 \cdot \frac{N' - n'}{N'n'} \cdot s_y^2,$$

$$s_y^2 = \frac{\sum_{i=1}^{n} \sum_{j=1}^{m} (y_{ij} - \bar{\bar{y}})^2}{nm-1}.$$

$$= \frac{\sum_i \sum_j (y_{ij} - \bar{y}_i)^2 + m \sum_{i=1}^{n} (\bar{y}_i - \bar{\bar{y}})^2}{nm-1}.$$

$$= \frac{n(m-1) \cdot s_w^2 + m(n-1) \cdot s_b^2}{mn-1}.$$

$$= \underline{\qquad\qquad\qquad}.$$

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