STOCHASTIC PROCESSES

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STOCHASTIC PROCESS

☆ **Stochastic Process:** A family or collection of RV's $\{X_\theta \mid \theta \in \Theta\}$ (where Θ is parameter space and all the events are defined on the same probability space $(\Omega, \mathcal{A}, P)$; where Ω is the sample space whose subsets constitute the σ-field $\mathcal{A}$ which generates events and P is a probability measure defined on Ω) is called a Stochastic Process.

Loosely speaking, a collection or family of RV's is called a random process or Stochastic Process.

Example:- Toss a coin indefinitely and let X_n denote the outcome of the nth throw $(n = 1, 2, \ldots)$; $\{X_n \mid n = 1, 2, \ldots\}$ is a stochastic process.

We shall usually consider stochastic process which can be represented as the model of a system that evolves randomly over time.

☆ **State Space: (S)** The set of all possible values of all the r.v.'s X_θ for $\theta \in \Theta$ is called the state space of the stochastic process $\{X_\theta \mid \theta \in \Theta\}$ and is denoted by S.

Note that for each $\theta \in \Theta$; $X_\theta : \Omega \to A_\theta$, where A_θ is the set of all possible values of X_θ. Thus $S = \bigcup_{\theta \in \Theta} A_\theta$.

Elements of S are called 'states'.

Note that S may be discrete (finite or countably infinite) or continuous (uncountably infinite); accordingly we have discrete state or continuous state process.

Also S may be one-dimensional or multidimensional.

When $S = \{0, 1, 2, \ldots\}$, we refer to the SP as a non negative integer valued process.

When $S = (-\infty, \infty) = \mathbb{R}$, we refer to the SP as a real valued process.

When $S \subseteq \mathbb{R}^k$, we refer to the SP as a k-valued process.

☆ Parameter Space: (Θ) The set of all possible values of the index parameter, i.e., Θ is called the parameter space of the stochastic process $\{X_\theta \mid \theta \in \Theta\}$.

Note that the parameter space can be discrete or continuous and accordingly, we have a discrete parameter or continuous parameter process. Also, Θ may be one-dimensional or multidimensional.

Example: (Where S or Θ is multidimensional)

[A]. Consider $X(t) = (m(t), M(t))$; $t \in [0, \infty)$, where $m(t)$ and $M(t)$ are minimum and maximum temperature of a place in $[0, t)$. $\{X(t) \mid t \in [0, \infty)\}$ is a SP in which S is two dimensional.

[B]. Consider the case of waves in oceans. We may regard the latitude and longitude coordinates (θ_1, θ_2) as the value of $\underset{\sim}{\theta}$ and define $X_{\underset{\sim}{\theta}}$ as the height of the wave at location $\underset{\sim}{\theta}$. $\{X_{\underset{\sim}{\theta}} \mid \underset{\sim}{\theta} \in \Theta\}$ is a SP in which Θ is two dimensional.

Time Parametric Stochastic Process:-

Stochastic process for which the index set Θ is:

Either $\Theta = \{0, 1, 2, \dots\}$ or $\Theta = [0, \infty)$

are of particular importance in application.

In such a case, we denote Θ by T and SP $\{X_t \mid t \in T\}$ is called a time parametric SP where t usually represents time parameter. (though it may represent such characters as distance/length etc)

(a) Discrete Time Parameter Stochastic Process (DTPSP):

$$\{X_n \mid n = 0, 1, 2, \dots\}$$

$$\Theta = T = \{0, 1, 2, \dots\}$$

Here T corresponds to discrete moments of time and X_t could denote the outcome of successive trials of a random Experiment.

Eg:- The result of successive tosses of a coin.

(b) Continuous Time Parameter Stochastic Process :- (CTPSP)

$$\{X_t \mid t \in [0,\infty)\}$$

Here $\Theta = T = [0,\infty)$

Here T corresponds to continuous moments of time and X_t could denote the outcome at different moments of a random experiment.

Eg: The total amount of data uploaded/downloaded from the internet in a computer during the time period $[0,t)$.

▨ The main elements distinguishing SP's are in the nature of the state space and the parameter space.

Case	State Space (S)	Parameter Space (Θ)	Nomenclature
1.	Discrete	Discrete	Dis. Valued Dis. Parameter SP (DVDPSP)
2.	Continuous	Continuous	Con. Valued Con. Parameter SP (CVCPSP)
3.	~~D~~Continuous	Discrete	CVDPSP
4.	Discrete	Continuous	DVCPSP

Examples:- $\boxed{1}$. (DVDPSP) Consider the expt. of throwing a true six faced dice.

$\boxed{\text{DIE ROLLING}}$

X_n = Outcome of the nth throw / Total no. of 6's in the first n throws

$\{X_n \mid n = 1, 2, \ldots\}$ is a DVDPSP with

$S = \{1,2,3,4,5,6\}$ / $\{0,1,2,3,\ldots\}$

and $\Theta = T = \{1, 2, \ldots\}$

Another eg: Let X_n: No. of claims submitted to an insurance company during the nth week. $\{X_n \mid n = 1, 2, \ldots\}$ is a DVDPSP with $S = \{0,1,2,\ldots\}$ and $\Theta = T = \{1,2,\ldots\}$.

$\boxed{2}$. (DVCPSP) Consider the Poisson Process.

$N(t)$ = Total no. of times a specified event '$\mathcal{E}$' occurs in $[0,t)$; $t>$

say, $N(t)$ is the total no. of telephone calls received by a person in $[0,t)$; where $t \geq 0$.

$\boxed{\text{TELEPHONE CALL AT TIME}}$

$\{N(t) \mid t \in [0,\infty)\}$ is a DVCPSP with $S = \{0,1,2,\ldots\}$ and $\Theta = T = [0,\infty)$.

[3]. (CVDPSP) Suppose $\{Z_1, Z_2, \dots\}$ denote the sequence of inter-arrival times in a queueing system. ④

Define $X_n = Z_1 + Z_2 + \cdots + Z_n$ (Time until the nth arrival); $n = 1, 2, \dots$

$\{X_n \mid n = 1, 2, \dots\}$ is a CVDPSP with $S = [0, \infty)$ and

$\Theta = T = \{1, 2, \dots\}$. $\boxed{\text{SUM OF INTER-ARRIVAL TIME}}$

[4]. (CVCPSP) Suppose $X(t)$ = Max. temperature recorded at a place in $[0, t)$. $\boxed{\text{TEMPERATURE AT A PLACE}}$

$\{X_t \mid t \in [0, \infty)\}$ is a CVCPSP with $S = [10^\circ C, 20^\circ C]$ and

$\Theta = T = [0, \infty)$.

A Simple Example of a Typical Problem in SP :-

X_n : The inventory of cars at a dealer at the end of the nth working day $(n = 1, 2, \dots)$

So, $\{X_n \mid n = 1, 2, \dots\}$ is a DVDPSP with $S = \{0, 1, 2, \dots\}$ and

$\Theta = T = \{1, 2, \dots\}$.

Suppose it costs Rs. 20/day to keep a car on the lot.

The study of the system may involve computing the total expected cost over the first three days, i.e., $E(20X_1 + 20X_2 + 20X_3)$.

One may also be interested in finding the long term average daily cost of inventory, quantified by $\lim_{n \to \infty} E\left[\frac{20(X_1 + X_2 + \cdots + X_n)}{n}\right]$

MARKOV PROCESS: Unless the contrary is stated explicitly, we shall assume that:

(1) The RV's X_θ, $\theta \in \Theta$ are all real valued, i.e., $S \subseteq \mathbb{R}$

(2) The parameter space is the time parametric space, i.e., $\Theta = T = \{0, 1, 2, \dots\}$ or $[0, \infty)$.

In the general definition of the SP, the independence assumption of the RV's X_θ, $\theta \in \Theta$ are not required. In such cases, we may consider a less stringent condition regarding the nature of dependence:

Future | Past and Present $\equiv$ Future | Present.

i.e., the probability of any future behaviour of the process depends on its present state and when the present state is known exactly, this probability is not altered by additional information concerning its past history.

Such processes are called Markov Processes and the stated property is called Markov Property.

A simple example of Markov Process is provided by $\{X_n \mid n=1,2\ldots\}$ where X_n = Total number of Heads in the first n tosses of a coin.

Note that,
$$\left.\begin{aligned} P(X_{n+1}=k+1 \mid X_n=k) &= p \\ P(X_{n+1}=k \mid X_n=k) &= 1-p \end{aligned}\right\} \text{ for } n=1,2,\ldots$$

A <u>markov process</u> is a random process in which the future is independent of the past, given the present.

Markov Chains:-

Suppose $\{X_n \mid n=0,1,2,\ldots\}$ is a sequence of RVs and if all the RV's X_n $(n=0,1,2,\ldots)$ are discrete, i.e., the state space S is discrete.

A markov process with a discrete state space is called a Markov Chain.

Thus, a Markov Process is a Stochastic Process where

$S \subseteq \mathbb{R}$ is discrete

$H = T = \{0,1,2,\ldots\}$ or $[0,\infty)$

which satisfies Markov Property.

When $H = T = \{0,1,2,\ldots\}$, we have Discrete time Parameter Markov Process (DTPMC);

$H = T = [0,\infty)$, we have CTPMC.

[From now on "Discrete time Parameter time homogeneous Markov Chain" will be abbreviated as "M.C".]

Many system have the property that given the present state, the past states have no influence on the future. The property is called Markov Property, and the system having this property are called Markov chains.

— The Markov Property is defined precisely by the requirement that

$$P(X_{n+1} = i_{n+1} \mid X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}, X_n = i_n)$$
$$= P(X_{n+1} = i_{n+1} \mid X_n = i_n)$$

whenever the LHS is defined, i.e., $P(X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i_n) > 0$.

implying that RHS is also defined for every $n \in \{0,1,2,\ldots\}$ and $i_0, i_1, \ldots, i_{n+1} \in S$; S is the discrete state space of the system. We will denote S by I for easy understanding. We shall label I by the non-negative integers $\{0,1,2,\ldots\}$ which is possible since I is denumerable.

The conditional probabilities $P(X_{n+1} = j \mid X_n = i)$ is called transition probabilities of the chain.

And if $P(X_{n+1} = j \mid X_n = i)$ is independent of n, then the MC is said to have stationary Transition Probabilities.

From now on, when we say that $X_n, n \geq 0$, forms a Markov chain, we mean that these random variables satisfy the Markov Property and have stationary transition probabilities (time homogeneous).

— A SP $\{X_n \mid n = 0,1,\ldots\}$ is said to be DTPMC if: $\forall$ $n = 0,1,2,\ldots$ and $i_0, i_1, \ldots, i_n, i_{n+1} \in I$

$$P(X_{n+1} = i_{n+1} \mid X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i_n) = P(X_{n+1} = i_{n+1} \mid X_n = i_n)$$

when LHS is defined implies that RHS is also defined.

☆ Transition Probability:-

Consider a DTPMC $\{X_n \mid n=0,1,2,\dots\}$ with state space I. The probability of X_{n+1} being in state j, given that X_n was in state i is denoted by $p_{ij}^{n,n+1}$.

$$p_{ij}^{n,n+1} = P(X_{n+1}=j \mid X_n=i) \; ; \; i,j \in I \text{ and } n=0,1,2,\dots$$

$p_{ij}^{n,n+1}$ is called the one-step transition probability from i to j at time n.

☆ Stationary transition Probability:-

A DTPMC $\{X_n \mid n=0,1,2,\dots\}$ is said to have stationary transition probabilities (or, said to be time Homogeneous MC) if for all $i,j \in I$; $p_{ij}^{n,n+1} = P(X_{n+1}=j \mid X_n=i)$ is independent of n whenever $p_{ij}^{n,n+1}$ is defined, i.e., $P(X_n=i)>0$.

Remark:- It may be possible that $P(X_{n+1}=j \mid X_n=i)$ is not defined for some values of n, for $i,j \in I$.

Consider a RV on $\{\dots,-2,-1,0,1,2,\dots\}$. A particle is executed s.t. starting from 0 it goes to states 1 and -1 with prob. $1/2$ each. Given that $X_0=0$; i.e. $P(X_0=0)=1$, $P(X_1=1)=P(X_1=-1)=\frac{1}{2}$. But $P(X_1=2)=0$. But $P(X_2=2)>0$. Thus $P(X_2=3 \mid X_1=2)$ is not defined but $P(X_3=3 \mid X_2=2)$ is defined.

Observations:-

1. Transition Prob. matrix (TPM) is a stochastic mtx.
2. The $(i+1)$st row of the TPM is the conditional p.d. of X_{n+1} under the condition $X_n=i$ $(i=0,1,2,\dots)$
3. We have a finite MC when $I=\{0,1,2,\dots,d\}$

$$P=\begin{pmatrix} p_{00} & p_{01} & \cdots & p_{0d} \\ p_{10} & p_{11} & \cdots & p_{1d} \\ \vdots & \vdots & & \vdots \\ p_{d0} & p_{d1} & \cdots & p_{dd} \end{pmatrix}$$

is a $(d+1)\times(d+1)$ square matrix.

<u>Initial Distribution:-</u> Consider a DTPMC $\{X_n \mid n=0,1,2,\dots\}$ with state space I. Then the probability distn. of the initial RV X_0, $\{\pi(i) \mid i\in I\}$; where;

$$\pi(i) = P(\{\omega : X_0(\omega)=i\}) = P(X_0=i);\ i\in I$$

is called the initial distribution of the MC.

Note that: (i) $0 \le \pi(i) \le 1 \ \forall\ i\in I$.

(ii) $\sum_{i\in I} \pi(i) = 1$.

☆ $\boxed{Q}$. (a) Suppose $\{X_n \mid n=0,1,\dots\}$ is a MC with state space $I=\{0,1,2,\dots\}$ and transition probability matrix, $P=((p_{ij}))_{i,j\in I}$ and Initial distn, $\pi=(\pi(i))_{i\in I}$. Find the expression for $P(X_0=i_0,\dots,X_n=i_n)$ i.e., for joint distribution of $X_0,\dots,X_n$ interms of initial distn. and transition prob. matrix.

(b) Using (a) prove that

$$P(X_0=i_0 \mid X_1=i_1,\dots,X_n=i_n) = P(X_0=i_0 \mid X_1=i_1)\ \forall$$

$n\in\{0,1,\dots\}$ and $i_0,i_1,\dots,i_n\in I$; such that LHS is defined.

<u>Sol.</u> (a) <u>Statement:</u>

$$P(X_0=i_0, X_1=i_1,\dots,X_n=i_n) = \pi(i_0)p_{i_0i_1}p_{i_1i_2}\cdots p_{i_{n-1}i_n} \quad (*)$$

$\forall\ n\in\{0,1,2,\dots\}$ and $i_0,i_1,\dots,i_n\in I$.

We will prove it by induction.

For $n=0$, it is trivial.

Assume $(*)$ is true for $n=k$.

We are to show that the result is true for $n=k+1$.

i.e., $P(X_0=i_0, X_1=i_1,\dots,X_{k+1}=i_{k+1}) = \pi(i_0)p_{i_0i_1}p_{i_1i_2}\cdots p_{i_ki_{k+1}}$ $\forall\ i_0,i_1,\dots,i_{k+1}\in I$.

Two cases are to be considered as follows:

Case I :- $P(X_0 = i_0, X_1 = i_1, \dots, X_k = i_k) > 0$

Then $P(X_0 = i_0, X_1 = i_1, \dots, X_k = i_k, X_{k+1} = i_{k+1})$

$= P(X_{k+1} = i_{k+1} \mid X_0 = i_0, X_1 = i_1, \dots, X_k = i_k) \, P(X_0 = i_0, X_1 = i_1, \dots, X_k = i_k)$

$= P(X_{k+1} = i_{k+1} \mid X_k = i_k) \, P(X_0 = i_0, X_1 = i_1, \dots, X_k = i_k)$ [By the defn of M.P.]

$= p_{i_k i_{k+1}} \cdot \pi(i_0) \cdot p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k}$

So, the proof is done for case I. (Proved)

Case II :- $P(X_0 = i_0, X_1 = i_1, \dots, X_k = i_k) = 0$ ——— (1)

Now, $\{X_0 = i_0, X_1 = i_1, \dots, X_k = i_k, X_{k+1} = i_{k+1}\} \subseteq \{X_0 = i_0, \dots, X_k = i_k\}$

$\therefore P(X_0 = i_0, X_1 = i_1, \dots, X_{k+1} = i_{k+1}) = 0$ ——— (2)

From (1) using induction hypothesis:

$P(X_0 = i_0, X_1 = i_1, \dots, X_k = i_k) = \pi(i_0) p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} = 0.$

Multiplying with $p_{i_k i_{k+1}}$

So, $\pi(i_0) p_{i_0 i_1} \cdots p_{i_k i_{k+1}} = 0$ ——— (3)

From (2) and (3); $P(X_0 = i_0, X_1 = i_1, \dots, X_k = i_k, X_{k+1} = i_{k+1})$

$= \pi(i_0) p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_k i_{k+1}}$. (Proved)

(b) $P(X_0 = i_0 \mid X_1 = i_1, \dots, X_n = i_n)$

$$= \frac{P(X_0 = i_0, X_1 = i_1, \dots, X_n = i_n)}{P(X_1 = i_1, \dots, X_n = i_n)}$$

$$= \frac{\pi(i_0) p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n}}{P(X_1 = i_1) p_{i_1 i_2} \cdots p_{i_{n-1} i_n}}$$

$$= \frac{P(X_0 = i_0, X_1 = i_1)}{P(X_1 = i_1)} = P(X_0 = i_0 \mid X_1 = i_1)$$

PROPOSITION:- $P(X_{n+k}=j \mid X_n=i) = \sum_{j_1\in I}\sum_{j_2\in I}\cdots\sum_{j_k\in I} p_{ij_1}p_{j_1j_2}\cdots p_{j_{k-1}j}$

for all $n=0,1,2,\dots;\ k=1,2,\dots;\ i,j\in I$ s.t. $P(X_n=i)>0$.

Proof:- $P(X_{n+k}=j \mid X_n=i) = P(X_n=i, X_{n+k}=j)/P(X_n=i)$

Now, $P(X_n=i, X_{n+k}=j) = \sum_{j_1\in I}\sum_{j_2\in I}\cdots\sum_{j_{k-1}\in I} P(X_n=i, X_{n+1}=j_1,\dots, X_{n+k-1}=j_{k-1}, X_{n+k}=j)$

We observe that:

$$P(X_n=i, X_{n+1}=j_1, X_{n+2}=j_2,\dots, X_{n+k-1}=j_{k-1}, X_{n+k}=j)$$

$$= \sum_{i_0\in I}\sum_{i_1\in I}\cdots\sum_{i_{n-1}\in I} P(X_0=i_0, X_1=i_1,\dots, X_{n-1}=i_{n-1}, X_n=i, X_{n+1}=j_1,\dots,X_{n+k-1}=j_{k-1}, X_{n+k}=j)$$

$$= \sum_{i_0\in I}\sum_{i_1\in I}\cdots\sum_{i_{n-1}\in I} \pi(i_0)p_{i_0i_1}p_{i_1i_2}\cdots p_{i_{n-1}i}p_{ij_1}p_{j_1j_2}\cdots p_{j_{k-1}j}$$

$$= p_{ij_1}p_{j_1j_2}\cdots p_{j_{k-1}j}\sum_{i_0\in I}\sum_{i_1\in I}\cdots\sum_{i_{n-1}\in I} \pi(i_0)p_{i_0i_1}p_{i_1i_2}\cdots p_{i_{n-1}i}$$

$$= p_{ij_1}p_{j_1j_2}\cdots p_{j_{k-1}j}\sum_{i_0\in I}\sum_{i_1\in I}\cdots\sum_{i_{n-1}\in I} f_{X_0,X_1,\dots,X_{n-1},X_n}(i_0,i_1,\dots,i_{n-1},i)$$

$$= p_{ij_1}p_{j_1j_2}\cdots p_{j_{k-1}j}\,P(X_n=i)$$

$$\therefore P(X_n=i, X_{n+k}=j) = P(X_n=i)\sum_{j_1\in I}\sum_{j_2\in I}\cdots\sum_{j_{k-1}\in I} p_{ij_1}p_{j_1j_2}\cdots p_{j_{k-1}j}$$

— Note that, according to the proposition, whenever the conditional probability $P(X_{n+k}=j \mid X_n=i)$ is defined, it does not depend upon the time of transition n and is a function of i,j and k alone. Also, there is at least one $n=0,1,2,\dots$ s.t. the conditional probability is defined.

— The analysis of MC concerns mainly the calculation of probabilities of the possible realizations of the chain.

☆ Transition Probability Matrix :-

$$p_{ij}^{(K)} = P(X_{n+K} = j \mid X_n = i) \text{ for some } n = 0,1,2,\ldots$$

s.t. $P(X_{n+K} = j \mid X_n = i)$ is defined, i.e., $P(X_n = i) > 0$.

[$i,j \in I$ and $K = 1,2,\ldots$]

For convenience, we define $p_{ij}^{(0)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Convention: $p_{ij}^{(1)}$ will be simply denoted by p_{ij}.

$\therefore$ $p_{ij}^{(K)}$ denotes the prob. that the M.C. goes from state i to state j in K transitions (steps).

Hence, $p_{ij}^{(K)}$ is called the K-step transition probability from i to j.

Define, $P^{(K)} = \left(\left(p_{ij}^{(K)}\right)\right)_{i,j \in I}$; $K = 1,2,\ldots$

$$P^{(0)} = I$$

Convension: $P^{(1)}$ will be denoted by P.

Hence, $P^{(K)}$ is called the K-step transition probability matrix of the MC.

☆ Q. (a) S.T. the K-step transition matrix of a MC is the Kth power of the one-step transition matrix.

(b) Using (a) prove Chapman-Kolmogrov equation.

Sol. (a) Clearly $P^{(0)} = I = P^0$. We are to show $P^{(K)} = P^K \; \forall K$.

Now $K = 1,2,\ldots$ and $i,j \in I$.

[where n is s.t. $P(X_n = i) > 0$]

$$p_{ij}^{(K)} = P(X_{n+K} = j \mid X_n = i) = \left[P^{(K)}\right]_{ij}$$

$$= \sum_{j_1 \in I} \sum_{j_2 \in I} \cdots \sum_{j_{K-1} \in I} p_{ij_1} p_{j_1 j_2} \cdots p_{j_{K-1} j}$$

$$= [P.P.\ldots P]_{ij}$$

$$= \left[P^K\right]_{ij}$$

Thus, $P^{(K)} = P^K$ for $K = 0,1,\ldots$

Also, $\pi^{(n)} = \pi . P^n$.

Note:- $p_{ij}^{m+n} = P(X_{m+n} = j \mid X_0 = i)$.

(b) Chapman - Kolmogrov Equation : $p_{ij}^{(m+n)} = \sum_{k \in I} p_{ik}^{(m)} p_{kj}^{(n)} \;\; \forall \; m,n = 0,1,\dots \;\&\; i,j \in I.$

Proof: For all $m, n = 0,1,2,\dots$

$$P^{(m+n)} = P^{m+n} = P^m . P^n = P^{(m)} P^{(n)}$$

Comparing $(i-j)$th entries of $P^{(m+n)}$ and $P^{(m)} \times P^{(n)}$; the result follows.

- Cor: (A) $p_{ij}^{(l)} = \sum_{k \in I} p_{ik}^{(r)} p_{kj}^{(s)} \;\; \forall \; r,s = 0,1,\dots$ and $r+s = l$ and $i,j \in I$.

  (B) $p_{ij}^{(n+1)} = \sum_{k \in I} p_{ik}^{(n)} p_{kj} = \sum_{k \in I} p_{ik} p_{kj}^{(n)} \;\; \forall \; n = 0,1,2,\dots$ and $i,j \in I$.

  (C) $P(X_n = i) = \pi_i^{(n)} = \sum_{j \in I} \pi_j p_{ji}^{(n)} \;;\; \forall \; n = 0,1,2,\dots$ and $i \in I$.

JOINT DISTRIBUTION OF NON - CONSECUTIVE RV'S :-

$$P(X_{t_0} = i_0, X_{t_1} = i_1, \dots, X_{t_n} = i_n) = P(X_{t_0} = i_0)\, p_{i_0 i_1}^{(t_1 - t_0)}\, p_{i_1 i_2}^{(t_2 - t_1)} \cdots p_{i_{n-1} i_n}^{(t_n - t_{n-1})}.$$

for all $n = 0,1,2,\dots$

$t_r \in \{0,1,2,\dots\}$ $(r = 0,1,2,\dots,n)$ s.t.

$t_0 < t_1 < t_2 < \dots < t_n$.

$i_r \in I$ $(r = 0,1,\dots,n)$.

Proof:-

$$LHS = \sum_{j_0, \dots, j_{t_0 -1}; j_{t_0+1}, \dots, j_{t_1 -1}; \dots; j_{t_{n-1}+1}, \dots, j_{t_n -1}} (A)$$

where; $A = P\big(X_0 = j_0, \dots, X_{t_0 - 1} = j_{t_0 -1};\; X_{t_0} = i_0;\; X_{t_1+1} = j_{t_0+1}, \dots,$
$X_{t_1 -1} = j_{t_1 -1};\; X_{t_1} = i_1;\; \dots\dots\dots;$
$X_{t_{n-1}} = i_{n-1};\; X_{t_{n-1}+1} = j_{t_{n-1}+1}, \dots, X_{t_n -1} = j_{t_n -1};\; X_{t_n} = i_n\big)$

$$= \pi(j_0)\, p_{j_0 j_1} \cdots p_{j_{t_0-1} i_0}\, p_{i_0 j_{t_0+1}} \cdots p_{j_{t_1 -1} i_1} \cdots p_{i_{n-1} j_{n-1+1}} \cdots \cdots p_{j_{t_n -1} i_n}.$$

The result follows after simplification.

Cor: $P(X_{t_{n+1}} = i_{n+1} | X_{t_0} = i_0, X_{t_1} = i_1, \ldots, X_{t_n} = i_n) = P(X_{t_{n+1}} = i_{n+1} | X_{t_n} = i_n)$

$\forall\ n = 0, 1, 2, \ldots$

$t_r \in \{0, 1, 2, \ldots\}$; $(r = 0, 1, \ldots, n+1)$ s.t. $t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1}$

$i_r \in I \quad (r = 0, 1, 2, \ldots, n+1)$

Whenever the LHS is defined.

Proof:- Since LHS is defined; we have:

$P(X_{t_0} = i_0, X_{t_1} = i_1, \ldots, X_{t_n} = i_n) > 0 \cdots\cdots (*)$

$\therefore\ P(X_{t_r} = i_r) > 0$ for $r = 0, 1, 2, \ldots, n$.

Hence; we can write:

$$p_{i_0 i_1}^{(t_1 - t_0)} = P(X_{t_1} = i_1 | X_{t_0} = i_0) = \frac{P(X_{t_0} = i_0, X_{t_1} = i_1)}{P(X_{t_0} = i_0)} > 0 \quad (\text{By } (*))$$

$$p_{i_1 i_2}^{(t_2 - t_1)} = P(X_{t_2} = i_2 | X_{t_1} = i_1) = \frac{P(X_{t_1} = i_1, X_{t_2} = i_2)}{P(X_{t_1} = i_1)} > 0 \quad (\text{By } (*))$$

$$\vdots$$

$$p_{i_{n-1} i_n}^{(t_n - t_{n-1})} = P(X_{t_n} = i_n | X_{t_{n-1}} = i_{n-1}) = \frac{P(X_{t_{n-1}} = i_{n-1}, X_{t_n} = i_n)}{P(X_{t_{n-1}} = i_{n-1})} > 0 \quad (\text{By } (*))$$

and, $p_{i_n, i_{n+1}}^{(t_{n+1} - t_n)} = P(X_{t_{n+1}} = i_{n+1} | X_{t_n} = i_n)$.

Now;

$$\text{LHS} = \frac{P(X_{t_0} = i_0, X_{t_1} = i_1, \ldots, X_{t_n} = i_n, X_{t_{n+1}} = i_{n+1})}{P(X_{t_0} = i_0, X_{t_1} = i_1, \ldots, X_{t_{n-1}} = i_{n-1}, X_{t_n} = i_n)}$$

$$= \frac{P(X_{t_0} = i_0)\, p_{i_0 i_1}^{(t_1 - t_0)}\, p_{i_1 i_2}^{(t_2 - t_1)} \cdots p_{i_{n-1} i_n}^{(t_n - t_{n-1})}\, p_{i_n i_{n+1}}^{(t_{n+1} - t_n)}}{P(X_{t_0} = i_0)\, p_{i_0 i_1}^{(t_1 - t_0)}\, p_{i_1 i_2}^{(t_2 - t_1)} \cdots\cdots p_{i_{n-1} i_n}^{(t_n - t_{n-1})}}$$

$$= p_{i_n i_{n+1}}^{(t_{n+1} - t_n)}$$

$$= P(X_{t_{n+1}} = i_{n+1} | X_{t_n} = i_n)$$

Remark:- In the special case: $t_0 = 0, t_1 = 1, t_2 = 2, \ldots, t_n = n, t_{n+1} = n+1$

The above cor. reduces to the MP.

According to the cor., the MP and the above prob. stated in the cor. are equivalent for a DTPTHMC (M.C.).

Hence, from now on the above property will be referred to as the M.P.

☆ Ex. 1. Mr and Mrs. Happy have planned to celebrate their 25th wedding anniversary in Honeymooner's paradise, a popular resort in Mussoorie. Counting today as first day, they are supposed to be there on the seventh and eighth day. They are thinking of buying a vacation insurance which promises to reimburse them for the entire vacation package cost of Rs. 2500 if it rains on both days and nothing is reimbursed otherwise. The insurance costs Rs. 200/-. Suppose that the weather at Mussoorie changes according to the following model:

The weather of Mussoorie is classified as: Sunny/Cloudy/Rainy

Assume: Tomorrow's weather depends only on today's weather.

Let X_n = Weather condition in Mussoorie on day n.

$$X_n = \begin{cases} 1 & \text{if it is sunny on day } n \\ 2 & \text{" " " cloudy " " "} \\ 3 & \text{" " " rainy " " "} \end{cases}$$

Given that $\{X_n \mid n = 0, 1, 2, \ldots\}$ is a MC with the following TPM

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.5 & 0.2 & 0.3 \\ 0.4 & 0.5 & 0.1 \end{pmatrix} \end{array}$$

Assume that it is sunny today in Mussoorie. Whether Mr. and Mrs. Happy will buy the insurance.

Solution:- Let R = Reimbursement that the couple gets from the insurance company.

Let X_n = Weather at Mussoorie at day n.

$n = 1, 2, \ldots$

Then given that $X_1 = 1$.

Also, $$R = \begin{cases} 2500 & \text{if } X_7 = X_8 = 3 \\ 0 & \text{ow} \end{cases}$$

$$\therefore E(R|X_1=1) = 2500\, P(X_7 = X_8 = 3 \mid X_1 = 1)$$
$$= 2500\, P(X_8=3 \mid X_7=3, X_1=1)\, P(X_7=3 \mid X_1=1)$$
$$= 2500\, \underset{(MP)}{P(X_8=3 \mid X_7=3)}\, P(X_7=3 \mid X_1=1)$$
$$= 2500\, \underset{(STP)}{P(X_1=3 \mid X_0=3)}\, P(X_6=3 \mid X_0=1)$$
$$= 2500 \times 0.1 \times \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.5 & 0.2 & 0.3 \\ 0.4 & 0.5 & 0.1 \end{pmatrix}^6$$
$$= 52.52$$
$$< 200$$

$\therefore$ The insurance is not worth it.

Ex. 2 Consider a MC with state space $I = \{1,2,3\}$; TPM is

$$P = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0 & 0.9 \\ 0.55 & 0 & 0.45 \end{pmatrix}$$

and I.D. is $\pi = (0.2, 0.3, 0.5)$. Compute the pmf of X_3.

Sol. $\pi^{(3)} = \pi P^3 = (0.2, 0.3, 0.5) \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0 & 0.9 \\ 0.55 & 0 & 0.45 \end{pmatrix}^3$

$$= (0.3626,\ 0.1207,\ 0.5167)$$

TPM:- A stochastic mtx describes a MC X_t over a finite state space S, if the prob. of moving from i to j in one time step is $P(j|i) = P_{ij}$, then the stochastic matrix ~~or~~ TPM is given by

$$P = \begin{pmatrix} P_{11} & P_{1,2} & \cdots & P_{1,s} \\ P_{2,1} & P_{2,2} & \cdots & P_{2,s} \\ \vdots & \vdots & & \vdots \\ P_{s,1} & P_{s,2} & \cdots & P_{s,s} \end{pmatrix}$$

where $\sum_{j=1}^{s} P_{i,j} = 1$.

☆ Q. Consider a MC with SS $I = \{0,1,2,\dots\}$ and transition probabilities given by $p_{i0} = p$ and $p_{i,i+1} = 1-p$ $\forall i \in I$, $p \in (0,1)$.

(a) S.T. $P(X_j \neq 0 \text{ for } j \in \{1,2,\dots,n-1\};\ X_n = 0 \mid X_0 = 0) = p(1-p)^{n-1}$

(b) Give interpretation of the result.

Solution:-

(a)

$$P(X_j \neq 0 \text{ for } j \in \{1,2,\dots,n-1\} : X_n = 0 \mid X_0 = 0\}$$

$$= P(X_1 \neq 0 \mid X_0 = 0)\, P(X_2 \neq 0 \mid X_1 \neq 0) \cdots\cdots P(X_{n-1} \neq 0 \mid X_{n-2} \neq 0)\, P(X_n = 0 \mid X_{n-1} \neq 0)$$

$$= \underbrace{(1-p)\cdots\cdots(1-p)}_{(n-1)\text{ times}}\, p \qquad [\because P(X_n = 0 \mid X_{n-1} \neq 0) = p]$$

$$= (1-p)^{n-1}\, p.$$

(b) $P(X_j \neq 0 \text{ for } j \in \{1,2,\dots,n-1\};\ X_n = 0 \mid X_0 = 0\} = (1-p)^{n-1} \cdot p$

This gives the geometric waiting time.

Suppose we toss a perfect coin repeatedly until a head turns up. Let X_n denote the number of tosses to appear a head, so that $\{X = n\}$ means $n-1$ tails before the first head. So, if we consider the coin is biased with prob. p for head and 1-p for tail. then

$$p_n = (q \cdots q)\, p = p q^{n-1}.$$

The RV X_n is called waiting time and the distn. is called geometric distn..

Existence Theorem:- Given a discrete set I; a sequence $\{\pi(i) \mid i \in I\}$ for all $i \in I$ and $\sum_{i\in I} \pi(i) = 1$; and a matrix $P = ((p_{ij}))_{i,j\in I}$ s.t. $p_{ij} \geq 0$ for all i,j and $\sum_{j\in I} p_{ij} = 1$; there exists a prob. space $(\Omega, \mathcal{A}, P)$ and a M.C. $\{X_n \mid n = 0,1,2,\ldots\}$ defined on it with state space I, ID (π) and TPM P.

FIRST PASSAGE PROBABILITIES:-

Convention:- Consider a MC $\{X_n \mid n = 0,1,2,\ldots\}$ with state space I and t.p.m. P.

suppose $i \in I$ and $A_1, A_2, \ldots, A_n \subseteq I$. Then,

$$P(X_{m+1} \in A_1, X_{m+2} \in A_2, \ldots, A_{m+n} \in A_n \mid X_m = i) = \sum_{j_1\in A_1}\sum_{j_2 \in A_2}\cdots\sum_{j_n\in A_n} p_{ij_1}p_{j_1j_2}\cdots p_{j_{n-1}j_n}$$

$\forall\ m,n = 0,1,2,\ldots$

s.t. L.H.S. is defined, i.e., $P(X_m = i) > 0$.

Thus, whenever the above conditional prob. is defined it doesn't depend upon m, also there is at least one $m = 0,1,2,\ldots$ s.t. the conditional prob. is defined. In view of this we will take $m=0$ in the above conditional prob. to mean its common value for m's for which it is defined.

i.e., $P(X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n \mid X_0 = i)$ is defined to be the common value of $P(X_{m+1} \in A_1, X_{m+2} \in A_2, \ldots, X_{m+n} \in A_n \mid X_m = i)$ for $m = 0,1,2,\ldots$ for which it is defined.

☆ **Definition:-** Suppose $A \neq \phi \subseteq I$.

Define T_A as follows: $T_A(\omega) = \begin{cases} \min\{n \geq 1 : X_n(\omega) \in A\}; & \text{if } \exists\ n \geq 1 \text{ s.t. } X_n(\omega) \in A \\ \infty & ;\text{ OW} \end{cases}$

T_A is a r.v. with p.v.'s $m = 1,2,\ldots,\infty$ and

$\{T_A = m\} \equiv \{X_1 \notin A, A_2 \notin A, \ldots, X_{m-1} \notin A; X_m \in A\}$; $m = 1,2,\ldots$

$\{T_A = \infty\} \equiv \{X_K \notin A$ for all $K = 1,2,\ldots\}$

T_A is thus the first instant of time after time 0 (i.e., the first passing time) the M.C. reaches (hits) A and is called the **Hitting time of A.**

☆ **DEFINITION [FIRST PASSAGE (RETURN) PROBABILITIES]:**

For $i, j \in I$ we define:

$$f_{ij}^{(m)} = P\left(T_j = m \mid X_0 = i\right) = \begin{cases} P\left(X_1 \neq j, X_2 \neq j, \ldots, X_{m-1} \neq j; X_m = j \mid X_0 = i\right) & ; m = 1, 2, \ldots \\ 0 & ; m = 0 \end{cases}$$

$$f_{ij}^{*} = P\left(T_j < \infty \mid X_0 = i\right) = \sum_{m=0}^{\infty} f_{ij}^{(m)}$$

where, $f_{ij}^{(m)}$ = Prob. that the first passage (return) to state j, starting from state i occurs at the m^{th} step, i.e. at the m^{th} transition.

$(i, j \in I$ and $m = 0, 1, 2, \ldots)$

$$f_{ij}^{*} = P\left(\bigcup_{m=1}^{\infty} \{\omega \in \Omega \mid X_m(\omega) = j\} \mid X_0(\omega) = i\right)$$

$= P\left(X_m^{*} = j \text{ for some } m = 1, 2, \ldots \mid X_0 = i\right)$

= Prob. that the MC will be in state j at least once (in positive time) starting from state i $(i, j \in I)$.

Note that, $f_{ij}^{*} = \sum_{m=1}^{\infty} f_{ij}^{(m)} \leq \sum_{m=1}^{\infty} p_{ij}^{(m)}$.

Also, $f_{ij}^{*} \geq p_{ij}^{(m)} \quad \forall\, m = 1, 2, \ldots$

Also, $f_{ij}^{*} \geq \sup_{m=1,2,\ldots} p_{ij}^{(m)}$.

<u>Remark:</u>-

1. $f_{ij}^{(m)} \leq p_{ij}^{(m)}$ for all $m = 0, 1, 2, \ldots$ and $i, j \in I$.

2. $\sup_{m=1,2,\ldots} p_{ij}^{(m)} \leq f_{ij}^{*} \leq \sum_{m=1}^{\infty} p_{ij}^{(m)} \quad \forall\, i, j \in I$.

☆ **Ehrenfest Model:-** Suppose we have two boxes, labelled 1 and 2 and d balls labelled $1, 2, \ldots, d$.

- Initially some of the balls are in Box 1 and rest are in Box 2.
- An integer is selected at random from $1, 2, \ldots, d$ and the ball labelled by that number is removed from its box and placed in the other box.
- The process is repeated indefinitely with selections being independent from trial to trial.

X_n = Number of balls in box 1 after n-th trial.

The state space $S = \{1, 2, \ldots, d\}$

Suppose at time n, box 1 has i balls, then the conditional probabilities will be

$$P(X_{n+1} = i+1 \mid X_0 = i_0, \ldots, X_n = i) = P(X_{n+1} = i+1 \mid X_n = i) = 1 - i/d.$$

$$P(X_{n+1} = i-1 \mid X_0 = i_0, \ldots, X_n = i) = P(X_{n-1} = i-1 \mid X_n = i) = i/d.$$

for all choices of $i_0, i_1, \ldots, i_{n-1}$.

This memoryless property is called the Markov Property, it says that the future depends on the past only through present.

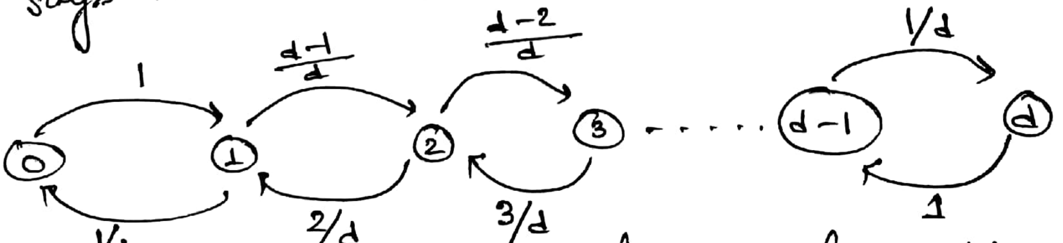


Fig: Transition diagram for Ehrenfest model.

When $d = 3$, TPM is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

When $d = 4$, TPM is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

If $d = n$;

$$P = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 1/n & 0 & 1-1/n & 0 & \cdots & 0 \\ 0 & 2/n & 0 & 1-2/n & 0 \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots 0 & \cdots & 1 & 0 \end{pmatrix}$$

☆ Q. A meter measuring a sequence of electric impulses records the heighest voltage that has passed through it up to any given time. Suppose that the impulse passing through the meter at the nth moment is Y_n.

$$P(Y_n = k) = \frac{1}{M+1} \text{ for } k = 0, 1, \ldots, M \quad (M \text{ is +ve integer})$$

(a) S.T. $\{X_n \mid n = 0, 1, 2, \ldots\}$ is a time homogeneous MC.

(b) Find the state space and transition matrix of the MC.

Sol. Here we will get the same probabilities if we condition on the history of the process.

And $P(Y_n = l \mid Y_n = k) = \frac{1}{M+1}$ is independent of n,

so it's called time homogeneous MC.

$$P(Y_n = l \mid Y_n = k) = \frac{P(Y_n = l, Y_n = k)}{P(Y_n = k)} = \frac{\left(\frac{1}{M+1}\right)^2}{\left(\frac{1}{M+1}\right)} = \left(\frac{1}{M+1}\right).$$

State space $= \{0, 1, 2, \ldots, M\} = S$.

TPM:-

$$\begin{bmatrix} 1/M+1 & 1/M+1 & \cdots & 1/M+1 \\ 0 & 2/M+1 & 1/M+1 & \cdots 1/M+1 \\ 0 & 0 & 3/M+1 & \frac{1}{M+1} \cdots \\ \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots \frac{M+1}{M+1} \end{bmatrix}$$

☆ **Basic Renewal Equation:-**

$$p_{ij}^{(n)} = \sum_{k=0}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)} \quad \text{for all } n = 1,2,3,\ldots \text{ and } i,j \in I.$$

[This equation is derived by decomposing the event from which $p_{ij}^{(n)}$ is computed according to the time of first entrance (return when $i=j$) into state j. This method is called "Method of decomposition by first entrance into state j"]

Proof:- $p_{ij}^{(n)} = P(X_n = j \mid X_0 = i)$

Note that $\{X_n = j\} = \bigcup_{k=1}^{n} E_k$ where;

$$E_k = \{X_1 \neq j, X_2 \neq j, \ldots, X_{k-1} \neq j, X_k = j, X_n = j\} = \{X_n = j \text{ and first entrance to } j \text{ at step } k\}$$
$(k = 1,2,\ldots,n)$

E_k's are disjoint events ; $k = 1,2,\ldots,n$.

$$p_{ij}^{(n)} = P\left(\bigcup_{k=1}^{n} E_k \,\middle|\, X_0 = i\right) = \sum_{k=1}^{n} P(E_k \mid X_0 = i) \quad [\because E_k\text{'s are disjoint } \forall\, k = 1,2,\ldots,n.]$$

Now, for $k = 1,2,\ldots,n$;

$$P(E_k \mid X_0 = i) = P(X_1 \neq j, X_2 \neq j, \ldots, X_{k-1} \neq j, X_k = j, X_n = j \mid X_0 = i)$$

$$= P(X_1 \neq j, X_2 \neq j, \ldots, X_{k-1} \neq j, X_k = j \mid X_0 = i) \, P(X_n = j \mid X_0 = i, X_1 \neq j, X_2 \neq j, \ldots, X_{k-1} \neq j, X_k = j)$$

[Using $P(A \cap B \mid C) = P(A \mid C) P(B \mid A \cap C)$]

But $P(X_1 \neq j, X_2 \neq j, \ldots, X_{k-1} \neq j, X_k = j \mid X_0 = i) = f_{ij}^{(k)}$.

Here, $\{X_0 = i, X_1 \neq j, \ldots, X_{k-1} \neq j, X_k = j\} = \bigcup_{\substack{\{i_1,\ldots,i_{k-1} \in I \text{ s.t.} \\ i_1 \neq j, \ldots, i_{k-1} \neq j\}}} \{X_0 = i, X_1 = i_1, X_2 = i_2, \ldots, X_{k-1} = i_{k-1}, X_k = j\}$

$$P(X_n = j \mid X_0 = i, X_1 = i_1, X_2 = i_2, \ldots, X_{k-1} = i_{k-1}, X_k = j) = P(X_n = j \mid X_k = j)$$
$$= p_{jj}^{(n-k)}$$

independent of $i, i_1, \ldots, i_{k-1}$.

[Using if C_i $(i = 1,2,\ldots)$ are disjoint events s.t. $P(A \mid C_i) = p$ independently of $i = 1,2,\ldots$ then $P(A \mid \bigcup C_i) = p$.]

Thus, $P(E_k \mid X_0 = i) = f_{ij}^{(k)} p_{jj}^{(n-k)}$ for $k = 1,2,\ldots,n$.

$$\therefore p_{ij}^{(n)} = \sum_{k=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)} = \sum_{k=0}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)}. \quad (\because f_{ij}^{(0)} = 0)$$

Remark:- (Basic Renewal Equation when $n=0$)

Suppose $n=0$. Then In BRE:

$$LHS = p_{ij}^{(0)} = \delta_{ij} = \begin{cases} 1 & \text{when } i=j \\ 0 & \text{when } i \neq j \end{cases}$$

$$RHS = f_{ij}^{(0)} p_{jj}^{(0)} = 0$$

$\therefore$ When $i=j$, BRE doesn't hold for $n=0$.

When $i \neq j$, BRE holds for $n=0$.

For $i \neq j$;

$$p_{ij}^{(n)} = \sum_{k=0}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)} \quad \text{for } n=0,1,2,\ldots$$

i.e. $\{p_{ij}^{(n)}\}_{n=0}^{\infty} = \{f_{ij}^{(n)}\}_0^{\infty} \times \{p_{jj}^{(n)}\}_0^{\infty}$.

Note that:

$$p_{ii}^{(n)} = \begin{cases} 1 + \sum_{k=0}^{n} f_{ii}^{(k)} p_{ii}^{(n-k)} & ; n=0 \\ \sum_{k=0}^{n} f_{ii}^{(k)} p_{ii}^{(n-k)} & ; n=1,2,\ldots \end{cases}$$

Proposition:- $f_{ij}^{(n+1)} = \sum_{\substack{k \in I \\ k \neq j}} p_{ik} f_{kj}^{(n)}$ for $n=1,2,\ldots\ldots$ and $i,j \in I$.

(Note: $f_{ij}^{(0)} = 0$, $f_{ij}^{(1)} = p_{ij}$, $f_{ij}^{(2)} = \sum_{\substack{k \in I \\ k \neq j}} p_{ik} p_{kj}$ etc.)

Proof:-

$$\begin{aligned}
f_{ij}^{(n+1)} &= P(X_1 \neq j, X_2 \neq j, \ldots\ldots, X_n \neq j, X_{n+1} = j \mid X_0 = i) \\
&= P\Big(\bigcup_{\substack{k \in I \\ k \neq j}} \{X_1 = k, X_2 \neq j, \ldots\ldots X_n \neq j, X_{n+1} = j\} \mid X_0 = i\Big) \\
&= \sum_{\substack{k \in I \\ k \neq j}} P(X_1 = k, X_2 \neq j, \ldots, X_n \neq j, X_{n+1} = j \mid X_0 = i) \\
&= \sum_{\substack{k \in I \\ k \neq j}} P(X_1 = k \mid X_0 = i)\, P(X_2 \neq j, \ldots, X_n \neq j, X_{n+1} = j \mid X_0 = i, X_1 = k) \\
&= \sum_{\substack{k \in I \\ k \neq j}} p_{ik} \cdot P(X_2 \neq j, \ldots, X_n \neq j, X_{n+1} = j \mid X_1 = k) \quad [\text{By MC}] \\
&= \sum_{\substack{k \in I \\ k \neq j}} p_{ik} \cdot f_{kj}^{(n)}
\end{aligned}$$

☆ Derive the relationship between the generating functions of the sequences $\{p_{ij}^{(n)}\}_{n=0}^{\infty}$ and $\{f_{ij}^{(n)}\}_{n=0}^{\infty}$.

Ans:

Definition [Related Generating functions]:-

Suppose $i, j \in I$.

(a) $P_{ij}(s) = \sum_{n=0}^{\infty} p_{ij}^{(n)} s^n$ for $|s|<1$ is the generating function of $\{p_{ij}^{(n)}\}_{n=0}^{\infty}$.

Observe that $P_{ij}(s) = \delta_{ij} + \sum_{n=1}^{\infty} p_{ij}^{(n)} s^n$; $|s|<1$.

(b) $F_{ij}(s) = \sum_{n=0}^{\infty} f_{ij}^{(n)} s^n$ for $|s|<1$ is the generating function of $\{f_{ij}^{(n)}\}_{n=0}^{\infty}$.

Observe that $F_{ij}(s) = \sum_{n=1}^{\infty} f_{ij}^{(n)} s^n$; $|s|<1$.

The relationship between the generating functions of $\{p_{ij}^{(n)}\}_{n=0}^{\infty}$ and $\{f_{ij}^{(n)}\}_{n=0}^{\infty}$ is given by:

for $i, j \in I$, $P_{ij}(s) = \delta_{ij} + F_{ij}(s) P_{jj}(s) = \begin{cases} F_{ij}(s) P_{jj}(s) \ ; & i \neq j \\ 1 + F_{jj}(s) P_{jj}(s) \ ; & i = j \end{cases}$

$(|s|<1)$

Proof:- By BRE, for $i, j \in I$;

$$p_{ij}^{(n)} = \sum_{k=0}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)} \ ; \ n = 1, 2, \ldots$$

Multiplying both sides by s^n $(|s|<1)$ and summing over $n = 1, 2, \ldots$

$$\sum_{n=1}^{\infty} p_{ij}^{(n)} s^n = \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)} \right) s^n$$

$$= \left(\sum_{k=0}^{\infty} \sum_{n=k}^{\infty} f_{ij}^{(k)} p_{jj}^{(n-k)} s^n \right)$$

$$= \left(\sum_{k=0}^{\infty} f_{ij}^{(k)} s^k \right) \left(\sum_{n=k}^{\infty} p_{jj}^{(n-k)} s^{n-k} \right)$$

$$= \left(\sum_{k=0}^{\infty} f_{ij}^{(k)} s^k \right) \left(\sum_{l=0}^{\infty} p_{jj}^{(l)} s^l \right)$$

So, the above eqn. is: $P_{ij}(s) = \delta_{ij} + F_{ij}(s) P_{jj}(s)$; $|s|<1$.

☆ PROBABILITY OF "AT LEAST m" and "INFINITELY MANY" VISITS :-

Ques:- Suppose $i \in I$. Define N_i = Total number of visits to state i at positive time points $n \in \{1, 2, \dots\}$.

(a) Show that $E(N_j | X_0 = i) = \sum_{n=0}^{\infty} p_{ij}^{(n)} - \delta_{ij}$ where $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

(b) Show that $\forall\, i, j \in I$,

$$P(N_j = m | X_0 = i) = \begin{cases} 1 - f_{ij}^* & \text{if } m = 0 \\ f_{ij}^* (f_{jj}^*)^{m-1}(1 - f_{jj}^*) & \text{if } m = 1, 2, \dots \\ g_{ij} & \text{if } m = \infty. \end{cases}$$

Solution:-

(a) N_i = Total number of visits to state i at positive time points, $n = 1, 2, \dots$

N_i is a random variable which can take values as $0, 1, 2, \dots, \infty$.

$N_i = \sum_{n=1}^{\infty} Y_n$ where; $Y_n = I\{X_n = i\} = \begin{cases} 1 & \text{if } X_n = i \;(n = 1, 2, \dots) \\ 0 & \text{if } X_n \neq i \end{cases}$

Y_n is thus 1 or 0 according as whether or not at time n, the MC is in state i.

$$E(N_j | X_0 = i) = \sum^{\infty} p_{ij}^{(n)} = \sum_{n=0}^{\infty} p_{ij}^{(n)} - \delta_{ij}$$

(b) $P(N_j = 0 | X_0 = i) = 1 - P(N_j \geq 1 | X_0 = i) = 1 - g_{ij}(1)$

$$= 1 - f_{ij}^*$$

Also for $m = 1, 2, \dots$

$$P(N_j = m | X_0 = i) = P(N_j \geq m | X_0 = i) - P(N_j \geq m+1 | X_0 = i)$$

$$= g_{ij}(m) - g_{ij}(m+1)$$

$$= f_{ij}^* (f_{jj}^*)^{m-1} - f_{ij}^* (f_{jj}^*)^m$$

$$= f_{ij}^* (f_{jj}^*)^{m-1} (1 - f_{jj}^*)$$

Finally, $P(N_j = \infty | X_0 = i) = g_{ij}$, by definition.

Definitions:- For $i,j \in I$, we define:

$g_{ij}(m) = P(N_j \geq m \mid X_0 = i) = P(X_n = j$ for at least m values of $n \geq 1 \mid X_0 = i)$; $m = 0, 1, 2, \ldots$

= Probability that the MC will visit (return to when $i=j$) state j at least m (positive) times starting from state i. ($i, j \in I$ and $m = 0, 1, 2, \ldots$)

$g_{ij} = P(N_j = \infty \mid X_0 = i) = P(X_n = j$ i.o., i.e., for infinitely many values of $n \geq 1 \mid X_0 = i)$

= Probability that the MC will visit (return to when $i=j$) state j infinitely often, i.e., infinitely many (positive) times starting from state i. ($i, j \in I$)

Note:

(i) $g_{ij}(0) = P(N_j \geq 0 \mid X_0 = i) = 1$; $g_{ij}(1) = P(N_j \geq 1 \mid X_0 = i) = P(T_j < \infty \mid X_0 = i) = f_{ij}^*$ for all $i, j \in I$.

(ii) $g_{ij}(m+1) \leq g_{ij}(m) \;\forall\; m = 0, 1, 2, \ldots$ and $i, j \in I$.

(iii) $g_{ij}(m) \leq f_{ij}^* \;\forall\; m = 1, 2, \ldots$ and $i, j \in I$.

(iv) $g_{ij} \leq f_{ij}^* \;\forall\; i, j \in I$.

(v) $g_{ij} = \lim_{m \to \infty} g_{ij}(m)$; $g_{ij} = f_{ij}^* g_{jj}$; $g_{ii} = \lim_{m\to\infty} (f_{ii}^*)^m$.

Proposition:- $g_{ij}(m+1) = f_{ij}^* g_{jj}(m)$ for all $m = 0, 1, 2, \ldots$ and $i, j \in I$.

Proof:- $g_{ij}(m+1) = P(X_n = j$ for at least $m+1$ values of $n \geq 1 \mid X_0 = i)$

$= P\left(\bigcup_{l=1}^{\infty} \{X_1 = j, \ldots, X_{l-1} \neq j, X_l = j, X_n = j \text{ for at least } m \text{ values of } n \geq l+1\} \mid X_0 = i \right)$

$= \sum_{l=1}^{\infty} P(X_1 \neq j, \ldots, X_{l-1} \neq j, X_l = j, X_n = j$ for at least m values $n \geq l+1 \mid X_0 = i)$

$= \sum_{l=1}^{\infty} P(X_1 \neq j, \ldots, X_{l-1} \neq j, X_l = j \mid X_0 = i) P(X_n = j$ for at least m values of $n \geq l+1 \mid X_0 = i, X_1 \neq j, \ldots, X_{l-1} = j, X_l = j)$

[Using $P(A \cap B \mid C) = P(A \mid C) P(B \mid A \cap C)$]

$= \sum_{l=1}^{\infty} f_{ij}^{(l)} P(X_n = j$ for at least m values of $n \geq l+1 \mid X_l = j)$

[Using M.P.]

$$= \sum_{l=1}^{\infty} f_{ij}^{(l)} P(X_n = j \text{ for at least } m \text{ values of } n \geq 1 \mid X_0 = j)$$

[Using stationarity]

$$= \sum_{l=1}^{\infty} f_{ij}^{(l)} g_{jj}(m)$$

$$= f_{ij}^{*} g_{jj}(m).$$

Proposition:- $g_{ij}(m+1) = f_{ij}^{*} (f_{ij}^{*})^{m}$ for all $m = 0, 1, 2, \ldots$ and $i, j \in I$.

Proof:- (By induction)

Trivial for $m = 0$.

Assume it is true for $m = m_0$ and all $i, j \in I$.

i.e., $g_{ij}(m_0+1) = f_{ij}^{*} (f_{ij}^{*})^{m_0}$ for all $i, j \in I$ (Induction Hypothesis)

Now, $g_{ij}(m_0+2) = f_{ij}^{*} g_{jj}(m_0+1)$ [By previous proposition]

$$= f_{ij}^{*} (f_{jj}^{*})^{m_0+1}$$

$\therefore$ True for $m = m_0 + 1$ and all $i, j \in I$.

Thus, the propn. is true for all $m = 0, 1, 2, \ldots$ and $i, j \in I$

(By iteration)

Trivial for $m = 0$, Assume $m = 1, 2, \ldots$

$$g_{jj}(m) = f_{jj}^{(m)} g_{jj}(m-1) \quad \text{(By previous propn.)}$$

$$= (f_{jj}^{*})^{2} g_{jj}(m-2)$$

$$\vdots$$

$$= (f_{jj}^{*})^{m} g_{jj}(0)$$

$$= (f_{jj}^{*})^{m}$$

$$\therefore g_{ij}(m+1) = f_{ij}^{*} (f_{jj}^{*})^{m}.$$

Communication among states:- Consider a MC $\{X_n \mid n=0,1,2\ldots\}$ with state space I and t.p.m. P. In this section various definitions regarding communication among different states will be given:

Definition:- Suppose $i, j \in I$.

(A) State i leads to state j (or, state j is accessible from state i) if: $p_{ij}^{(m)} > 0$ for some ptg. $m = 1,2,3,\ldots$

i.e., if there is a positive probability of reaching state j from state i in a finite number of transitions.

State i does not lead to state j (or, state j is not accessible from state i) if: $p_{ij}^{(m)} = 0$ for all ptg. $m = 1,2,3,\ldots$

i.e., if the probability of reaching state j from state i in a finite no. of transitions is zero.

We write $i \rightarrow j$ when i leads to j and $i \not\rightarrow j$ when i doesn't lead to j.

Note: Every state leads to some state (may be itself) because $\sum p_{ij} = 1$.

(B) State i communicates with state j (or, state i and j communicate) if: $i \rightarrow j$ and $j \rightarrow i$.

i.e. $p_{ij}^{(m)} > 0$ and $p_{ij}^{(n)} > 0$ for ptgs. $m, n = 1,2,3,\ldots$

i.e. if there is a positive probability of reaching state j from state i in a finite no. of transitions and vice versa.

State i does not communicate with state j (or, state i and j do not communicate) if: $i \not\rightarrow j$ and $j \not\rightarrow i$

i.e. $p_{ij}^{(m)} = 0$ for all ptgs $m = 1,2,3\ldots$ or $p_{ij}^{(n)} = 0$ $\forall$ ptgs $n = 1,2,3,\ldots$

We write $i \leftrightarrow j$ when i and j communicate and $i \not\leftrightarrow j$ when i and j do not communicate.

Observations: $i \rightarrow j$ iff $f_{ij}^{*} > 0$

$i \leftrightarrow j$ iff $f_{ij}^{*} f_{ji}^{*} > 0$

Question:- Show that the relationship of communication is an equivalence relation.

Solution:-

It means that we have to show —

- every state communicates with itself (Reflexive)
- if $i \leftrightarrow j$ then $j \leftrightarrow i$ (Symmetric)
- if $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$ (transitive)

Proof:- Since $i \to j$ and $j \to k$, $\exists$ ptgs $m, n = 1, 2, \dots$

such that $p_{ij}^{(m)}$; $p_{jk}^{(n)} > 0$.

Then $$p_{ik}^{(m+n)} = \sum_{l \in I} p_{il}^{(m)} p_{lk}^{(n)} \geq p_{ij}^{(m)} p_{jk}^{(n)} > 0.$$

Since $(m+n)$ is a ptg., result follows.

So, the relation $\leftrightarrow$ is transitive.

$i \leftrightarrow j$ and $j \leftrightarrow k \Rightarrow i \leftrightarrow k$ (for all $i, j, k \in I$)

Definition (Class Property) :-

We can partition the set of return state I' into equivalence classes w.r.t. E.R. (equivalence relation) $\leftrightarrow$ s.t.

(i) The EC's are disjoint.

(ii) Within EC states (disjoint or not) communicate with each other.

(iii) Two states belonging to different EC's don't communicate.

By a 'class' we mean:

Either one of the EC's mentiontioned above.

or, the stngleton $\{i\}$ for $i \in I - I'$.

The class containing the state $i_0 \in I$ is denoted by $C(i_0)$

$$C(i_0) = \begin{cases} [i_0] = \text{EC of } i_0 \text{ w.r.t. } \leftrightarrow ; & \text{if } i_0 \in I' \\ \{i_0\} ; & \text{if } i_0 \in I - I' \end{cases}$$

A property P is defined for all states in I is said to be a "class property" if its possession by any one state in any class implies its possession by all states in the same class, i.e., if for each class either all states in the class possess P or no state in the class possess P.

Question:- A professor continuously gives exams to the students. He can give three possible types of exams and his class is graded as either having done well or badly. Let P_i denote the prob. that the class does well on type i exam and assume that $P_1 = 0.3$, $P_2 = 0.6$, $P_3 = 0.9$. If the class does well on an exam, then the next exam is equally likely to be any of the three types. However if the class does badly, the next exam is always type 1. What proportion of exams are of type i $(i=1,2,3)$?

Solution:-

Consider the MC whose state at time n is the type of exam number n, the transition probs. of this MC are obtained by conditioning on the performance of the class. this gives

$$P_{11} = 0.3(1/3) + 0.7 \times 1 = 0.8$$

$$P_{21} = 0.6(1/3) + 0.4 \times 1 = 0.6$$

$$P_{31} = 0.9(1/3) + 0.1 \times 1 = 0.4$$

$$P_{12} = P_{13} = 0.3(1/3) = 0.1$$

$$P_{22} = P_{23} = 0.6(1/3) = 0.2$$

$$P_{32} = P_{33} = 0.9(1/3) = 0.3$$

Let π_i denote the proportion of exams that are of type I. $(i=1,2,3)$

$$\left.\begin{aligned} \pi_1 &= 0.8\pi_1 + 0.6\pi_2 + 0.4\pi_3 \\ \pi_2 &= 0.1\pi_1 + 0.3\pi_2 + 0.3\pi_3 \\ \pi_1 + \pi_2 + \pi_3 &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} \pi_1 &= 5/7 \\ \pi_2 &= 1/7 \\ \pi_3 &= 1/7 \end{aligned}$$

Definition (Irreducible MC) :-

A MC is said to be irreducible if the ER communication $(\leftrightarrow)$ induces exactly EC, namely I.

Thus, if MC is irreducible iff:

$i \to j$ for all choices of i and $j \in I$.

i.e., every state leads back to itself and also to every other state.

i.e. all pairs of states intercommunicate.

Definition:- (Essential States and Inessential States)

Suppose $i \in I$.

i is essential if : $\forall\ j \in I$, $i \to j \Rightarrow j \to i$

i is inessential if : i is not essential.

Thus a state that communicates with every state is called essential, otherwise inessential.

→ We can show that an essential state can't lead to inessential state. [$j \to k$ $(k \in I)$. Show $k \to j$.

Now $i \to j$ and $j \to k$. So, $i \to k$

$k \to i$ since i is essential

Thus $k \to i$ and $i \to j$ gives $k \to j$.

$\therefore$ j is essential.]

Cor The property of being essential (inessential) is a class property.

[Hints: class : single state, mutually communicating states]

Definition (Recurrent and Transient States) :-

Suppose $i \in I$ and f_{ii}^{*} = Prob. that MC will be in state i at least once (in positive time) starting from state i $(i \in I)$.

i is recurrent if : $f_{ii}^{*} = 1$.

i is transient if : i is not recurrent, i.e., $0 \le f_{ii}^{*} < 1$.

When i is recurrent, a MC starting in i will return to i after finite length of time with probability 1.

When i is transient, a MC starting in i has positive prob. $1 - f_{ii}^{*}$ of never returning to i.

[Criterion for Recurrence]

Ques:- Suppose $i \in I$ where I is the state space of MC. Show that state i is recurrent iff $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$

and state i is transient iff $\sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty$.

Solution:- We know $P_{ii}(s) = \dfrac{1}{1-F_{ii}(s)}$; $|s|<1$.

Taking limits as $s \to 1-$ we obtain:

$$\lim_{s\to 1-} P_{ii}(s) = \lim_{s \to 1-}\left(\frac{1}{1-F_{ii}(s)}\right) = \frac{1}{1-\lim\limits_{s\to 1-} F_{ii}(s)}$$

By Abel's theorem;

$$\lim_{s\to 1-} P_{ii}(s) = P_{ii}(1) = \sum_{n=0}^{\infty} p_{ii}^{(n)}$$

$$\lim_{s\to 1-} F_{ii}(s) = F_{ii}(1) = \sum_{n=0}^{\infty} f_{ii}^{(n)} = f_{ii}^{*}$$

$$\therefore \sum_{n=0}^{\infty} p_{ii}^{(n)} = \frac{1}{1-f_{ii}^{*}}$$ (with a sense that either both sides are finite and equal or both sides are ∞)

$\therefore$ i is recurrent iff $f_{ii}^{*} = 1$ iff $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$.

i is transient iff $f_{ii}^{*} < 1$ iff $\sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty$.

So, i is recurrent or transient according as the infinite series $\sum_{n=0}^{\infty} p_{ii}^{(n)}$ diverges or converges.

Remark:- Recall that:

Expected no. of visits to i at positive time $n=1,2,\dots$ starting from i.

Expected no. of returns to i starting from i

$=$ Expected no. of

$$= E(N_i \mid X_0 = i) = \sum_{n=1}^{\infty} p_{ii}^{(n)} = \sum_{n=0}^{\infty} p_{ii}^{(n)} - 1$$

$\therefore$ when i is recurrent $E(N_i \mid X_0 = i) = \infty$.

When i is transient $E(N_i \mid X_0 = i) = \dfrac{f_{ii}^{*}}{1-f_{ii}^{*}} = P$; $0 \le P < \infty$.

We can interpret $\sum_{n=0}^{\infty} p_{ii}^{(n)}$ as the total no. of visits to i starting from i including time 0. Also,

i is recurrent if $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$, i is transient if $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \dfrac{1}{1-f_{ii}^{*}} < \infty$, Ques:- S.T. one dimensional simple random walk on the integers $I = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ with transition probs.

$$p_{ij} = \begin{cases} p & \text{if } j = i+1 \\ q & \text{if } j = i-1 \\ 0 & \text{ow} \end{cases} \qquad (0 < p, q < 1,\ p+q = 1)$$

is recurrent iff $p = q = 1/2$.

Solution:- $p_{00}^{(1)} = p_{00} = 0$; $p_{00}^{(2)} = pq + qp = 2pq$

$p_{00}^{(3)} = 0$; $p_{00}^{(n)} = 0$ if n is odd.

$$p_{00}^{(2n)} = \binom{2n}{n} p^n q^n = \frac{(2n)!}{n!\,n!} p^n q^n; \quad \text{Take } S = \sum_{n \geq 1} \binom{2n}{n} p^n q^n.$$

$$\left[k! \sim \sqrt{2\pi}\, e^{-k} k^{k+1/2} \Rightarrow \frac{k!}{\sqrt{2\pi}\, e^{-k} k^{k+1/2}} \to 1 \text{ as } k \to 1. \right.$$

$$\frac{(2n)!}{(n!)^2} \sim \frac{\sqrt{2\pi}\, e^{-2n} (2n)^{2n+1/2}}{2\pi\, e^{-2n}\, n^{2n+1}} = \frac{1}{\sqrt{\pi}} \cdot 2^{2n} \cdot \frac{1}{\sqrt{n}}$$

$$S \sim \sum \frac{1}{\sqrt{\pi}} (4pq)^n \frac{1}{\sqrt{n}} \quad \text{diverges if } p = q = \tfrac{1}{2}.$$

We know when it is recurrent $\sum_{n \geq 1} p_{00}^{(n)}$ diverges.

Ques:- ST Recurrence is a class property.

Sol. Assume that $i \leftrightarrow j$ for two states $i, j \in E$ and i is null recurrent, then there are two numbers $m, n \in N$ with $p_{ij}^{n} > 0$ and $p_{ij}^{m} > 0$.

Now, $$E\left(N_i(k) \mid X_0 = i\right) = \sum_{l=0}^{\infty} p_{ii}^{(l)}$$

We obtain
$$0 = \lim_{k \to \infty} \frac{\sum_{l=0}^{k} p_{ii}^{(l)}}{k} \geq \lim_{k \to \infty} \frac{\sum_{i=0}^{k-m-n} p_{ij}^{(l)} p_{ij}^{(n)} p_{ji}^{(m)}}{k}$$

$$= \lim_{k \to \infty} \frac{\sum_{l=0}^{k} p_{ij}^{(l)} p_{ij}^{n} p_{ji}^{m}}{k}$$

$$= \frac{p_{ij}^{n}\, p_{ji}^{m}}{m_j}$$

When $m_j = \infty$ it signifies null recurrence of j.

Ques:- Show that a recurrent state is necessarily essential.

Sol. Suppose i is recurrent.
We will show i is essential.
Suppose towards a contradiction, that i is inessential.
Then $\exists$ state $j \in I$ s.t. $i \to j$ but $j \not\to i$

$\therefore \exists \; n^* \in \{1, 2, \dots\}$ s.t. $p_{ij}^{(n^*)} > 0$.

Also, $p_{ji}^{(m)} = 0 \;\; \forall \; m = 1, 2, \dots$

This implies $f_{ji}^{(m)} = 0 \;\; \forall \; m = 1, 2, \dots \quad \left(\because f_{ji}^{(m)} \le p_{ji}^{(m)} \;\; \forall \; m = 1, 2, \dots\right)$

Thus $f_{ji}^{*} = 0 \;\; \therefore \; g_{ji} = 0 \quad (\because g_{ji} \le f_{ji}^{*})$

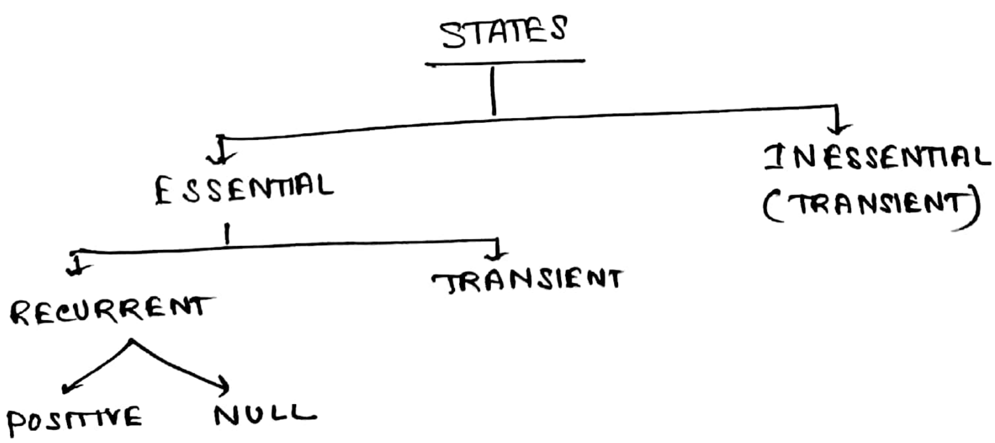
Now, $$g_{ii} = \sum_{k \in I} p_{ik}^{(n^*)} g_{ki} = \sum_{\substack{k \in I \\ k \ne j}} p_{ik}^{(n^*)} g_{ki} + p_{ij}^{(n^*)} g_{ji}$$

$$= \sum_{\substack{k \in I \\ k \ne j}} p_{ik}^{(n^*)} g_{ki} \quad (\because g_{ji} = 0)$$

$$\le \sum_{\substack{k \in I \\ k \ne j}} p_{ik}^{(n^*)} = 1 - p_{ij}^{(n^*)} < 1.$$

$\therefore g_{ii} = 0$ ($\because g_{ii}$ is either 0 or 1) is necessarily essential.

Classification of states:-



Ques:- State the definition of g_{ii} and prove that it can only take two values 0 or 1.

Solution:-

[Read Page: 25]

$$g_{ij}(m+1) = f_{ij}^{*} \, g_{jj}(m) = f_{ij}^{*} \left(f_{jj}^{*}\right)^{m} \quad \forall \; m = 0,1,2,\ldots \text{ and } i,j \in I.$$

Taking limits as $m \to \infty$ gives $g_{ij} = f_{ij}^{*} \, g_{jj}$. | then using $g_{ii}(m) = \left(f_{ii}^{*}\right)^{m}$ $\forall \; m = 0,1,2,\ldots$ and $i \in I$,

we get $g_{ii} = \lim_{m \to \infty} \left(f_{ii}^{*}\right)^{m}$.

Now, we have $g_{ii} = \lim_{m \to \infty} \left(f_{ii}^{*}\right)^{m}$

Since $0 \le f_{ii}^{*} \le 1$ it follows that g_{ii} is either 0 or 1.

$g_{ii} = 1$ iff $f_{ii}^{*} = 1$ iff i is recurrent.

$g_{ii} = 0$ iff $f_{ii}^{*} < 1$ iff i is transient.

Ques:- Write the following Definitions:-

(a) Positive and Null Recurrence.

(b) Mean Recurrence time of a recurrent state.

(c) Recurrence time distribution.

Ans:- suppose $i \in I$ is a recurrent state.

Then the p.v's of the r.v. T_i given $X_0 = i$ are 1, 2, ... Conditional distn. of $T_i \mid X_0 = i$ is given by the sequence $\{f_{ii}^{(n)}\}_{n=1}^{\infty}$

(As, $0 \leq f_{ii}^{(n)} \leq 1$ and $\sum_{n=1}^{\infty} f_{ii}^{(n)} = f_{ii} = 1$)

$f_{ii}^{(n)} = P(T_i = n \mid X_0 = i)$ = Prob. that starting in i, first return to i occurs in exactly n steps. $(n = 1, 2, \dots)$

— The pdf $\{f_{ii}^{(n)}\}_{n=1}^{\infty}$ is called first return time distn. from i to i or recurrence time distn.

— The conditional expectation of $T_i \mid X_0 = i$, i.e., the expectation of the recurrence time distn. is called mean time of return at i or the mean recurrence time of i and is denoted by m_{ii}.

$$m_{ii} = E(T_i \mid X_0 = i) = \sum_{n=1}^{\infty} n f_{ii}^{(n)} = F_{ii}'(1).$$

— Note that i being recurrent, return to i starting from i is certain, i.e., $P(T_i < \infty \mid X_0 = i) = 1$. But the mean time of return to i starting from i may be finite or infinite.

— Suppose $i \in I$ is a recurrent state.

i is positive recurrent if $m_{ii} < \infty$.

i is null recurrent if $m_{ii} = \infty$.

APPLIED

Stochastic Process :-

Using technical terminalogy, it is a family of random variables (RV) $\{X(t) : t \in T\}$ defined over time, indexed by time t.

Difference between SP and RV :-

X : Number of tubelight failures in a week. (RV)
0, 2, 5 $\rightarrow$ Realisation values, these are independent and randomness exists.

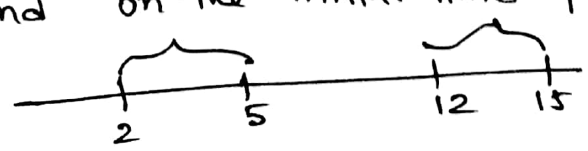
Y : Temperature recorded over time in a year. (SP)
$\rightarrow$ Here dependency among realisations are over time.

- For a given time t, $X(t)$ denotes the behaviour, or, state, say, s, i.e., $X(t) = s$, $s \in S$ (State space).
- If $S \subseteq \mathbb{R}$, the stochastic process is said to be real valued.
  If $S \subseteq \mathbb{R}^k$, the SP is said to be vector valued.
  (e.g. - consider no. of arrivals and departures in a queue at some instant)

(a) Process with independent increments :-

An SP is said to have independent increment if for all $t_0 < t_1 < \ldots < t_n$, the RV's $X(t_1) - X(t_0)$, $X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are independent.
i.e., number of occurances over non-overlapping intervals are independent.

(b) Process with stationary increments :- An SP is said to posses stationary increments if $X(t) - X(s)$ has the same distn. as that of $X(t-s)$ $\forall\ s \le t$.
i.e., $X(5) - X(2) \simeq X(3)$; $X(15) - X(12) \simeq X(3)$.
They are same as both has the same distn. as of length 3 on the real line.
i.e., number of occurances becomes same for the same length of interval.
i.e., no. of intervals over an interval of certain length doesn't depend on the initial time point of the interval.



(c) **Process with Markov Property**:- An SP is said to have MP if $\forall\ t_1 < t_2 < \dots < t_n$

$$P\left[X(t) \le x \,\middle|\, X(t_n) = x_n, \dots, X(t_1) = x_1\right] = P\left[X(t) \le x \,\middle|\, X(t_n) = x_n\right]$$

for every $x_1, \dots, x_n \in S$.

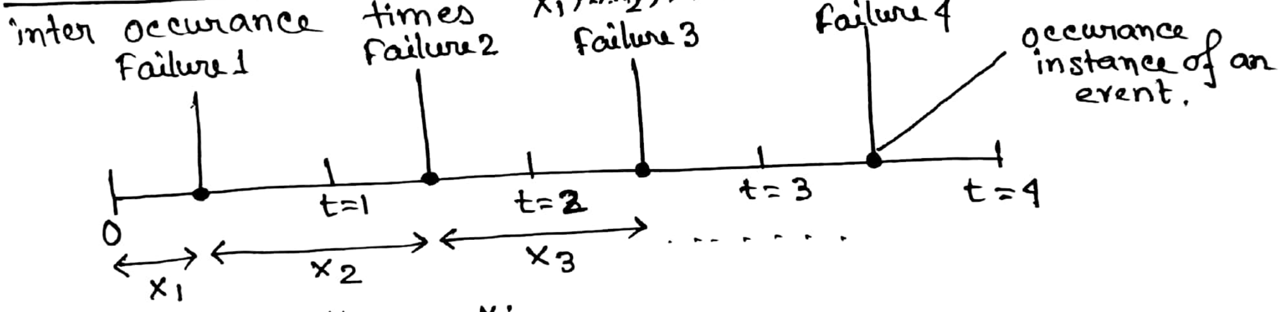
Some Stochastic Processes:-

1. **Markov Chain (MC)**:- An SP having discrete time and satisfying markov property is called a MC.

2. **Counting Process (CP)**:- An SP $\{N(t), t \in [0, \infty)\}$ is said to be a counting process if $N(t)$ is non-negative integer valued and a non-decreasing function of t.
[N(t) gives the count occurances of a given event over the interval $[0, t]$]

e.g.- Number of occurances in the interval 0-5 if say 5, then no. of occurances in the interval 0-10 should be greater than equal to 0-5.

3. **Renewal Process (RP)**:- A CP is called an RP if the inter occurance times $X_1, X_2, \dots$ are iid RVs.



inter occurance time $= X_i$

$T_1, T_2, T_3, \dots$ are occurance instancer.

$X_1 = T_1,\ X_2 = T_2 - T_1,\ X_3 = T_3 - T_2 \dots$

→ RP has implication on non-repairable system.

4. **Poisson Process (PP)**:- An RP is called a PP if $X_1, X_2, \dots, X_n$ are each exponential random variable.

Application of Stochastic Process:-

1. Queueing Theory
2. Reliability and Warranty Analysis.
3. Time Series
4. Economics (Stock Price & Movement)
5. Signal Processing
6. Video Processing.

SP : $\{X(t),\ t \in T\}$

CP : $\{N(t), t \in [0,\infty)\}$

Poisson Process : $N(t)$: Number of occurances of a specified event in the interval $[0,t]$.

Definition:1 (Axiomatic way)

A CP $\{N(t), t \geq 0\}$ is said to be a Poisson Process with intensity / rate $\lambda (>0)$ if the following conditions hold:

C1 : $N(0) = 0$

C2 : The process has independent increment.

C3 : The no. of occurance of a time interval of length t has Poisson distn with rate λt, i.e.,

$$P[N(s+t) - N(s) = n] = e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!} \; ; \; n = 0,1,2,\ldots \quad s,t \geq 0$$

Remark: C3 implies that the process has stationary increments.

Note: A function $f : \mathbb{R} \to \mathbb{R}$ is said to be of order h, $o(h)$ if $\lim_{h\to 0} \frac{f(h)}{h} = 0$.

e.g. $f(x) = x^2$ and $\lim_{x\to 0} \frac{f(x)}{x} = 0$.

Definition: 2 : A CP $\{N(t) : t \geq 0\}$ is called a Poisson Process with intensity $\lambda (>0)$ if

C1 : $N(0) = 0$

C2 : $N(\cdot)$ is a non-decreasing process. $\Big\}$ → Counting process.

C3 : $P(N(t+s) - N(s) = n) = P(N(t) = n) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}$, $n = 0,1,2.$

i.e., $N(t)$ has Poisson distn with parameter λt.

C4 : The process has independent and stationary increments.

i.e., if $0 < t_1 < \cdots < t_n < \infty$, then $N(t_1), N(t_2) - N(t_1), \ldots\ldots, N(t_n) - N(t_{n-1})$ are independent RVs.

and for $0 \leq s \leq t$, $N(t) - N(s) = N(t+h) - N(s+h)$.

C5 : $\lim_{h\downarrow 0} \frac{1}{h} P(N(h) \geq 2) = 0$

i.e., Prob. of no. of occurances to be 2 or more in a small interval of length h is zero.

C6 : $\exists$ a constant $\lambda > 0$ s.t. $P(N(h) = 1) = \lambda h + o(h)$, where $\lim_{h\downarrow 0} \frac{1}{h} o(h) = 0$.

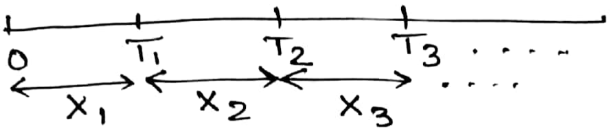
i.e., probability of no. of occurances to be 1 is λh.

Definition :3:- (The constructive way)

A CP $\{N(t), t \geq 0\}$ is called a Poisson Process with rate (intensity) λ if

$$N(t) = \sum_{n=1}^{\infty} I_{(0,t]}(T_n)$$

where, $\{T_n\}_{n\geq 1}$ has iid increments $X_1, X_2, \ldots$ as $Exp(\lambda)$ as inter-occurance times are exponential.

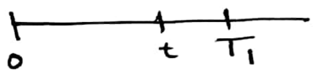


T_n is the time point when n^{th} occurance has taken place.

Theorem:- Show that definition 1 and 3 are equivalent.

Proof:- Definition 1 $\Rightarrow$ Definition 3.

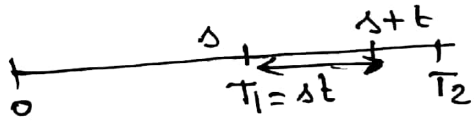
$\rightarrow$ $\bar{F}_{X_1}(t) = P[X_1 > t] = P[N(t) = 0] = e^{-\lambda t}$



1^{st} occurance happens beyond $(0, t)$.

$f_{X_1}(t) = -\lambda e^{-\lambda t}$

$\therefore X_1 \sim Exp(\lambda)$



$$\bar{F}_{X_2}(t) = P[X_2 > t \mid X_1 = s]$$

$= P[$No occurance in the time interval $(s, s+t]]$

$= P[$No occurance in the time interval $(0, t)]$

$= P[N(t) = 0]$

$= e^{-\lambda t}$.

$\therefore X_2 \sim Exp(\lambda)$ and is independent of X_1.

So, successive interoccurance times are independent of all the previous ones.

Definition 3 $\Rightarrow$ Definition 1.

$\rightarrow$ $P\{N(0)>0\}=0 \Rightarrow P\{X_1=0\}=0$ [X_1 is continuous RV (exponential)]

Starting with time point $t=0$ and till time point $t=0$, one or more occurance has taken place which is impossible.

So, $N(0)=0$ holds [C1 holds]

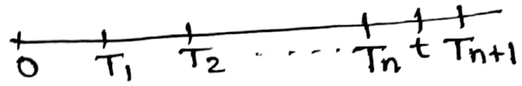
Now, we need to compute $P(N(t)=n)$.

$T_1 = X_1$

$T_2 = X_2 + X_1$

$\vdots$

$$T_n = \sum_{i=1}^{n} X_i$$



So, $T_n \sim \text{Gamma}(n, \lambda)$

$$f_{T_n}(x) = \frac{\lambda^n}{\Gamma n} e^{-\lambda x} x^{n-1} \quad ; x>0$$

So, $P(N(t)=n) = P[T_n \le t \le T_{n+1}]$

$$= P[T_n \le t < T_n + X_{n+1}]$$

$$= \int_0^\infty P[T_n \le t < T_n + X_{n+1} \mid T_n = x] f_{T_n}(x)\,dx$$

$$= \int_0^t P[X_{n+1} > t-x] f_n(x)\,dx$$

$$= \int_0^t e^{-\lambda(t-x)} \frac{\lambda^n}{\Gamma n} e^{-\lambda x} x^{n-1}\,dx$$

$$= \frac{e^{-\lambda t}\lambda^n}{\Gamma n} \int_0^t x^{n-1}\,dx$$

$$= \frac{e^{-\lambda t}(\lambda t)^n}{n!} \quad ; \quad \text{So, } N(t) \sim \text{Pois}(\lambda t).$$

Increments and intervals are iid. So, C2 follows.

— Also, as exponential distribution has memoryless property, so we can write $P(Y > s+t \mid Y > s) = P(Y>t)$.

which gives, $P(N(t)=n) = P(N(t) - N(0) = n)$

$$= P(N(s+t) - N(s) = n) \quad \forall s \ge 0.$$

Theorem:- Show that definition 1 and 2 are equivalent.

Proof:- Definition 1 $\Rightarrow$ Definition 2

$$P[N(h)=1] = P[N(h)-N(0)=1] \qquad [\because N(0)=0]$$
$$= e^{-\lambda h}\frac{\lambda h}{1!}$$
$$= \lambda h\left(1-\lambda h+\frac{(\lambda h)^2}{2!}-\frac{(\lambda h)^3}{3!}+\cdots\cdots\right)$$
$$= \lambda h + o(h)$$

$$P(N(h)\geq 2) = 1-P[N(h)=0]-P[N(h)=1]$$
$$= 1-e^{-\lambda h}-\lambda h+o(h)$$
$$= 1-\lambda h\left(1-\lambda h+\frac{(\lambda h)^2}{2!}-\cdots\right)+o(h)$$
$$= o(h).$$

Definition 2 $\Rightarrow$ Definition 1

$$p_n(t) = P[N(t)=n] = P[N(s+t)-N(s)=n] \quad [\text{Due to stationary increments}]$$

$p_0(t)$ = No occurance take place in $(0,t)$.

$$p_0(t+h) = P[N(t+h)=0]$$
$$= P[N(t)=0,\ N(t+h)-N(t)=0]$$
$$= P[N(t)-N(0)=0]\,P[N(t+h)-N(t)=0]$$
$$= P[N(t)=0]\,P(N(h)=0) \quad [\text{By independent increment}]$$
$$= p_0(t)\,p_0(h)$$
$$= p_0(t)\,[1-P(N(h)\geq 1)]$$
$$= p_0(t)\,[1-\lambda h+o(h)]$$

$$\Rightarrow \lim_{h\to 0}\frac{p_0(t+h)-p_0(t)}{h} = -\lambda p_0(t)$$

$$\Rightarrow p_0'(t) = -\lambda p_0(t) \Rightarrow p_0(t) = ke^{-\lambda t}; \quad \text{when } t=0,\ k=1.$$

$$\Rightarrow p_0(t) = e^{-\lambda t}.$$

For $n \geq 1$,

$$p_n(t+h) = p_n(t)\,p_0(h) + p_{n-1}(t)\,p_1(h) + \sum_{r \geq 2} p_{n-r}(t)\,p_r(h)$$

$$= [1 - \lambda h + o(h)]\,p_n(t) + \lambda h\,p_{n-1}(t) + o(h)$$

$$= p_n(t) - \lambda h\,p_n(t) + \lambda h\,p_{n-1}(t) + o(h)$$

$$\therefore \lim_{h \to 0} \frac{p_n(t+h) - p_n(t)}{h} = -\lambda p_n(t) + \lambda p_{n-1}(t)$$

i.e., $p_n'(t) = -\lambda p_n(t) + \lambda p_{n-1}(t) \;;\; n \geq 1$.

$$\therefore e^{\lambda t} p_n'(t) + \lambda e^{\lambda t} p_n(t) = \lambda e^{\lambda t} p_{n-1}(t)$$

$$\therefore \frac{d}{dt}\left(e^{\lambda t} p_n(t)\right) = \lambda e^{\lambda t} p_{n-1}(t) \quad \text{———} \; (*)$$

Putting $n=1$,

$$\frac{d}{dt}\left(e^{\lambda t} p_1(t)\right) = \lambda e^{\lambda t} p_0(t) = \lambda .$$

$$\Rightarrow e^{\lambda t} p_1(t) = \lambda t + k \quad ; \quad \text{For } t=0,\; p_1(0) = k = 0 .$$

$$\Rightarrow p_1(t) = e^{-\lambda t} \cdot \frac{(\lambda t)}{1!}$$

Now putting $n = m+1$, in $(*)$,

$$\frac{d}{dt}\left(e^{\lambda t} p_{m+1}(t)\right) = \lambda e^{\lambda t} p_m(t)$$

$$= \lambda e^{\lambda t} \cdot e^{-\lambda t} \cdot \frac{(\lambda t)^m}{m!}$$

$$= \frac{\lambda^{m+1} t^m}{m!}$$

$$\Rightarrow e^{\lambda t} p_{m+1}(t) = \frac{\lambda^{m+1} t^{m+1}}{(m+1)!} + k$$

Put $t=0$, then $k=0$.

$$\therefore p_{m+1}(t) = \frac{e^{-\lambda t} (\lambda t)^{m+1}}{(m+1)!}$$

Properties of Poisson Process:-

If $N(t) \sim Poi(\lambda t)$, then

(i) $E(N(t)) = \lambda t$

(ii) $V(N(t)) = \lambda t$

(iii) $Cov(s,t) = C_N(s,t) = Cov[N(s), N(t)]$, $s<t$

$$= E[N(s)N(t)] - E[N(s)]E[N(t)]$$

$$= E[N(s)(N(t)-N(s)) + N^2(s)] - \lambda^2 st$$

$$= E[\{N(s)-N(0)\}\{N(t)-N(s)\}] + E[N(s)]^2 - \lambda^2 st$$

$$= E[N(s)-N(0)]E[N(t)-N(s)] + Var(N(s)) + [E\{N(s)\}]^2 - \lambda^2 st$$

$$= \lambda s \cdot \lambda(t-s) + \lambda s + \lambda^2 s^2 - \lambda^2 st$$

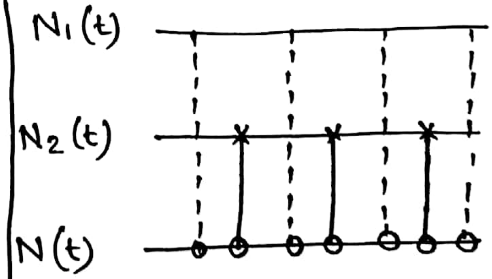
$$= \lambda s.$$

(iv) $$\rho_N(s,t) = \frac{cov(s,t)}{\sqrt{V(N(s))\,V(N(t))}} = \frac{\lambda s}{\sqrt{\lambda s . \lambda t}} = \sqrt{\frac{s}{t}} \quad \text{as } s<t$$

So, $0 < \rho_N(s,t) < 1$.

(v) Sum of two independent poisson processes is also a Poisson Process.

Proof:- Let $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ be two independent poisson processes with rates λ_1 and λ_2, respectively. Write $N(t) = N_1(t) + N_2(t)$ for all t. Then we are going to show $\{N(t), t \geq 0\}$ is a Poisson Process with rate $\lambda_1 + \lambda_2$.



(Combined Process)

C_1: $N(0) = N_1(0) + N_2(0) = 0$

C_2: $N(t_2) - N(t_1)$

$$= N_1(t_2) - N_1(t_1) + N_2(t_2) - N_2(t_1)$$

$N(t_4) - N(t_3)$

$$= N_1(t_4) - N_1(t_3) + N_2(t_4) - N_2(t_3)$$

$(N_1(t_2) - N_1(t_1))$ and $(N_1(t_4) - N_1(t_3))$ are independent.

Also, $(N_2(t_2) - N_2(t_1))$ and $(N_2(t_4) - N_2(t_3))$ are independent. (since $N_1(t)$ is a PP)

So, $(N(t_2) - N(t_1))$ and $(N(t_4) - N(t_3))$ are independent.

PGF of $N_i(t)$ is $E\left(s^{N_i(t)}\right) = e^{\lambda_i t(s-1)}$; $i=1,2,\ldots$

$$\left[X \sim Pois(\lambda),\ P_X(s) = E(s^X) = \sum_{x=0}^{\infty} s^x \cdot \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \cdot e^{\lambda s} = e^{\lambda(s-1)}. \right]$$

PGF of $N(t)$ is —

$$E\left[s^{N(t)}\right] = E\left[s^{N_1(t)+N_2(t)}\right] = P_{N_1(t)}(s) P_{N_2(t)}(s)$$ [$\because$ $N_1(t)$ and $N_2(t)$ are independent]

$$= e^{(\lambda_1+\lambda_2)t(s-1)}$$

$\therefore$ $N(t) \sim Pois((\lambda_1+\lambda_2)t)$ [since PGF uniquely defines a distribution]

Stationary Increment:-

Enough to show $N(t) - N(s) \simeq N(t-s)$.

$\rightarrow$ For $s,t \geq 0$

$$N(t) - N(s) = N_1(t) + N_2(t) - N_1(s) - N_2(s)$$

$$= N_1(t-s) + N_2(t-s)$$ [As both $N_1(t)$ and $N_2(t)$ are P.P.]

$$= N(t-s).$$

So, $\{N(t),\ t \geq 0\}$ is a PP with rate $(\lambda_1+\lambda_2)$.

Ques:- (Continued) For a combined Poisson Process $N(t) = N_1(t) + N_2(t)\ \forall t$, the probability that the first combined process comes from occurance of the $\{N_1(t),\ t \geq 0\}$ is $\frac{\lambda_1}{\lambda_1+\lambda_2}$.

Proof:-

$$P(N_1(t)=1 \mid N(t)=1) = \frac{P(N_1(t)=1, N(t)=1)}{P(N(t)=1)}$$

$$= \frac{P(N_2(t)=0, N_1(t)=1)}{P(N(t)=1)}$$

$$= \frac{P(N_2(t)=0)\, P(N_1(t)=1)}{P(N(t)=1)}$$, due to independence.

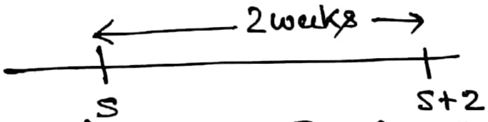
$$= \frac{e^{-\lambda_1 t}\lambda_1 t \cdot e^{-\lambda_2 t}}{e^{-\lambda t} \cdot \lambda t} = \frac{\lambda_1}{\lambda_1+\lambda_2}.$$

Example 1:- A m/c goes out of order whenever a particular component fails. The failure of this component is in accordance with a poisson process having rate of 1/week.

(a) What is the prob. that two consecutive weeks have elapsed since the last failure?

(b) Suppose there are five spares of the conponent in the inventory and that the next supply, not due in the weeks. What's the prob. the m/c will not be out of order in the next 10 weeks?

Solution:-



(a)

We want to find $P_r\left[N(s+2)-N(s)=0\right]$.

$\{N(t): t\geq 0\}$ is a Poisson process with rate 1 per week., i.e., $PP(1)$.

$N(t)$ = Number of failures occur $[0,t]$ in weeks.

$$P\left[N(s+2)-N(s)=0\right] = P\left[N(2)-N(0)=0\right]$$
$$= P\left[N(2)=0\right]$$
$$= e^{-2}.$$

(b) $$P\left(N(10)\leq 5\right) = \sum_{x=0}^{5}\frac{e^{-10}\,10^{x}}{x!}$$

$\lambda = 1/\text{week}.$
$t = 10$ weeks.

$$= 0.671.$$

Example 2 :- Let $N(t)$ be the number of failures of a computer system in the interval $[0,t]$. We suppose that $\{N(t), t \geq 0\}$ is PP with rate $\lambda = 1/\text{week}$.

(a) Calculate the prob. that the system operate without failures for 2 consecutive weeks.

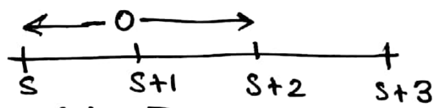
(b) Calculate that the system will have exactly two failures during a given week knowing that it operates without failure previous two weeks.

(c) Calculate the prob. that two weeks elapse before the third failure occurs.

Sol.

(a) $P[N(2)=0] = e^{-2}$.

(b)



$$\Pr[N(s+3)-N(s+2)=2 \mid N(s+2)-N(s)=0]$$

$$= P[N(s+3)-N(s+2)=2] \quad [\text{Due to independence increments}]$$

$$= P[N(1)=2]$$

$$= \frac{e^{-\lambda t}(\lambda t)^2}{2!} \quad ; \quad t=1, \lambda=1.$$

$$= \frac{1}{2e}.$$

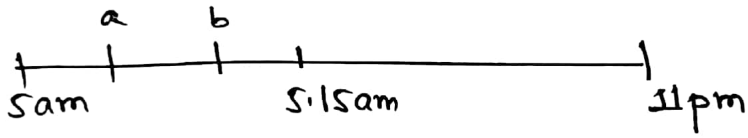
(c) $P[X_3 > 2]$ $\qquad$ X_3: Interoccurance time

$$= e^{-2\lambda} \quad [\text{since } X_3 \sim Exp(\lambda)]$$

$$= e^{-2}.$$

Example 3:- City buses arrive at a certain street corner between 5am and 11pm according to Poisson process with rate $\lambda = 4/hr$. Let T_1 be the waiting time (in minutes) until the first bus (after 5am) and M be the total number of buses arrive at this corner between 5 am and 5:15 am, Calculate the probability $P[T_1 \in [a,b] \mid M=1]$, when the interval $[a,b]$ is included in $[0,15]$.

Sol.



$N(t)$ = No. of buses arrive at the corner during $[0,t]$; t is in minutes.

$\{N(t) : t \geq 0\}$ is a PP $(1/15)$

$$P[T_1 \in [a,b] \mid M=1] = P[N(a)-N(0)=0,\ N(b)-N(a)=1,\ N(15)-N(b)=0]$$

$$= P[N(a)-N(0)=0]\,P[N(b)-N(a)=1] \times P[N(15)-N(b)=0]$$ [PP has independent increments]

$$= P[N(a)=0]\,P(N(b-a)=1)\,P(N(15-b)=0)$$ [PP has stationary increments]

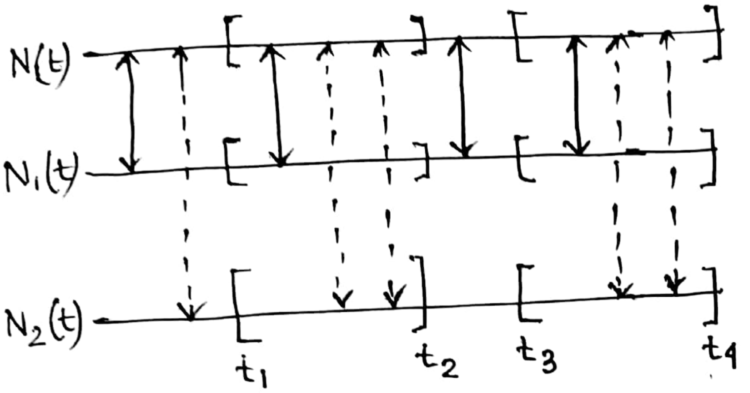
$$= e^{-\lambda a}\, e^{-\lambda(b-a)}\, \lambda(b-a) \times e^{-\lambda(15-b)}$$ [$N(t) \sim PP$]

$$= e^{-\lambda(a+b-a+15-b)} \cdot \lambda(b-a) = \lambda(b-a)\, e^{-\lambda \cdot 15}$$

$$= \frac{b-a}{15}\, e^{-1}$$

Example 4:- (Decomposition of PP)

Let $\{N(t), t \geq 0\}$ be a $PP(\lambda)$. In this process as and when an occurance takes place, it is classified as Type-I with probability $p\ (0<p<1)$ and as Type-II with probability $q\ (=1-p)$. Call the two streams of occurances as $\{N_1(t), t\geq 0\}$ and $\{N_2(t), t \geq 0\}$



Note that,
$N(t) = N_1(t) + N_2(t)$, $t \geq 0$
Then the processes $\{N_i(t), t\geq 0\}$; $i=1,2$, are independent Poisson Process with rates λp and λq respectively.

Proof:-

(i) $0 \leq N_i(0) \leq N(0) = 0$
$\Rightarrow N_i(0) = 0$, $i=1,2$.

(ii) Because $\{N(t), t\geq 0\}$ has independent increments.
So, $\{N(t_i), t \geq 0\}$; $i=1,2$ will have independent increments.

(iii) We first show that, $N_1(s+t)-N_1(s)$ and $N_2(s+t)-N_2(s)$ are independent poisson.

Consider for non-negative integers m, n,

$P[N_1(s+t)-N_1(s)=m,\ N_2(s+t)-N_2(s)=n]$

$= P[N(s+t)-N(s) = m+n$, out of $(m+n)$ occurances, m are of type I$]$

$$= \frac{e^{-\lambda t}(\lambda t)^{m+n}}{(m+n)!}\binom{m+n}{m}p^m q^n = \left\{\frac{e^{-\lambda pt}(\lambda pt)^m}{m!}\right\}\left\{\frac{e^{-\lambda qt}(\lambda qt)^n}{n!}\right\} \quad ---- (*)$$

from (*),

$$\sum_{n=0}^{\infty} P[N_1(s+t)-N_1(s)=m,\ N_2(s+t)-N_2(s)=n] = \sum_{n=0}^{\infty}\frac{e^{-\lambda pt}(\lambda pt)^m}{m!}\times\frac{e^{-\lambda qt}(\lambda qt)^n}{n!} \quad \text{——— (1)}$$

$$\Rightarrow P[N_1(s+t)-N_1(s)=m] = \frac{e^{-\lambda pt}(\lambda pt)^m}{m!} \quad \text{——— (2)}$$

Similarly, $$P[N_2(s+t)-N_2(s)=n] = \frac{e^{-\lambda qt}(\lambda qt)^n}{n!} \quad \text{——— (3)}$$

Putting $s=0$ in (1), (2), (3); we get

(a) joint distn. = product of marginal distns.

(b) $N_1(t) \sim \text{Pois}(\lambda p)$

$N_2(t) \sim \text{Pois}(\lambda q)$

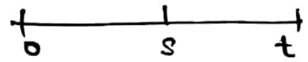
(c) Both $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ have stationary increments.

Theorem:- Suppose that one occurrance has taken place by time t in a PP. Then the distribution of this occurance time is $U(0,t)$.

$$[T_1 \mid N(t)=1] \sim U(0,t)$$

Proof:- We have to find the distn. of $T_1 \mid N(t)=1$.

Consider the distn. function:-



$$F_{T_1|N(t)}(s) = P[T_1 \leq s \mid N(t)=1]$$

$$= \frac{P[T_1 \leq s, N(t)=1]}{P[N(t)=1]}$$

$$= \frac{P[N(s)-N(0)=1, N(t)-N(s)=0]}{P[N(t)=1]}$$

$$= \frac{P[N(s)=1]\, P[N(t-s)=0]}{P[N(t)=1]}$$

$$= \frac{\{e^{-\lambda s}\lambda s\}\{e^{-\lambda(t-s)}\}}{e^{-\lambda t}\lambda t}$$

$$= \frac{s}{t}\ ; \quad 0<s<t.$$

So, the density function is $f_{T_1|N(t)}(s) = \frac{1}{t}\ ; \quad 0<s<t.$

Lemma:- Let $Y_1, Y_2, \ldots, Y_n \overset{iid}{\sim} U(0,t)$. Then the joint distn. of order statistics corresponding to $Y_1, Y_2, \ldots, Y_n$ denoted by $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ is given by

$$f_{Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}}(y_1, y_2, \ldots, y_n) = \frac{n!}{t^n} \; ; \; 0 < y_1 < y_2 < \cdots < y_n < t$$

Theorem:- For a given PP, let $N(t) = n$. Then the joint distn. of the n occurance times $T_1 < T_2 < \ldots < T_n$ is same as that of order statistics corresponding to n iid $U(0,t)$ r.v.s.

Proof:- Let $0 < t_1 < \cdots < t_n < t_{n+1} = t$ and $t_i + h_i < t_{i+1}$ for $i = 1(1)n$, where $h_i > 0 \; \forall i$

Consider $P\left[T_i \in [t_i, t_i + h_i],\; i = 1(1)n \,\middle|\, N(t) = n\right]$

$$= \frac{P\left[T_i \in [t_i, t_i+h_i], i=1(1)n, N(t)=n\right]}{P[N(t)=n]}$$

$$= \frac{P\left[\text{exactly one occurance in } [t_i, t_i+h_i] \text{ for } i=1(1)n, \text{ no occurance elsewhere}\right]}{P[N(t)=n]}$$

$$= \frac{\left[\prod_{i=1}^{n}\left\{e^{-\lambda h_i}\lambda h_i\right\}\right] e^{-\lambda t_1} e^{-\lambda(t_2 - t_1 - h_1)} \cdots e^{-\lambda(t - t_n - h_n)}}{\frac{e^{-\lambda t}(\lambda t)^n}{n!}}$$ [Due to independence and stationarity]

$$= \frac{e^{-\lambda\sum_{i=1}^{n} h_i}\lambda^n \prod_{i=1}^{n} h_i \, e^{-\lambda t + \lambda\sum_{i=1}^{n} h_i}}{\frac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$= \frac{n! \prod_{i=1}^{n} h_i}{t^n}$$

$$\therefore \lim_{\substack{h_i \to 0 \\ \forall i}} \frac{P\left[T_i \in [t_i, t_i+h_i]; i=1(1)n \mid N(t)=n\right]}{\prod_{i=1}^{n} h_i} = \frac{n!}{t^n}, \quad 0<t<1.$$

$$\therefore f_{(T_1, T_2, \ldots, T_n) | N(t)=n}(t_1, t_2, \ldots, t_n) = \frac{n!}{t^n} \; ; 0 < t < 1.$$

Theorem:- In a Poisson Process, let $0 < T_1 < T_2 < \cdots < T_n$ be the occurrance times. Given $T_n = t_n$, the joint distn. of $(T_1, T_2, \ldots, T_{n-1})$ is same as that of the order statistics corresponding to $(n-1)$ $U(0, t_n)$ r.v.s.

Proof:- Let $0 < t_1 < t_2 < \cdots < t_{n-1} < t_n$. Then the conditional joint density function of $(T_1, T_2, \ldots, T_{n-1})$ given $T_n = t_n$ is

$$f_{(T_1, \ldots, T_{n-1}) | T_n = t_n}(t_1, \ldots, t_{n-1}) = \frac{h(t_1, \ldots, t_n)}{g(t_n)}.$$

where, $h(t_1, \ldots, t_n)$ is the joint pdf of $T_1, \ldots, T_n$ and $g(t_n)$ is the pdf of T_n.

Now,

$$h(t_1, \ldots, t_n) = \left\{\lambda e^{-\lambda t_1}\right\}\left\{\lambda e^{-\lambda(t_2 - t_1)}\right\}\left\{\lambda e^{-\lambda(t_3 - t_2)}\right\} \cdots \left\{\lambda e^{-\lambda(t_n - t_{n-1})}\right\}$$

$$= \lambda^n e^{-\lambda t_n}$$

$$\therefore f_{(T_1, \ldots, T_{n-1}) | T_n = t_n}(t_1, \ldots, t_n) = \frac{\lambda^n e^{-\lambda t_n}}{\frac{\lambda^n}{\Gamma n} e^{-\lambda t_n} t_n^{n-1}}$$

$$= \frac{(n-1)!}{t_n^{n-1}} \; ; \; 0 < t_1 < \cdots < t_n.$$

Non-Homogeneous Poisson Process (NHPP) :-

We generalise the Poisson Process by allowing the intensity function (or, state) at time t to be a function of t.

Definition:- A counting process $\{N(t), t \geq 0\}$ is said to be a non-homogeneous or non-stationary PP with intensity function $\lambda(t)$, $t \geq 0$ if

C1 : $N(0) = 0$

C2 : The process has independent increments.

C3 : $P[N(t+h) - N(t) = 1] = \lambda(t)h + o(h)$.

C4 : $P[N(t+h) - N(t) \geq 2] = o(h)$.

Theorem:- If $\{N(t), t \geq 0\}$ is a NHPP then $N(t) \sim Poi(\Lambda(t))$

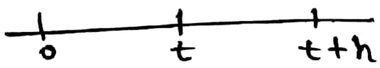
i.e., $P[N(t) = n] = e^{-\Lambda(t)} \dfrac{[\Lambda(t)]^n}{n!}$, $n = 0, 1, 2, \ldots$

where, $\Lambda(t) = \int_0^t \lambda(x)dx$ = Mean function.

Proof:- Let $P_n(t) = P[N(t) = n]$ for $n = 0, 1, \ldots$

Note that ① $P_0'(t) = -\lambda(t) P_0(t)$

② $P_n'(t) = -\lambda(t) P_n(t) + \lambda(t) P_{n-1}(t)$, $n = 1, 2, \ldots$



① $P_0(t+h) = P\{N(t+h) = 0\}$

$= P\{N(t) = 0, N(t+h) - N(t) = 0\}$

$= P\{N(t) = 0\} P\{N(t+h) - N(t) = 0\}$

$= P_0(t)[1 - \lambda(t)h + o(h)]$

$P_n(t+h) = P\{N(t+h) = n\}$

$= P_n(t)[1 - \lambda(t)h + o(h)] + P_{n-1}(t)[\lambda(t)h + o(h)] + \sum_{r \geq 2} P_{n-r}(t) o(h)$

From ①, $P_n(t) = e^{-\Lambda(t)}$ $[\because P_0(0) = 1]$

From (11), $\frac{d}{dt}\left[e^{\Lambda(t)} P_n(t)\right] = \lambda(t) e^{\Lambda(t)} P_{n-1}(t)$

$$= \lambda(t) e^{\Lambda(t)} e^{-\Lambda(t)}$$

$$= \lambda(t).$$

$\Rightarrow e^{\Lambda(t)} P_1(t) = \Lambda(t) + K \quad [P_1(0) = 0 \Rightarrow K = 0]$

$\Rightarrow P_1(t) = e^{-\Lambda(t)} \Lambda(t)$

Apply induction to complete the proof.

<u>Theorem</u>:- For an NHPP $\{N(t), t \geq 0\}$ having mean functn $\Lambda(t)$, for $0 < a < b$,

$N[a, b] = N(b) - N(a) \sim \text{Poi}\left(\Lambda(b) - \Lambda(a)\right)$; where

$$\Lambda(b) - \Lambda(a) = \int_a^b \lambda(x)\,dx.$$

<u>Proof</u>:- $X \sim \text{Poi}(\lambda)$, MGF is

$$M_X(s) = E\left(e^{sX}\right) = \sum_{x=0}^{\infty} e^{sx} \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!} = e^{-\lambda} \cdot e^{\lambda e^s} = e^{\lambda[e^s - 1]}$$

$$M_{N(b)}(s) = E\left[e^{sN(b)}\right] = E\left[e^{s\{N(a) + N(b) - N(a)\}}\right]$$

$$= E\left[e^{sN(a)}\right] E\left[e^{sN(a,b]}\right]$$

$$= M_{N(a)}(s) M_{N(a,b]}(s).$$

$$\Rightarrow M_{N(a,b]}(s) = \frac{M_{N(b)}(s)}{M_{N(a)}(s)} = \frac{e^{\Lambda(b)[e^s - 1]}}{e^{\Lambda(a)[e^s - 1]}}$$

$$= e^{[\Lambda(b) - \Lambda(a)][e^s - 1]}$$

$\therefore N(a, b] \sim \text{Poi}\left(\Lambda(b) - \Lambda(a)\right).$

$\lambda(t)$: Intensity function.

$\Lambda(t)$: mean function.

Theorem:- If an NHPP has the intensity function $\lambda(t)$ then the density function of $[T_1 \mid N(t)=1]$ is given by

$$f(s) = \frac{\lambda(s)}{\Lambda(t)} , \quad 0 < s < t.$$

Proof:-

$$F_{T_1 | N(t)=1} = P\left[T_1 \le s \mid N(t)=1\right] \quad ; \; 0 \le s \le t$$

$$= \frac{P[T_1 \le s, N(t)=1]}{P[N(t)=1]}$$

$$= \frac{P[N(s)=1, N(t)-N(s)=0]}{P[N(t)=1]}$$

$$= \frac{P[N(s)=1]\, P[N(t)-N(s)=0]}{P[N(t)=1]}$$

$$= \frac{\left\{e^{-\Lambda(s)}\Lambda(s)\right\}\left\{e^{-[\Lambda(t)-\Lambda(s)]}\right\}}{e^{-\Lambda(t)}\Lambda(t)}$$

$$= \frac{\Lambda(s)}{\Lambda(t)}.$$

$$\therefore f_{T_1 | N(t)=1}(s) = \frac{\lambda(s)}{\Lambda(t)} , \quad 0 < s < t.$$

Theorem:- For an NHPP having intensity functn $\lambda(t)$, the joint pdf of the occurance times $T_1, T_2, \ldots, T_n$ is given by

$$f(t_1, \ldots, t_n) = \left\{\prod_{i=1}^{n} \lambda(t_i)\right\} e^{-\Lambda(t_n)} \quad ; \quad 0 < t_1 < t_2 < \cdots < t_n.$$

Proof:- We know that

$$f(t_1, t_2, \ldots, t_n) = f_1(t_1) f_2(t_2|t_1) f_3(t_3|t_1 t_2) \cdots f_n(t_n|t_1 t_2 \cdots t_{n-1})$$

$$= f(t_1) \cdot \frac{g_2(t_1, t_2)}{f_1(t_1)} \cdot \frac{g_3(t_1, t_2, t_3)}{g_2(t_1, t_2)} \cdots\cdots$$

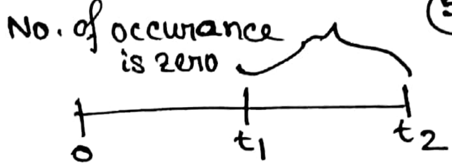
$$\bar{F}_1(t_1) = P[T_1 > t_1]$$

$$= P[N(t_1)=0]$$

$$= e^{-\Lambda(t_1)}$$

$$\Rightarrow f_1(t_1) = \lambda(t_1) e^{-\Lambda(t_1)}.$$

$$\bar{F}_2(t_2|t_1) = P[T_2 > t_2 | T_1 = t_1]$$



$$= P[N(T_2) - N(t_1) = 0]$$

$$= e^{-[\Lambda(t_2) - \Lambda(t_1)]}$$

$$\therefore f_2(t_2|t_1) = \lambda(t_2) e^{-[\Lambda(t_2) - \Lambda(t_1)]}$$

$$\therefore f_3(t_3|t_1, t_2) = \lambda(t_3) e^{-[\Lambda(t_3) - \Lambda(t_2)]}$$

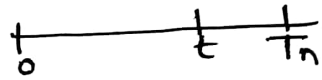
$$\therefore f(t_1, t_2, \dots, t_n) = \left\{ \prod_{i=1}^{n} \lambda(t_i) \right\} e^{-\Lambda(t_n)}$$

Lemma:- The pdf of T_n is given by

$$f_{T_n}(t) = \lambda(t) e^{-\Lambda(t)} \frac{[\Lambda(t)]^{n-1}}{(n-1)!}$$

Proof:- Consider the event

$$\bar{F}_{T_n}(t) = P[T_n > t] = P[N(t) < n]$$



$$= \sum_{r=0}^{n-1} e^{-\Lambda(t)} \cdot \frac{[\Lambda(t)]^r}{r!}$$

$$\therefore -f_{T_n}(t) = -\lambda(t) \sum_{r=0}^{n-1} e^{-\Lambda(t)} \frac{[\Lambda(t)]^r}{r!} + \lambda(t) \sum_{r=0}^{n-2} e^{-\Lambda(t)} \cdot \frac{[\Lambda(t)]^r}{r!}$$

$$= -\lambda(t) e^{-\Lambda(t)} \frac{[\Lambda(t)]^{n-1}}{(n-1)!}$$

$$\therefore f_{T_n}(t) = \lambda(t) e^{-\Lambda(t)} \frac{[\Lambda(t)]^{n-1}}{(n-1)!}$$

Theorem (Failure censoring):- The joint distn. of $(T_1, \dots, T_{n-1})$ given $T_n = t_n$ is same as that of $(n-1)$ order statistics from the distn. having CDF $G(y) = \frac{\Lambda(y)}{\Lambda(t_n)}$, $0 < y < t_n$.

Theorem (Time censoring):- Given $N(t) = n$, the joint distn. of $(T_1, T_2, \dots, T_n)$ is same as that of n order statistics of the distn. whose distn. function (CDF) is given by

$$G(y) = \frac{\Lambda(y)}{\Lambda(t)}; \quad 0 < y < t.$$

Example :- The failure of a certain device occur according to NHPP whose intensity function $\lambda(t)$ is given by

$$\lambda(t) = \begin{cases} 0.2 & 0 < t \leq 10 \\ 0.3 & t > 10 \end{cases}$$

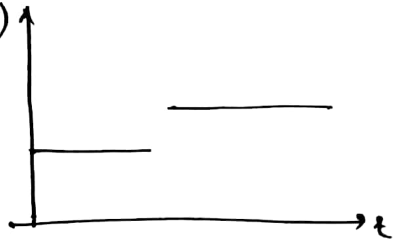
where, t is the age (in years) of the device.

(a) Calculate the prob. that a 5 years old device (without failure) will have exactly two failures over next 10 years.

(b) Knowing that the device had exactly one failure in the course of first 5 years of the 10 years considered in (a), what is the prob. that this failure took place during it's sixth year of use?

Solution:-

(a)



$N(t)$ = No. of failures in $(0,t]$

$N(t) \sim Poi(\Lambda(t))$

$P[N(15) - N(5) = 2 \mid N(5) = 0]$

$= P[N(15) - N(5) = 2]$ [Due to independent increments]

$$= e^{-[\Lambda(15) - \Lambda(5)]} \cdot \frac{[\Lambda(15) - \Lambda(5)]^2}{2!}$$

$$= \frac{e^{-2.5}(2.5)^2}{2}$$

$\Lambda(15)$

$$= \int_0^{10} 0.2\,dx + \int_{10}^{15} 0.3\,dx$$

$= 2 + 1.5 = 3.5$

$\Lambda(5) = 1$.

(b) $P[N(6) - N(5) = 1 \mid N(10) - N(5) = 1]$

$$= \frac{P[N(6) - N(5) = 1,\ N(10) - N(5) = 1]}{P[N(10) - N(5) = 1]}$$

$$= \frac{P[N(6) - N(5) = 1,\ N(10) - N(6) = 0]}{P[N(10) - N(5) = 1]} = \frac{P[N(6) - N(5) = 1]\,P[N(10) - N(6) = 0]}{P[N(10) - N(5) = 1]}$$

$$= \frac{e^{-[\Lambda(6) - \Lambda(5)]}[\Lambda(6) - \Lambda(5)]\,e^{-[\Lambda(10) - \Lambda(6)]}}{e^{-[\Lambda(10) - \Lambda(5)]}[\Lambda(10) - \Lambda(5)]}$$

$$= \frac{\Lambda(6) - \Lambda(5)}{\Lambda(10) - \Lambda(5)} = 0.2.$$

$X(t)$: General Stochastic Process
$N(t)$: Counting Process (HPP, NHPP)
$Y(t)$: Compound Poisson Process

Compound Poisson Process:-

Let $\{N(t), t \geq 0\}$ be a PP with rate λ. Also let that $X_1, X_2, \ldots$ are iid non-negative RVs that are independent of the Poisson Process. Then $\{Y(t), t \geq 0\}$ is called compound Poisson process, where $Y(t) = \sum_{K=1}^{N(t)} X_K$ with $Y(t) = 0$ if $N(t) = 0$.

Is $\{Y(t), t \geq 0\}$ a counting process?

→ Need not be.

Compute $E[Y(t)]$ and $Var(Y(t))$:-

$$M_{Y(t)}(s) = E\left[e^{sY(t)}\right] = E\left[e^{s\sum_{K=1}^{N(t)} X_K}\right]$$

$$= \sum_{n=0}^{\infty} E\left[e^{s\sum_{K=1}^{N(t)} X_K} \,\Big|\, N(t) = n\right] P[N(t) = n]$$

$$= \sum_{n=0}^{\infty} E\left[e^{s\{X_1 + X_2 + \cdots + X_n\}}\right] P[N(t) = n]$$

$$= \sum_{n=0}^{\infty} \left\{\prod_{K=1}^{n} E\left(e^{sX_K}\right)\right\} P[N(t) = n]$$

$$= \sum_{n=0}^{\infty} \{M_{X_1}(s)\}^n \cdot \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= e^{\lambda t[M_{X_1}(s) - 1]}$$

$$M'_{Y(t)}(s) = (\lambda t) M'_{X_1}(s)\, e^{\lambda t[M_{X_1}(s) - 1]}$$

$$\therefore E[Y(t)] = M'_{Y(t)}(s)\Big|_{s=0} = (\lambda t) E[X_1]$$

$$M''_{Y(t)}(s) = (\lambda t) M''_{X_1}(s)\, e^{\lambda t [M_{X_1}(s)-1]} + (\lambda t)^2 \{M'_{X_1}(s)\}^2 e^{\lambda t [M_{X_1}(s)-1]}$$

$$\therefore E[Y^2(t)] = M''_{Y(t)}(s)\Big|_{s=0}$$

$$= (\lambda t) E(X_1^2) + (\lambda t)^2 E^2(X_1)$$

$$\therefore V(Y(t)) = E[Y^2(t)] - E^2[Y(t)]$$

$$= \lambda t\, E(X_1^2)$$

Example 1:- The number of accidents in a town is in accordance with a PP having rate 2/day, and the no. X_K of people involved in the K^{th} accident has the distn.

$$P[X_K = i] = \frac{1}{2^i}, \quad i = 1, 2, \ldots$$

[Assume that, X_K's are independent of themselves and are independent of the Poisson Process]

Find the MGF $Y(t)$ and hence find its mean and variance.

Solution:- $Y(t)$: total no. of people involved in the accidents during $(0, t]$.

$$E[Y(t)] = \lambda t E(X_1) \quad \text{and} \quad Var(Y(t)) = \lambda t\, E(X_1^2).$$

$$M_{X_1}(s) = E[e^{sX_1}] = \sum_{x=1}^{\infty} e^{sx} 2^{-x} = \sum_{x=1}^{\infty} \left(\frac{e^s}{2}\right)^x$$

$$= \frac{e^s/2}{1 - e^s/2} = \frac{e^s}{2 - e^s}.$$

$$E(X_1) = \frac{d}{ds} M_{X_1}(s)\Big|_{s=0}$$

$$M_{Y(t)}(s) = e^{\lambda t [M_{X_1}(s) - 1]}$$

$$= e^{2t\left(\frac{e^s}{2-e^s} - 1\right)} = e^{2t\left(\frac{2e^s - 2}{2 - e^s}\right)}$$

$$M'_{Y(t)}(s)\Big|_{s=0} =$$

Example 2:- Customers arrive at a store in group consisting of 1 or 2 individuals with equal probability and arrivals of groups are in accordance with PP with rate $\lambda = 1/\text{min}$.

(a) Find $E[Y(t)]$.

(b) For $\lambda = 1/2\ /\text{min}$, find the prob. that no. of arrivals in 4 mins is exactly 4.

Solution:- $N(t)$: Number of groups arrived in $(0,t]$

$\{N(t), t \geq 0\}$ is Poisson Process.

$Y(t)$: Total no. of individuals arrive during $(0,t]$

$$Y(t) = \sum_{k=1}^{N(t)} X_K$$ where X_K = Number of individuals in group k

$$= \begin{cases} 1 & \text{with prob } 1/2 \\ 2 & \text{with prob } 1/2 \end{cases}$$

(a) $E[Y(t)] = \lambda t E(X_1) = \frac{3t}{2}$.

(b) Calculate $P[Y(4) = 4]$

$$P_{Y(t)}(s) = E\left[s^{Y(t)}\right]$$

$$= E\left[s^{\sum_{k=1}^{N(t)} X_K}\right]$$

$$= \sum_{n=0}^{\infty} E\left[s^{\sum_{1}^{N(t)} X_K} \,\Big|\, N(t) = n\right] P[N(t) = n]$$

$$= \sum_{n=0}^{\infty} \{P_{X_1}(s)\}^n \cdot e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!}$$

$$\therefore P_{Y(t)}(s) = e^{\lambda t[P_{X_1}(s) - 1]} = e^{\lambda t\left[\frac{s^2+s}{2} - 1\right]} \quad ; \quad \lambda = 1/2, t = 4.$$

$$\therefore P_{Y(4)}(s) = e^{(s^2+s)-2} = e^{-2}\left\{\sum_{n=0}^{\infty} \frac{(s^2+s)^n}{n!}\right\}$$

$P[Y(4) = 4]$ = coefficient of s^4 in the expansion of $P_{Y(4)}(s)$.

$$= e^{-2}\left[\frac{1}{2} + \frac{3}{3!} + \frac{1}{4!}\right]$$

$$= \frac{25}{24} e^{-2} = 0.141$$

PROBLEMS WITH HINTS & SOLUTIONS

POISSON PROCESS

1. Eight light bulbs are turned on at time $t=0$. The lifetime of any particular bulb is independent of lifetime of all other bulbs and is described by $Exp(\lambda)$ having mean $1/\lambda$. Let Y be the time of third failure. Find $E(Y)$ and $V(Y)$?

Solution:-

$B_1 \sim P(\lambda)$
$B_2 \sim P(\lambda)$
$\vdots$
$B_8 \sim P(\lambda)$

So, $N_8(t) \sim Poi(8\lambda t)$

Interoccurance time $X_1 \sim exp(8\lambda)$, $X_2 \sim exp(7\lambda)$, $X_3 \sim exp(6\lambda)$

X_1: first failure time

$$Z = X_1 + X_2 + X_3$$

$$E(Z) = E(X_1) + E(X_2) + E(X_3)$$

$$= \frac{1}{6\lambda} + \frac{1}{7\lambda} + \frac{1}{8\lambda}$$

$$V(Z) = V(X_1) + V(X_2) + V(X_3)$$

$$= \frac{1}{64\lambda^2} + \frac{1}{49\lambda^2} + \frac{1}{36\lambda^2}.$$

2. Let $\{N(t), t \geq 0\}$ be a Poisson Process with rate λ. Define

$$X_t = N(t) - \frac{t}{c} N(c) \quad , 0 \leq t \leq c \; ; \; c > 0 \text{ is a constant.}$$

(a) Calculate $E(X(t))$.

(b) Calculate autocovariance function of $X(t)$.

Hints:-

$$E(X(t)) = \lambda t - \frac{t}{c} \lambda c$$

$$Cov(s,t) = E\left(X(s) X(t)\right) - E\left(X(s)\right) E\left(X(t)\right)$$

$$= E\left[\left\{N(s) - \frac{s}{c} N(c)\right\}\left\{N(t) - \frac{t}{c} N(c)\right\}\right]$$

$$\left[\begin{array}{l} E[N(t)] = \lambda t \\ E[N(c)] = \lambda c \\ E[X(t)] = 0 \end{array}\right]$$

3. Travellers arrive at a bus station from 6 am. According to a Poisson Process with rate $\lambda = 1$ per minute. The first bus leaves T minutes after 6 am.

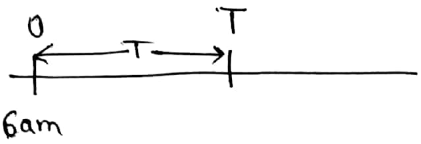
(a) Calculate the mean and variance of the number of travellers ready to board the bus if

(i) T has exponential distn. with mean = 15 mins.

(ii) T has uniform distn. (0, 20 min).

(b) Calculate the average number of passengers on the bus if it leaves at 6.15 am and if its capacity is 20 passengers.

Solution:-



(a)

(i) $N(T) \sim Pois(\lambda t)$ $\qquad T \sim Exp(\theta)$

$E(T) = 15 min$

$$E(N(T)) = E(E(N(T)|T)) = E(\lambda T) = \lambda E(T) = 15\lambda.$$

$$Var(N(T)) = E[Var(N(T)|T)] + Var[E(N(T)|T)]$$

(ii) $E(T) = 10 min$

(b) X: No. of traveller in the bus leaving at 6.15

N: No. of traveller at time t arriving at time point t.

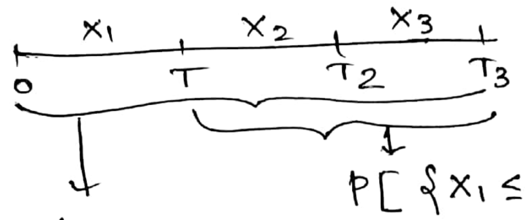
$$X = \begin{cases} x & \text{w.p. } P[N(15) = x] \quad \text{for } x = 0, 1, \dots, 20 \\ 20 & \text{w.p. } P[N(15) \geq 21] \end{cases}$$

$$E[X] = \sum_{x=0}^{20} x P[X = x] + \sum_{x=21}^{\infty} 20 . P[N(15) = x]$$

$$= \sum_{x=0}^{20} \frac{x . e^{-15} . 15^x}{x!} + 20 . \sum_{x=21}^{\infty} \frac{e^{-15} (15)^x}{x!}$$

$\boxed{4}$. Let $\{N(t), t \geq 0\}$ be a poisson process with rate $\lambda = 2$ per minute. What's the probability that the time elapsed between at least two of the first three events of the process is smaller than or equal to one minute?

Sol.



X : Time between first failure

$X_i \overset{iid}{\sim} Exp.$

$P[\{X_1 \leq 1\} \cup \{X_2 \leq 1\}]$

$$= \binom{3}{2} P^2(X_1 \leq 1) P(X_1 > 1) + P^3(X_1 \leq 1).$$

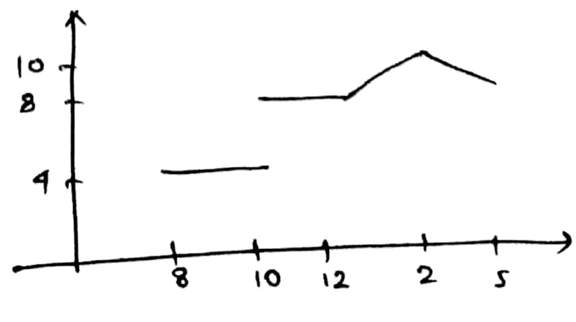
NHPP and COMPOUND PP

$\boxed{5}$. A store opens at 8AM. From 8 until 10, customers arrive at a Poisson rate of four an hour. Between 10 and 12, they arrive at Poisson rate of eight an hour. From 12 to 2, the arrival rate increases steadly from eight per hour, at 12 noon to 10 per hour at 2PM; and from 2 to 5, the arrival rate drops steadily from ten per hour to four per hour. Determine the prob. distn. of the number of customers that arrive the store on a given day.

Sol. $N(t) \sim Poi(\Lambda(t))$

$$P[N(t) = n] = \frac{e^{-\Lambda(t)} [\Lambda(t)]^n}{n!}$$

$$\lambda(t) = \begin{cases} 4 & 8am \leq t \leq 10am \\ 8 & 10am \leq t \leq 12 noon \\ \vdots & \text{-- -- -- --} \\ \vdots & \text{-- -- -- --} \end{cases}$$



6. Suppose that (a) customer arrives at a restaurant in batches of size 1, 2, 3, 4, 5 and 6,
(b) the batches themselves arrive according to a Poisson Process having rate λ. The successive batch sizes X_k's $(k=1,2,\ldots)$ are iid RVs having pmf:

$$P[X=x] = \begin{cases} 0.10 & \text{if } x=1,3 \\ 0.25 & \text{if } x=2,4 \\ 0.15 & \text{if } x=5,6. \end{cases}$$

Compute the mean and variance of the number of customers who arrive at the restaurant during $(0,t]$.

Hints:- $Y(t) = \sum_{k=1}^{N(t)} X_k$

$$E(Y(t)) = \lambda t E(X_1)$$
$$V(Y(t)) = \lambda t E(X_1^2).$$

7. Customer arrivals to a shop is governed by Poisson process with time dependent rate

$$\lambda(t) = \begin{cases} 4+2t & \text{if } 0 \le t \le 4 \\ 24-3t & \text{if } 4 \le t \le 8. \end{cases}$$

(a) Find $\Lambda(t)$.
(b) Determine the expected number of arrivals in the last 5 hrs (i.e., in the interval $(3,8]$), given that 15 customers have arrived in the last 3 hrs (i.e., in $(5,8]$).

Hints:- (a) $\Lambda(t) = \begin{cases} 4t+t^2 & \text{if } 0 \le t \le 4 \\ \int_0^4 (4t+t^2)\,dt + \int_4^t (24-3s)\,ds & \text{if } 4 \le t \le 8 \end{cases}$

(b) $E\left[N(3,8] \mid N(5,8]=15\right]$

$= 15 + E\left[N(3,5]\right]$

$= 15 + \sum_{r=0}^{\infty} r P\left[N(3,5]=r\right]$

8. Let $X_1, X_2, \ldots$ denote the interoccurrance times of events of a nonhomogeneous PP having $\lambda(t)$ intensity functn.
(a) Find the distn. of T_1. Then find d.f. of T_2.
(b) Are X_i's independent.

Sol.
(a)

$$\bar{F}_{T_1}(t) = P[T_1 > t] = P[N(t) = 0] = e^{-\lambda(t)}$$

$$f_{T_1}(t) = \lambda(t) e^{-\Lambda(t)}$$

$$\bar{F}_{T_2}(t) = P(T_2 > t) = \int_{s=0}^{t} P[T_2 > t \mid T_1 = s] f_{T_1}(s)\, ds$$

$$= \int_{s=0}^{t} P[N(t) - N(s) = 0] f_{T_1}(s)\, ds$$

$$= \int_{s=0}^{t} e^{-[\Lambda(t) - \Lambda(s)]} \cdot \lambda(s) e^{-\Lambda(s)}\, ds$$

$$= \Lambda(t) e^{-\Lambda(t)}.$$

(b)

$$P[T_2 > t \mid T_1 = s] = e^{-[\Lambda(t) - \Lambda(s)]}$$

$$\therefore f_{T_2 \mid T_1 = s}(t) = \lambda(t) e^{-[\Lambda(t) - \Lambda(s)]}$$

$\Rightarrow$ T_1 and T_2 are not independent, as the pdf of $T_2 \mid T_1 = s$ depends on s.

9. An office has two clerks, and three people, A, B and C, enter simultaneously. A and B begin service at the two clerks, while C waits for the first available clerk. Assume that the service time is Exponential (λ).
(a) Compute the probability that A is the last to finish the service. (b) Compute the expected time before C is done (i.e., C is combined waiting and service time).

Sol. (a) This is the probability that two events happen in a rate λ Poisson process before a single one in a independent rate λ process, i.e., $1/4$.

(b) First C has to wait for the first event in two combined Poisson Processes, which is a single process with rate 2λ, and then for the service time; the answer is $\frac{1}{2\lambda} + \frac{1}{\lambda}$.

10. A car wash has two stations, 1 and 2, with $Exp(\lambda_1)$ and $Exp(\lambda_2)$ service times. A car enters at station 1. Upon completing the service at station 1, the car then proceeds to station 2, provided station 2 is free; otherwise, the car has to wait at station 1, blocking the entrance of other cars. The car exits the wash after service at station 2 is completed. When you arrive at the wash there is a single car at station 1. Compute your expected time before you exit.

Sol. T_1 : Time the other car spends at station 1
T_2 : Time you spend at station 2.

$T_1 \sim Exp(\lambda_1)$, $T_2 \sim Exp(\lambda_2)$.

Total time $= T_1 + T_2 + \max\{T_1, T_2\}$.

$$\therefore E(T) = E(T_1) + E(T_2) + E\left[\max\{T_1, T_2\}\right]$$

$$= E(T_1) + E(T_2) + E\left[T_1 + T_2 - \min\{T_1, T_2\}\right]$$

$$= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}.$$

$$= \frac{2}{\lambda_1} + \frac{2}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}. \quad \left[\because \min\{T_1, T_2\} \sim Exp(\lambda_1 + \lambda_2)\right]$$

11. Assume that certain events (say, power surges) occur as a Poisson Process with rate 3 per hour. These events cause damage to certain system (say, a computer), thus a special protecting unit has been designed. That unit now has to be removed from the system for 10mins for service.

(a) Assume that a single event occurring in the service period will cause the system to crash. What is the probability that the system will crash?

(b) Assume that the system will survive a single event, but two events occurring in the service period will cause it to crash. What is now the prob. that the system will crash?

(c) Assume that crash will not happen unless there are two events within 5 mins of each other. Compute the prob. that the system will crash.

<u>Sol.</u> Assume the time units is 10 mins, $\frac{1}{6}$ of an hour.

(a) $$P\left(N\left(\tfrac{1}{6}\right) \geq 1\right) = 1 - e^{-1/2}.$$

(b) $$P\left(N\left(\tfrac{1}{6}\right) \geq 2\right) = 1 - \frac{3}{2} e^{-1/2}.$$

(c) If there are 0 or 1 event in the 10mins, there will be no crash, but 3 or more events in the 10mins will cause a crash. The final possibility is exactly two events, in which case the crash will happen with probability

$$\underbrace{P\left(|U_1 - U_2| < \tfrac{1}{2}\right)}_{=\frac{3}{4}}, \qquad U_1 \sim U(0,1), \quad U_2 \sim U(0,1).$$

$$P(\text{crash}) = P(X > 2) + P(\text{crash} \mid X = 2)\, P(X = 2)$$

$$= 1 - e^{-1/2} - \frac{1}{2} e^{-1/2} - e^{-1/2} \cdot \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{3}{4} \cdot \frac{\left(\frac{1}{2}\right)^2}{2} \cdot e^{-1/2}$$

$$= 1 - \frac{49}{32} e^{-1/2}.$$

$\boxed{12}$. A m/c needs frequent maintenance to stay on. The maintenance times occur as a Poisson process with rate μ. Once the m/c receives no maintenance for a time interval of length h, it breaks down. It then needs to be repaired, which takes an Exp(λ) time, after which it goes back on.

(a) After the m/c is started, find the prob. that the m/c will break down before receiving its first maintenance.

(b) Find the expected time for the first breakdown.

(c) Find the proportion of time the m/c is on.

Soln. :-

(a) $e^{-\mu h}$.

(b) W : the waiting time for a maintenance $\ni$ the next maintenance is at least h in the future,

T_1 : the time of the first maintenance.

$t < h$, $E(W / T_1 = t) = t + E(W)$

$$E(W) = \frac{1 - \mu h e^{-\mu h} - e^{-\mu h}}{e^{-\mu h}}.$$

The answer is = the m/c waits for h more units before it breaks down = $E(W) + h$.

(c)

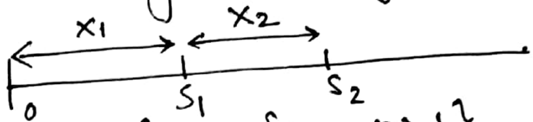
Ans. is $\dfrac{E(W) + h}{E(W) + h + \frac{1}{\lambda}}$.

Renewal Process

Let $\{X_n, n \geq 1\}$ be a sequence of non-negative iid RVs with $P[X_1 = 0] < 1$, $\mu = E[X_1]$ and cdf $F(x) = P[X_1 \leq x]$. Suppose that $S_n = X_1 + X_2 + \cdots + X_n$ for $n \geq 1$ with DF $F_n(x) = P[S_n \leq x]$ (also called the nth convolution of X_1 with itself)

Definition:- Let No. of occurances in the interval $(0,t)$, $N(t) = \max\{n : S_n \leq t\}$. Then the process $\{N(t), t \geq 0\}$ is called a renewal process (RP) induced or generated by F.

S_n: Tim epoch of nth renewal.



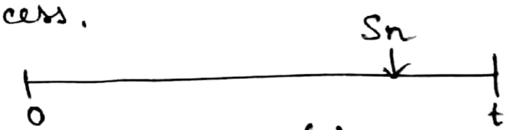
Typically the sequence of RVs $\{X_n, n \geq 1\}$ or $\{S_n, n \geq 1\}$ are also refered to as RP induced by F. If for some n, $S_n = t$, then we say that a renewal has taken place at time-epoch t or S_n. $N(t)$ gives the number of renewal in $(0, t)$. X_1 is called the inter renewal time or inter arrival time.

X_n is called the inter renewal time between the $(n-1)$th and nth renewal.

Application:-

1. Reliability : replacement of failed component.
2. Queues : Arrival of customers.
3. Traffic Management : Flow of Vehicle.
4. Insurance : claim process.

Distribution of $N(t)$:-



$$S_n \leq t \Leftrightarrow N(t) \geq n$$

Consequently,

$$P[N(t) = n] = P[N(t) \geq n] - P[N(t) \geq n+1]$$
$$= P[S_n \leq t] - P[S_{n+1} \leq t]$$
$$= F_n(t) - F_{n+1}(t).$$

[Simplified form is ruled out. If exponential X_i's, then Poisson]

Theorem 1:- (Renewal Equation)

The renewal function $m(t)$ is given by $m(t) = E(N(t))$
$$= \sum_{n=1}^{\infty} F_n(t).$$

Proof:- $m(t) = E[N(t)]$

$$= \sum_{n=0}^{\infty} n P[N(t) = n]$$

$$= \sum_{n=1}^{\infty} n[F_n(t) - F_{n+1}(t)]$$

$$= F_1(t) - F_2(t) + 2F_2(t) - 2F_3(t) + 3F_3(t) - 3F_4(t) + \cdots$$

$$= \sum_{n=1}^{\infty} F_n(t).$$

Laplace Transformation (LT) of $m(t)$:-

Let $f(x) = F'(x)$ be the pdf of X_1 and $g^*(s)$ be the LT of $g(t)$,

i.e., $g^*(s) = \int_0^\infty e^{-st} g(t)\,dt$

$F_n^*(s)$: LT of distn function of $X_1 + X_2 + \cdots + X_n$.

$f_n^*(s)$: LT of density functn of $X_1 + X_2 + \cdots + X_n$.

$$\boxed{f_n^*(s) = s F_n^*(s)}$$

So, $m^*(s) = \int_0^\infty e^{-st} m(t)\,dt$

$$= \int_0^\infty e^{-st} \sum_{n=1}^{\infty} F_n(t)\,dt$$

$$= \sum_{n=1}^{\infty} \int_0^\infty e^{-st} F_n(t)\,dt$$

$$= \sum_{n=1}^{\infty} F_n^*(s), \quad F_n^*(s) \text{ is the LT of } F_n(t).$$

$$= \frac{1}{s} \sum_{n=1}^{\infty} f_n^*(s)$$

$$= \frac{1}{s} \sum_{n=1}^{\infty} [f^*(s)]^n = \boxed{\frac{1}{s} \cdot \frac{f^*(s)}{1 - f^*(s)} = m^*(s)}.$$

Example:- $X_1 \sim \text{Gamma}(\alpha, \lambda)$

$$f(x) = \frac{\lambda^\alpha}{\Gamma\alpha} e^{-\lambda x} x^{\lambda - 1}$$

$$f^*(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-st} \cdot \frac{\lambda^\alpha}{\Gamma\alpha} e^{-\lambda t} t^{\alpha-1}\, dt$$

$$= \frac{\lambda^\alpha}{\Gamma\alpha} \int_0^\infty e^{-(s+\lambda)t} t^{\alpha-1}\, dt$$

$$= \frac{\lambda^\alpha}{\Gamma\alpha} \cdot \frac{\Gamma(\alpha)}{(s+\lambda)^\alpha}$$

$$= \left(\frac{\lambda}{s+\lambda}\right)^\alpha.$$

(a) $\alpha = 1$, $f^*(s) = \frac{\lambda}{s+\lambda}$

$$\Rightarrow m^*(s) = \frac{1}{s} \cdot \frac{\frac{\lambda}{s+\lambda}}{1 - \frac{\lambda}{s+\lambda}}$$

$$= \frac{\lambda}{s^2}$$

$$\Rightarrow m(t) = \lambda t$$

(b) $\alpha = 2$, $f^*(s) = \left(\frac{\lambda}{s+\lambda}\right)^2$

$$\Rightarrow m^*(s) = \frac{1}{s} \cdot \frac{\left(\frac{\lambda}{s+\lambda}\right)^2}{1 - \left(\frac{\lambda}{s+\lambda}\right)^2}$$

Theorem 2: (Renewal Equation)

Every renewal function $m(t)$ satisfies the renewal equation

$$m(t) = F(t) + \int_0^t m(t-x)\, dF(x)$$

Proof:- $m(t) = E(N(t)) = \int_0^\infty E[N(t) \mid X_1 = x] f(x)\, dx$

Observe that $E[N(t) \mid X_1 = x] = \begin{cases} 0 & \text{if } x > t \\ 1 + E(N(t-x)) & \text{if } x \le t \end{cases}$

$$\therefore m(t) = \int_0^t \{1 + E(N(t-x))\} f(x)\, dx$$

$$= F(t) + \int_0^t m(t-x)\, dF(x).$$

Example 1:- $X_1 \sim Exp(\lambda)$

$f(x) = \lambda e^{-\lambda x}$

$F(x) = 1 - e^{-\lambda x}$; $0 < x < \infty$

$$m(t) = 1 - e^{-\lambda t} + \int_0^t m(t-x)\lambda e^{-\lambda x}\,dx$$

$$m(t) = 1 - e^{-\lambda t} + \lambda e^{-\lambda t}\int_0^t m(y) e^{\lambda y}\,dy$$

$$m'(t) = \lambda e^{-\lambda t} - \lambda^2 e^{-\lambda t}\int_0^t m(y) e^{\lambda y}\,dy + \lambda e^{-\lambda t}\left[m(t)e^{\lambda t}\right]$$

$$= \lambda e^{-\lambda t} - \lambda^2 e^{-\lambda t}\int_0^t m(y) e^{\lambda y}\,dy + \lambda m(t)$$

$$m'(t) = \lambda m(t) + \lambda e^{-\lambda t} - \lambda\left[m(t) - 1 + e^{-\lambda t}\right]$$

$$= \lambda m(t) + \lambda e^{-\lambda t} - \lambda m t - \lambda - \lambda e^{-\lambda t}$$

$$= \lambda$$

$$m(t) = \lambda t + k$$

At $t = 0$, $m(0) = 0 = k$

$\therefore m(t) = \lambda t$.

Example 2:- $X_1 \sim U(0,1)$

$f(x) = 1$ if $0 \le x \le 1$

$F(x) = x$ if $0 \le x \le 1$

$$F(t) = \begin{cases} t & 0 \le t \le 1 \\ 1 & t \ge 1 \end{cases}$$

Suppose $0 \le t \le 1$.

$$m(t) = F(t) + \int_0^t m(t-x)\,dF(x)$$

$$= t + \int_0^t m(y)\,dy$$

$$\Rightarrow m'(t) = 1 + mt$$

$$\Rightarrow \left[m'(t) - mt\right]e^{-t} = e^{-t}$$

$$\Rightarrow \frac{d}{dt}\left[m(t)e^{-t}\right] = e^{-t} = \frac{d}{dt}\left[k - e^{-t}\right]$$

$$\Rightarrow m(t)e^{-t} = k - e^{-t}$$

At $t = 0$, $m(t) = 0 \Rightarrow k = 1$.

$\therefore m(t) = ke^{t} - 1 = e^{t} - 1$.

Theorem 3 (Renewal Type Equations):-

Let g be any known function and F be the cdf of X_1. Suppose that function v is unknown. But

$$v(t) = g(t) + \int_0^t v(t-x)\,dF(x)$$

Then
$$v(t) = g(t) + \int_0^t g(t-x)\,dm(x),$$

$m(x)$ is renewal function corresponding to F (i.e., $m(t) = E[N(t)]$)

Proof:- Since $v(t) = g(t) + \int_0^t v(t-x)\,dF(x)$

$$v^*(s) = g^*(s) + v^*(s) f^*(s)$$

$$m^*(s) = \frac{1}{s} \cdot \frac{f^*(s)}{1-f^*(s)} \quad \Rightarrow \quad f^*(s) = \frac{s m^*(s)}{1 + s m^*(s)}$$

$$v^*(s) = \frac{g^*(s)}{1-f^*(s)} = g^*(s)\left[1 + s m^*(s)\right]$$
$$= g^*(s) + s g^*(s) m^*(s).$$

By inverse LT,
$$v(t) = g(t) + \int_0^t g(t-x)\,dm(x).$$

(Some Limit Results)

Theorem 4:- With probability 1, $\underbrace{\frac{N(t)}{t}}_{\text{No. of renewals per unit time (Rate of Renewals)}} \to \underbrace{\frac{1}{\mu}}_{\mu:\text{ mean life}}$ as $t \to \infty$

$$\boxed{P\left[\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}\right] = 1}$$

Proof:-

$$0 \qquad S_{N(t)} \qquad t \qquad S_{N(t)+1}$$

$$S_{N(t)} \le t < S_{N(t)+1}$$

$$\Rightarrow \frac{S_{N(t)}}{N(t)} \le \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)} \quad \cdots\cdots (1)$$

$$S_{N(t)} = \sum_{i=1}^{N(t)} X_i$$

As $t \to \infty$, $N(t) \to \infty$ by SLLN

With Prob. 1, $\dfrac{\sum_{i=1}^{N(t)} X_i}{N(t)} \to \mu$ as $t \to \infty$.

i.e., $\dfrac{S_{N(t)}}{N(t)} \to \mu$ as $t \to \infty$.

Next, $\dfrac{S_{N(t)+1}}{N(t)} = \dfrac{S_{N(t)+1}}{N(t)+1}\left[\dfrac{N(t)+1}{N(t)}\right] = \dfrac{S_{N(t)+1}}{N(t)}\left[1+\dfrac{1}{N(t)}\right]$

for similar argument

$$\frac{S_{N(t)+1}}{N(t)} \to \mu \text{ as } t \to \infty.$$

from ①: $\mu \le \dfrac{t}{N(t)} \le \mu$ w.p.1.

i.e. $\dfrac{N(t)}{t} \to \dfrac{1}{\mu}$ w.p.1.

Definition:- (Stopping time)

An integer valued RV N is called stopping time for the sequence of non-negative iid RVs $\{X_n, n \ge 1\}$ if the event $\{N=n\}$ does not depend on $X_{n+1}, X_{n+2}, \ldots$

Example:- Consider the RP $\{N(t), t \ge 0\}$ induced by $\{X_n, n \ge 1\}$.

Also consider the integer-valued RV $N(t)$ for any t.

Is $N(t)$ a stopping time for $\{X_n, n \ge 1\}$. look at the event $\{N(t)=n\}$

See that $\{N(t)=n\} \Leftrightarrow \{S_n \le t < S_{n+1}\}$

$\equiv \{S_n \le t$ and $S_n + S_{n+1} > t\}$ depends on X_{n+1}.

Hence $N(t)$ is not stopping time for $\{X_n, n \ge 1\}$.

Next, what about $N(t)+1$?

Consider the event $\{N(t)+1=n\}$

$\equiv \{N(t)=n-1\} \Leftrightarrow \{S_{n-1} \le t$ and $S_n > t\}$

$\therefore \{S_{n-1} \le t,\ S_{n-1}+X_n > t\}$ doesn't depend on $X_{n+1}, X_{n+2}, \ldots$

So, $N(t)+1$ is stopping time for $\{X_n, n \ge 1\}$.

[Wald's Equation]

Lemma 1: Let N be a stopping time for $\{X_n, n \geq 1\}$ and $E[N] < \infty$. Then $E\left[\sum_{i=1}^{N} X_i\right] = E(X_1)\, E(N)$

Proof: Define $Z_i = \begin{cases} 1 & N \geq i \\ 0 & N < i \end{cases}$

$$= I_{\{N \geq i\}} = 1 - I_{\{N < i\}}$$

Since N is stopping time for $\{X_n, n \geq 1\}$

$\{N < i\}$ is independent of $X_i, X_{i+1}, \ldots\ldots$ which implies, in particular, X_i is independent of Z_i for any i.

Now, observe that,

$$\sum_{i=1}^{N} X_i = \sum_{i=1}^{\infty} X_i Z_i$$

$$\Rightarrow E\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{\infty} E(X_i)\, E(Z_i) = E(X_1) \sum_{i=1}^{\infty} E(Z_i)$$

$$= E(X_1) \sum_{i=1}^{\infty} P(N \geq i)$$

$$= E(X_1)\, E(N)$$

[$P(N \geq 1) + P(N \geq 2) + P(N \geq 3) + \ldots\ldots$

$= (p_1 + p_2 + \ldots\ldots) + (p_2 + p_3 + \ldots\ldots) + (p_3 + p_4 + \ldots\ldots)$

$= p_1 + 2p_2 + 3p_3 + \ldots\ldots\ldots$

$= E[N]$.]

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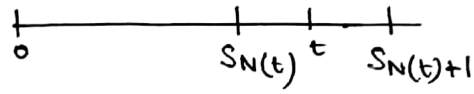
Theorem 5:- (Elementary Renewal Theorem)

As $t \to \infty$, $\dfrac{m(t)}{t} \to \dfrac{1}{\mu}$

Proof:- Clearly, $t < S_{N(t)+1}$



Taking expectation on both sides,

$$t < E\left[S_{N(t)+1}\right] = E\left[\sum_{i=1}^{N(t)+1} x_i\right] = E(X_1)E(N(t)+1) \quad \left[\text{By Wald's Equation}\right]$$

$$= \mu\left(m(t)+1\right)$$

$$\Rightarrow \frac{m(t)}{t} > \frac{1}{\mu} - \frac{1}{t} \quad \text{.......} \; (1)$$

Define $\tilde{X}_i = \begin{cases} x_i & x_i \le a \quad \text{for } a > 0 \\ a & x_i > a \end{cases}$

$= \min\{x_i, a\}$ $\quad \left[E(\tilde{X}_i) < E(X_i)\right]$

$\tilde{X}_1, \tilde{X}_2, \ldots$ form a sequence of non-negative iid rvs.

Let $\{\tilde{N}(t), t \geq 0\}$ be the corresponding RP with $\tilde{S}_n = \tilde{X}_1 + \cdots + \tilde{X}_n$.

If $\tilde{\mu} = E(\tilde{X}_1)$ then $\tilde{\mu} \le \mu$ and $\tilde{m}(t) \geq m(t)$

Consider $\tilde{\mu}\left(m(t)+1\right) \le \tilde{\mu}\left(\tilde{m}(t)+1\right) = E\left[\tilde{S}_{\tilde{N}(t)+1}\right]$

$$= E\left[\sum_{1}^{\tilde{N}(t)+1} \tilde{X}_i\right]$$

$$\le t + a.$$

$$\therefore E\left[\tilde{S}_{\tilde{N}(t)+1}\right] = E\Big[\underbrace{\tilde{S}_{\tilde{N}(t)}}_{\le t} + \underbrace{\tilde{X}_{\tilde{N}(t)+1}}_{\le a}\Big]$$

$$\Rightarrow \frac{m(t)}{t} \le \frac{1}{t\tilde{\mu}}(t+a) - \frac{1}{t} = \frac{1}{\tilde{\mu}} + \frac{a}{t\tilde{\mu}} - \frac{1}{t} \quad \text{......} \; (2)$$

Combining (1) and (2): $\dfrac{1}{\mu} - \dfrac{1}{t} < \dfrac{m(t)}{t} \le \dfrac{1}{\tilde{\mu}} + \left(\dfrac{a}{\tilde{\mu}} - 1\right)\dfrac{1}{t}$.

Now, letting $a = \sqrt{t}$,

$$\frac{1}{\mu} - \frac{1}{t} < \frac{m(t)}{t} \le \frac{1}{\tilde{\mu}} + \frac{1}{\tilde{\mu}\sqrt{t}} - \frac{1}{t}$$

As, $t \to \infty$, $\quad \dfrac{1}{\mu} \le \dfrac{m(t)}{t} \le \dfrac{1}{\mu}$

So, $\displaystyle\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}$.

Ex. 1. A transister radio works on a single battery. As soon as the battery in use fails, it is immediately replaced by a new one. If the lifetime (hrs) of batteries is distributed as $U(30, 60)$ then at what rate battery need to be replaced during a year.

Sol. Renewal: Replacement of battery.

$$\text{Rate of Renewals} = \frac{1}{\text{Expected life of a single battery}} = \frac{1}{45 \text{hrs}}$$

[$\because U(30,60)$ hours during a year. $t \to \infty$]

Ex. 2. Suppose that potential customers arrive at a single-server bank in accordance with $PP(\lambda)$. However, the potential customer enters the bank only if the server is free. If we assume that the amount of time spent in the bank by an entering customer is a rv having distr. G, then

(a) find the rate at which customers enter the bank.

(b) What is the proportion of potential customers that can't join the bank?

(c) What is the utilization of the bank?

Sol. (a) Renewal = Entering the bank. Mean service time $= \mu_G$. Rate of customer arriving $= \lambda$.

$$\text{Mean inter-arrival time} (\mu) = \frac{1}{\lambda} + \mu_G.$$

$$\therefore \text{Rate of entering customer} = \frac{1}{\mu} = \frac{1}{\frac{1}{\lambda} + \mu_G} = \frac{\lambda}{1 + \lambda\mu_G}.$$

(b) Proportion of lost customer = 1 − Proportion of entering customers

$$= 1 - \frac{1/\mu}{\lambda} = \frac{\lambda\mu_G}{1 + \lambda\mu_G}.$$

(c) $$\text{Utilization} = \frac{\text{Mean Service time}}{\text{Mean inter arrival time}}$$

$$= \frac{\mu_G}{\frac{1}{\lambda} + \mu_G} = \frac{\lambda\mu_G}{1 + \lambda\mu_G}.$$

Theorem 6:- (Renewal Central limit theorem)

Let $E(X_1) = \mu$, $V(X_1) = \sigma^2$ (finite). Then $P\left[\frac{N(t) - t/\mu}{\sigma\sqrt{t/\mu^3}} < y\right] \to \Phi(y)$ as $t \to \infty$.

$N(t) \to$ Renewal process.

i.e., $N(t) \overset{a}{\sim} N\left(t/\mu, \frac{\sigma^2 t}{\mu^3}\right)$ as $t \to \infty$

Proof:- Write: $r_t = \frac{t}{\mu} + y\sigma\sqrt{\frac{t}{\mu^3}}$

Then $P\left[\frac{N(t) - t/\mu}{\sigma\sqrt{t/\mu^3}} < y\right] = P\left[N(t) < r_t\right] = P\left[S_{r_t} > t\right]$

$$= P\left[\frac{S_{r_t} - E(S_{r_t})}{\sqrt{V(S_{r_t})}} > \frac{t - \mu r_t}{\sigma\sqrt{r_t}}\right]$$

As $t \to \infty$ by CLT $\frac{S_{r_t} - E(S_{r_t})}{\sqrt{V(S_{r_t})}} \to N(0,1)$

So, $P\left[\frac{N(t) - \frac{t}{\mu}}{\sigma\sqrt{t/\mu^3}} < y\right] \xrightarrow[t\to\infty]{} P\left[N(0,1) > -y\right] = \Phi(-y)$.

$S_{r_t} = \sum_{i=1}^{r_t} X_i$

$E(S_{r_t}) = \mu r_t$

$V(S_{r_t}) = \sigma^2 r_t$.

And $\frac{t - \mu r_t}{\sigma\sqrt{r_t}}$

$$= -y\left[1 + \frac{y\sigma}{\sqrt{t\mu}}\right]^{-1/2}$$

So, $t \to \infty$, $\frac{t - \mu r_t}{\sigma\sqrt{r_t}} \to -y$

Ex. 3. Processing times by machine M_1 and M_2 have Gamma(4,2) and $U(0,4)$, respectively. Approximate the prob. that the two maching together complete at least 90 jobs by time $t = 100$.

Solution:- Renewal: Completion of processing a job.

$N_1(t)$: No. of renewals in $[0,t]$ corresponding to interarrival time Gamma(4,2)

$N_2(t)$: " " " " " ...... U(0,4)

$\{N_1(t)\}$ and $\{N_2(t)\}$ are two RPs. $(t \geq 0)$.

$\mu_1 = \frac{4}{2} = 2$; $\mu_2 = 2$

$\sigma_1^2 = 1$; $\sigma_2^2 = \frac{16}{12}$

$$\left[\mu_1 = \frac{\alpha}{\lambda}, \mu_2 = \frac{a+b}{2}; \quad \sigma_1^2 = \frac{\alpha}{\lambda^2}, \sigma_2^2 = \frac{(b-a)^2}{4}\right]$$

$N_i(t) \overset{a}{\sim} N\left(\frac{t}{\mu_i}, \frac{\sigma_i^2 t}{\mu_i^3}\right)$; $i = 1, 2$.

$N(t) = N_1(t) + N_2(t) \overset{a}{\sim} N\left(\frac{t}{\mu_1} + \frac{t}{\mu_2}, t\left(\frac{\sigma_1^2}{\mu_1^3} + \frac{\sigma_2^2}{\mu_2^3}\right)\right)$

$P\left[N(100) \geq 90\right] = P\left[N(0,1) \geq \frac{90 - 100}{\sqrt{\frac{100 \times 7}{24}}}\right]$

$$\left[E\left[N(100)\right] = \frac{100}{2} + \frac{100}{2}; \quad V(N(100)) = 100\left[\frac{1}{8} + \frac{4}{3}\cdot\frac{1}{8}\right]\right]$$

BIRTH - DEATH PROCESS

In HPP, we had, for very small $h > 0$

$$P\left[N(t+h) - N(t) = k \mid N(t) = n\right] = P\left[N(t+h) - N(t) = k\right]$$

(because of independent increment)

$$= P[N(h) = k] \quad \text{(because of stationary increment)}$$

$$= \begin{cases} \lambda h + o(h) & , k = 1 \\ o(h) & , k \geq 2 \end{cases}$$

which is independent of
(a) n – population size at the start of the time interval, and
(b) t – starting time point of interval.

We can generalize PP by allowing λ to be function of n or t or both. The resulting process still remains markovian in character.

In NHPP, we have, λ as function of time t. We now consider a generalization as follows which has application in demography, queuing, reliability, biology, etc.

(a) So far we have dealt with only one type of transition – birth (increase in state variable by one). Additionally, we will have the new transition type - death (decrease state variable by one).

(b) The birth and death rates are given by λ_n and μ_n (function of n). Specifically, we take

$$P\left[N(t+h) - N(t) = k \mid N(t) = n\right] = \begin{cases} \lambda_n h + o(h) & , k = 1 \\ \mu_n h + o(h) & , k = -1 \\ o(h) & , |k| \geq 2 \end{cases}$$

where, λ_n is defined for $n = 0, 1, 2, \ldots\ldots$ and
μ_n is defined for $n = 1, 2, \ldots\ldots$

$$\boxed{\mu_n = 0 \text{ for } n = 0}$$

Consequently,

$$P_0(t+h) = \{1-\lambda_0 h\} P_0(t) + \mu_1 h P_1(t) + o(h)$$

$[P_n(t) = P[N(t) = n]]$

and,

$$P_n(t+h) = \{1-\lambda_n h - \mu_n h\} P_n(t) + \lambda_{n-1} h P_{n-1}(t) + \mu_{n+1} h P_{n+1}(t) + o(h)$$

for $n \geq 1$

This implies

$$\left.\begin{aligned} P_0'(t) &= -\lambda_0 P_0(t) + \mu_1 P_1(t) \\ P_n'(t) &= -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t) \\ &\quad - \mu_n P_n(t) + \mu_{n+1} P_{n+1}(t) \end{aligned}\right\} \text{kolmogrov DE.}$$

General solution to this system is intactible.

Initial condition : $P_{N(0)}(0) = 1$, $P_n(0) = 0$ for $n \neq N(0)$

$N(0)$: Initial Population size.

Examples of Birth-Death Process :-

1. $(M/M/1)$:

$\lambda_n = \lambda$, $n = 0, 1, 2, \ldots$

$\mu_n = \mu$, $n = 1, 2, \ldots$

2. $(M/M/s)$:

$\lambda_n = \lambda$, $n = 0, 1, \ldots$

$$\mu_n = \begin{cases} n\mu, & n = 1, 2, \ldots, s \\ s\mu, & n = s+1, s+2, \ldots \end{cases}$$

Two Important special cases :-

(i) (Pure) Birth Process $\mu_n = 0$, $n = 1, 2, \ldots$

(ii) (Pure) Death Process $\lambda_n = 0$, $n = 0, 1, 2, \ldots$

(Pure) Birth Process : See that the differential equations are;

$$P_0'(t) = -\lambda_0 P_0(t)$$

$$P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t) \quad , n \geq 1.$$

Yule - Furry Process:- [member gives birth, but never dies]

Consider a population whose members are physical/biological entity. Each member gives birth (by splitting or otherwise) to a new member (which is a replica of original member), but never dies.

Assume that in a very small time interval of length h, the probability that a member gave birth to another member is λh.

Consequently, when n members are there in the population, the prob. of a birth in a small time-interval of length h, is $n\lambda h$.

Hence $\lambda_n = n\lambda$.

The corresponding DE are

$P_0'(t) = 0$

For $n \geq 1$, $P_n'(t) = -n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t)$

For $n=1$, $P_1'(t) = -\lambda P_1(t)$

$P_1(t) = c \cdot e^{-\lambda t}$

At $t=0$, $c = P_1(0) = 1$.

$\Rightarrow P_1(t) = e^{-\lambda t}$.

For $n=2$,

$P_2'(t) = -2\lambda P_2(t) + \lambda P_1(t) = -2\lambda P_2(t) + \lambda e^{-\lambda t}$.

$e^{2\lambda t} P_2'(t) + 2\lambda e^{2\lambda t} P_2(t) = \lambda e^{\lambda t}$

$P_2(t) e^{2\lambda t} = e^{\lambda t} + K$

At $t=0 \Rightarrow P_2(0) = 1+K \Rightarrow K = -1$.

$P_2(t) = \left[e^{\lambda t} - 1\right] e^{-2\lambda t} = e^{-\lambda t}\left[1 - e^{-\lambda t}\right]$

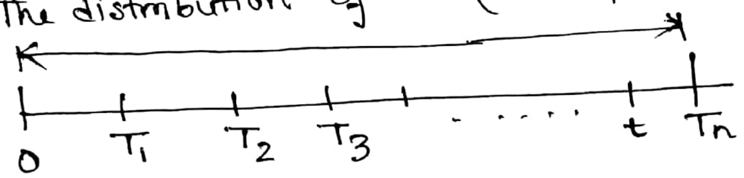
By induction, see that

$$P[N(t)=n] = P_n(t) = e^{-\lambda t}\left[1-e^{-\lambda t}\right]^{n-1}, \quad n=1,2,\ldots\ldots$$

$$N(t) \sim Geo(p), \quad p = e^{-\lambda t}$$

$$E[N(t)] = \frac{1}{p}, \quad Var[N(t)] = \frac{q}{p^2}$$

Result:- The distribution of T_n (time point of nth occurence)



Suppose, $P[T_n > t] = \bar{F}_{T_n}(t) = P[N(t) < n]$

$$= 1 - P[N(t) \geq n]$$

$$= 1 - \sum_{r=n}^{\infty} e^{-\lambda t}\left[1-e^{-\lambda t}\right]^{r-1}$$

$$= 1 - \sum_{r=n}^{\infty} p q^{r-1}$$

$$= 1 - \frac{p q^{n-1}}{p}$$

$$= 1 - q^{n-1}$$

$$= 1 - \left[1-e^{-\lambda t}\right]^{n-1}$$

The PDF of T_n is

$$f_{T_n}(t) = (n-1)\left[1-e^{-\lambda t}\right]^{n-2} \lambda e^{-\lambda t}$$

$$= (n-1)\lambda e^{-\lambda t}\left[1-e^{-\lambda t}\right]^{n-2}, \quad t > 0.$$

$N(t) = K$: The system is in state K at time t.
(no. of inviduals is K in the system at time t)

λ_n : rate of birth, μ_n : rate of death.

$$P\left[N(t+h) - N(t) = K \mid N(t) = n\right] = \begin{cases} \lambda_n h + o(h) & , K = +1 \\ \mu_n h + o(h) & , K = -1 \\ o(h) & , |K| \geq 2 \end{cases}$$

$\lambda_n \geq 0 \ \forall\ n \geq 0$, $\mu_0 = 0$, $\mu_n \geq 0 \ \forall\ n \geq 1$

Notation: $P_n(t) = P\left[N(t) = n\right]$, $n = 0, 1, \ldots$

Kolmogrov Differential Equation:

$$P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t)$$

$$P_n'(t) = -(\lambda_n + \mu_n) P_n(t) + \lambda_{n-1} P_{n-1}(t) + \mu_{n+1} P_{n+1}(t)$$

1. Pure Birth Process: $\mu_n = 0 \ \forall\ n \geq 0$

Yule - Furry Process: $N(0) = 1$, $\lambda_n = n\lambda$

$$N(t) \sim \text{Geo}\left(e^{-\lambda t}\right)$$

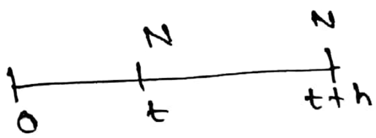
$$E\left[N(t)\right] = \frac{1}{e^{-\lambda t}} = e^{\lambda t}$$

$$V(N(t)) = \frac{1 - e^{-\lambda t}}{e^{-2\lambda t}} = e^{2\lambda t} - e^{\lambda t}$$

2. Pure Death Process:- $\lambda_n = 0$

$\mu_n = n\mu$, $n = 1, 2, \ldots$

Assume: $N(0) = N$, $\mu_n = n\mu$



$$P_N(t+h) = \left\{1 - \mu_N (h)\right\} P_N(t)$$

$0 \leq n \leq N+1$:

$$P_n(t+h) = \left\{1 - \mu_n h\right\} P_n(t) + \mu_{n+1} h P_{n+1}(t) + o(h)$$

$\Rightarrow$ Kolmogrov Differential Equation:

$$P_N'(t) = -N\mu P_N(t)$$

$0 \leq n \leq N-1$;

$$P_n'(t) = -n\mu P_n(t) + (n+1)\mu P_{n+1}(t)$$

For $n = N$,

$$P_N'(t) = -N\mu P_N(t)$$

$$\Rightarrow P_N(t) = ke^{-N\mu t}$$

$t = 0$, $\Rightarrow k = 1$.

Hence, $\boxed{P_N(t) = e^{-N\mu t}}$.

For $n = N-1$

$$P_{N-1}'(t) = -n\mu P_{N-1}(t) + N\mu P_N(t)$$

$$= -(N-1)\mu P_{N-1}(t) + N\mu e^{-N\mu t}$$

$$\Rightarrow e^{(N-1)\mu t} P_{N-1}'(t) + (N-1)\mu e^{(N-1)\mu t} P_{N-1}(t) = N\mu e^{-\mu t}$$

$$\Rightarrow \frac{d}{dt}\left[e^{(N-1)\mu t} P_{N-1}(t)\right] = N\mu e^{-\mu t} = N\frac{d}{dt}\left(-e^{-\mu t}\right)$$

$$\Rightarrow e^{(N-1)\mu t} P_{N-1}(t) = N\left[-e^{-\mu t} + k\right]$$

$t = 0$, $\Rightarrow k = 1$.

$$P_{N-1}(t) = N\left[-e^{-\mu t} + 1\right] e^{-(N-1)\mu t}$$

$$= Ne^{-(N-1)\mu t}\left[1 - e^{-\mu t}\right]$$

For $n = N-2$

$$P_{N-2}'(t) = -(N-2)\mu P_{N-2}(t) + (N-1)\mu P_{N-1}(t)$$

$$= -(N-2)\mu P_{N-2}(t) + (N-1)\mu N e^{-(N-1)\mu t}\left[1 - e^{-\mu t}\right]$$

By multiplying $e^{(N-2)\mu t}$

$$e^{(N-2)\mu t} P_{N-2}'(t) + (N-2)\mu e^{(N-2)\mu t} P_{N-2}(t)$$

$$= N(N-1)\mu e^{-\mu t}\left[1 - e^{-\mu t}\right]$$

$$\frac{d}{dt}\left[e^{(N-2)\mu t}\cdot P_{N-2}(t)\right] = \frac{N(N-1)}{2}\cdot\frac{d}{dt}\left[1-e^{-\mu t}\right]^2$$

$$e^{(N-2)\mu t} P_{N-2}(t) = \frac{N(N-1)}{2}\left[\left(1-e^{-\mu t}\right)^2 + K\right]$$

$t=0 \Rightarrow K=0$

$$e^{(N-2)\mu t} P_{N-2}(t) = \frac{N(N-1)}{2}\left(1-e^{-\mu t}\right)^2$$

$$\therefore\ P_{N-2}(t) = \frac{N(N-1)}{2}\, e^{-(N-2)\mu t}\left(1-e^{-\mu t}\right)^2.$$

See that in general,

$$P_n(t) = P[N(t)=n] = \binom{N}{n}\left(e^{-\mu t}\right)^n\left[1-e^{-\mu t}\right]^{N-n} \quad ;\ n=0,1,2,\dots,N$$

$$N(t) \sim Bin\left(N, e^{-\mu t}\right)$$

$$E[N(t)] = Ne^{-\mu t}\ ,\quad V[N(t)] = Ne^{-\mu t}\left[1-e^{-\mu t}\right].$$

<u>Steady State Probabilities</u>:-

$P[N(t)=n] = P_n(t) =$ Prob. that at time t, the system is in state n. $n=0,1,2,\dots$

We have all λ_n and $\mu_n > 0$, then all the states could communicate among themselves under this situation.

$$\lim_{t\to\infty} P_n(t) \text{ exists } \forall\, n.$$

and we denote it by P_n (Prob. that in the long run system is in state n)

Hence from the kolmogrov differential equation:

$$0 = -\lambda_0 P_0 + \mu_1 P_1$$

$$0 = -(\lambda_n + \mu_n)P_n + \lambda_{n-1}P_{n-1} + \mu_{n+1}P_{n+1}\ ,\ n\geq 1$$

Solve recurssively,

$$P_1 = \frac{\lambda_0}{\mu_1}P_0.$$

$$P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0$$

$$P_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} P_0$$

Since $\sum_{n=0}^{\infty} P_n = 1$, then $P_0 = S^{-1}$.

where $S = \left[1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} + \cdots \right] = \left[1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} \right]$

Example 1:- $(M/M/1)$ queue

$\lambda_n = \lambda \qquad n \geq 0$

$\mu_n = \mu \qquad n \geq 1$

Then $P_n = \frac{\lambda^n}{\mu^n} P_0 = \rho^n P_0$

$$S = \left[1 + \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \right] = \frac{1}{1 - \frac{\lambda}{\mu}} = \frac{1}{1-\rho}. \qquad \boxed{\lambda < \mu, \ \rho < 1}$$

Hence $P_0 = 1 - \rho$ and $P_n = (1-\rho)\rho^n$, $n = 1, 2, \ldots$

Example 2:- $(M/M/s)$ servers

$\lambda_n = \lambda \qquad n \geq 0$

$$\mu_n = \begin{cases} n\mu & n \leq s \\ s\mu & n > s \end{cases}$$

$$P_n = \begin{cases} \dfrac{\lambda^n}{n!\, \mu^n}, & n \leq s \\[2ex] \dfrac{\lambda^n}{s!\, \mu^s (s\mu)^{n-s}}, & n > s \end{cases}$$

$$S = \left[1 + \sum_{n=1}^{s} \frac{\lambda^n}{n!\, \mu^n} + \sum_{n=s+1}^{\infty} \frac{\lambda^n}{s!\, \mu^n s^{n-s}} \right]$$

$$\frac{\lambda}{s\mu} < 1 \Rightarrow \boxed{\lambda < s\mu}.$$

CONTINUOUS TIME MARKOV CHAIN

Consider a continuous time SP $\{X(t), t \geq 0\}$ taking values in the set S of non-negative integers. We say that $\{X(t), t \geq 0\}$ is continuous time MC (CTMC) if for $s, t \geq 0$ and for non-negative integers $i, j, x(u)$ for $0 \leq u < s$.

$$P\left[X(s+t) = j \mid X(s) = i, X(u) = x(u), 0 \leq u < s\right]$$

$$= P\left[X(s+t) = j \mid X(s) = i\right] \quad \text{[By Markov Property]}$$

If, in addition, the transition probability $P\left[X(s+t) = j \mid X(s) = i\right]$ is independent of s then the CTMC is said to have stationary or homogeneous transition probability.

We shall consider only CTMC having stationary transition probability. Denote

$$p_{ij}(t) = P\left[X(s+t) = j \mid X(s) = i\right] = P\left[X(t) = j \mid X(0) = i\right]$$

$\forall\, i, j \in S$ and $t \in T$.

and $\underset{|S| \times |S|}{P_t} = ((p_{ij}(t)))$, where,

$0 \leq p_{ij}(t) \leq 1$ and $\sum_{j \in S} p_{ij}(t) = 1$.

See that $P_0 = I$ $\begin{cases} p_{ij}(0) = 1 \\ p_{ij}(0) = 0 \end{cases}$

Write $P_j(t) = P[X(t) = j] = \sum_{i \in S} P\left[\underbrace{X(0) = i}_{B}, \underbrace{X(t) = j}_{A}\right]$

$$= \sum_{i \in S} P\left(X(0) = i\right) P\left[X(t) = j \mid X(0) = i\right]$$

$$= \sum_{i \in S} P_i(0)\, p_{ij}(t) \quad \text{and of course } \sum_{j \in S} P_j(t) = 1.$$

Lemma 1:- (Chapman Kolmogrov Equation)

$$P_{s+t} = P_s P_t = P_t P_s$$

Proof: Consider,

$$p_{ij}(s+t) = P\left[X(s+t)=j \mid X(0)=i\right]$$

$$= \sum_{k\in S} P\left[\underbrace{X(s+t)=j}_{A},\ \underbrace{X(s)=k}_{B} \mid \underbrace{X(0)=i}_{C}\right]$$

$$= \sum_{k\in S} P\left[X(s+t)=j \mid X(s)=k, X(0)=i\right] P\left[X(s)=k \mid X(0)=i\right]$$

$$= \sum_{k\in S} P\left[X(s+t)=j \mid X(s)=k\right] P\left[X(s)=k \mid X(0)=i\right]$$

$$= \sum_{k\in S} p_{ik}(s)\,p_{kj}(t) \quad \forall\ i,j\in S,\ t\in T.$$

$$\Rightarrow P_{s+t} = P_s P_t .$$

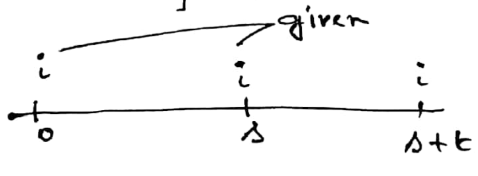
$$p_{ij}(s+t) = P\left[X(s+t)=j \mid X(0)=i\right]$$

$$= \sum_{k\in S} P\left[X(s+t)=j,\ X(t)=k \mid X(0)=i\right]$$

$$= \sum_{k\in S} p_{ik}(t)\,p_{kj}(s) = P_t P_s .$$

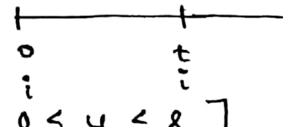
Time spent by the system on a state:-

Suppose that $X(0)=i$ and it continues to remain in i for T_i units of time before it moves to another state j $(j\neq i)$. We need to find the distr. of T_i.



Let $s,t\geq 0$ and consider

$$P\left[T_i > s+t \mid T_i > s\right]$$



$$= P\left[X(v)=i \text{ for } 0\leq v\leq s+t \mid X(u)=i \text{ for } 0\leq u\leq s\right]$$

$$= P\left[X(v)=i,\ s< v\leq s+t \mid X(u)=i,\ 0\leq u\leq s\right]$$

$= P[X(v) = i,\ s < v \le s+t \mid X(s) = i]$ (using Markov Property)

$= P[X(v) = i,\ 0 < v \le t \mid X(0) = i]$ (using stationarity)

$= P[T_i > t]$

$\therefore$ T_i has memoryless property.

Hence T_i is an exponential RV.

Define the rate of transition from state i to state j by

$$a_{ij} = \lim_{h \to 0} \frac{P[X(t+h) = j \mid X(t) = i]}{h} \quad \text{(if it exists)}$$

$$= \lim_{h \to 0} \frac{p_{ij}(h)}{h} \quad \text{for } i \neq j$$

Let us assume that right hand derivative of the transition probability $p_{ij}(t)$ as t goes to zero exists.

Look at the matrix of derivatives

$$G = \lim_{t \to 0} \frac{P_t - P_0}{t}$$

$$g_{ij} = \lim_{t \to 0} \frac{p_{ij}(t) - p_{ij}(0)}{t}$$

G is called the generator of CTMC (plays similar role as that of transition prob. mtx. in DTMC).

Observe that $g_{ij} = a_{ij}$ for $i \neq j$ $[\because p_{ij}(0) = 0,\ i \neq j]$

and $g_{ij} \geq 0$ $\forall\ i \neq j$, $g_{ii} \le 0$ $[p_{ii}(0) = 1]$

Since $\sum_{j \in S} p_{ij}(t) = 1$ $\forall\ i, j \in S,\ t \in T$

We have $\sum_{j \in S} g_{ij} = 0$.

Hence $g_{ii} = -\sum_{j \in S, j \neq i} g_{ij} = -\sum_{\substack{j \in S \\ j \neq i}} a_{ij} = -a_i$

If we know the generators with state space, process can be found (we get all the information).

Theorem: (Kolmogrov Differential Equation):—

(a) Forward : $P_t' = P_t G$

(b) Backward: $P_t' = G P_t$

Proof:- $\dfrac{P_{t+h} - P_t}{h} = \dfrac{P_t P_h - P_0}{h}$

Taking $h \to 0$

$\Rightarrow \boxed{P_t' = P_t G}$ & $\boxed{P_t' = G P_t}$

We solve the above system (KDE) using the boundary condition ($P_0 = I$).

Example:- (Two State Process)

Consider a system that can be in 2 states ("operating" and "under repair") denoted by 1 and 0 respectively. Suppose that the lengths of operating time and under repair time are independent exponential variables with mean $\frac{1}{\mu}$ and $\frac{1}{\lambda}$ respectively.

$X(t)$ = State of the system at time t

$$= \begin{cases} 1 \\ 0 \end{cases}$$

$\{X(t), t \geq 0\}$ CTMC with $p_{01}(h) = \lambda h + o(h)$

$p_{10}(h) = \mu h + o(h)$

$a_{01} = \lambda \Rightarrow a_0 = \lambda$

$a_{10} = \mu \Rightarrow a_1 = \mu$

$$G = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} \end{array}$$

$P_t' = P_t G$

$$\begin{bmatrix} p'_{00}(t) & p'_{01}(t) \\ p'_{10}(t) & p'_{11}(t) \end{bmatrix} = \begin{bmatrix} p_{00}(t) & p_{01}(t) \\ p_{10}(t) & p_{11}(t) \end{bmatrix} \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

$$\Rightarrow \quad p_{00}'(t) = -\lambda p_{00}(t) + \mu p_{01}(t)$$
$$p_{01}'(t) = \lambda p_{00}(t) - \mu p_{01}(t)$$
$$p_{10}'(t) = -\lambda p_{10}(t) + \mu p_{01}(t)$$
$$p_{11}'(t) = \lambda p_{10}(t) - \mu p_{11}(t)$$

$$p_{00}(t) + p_{01}(t) = 1$$
$$p_{10}(t) + p_{11}(t) = 1$$
$$p_{00}(t) = \frac{1}{\lambda+\mu}\left[\mu + \lambda e^{-(\lambda+\mu)t}\right]$$
$$p_{11}(t) = \frac{1}{\lambda+\mu}\left[\lambda + \mu e^{-(\lambda+\mu)t}\right]$$

Theorem 2: $P_t = e^{Gt} \left[\equiv \sum_{n=0}^{\infty} \frac{(tG)^n}{n!} = I + tG + \frac{(tG)^2}{2!} + \frac{(tG)^3}{3!} + \cdots\right]$ where e^{Gt} is just a shorthand notation for the infinite sum.
is a solution to the KDEs

Proof:- We simply check that the given P_t satisfies KDEs

$$P_t' = G + \frac{2tG^2}{2!} + \frac{3t^2G^3}{3!} + \cdots\cdots$$
$$= \left[I + tG + \frac{(tG)^2}{2!} + \cdots\cdots\right] G$$
$$= P_t G.$$

This result is useful for numerical approximation when $|S|$ is finite.

Given state space $S = \{0, 1, 2, \ldots\}$, generate G and $P_t = ((p_{ij}(t)))$ as matrix of transition prob., a row vector $\pi = [\pi_0, \pi_1, \ldots, \pi_{|S|}]$ is called a stationary distn of the CTMC if $\pi = \pi P_t$ $(\pi G = 0)$. (**Prove it**)