NOTES ON RELIABILITY THEORY

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RELIABILITY

Reliability is defined to be the probability that a component or system will perform a required function for a given period of time when used under stated operating conditions. It is the probability of a nonfailure over time.

Maintainability is defined to be the probability that a failed component or system will be restored or repaired to a specified condition within a period of time when maintenance is performed in accordance with prescribed procedures. Maintainability is the probability of repair in a given time.

Availability is defined as the probability that a component or system is performing its required function at a given point in time when used under stated operating conditions. Availability may also be interpreted as the percentage of time a component or system is operating over a specified time interval or the percentage of components operating at a given time.

Reliability Vs Quality

Reliability is closely associated with the quality of a product & is often considered as a subset of quality. Quality can be defined qualitatively as the amount by which the product satisfies the customers' requirements. Product quality is in part a function of design & conformance to design specifications. It is also depended on the production system and on adherence to manufacturing procedures & tolerances. Quality is achieved through a good quality assurance program. Quality assurance is a planned set of processes & procedures necessary to achieve high product quality.

On the other hand, reliability is concerned with how long the product continues to function once it becomes operational. A poor-quality product will likely have poor reliability, and a high-quality product will have a high reliability. However, reliability may depend upon some external factors & not just the quality of the product itself. Nevertheless, reliability may be viewed as the quality of the product's operational performance over time, and as such it extends quality into the time domain.

THE RELIABILITY FUNCTION :- Reliability is defined as the probability that a system (component) will function over some time period t.

- T : A non-negative continuous RV that represent the time to failure of the system (component);

Then the prob. of failure can be defined as

$$F(t) = P(T \le t)$$, where T denotes the failure time.

Then $F(t)$ is the probability that the system will fail by time t.
In other words, $F(t)$ is the failure distribution function (also called the unreliability function).

- If we define reliability as the probability of success, or the probability that the system will perform its intended function at a certain time t, then we write

$$R(t) = P(T > t) = 1 - F(t)$$

where, $R(t)$ is the reliability function.

- Failure density function $= f(t) = F'(t)$, then

$$R(t) = 1 - F(t) = 1 - \int_0^t f(k)\,dk = \int_t^\infty f(k)\,dk.$$

Here, $F(0) = 0$, $\lim_{t\to\infty} F(t) = 1$.

$$f(t) = \frac{dF(t)}{dt} = -\frac{dR(t)}{dt}$$ is the probability density function

- Properties:- $f(t) \ge 0$ and $\int_0^\infty f(t)\,dt = 1$.

Given the PDF, $f(t)$ then $F(t) = \int_0^t f(k)\,dk$, $R(t) = \int_t^\infty f(k)\,dk$.

Also, $0 \le R(t) \le 1$ and $0 \le F(t) \le 1$.

- The probability of a failure occuring within some interval of time $[a, b]$ may be found using any of the three prob. functions, since

$$P_r\{a \le T \le b\} = F(b) - F(a) = R(a) - R(b) = \int_a^b f(t)\,dt.$$

- Note:- $f(t)dt$ referred to as probability element. This is dimensionally correct since dt has the unit time & $f(t)$ is the failure probability per unit time.

Ex. I. Suppose the time to failure has exponential density function, i.e. $f(t) = \frac{1}{\theta} e^{-t/\theta}$, $t \geq 0, \theta > 0$ then find the reliability function.

Solution:- $$R(t) = \int_t^\infty \frac{1}{\theta} e^{-t'/\theta} dt' = e^{-t/\theta} \ ; t \geq 0.$$

Ex.2. A machine has a useful life well described by the normal distribution $N(500, 100)$.

(i) What is the prob. that a new machine of this type will last at least 600 hrs.

(ii) What is the probability that a machine of this type that has already function for 500 hrs will function for at least 100 hrs more.

Solution:- (i)
$$\begin{aligned} P(T \geq 600) &= P\left(Z \geq \frac{600-\mu}{\sigma}\right) \\ &= P\left(Z \geq \frac{600-500}{100}\right) \\ &= P(Z \geq 1) \\ &= 1 - P(Z \leq 1) \\ &= 0.15860 \end{aligned}$$

(ii)
$$\begin{aligned} P(T \geq 600 \mid T \geq 500) &= \frac{P(T \geq 600)}{P(T \geq 500)} \\ &= \frac{1 - P(T \leq 600)}{1 - P(T \leq 500)} \\ &= \frac{0.15866}{0.5} \\ &= 0.31726. \end{aligned}$$

- **Note:-** t = TBF = Time between failure = for Repairable system
  = TTF = Time to Failure = for non-repairable system

$\infty \uparrow$ HTB = Higher the Better, e.g. milege, life. (Usually -vely skewed)

$0 \downarrow$ LTB = Lower the Better, e.g. waiting time. (Usually +vely skewed)

$\bigcirc$ NTB = Nominal the Best, e.g. Room Temp. (Usually symmetric)

Ex.3. Given the following PDF for the RV. T, the time to failure of a compressor, what is reliability for a 100-hrs operating life?

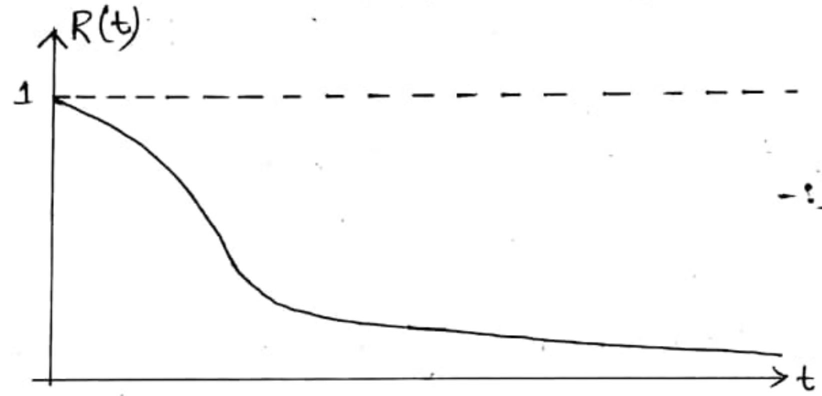
$$f(t) = \begin{cases} \dfrac{0.001}{(0.001t+1)^2} & , t \geq 0 \\ 0 & , \text{OW} \end{cases}$$

Solution:-

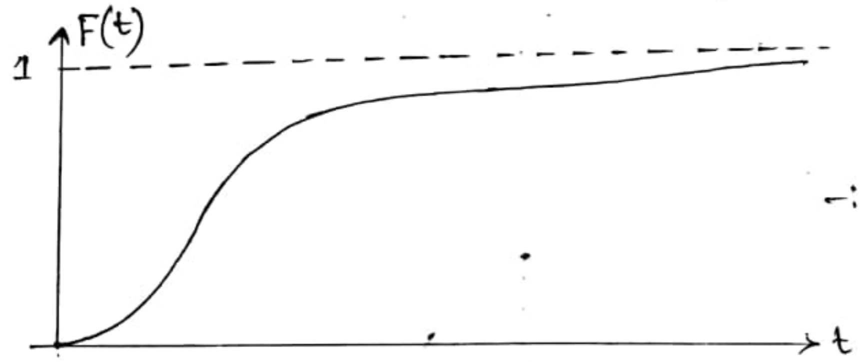
$$R(t) = \int_t^\infty \frac{0.001}{(0.001t'+1)^2}\,dt' = \left.\frac{-1}{(0.001t'+1)}\right]_t^\infty = \frac{1}{0.001t+1}$$

$$F(t) = 1 - R(t) = \frac{0.001t}{0.001t+1}$$

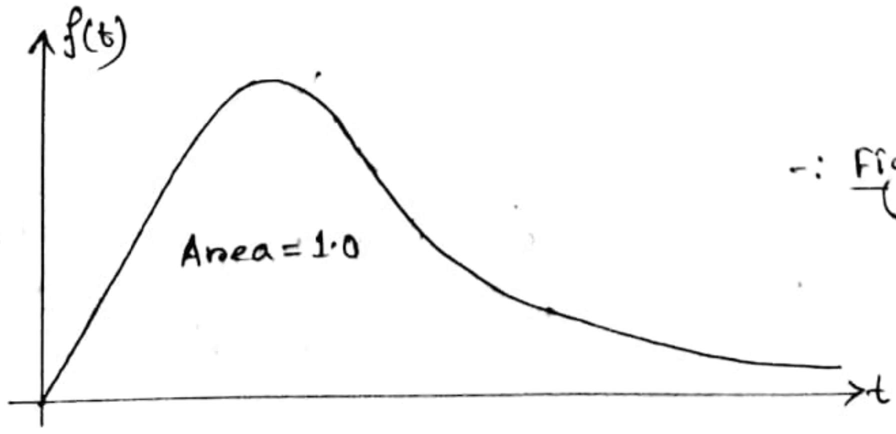
Then $R(100) = \dfrac{1}{0.1+1} = 0.909$



-: Figure of Reliability functn



-: Figure of CDF :-



-: Figure of PDF :-

Remark:- A design life is defined to be the time to failure t_R that corresponds to a specified reliability R, i.e., $R(t_R) = R$. To find the design life if a reliability of 0.95 is desired, we set

$$R(t_R) = \frac{1}{0.001 t_R + 1} = 0.95$$

Solving for t_R, $t_R = 1000\left(\frac{1}{0.95} - 1\right) = 52.6$ hrs.

THE EXPECTED LIFE :- Expected time during which a component will perform successfully is defined as

$$\text{MTTF/MTBF} = E(T) = \int_0^\infty t f(t)\,dt = \int_0^\infty R(t)\,dt \quad , 0 < t < \infty$$

$$= \left[t.R(t)\right]_0^\infty + \int_0^\infty t f(t)\,dt = \text{Mean time to failure/ Mean time between failure.}$$

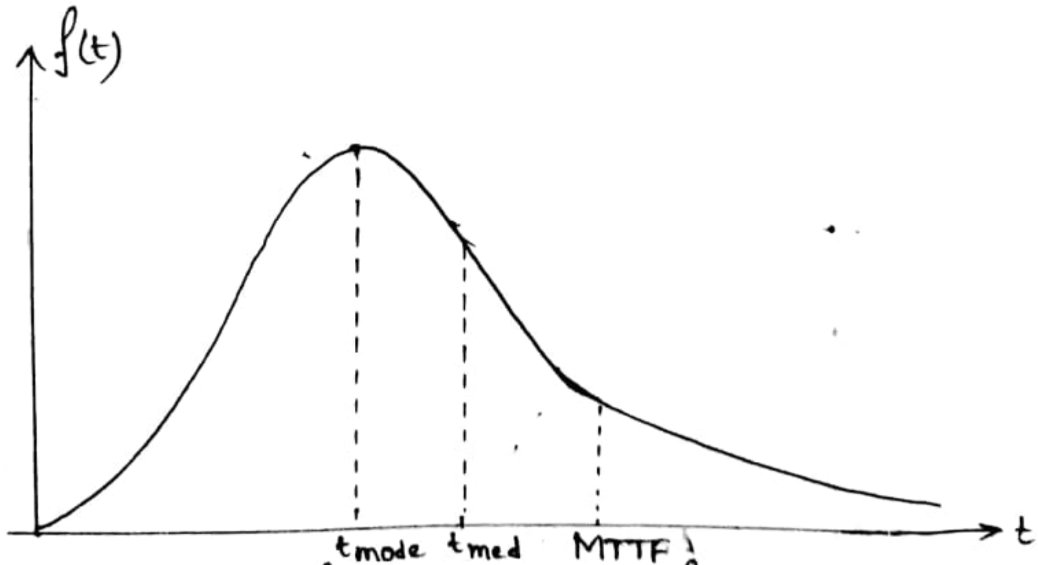
Median time to failure is given by,

$$R(t_{med}) = 0.5 = \Pr\{T > t_{med}\}$$

The median may be preferred to the mean when the distn is highly skewed.

The most likely observed failure time is given by,

$$f(t_{mode}) = \max_{0 \le t < \infty} f(t)$$



-: Comparison of the measures of central tendency for a skewed distribution :-

Ex.1. Consider the PDF $f(t) = \begin{cases} 0.002e^{-0.002t}, & t \geq 0 \\ 0 & \text{OW} \end{cases}$

with t in hours. Find MTTF, $R(t_{med})$ & $f(t_{mode})$?

Solution:-

$$R(t) = \int_t^{\infty} 0.002e^{-0.002t}\,dt = e^{-0.002t}$$

$$MTTF = \int_0^{\infty} e^{-0.002t}\,dt = \left[\frac{e^{-0.002t}}{-0.002}\right]_0^{\infty} = \frac{1}{0.002} = 500\text{ hrs.}$$

To find the median time to failure, set

$$R(t_{med}) = e^{-0.002t_{med}} = 0.5$$

Then solving for t_{med}, $t_{med} = \dfrac{\ln 0.5}{-0.002} = 346.6\text{ hrs}$

To find the mode, we observe that the function $f(t)$ is monotonically decreasing and positive. Therefore its maximum value occurs at $t = 0$, and $t_{mode} = 0$.

Ex.2. Justify: MTTF alone will not uniquely characterize a failure distribution.

Solution:- Considering two reliability functions,

$R_1(t) = e^{-0.002t}$, $t \geq 0$ and $R_2(t) = \dfrac{1000-t}{1000}$, $0 \leq t \leq 1000$

So, here $MTTF_1 = 500\text{ hrs}$ (shown above)

$$MTTF_2 = \int_0^{1000}\left(1 - \frac{t}{1000}\right)dt = \left[t - \frac{t^2}{2000}\right]_0^{1000} = 500\text{ hrs.}$$

Now if we compute their reliabilities for an operating time of 400 hrs, for $R_1(t)$ we obtain $R_1(400) = e^{-0.002(400)} = 0.449$

For $R_2(t)$ we obtain $R_2(400) = \dfrac{1000-400}{1000} = 0.60$

So, MTTF alone will not uniquely characterize a failure distn. One measure that is often used to further describe a failure distn. is its variance σ^2, defined by

$$\sigma^2 = \int_0^{\infty} (t - MTTF)^2 f(t)\,dt = \int_0^{\infty} t^2 f(t)\,dt - (MTTF)^2$$

So, from the first failure distribution, we have

$$\sigma^2 = \int_0^\infty t^2 \left(0.002e^{-0.002t}\right) dt - (500)^2 = 250,000$$

and $\sigma = 500$.

From the second with $f(t) = -\dfrac{dR(t)}{dt} = \dfrac{1}{1000}$,

$$\sigma^2 = \int_0^{1000} t^2 \left(\frac{1}{1000}\right) dt - (500)^2$$

$$= \left[\frac{t^3}{3000}\right]_0^{1000} - (500)^2$$

$$= 83,333.33$$

and $\sigma = 288.67$.

Therefore, although their MTTFs are identical, they have considerably different standard deviations, from which we conclude that their reliability distribution should be inherently different. We would generally prefer the distribution having the small variance.

- Failure Rate :- The rate at which failures occur in a certain time interval $[t_1, t_2]$ is called the failure rate during that interval. It is defined as the probability that a failure per unit time occurs in the interval, given that a failure has not occured prior to t_1, the beginning of the interval.

Thus the failure rate is $\dfrac{R(t_1) - R(t_2)}{(t_2 - t_1)\, R(t_1)}$.

Note that the failure rate is a function of time.

Now if we redefine the interval as $[t, t+\Delta t]$, the previous expression becomes

$$\frac{R(t) - R(t+\Delta t)}{\Delta t . R(t)}$$

The 'rate' in the above definition is expressed as failure per unit time, in reality the time units might be km, revolutions, cycles, etc.

• **Hazard Function:-** This is defined as the limit of the failure rate as the interval approaches '0'. Thus the hazard function is the instantaneous failure rate. The hazard function $h(t)$ is defined by

$$h(t) = \lim_{\Delta t \to 0} \frac{R(t) - R(t+\Delta t)}{\Delta t \cdot R(t)}$$

$$= \frac{1}{R(t)}\left[-\frac{d}{dt}R(t)\right]$$

$$= \frac{-R'(t)}{R(t)}$$

$$= \frac{f(t)}{R(t)}$$

The quantity $h(t)\,dt$ represents the probability that a device of age t will fail in the small interval of time t to $t+dt$. The importance of hazard function is that it indicates the change in the failure rate over the life of a population of devices. For example, two designs may provide the same reliability at a specific point of time; however the failure rates up to this point in time can differ.

• **Alternative way of developing Hazard Function:-**

Q. Consider a device that has survived an interval of time $[0,t]$. What is the probability that it will fail over the next interval of time $(t, t+\Delta t)$?

Solution:- The conditional probability of a failure in the time interval from t to $t+\Delta t$ given that the system has survived to time t is given by

$$P[t \le T \le t+\Delta t \mid T > t] = \frac{P[(t<T\le t+\Delta t)\cap(T>t)]}{P(T>t)}$$

$$= \frac{P(t<T\le t+\Delta t)}{P(T>t)}$$

$$= \frac{F(t+\Delta t) - F(t)}{1-F(t)}$$

$$= \frac{F(t+\Delta t) - F(t)}{R(t)}.$$

$$\therefore \lim_{\Delta t \to 0} \frac{P[t<T\le t+\Delta t \mid T>t]}{\Delta t}$$

$$= \frac{1}{R(t)} \cdot \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t}$$

$$= \frac{F'(t)}{R(t)}$$

$$= \frac{f(t)}{R(t)} = h(t) \text{ is the hazard rate.}$$

Remark:- $h(t)$ = Rate of change of the conditional probability of failure given that the system has survived upto time t.

$f(t)$ = time rate of change of the ordinary probability of failure.

- If any of the four quantity $f(t), F(t), R(t)$ and $h(t)$ are given, then the other three may be obtained from it.

Case I: Assume $f(t)$ is given

$$F(t) = \int_0^t f(x)\,dx$$

$$R(t) = 1 - F(t)$$

$$h(t) = \frac{f(t)}{R(t)}$$

Case II: Assume $F(t)$ is given

$$f(t) = F'(t)$$

$$R(t) = 1 - F(t)$$

$$h(t) = \frac{f(t)}{R(t)} = \frac{F'(t)}{1-F(t)}.$$

Case III: Assume $R(t)$ is given

$$F(t) = 1 - R(t)$$

$$f(t) = -R'(t)$$

$$h(t) = \frac{f(t)}{R(t)} = -\frac{R'(t)}{R(t)}$$

Case IV: Assume $h(t)$ is given

$$R(t) = e^{-\int_0^t h(x)dx}$$

$$F(t) = 1 - R(t)$$

$$f(t) = R(t) \times h(t)$$

Q. Justify: "A particular hazard rate function will uniquely determine a reliability function".

Solution:-

$$h(t) = -\frac{dR(t)}{dt} \cdot \frac{1}{R(t)}$$

$$\therefore h(t)dt = \frac{-dR(t)}{R(t)}.$$

Integrating,
$$\int_0^t h(t')dt' = \int_1^{R(t)} \frac{-dR(t')}{R(t')}$$

where, $R(0) = 1$ establishes the lower limit in the integral on the RHS.

Then
$$-\int_0^t h(t')dt' = \ln R(t)$$

or,
$$R(t) = \exp\left[-\int_0^t h(t')dt'\right]$$

This equation can then be used to derive the reliability function from a known hazard rate function.

Ex. 1. Given the linear hazard rate function $h(t) = 5\times10^{-6}t$, where t is measured in operating hours, what is the design life if a 0.98 reliability is desired? Also find cumulative failure rate & $AFR(t)$?

Solution:- $R(t) = \exp\left[-\int_0^t 5\times10^{-6}t'dt'\right] = \exp\left[-2.5\times10^{-6}t^2\right] = 0.98$

or, $t_{0.98} = \sqrt{\dfrac{0.98}{-2.5\times10^{-6}}} = 89.89 \approx 90$ hr.

- **Cumulative & Average Failure Rate:-** The cumulative failure rate over a period of time t is defined by

$$L(t) = \int_0^t h(t')dt'$$

A related and useful concept is the average failure rate, defined between two times t_1 and t_2:

$$AFR(t_1, t_2) = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} h(t')dt' = \frac{\ln R(t_1) - \ln R(t_2)}{t_2 - t_1}$$

Putting $t_1 = 0$, $t_2 = t$, we get

$$AFR(t) = \frac{\ln R(0) - \ln R(t)}{t-0} = -\frac{\ln R(t)}{t} = \frac{L(t)}{t}.$$

$AFR(t)$ is a nondecreasing function.

Here, $L(t) = \int_0^t 5\times10^{-6}t'dt' = 2.5\times10^{-6}t^2$ is the cumulative failure rate, and the average failure rate from time 0 to t is

$$AFR(t) = \frac{2.5\times10^{-6}t^2}{t} = 2.5\times10^{-6}t.$$

- **Note:-** $f(t)$ and $h(t)$ are two entirely different functions in mathematical terms & concepts.

$f(t)$: Unconditional prob. that the component will fail in the time interval dt between t and $t+dt$.

$h(t)$: Conditional prob. that the component will fail in the same time interval given it has reached age T without failure.

By analogy, the prob. that that a new born child will die at an age between 108 and 109 years (corresponding to $f(t)dt$) is obviously small.

But the prob. of dying in the same period, provided the person has lived to the age of 108 (corresponding to $h(t)dt$) is much greater.

Ex.2. A component has a reliability function given by

$$R(t) = 1 - \frac{t^2}{a^2} \quad \text{for } 0 \le t \le a$$

where a is the parameter of the distn. representing the component's maximum life. Find $h(t)$, MTTF, t_{med}, $AFR(t)$ and $AFR(MTTF)$? Find the conditional Reliability?

Solution:- $f(t) = \frac{2t}{a^2}$

$$\lambda(t) = \frac{2t/a^2}{(a^2-t^2)/a^2} = \frac{2t}{a^2-t^2}, \quad 0 \le t \le a$$

$$MTTF = \int_0^a \left(1 - \frac{t^2}{a^2}\right) dt = \left[t - \frac{t^3}{3a^2}\right]_0^a = \frac{2}{3}a.$$

$$1 - \frac{t_{med}^2}{a^2} = 0.5$$

$$\therefore t_{med} = \sqrt{0.5a^2} = 0.707a$$

$$AFR(t) = \frac{-\ln(1 - t^2/a^2)}{t}$$

The average failure rate up to the MTTF is

$$AFR(MTTF) = AFR\left(\frac{2}{3}a\right) = -\ln\left(1 - \frac{(4/9)a^2}{a^2}\right) \Big/ \left(\frac{2}{3}a\right)$$

$$= \frac{0.8817}{a}$$ [See Page: 20]

Empirical concept of Reliability:-

Survivor function, $n(t)$ = # of survivors at time t.

Fraction of survivors, $R^*(t) = \frac{n(t)}{N_0}$

$R^*(0) = 1$, $R^*(\infty) = 0$.

$R^*(t)$ = the empirical or data based reliability function.

Increment of the survivor function $-\Delta n = n(t) - n(t+\Delta t)$

$$Rate = \frac{n(t) - n(t+\Delta t)}{\Delta t} = \frac{-\Delta n}{\Delta t}$$

This is called failure count per unit time and then to standardize it by the initial number of units.

$$f^*(t) = \frac{n(t_i) - n(t_i + \Delta t)}{\Delta t_{i+1} \cdot N_o} = \frac{-\Delta n_{i+1}}{\Delta t_{i+1} \times N_o} \; ; \; t_i < t < t_{i+1}$$

This is called empirical estimate of probability of failure in the time interval $(t, \Delta t + t)$.

- Case I: For Non-repairable :-

Example:- 10 hypothetical electronic components are placed on life test. Failure times for the components are {5, 10, 17.5; 30, 40, 55, 67.5, 82.5, 100, 117.5}

(i) Plot histograms & polygon graphs for $f^*(t)$ & $h^*(t)$ for the data.
(ii) Plot the reliability and unreliability function $R^*(t)$ & $F^*(t)$.

Solution:-

Function	Operating time	Failure Density $N_o = 10 \times f^*(t)$	Hazard Rate $h(t)$	$R^*(t)$
1	0 – 5	$1/(5 \times 10)$	$1/(5 \times 10)$	9/10
2	5 – 10	$1/(5 \times 10)$	$1/(5 \times 9)$	8/10
3	10 – 17.5	$1/(7.5 \times 10)$	$1/(7.5 \times 8)$	7/10
4	17.5 – 30	$1/(12.5 \times 10)$	$1/(12.5 \times 7)$	6/10
5	30 – 40	$1/(10 \times 10)$	$1/(10 \times 6)$	5/10
6	40 – 55	$1/(10 \times 15)$	$1/(15 \times 5)$	4/10
7	55 – 67.5	$1/(12.5 \times 10)$	$1/(12.5 \times 4)$	3/20
8	67.5 – 82.5	$1/(15 \times 10)$	$1/(15 \times 3)$	2/10
9	82.5 – 100	$1/(17.5 \times 10)$	$1/(17.5 \times 2)$	1/10
10	100 – 117.5	$1/(17.5 \times 10)$	$1/17.5$	

Censoring — Right / Left / Interval

Right:- Test units that are removed from a reliability test prior to failure, or that are still operating at the conclusion test.

This is of two types: Type I Censoring (non-repairable)
Type II Censoring (Decide in advance that you want to see exactly failure time & then test until they occur).

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<u>Example:-</u> 800 hypothetical components are placed on life test. The system is observed at 3 hrs, 6, 9, ..., 30 hrs and the no. of survivors is noted. Find $f^*(t)$ & $h(t)$?

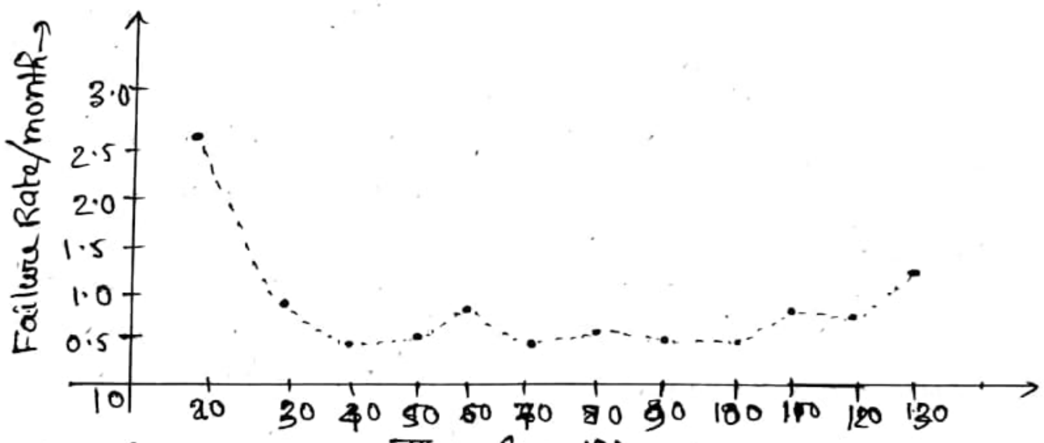
Time Interval (hrs)	No. of failure in interval	Failure density ($f^*(t)$)	$h(t)$
0-3	185	185/(3X800)	185/(3X800)
3-6	42	42/(3X800)	42/(3X615)
6-9	36	36/(3X800)	36/(3X573)
9-12	30	30/(3X800)	30/(3X537)
12-15	17	17/(3X800)	17/(3X507)
15-18	8	8/(3X800)	8/(3X490)
18-21	14	14/(3X800)	14/(3X482)
21-24	9	9/(3X800)	9/(3X468)
24-27	6	6/(3X800)	6/(3X459)
27-30	3	3/(3X800)	3/(3X453)

- Case II : For Repairable Type :— Here hazard rate & failure rate i.e., $h(t)$ and $f^*(t)$ are same.

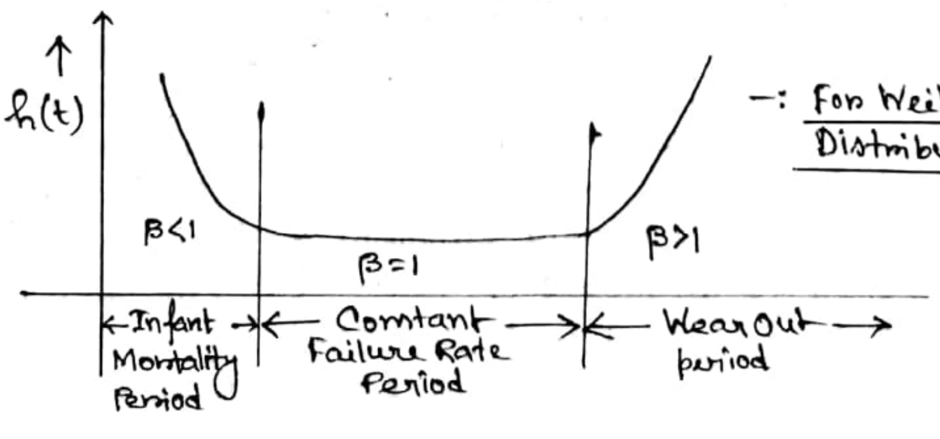
Q. No. 1.

month	No. of failures	Hazard Rate
10-20	27	2.7
20-30	10	1
30-40	4	0.4
40-50	5	0.5
50-60	8	0.8
60-70	5	0.5
70-80	6	0.6
80-90	5	0.5
90-100	5	0.5
100-110	8	0.8
110-120	8	0.8
120-130	12	1.2

Solution:-



This type of curve is called Bath-Tub Curve.



—: For Weibull Distribution :-

Bath-Tub curve is modelled through diffeuent statistical method.

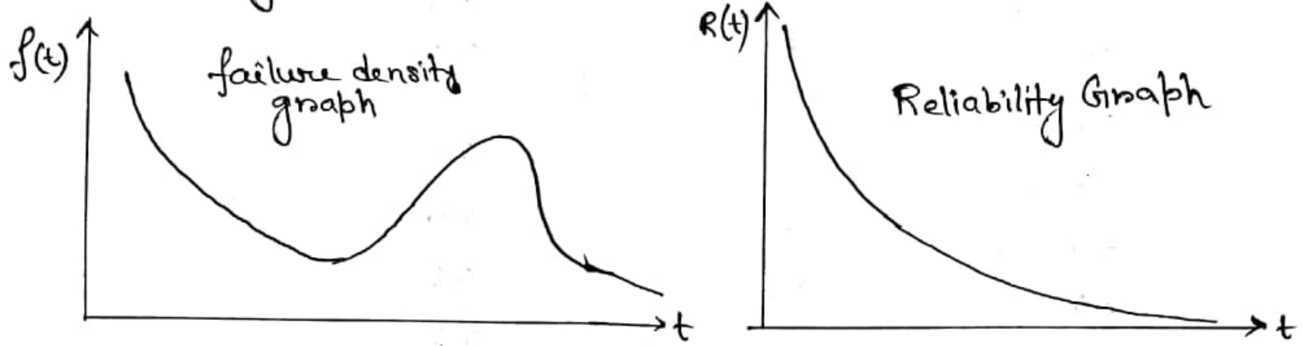
So, we have $f(t)$ = failure density function

$h(t)$ = Hazard rate

$R(t)$ = Reliability

$F(t) = 1 - R(t)$ = Unreliability.

Generally $f(t)$ or $f^*(t)$ & $R(t)$ graphs look like this:



Repairable : Up — Down — Up = TBF — Down — Up — Down

Non-repairable: ① Up = TTF

② Up = TTF

TTF :- Time to Failure

TBF :- Time b/w Failure

BATHTUB CURVE :- The decreasing hazard rate corresponds to the time t_0 to t_1 in the hazard function plot and suggests initial failure rate. Generally, after a system is produced, or assembled, and placed in operation, the initial failure rate is higher than that encountered later. The initial failures may be due to various manufacturing & assembling defects that escape detection by the quality control system. As the defective parts are replaced with new ones, the reliability improves; this phenomenon is sometimes facetiously referred to as servicing-in reliability.

An important form of the hazard rate function is shown in the figute below. Because of its shape, it is commonly referred to as the bathtub curve. Systems having this hazard rate function experience decreasing failure rates early in their life cycle (infant mortality), followed by a nearly constant failure rate (useful life), followed by an increasing failure rate (wearout). This curve may be obtained as a composite of several failure distn., or as a function of piecewise linear & constant failure rates.

Example: A simplified form of the bathtub curve is based upon linear & constant hazard rates:

$$h(t) / \lambda(t) = \begin{cases} c_0 - c_1 t + \lambda, & 0 \le t \le \frac{c_0}{c_1} \\ \lambda, & \frac{c_0}{c_1} < t \le t_0 \\ c_2 (t - t_0) + \lambda, & t_0 < t \end{cases}$$

where, $\lambda > 0$. This hazard function linearly decreases to λ at time c_0/c_1, remains constant until time t_0, and then linearly increases. The resulting density function must be defined over three regions, so

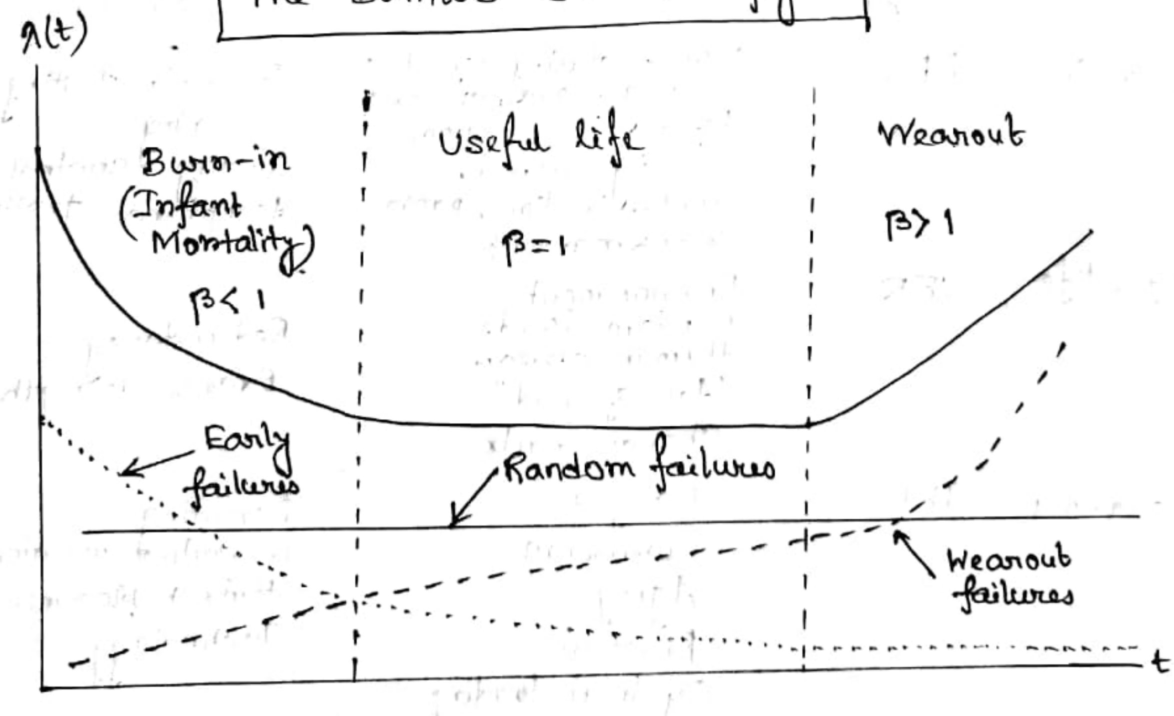
$$f(t) = \begin{cases} (c_0 + \lambda - c_1 t) \exp\left\{-\left[(c_0+\lambda)t - c_1 (t^2/2)\right]\right\}, & 0 \le t \le c_0/c_1 \\ \lambda \exp\left\{-(\lambda t + c_0^2/2c_1)\right\}, & c_0/c_1 < t \le t_0 \\ \left[c_2(t - t_0) + \lambda\right] \exp\left\{-\left[(c_2/2)(t-t_0)^2 + (c_0^2/2c_1) + \lambda t\right]\right\}, & t_0 < t \end{cases}$$

Then the reliability function is defined over the three regions as

$$R(t) = \begin{cases} \exp\left\{-\left[(c_0+\lambda)t - c_1 (t^2/2)\right]\right\}, & 0 \le t \le c_0/c_1 \\ \exp\left\{-\left(\lambda t + \frac{c_0^2}{2c_1}\right)\right\}, & c_0/c_1 < t \le t_0 \\ \exp\left\{-\left[\left(\frac{c_2}{2}\right)(t-t_0)^2 + \lambda t + \left(\frac{c_0^2}{2c_1}\right)\right]\right\}, & t_0 < t \end{cases}$$

where c_0, c_1, c_2 and t_0 are constants to be determined.

The bathtub curve :- figure



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Intermediate Calculations for the Linear Bathtub Curve:-

For $t \leq \frac{c_0}{c_1}$:

$$R(t) = \exp\left[-\int_0^t (c_0 - c_1 t' + \lambda)\, dt'\right]$$

$$= \exp\left[-\left(c_0 t + \lambda t - \frac{c_1 t^2}{2}\right)\right]$$

For $\frac{c_0}{c_1} < t \leq t_0$:

$$R(t) = R\left(\frac{c_0}{c_1}\right) \exp\left[-\int_{c_0/c_1}^{t} \lambda\, dt'\right]$$

$$= \exp\left(-\frac{c_0^2}{c_1} + c_1 \frac{c_0^2}{2c_1^2} - \lambda \frac{c_0}{c_1}\right) \exp -\left(\lambda t - \lambda \frac{c_0}{c_1}\right)$$

$$= \exp -\left(\lambda t + \frac{c_0^2}{2c_1}\right)$$

For $t_0 < t$:

$$R(t) = R(t_0) \exp\left[-\int_{t_0}^{t} [c_2 (t' - t_0) + \lambda]\, dt'\right]$$

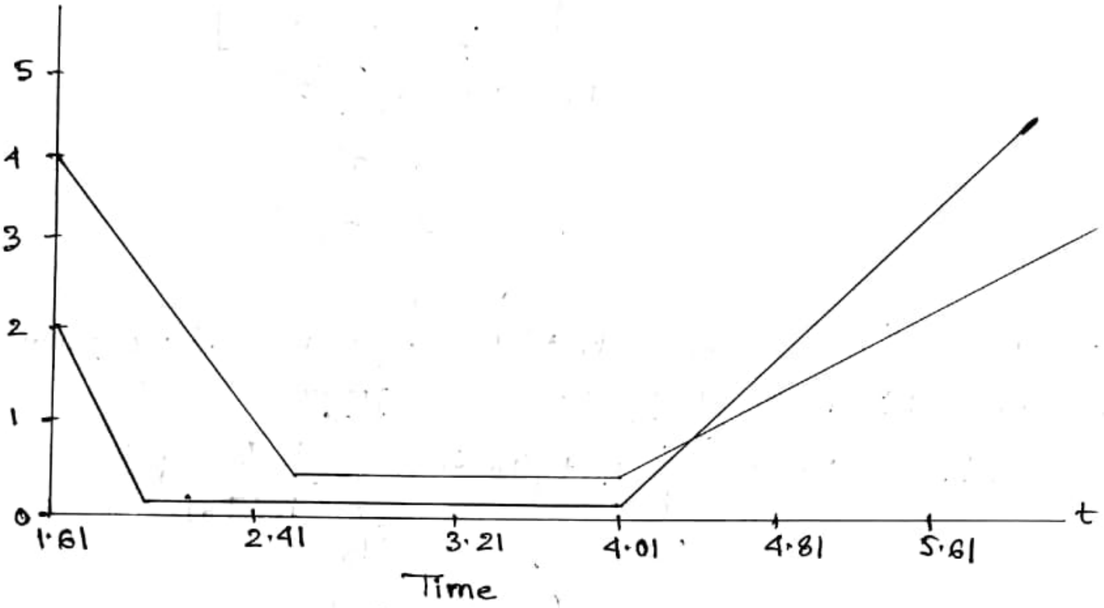
$$= \exp -\left(\lambda t_0 + \frac{c_0^2}{2c_1}\right) \exp -\left(\frac{c_2}{2} (t - t_0)^2 + \lambda t - \lambda t_0\right)$$

$$= \exp -\left(\frac{c_2}{2} (t - t_0)^2 + \lambda t + \frac{c_0^2}{2c_1}\right)$$

The Bathtub Curve :-

	Characterized by	Caused by	Reduced by
Burn-in	DFR	Manufacturing defects : welding flaws, cracks, defective parts, poor quality control, contamination, poor workmanship	Burn-in testing Screening Quality control Acceptance testing
Useful life	CFR	Environment Random loads Human error "Act of God" Chance events	Redundancy Excess strength
Wear-out	IFR	Fatigue Corrosion Aging Friction Cyclical loading	Derating Preventive maintenance Parts replacement Technology

Conditional Reliability :— Conditional reliability is useful in describing the reliability of a component or system following a burn-in period T_0 or after a warranty period T_0. We define conditional reliability as the reliability of a system given that it has operated for time T_0 :



— : A piecewise linear bathtub curve :-

$$R(t|T_0) = Pr\{T > T_0 + t \,|\, T > T_0\}$$

$$= \frac{Pr\{T > T_0 + t\}}{Pr\{T > T_0\}} = \frac{R(T_0 + t)}{R(T_0)}$$

$$= \frac{\exp\left[-\int_0^{T_0+t} \lambda(t')dt'\right]}{\exp\left[-\int_0^{T_0} \lambda(t')dt'\right]} = \exp\left[-\int_{T_0}^{T_0+t} \lambda(t')dt'\right]$$

Therefore
$$\frac{dR(t|T_0)}{dT_0} = \exp\left[-\int_{T_0}^{T_0+t} \lambda(t')dt'\right]\frac{d}{dT_0}\left[\int_{T_0}^{t+T_0} \lambda(t')dt'\right]$$

$$= R(t|T_0)\left[-\lambda(t+T_0) + \lambda(T_0)\right]$$

In order for the reliability to improve as a function of T_0, the derivative, or slope, of $R(t|T_0)$ with respect to T_0 must be positive. From the above result, this will occur only if $\lambda(T_0) > \lambda(t+T_0)$. In other words, the failure rate must be decreasing.

Example:- In Ex.2.(Page 12),

$$R(t|T_0) = \exp\left[-\int_{T_0}^{T_0+t} \lambda(t')dt'\right]$$

$$= \exp\left[-\int_{T_0}^{T_0+t} \frac{2t'dt'}{a^2-t'^2}\right]$$

$$= \frac{1-(t+T_0)^2/a^2}{1-T_0^2/a^2}$$

$$= \frac{a^2-(t+T_0)^2}{a^2-T_0^2}.$$

Residual MTTF:- Since $R(t|T_0)$ is a reliability function, a residual MTTF may be obtained from

$$MTTF(T_0) = \int_0^\infty R(t|T_0)dt = \int_{T_0}^\infty \frac{R(t')}{R(T_0)}dt'$$

$$= \frac{1}{R(T_0)}\int_{T_0}^\infty R(t')dt'$$

where $t' = t+T_0$.

For those units having survived to time T_0, the above equation determines their mean remaining lifetime.

Ex. For the reliability function $R(t) = \frac{a^2}{(a+t)^2}$, $t \geq 0$, ($a>0$ is a parameter of the distn.) find out residual MTTF?

Solution:- Hazard rate function is given by

$$\lambda(t) = \frac{2}{(a+t)}$$ which is decreasing.

The residual MTTF is

$$MTTF(T_0) = \frac{(a+T_0)^2}{a^2}\int_0^\infty \frac{a^2}{(a+t')^2}dt'$$

$$= \frac{(a+T_0)^2}{a^2}\cdot\left[\frac{-a^2}{a+t'}\right]_{T_0}^\infty$$

$$= a+T_0.$$

which has the ~~increasing~~ interesting property that the residual mean increases by the amount of the current age.
If $T_0 = 0$, the unconditional mean, $MTTF = a$, is obtained.

DERIVATION OF MTTF FORMULAE:-

By definition, $MTTF = \int_0^\infty t f(t) dt$

$$= \int_0^\infty t \cdot \frac{dF(t)}{dt} dt$$

$$= \int_0^\infty t \cdot -\frac{dR(t)}{dt} dt$$

Using integration by parts,

$$MTTF = -tR(t)\Big|_0^\infty + \int_0^\infty R(t) dt$$

$$= \int_0^\infty R(t) dt, \text{ since } \lim_{t \to \infty} tR(t) = 0 \text{ \& } 0.R(0)=0$$

DERIVATION OF VARIANCE FORMULAE:-

$$\sigma^2 = \int_0^\infty (t - MTTF)^2 f(t) dt$$

$$= \int_0^\infty \left[t^2 - 2t.MTTF + (MTTF)^2\right] f(t) dt$$

$$= \int_0^\infty t^2 f(t) dt - 2MTTF \int_0^\infty t f(t) dt + (MTTF)^2 \int_0^\infty f(t) dt$$

$$= \int_0^\infty t^2 f(t) dt - 2(MTTF)^2 + (MTTF)^2$$

$$= \int_0^\infty t^2 f(t) dt - (MTTF)^2.$$

Since, $\int_0^\infty t f(t) dt = MTTF$ and $\int_0^\infty f(t) dt = 1$.

SUMMARY

A failure process, represented by the random variable T (time to failure), may be uniquely characterized by any of the following four functions:

$f(t)$, the probability density function (PDF);
$F(t)$, the cumulative distribution function (CDF);
$R(t)$, the reliability function;
$\lambda(t)$, the hazard rate function;

The following relationships hold.

$$F(t) = \int_0^t f(t')\,dt'$$

$$R(t) = \int_t^\infty f(t')\,dt'$$

$$R(t) = 1 - F(t);$$

$$f(t) = -\frac{dR(t)}{dt} = \frac{dF(t)}{dt}$$

$$\lambda(t) = \frac{f(t)}{R(t)}$$

$$R(t) = \exp\left[-\int_0^t \lambda(t')\,dt'\right]$$

$$MTTF = \int_0^\infty R(t)\,dt$$

$$\sigma^2 = \int_0^\infty t^2 f(t)\,dt - (MTTF)^2$$

$$R(t|T_0) = \frac{R(t+T_0)}{R(T_0)}$$

$$L(t) = \int_0^t \lambda(t')\,dt'$$

$$AFR(t_1, t_2) = \frac{\int_{t_1}^{t_2} \lambda(t)\,dt}{t_2 - t_1} = \frac{\ln R(t_1) - \ln R(t_2)}{t_2 - t_1}$$

$$MTTF(T_0) = \frac{1}{R(T_0)} \int_{T_0}^\infty R(t')\,dt'.$$

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CONSTANT FAILURE RATE MODEL

A failure distribution that has a constant failure rate is called an exponential probability distribution. The exponential distn. is one of the most important reliability distribution. Many systems exhibit constant failure rates, and the exponential distribution is in many respects the simplest reliability distn. to analyze.

- Exponential Distribution:– This distribution is very useful in analysis of failure rates of complete system, subsystem or assemblies. For example,

In the case of aircraft pump, the probability of the failure of component such as shaft or bearing can be described by a Log-normal distribution. However, for the pump as a whole (as a system) exponential distn. is a better choice.

Here the pdf of Exp. distn. is given by

$$f(t) = \frac{1}{\theta} e^{-t/\theta} \quad , t \geq 0 \text{, where } \theta \text{ is the parameter } \ni \theta > 0.$$

$$\theta = \text{characteristic life} = \frac{B_{50}}{\ln 2}$$

$$R(t) = \exp\left[- \int_0^t \lambda \, dt' \right]$$

$$= e^{-t/\theta}$$

[Note that, Exp. distn. is a special case of the Weibull Distribution when $\beta = 1$, $\gamma = 0$]

$$F(t) = P(X \leq t)$$

$$= \int_0^t f(x) dx$$

$$= \int_0^t \frac{1}{\theta} e^{-\frac{x}{\theta}} dx$$

$$= 1 - e^{-t/\theta}$$

$$R(t_R) = e^{-t_R/\theta}$$

then $t_R = -\ln R . \theta$

When $R = 0.5$, the median is

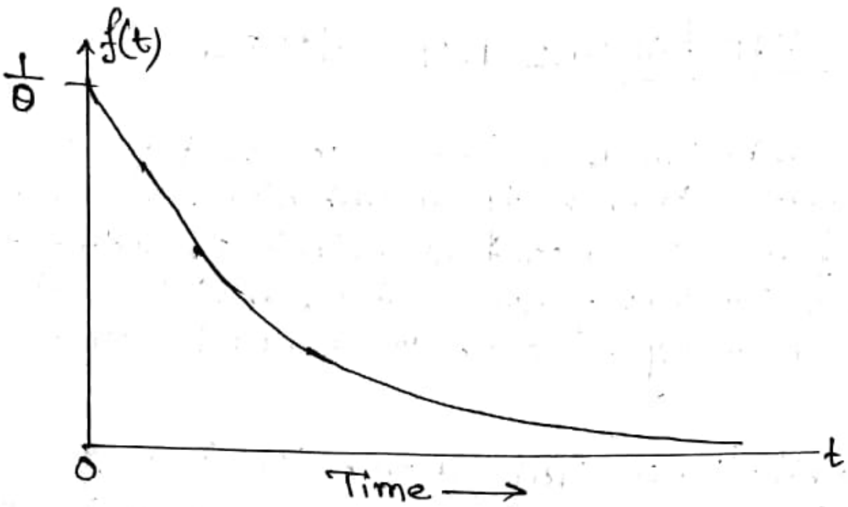
$t_{med} = -$

$$h(t) = \frac{f(t)}{R(t)} = \frac{1}{\theta}.$$

$$E(t) = MTTF/MTBF = \int_0^\infty t f(t) dt = \theta$$

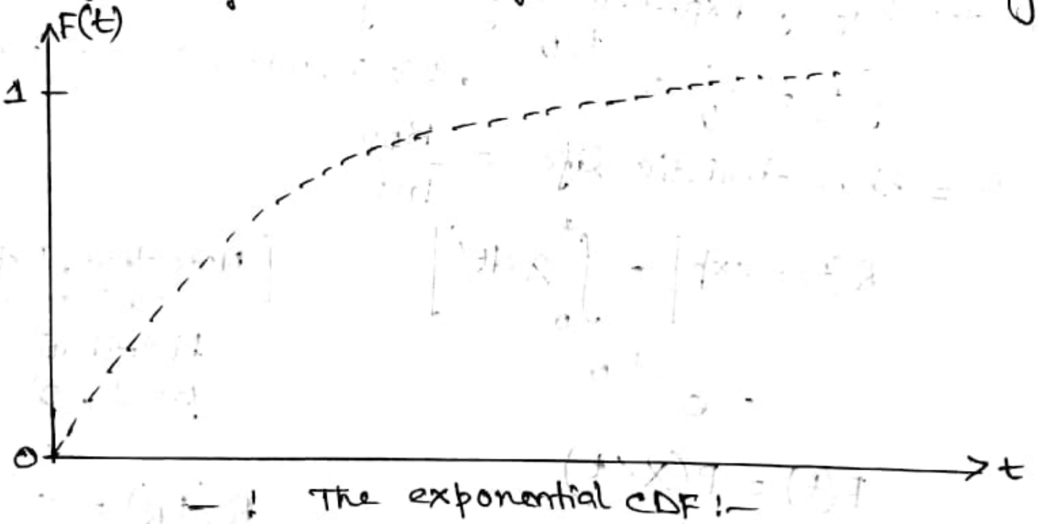
$$V(t) = \theta^2$$

and the standard deviation is θ = MTTF. This is an interesting result since it implies that the variability of failure time increases as the reliability (MTTF) increases.

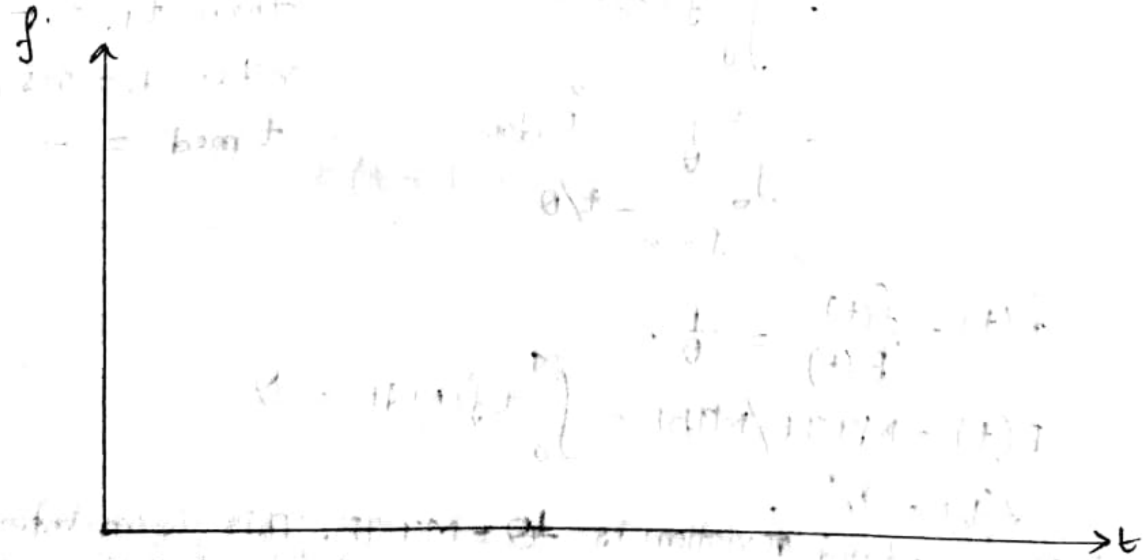


–: The exponential failure density function:–
(~~Reliability function~~)

It should also be noted that the mean time to failure is the reciprocal of the failure rate. Although $1/h(t)$ is always in units of time per (between) failures, it is the mean of the failure distn. for the CFR model only.



– : The exponential CDF :–



Ex.1. A radder set has a mean time between failures of 240 hrs based on exponential distribution. Suppose a certain mission requires failure free operation for the set of 24 hours. What's the chance that the set will complete the mission without failure?

Solution:-

$$E(T) = 240 \text{hrs} = \theta$$

$$t = 24.$$

$$R(24 \text{hrs}) = e^{-\frac{t}{\theta}} = e^{-0.1} = 0.9048$$

Ex.2. The average life of sub-assembly 'A' is 2000 hrs. Data indicate that this life characteristic is exponentially distributed.

(a) What % of the sub-assembly in the assembly will last at least 200 hrs?

(b) The average life of sub-assembly 'B' is 1000 hrs and the life is exponentially distributed. What % of the sub-assembly in the population will be at least 200 hrs?

(c) These sub-assembly are independently manufactured and then connected in series to form the total assembly. What % of assembly in the population will last at least 200 hrs?

Solution:- (a) $MTBF = 2000 \text{hrs} = \theta$

$$R(200) = P(T > 200 \text{hrs}) = e^{-\frac{t}{\theta}} = e^{-\frac{1}{10}} = 0.9048$$

(b) $MTBF = 1000 \text{hrs} = \theta$

$$R(200) = e^{-t/\theta} = e^{-200/1000} = 81.8\%$$

(c)

Memorylessness:- A well-known characteristic of the CFR model, is its lack of memory. That is, the time to failure of a component is not dependent on how long the component has been operating. There is no aging or wearout effect. The prob. that the component will operate for the next 1000 hrs is the same regardless of whether the component is brand new, has been operating for several hundred hrs, or has been operating for several hundred hours. This property is consistent with the completely random & independent nature of the failure process.

This property can be demonstrated mathematically using conditional ~~probabili~~ reliability:

$$R(t|T_0) = \frac{R(t+T_0)}{R(T_0)} = \frac{e^{-(t+T_0)/\theta}}{e^{-T_0/\theta}}$$

$$= e^{-t/\theta} = R(t)$$

In other words, a burn-in period T_0 has no subsequent effect on reliability and will not improve the component's reliability. Time to failure depends only on the length of the observed operating time (t) and not on its current age (T_0).

Relationship between Exponential Distribution & Poisson Distn. :-

If we consider the Poisson distribution as a model of the number of occurance of some event in the time interval $(0,t]$.

Then $P(X=x) = \dfrac{e^{-\lambda t}(\lambda t)^x}{x!}$

Now $x=0$, there are no occurances of the event in $(0,t]$ and $P[X=0] = e^{-\lambda t}$.

We may think $P(X=0)$ as the probability that the interval to the first occurance is greater than 't', or

$$P(Y>t) = e^{-\lambda t} = P(X=0)$$

where, Y is the continuous RV denoting the interval to the first occurance.

Since $F(t) = P(Y \le t) = 1 - e^{-\lambda t}$; so $f(y) = \lambda e^{-\lambda y}$, as the distribution of the interval to the first occurance.

This distn. is an Exponential distribution with parameter λ.

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Therefore, if the number of occurance of an event has a Poisson Distribution with parameter λ then the distribution of the interval between occurances is Exponential Distn. with parameter λ. This is also known as Poisson Process.

Ex. In a large corporate computer network, user log ons of the system can be modelled as Poisson distribution with a mean 25 log ons per hour.

(a) What is the probability that there are no log ons in an interval of 6 minute?

(b) What is the probability that the time until the next log on is between 2 and 3 minutes?

(c) Determine the interval of time such that the prob. that no log on in the interval is 0.90. Find out the length of the interval.

(d) What's the mean time until the next log on.

Solution:- (a) $X \sim Exp(25 \text{ log ons/hr})$

$$\therefore P(X > 6\text{min}) = e^{-\lambda t} = e^{-6 \times 0.417} = e^{-2.5}$$

(b) $P(2 < X < 3) = F(3) - F(2)$

$$= \cancel{1 - e^{-3\times 0.417}}$$

$$= R(0.0333) - R(0.05)$$

$$= 0.151$$

(c) $P(X=0) = e^{-\lambda t}$

$$= e^{-25 \times t} = 0.90$$

$$\Rightarrow -25 \times t = \log(.9)$$

$$\Rightarrow t = 0.0042$$

(d) $T \sim Exp(\lambda)$

$$E(T) = \frac{1}{\lambda} = \frac{1}{25} = 0.04$$

$$V(T) = \frac{1}{\lambda^2} = \frac{1}{625}$$

If the log ons are well modelled by a Poisson process, the probability that the 1st log on afternoon occurs after 12.06PM is the same as the probability that the 1st log on after 3PM occurs after 3.06PM, and if some one log on at 2.22PM. The probability that the next log on occurs at 2.28PM is still 0.082.

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- The Two-parameter Exponential Distribution:-

If a failure will never occur prior to some specified time t_0, then t_0 is a minimum, or threshold, time. It is also known as the guaranteed lifetime. The parameter t_0 is a location parameter that shifts the distribution an amount equal to t_0 to the right on the time (horizontal) axis.

The probability density function becomes

$$f(t) = -\frac{dR(t)}{dt} = \lambda e^{-\lambda(t-t_0)}, \quad 0 < t_0 \leq t < \infty$$

and $R(t) = e^{-\lambda(t-t_0)}, \quad t \geq t_0$

Falure rate is $\lambda(t) = \frac{f(t)}{R(t)} = \lambda$.

$$MTTF = \int_0^\infty \lambda t e^{-\lambda(t-t_0)}\, dt = t_0 + \frac{1}{\lambda} = \text{Mean}$$

$$R(t_{med}) = e^{-\lambda(t_{med}-t_0)} = 0.5$$

$$\Rightarrow t_{med} = t_0 + \frac{\ln 0.5}{-\lambda} = t_0 + \frac{0.69315}{\lambda}$$

Also $t_R = t_0 + \frac{\ln R}{-\lambda}$.

The mode occurs at t_0.

And the variance & s.d. of the two parameter exponential distn. are not affected by the location parameter. Therefore $\sigma = 1/\lambda$.

Ex. Let $\lambda = 0.001$ and $t_0 = 200$. Then $R(t) = e^{-0.001(t-200)}, \quad t \geq 200$.

Find MTTF, t_{med} and σ?

Solution:-

$$MTTF = 200 + \frac{1}{0.001} = 1200$$

$$t_{med} = 200 + \frac{0.69315}{0.001} = 893.15$$

$$t_{0.95} = 200 - \frac{\ln 0.95}{0.001} = 251.3$$

$$\sigma = \frac{1}{0.001} = 1000.$$

Ex. An engine shaft has a failure rate of 0.5×10^{-7}/hr. The seals used with the shaft has a failure rate of 2.5×10^{-7}/hr. If a given company has 5000 engines with these shafts, seals and each engine operates 350 days/yr in useful life, estimate the no. of shaft and seals that must be replaced annually.

Solution:- $\lambda_{shaft} = 0.5 \times 10^{-7}$ hr.

$\lambda_{seal} = 2.5 \times 10^{-7}$ hr.

$$R(350 \times 24) = R(8400\,hr) = e^{-\lambda t} = e^{-8400 \times 0.5 \times 10^{-7}}$$
$$= 0.99958$$

$$F(350 \times 24) = 1 - 0.99958$$
$$= 0.00042$$

So, No. of shafts $= 5000 \times 0.00042 \cong 2$.

Do same thing for seals.

EXAMPLE OF POISSON PROCESS:-

Ex. A specially designed welding machine has a nonrepairable motor with a constant failure rate of 0.05 failure per year. The company has purchased two spare motors. If the design life of the welding machine is 10 yr, what is the probability that the two spares will be adequate?

Solution:- The expected number of failures over the life of the machines is $\lambda t = 0.05(10) = 0.5$.

$$R_2(10) = \sum_{n=0}^{2} \frac{e^{-0.5}(0.5)^n}{n!}$$

$$= e^{-0.5}\left(1 + 0.5 + \frac{0.25}{2}\right) = 0.9856.$$

is the probability of 2 or fewer failures occuring over the 10 yrs. Let Y_3 be the time of the third failure. Y_3 has a gamma distr. with $k = 3$ and $\lambda = 0.05$. Therefore the mean time to obtain 3 failures is $3/0.05 = 60$ yrs. The probability that the third failure will occur within 10 yrs is obtained from

$$F_{Y_3}(10) = 1 - e^{-0.05 \times 10}\left(1 + 0.05 \times 10 + \frac{(0.05 \times 10)^2}{2!}\right) = 0.0144$$

Observe that $0.0144 = 1 - 0.9856$ since the probability of two or fewer failures in 10 yr is complementary to the event that the third failure occurs within 10 yr.

TIME-DEPENDENT FAILURE MODELS

- **The Weibull Distribution:-** The distribution well describes the characteristic (i.e. life) of parts or components. The three parameter Weibull failure density function is

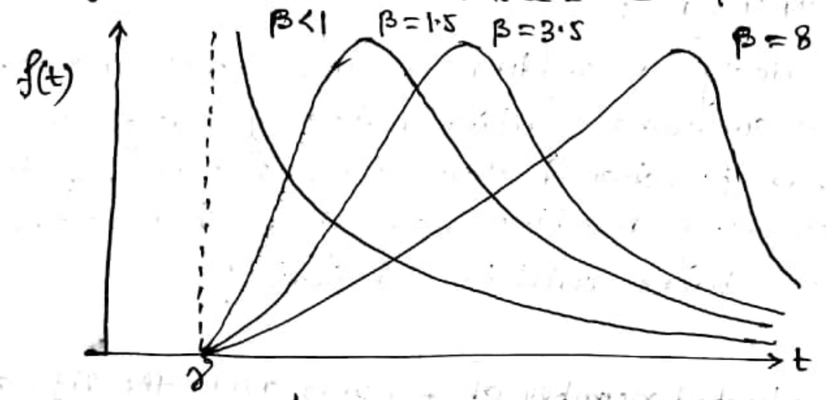
$$f(t) = \frac{\beta (t-\gamma)^{\beta-1}}{(\theta-\gamma)^{\beta}} \exp\left[-\left(\frac{t-\gamma}{\theta-\gamma}\right)^{\beta}\right] \; ; \quad t \geq \gamma \geq 0, \; \beta > 0, \theta > 0$$

where, the parameters, usually determined (or, estimated) experimentally are γ, β, θ.

γ = Expected minimum value of 't' = LOCATION PARAMETER

β = Shape parameter = Weibull slope.

θ = characteristic value = Scale parameter



$$F(t) = \int_{\gamma}^{t} f(x)\,dx$$

$$= \int_{\gamma}^{t} \frac{\beta}{(\theta-\gamma)} \left(\frac{x-\gamma}{\theta-\gamma}\right)^{\beta-1} \exp\left[-\left(\frac{x-\gamma}{\theta-\gamma}\right)^{\beta}\right] dx$$

Take $y = \left(\frac{x-\gamma}{\theta-\gamma}\right)^{\beta}$

$$= \int_{0}^{\left(\frac{t-\gamma}{\theta-\gamma}\right)^{\beta}} e^{-y}\,dy$$

$$= 1 - \exp\left[-\left(\frac{t-\gamma}{\theta-\gamma}\right)^{\beta}\right]$$

$$\therefore R(t) = \exp\left[-\left(\frac{t-\gamma}{\theta-\gamma}\right)^{\beta}\right]$$

$$\therefore h(t) = \frac{f(t)}{R(t)} = \frac{\beta (t-\gamma)^{\beta-1}}{(\theta-\gamma)^{\beta}}.$$

$$E(T) = \int_0^\infty t f(t)\,dt = \int_0^\infty R(t)\,dt$$

$$= \int_0^\infty \exp\left[-\left(\frac{t-\gamma}{\theta-\gamma}\right)^\beta\right] dt$$

$$= \gamma + (\theta-\gamma)\Gamma\left(1+\frac{1}{\beta}\right)$$

$$V(T) = (\theta-\gamma)^2\left[\Gamma\left(1+\frac{2}{\beta}\right) - \left\{\Gamma\left(1+\frac{1}{\beta}\right)\right\}^2\right]$$

Ex. The failure time of a certain component has a Weibull distn. with $\beta=4$, $\theta=2000$, and $\gamma=1000$. Find the reliability of the component and the hazard rate for an operating time of 1,500 hours.

Solution:- $t = 1500$ hrs.

$$R(1500) = \exp\left[-\left(\frac{1500-1000}{2000-1000}\right)^4\right]$$

$$= \exp(-0.0625)$$

$$= 0.939.$$

$$h(1500) = \frac{4(1500-1000)^{4-1}}{(2000-1000)^4} = 0.0005 \text{ failures/hour.}$$

$$E(t) = 1000 + 1000\,\Gamma\left(1+\frac{1}{4}\right)$$

$$= 1000\left(1+\Gamma\left(\frac{5}{4}\right)\right)$$

$$= 1000(1+0.9064)$$

$$= 1906.4 \text{ hrs}$$

Two Parameter Weibull Distribution:-

Assume $\gamma = 0$ = Minimum life

$$f(t) = \frac{\beta}{\theta^{\beta}} t^{\beta-1} \exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right]$$

$$F(t) = 1 - \exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right]$$

$$\frac{1}{1-F(t)} = \exp\left[+\left(\frac{t}{\theta}\right)^{\beta}\right]$$

$$\therefore \ln \frac{1}{1-F(t)} = \left(\frac{t}{\theta}\right)^{\beta}$$

$$\therefore \ln\left(\ln \frac{1}{1-F(t)}\right) = \beta \ln t - \beta \ln \theta$$

$$\Rightarrow Y = bX + C$$

which is a straight line with slope 'b' & intercept 'c' on the cartesian co-ordinate.

Here a plot of $\ln\left(\ln\left(\frac{1}{1-F(t)}\right)\right)$ against $\ln t$ will also be a straight line with slope 'β'.

Note:- The reason for calling 'β' the Weibull slope is therefore apparent.

Characteristic life θ:- This can be related to the median life which is 50% life or B_{50} life, is the life corresponding to 50% of the population. That is, 50% of the population is expected to have life less than or equal to B_{50} life.

Since $F(t) = 0.50 = 1 - \exp\left[-\left(\frac{B_{50}}{\theta}\right)^{\beta}\right]$

$$\Rightarrow \frac{1}{2} = \frac{1}{\exp\left[\left(\frac{B_{50}}{\theta}\right)^{\beta}\right]}$$

$$\therefore \ln 2 = \left(\frac{B_{50}}{\theta}\right)^{\beta}$$

$$\therefore \theta = \frac{B_{50}}{(\ln 2)^{1/\beta}}.$$

Now, $F(\theta) = P(t \le \theta)$

$$F(t) = 1 - \exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right]$$

$$F(\theta) = 1 - e^{-1}$$

$$= 0.632$$

$$\equiv 63.2\% \text{ at } t = \theta,$$

It means θ is the life by which 63.2% of the parts will have failed.

MTTF:-

$$E(T) = \int_0^\infty \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} e^{-\left(\frac{t}{\theta}\right)^\beta} t\,dt \quad ; \quad y = \left(\frac{t}{\theta}\right)^\beta$$

$$dy = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} dt$$

$$= \int_0^\infty t e^{-y} dy$$

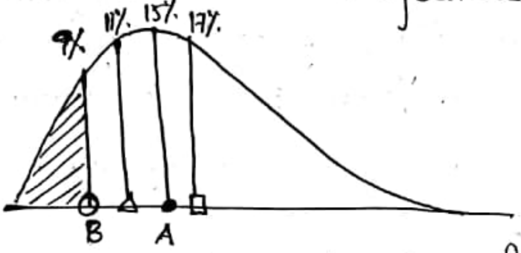
$$= \theta \int_0^\infty y^{1/\beta} e^{-y} dy$$

$$= \theta\, \Gamma\left(1+\frac{1}{\beta}\right)$$

Concept of Median Ranks:-

$$MR = \frac{i - 0.3}{n + 0.4} \times 100$$

i = failure order no, n = sample size.



In sample 1 (of 5 items) the first failure may have occurred at A where 15% of the population has values less than A. In another sample of 5 and the test repeated, the lowest life B might represent only 9% of the population.

This formula is an approximate formula for the calculation of median ranks, may be used when the tables are not available, or when sample size is beyond the range covered by the table.

About Weibull Distribution:-

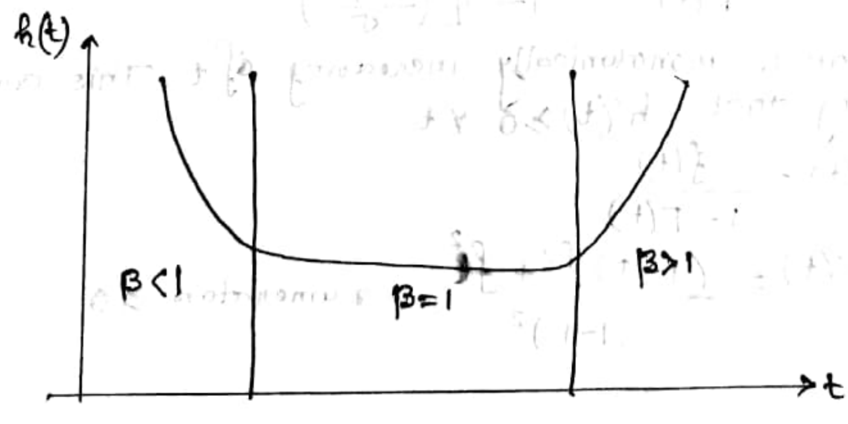
When $\beta < 1$, the Weibull distn takes a hyper exponential shape & hazard function is decreasing.

When $\beta > 1$, $h(t)$ is increasing.

When $\beta = 1$, $h(t)$ is constant.

When $\beta = 3.5$, the distn is approximately symmetrical.

When $\beta > 3.5$, the distn becomes negatively skewed.



—: Bath-tub curve :-

- The Normal Distribution:-

The normal distribution has been used successfully to model fatigue and wearout phenomena. The formula for the PDF;

$$f(t) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\cdot\frac{(t-\mu)^2}{\sigma^2}\right] \quad \text{for } -\infty < t < \infty, \; \sigma > 0$$

$$E(T) = \mu, \quad V(T) = \sigma^2$$

The reliability function for this distn. is determined from

$$R(t) = \int_t^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\cdot\frac{(t'-\mu)^2}{\sigma^2}\right] dt'$$

However, there is no closed-form solution to this ~~solution~~ integral, and it must be evaluated numerically. If the transformation $z = \frac{T-\mu}{\sigma}$ is made, then z will be normally distributed with a mean of zero and a variance of one. The PDF for z is given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

$$\Phi(z) = \int_{-\infty}^{z} \phi(z')\,dz'$$

$$F(t) = \Pr\{T \le t\} = \Pr\left\{\frac{T-\mu}{\sigma} \le \frac{t-\mu}{\sigma}\right\}$$

$$= \Pr\left\{z \le \frac{t-\mu}{\sigma}\right\}$$

$$= \Phi\left(\frac{t-\mu}{\sigma}\right)$$

$$R(t) = 1 - \Phi\left(\frac{t-\mu}{\sigma}\right)$$

$$\therefore h(t) = \frac{f(t)}{R(t)} = \frac{f(t)}{1-\Phi\left(\frac{t-\mu}{\sigma}\right)}$$

This function is monotonically increasing of t. This can be shown by proving that $h'(t) \ge 0 \; \forall t$.

$$h(t) = \frac{f(t)}{1-F(t)}$$

$$h'(t) = \frac{(1-F)f' + f^2}{(1-F)^2} \quad \text{numerator} \ge 0$$

Ex.1. A component has a Normal distribution of failure times, i.e., $FT \sim N(20000, 2000^2)$. Find $R(19000)$?

Solution:-

$$R(19000) = P(t \geq 19000)$$
$$= P\left(\frac{t-\mu}{\sigma} \geq \frac{19000-20000}{2000}\right)$$
$$= 1 - \Phi(-0.5)$$
$$= 0.69$$

$$h(t) = \frac{f(t)}{R(t)} = \frac{\frac{1}{2000\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{19000-20000}{2000}\right)^2\right)}{0.69} = \frac{0.0001760}{0.69146}$$
$$= 0.000254 \text{ number per cycle.}$$

Ex.2. Wearout (failure) of an oil-drilling bit is normally distributed with a mean of 120 drilling hours & s.d. of 14 drilling hours. Drilling occurs for 12 hrs each day. How many days should drilling continue before the operation is stopped in order to replace the drill bit? A 95% reliability is desired.

Solution:- Find $t_{0.95}$ $\ni$ $Pr\{T \geq t_{0.95}\} = 0.95$.

Standardizing, $Pr\left\{Z \geq \frac{t_{0.95}-120}{14}\right\} = 1 - \Phi\left(\frac{t_{0.95}-120}{14}\right) = 0.95$

Using Normal table, $t_{0.95} = 96.97$ hrs $\simeq 97$ hrs

So, 8 days are required.

Ex.3. Five percent of a certain grade of tires wear out before 25,000 miles, and another 5% of the tires exceed 35,000 miles. Determine the tire reliability at 24,000 miles if wearout is normally distributed.

Solution:- $Pr\{25,000 \leq T \leq 35,000\} = 0.90$

$$Pr\left\{\frac{25,000-\mu}{\sigma} \leq Z \leq \frac{35,000-\mu}{\sigma}\right\} = 0.90$$

$Pr\{-1.645 \leq Z \leq 1.645\} = 0.90$, from the normal tables & the symmetry of the distn..

or, $\frac{25000-\mu}{\sigma} = -1.645$, $\frac{35000-\mu}{\sigma} = 1.645$

Solving, $\mu = 30,000$, $\sigma = 3039.5$

$$\therefore R(24,000) = 1 - \Phi\left(\frac{24000-30000}{3039.5}\right) = 1 - \Phi(-1.97) = 0.9756.$$

- **The Lognormal Distribution:-**

If $L \sim N(\mu, \sigma^2)$

$T = e^L \sim LN(\mu, \sigma^2)$

Alternatively, if $T \sim LN(\mu, \sigma^2)$

$L = \ln T \sim N(\mu, \sigma^2)$.

$$f(t) = \frac{1}{t\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ln t - \mu)^2}{2\sigma^2}\right], \text{ for } 0 < t < \infty, \sigma > 0.$$

$$E(T) = MTBF = \exp(\mu + \sigma^2/2)$$

$$V(T) = \exp(2\mu + \sigma^2)(e^{\sigma^2} - 1)$$

The parameters of lognormal distn. are μ, σ^2, but care is needed to interpret that these are the mean & variance of Normal distn. also.

μ = Location parameter

σ = scale parameter

$$F(t) = Pr\{T \le t\} = Pr\{\ln T \le \ln t\}$$

$$= Pr\left\{z \le \frac{\ln t - \mu}{\sigma}\right\}$$

$$= \Phi\left(\frac{\ln t - \mu}{\sigma}\right)$$

$$R(t) = 1 - \Phi\left(\frac{\ln t - \mu}{\sigma}\right)$$

$$h(t) = \frac{f(t)}{R(t)} = \frac{\phi\left(\frac{\ln t - \mu}{\sigma}\right)}{t\sigma R(t)}.$$

Ex. The failure time of a certain component is log normally distributed with $\mu = 5, \sigma = 1$. Find the reliability of the component & the hazard rate for a life of 150 time units.

Solution:-

$$R(150) = P\left(z > \frac{\ln 150 - \mu}{\sigma}\right) = P\left(z > \frac{\ln 150 - 5}{1}\right)$$

$$= P(z > 0.01)$$

$$= 0.496.$$

$$h(150) = \frac{\phi\left(\frac{\ln 150 - 5}{1}\right)}{(150)(1)(0.496)} = \frac{\phi(0.01)}{(150)(0.496)} = 0.0053 \text{ failures/unit time.}$$

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Ex. The lifetime of a semiconductor laser has a lognormal distn. with $\mu = 10$ hrs and $\sigma = 1.5$ hrs.

(a) What is the probability that lifetime exceeds 10000 hrs?

(b) What lifetime is exceeding by 99% of the lasers?

(c) Determine the mean & s.d. of lifetime?

Solution:- (a) $R(10000) = 1 - \Phi\left(\frac{\ln 10000 - 10}{1.5}\right)$

$= 0.7$

(b) $B_{99} = P(T > t) = 0.99$

$\Rightarrow 1 - \Phi\left(\frac{\ln t - 10}{1.5}\right) = 0.99$

$\Rightarrow \frac{\ln t - 10}{1.5} = -2.32$

$\Rightarrow t = 672$ hrs

(c) $E(T) = \exp\left(\mu + \frac{\sigma^2}{2}\right)$

$= 67846.24$ hrs.

$\text{s.d} = \sqrt{V(t)} = 197661$ hrs.

NOTE:- The lifetime of a product that degrades overtime is often modelled by lognormal distribution.

For example, this is a common example of lifetimes of semi-conductor lasers.

- **Gamma Distribution:–** The failure density function for a gamma distn. is

$$f(t) = \frac{\lambda^n}{\Gamma n} t^{n-1} e^{-\lambda t}, \quad t \geq 0, \; n > 0, \; \lambda > 0$$

where, n is the shape parameter
and λ is the scale parameter.

Hence, $$F(t) = \int_0^t \frac{\lambda^n}{\Gamma n} \cdot t'^{\,n-1} e^{-\lambda t'} dt'$$

If n is an integer, it can be shown by successive integration by parts that

$$F(t) = \sum_{k=n}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

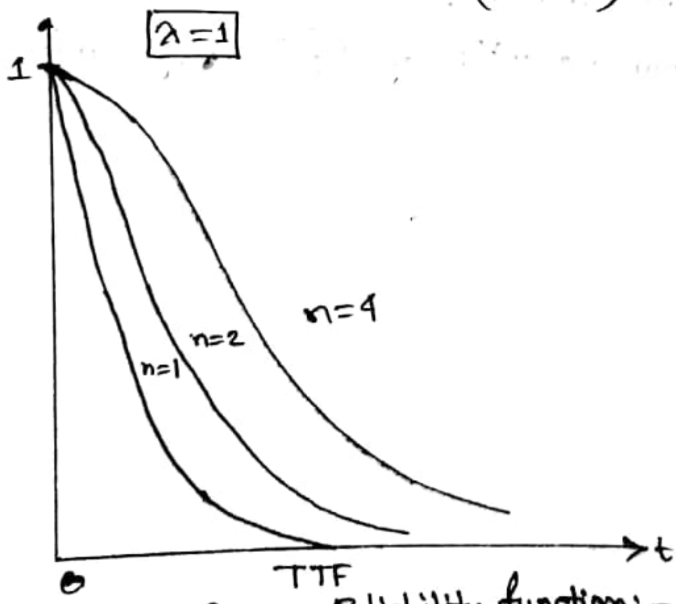
Then $$R(t) = 1 - F(t) = \sum_{k=0}^{n-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

and $$h(t) = \frac{f(t)}{R(t)} = \frac{\frac{\lambda^n}{n!} t^{n-1} e^{-\lambda t}}{\sum_{k=0}^{n-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!}}$$

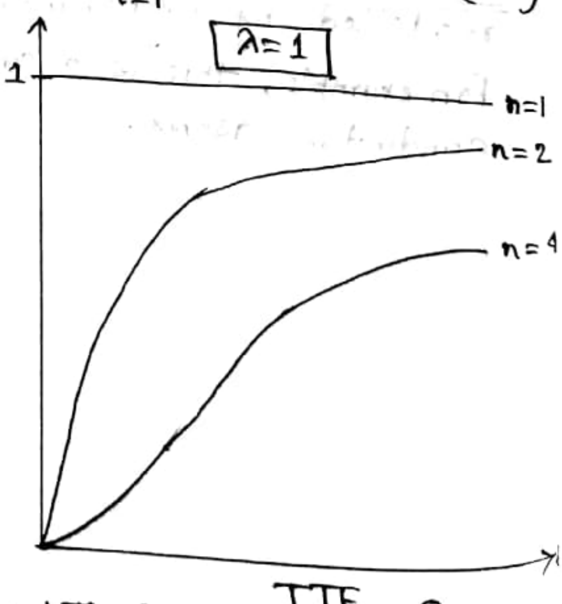
The gamma failure density has very similar shapes to the Weibull distribution.

The gamma distn. can also be used to model the time to the nth failure of a system if the underlying failure distn. is exponential. That is,

$$X_i \sim \text{Exp}\left(\theta = 1/\lambda\right), \text{ then } T = \sum_{i=1}^{n} X_i \sim \text{Gamma}(\lambda, n).$$



→ : The Gamma Reliability function:–



→ : The Gamma Hazard function:–

Ex. The time to failure of a component has a gamma distn. with $n=3$ and $\lambda=0.05$. Determine the reliability of the component and the hazard rate at 24 time units.

Solution:-
$$R(t) = \sum_{k=0}^{n-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

$$R(24) = \sum_{k=0}^{2} \frac{(0.05 \times 24)^k \exp[-0.05 \times 24]}{k!} = 0.88$$

$$f(t) = \frac{(0.05)^3 (24)^2 \exp[-0.05 \times 24]}{\Gamma(3)} = 0.011$$

Then $h(24) = \dfrac{f(24)}{R(24)} = \dfrac{0.011}{0.88} = 0.0125$ failures/unit time.

- Erlang Distribution:– The failure of CPUs of large Computer system are often modeled as a Poisson distn.. Typically failures are not caused by components wearing out but by more random failures of the large number of semiconductor circuits in the units.

Ex:- Assume that the units that fail are immediately repaired and the mean number of failures/hrs = 0.0001.

Let N: # failures in 40,000 hrs of operation

P(X

The time until 4 failures occur exceeds 40,000 hrs iff the no. of failures in 40,000 hrs are 3 or less.

$$\therefore P(X > 40{,}000 \text{ hrs}) = P(N \leq 3)$$

Assuming N has poisson distn with mean

$$E(N) = 40{,}000 \times 0.0001 = 4 \text{ failures}$$

$$P(X=r) = \frac{e^{-\lambda} \cdot \lambda^r}{r!} ; \therefore E(X) = \lambda = V(X)$$

$$\therefore P(X > 40{,}000 \text{ hrs}) = P(N \leq 3) = \sum_{k=0}^{3} \frac{e^{-4} 4^k}{k!} = 0.433$$

This example can be generalised to show that if T is the time until the n^{th} event in a Poisson Process, then

$$P(T > t) = \sum_{k=0}^{n-1} \frac{e^{-\lambda t} (\lambda t)^k}{k!} = 1 - F(t)$$

so, $f(t) = \dfrac{\lambda^n}{(n-1)!} \cdot e^{-\lambda t} t^{n-1}$ for $t > 0$ & $n = 1, 2, 3, \ldots$

This is the pdf of Erlang Distribution.

For $n=1$, $f(t) = \lambda e^{-\lambda t}$, i.e., Exponential Distn..

$$E(T) = \frac{n}{\lambda}, \quad V(T) = \frac{n}{\lambda^2}.$$

Note:– The ERLANG Distn. is a special case of Gamma distn.. If the parameter λ of Erlang distn. is not an integer but $n > 1$ the RV has a Gamma distn.

Gamma function can be interpreted as a generalisation to non-integer values of n of the term $(n-1)!$, i.e., used in Erlang probability density.

SYSTEM RELIABILITY

System — Subsystem – Assembly — Subassembly — Components – Parts — material

Components within a system may be related to one another in two primary ways: in either a serial or a parallel configuration. In series all components must function for the system to function. In a parallel, or redundant, configuration, at least one component must function for the system to function.

Series System:-

SERIES SYSTEM

$$\circ\!-\!\boxed{1}\!-\!\boxed{2}\!-\cdots-\!\boxed{n}\!-\!\circ$$

Assumption:- 1, These components are independent.
2, The system will survive for mission time 't' (with $R(0)=1$) iff each one of the components survive to time 't'.

Since reliability is a probability, a system reliability R_s may be determined from the component reliabilities in the following way.

E_1 = the event that component 1 does not fail

E_2 = " " " " 2 " " "

so, $P(E_1)=R_1$, $P(E_2)=R_2$

Therefore $R_s = P(E_1 \cap E_2) = P(E_1)P(E_2) = R_1(R_2)$

assuming that two components are independent (i.e., the failure or nonfailure of one component does not change the reliability of the other component)

Generalizing to n mutually independent components in series,

$$R_s(t) = R_1(t)\times R_2(t)\times\cdots\times R_n(t) = \prod_{i=1}^{n} R_i(t)$$

$$= \prod_{i=1}^{n}\exp\left\{-\int_0^t h_i(t')\,dt'\right\} = \exp\left\{-\int_0^t \sum_{i=1}^{n} h_i(t')\,dt'\right\},$$

where, $h_i(t')$ is the hazard rate for i^{th} component.

If each component has a constant failure rate of λ_i, then

$$R_s(t) = \prod_{i=1}^{n} R_i(t) = \prod_{i=1}^{n}\exp(-\lambda_i t) = \exp\left(-\sum_{i=1}^{n}\lambda_i t\right) = e^{-\lambda_s t},$$

where, $\lambda_s = \sum_{i=1}^{n}\lambda_i$.

Thus the effective failure rate of a system formed from non-redundant components is equal to the sum of the failure rate of the individual components. The components need not to be identical.

For given $h_i(t)$,

$$f_s(t) = \sum_{i=1}^{n} h_i(t) \exp\left[-\int_0^t \left\{\sum_{i=1}^{n} h_i(t')\,dt'\right\}\right]$$

$$MTBF = \int_0^\infty R_s(t)\,dt = \int_0^\infty \exp\left\{-\int_0^t \left[\sum_{i=1}^{n} h_i(t')\,dt'\right] dt\right\}$$

Now, for useful life, with all failure rates λ_i, constant,

$$MTBF = \frac{1}{\sum_{i=1}^{n} \lambda_i} = \frac{1}{\sum_{i=1}^{n} 1/MTTF_i}$$, where $MTTF_i$ = mean time to failure of the i^{th} component.

<u>Ex.</u> If component failures are governed by the Weibull failure law, then

$$R_s(t) = \prod_{i=1}^{n} \exp\left[-\left(\frac{t}{\theta_i}\right)^{\beta_i}\right] = \exp\left[-\sum_{i=1}^{n}\left(\frac{t}{\theta_i}\right)^{\beta_i}\right]$$

and
$$\lambda(t) = \frac{\exp\left\{-\sum_{i=1}^{n}\left(\frac{t}{\theta_i}\right)^{\beta_i}\right\}\left[\sum_{i=1}^{n}\frac{\beta_i}{\theta_i}\left(\frac{t}{\theta_i}\right)^{\beta_i-1}\right]}{\exp\left[-\sum_{i=1}^{n}\left(\frac{t}{\theta_i}\right)^{\beta_i}\right]}$$

$$= \sum_{i=1}^{n}\frac{\beta_i}{\theta_i}\left(\frac{t}{\theta_i}\right)^{\beta_i-1}$$

This indicates that <u>the system does not exhibit Weibull-type failures although every component has a Weibull failure distribution</u>.

<u>Ex.1.</u> Consider a four-component system of which the components are i.i.d with CFR. If $R_s(100) = 0.95$ is the specified reliability, find the individual component MTTF?

<u>Solution:-</u> $R_s(100) = e^{-100\lambda_s} = e^{-100(4)\lambda} = 0.95$

or, $\lambda = \frac{-\ln 0.95}{400} = 0.000128$

$MTTF = \frac{1}{0.000128} = 7812.5$

Ex.2. (i) A system consists of 4 independent components in series, each having a reliability of 0.970. then what's the reliability of the system?

(ii) If the system complexity is increased so that it contains 8 of these components, what's the new reliability?

(iii) If the more complex system is required to have the same reliability of the simpler system, i.e., answer of (i), what must be the reliability of each component in (ii)?

Solution:- (i) $R_S(t) = (0.97)^4 = 0.885$

(ii) $R_S(t) = (0.97)^8 = 0.784$

(iii) $x^8 = (0.97)^4$

$\therefore x = 0.984$

Ex.3. (i) An airbone electronic system has a rader, a computer and an auxilliary unit with MTBF's of 83, 167 and 500 hours, respectively. Find the system MTBF and the reliability for a 5 hrs operating time.

(ii) Find the individually the reliability?

Solution:- (i) $t = 5$,

$$R_S(t) = e^{-\sum_{i=1}^{n} \frac{t}{MTBF_i}} = e^{-\left(\frac{5}{83} + \frac{5}{167} + \frac{5}{500}\right)} = 0.9046.$$

$$MTBF = \frac{1}{\frac{1}{83} + \frac{1}{167} + \frac{1}{500}} = 49.9$$

(ii) Rader: $e^{-5/83} = 94.1\%$

Comp: $e^{-5/167} = 97\%$

AU: $e^{-5/500} = 99\%$

Ex. 4. A system consists of three subsystems A, B & C. The system is primarily used on a certain mission that lasts 8 hours. The informations are given below:

Subsystem	Required operating time during mission, in hours	Type of failure distn.	Reliability Information
A	8	Exponential	50% of subsystem will last at least 14 hrs.
B	3	Normal	Avg. life is 6 hr with s.d. of 1.5 hours.
C	4	Weibull with $\beta = 1.0$	Avg. life is 40 hrs.

Assuming independence of the subsystems, calculate the reliability for a mission.

Solution:- A: $R(14) = 0.50 = e^{-\lambda \times 14} = \frac{1}{2}$

$\therefore \lambda = 0.0493$

$\therefore R(8) = e^{-0.0493 \times 8} = 0.693.$

B: $R(3) = P(T > 3) = P\left(Z > \frac{3-6}{1.5}\right) = P(Z > -2) = 0.977$

C: $R(t) = \exp\left\{-\left(\frac{t}{\theta}\right)^{\beta}\right\}$

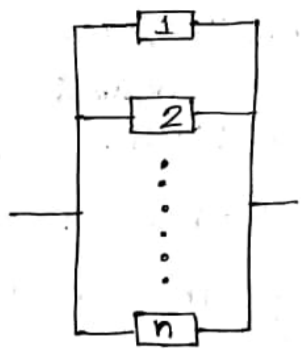
$E(t) = \theta\,\Gamma\left(1 + \frac{1}{\beta}\right) = 40$

$\Rightarrow \theta = \frac{40}{\Gamma(2)} = 40, \ \beta = 1$

$\therefore R(t) = e^{-0.1} = 0.9048$

$\therefore R_s(t) = 0.693 \times 0.977 + 0.9048 = 0.595$

PARALLEL SYSTEM



Schematic system with units in parallel.

A parallel system is a system that is not considered to have failed unless all components have failed.

If we define $F_s(t)$ = unreliability of the system, then

$$F_s(t) = P(A_1 \cap A_2 \cap \cdots \cap A_n)$$; A_i = Failure of the i^{th} component & $i=1(1)n$ and assuming that n components are independent

$$= P(A_1)P(A_2) \cdots\cdots P(A_n)$$

$$= F_1(t)F_2(t) \cdots\cdots F_n(t)$$

$$= (1-R_1(t))(1-R_2(t)) \cdots\cdots (1-R_n(t))$$

$$= \prod_{i=1}^{n}(1-R_i(t)) \quad ; \quad f_s(t) = \sum_{j=1}^{n} f_j(t) \prod_{i \neq j = 1}^{n} (1-R_i(t))$$

$$R_s(t) = 1 - \prod_{i=1}^{n}(1-R_i(t)).$$

It is always true that $R_s(t) \geq \max\{R_1(t), R_2(t), \ldots, R_n(t)\}$ since $\prod_{i=1}^{n}(1-R_i(t))$ must be less than the failure probability of the most reliable component.

There is an implied assumption that all the components are working simultaneously.

For a redundant system, where 'n' components are connected in parallel and having constant failure rates λ_i, $i=1(1)n$, then

$$R_s(t) = 1 - \prod_{i=1}^{n}(1-R_i(t))$$

$$= 1 - \prod_{i=1}^{n}(1-e^{-\lambda_i t})$$

For a two-component system in parallel & having CFR,

$$R_s(t) = 1 - (1-e^{-\lambda_1 t})(1-e^{-\lambda_2 t}) = e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}.$$

and $$\text{MTTF} = \int_0^\infty R_s(t)\,dt = \int_0^\infty e^{-\lambda_1 t}dt + \int_0^\infty e^{-\lambda_2 t}dt - \int_0^\infty e^{-(\lambda_1+\lambda_2)t}dt$$

$$= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1+\lambda_2}$$

If all λ_i's are equal, then $\text{MTTF} = \frac{1}{\lambda} + \frac{1}{2\lambda}$

So, for n-component system, if the failure rate of all the units are identical and constant then

$$\text{MTTF} = \frac{1}{\lambda} + \frac{1}{2\lambda} + \cdots\cdots + \frac{1}{n\lambda}.$$

Ex. Two parallel, identical and independent components have CFR. If it is desired that $R_S(1000) = 0.95$, find the component & system MTTF.

Solution:- We have $R_S(1000) = 2e^{-1000\lambda} - e^{-2000\lambda} = 0.95.$

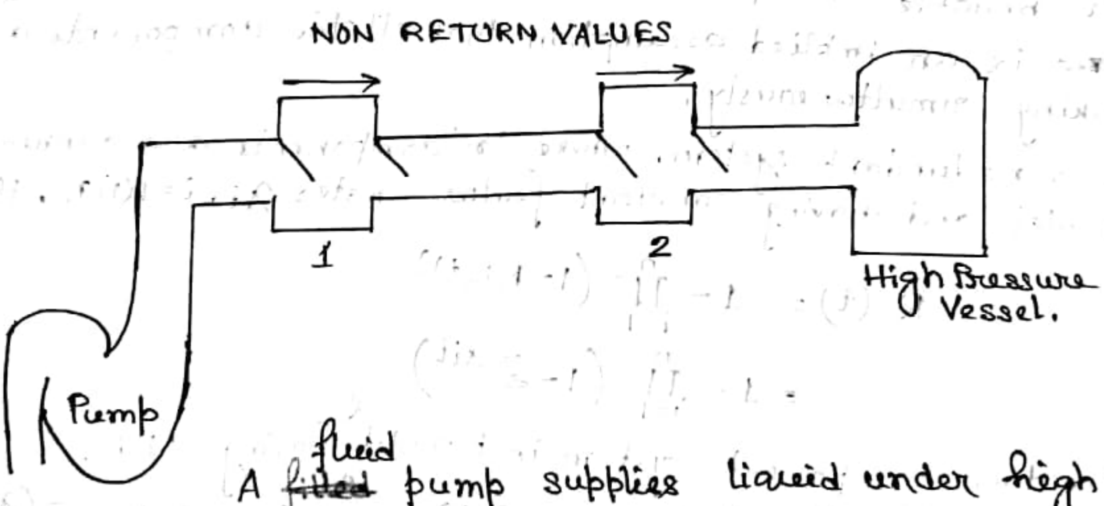
Then by trial & error method

λ	$R_S(1000)$
0.001	0.600
0.0005	0.845
0.0002	0.967
0.00025	0.951

Therefore $MTTF = \frac{1}{0.00025} =$

$$MTTF_S = \frac{2}{0.00025} - \frac{1}{2 \times 0.00025} =$$

Physical Diagram Vs Reliability Block Diagram (Logic Diagram):



A ~~filled~~ fluid pump supplies liquid under high pressure to a vessel. It is essential that the liquid be permitted to flow in one direction, i.e., from the pump to the high pressure vessel. Otherwise if the pump fails ~~there~~ will be the danger of the liquid flowing back to the pump. These can be prevented by providing a non-return value in the pipe system. ~~To~~ assure greater reliability, two non-return values are used in series between the pump and the high pressure vessels.

SERIES-PARALLEL SYSTEM

Now let us assume that we have an n component system. We can either provide redundant components, which gives a system block diagram as shown in Fig. 1, or provide a total redundant system as shown in Fig. 2. The component level redundancy is termed as low-level redundancy and the system level redundancy is termed as high-level redundancy. Here we assume that all components have the same reliability R. Now the question is how ~~can~~ do these two levels of redundancy compare?

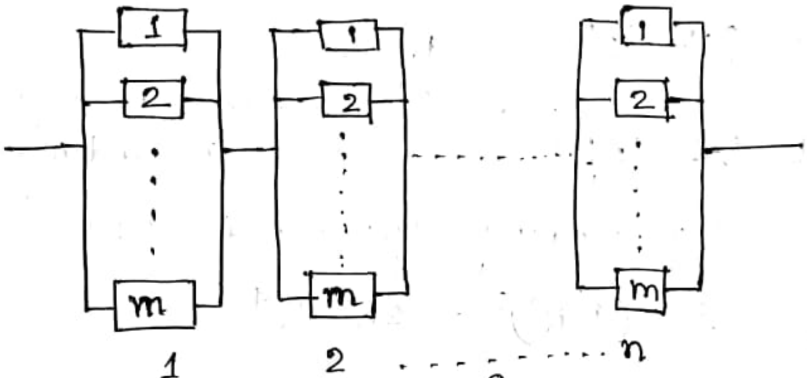


Fig 1: Low-level redundancy of components

For the low-level redundancy system, the equivalent reliability for a parallel bank is

$$R_{EQ} = 1-(1-R)^m$$

Then the EQ components are in series, which gives a system reliability of

$$R_{S,low} = \left[1-(1-R)^m\right]^n$$

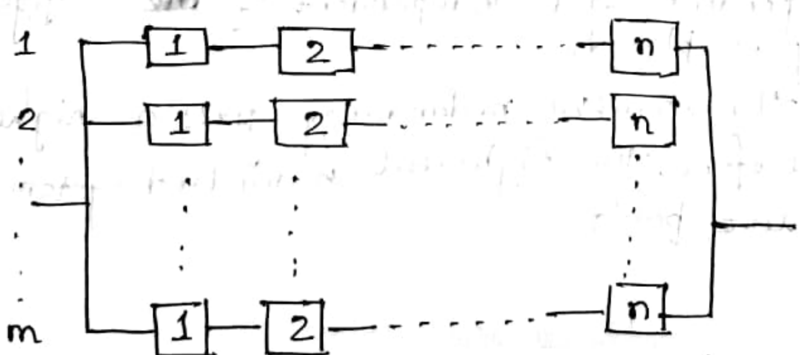


Fig 2: High-level redundancy of systems

Now the high-level redundancy gives an equivalent path reliability of

$$R_{EQ} = R^n \text{ for each parallel path.}$$

Thus the system reliability is

$$R_{S,high} = 1-(1-R^n)^m$$

High level Vs. Low-level Redundancy:-

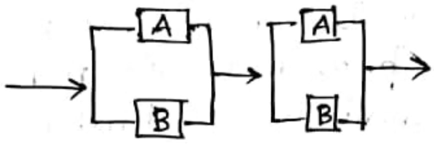


Fig:- Two components in low level redundancy

It is assumed that both components have the same reliability R.

So, for the case of low-level redundancy, the system reliability is

$$R_{low} = \left[1 - (1-R)^2\right]^2 = (2R-R^2)^2$$

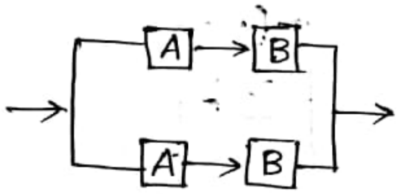


Fig:- Two components in high level redundancy

For high-level redundancy, system reliability is

$$R_{high} = 1-(1-R^2)^2 = 2R^2-R^4$$

By comparing the two reliabilities,

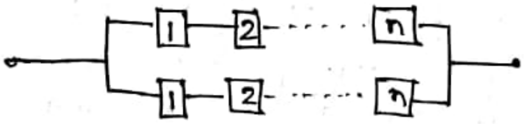
$$R_{low} - R_{high} = (2R-R^2)^2-(2R^2-R^4)$$
$$= R^2(2-R)^2 - R^2(2-R^2)$$
$$= 2R^2(R-1)^2 \geq 0$$

'=' holds when $R=1$.

The equation holds when components reliabilities are mutually independent and independent of ~~the~~ configuration in which they are placed.

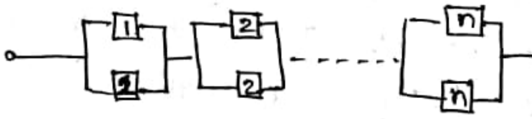
In general, the low level redundancy gives a higher system reliability. Therefore, the high-level redundant system has additional failure paths.

Ex.1. Series parallel system:



(i)

Parallel series system:



(ii)

Which configuration give the better reliability from the figures? Is the components duplicated at component level or at subsystem level?

Solution:- Let 'r' be the reliability for each component.

For (i), $R_1 = r^n = R_2$

$F_1 = 1 - r^n = F_2$

$F_S = (1-r^n)^2$

$R_S = 1 - (1-r^n)^2$

$= r^n(2-r^n)$

For (ii) $R_1 = 1-(1-r)^2$

$\therefore R_S' = \left[1-(1-r)^2\right]^n$

$= r^n(2-r^n)$

$$\therefore \frac{R_S'}{R_S} = \frac{(2-r)^n}{(2-r^n)} > 1 \text{ if } 0 \le r < 1$$

'r' out of 'n' configuration:-

In many practical situation an 'r' out of 'n' configuration serves as a useful system. Then such a system consist of 'n' components in which 'r' of the 'n' components must be good for the system to operate where $r<n$.

Ex. (1) In a 8 cylinder automobile it may be possible to drive the car, if only 4 cylinders are firing representing (4,8) system.

(2) In a communication system 2 out of 3 transmitter should be operating otherwise critical messages are lost. Here the system functions as (2,3).

Assumption:- 'n' components are identical and independent.

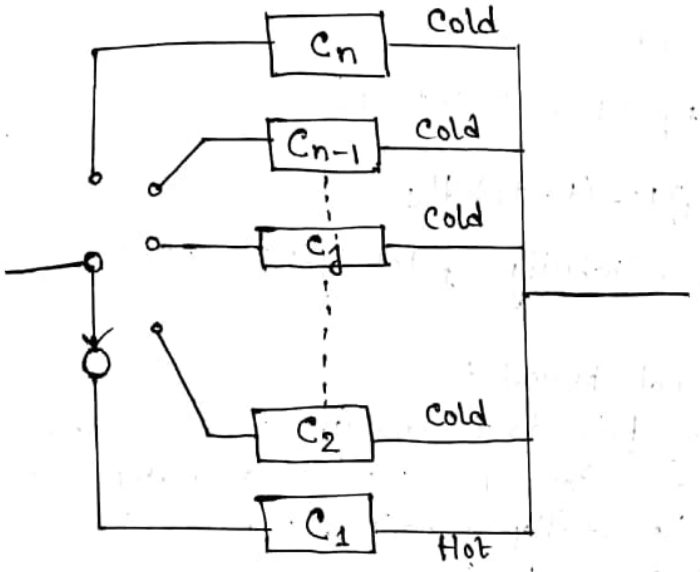
∴ For a constant failure rate, the reliability for an 'r' out of 'n' configuration is expressed as

$$R_s(t) = \sum_{k=r}^{n} \binom{n}{k} e^{-k\lambda t}\left[1-e^{-\lambda t}\right]^{n-k}$$

Remark:- If the components differ or are dependent, then model approach is used.

$$MTTF = \int_0^{\infty} R_s(t) = \frac{1}{\lambda}\sum_{k=r}^{n}\frac{1}{k}$$

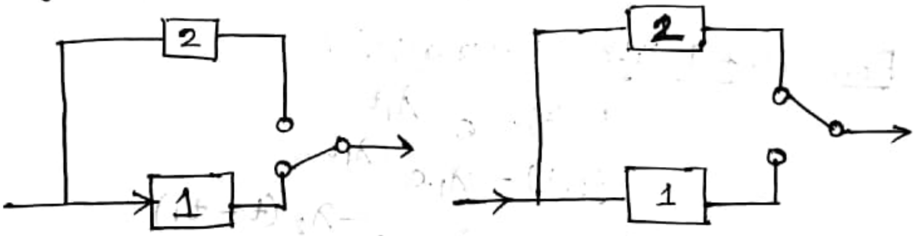
STANDBY SYSTEMS



C_1 is operating.
There are no. of spare components (subsystems) which may be automatically switched to take over the system requirements when the operating components fail.
Thus when C_1 fails, C_2 is switched into take its place, when C_2 fails C_3 is switched into take its place and so on, until C_n is supplying the system requirement.

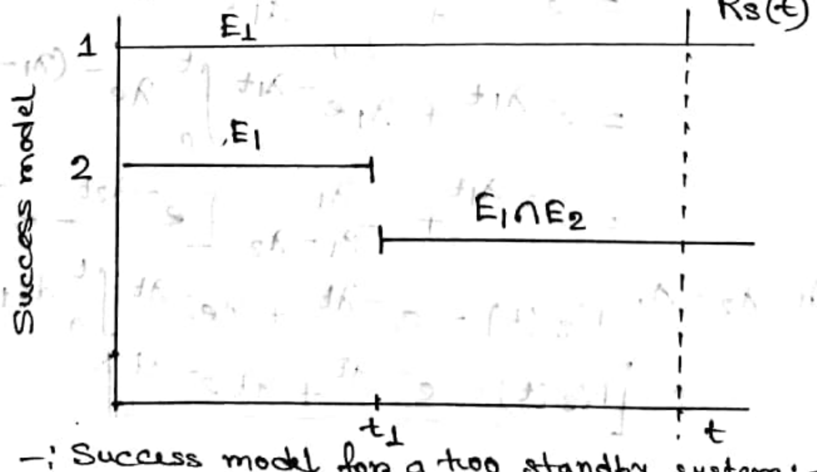
The entire system doesn't fail until C_n fails, then the system has one component originally operating by $(n-1)$ standby components. The standby don't operate until there individually turns come to replace the previously operating component. These standbys are also known as Cold spares.

- Two-Unit Standby System:-



(a)

A two unit standby system function successfully when the functioning unit does not fail [e.g. (a)], or, if functioning unit fails during operating time 't', then sensing and switching unit functions properly, and the standby unit (not having failed while idle) functions properly from the remainder of the mission.



—: Success model for a two standby system: —

$$R_S(t) = P\left[(t_1 > t) \cup (t_1 \le t \cap t_2 > t - t_1) \right]$$

Because of mutually exclusive success model

$$R_S(t) = P(t_1 > t) + P(t_1 \le t \cap t_2 > t - t_1)$$

$$= R_1(t) + \int_0^t f_1(t_1) R_2(t - t_1)\, dt_1$$

In probabilistic ~~mod~~term, the reliability of the system is the probability that

(i) Unit 1 succeed for the whole period t or

(ii) Unit 1 fails at some time t_1 prior to t and the sensing and switching unit does not fail by t_1, and the standby unit doesn't fail by t_1 and sucessfully functions for the remainder of the mission.

This can be written (assuming 100% reliability of sensing and switching unit and of unit 2 while ideling) as

$$R_S(t) = R_1(t) + \int_0^t f_1(t_1) R_2(t - t_1)\, dt_1 \quad \text{, where } t_1 < t.$$

The time 't_1' can be any value from zero (immediate failure of Unit 1) to t (no failure of unit 1).

For Useful life (Exponential):-

$$R_1(t) = e^{-\lambda_1 t}$$

$$f_1(t) = \lambda_1 e^{-\lambda_1 t}$$

$$R_2(t - t_1) = e^{-\lambda_2 (t - t_1)}$$

$$R_S(t) = R_1(t) + \int_0^t f_1(t_1) R_2(t - t_1)\, dt_1$$

$$= e^{-\lambda_1 t} + \int_0^t \lambda_1 e^{-\lambda_1 t} \cdot e^{-\lambda_2 (t - t_1)}\, dt_1$$

$$= e^{-\lambda_1 t} + \int_0^t \lambda_1 e^{-(\lambda_1 - \lambda_2) t_1} \cdot e^{-\lambda_2 t}\, dt_1$$

$$= e^{-\lambda_1 t} + \lambda_1 e^{-\lambda_1 t} \int_0^t \lambda e^{-(\lambda_1 - \lambda_2) t_1}\, dt_1$$

$$= e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_1 - \lambda_2} \left[e^{-\lambda_2 t} - e^{-(\lambda_1 - \lambda_2) t} \right]$$

Now, if $\lambda_1 = \lambda_2 = \lambda$, $R_S(t) = e^{-\lambda t} + \lambda e^{-\lambda t} \int_0^t dt_1$

$$\therefore \boxed{R_S(t) = e^{-\lambda t} + \lambda t e^{-\lambda t}}$$

The case of 2 unit standby system with the same failure rate for both units can be viewed as a situation in which the probability of system success is the probability of 1 failure or less. Using Poisson distribution, the reliability is

$$R_s(t) = \sum_{K=0}^{1} \frac{e^{-\lambda t}(\lambda t)^K}{K!} = e^{-\lambda t} + \lambda t e^{-\lambda t}; \ \lambda: \text{failure rate.}$$

For 'n' components with equal failure rate

$$R_s(t) = e^{-\lambda t} \sum_{r=0}^{n-1} \frac{(\lambda t)^r}{r!}$$

<u>Ex.1.</u> The life (in hrs.) of a magnetic reasonance imagining Machine (MRI) is modelled by a Weibull distribution with parameter $\beta = 2$, $\eta = 500$.

(i) Determine the mean life of MRI.

(ii) Determine the Variance of life of MRI.

(iii) What's the probability that MRI fails before 250 hrs.

<u>Solution:-</u> (i) $\eta = 500$, $\beta = 2$, $\gamma = 0$

$$E(t) = \eta \Gamma\left(1 + \frac{1}{\beta}\right) = 500 \Gamma\left(1 + \frac{1}{2}\right) = 443.113$$

(ii) $$V(t) = \eta^2 \left\{ \Gamma\left(1 + \frac{2}{\beta}\right) - \left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2 \right\}$$

$$= (500)^2 \left\{ \Gamma(2) - \left(\frac{1}{2}\sqrt{\pi}\right)^2 \right]$$

$$= 53650.5$$

(iii) $$F(250) = 1 - e^{-\left(\frac{250}{500}\right)^2}$$

Ex.2. The time between calls to a corporate office is Exponentially distributed with mean of 10 mins.

(i) What's the probability that there are more than 3 calls in 1/2 hrs?

(ii) What's the probability that there are no calls within 30 mins?

(iii) Determine X such that the probability that there are no calls within X hrs is 0·01.

(iv) What's the probability that there are no calls within a 2 hr interval.

(vi) If 4 non-overlapping half hours are selected, what's the probability that no of the intervals contain any calls.

(vii) Explain the relationship between the results in part (a) & (b)?

Solution:-

$$E(t) = MTTF = \frac{10}{60} \text{ hrs} = \frac{1}{\lambda}$$

$\therefore \lambda = 6$ = one call per 10 minute.

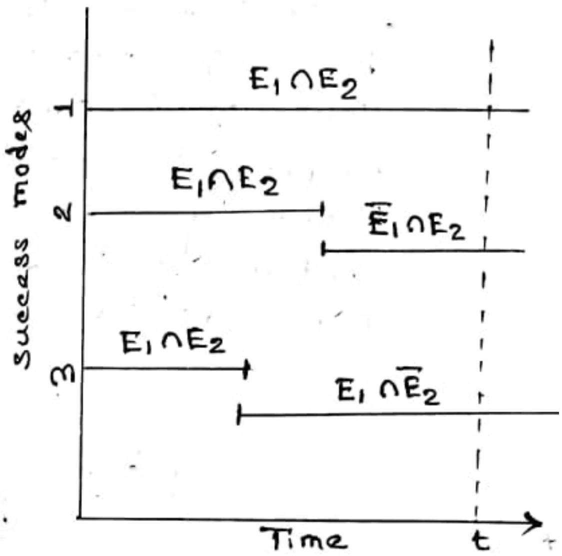
$$P(X \geq 3) = 1 - \sum_{r=0}^{3} \frac{e^{-3} 3^r}{r!}$$

$$P(X=0) = \frac{e^{-\lambda t} (\lambda t)^x}{x!} = 0 \cdot 01$$

$t = 2 \text{hrs} = 120 \text{ min}; \ \lambda t = 12$

$$P(X=0) = e^{-12}$$

LOAD-SHARING SYSTEM



In this configuration, the parallel subsystems equally share the load, and as a subsystem fails, the surviving subsystems must sustain an increased load. Thus as successive subsystems fail, the failure rate of the surviving components increases. An example of a shared parallel configuration would be when bolts are used to hold a machine member; if one bolt breaks the remainder must support the load.

Define λ_h = half load failure rate; λ_f = full load failure rate

Here R(t) is given by,

$$R(t) = e^{-2\lambda_h t} + \frac{2\lambda_h}{(2\lambda_h - \lambda_f)}\left[e^{-\lambda_f t} - e^{-2\lambda_h t}\right],\ t \geq 0$$

which is the system reliability for a two-unit shared parallel system.

$$MTTF = \int_0^\infty R(t)\,dt = \frac{1}{2\lambda_h} + \frac{2\lambda_h}{2\lambda_h - \lambda_f}\left[\frac{1}{\lambda_f} - \frac{1}{2\lambda_h}\right]$$

Ex. Two generators provide needed electrical power. If either fails, the other can provide electrical power. However, the increased load results in a higher failure rate for the remaining generator. If $\lambda_h = 0.01$ failure/day & $\lambda_f = 0.10$ failure/day. Determine the system reliability for a 10-day contingency operation and determine the system MTTF.

Solution:- $$R(10) = e^{-2\times 0.01\times 10} + \frac{0.02}{-(0.08)}\left[e^{-1} - e^{-0.2}\right] = 0.9819$$

$$MTTF = \frac{1}{0.02} + \frac{0.02}{-0.08}\left[\frac{1}{0.10} - \frac{1}{0.02}\right] = 60 \text{ days}.$$

FAULT-TREE ANALYSIS

The fault-tree technique is another approach to reliability analysis of complex system. This method is based on the events that will lead to system failure. A fault-tree (FT) is a diagrammatic representation of all possible fault events, their logical combinations, and their relationship to the system failure. The faults at the lowest level of the system are normally represented at the bottom of the tree and the system fault at the top. The events at the lowest level are known as "basic events". Consider the following system:

This system fails when C_1 fails, or both the components C_2, C_3 fail, or all 3 fail. The logical AND and OR gates used to combine two or more events. ~~The logical AND and OR~~

An AND gate is a multi-input device which causes an ~~input~~ output signal only when all its inputs exists. If $X_1, X_2, \ldots$ are the input events and Y represents the output, then

$$Y = X_1 \cap X_2 \cap X_3 \cap \ldots \cap X_n.$$

An OR gate produces an output even when one of its input is presented,

$$Y = X_1 \cup X_2 \cup \ldots \cup X_n.$$

E_1, E_2 and E_3 are basic events, E_4 is an intermediate event which occurs when both E_2 and E_3 occur. Therefore,

$$E_4 = E_2 \cap E_3$$

The top event, i.e., system failure, occurs when either E_1 or E_4 occurs; The top event $= E_1 \cup E_4 = E_1 \cup (E_2 \cap E_3)$.

The prob. of system failure is

$$P(\text{Top event}) = P(E_1 \cup (E_2 \cap E_3)) = P(E_1) + P(E_2 \cap E_3) - P(E_1)P(E_2 \cap E_3)$$

If E_2 and E_3 are independent,

$$P(\text{Top event}) = P(E_1) + P(E_2)P(E_3) - P(E_1)P(E_2)P(E_3)$$

Let; q_i = Prob. of occuring E_i, the system unreliability can be expressed as $Q = Pr(\text{Top event})$

$$= q_1 + q_2 q_3 - q_1 q_2 q_3$$

Solved Examples :-

Q.1. λ = failure/1000hr, MTBF = $1/\lambda$
$t = 6000$ hrs, $R(t) = 0.95$. Find $1/\lambda$?

Ans:- $R(t) = e^{-\lambda t}$

$\Rightarrow 0.95 = e^{-\lambda . 6000}$

$\Rightarrow \lambda = 8.54 \times 10^{-6}$

$\Rightarrow$ MTBF $= \frac{1}{\lambda} = 7310$ days $= 20$ years

Q.2. 15000 components (series model)
Independent failures, Constant Failure rate
Identical failure rate for all components (λ)
Life expectancy (5yrs) ; $R(5) = 0.95$, find λ ?

Ans:- $\lambda_{satellite} = 15,000 \times \lambda_{component}$

$R(5) = 0.95 = e^{-\lambda_{sat} \times 5}$

$\Rightarrow \lambda_{sat} = 0.0102586$

$\therefore \lambda_{component} = 6.84 \times 10^{-7}$ failure/year.

$\therefore$ MTBF $= \frac{1}{\lambda_{comp}} = 1462179$ years

Example:- $R(t) = 0.9$
2 components with same failure rate & independent.

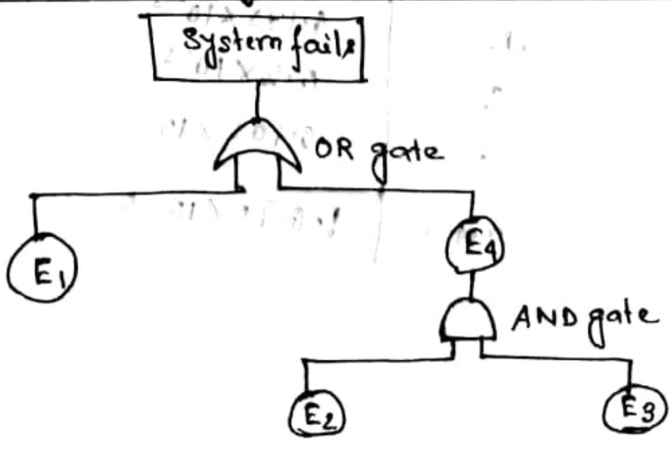
$R_S = 1 - (1-0.9)(1-0.9)$
$= 1 - 0.01 = 0.99$

Put 1 more component,

$R_S = 1 - 0.001 = 0.999$

i.e. increase in reliability.

Fault-tree diagram



Q.3. An electronic ~~equipme~~ measuring equipment comprises of the following components with given failure rates

Component	Quantity	Failure Rate
Transistors	5	0.015×10^{-5} failure/hrs
Diode	8	0.055×10^{-5} "
Resistors	25	0.007×10^{-5} "
Capacitors	12	0.025×10^{-5} "
Shoulder Joints	70	0.0015×10^{-5} "

Estimate the reliability of equipment for a period of 1000 hrs. State the assumptions you made.

Solution:-

Assumptions are;
(i) all the 5 transistors have same λ
(ii) they are independent
(iii) All components are in series.

$$R(1000) = [R_T(1000)]^5 \times [R_D(1000)]^8 \times [R_R(1000)]^{25} \times [R_C(1000)]^{12} \times [R_S(1000)]^{70}$$

$$= 0.989$$

Alt, $R(1000) = e^{-1000 \times \text{total failure rate}}$

$$= e^{-1.095 \times 1000 \times 10^{-5}}$$

$$= 0.9891.$$

Total failure rate

T	0.075×10^{-5}
D	0.44×10^{-5}
R	0.175×10^{-5}
C	0.3×10^{-5}
S	0.105×10^{-5}
Total	1.095×10^{-5}

Q.4. (a). An equipment consists of 3 subsystem A, B, C in series with failure rate as given below. (58)

$$\lambda_A = 0.95\times10^{-5}\ ;\quad \lambda_B = 0.06\times10^{-5}\ ;\quad \lambda_C = 0.05\times10^{-5}$$

Determine the system failure rate and reliability for an operating time of 1000 hrs. Would this equipment is suitable for application that requires 10,00,00 hrs.

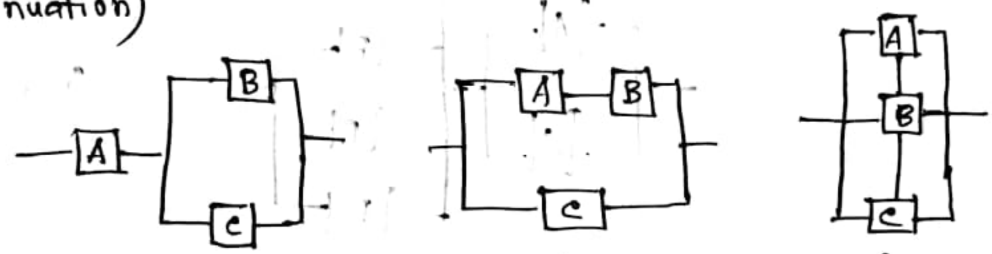
Sol. $\lambda_R = (0.95+0.06+0.05)\times10^{-5} = 1.06\times10^{-5}$.

$R(1000) = e^{-\lambda_R t} = e^{-1.06\times10^{-5}\times1000} = 0.98945$

$MTBF = \dfrac{1}{\lambda_R} = 94339.62 < 100000$

i.e. not suitable for the application.

(b) (Continuation)



α β γ

$R_A = e^{-0.95\times10^{-5}\times1000} = e^{-0.0095}$

$R_B = e^{-0.06\times10^{-5}\times1000} = e^{-0.0006}$

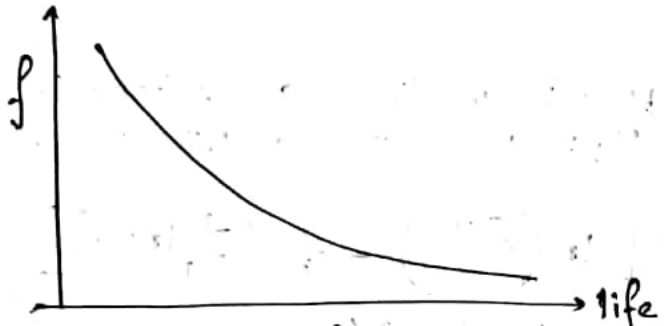
$R_C = e^{-0.05\times10^{-5}\times1000} = e^{-0.0005}$

$R_\alpha = R_A\{1-(1-R_B)(1-R_C)\} = 0.9905$

$R_\beta = 1-\{(1-R_A R_B)(1-R_C)\} = 0.999\ldots$

$R_\gamma = 1-\{(1-R_A)(1-R_B)(1-R_C)\} = 0.999.$

Life Tests



- All to fail
- Censored — time / failure
- With replacement
- without replacement

Life estimation is done for two purpose :—

— to improve a existing design (i.e., to compare)

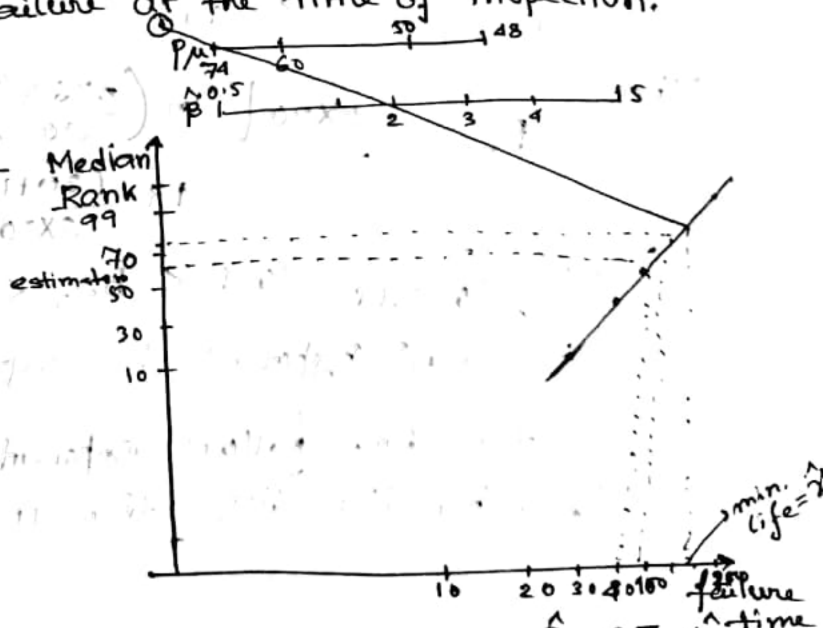
— Demonstration (i.e., to show that the product meet the expectation)

- **Weibull Probability Plot :-**

1. Arrange data in ascending order (i)
2. Calculate the Median rank $\mu_R = \dfrac{i - 0.3}{n + 0.4}$
3. Plot the data life (in hours) on X-axis and median rank (μ_R) on Y-axis.
4. Fit a straight line to the data Points.
5. Draw a $\perp$ to the fitted line passing through estimation point.
6. Estimate β at the intersection value of $\perp$ with the β-scale.
7. Estimate η (characteristic life) as the life value corresponding to the interaction point of estimation, dotted line to the fitted line.

Suspended life :- No failure at the time of inspection.

i	Failure time, hrs	$\mu_R \times 100$
1	32	13
2	51	31
3	74	50
4	90	69
5	120	87



$\hat{\beta} = 2$ $N = 5$ $\hat{\theta} = 85 = \hat{\mu}$ $\hat{A} = 40$

Testing the Goodness of Fit for Exponential distribution:-

• **Bartlett's Test:-**

The hypothesis are. H_0 : Failure times are exponential.
H_1 : " " " NOT " .

the test statistic is

$$B_n = \frac{2n\left[\ln\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) - \frac{1}{n}\left(\sum_{i=1}^{n} \ln x_i\right)\right]}{1+(n+1)/6n} \sim \chi^2_{n-1}, \text{ under } H_0.$$

where n = number of failures,
x_i = life to failure ;

Here null hypothesis is accepted when

$$\chi^2_{1-\alpha/2,\, n-1} < B_n < \chi^2_{\alpha/2,\, n-1}.$$

Example:- Thirty units were placed on the test until 20 failures were observed. the following failure times were obtained in accelerated test hours :

50.1	20.9	31.1	96.5	36.3	99.1	42.6	84.9	6.2	32.0
30.4	87.7	14.2	4.6	2.5	1.8	11.5	84.6	88.6	10.7

Solution:- A constant failure rate is assumed. Therefore

H_0 : Failures are exponential
H_1 : Failures are not exponential

Let $\alpha = 0.10$ with $n = 20$.

$$\sum_{i=1}^{20} x_i = 836.3 \; ; \quad \sum_{i=1}^{20} \ln x_i = 63.93848$$

Then

$$B_{20} = \frac{2\times 20\left[\ln\left(\frac{836.3}{20}\right) - \frac{1}{20}(63.93848)\right]}{1+\frac{(20+1)}{6\times 20}} = 18.258$$

Since $\chi^2_{0.95,19} = 10.1 < B_{20} < \chi^2_{0.05,19} = 30.1$,

So, the null hypothesis is accepted.

So, the data follows exponential distribution.

If exponential distn, then it is useful life.

- **Chi-Square Goodness-of-Fit Test :-**

This test is applicable for both discrete and continuous data. The test is valid for large sample sizes only; however it will accommodate single censored data. The data must be grouped into classes. The test statistic is

$$\chi_c^2 = \sum_{i=1}^{K} \frac{(O_i - E_i)^2}{E_i}$$

where K = number of classes
O_i = Observed number of failures in the i^{th} class
E_i = Expected " " " " " " " $= np_i$
n = sample size
p_i = prob. of a failure occuring in the i^{th} class if H_0 is true.

Here we accept H_0 if $\chi_c^2 < \chi^2_{\alpha, K-2}$.

Example:- The following 35 failure times (in operating hours) were obtained from field data over a 6-month period.

1476	300	98	221	157	182	499
552	1563	36	246	442	20	796
31	47	438	400	279	247	210
284	553	767	1297	214	428	597
2025	185	467	401	210	289	1024

Solution:- $MTTF = \sum_{i=1}^{35} x_i / 35 = 485.4$

MLE for the parameter is $\hat{\lambda} = \frac{1}{MTTF} = 0.00206$.

Cell	Upper Bound	Frequency
1	354	18
2	688	10
3	1022	2
4	1356	2
5	1690	2
6	2026	1

$\}$ 7 (cells 3–6)

[Cells 3 through 6 are combined to ensure that the expected cell count will be at least 5.]

$E_1 = np_1 = 35 \times \left[1 - e^{-354/485.4}\right] = 18.120$

$E_2 = np_2 = 35 \times \left[1 - e^{-688/485.4}\right] = 8.396$

$E_3 = np_3 = 35 \left[1 - p_1 - p_2\right] = 8.483$

(62)

The hypothesis are

H_0: Failure times are exponential with $\lambda = 0.00206$

H_1: " " " NOT " " "

The level of significance $\alpha = 0.10$

$K = 3$

Upper Bound	Oi (Freq)	Ei	$(O_i - E_i)^2/E_i$
354	18	18.1204	0.0008
688	10	8.3966	0.3062
∞	7	8.4830	0.2593

$$\chi_c^2 = 0.5663$$

Now, $\chi^2_{\alpha, k-2} = \chi^2_{0.10, 1} = 2.71 > \chi_c^2$

So, we accep H_0; the life data is exponential.

<u>Testing for Abnormally Short Failure Times:-</u>

Let $(x_1, x_2, \dots, x_n)$ be a sequence of n iid exponential r.v.s and θ is the average life.

Then $\frac{2x_i}{\theta} \sim \chi^2_2$,

$$\frac{2\sum_{i=2}^{n} x_i}{\theta} \sim \chi^2_{2n-2}$$

Thus, $$F_{2,2n-2} = \frac{\frac{2x_1}{\theta}(n-1)}{\frac{2\sum_{i=2}^{n} x_i}{\theta}} = \frac{\chi_2^2/2}{\chi^2_{2n-2}/2n-2}$$

$$= \frac{(n-1)x_1}{\sum_{i=2}^{n} x_i}$$

We assumed that x_1 is the short failure time.

If x_1 is significantly small, then

$$F_{1-\alpha, 2, 2n-2} > \frac{(n-1)x_1}{\sum_{i=2}^{n} x_i}$$

then there is evidence that x_1 represents an abnormally early failure.

Alt. hypothesis is rejected if $F_{\alpha, 2n-2, 2} < \frac{\sum_{i=2}^{n} x_i}{(n-1)x_1}$

Example: Consider the following data for 20 turbine blades.

Kilocycles to failure for turbine blades

193	1793	3479	5310	1582
2028	4235	6809	1637	2260
4264	8317	1658	2272	4635
9728	1786	2700	4919	10,700

The sample F value is calculated as follows:

$$\sum_{i=2}^{20} x_i = 80,112$$

$$x_1 = 193$$

$$F_C = \frac{80,112}{(20-1)\,193} = 21.8$$

The critical F value is $F_{0.05,38,2} = 19.47$ indicating that the first failure time of 193kms is not representative of the rest of the data.

Ex. Consider the heater switch test data as given below:

Cycles to failure for 20 heater switches

100	7120	24110	36860
340	12,910	28570	38540
1940	13,670	31620	42110
5670	19,490	32800	43970
6010	23,700	34910	64730

Let us run a test to determine the validity of the hypothesis that the first two failures are from the same population as the remaining failures.

$$\frac{2(x_1+x_2)}{\theta} \sim \chi^2_4$$

$$\frac{2\left(\sum_{i=3}^{20} x_i\right)}{\theta} \sim \chi^2_{40-4}$$

$$F_{4,36} = \frac{(x_1+x_2)/4}{\left(\sum_{i=3}^{20} x_i\right)/36} = \frac{9(x_1+x_2)}{\sum_{i=3}^{20} x_i}$$

Here, $\sum_{i=3}^{20} x_i \Big/ 9(x_1+x_2) = \frac{468,730}{9(440)} = 118.37$

and the critical F value is $F_{0.05,36,4} = 5.74$.

Thus there is evidence that the first two failures occured abnormally early.

Estimation of Mean Life :-

Estimating the mean life in the case of the exponential is a straight forward computation.

$$\text{Average life} = \hat{\theta} = \frac{\text{Total test time}}{\text{Total no of failure}} = \frac{T}{r}$$

$\hat{\theta}$ is an MLE estimator and also unbiased, minimum variance, efficient and sufficient.

1. Time Censored - Type-I Censoring

Counting the time to failure.

With Replacement :- Total test time = Test item × Test time

$$T = n \times t$$

$$\text{No. of failures} = r$$

$$\theta = \frac{n \times t}{r}$$

Without Replacement :- $\theta = \dfrac{\sum_{i=1}^{r} x_i + (n-r) \times t}{r}$, where,

x_i is the time to i^{th} failure.

2. Failure censored - Type-II Censoring

Counting number of failures.

With Replacement :- $\hat{\theta} = \dfrac{n \times t_r}{r}$

WOR :- $\hat{\theta} = \dfrac{\sum_{i=1}^{r} x_i + (n-r) x_r}{r}$

x_r : time to r^{th} failure.

These assumptions are for exponential distn. with min. life 0

Confidence limits for Mean life (θ) :- (Failure time are recorded)

$$\theta = \frac{\sum_{i=1}^{r} x_i + (n-r)t}{r}$$

(WR cases)

$$\frac{2x_i}{\theta} \sim \chi^2_2 \quad ; \quad \frac{2\sum_{i=1}^{r} x_i}{\theta} \sim \chi^2_{2r}$$

$$\therefore \frac{2\sum_{i=1}^{r} x_i}{\chi^2_{\alpha/2, 2r}} \leq \theta \leq \frac{2\sum_{i=1}^{r} x_i}{\chi^2_{1-\alpha/2, 2r}}$$

(No of failures are recorded) $\dfrac{2\sum_{i=1}^{r} x_i}{\chi^2_{\alpha/2, 2(r+1)}} \leq \theta \leq \dfrac{2\sum_{i=1}^{r} x_i}{\chi^2_{1-\alpha/2, 2r}}$

Ex. 1. Eight leaf springs were cycle tested to failure on an accelerated life test. The result follows:

8,712	39,400	79,000	151,208
21,915	54,613	110,200	204,312

The mean life is estimated by

$$\hat{\theta} = \sum_{i=1}^{8} x_i / 8 = 83,670 \text{ cycles} = \frac{669,360}{8}$$

95% two-sided C.I. on the life time,

$$\frac{2\sum_{i=1}^{8} x_i}{\chi^2_{0.025,16}} \le \theta \le \frac{2\sum_{i=1}^{8} x_i}{\chi^2_{0.975,16}}$$

$$\Rightarrow \frac{2 \times 669,360}{28.84} \le \theta \le \frac{2 \times 669,360}{6.91}$$

$$\Rightarrow 46,419 \le \theta \le 193,736$$

Ex. 2.

Car No	Censored	r_i (failure)
1	8012	4
2	6147	0
3	9002	3
4	11,000	5
5	4651	1
6	8012	4
7	12718	2
Total	56,542	19

$$\theta = \frac{T}{r} = \frac{\sum x_i}{19} = \frac{56542}{19} = 5951.78$$

$$\theta_L = \frac{2\sum x_i}{\chi^2_{\alpha/2,\,2r+2}} = \frac{2 \times 56542}{\chi^2_{0.025,40}} = \frac{2 \times 56542}{59.342}$$

$$= 1905.6317$$

$$\theta_U = \frac{2\sum x_i}{\chi^2_{0.975,38}} = \frac{2 \times 56542}{22.8784} \doteq 4942.8281$$

Detecting Changes in the failure Rate:-

$$F_{\alpha, n_1, n_2} = \frac{\sum_{i=1}^{n_1+1} x_i / 2(n_1+1)}{\sum_{i=1}^{n_2+1} x_i / 2(n_2+1)}$$

Example:- x_1 : 10, 20, 25
x_2 : 15, 16, 20, 18, 26

$$\frac{\sum_{i=1}^{3} x_{1i}/2\times 3}{\sum_{i=1}^{5} x_{2i}/2\times 5} = \frac{55/6}{95/10} = 0.9649$$

$F_{0.025,2,4} = 10.65$

$F_{0.975,2,4} = 0.025$

Hence we donot detect a significant change.

Reliability & Parameter Estimation with Confidence Limits:-

Confidence limits for reliability based on confidence limit of θ

$$e^{-t/\theta_L} \le R(t) \le e^{-t/\theta_U}$$

Two parameter Exponential Distn:-

$$f(t) = \lambda e^{-\lambda(t-\delta)} = \frac{1}{\theta} e^{-\left(\frac{t-\delta}{\theta}\right)}$$

Estimate of average life,

$$\hat{\theta} = \frac{1}{r-1}\left\{\sum_{i=2}^{r}(x_i - x_1) + (n-r)(x_r - x_1)\right\} \ge 0.$$

and $\hat{\delta} = x_1 - \frac{\hat{\theta}}{n}$ is the estimate of minimum life.

Reliability function is estimated by

$$\hat{R}(x) = e^{-\left(\frac{x-\hat{\delta}}{\hat{\theta}}\right)}, \quad x \ge \hat{\delta}$$

Ex. Consider the data which represents cycles to failure for throttle return springs: 20 springs were tested under conditions similar to those encountered in actual use. The test was truncated at the time of the tenth failure.

Throttle return spring data

Cycles to failure (x_i)	($x_i - x_1$)
190,437	0
245,593	55,156
277,761	87,324
432,298	241,861
530,100	339,663
626,300	435,863
1,043,307	852,870
1,055,528	865,091
1,221,393	1030,956
2,099,199	1908,762

$$\Sigma = 5,817,546$$

$$\hat{\theta} = \frac{\sum_{i=2}^{10} (x_i - x_1) + (n-r)(x_r - x_1)}{r-1}$$

$$= \frac{5,817,546 + (20-10)(1,908,762)}{9}$$

$$= 2,767,241 \text{ cycles}$$

$$\hat{\delta} = x_1 - \frac{\hat{\theta}}{n}$$

$$= 1,90,437 - \frac{2,767,241}{20}$$

$$= 52,075 \text{ cycles}$$

$$\hat{R}(x) = \exp\left[-\frac{(x - 52,075)}{2,767,241}\right], \quad x \geq 52,075.$$

Confidence Intervals for Mean Life, Minimum Life and Reliability:

C.I. for θ:- $\quad \dfrac{2(n-1)\hat{\theta}}{\theta} \sim \chi^2_{2n-2}$,

$$P\left[\chi^2_{1-\alpha/2,\,2(n-1)} \leq 2(n-1)\hat{\theta}/\theta \leq \chi^2_{\alpha/2,\,2(n-1)}\right] = 1-\alpha$$

So, $$\frac{2(n-1)\hat{\theta}}{\chi^2_{\alpha/2,\,2n-2}} \leq \theta \leq \frac{2(n-1)\hat{\theta}}{\chi^2_{1-\alpha/2,\,2n-2}}.$$

For the above example, $n=10$, $\alpha=0.10$,

$$\chi^2_{0.95,18} = 9.390, \quad \chi^2_{0.05,18} = 28.869$$

Hence, $$\frac{2\times 9\times 2,767,241}{28.869} \leq \theta \leq \frac{2\times 9\times 2,767,241}{9.390}$$

or, $$1,725,392 \leq \theta \leq 5,304,615$$

C.I. for δ:- $\quad F_{2,\,2n-2} = \dfrac{n(x_1-\delta)}{\hat{\theta}}$

$$P\left[0 \leq \frac{n(x_1-\delta)}{\hat{\theta}} \leq F_{\alpha,\,2,\,2n-2}\right] = 1-\alpha$$

So, $$x_1 - \frac{\hat{\theta}}{n} F_{\alpha,\,2,\,2n-2} \leq \delta \leq x_1$$

For the above example, $F_{0.10,2,18} = 2.62$.

So, $$1,90,437 - \frac{(2,767,241)(2.62)}{20} \leq \delta \leq 1,90,437$$

or, $$0 \leq \delta \leq 1,90,437.$$

C.I. for Reliability:-

$$e^{-\frac{(x-L)}{L'}} \leq R(x) \leq e^{-\frac{(x-U)}{U'}}$$

For the above example,

$$R(1000) = e^{-\left(\frac{1000-\hat{\delta}}{\hat{\theta}}\right)} = e^{-\left(\frac{1000-50,075}{2,767,241}\right)} =$$

$$R_L(1000) = e^{-\left(\frac{1000-0}{1,725,392}\right)} =$$

$$R_U(1000) = e^{-\left(\frac{1000-1,90,437}{5,304,615}\right)} =$$

Hypothesis Testing:-

Test for θ (mean life):- $H_0: \theta \le \theta_0$
$H_1: \theta > \theta_0$

$$\frac{2n\hat{\theta}}{\theta} \sim \chi^2_{2n}$$

$$P\left[\frac{2n\hat{\theta}}{\theta_0} \le \chi^2_{\alpha,2n} \,\middle|\, \theta = \theta_0\right] = 1-\alpha$$

Thus the procedure is:

1. Calculate $\chi_c^2 = \frac{2n\hat{\theta}}{\theta_0}$
2. Reject Ho if $\chi_c^2 > \chi^2_{\alpha,2n}$

Example:- For the above example,
$H_0: \theta \le 1,000,000$ cycles
$H_1: \theta > 1,000,000$ cycles

$$\hat{\theta} = 2,871,391 \text{ cycles}$$

$$\chi_c^2 = \frac{2 \times 10 \times 2,871,391}{1,000,000} = 57$$

For $\alpha = 0.05$, $\chi^2_{0.05,20} = 31.41$

Thus, there is sufficient evidence to reject Ho and conclude that $\theta > 1,000,000$ cycles.

Test for δ (minimum life) :-

$H_0: \delta = 0$

$H_1: \delta > 0$

The basic procedure is

1. Calculate $F_c = \dfrac{n x_1}{\hat{\theta}}$

2. Reject H_0 if $F_c > F_{\alpha, 2, 2r-2}$.

Example:- $F_c = \dfrac{n x_1}{\hat{\theta}} = \dfrac{15 \times 1,90,437}{2,767,421} = 1.03$

$F_{0.05, 2, 18} = 3.55$

Comparison of Two designs:- n_1, n_2 are sample sizes.

$S_1 = (x_{11}, x_{12}, \ldots, x_{1r_1})$, $S_2 = (x_{21}, x_{22}, \ldots, x_{2r_2})$

Consider the hypothesis $H_0: \theta_1 = \theta_2$

$H_1: \theta_1 \neq \theta_2$

the test procedure is

1. Calculate $$C = \frac{\sum_{i=1}^{r_2} (x_{2i} - x_{21}) + (n_2 - r_2)(x_{2r_2} - x_{21})}{\sum_{i=1}^{r_1} (x_{1i} - x_{11}) + (n_1 - r_1)(x_{1r_1} - x_{11})}$$

2. Calculate $F_c = \dfrac{(r_1 - 1)}{(r_2 - 1)} C$

3. Reject H_0 if either $F_c > F_{\alpha/2, 2r_2 - 2, 2r_1 - 2}$.

Test for minimum life:- Hypothesis is $H_0: \delta_1 = \delta_2$

$H_1: \delta_1 \neq \delta_2$

The test procedure is

1. Calculate $$d = \frac{n_2 (x_{21} - x_{11})}{\sum_{i=1}^{r_2} (x_{2i} - x_{21}) + (n_2 - r_2)(x_{2r_2} - x_{21})}$$

2. Calculate $F_c = \dfrac{(2r_2 - 2)}{2} d$

3. Reject H_0 if $F_c > F_{\alpha, 2, 2r_2 - 2}$.

Expected time to complete testing :-

n = products = 20

r = failures = 10

$$E(\text{test time}) = \theta \sum_{i=1}^{r} \frac{1}{n-i+1}$$

$$= 2{,}767{,}241 \left[\frac{1}{20} + \frac{1}{19} + \cdots + \frac{1}{11} \right]$$

$$=$$

The variance is given by

$$V(t_r) = \theta^2 \sum_{i=1}^{r} \frac{1}{(n-i+1)^2}.$$

FAILURE MODE AND EFFECT ANALYSIS :-

Any FMEA conducted properly & appropriately will provide the practitioner with useful information that can reduce the risk load in the system, design, process, and service.

A good FMEA :-

- Identifies known & potential failure modes
- Identifies the cause and effects of each failure mode
- Prioritizes the identified failure modes according to the risk priority number (RPN) — the product of frequency of occurrence, severity, and detection.
- provides for problem follow-up and corrective action

FMEA is a team function & can't be done on an individual basis.

FMEA program should start :

- when new systems, designs, products, processes or services are designed and also existing things are about to change.
- When improvements are considered for the existing systems, designs, products, processes and services.

Conducting an FMEA: Basic Steps:-

1. Define the scope
2. Develop a detailed understanding of the current process
3. Brainstorm potential failure modes
4. List potential effects of failures and causes of failures.
5. Assign severity, occurrence and detection ratings.
6. Calculate risk priority number (RPN) for each cause.
7. Rank causes.
8. Take action on high risk failure modes.
9. Recalculate RPN numbers.

Design FMEA (DFMEA):-

- Define & Scope: Boundary of the FMEA analysis. What is included or excluded based on the system, subsystem or component FMEA.
- Define customer – End user, OEM assembly, manufacturing centres, supply chain manufacturing.
- Identify functions, requirements & specifications.

Identification
- Identify the Failure modes
- Identify Potential effects
- Identify Potential causes.

Control
- Design control
- Prevention controls
- Detection controls (DOE)

- DFMEA does not rely on process controls to overcome the potential design weaknesses but it does take the technical & physical limits of a manufacturing process into consideration (eg. process capability)

RPN:- RPN values are calculated by multiplying together the severity, occurrence and detection (SOD) values associated with each cause-and-effect item identified for each failure mode.

Sequential Life Testing:-

It provides an efficient method for accepting or rejecting a statistical hypothesis when the sample is highly favourable to one of the two decisions.

- Sequential sampling is an extension to double sampling to multiple sampling to sequential.

Let us consider the simple null hypothesis

$$H_0: \theta = \theta_0$$
$$\text{Vs} \quad H_1: \theta = \theta_1 \; (> \theta_0)$$

$$P(H_1 | H_0) = \alpha \quad ; \quad P(H_0 | H_1) = \beta$$

where $P(H_1|H_0)$ is the probability of accepting H_1 when H_0 is true.

The values of A and B are computed so that the specified probabilities of making a Type I and Type II error are approximated.

$$A = \frac{\beta}{1-\alpha} \text{ and } B = \frac{1-\beta}{\alpha}.$$

- **Exponential Case:-** $f(x) = \frac{1}{\theta} e^{-x/\theta}, \quad x \geq 0$

where θ = mean time to failure.

The hypotheses are:

$$H_0: \theta \geq \theta_0$$
$$H_1: \theta < \theta_0 \qquad \text{(SPR)}$$

The sequential probability ratio becomes

$$\prod_{i=1}^{n} \frac{\left(\frac{1}{\theta_1}\right)\exp(-x_i/\theta_1)}{\left(\frac{1}{\theta_0}\right)\exp(-x_i/\theta_0)} = \left(\frac{\theta_0}{\theta_1}\right)^n \exp\left[-\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)\right]\sum_{i=1}^{n} x_i$$

Substituting into inequality:-

$$\frac{\beta}{1-\alpha} < \left(\frac{\theta_0}{\theta_1}\right)^n \exp\left[-\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)\right]\sum_{i=1}^{n} x_i < \frac{1-\beta}{\alpha}$$

Taking the natural logarithm & rearranging

$$\frac{n\log\left(\frac{\theta_0}{\theta_1}\right) - \ln\left(\frac{1-\beta}{\alpha}\right)}{\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)} < \sum_{i=1}^{n} x_i < \frac{n\ln\left(\frac{\theta_0}{\theta_1}\right) + \ln\left(\frac{1-\alpha}{\beta}\right)}{\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)}$$

In the rearrangement note that $\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right) > 0$.

The OC curve for the test is, the prob. of accepting H_0 when θ is the true parameter value denoted as, $P(\theta)$ is calculated from the pair of eauations

$$P(\theta) = \frac{B^h - 1}{B^h - A^h} \quad \text{and} \quad \theta = \frac{\left(\frac{\theta_0}{\theta_1}\right)^h - 1}{h\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)};$$

where h can be any real number, and meaningless selections are made by trial and error.

- Ex. Suppose that we are interested in life testing a new product to see if it meets a standard of 1000 hrs. We select $\theta_0 = 1000$ hrs with $\alpha = 0.05$ and decide on $\theta_1 = 500$ hrs with $\beta = 0.10$.

$$\rightarrow \quad \frac{n \ln 2 - \ln(0.90/0.05)}{\left(\frac{1}{500} - \frac{1}{1000}\right)} < \sum_{i=1}^{n} x_i < \frac{n \ln 2 + \ln\left(\frac{0.95}{0.10}\right)}{\left(\frac{1}{500} - \frac{1}{1000}\right)}$$

$$\Rightarrow \quad 693n - 2890 < \sum x_i < 693n + 2251.$$

Application to the Exponential Distn.: Recording total test time Vs. Total Number of failures:-

If the time-to-failure distribution is exponential with failure rate λ, then for an interval of time of length T, the number of failures r is a Poisson RV with parameter λT. The pdf of r is

$$P(r|\lambda) = \frac{(\lambda T)^r e^{-\lambda T}}{r!}, \quad r = 0, 1, 2, \ldots$$

SPR could be $P(r|\lambda_1)$ and $P(r|\lambda_0)$.

To test $H_0: \lambda \geq \lambda_0$ Vs. $H_1: \lambda < \lambda_0$.

In this situation, the rejection of H_0 implies that the failure rate is significantly less than λ_0.

To define the sequential life test the values α, β and λ_1 must also be specified. Here λ_1 is a value $\ni$ $\lambda_1 < \lambda_0$.

The sequential prob. ratio (SPR) is

$$\left(\frac{\lambda_1}{\lambda_0}\right)^r \exp\left[-T(\lambda - \lambda_0)\right]$$ and the continue

region is

$$\frac{T(\lambda_0-\lambda_1) - \ln\left(\frac{1-\beta}{\alpha}\right)}{\ln\left(\frac{\lambda_0}{\lambda_1}\right)} < r < \frac{T(\lambda_0-\lambda_1) + \ln\left(\frac{1-\alpha}{\beta}\right)}{\ln(\lambda_0/\lambda_1)}$$

where T = total test time.

The OC curve for the test is $P(\lambda) = \frac{B^h - 1}{B^h - A^h}$

with $A = \frac{\beta}{1-\alpha}$ and $B = \frac{1-\beta}{\alpha}$ and

$$\lambda = \frac{h(\lambda_0 - \lambda_1)}{1 - \left(\frac{\lambda_1}{\lambda_0}\right)^h};$$

Here $P(\lambda)$ is the prob. of accepting H_0 when λ is the true value of the failure rate. Again these equations are solved for arbitrary chosen h.

- **Ex.** Let us assume that we are interested in setting up a sequential test for the hypothesis $H_0: \lambda \geq 0.002$ and $H_1: \lambda < 0.002$.

We specify $\alpha = 0.05$, $\beta = 0.10$ and $\lambda_1 = 0.001$.

$$\rightarrow \frac{T(0.002 - 0.001) - \ln\left(\frac{1-0.10}{0.05}\right)}{\ln\left(\frac{0.002}{0.001}\right)} = \frac{T \times 0.001 - 2.89}{0.693}$$

$$= 1.44 \times 10^{-3} T - 4.17$$

So, $1.44 \times 10^{-3} T - 4.17 < r < 1.44 \times 10^{-3} T + 3.25$

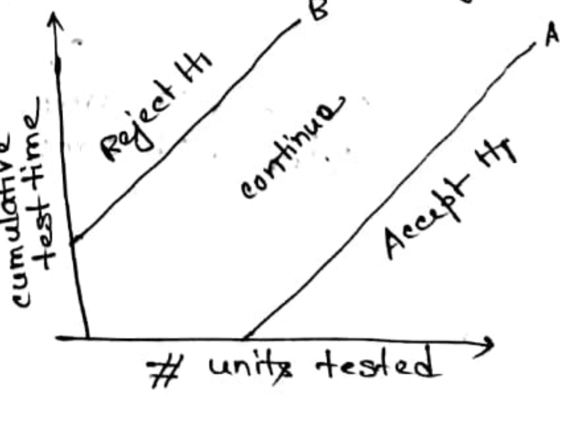
Here $\lambda = \frac{0.001h}{1-(0.5)^h}$

and $P(\lambda) = \frac{18^h - 1}{18^h - 0.105^h}$. O.C. curve is done in next page.

Note:- Condition:- $\theta_1 < \theta_0$

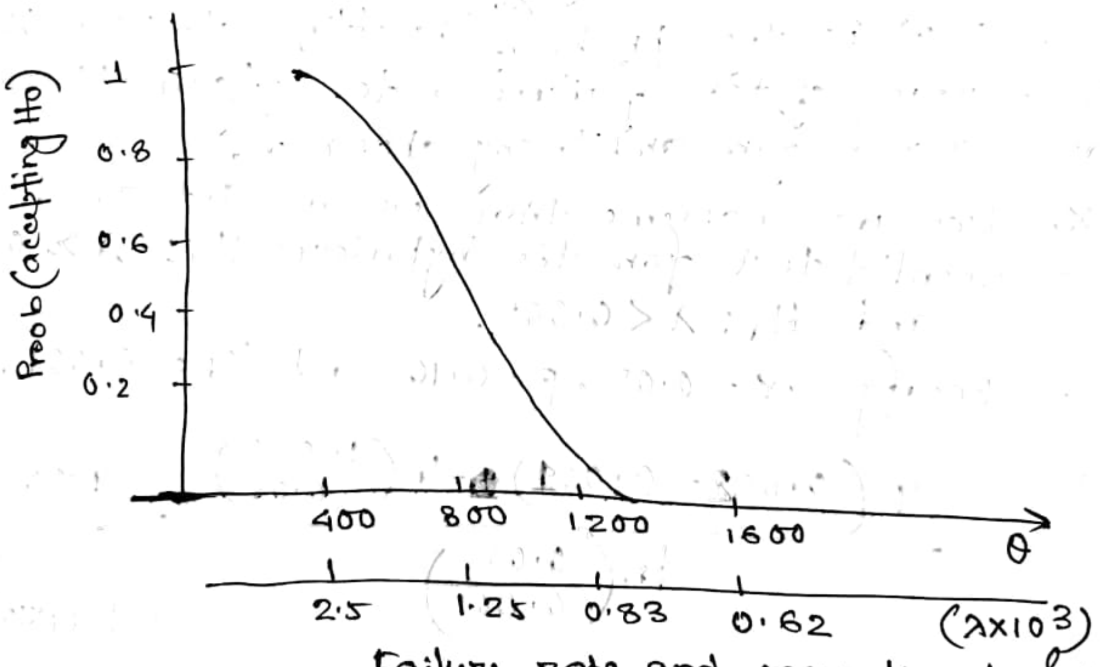
$\alpha = P(H_1 | H_0)$

$\beta = P(H_0 | H_1)$



Calculation of OC curve:-

h	$P(\lambda)$	λ
2	0.9969	0.0027
1	0.9500	0.0020
0.5	0.8275	0.0017
0.25	0.7110	0.0016
0.1	0.6242	0.0015
0.001	0.5625	0.0014
−0.5	0.2681	0.0012
−1	0.1000	0.0010
−2	0.0110	0.0007



Failure rate and mean time to failure

Non-parametric Case:-

$$F(t) = \int_0^t f(t)\,dt$$

$$F(t_i) = \frac{\sum d_i}{n} \quad ; \; d_i = \text{No. of failure in } i^{th} \text{ time interval.}$$

n = no. of items on test

$$R(t_i) = 1 - F(t_i)$$

Let $\hat{F} = F(t_i)$ then

$$\left\{1 + \frac{(n - n\hat{F} + 1)}{n\hat{F}} \cdot F_{1-\alpha/2,\, 2n-2n\hat{F}+2,\, 2n\hat{F}}\right\}^{-1} \le \hat{F}(t_i) \le$$

$$\left\{1 + \frac{n - n\hat{F}}{n\hat{F}+1} \cdot F_{\alpha/2;\, 2n-2n\hat{F};\, 2n\hat{F}+2}\right\}^{-1}.$$

$$\sigma_{\hat{F}(t_i)} = \sqrt{\frac{\hat{F}(t_i)\{1 - \hat{F}(t_i)\}}{n}}$$

$$Z_i = \frac{\hat{F}(t_i) - F(t_i)}{\sigma_{\hat{F}(t_i)}}$$

d_i = failed in i^{th} interval

c_i = censored in i^{th} interval

F_i = cumulative failure = $\frac{d_i}{n_i}$

n_i' = number of risk

$n = 100$

Time (i)	d_i	c_i	n_i	n_i'	F_i	$1-F_i$
1	0	0	100	100	0	
2	1	0	100	100	$1/100 = 0.01$	
3	0	2	99	$99 - 2/2$	$1/100 = 0.01$	
4	0	3	97	$97 - 3/2$	$1/100 = 0.01$	
5	1	1	94	$94 - 1/2$	$2/93.5 + 1/100 = 0.021$	
6	0	5	92	$92 - 5/2$	$1/100 + 2/93.5 = 0.021$	
7	2	0	87	87	$4/87 + 2/93.5 + 1/100 = 0.045$	
8	5	1	85	$85 - 1/2$	$9/84.5 + 4/87 + 2/93.5 + 1/100 = 0.1$	
9	0	10	79	$79 - 10/2$	$9/84.5 + 4/87 + 2/93.5 + 1/100 = 0.107$	

$\alpha = 0.05$

$$\left[1 + \frac{(n - n\hat{F} + 1)\, F_{1-\alpha/2,\ 2n-2n\hat{F}+2,\ 2n\hat{F}}}{n\hat{F}}\right]^{-1} < F <$$

$$\left[1 + \frac{(n - n\hat{F})}{(n\hat{F}+1)\, F_{1-\alpha/2,\ 2n\hat{F}+2,\ 2n-2n\hat{F}}}\right]^{-1}$$

$$\Rightarrow \left[1 + \frac{(100-2+1)}{2} \cdot F_{0.975,\,148,\,4}\right]^{-1} < F < \left[1 + \frac{(100-2)}{(2+1) F_{0.975,\,6,\,196}}\right]^{-1}$$

$$\Rightarrow \left[1 + \frac{99 \times 8.2}{2}\right]^{-1} < F < \left[\frac{3 \times 2.41}{3 \times 2.41 + 9.8}\right]^{-1}$$

$$\Rightarrow 0.00245 < F < 0.066$$

Reliability, Availability, and Maintainability :- (RAM)

RAM are system design attributes that have significant impacts on the sustainment or total Life Cycle Costs (LCC) of a developed system. Additionally, the RAM attributes impact the ability to perform the intended misson and affect overall mission success. The standard definition of Reliability is the probability of zero failures over a defined time interval, where as Availability is defined as the percentage of time a system is considered ready to use when tasked. Maintainability is a measure of the ease and rapidly with which a system or equipment can be restored to operational status following a failure.

Reliability, Availability and Serviceability (RAS) :-

RAS is a set of related attributes that must be considered when designing, manufacturing, purchasing or using a computer product or component. Some of the key elements of RAS are :

- Over-engineering, which is designing systems to specifications better than minimum requirements.
- Duplication, which is extensive use of redundant system and components.
- Recoverability, which is the use of fault-tolerant engineering methods.

Remember - RAM is a risk reduction activity :-

RAM activities and engineering processes are a risk mitigation activity used to ensure that performance needs are achieved for mission success that the LCC are bounded and predictable.

Weibull Plotting of Suspended Data:-

Suspended data :- Data associated with items that have not failed

- Increment in Rank number (N_{ti}) :- ((N+1) − Previous rank order number) / (1 + Number of items following suspended item)
- Cumulative Rank Number (i_{ti}) :- $i_{t(i-1)} + N_{ti}$

Sl. No.	Life	No. Survived	N_{ti}	i_{ti}	Median Rank %
1	2	1	1.016	1.016	1.15
2	6.6	9	1.148	2.144	2.99
3	8.2	12	1.193	3.357	4.90
4	17	27	1.657	5.014	7.55
5	20	31	1.812	6.826	10.46
6	29.2	36	2.081	8.907	13.79
7	35	38	2.164	11.071	17.26
8	45	41	2.360	13.431	21.04
9	53.5	49	3.541	16.972	26.72
10	65	58	9.206	26.178	41.47
11	76.2	59	9.206	35.384	56.22

$\beta = 1.23$, $\eta = 123.0$, $\gamma = 0$, MTBF = 113.0, $R(10.0) = 0.955$

Accelerated Life Testing:- Life tests are done to get data about life, to predict reliability of system/compononent. If we know component reliability, we can calculate system reliability. Life tests is also done to compare 2 designs, 2 brands.

Sequential life testing:- is useful because the minimum number of samples can be taken.

Accelerated Life testing:- The part is subjected to increased stress. Here test conditions are accelerated.

Accelerated Life Testing :-

Tests :- Dry heat, Temp Cycling, Humidity, Vibration, etc.

Severity :- Leading to same failure as in service.

Time :- No. of hours tested, No. of failures to be observed.

Sample :- No. of products to be tested.

Stress and strain are critical measures in accelerated test.

Evaluation of Test :- — Photo type (during designs)

— Reliability audit (after making the product)

- **Arrhenius Law :-** Let t_n = time to failure under normal stress

t_s = time to failure at high stress level

Then $t_n = AF \times t_s$

where AF is an acceration factor to be determined.

When failures are accelerated primarity as a result of an increase in temperature, then

$$r = Ae^{-B/T}$$

where, r is the reaction or process rate, A and B are constants, and T is temperature.

$$AF = \frac{Ae^{-B/T_2}}{Ae^{-B/T_1}} = \exp\left[B\left(\frac{1}{T_1} - \frac{1}{T_2}\right)\right]$$

alternatively, $\frac{t_{amb}}{t_{stressed}} = \exp\left[1.1 \times \left(\frac{1}{t_{amb}} - \frac{1}{t_{stressed}}\right) / 8.63 \times 10^{-5}\right]$

1. **Arrhenius Model :-**

$$\text{Reaction Rate} = \lambda = \gamma \, \text{Exp}\left[\frac{-Ea}{k_B \, \text{Temp}K}\right]$$

where, k_B = Boltzman constant $= 8.617 \times 10^{-5} = \frac{1}{11605}$

Ea = Activation Energy

γ = constant to material properties

$\text{Temp}K = \text{Temp}\,^\circ C + 273$

$\text{Temp Usage} = \text{Temp}_U$

$\text{Temp accelerated} = \text{Temp}_A$

$$\text{Acceleration Factor} = AF = \frac{\lambda_{usage}}{\lambda_{accelerated}}$$

$$= \text{Exp}\left[Ea\left(\frac{11605}{\text{Temp}_A} - \frac{11605}{\text{Temp}_B}\right)\right]$$

$$\text{Reaction Rate} = \lambda = \gamma e^{-\left[\frac{Ea}{K_B \, Temp K}\right]}$$

$$\lambda_U = \gamma e^{-\left[\frac{Ea}{K_B \, Temp K_U}\right]}$$

$$\lambda_A = \gamma e^{-\left[\frac{Ea}{K_B \, Temp K_A}\right]}$$

$$AF = \frac{e^{+\left[\frac{Ea}{K_B \, Temp K_U}\right]}}{e^{+\left[\frac{Ea}{K_B \, Temp K_A}\right]}}$$

$$= e^{+Ea\left[\frac{11605}{Temp K_U} - \frac{11605}{Temp K_A}\right]}$$

2. Eyring Model:-

$$\text{Reaction Rate} = \gamma \times A(Temp) \times Exp\left[\frac{-Ea}{K_B . Temp K}\right]$$

$A(Temp)$ = Function related to material properties

$= (Temp)^m$; $m = 0$ ~~to~~ 2

$m = 0$, it becomes arrhenious model.

$$AF = \frac{A(Temp_A)}{A(Temp_U)} . Exp\left[+Ea\left(\frac{11605}{Temp K_U} - \frac{11605}{Temp K_A}\right)\right]$$

3. Inverse Power Model:-

$$AF = \left(\frac{Volt\,A}{Volt\,B}\right)^{-B_1}$$

$T(Volt_A)$ = Failure time at accelerated condition

B_1 = constant related to the material.

$$\text{Safety Factor (SF)} = \frac{\text{Minimum Strength}}{\text{Maximum Stress}}$$

$$\therefore SF = \frac{\overline{Strength} - 3\sigma_{strength}}{\overline{Stress} + 3\sigma_{stress}}$$

$$\text{Derating Factor (DF)} = \frac{\text{Rated Capacity}}{\text{Used capacity}}$$

$$\therefore DF = \frac{1A}{0.1A} = 10.$$

Scanned by CamScanner

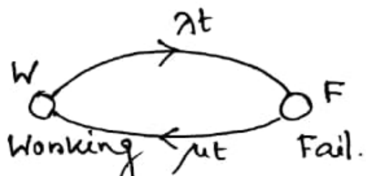
Maintainability, Availibility, Serviceability, Operational Readiness:-

Assumptions:- 1. After repair device works as new.

2. For a small period of time (δt) it can't take more than one transition.

3. Rate of entry into working state = Rate of entry into failed state.

$$P_{up} \times \lambda = P_{Down} \times \mu$$



$$\Rightarrow P(W) \times \lambda = P(F) \times \mu$$

$$\Rightarrow P(W) \times \lambda = (1 - P(W)) \times \mu$$

$$\Rightarrow P(W)(\mu + \lambda) = \mu$$

$$\Rightarrow P(W) = \frac{\mu}{\mu + \lambda}$$

$P(W) + P(F) = 1.$

μ = Repair Rate

λ = failure Rate

MTBF : Mean time between failure

MTTR : Mean time to repair

$$\mu = \frac{1}{MTTR}, \quad \lambda = \frac{1}{MTBF}$$

$$\therefore P(W) = \frac{MTBF}{MTBF + MTTR} = \frac{\text{Up time}}{\text{Total time}} = \text{Availibility}$$

Maintanance Types:-

1. Preventive Maintanance
2. Scheduled Maintenance
3. Corrective Maintenance
4. Predictive Maintenance

Preventive Maintenance should be done only when product is in Warhouse zone. It should not be done in Infant mortality zone because it will make it worst.

Ex. 1. $\lambda = 0.001/1000\,kms$ = failure rate of engine

$n = 1{,}00{,}000$ cars ;

5% risk of shortage of spare engine;

No. of Kms runs in a month = 2000 kms ;

failure rates are constant.

Find the no. of spare engine required in a month ?

Solution:- No. of expected failures in 2000 kms $= \{1 - R(2000)\} \times n$

$$= \left(1 - e^{-2000 \times \frac{0.001}{1000}}\right) \times 1{,}00{,}000$$

$$= 200$$

λt = average no. of failures = 2 (suppose)

$$P(r) = \frac{e^{-\lambda t} \cdot (\lambda t)^r}{r!}$$

$$= \frac{e^{-2} \times 2^r}{r!} = 0.95$$

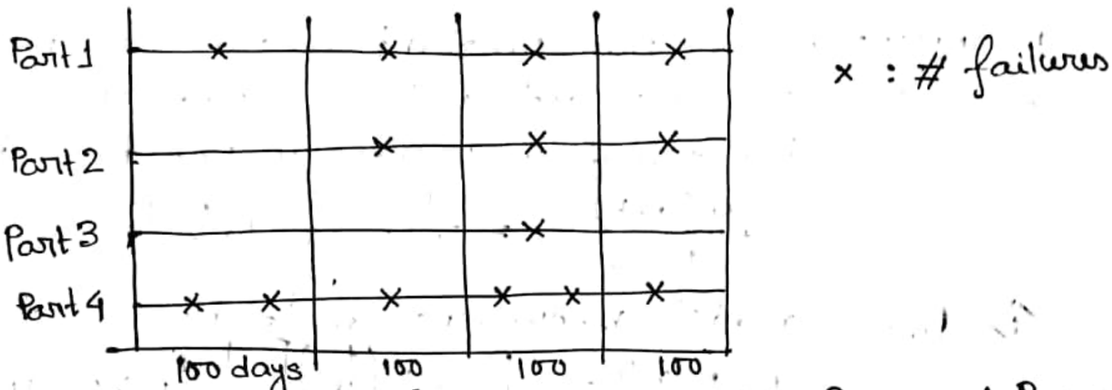
find $r = ?$

[Here ready to take 5% risk, so cumulative probability is $P(r) = 0.95$.]

Recurrent Process:- Here instead of looking at time at which failed we divide the whole time into several time intervals and find no. of failures in each time interval. So this is reccurrence data i.e., no. of failures observed in K^{th} interval.

This is no. of failures on a continuous time scale. this is Poisson process

Recurrent Process:—



Data collected in this format are called Recurrent Process.
[counting no. of failures in a time interval for different parts]

3-types of Recurrent Process:-

1. Homogeneous Poisson Process (HPP)
2. Non-homogeneous Poisson Process (NHPP)
3. Renewal Process.

Properties for Recurrent Process:-

$N(K)$ = No. of recurrences in K^{th} interval.

1) $N(0) = 0$
2) Number of recurrences in independent time interval are statistically independent (independent increments).
3) Process Recurrence Rate $\gamma(t)$ is +ve and average recurrence rate within (a,b) is $\mu(a,b) = E(N(a,b)) = \int_a^b N(a,b)P(d)$; $P(d) = \frac{\mu(a,b)^d}{d!} e^{-\mu(a,b)}$

Properties of H.P.P:—

1) $N(a,b)$ has a Poisson distribution with $\mu(a,b) = \frac{b-a}{\theta}$,
   Failure rate $= \frac{1}{\theta}$;
2) Expected no. of occurance in (a,b) is $\mu(a,b)$.
   Expected no. of occurence in unit time is $1/\theta$.
3) Inter occurence times are IID.
4) Time for K^{th} occurence has a gamma distn. $g(\theta, K)$.

NHPP Properties:-

1) NHPP is a non-constant recurrence rate $\gamma(t)$.
2) Inter occurence intervals are not IIDs.
3) Expected no. of occurences in unit time is

$$\frac{\mu(a,b)}{b-a} = \frac{1}{b-a}\int_a^b \gamma(t)\,dt$$

4) Power model recurrence rate is

$$\gamma(t,\beta,\eta) = \frac{\beta}{\eta} e^{-\left(\frac{t}{\eta}\right)^{\beta-1}};\ \beta>0,\ \eta>0$$

Mean recurrence over $(0,t)$ is $\left(\frac{t}{\eta}\right)^{\beta}$.

Warranty Analysis:-

Warranty requires life data of number of failures.
We count no. of failures in a time interval.
This is a stochastic process data.
We find the status in time intervals.

For NPP, failure rate is constant
IID

For NHPP, failure rate is not constant
not IID

$$\text{Safety factor} = \frac{\text{Minimum strength}}{\text{Maximum stress}}$$, as high as possible.

Definition of Warranty:- Warranty is a contractual abligation incurred by a manufacturer (or vendor/seller) in connection with the sell of a product.
Purpose of a warranty is to establish liability in the event of a premature failure or inability of the item to perform its intended function.

The contract specify the promised product performance and when it is not met the redress available to the buyer as a compensation of the failure.

Life Cycle Cost:- It is incurred by buyer. It includes Purchase Price + Running cost + maintanance cost + Disposal cost (outside India) + Other cost.

Free-replacement Warranty:- This is one maintainance model known as FRW. Manufacturer agrees to repay as possible replacement for failed items free of charge upto a time W from the time of initial purchase.

Pro-rated Warranty:- Manufacturer agrees to refund a fraction of the purchased price of the item failed before the warranty period.

PRW:- C_s = Cost of seller ; C_b = Price (buyers)
W = Warranty period ; θ = Average Life

$$\text{Seller's cost} = C_s + C_b\left[1 - \left(1 - e^{-\frac{\omega}{\theta}}\right)\frac{\theta}{\omega}\right]$$

$$\text{Seller's average profit} = C_b\left[\frac{\theta}{\omega}\left(1 - e^{-\frac{\omega}{\theta}}\right)\right] - C_s.$$

$1 - e^{-W/\theta}$ = unreliability.

ω	θ	$1-e^{-\omega/\theta}$	$\left(1-e^{-\frac{\omega}{\theta}}\right)\frac{\theta}{\omega}$	$1-\left(1-e^{-\omega/\theta}\right)\frac{\theta}{\omega}$	$\frac{\omega}{\theta}$
1 Yr	5 Yr	0.18	0.906	0.043	0.2
1	10 Yr	0.96	0.951	0.048	0.1
1	2 Yr	0.393	0.786	0.213	0.5
1	1 Yr	0.632	0.63	0.368	1
1	0.5 Yr	0.864	0.43	0.568	2.0

$$\left[\text{Seller's cost} = C_s\left[1 + \frac{\omega}{\theta}\right]\right]$$

For Repairable System:- → **FRW**

$$\text{Seller's cost} = C_s + C_r[\lambda\omega]$$

$$\text{Profit} = C_b - C_s - C_r(\lambda\omega)$$

Pre-Replacement Warranty:- (PRW)

$$\text{Refund} = [\alpha(W - X_1)/W]\, C_f$$

$$0 < \alpha < 1$$

C_f = Cost to buyer
X_1 = failure time

Non-linear case: $\left[\dfrac{W - X_1}{W}\right]^2 C_f.$

FRW (for repairable, non-repairable)

	Non-repairable	
	1st failure	Repeat failure
Seller's cost:	$C_s[1+F(W)]$	$C_s[1+\mu/w]$
Profit to seller:	$C_b - C_s[1+\mu(w)]$	$C_s[1+\lambda w]$

$F(W)$ = Cummulative Probability of failures till warranty

C_s = cost of the product to the seller

C_r = cost of repair

C_b = Price

	Repairable System	
	1st failure	Repeat failure
Seller's Cost	$C_s + C_r(\mu/w)$	$C_s + C_r(\lambda w)$

Maintenance Type:- There are three types of maintenance in use:

- Preventive Maintenance:- where equipment is maintained before break down occurs. It is maintenance performed in an attempt to avoid failures, unnecessary production loss and safety violations.

- Operational Maintenance:- where equipment is maintained in using.

- Corrective Maintenance:- where equipment is maintained after break down. This maintenance is often most expensive because worn equipment can damage other parts and cause multiple damages.

Stress Strength Models

INTRODUCTION

The purpose of studying Stress-Strength Models is to determine the probability that a component, a subsystem, or a system fails when the stress, in general, exceeds the strength.

In order to compute the reliability we have to know the nature of stress (S) and strength (T) random variables. Our focus of this session is to show how to compute reliability of a component when the density functions for the stress and the strength are known.

Stress-strength analysis is highly useful in mechanical component design.

Our objective is to find the reliability of a component when the density functions for the stress and strength random variables are known.

We can find expression for reliability when stress & strength following different distributions such as Normal, Exponential, Lognormal, Gamma, Weibull distribution.

Examples of stress-related failures include the following:

1. Misalignment of a journal bearing, lack of lubricants, or incorrect lubricants generate an internal load (mechanical or thermal stress) that causes the bearing to fail.

2. The voltage applied to transistor gate is too high, causing a high temperature that melts the transistor's semiconductor material.

3. Cavitation causes pump failure, which in turn causes a violent vibration that ultimately breaks the rotor.

4. Lack of heat removal from a feed pump in a power plant results in overheating of the pump seals, causing the seals to break.

General Expression for Reliability

Let the density function for the stress (S) be denoted by$f_S(.)$, and that for strength (T) by$f_T(.)$. Then by definition,

$$Reliability = R = P(T > S) = P(T - S > 0) \qquad (1)$$

- The probability that the strength T is greater than certain stress s_0 is given by

$$P(T > s_0) = \int_{s_0}^{\infty} f_T(t)dt$$

- Now the reliability of the component is the probability that the strength T is greater than the stress S for all possible values of the stress S and is given by

$$R = \int_{-\infty}^{\infty} f_S\,(s)\left[\int_{s}^{\infty} f_T(t)dt\right]ds \qquad (2)$$

- Reliability can also be computed on the basis that the stress remains less than the strength and the probability of the stress being less than t_0 is given by

$$P(S < t_0) = \int_{-\infty}^{t_0} f_S\,(s)ds$$

- Hence the reliability of the component for all the possible values of the strength T is

$$R = \int_{-\infty}^{\infty} f_T\,(t)\left[\int_{-\infty}^{t} f_S(s)ds\right]dt \qquad (3)$$

- Some other expressions of unreliability. Let unreliability be denoted by $\bar{R}$. $\bar{R} = probability\ of\ failure = 1 - R = P(T < s)$.

Substituting for R from equation (2) we have

$$\bar{R} = P(T < s) = 1 - \int_{-\infty}^{\infty} f_S\,(s)\left[\int_{s}^{\infty} f_T(t)dt\right]ds$$
$$= 1 - \int_{-\infty}^{\infty} f_S\,(s)[1 - F_T(s)]ds$$
$$= \int_{-\infty}^{\infty} F_T(s).\,f_S\,(s)ds \qquad (4)$$

Alternatively using equation (3) we have

$$\bar{R} = \int_{-\infty}^{\infty}[1 - F_S\,(t)f_T(t) \qquad (5)$$

- Define $Y = T - S$. Y is called the interference random variable. We can define reliability as

$$R = P(Y > 0)$$

Assuming T and S are independent random variables and greater than equal to zero the density of y is given by

$$f_Y(y) = \int_S f_T(y + s).\,f_S\,(s)ds$$
$$= \begin{cases} \displaystyle\int_0^{\infty} f_T(y + s).\,f_S(s)ds, & y \geq 0 \\ \displaystyle\int_{-y}^{\infty} f_T(y + s).\,f_S\,(s)ds, & y \leq 0 \end{cases} \qquad (6)$$

Hence the probability of failure is given by

$$\bar{R} = \int_{-\infty}^{0} f_Y(y)dy = \int_{-\infty}^{0}\int_{-y}^{\infty} f_T\,(y + s).\,f_S(s)ds\,dy \qquad (7)$$

and the reliability by

$$R = \int_0^\infty f_Y\,(y)dy = \int_0^\infty \int_0^\infty f_T(y+s).\,f_S(s)ds\,dy \qquad (8)$$

Reliability Computation for Normally Distributed Strength and Stress

The probability density function for a normally distributed stress S is given by

$$f_S(s) = \frac{1}{\sigma_s\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{s-\mu_s}{\sigma_s}\right)^2\right], -\infty < s < \infty \qquad (9)$$

The probability density function for a normally distributed stress S is given by

$$f_T(t) = \frac{1}{\sigma_T\sqrt{2\pi}}exp\left[-\frac{1}{2}\left(\frac{T-\mu_T}{\sigma_T}\right)^2\right] - \infty < T < \infty \qquad (10)$$

where

$$\mu_s = mean\ value\ of\ the\ stress$$
$$\sigma_s = standard\ deviation\ of\ the\ stress$$
$$\mu_T = mean\ value\ of\ the\ strength$$
$$\sigma_T = standard\ deviation\ of\ the\ strength$$

Let us define Y=T-S. It is well known that the random variable Y is normally distributed with a mean of

$$\mu_Y = \mu_T - \mu_S$$

and a standard deviation of

$$\sigma_y = \sqrt{\sigma_T^2 + \sigma_S^2}$$

The reliability R can now be expressed in terms of Y as

$$R = P(Y > 0)$$
$$= \int_0^\infty \frac{1}{\sigma_Y\sqrt{2\pi}}exp\left[-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]dy$$

If we let $z = (y-\mu_y)/\sigma_y$ then $\sigma_y dz = dy$.

When $y = 0$, the lower limit of z is given by

$$z = \frac{0-\mu_y}{\sigma_y} = -\frac{\mu_T - \mu_S}{\sqrt{\sigma_T^2+\sigma_S^2}} \qquad (11)$$

and when $y \to +\infty$, the upper limit of $z \to +\infty$. Therefore,

$$R = \frac{1}{\sqrt{2\pi}}\int_{-\frac{\mu_T-\mu_S}{\sqrt{\sigma_T^2+\sigma_S^2}}}^{\infty} e^{-z^2/2}\,dz \qquad (12)$$

Clearly the random variable $z = (y-\mu_y)/\sigma_y$ is the standard normal variable. Clearly reliability can be found by merely referring to the normal tables.

Equation (12) can be rewritten as

$$R = 1 - \Phi\left(-\frac{\mu_T - \mu_S}{\sqrt{\sigma_T^2 + \sigma_S^2}}\right) \tag{13}$$

Example 1

An automotive component has been designed to withstand certain stresses It is known from the past experience that, because of variation in loading, the stress on the component is normally distributed with a mean of 30,000 kPa and a standard deviation of 3000 kPa. The strength of the component is also random because of variations in the material characteristics and the dimensional tolerances. It has been found that the strength is normally distributed with a mean of 40,000 kPa and a standard deviation of 4000 kpa. Determine the reliability of the component.

Sol: We are given that $T \sim N(40{,}000, 4000)\ kPa$

$$S \sim N(30{,}000, 3000) kPa$$

Then the lower limit of the integral for R is given by

$$z = -\frac{40{,}000 - 30{,}000}{\sqrt{4000^2 + 3000^2}} = -\frac{10{,}000}{5000} = -2.0$$

and hence from the normal tables R=0.977

Example 2

The stress developed in an engine component is known to be normally distributed with a mean of 350.00 Mpa and a standard deviation of 40.00 Mpa. The material strength distribution, based on the expected temperature range and various other factors, is known to be normal with a mean of 820.00 Mpa and a standard deviation of 80.00 Mpa.

Sol: Conventional factor of safety, defined as the ratio of mean strength to mean stress, is given by

$$F.S. = \frac{\mu_T}{\mu_S} = \frac{820.00}{350.00} = 2.34$$

To compute the reliability of the component we use the coupling equation:

$$z = -\frac{\mu_T - \mu_S}{\sqrt{\sigma_T^2 + \sigma_S^2}} = -\frac{820.00 - 350.00}{\sqrt{40.00^2 + 80.00^2}} = -\frac{470.00}{89.44} = -5.25$$

Hence the reliability of the component is 0.9999999.

Now, suppose that poor heat treatment and larger variations in the environmental temperatures cause the standard deviation for the strength of the component to increase to 150.00 Mpa. In that case the factor of safety as defined before remains unchanged, but the reliability is altered. Using the coupling equation,

$$z = -\frac{\mu_T - \mu_S}{\sqrt{\sigma_T^2 + \sigma_S^2}} = -\frac{820.00 - 350.00}{\sqrt{40.00^2 + 150.00^2}} = -\frac{470.00}{155.24} = -3.03$$

and the reliability of the component is found to be 0.99877. Thus we witness a downgrading of reliability resulting from an increased variability in the strength of the component.

Example 3

A new component is to be designed. A stress analysis revealed that the component is subjected to a tensile stress. But there are variations in the load and the tensile stress is found to be normally distributed with a mean of 35,000 psi and a standard deviation of 4000 psi. The manufacturing operations create a residual compressive stress that is normally distributed with a mean of 10,000 psi and a standard deviation of 1500 psi. A strength analysis of the component showed that the mean value of the significant strength is 50,000 psi. The variations introduced by various by various strength factors are not clear at the present time. The engineer wants to know the maximum value of the standard deviation for the strength that will ensure that the component reliability does not drop below 0.999.

Sol: We are given that

$$S_t \sim N(35{,}000{,}4000)\ psi$$

$$S_c \sim N(10{,}000{,}15{,}000)\ psi$$

whereS_t is the tensile stress and S_c is the residual compressive stress.

The mean effective stress $\bar{S}$ is obtained by

$$\bar{S} = \bar{S}_t - \bar{S}_c = 35000 - 10000 = 25000\, psi$$

and its standard deviation by

$$\sigma_s = \sqrt{(\sigma_{s_t})^2 + (\sigma_{s_c})^2}$$

$$= \sqrt{4000^2 + 1500^2}$$

$$= 4272\, psi$$

From the normal tables, we find the value of z associated with a reliability of 0.999 to be -3.1. Substituting in the coupling equation yields

$$-3.1 = -\frac{50000 - 25000}{\sqrt{\sigma_T^2 + 4272^2}}$$

Solving for σ_T we get

$$\sigma_T = 6840\, psi$$

Reliability Computation for Log Normally Distributed Strength and Stress

The standard form of a log normal density function is

$$f_Y(y) = \frac{1}{y\sigma\sqrt{2\pi}} exp\left[-\frac{1}{2\sigma^2}(lny-\mu)^2\right], \qquad y > 0 \qquad (14)$$

Where Y is the random variable. The parameters μ and σ are the mean and the standard deviation, respectively, of the variable $\ln Y$, which is normally distributed. First we develop those relationships for the log normal distribution that are needed later in the analysis.

Let $X = \ln Y$. Then $dx = \left(\frac{1}{y}\right)dy$. From equation (14) we have

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right], \quad -\infty < x < \infty$$

and hence,

$$E(X) = E(\ln Y) = \mu$$

and

$$V(X) = \sigma^2 = V(\ln Y) = \sigma^2_{\ln Y}$$

Now considering the exponent of e in the expression

$$E(Y) = E(e^X) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^x\, exp\left\{-\left(\frac{1}{2}\right)\left(\frac{x-\mu}{\sigma}\right)^2\right\} dx$$

We have

$$x - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 = x - \frac{1}{2\sigma^2}(x^2 - 2x\mu + \mu^2)$$

$$= -\frac{1}{2\sigma^2}(x^2 - 2x\mu - 2\sigma^2 x + \mu^2)$$

$$= -\frac{\mu^2}{2\sigma^2} + \frac{(\mu+\sigma^2)^2}{2\sigma^2} - \frac{1}{2\sigma^2}[x^2 - 2x(\mu+\sigma^2) + (\mu+\sigma^2)^2]$$

Therefore

$$E(Y) = exp\left(\mu + \frac{\sigma^2}{2}\right)\int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{\{x-(\mu+\sigma^2)^2\}}{2\sigma^2}\right]dx \qquad (15)$$

$$= exp\left(\mu + \frac{\sigma^2}{2}\right)$$

To compute the variance of Y we observe that

$$E(Y^2) = \int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}} exp\left[2x - \frac{1}{2\sigma^2}(x-\mu)^2\right]dx$$

Considering the exponent of e in the expression for $E(Y^2)$, we have

$$2x - \frac{1}{2\sigma^2}(x-\mu)^2$$

$$= -\frac{1}{2\sigma^2}(-4\sigma^2 x + x^2 - 2\mu x + \mu^2)$$
$$= -\frac{1}{2\sigma^2}[x^2 - 2x(\mu + 2\sigma^2) + (\mu + 2\sigma^2)^2] - \frac{\mu^2}{2\sigma^2} + \frac{(\mu + 2\sigma^2)^2}{2\sigma^2}$$
$$= -\frac{1}{2\sigma^2}[x - (\mu + 2\sigma^2)]^2 + 2\mu + 2\sigma^2$$

which, when substituted back and simplified as before yields,
$E(Y^2) = exp[2(\mu + \sigma^2)]$

Hence by the definition of variance we may write
$$V(Y) = exp[2(\mu + \sigma^2)] - \left\{exp\left[\mu + \frac{\sigma^2}{2}\right]\right\}^2$$
$$= [exp(2\mu + \sigma^2)][exp(\sigma^2) - 1] \qquad (16)$$

We now observe that
$$\frac{V(Y)}{[E(Y)]^2} = e^{\sigma^2} - 1$$

which, after rearranging, leads to
$$\sigma^2 = ln\left[\frac{V(Y)}{[E(Y)]^2} + 1\right] \qquad (17)$$

And the expression for μ is given by

$$\mu = lnE(Y) - \frac{\sigma^2}{2}$$

So the Reliability R can be expressed as

$$R = \Phi\left(\frac{\mu_T - \mu_S}{\sqrt{\sigma_T{}^2 + \sigma_S{}^2}}\right)$$

Example:
The strength T and stress S are log normally distributed for a component with the following parameters:
E(T) = 100,000 kPa Standard Deviation of T = 10,000 kPa
E(S) = 60,000 kPa Standard Deviation of T = 20,000 kPa
Compute the Reliability of the component.

Solution:

$$\sigma_T{}^2 = \ln\left[\frac{V(T)}{(E[T])^2} + 1\right] = \ln\left[\frac{10^8}{10^{10}} + 1\right] = 0.00995$$

$$\mu_T = lnE(T) - \frac{\sigma_T{}^2}{2} = ln100{,}000 - \frac{0.00995}{2} = 11.50795$$

Similarly for stress S we have $\sigma_S{}^2$ = 0.10535 & μ_S = 10.94942

Therefore, $R = \Phi\left(\frac{\mu_T - \mu_S}{\sqrt{\sigma_T{}^2 + \sigma_S{}^2}}\right) = \Phi(1.64) = 0.9495$ (Using Normal Table).

RELIABILITY COMPUTATION FOR EXPONENTIALLY DISTRIBUTED S & T

The probability density function for an exponentially distributed stress S is given by

$$f_S(s) = \lambda_S e^{-\lambda_S s}, \quad 0 \le s < \infty$$

The probability density function for an exponentially distributed strength T is given by

$$f_T(t) = \lambda_T e^{-\lambda_T t}, \quad 0 \le t < \infty$$

The reliability R can now be expressed

$$R = \int_0^\infty f_S(s)\left[\int_s^\infty f_T(t)dt\right]ds = \frac{\lambda_S}{\lambda_T + \lambda_S}.$$

Accelerated Life Testing

INTRODUCTION

The development of new products in a short time has motivated the development of new methods such as robust design, just−in−time manufacturing, and design for manufacturing and assembly.

More importantly, both producers and customers expect that the product will perform the intended functions for extended periods of time. Hence, extended warranties and similar assurances of product reliability have become standard features of the product.

These requirements have increased the need for providing more accurate estimates of reliability by performing testing of materials, components, and systems at different stages of product development. Testing under normal operating conditions requires a very long time possibly years and the use of an extensive number of units under test, so it is usually costly and impractical to perform reliability testing under normal conditions.

This has led to the development of accelerated life testing (ALT), where units are subjected to a more severe environment (increased or decreased stress levels) than the normal operating environment so that failures can be induced in a short period of test time.

Information obtained under accelerated conditions is then used in conjunction with a reliability prediction (inference) model to relate life to stress and to estimate the characteristics of life distributions at design conditions (normal operating conditions).

Conducting an accelerated life test requires careful allocation of test units to different stress levels so that accurate estimation of reliability at normal conditions can be obtained using relatively small units and short test durations.

Design of Accelerated Life Testing Plans

A detailed test plan is usually designed before conducting an accelerated life test. The plan requires determination of the type of stress, methods of applying stress, stress levels, the number of units to be tested at each stress level, and an applicable accelerated life testing model that relates the failure times at accelerated conditions to those at normal conditions.

Stress loadings

Stress in ALT can be applied in various ways. Typical loadings include constant, cyclic, step, progressive, random stress loading, and combinations of such loadings.

Typical accelerated testing plans allocate equal units to the test stresses. However, units tested at stress levels close to the design or operating conditions may not experience enough failures that can be effectively used in the acceleration models.

Therefore, it is preferred to allocate more test units to the low stress conditions than to the high stress conditions so as to obtain an equal expected number of failures at both conditions.

Types of Stress

The type of applied stress depends on the intended operating conditions of the product and the potential cause of failure. We classify the types of the stresses as follows:

1. Mechanical stresses: Fatigue stress is the most commonly used accelerated test for mechanical components. When the components are subject to elevated temperature, then creep testing (which combines both temperature and load) should be applied. Shock and vibration testing is suitable for components or products subject to such conditions as in the case of bearings, shock absorbers, tires and circuit boards in airplanes and automobiles.

2. Electrical stresses: These include power cycling, electric field, current density, and electromigration. Electric field is one of the most common electrical stresses, as it induces failures in relatively short times; its effect is also significantly higher than other types of stress.

3. Environmental stresses: Temperature and thermal cycling are commonly used for most products. Of course, it is important to use appropriate stress levels that do not induce different failure mechanisms than those under normal conditions. Humidity is as critical as temperature, but its application usually requires a very long time before its effect is noticed. Other environmental stresses include ultraviolet light, sulfur dioxide, salt and fine particles, and alpha and gamma rays.

Accelerated Life Testing Models

Elsayed classified the inference procedures (or models) that relate life under stress conditions to life under normal or operating conditions into three types:

- statistical−based models
- physics−statistics−based models
- physics−experimental−based models

The underlying assumption in relating the failure data, when using any of the models, is that the components/products operating under normal conditions experience the same failure mechanism as those occurring at the accelerated conditions.

The statistics−based models are further classified as parametric models and non−parametric models. We are here going to discuss about Parametric-Statistics-based Models.

Statistics−based models are generally used when the exact relationship between the applied stresses and the failure time of the component or product is difficult to determine based on physics or chemistry principles. In this case, components are tested at different stress levels and the failure times are then used to determine the most appropriate failure time distribution and its parameters.

The most commonly used failure time distributions are the exponential, Weibull, normal, lognormal, gamma, and the extreme value distributions. The failure times follow the same general distributions for all different stress levels, including the normal operating conditions.

Parametric Statistics−based Models

As stated above, statistics−based models are generally used when the exact relationship between the applied stresses (temperature, humidity, voltage, etc.) and the failure time of the component (or product) is difficult to determine based on physics or chemistry principles. In this case, components are tested at different accelerated stress levels $s_1, s_2, \dots, s_n$. The failure times at each stress level are then used to determine the most appropriate failure time probability distribution, along with its parameters. Under the parametric statistics−based model assumptions, the failure times at different stress levels are linearly related to each other.

Moreover, the failure time distribution at stress level s_1 is expected to be the same at different stress levels $s_2, s_3, \dots$ as well as under the normal operating conditions. In other words, the shape parameters of the distributions are the same for all stress levels (including normal conditions) but the scale parameters may be different.

- **Failure times**

$$t_o = A_F t_s \qquad (22.1)$$

Where t_0 is the failure time under operating conditions, t_s is the failure time under operating conditions, t_s is the failure time under stress conditions, and A_F is the acceleration factor (the ratio between product life under normal conditions and life under accelerated conditions);

- **Cumulative distribution functions (CDFs)**

$$F_O(t) = F_S\left(\frac{t}{A_F}\right) \qquad (22.2)$$

- **Probability density functions**

$$f_o(t) = \left(\frac{1}{A_F}\right) f_s\left(\frac{t}{A_F}\right) \qquad (22.3)$$

- **Failure rates**

$$h_o(t) = \left(\frac{1}{A_F}\right) h_s\left(\frac{t}{A_F}\right) \qquad (22.4)$$

The most widely used parametric models are the exponential and Weibull models. Therefore, we derive the above equations for both models and demonstrate their use.

Acceleration Model for the Exponential Model

This is the case where the time to failure under stress conditions is exponentially distributed with a constant failure rate λ_s. The CDF at stress s is

$$F_s(t) = 1 - e^{-\lambda_s t} \qquad (22.5)$$

And the CDF under normal conditions is

$$F_O(t) = F_S\left(\frac{t}{A_F}\right) = 1 - e^{-\frac{\lambda_S t}{A_F}} \qquad (22.6)$$

The failure rates are related as $\lambda_O = \frac{\lambda_S}{A_F}$ (22.7)

Table 22.1 Failure times of the capacitors in hours

Temperature 145℃	Temperature 240℃	Temperature 305℃
75	179	116
359	407	189
701	466	300
722	571	305
738	755	314
1015	768	403
1388	1006	433
2285	1094	440
3157	1104	468
3547	1493	609
3986	1494	634
4077	2877	640
5447	3001	644
5735	3160	699
5869	3283	781
6242	4654	813
7804	5259	860
8031	5925	1009
8292	6229	1176
8506	6462	1184
8584	6629	1245
11512	6855	2071
12370	6983	2189
16062	7387	2288
17790	7564	2637
19767	7783	2841
20145	10067	2910
21971	11846	2954
30438	13285	3111
42004	28762	4617
Mean = 9287	Mean = 5244	Mean = 1295

Table 22.2 Temperatures and the 50th percentiles

Temperature (℃)	145	240	305
50th percentile	6437	3635	898

Example1. In recent years, silicon carbide (SiC) is used as an optional material for semiconductor devices, especially for those devices operating under high temperatures and high electric fields conditions. An extensive accelerated life experiment is conducted by subjecting 6H−SiC metal−oxide−silicon (MOS) capacitors to temperatures of 145, 240, and 305℃. The failure times are recorder in Table 22.1. Determine the mean time to failure (MTTF) of the capacitors at 25℃ and plot the reliability function.

Solution: The data for every temperature are fitted using an exponential distribution and the means are shown in Table 22.1. In order to estimate the acceleration factor we chose some percentile of the failed population, which can be done non−parametrically using the rank distribution or a parametric model. In the example, the exponential distribution is used and the time at which 50% of the population fails is

$$t = \lambda(-\ln 0.5) \qquad (22.8)$$

The 50th percentiles are given in Table 22.2.

We use the Arrhenius model to estimate the acceleration factor

$$t = k\, e^{C/T}$$

Where t is the time at which a specified portion of the population fails, k and c are constants and T is the absolute temperature (measured in degrees Kelvin). Therefore

$$\ln t = \ln k + \frac{c}{T}$$

Using the values in table 22.2 and least−squares regression we obtain $c = 2730.858\ and\ k = 15.84432$. Therefore, the estimated 50th percentile at 25°C is

$$t_{25°C} = 15.84432\, e^{2730.858/(25+273)} = 151261\, h$$

The acceleration factor at 25°C is

$$A_F = \frac{151261}{6437} = 23.49$$

And the failure rate under normal operating conditions is $1/(1295 \times 23.49) = 3.287 \times 10^{-5}$ failures/h, the mean time to failure is 30419h and the plot of the reliability function is shown in Figure 22.3.

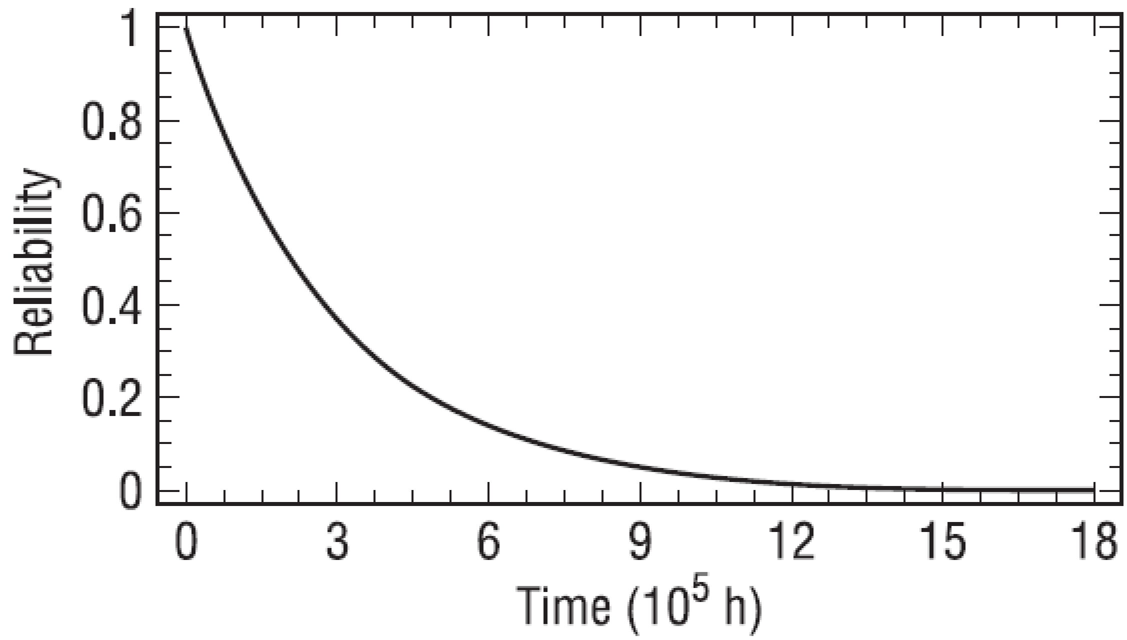


Figure 22.3. Reliability function for the capacitors

Acceleration Model for the Weibull Model

Again, we consider the true linear acceleration case. Therefore, the relationships between the failure time distributions at the accelerated and normal conditions can be derived using Equations 22.2–22.4. Thus

$$F_s(t) = 1 - e^{-\left(\frac{t}{\theta_s}\right)^{\gamma_s}} \quad t \geq 0, \gamma_s \geq 1, \theta_s > 0 \qquad (22.9)$$

where γ_s is the shape parameter of the Weibull distribution under stress conditions and θ_s is the scale parameter under stress conditions. The CDF under normal operating conditions is

$$F_o(t) = F_s\left(\frac{t}{A_F}\right) = 1 - e^{-\left[\frac{t}{A_F \theta_s}\right]^{\gamma_s}}$$

$$= 1 - e^{-\left[\frac{t}{\theta_o}\right]^{\gamma_0}} \qquad (22.10)$$

The underlying failure time distributions under both the accelerated stress and operating conditions have the same shape parameters, i.e., $\gamma_s = \gamma_o, and\ \theta_o = A_F \theta_s$. If the shape parameters at different stress levels are significantly different, then either the assumption of true linear acceleration is invalid or the Weibull distribution is inappropriate to use for analysis of such data.

Let $\gamma_s = \gamma_o = \gamma \geq 1$. Then the probability density function under normal operating conditions is

$$f_o(t) = \frac{\gamma}{A_F\theta_s}\left(\frac{t}{A_F\theta_s}\right)^{\gamma-1} e^{-\left[\frac{t}{A_F\theta_s}\right]^{\gamma}}$$
$$t \geq 0, \theta_s \geq 0 \qquad (22.11)$$

The MTTF under normal operating conditions is

$$MTTF_O = \theta_O^{1/\gamma}\,\Gamma\left(1+\frac{1}{\gamma}\right) \qquad (22.12)$$

The failure rate under normal operating conditions is

$$h_0(t) = \frac{\gamma}{A_F\theta_S}\left(\frac{t}{A_F\theta_S}\right)^{\gamma-1} = \frac{h_s(t)}{A_F^{\gamma}} \qquad (22.13)$$

Table 22.3. Time (hours) to detect leak

100 psi	120 psi	140 psi
1557	1378	215
4331	2055	426
5725	2092	431
5759	2127	435
6207	2656	451
6529	2801	451
6767	3362	496
6930	3377	528
7146	3393	565
7277	3433	613
7346	3477	651
7668	3947	670
7826	4101	708
7885	4333	710
8095	4545	743
8468	4932	836
8871	5030	865
9652	5264	894
9989	5355	927
10471	5570	959
11458	5760	966
11728	5829	1067
12102	5968	1124
12256	6200	1139
12512	6783	1158
13429	6952	1198
13536	7329	1293
14160	7343	1376
14997	8440	1385
17606	9183	1780

Table 22.4 Percentiles at different pressures

Pressure (psi)	100	120	140
50th percentile	9050	4681	821

Example2. A manufacturer of Bourdon tubes (used as a part of pressure sensors in avionics) wishes to determine its MTTF. The manufacturer defines the failure as a leak in the tube. The tubes are manufactured from 18 Ni (250) maraging steel and operate with dry 99.9% nitrogen or hydraulic fluid as the internal working agent. Tubes fail as a result of hydrogen embrittlement arising from the pitting corrosion attack. Because of the criticality of these tubes, the manufacturer decides to conduct ALT by subjecting them to different levels of pressures and determining the time for a leak to occur. The units are continuously examined using an ultrasound method for detecting leaks, indicating failure of the tube. Units are subjected to three stress levels of gas pressures and the times for tubes to show leak are recorded in Table 22.3.

Determine the mean lives and plot the reliability function for design pressures of 80 and 90 psi.

Solution: We fit the failure times to Weibull distributions, which results in the following parameters for pressure levels of 100, 120, and 140 psi.

For 100 psi: $\gamma_1 = 2.87, \theta_1 = 10392$

For 120 psi: $\gamma_2 = 2.67, \theta_2 = 5375$

For 140 psi: $\gamma_3 = 2.52, \theta_3 = 943$

Since $\gamma_1 = \gamma_2 = \gamma_3 \cong 2.65$, then the Weibull model is appropriate to describe the relationship between failure times under accelerated conditions and normal operating conditions. Moreover, we have a true linear acceleration. Following Example 1, we determine the time at which 50% of the population fails as

$$t = \theta[-\ln(0.5)]^{1/\gamma}$$

The 50th percentiles are shown in Table 22.4.

The relationship between the failure time t and the applied pressure P can be assumed to be similar to the Arrhenius model;

Thus

$$t = k\, e^{c/P}$$

Where k and c are constants. By making a logarithmic transformation, the above expression can be written as

$$\ln t = \ln k + \frac{c}{P}$$

Using a linear regression model, we obtain $k = 3.319\ and\ c = 811.456$. The estimated 50th percentiles at 80 psi and 90 psi are 84361 h and 27332 h respectively. The corresponding acceleration factors are 9.32 and 3.02. The failure rates under normal operating conditions are

$$h_o(t) = \frac{\gamma}{A_F\theta_s}\left(\frac{t}{A_F\theta_s}\right)^{\gamma-1} = \frac{h_s(t)}{A_F^{\gamma}}$$

Or

$$h_{80}(t) = \frac{2.65}{1.63382 \times 10^{13}}\ t^{1.65}$$

And

$$h_{90}(t) = \frac{2.65}{8.24652 \times 10^{13}}\ t^{1.65}$$

The reliability functions are shown in Figure 22.4.

The MTTFs for 80 and 90 psi are calculated as

$MTTF_{80} = \theta^{\frac{1}{\gamma}}\Gamma\left(\frac{1}{\gamma}\right) = (1.63382 \times 10^{13})^{1/2.65}\,\Gamma\left(1 + \frac{1}{2.65}\right) = 96853.38 \times 0.885 = 85715\ h$

And

$$MTTF_{90} = 31383.829 \times 0.885 = 27775\ h$$

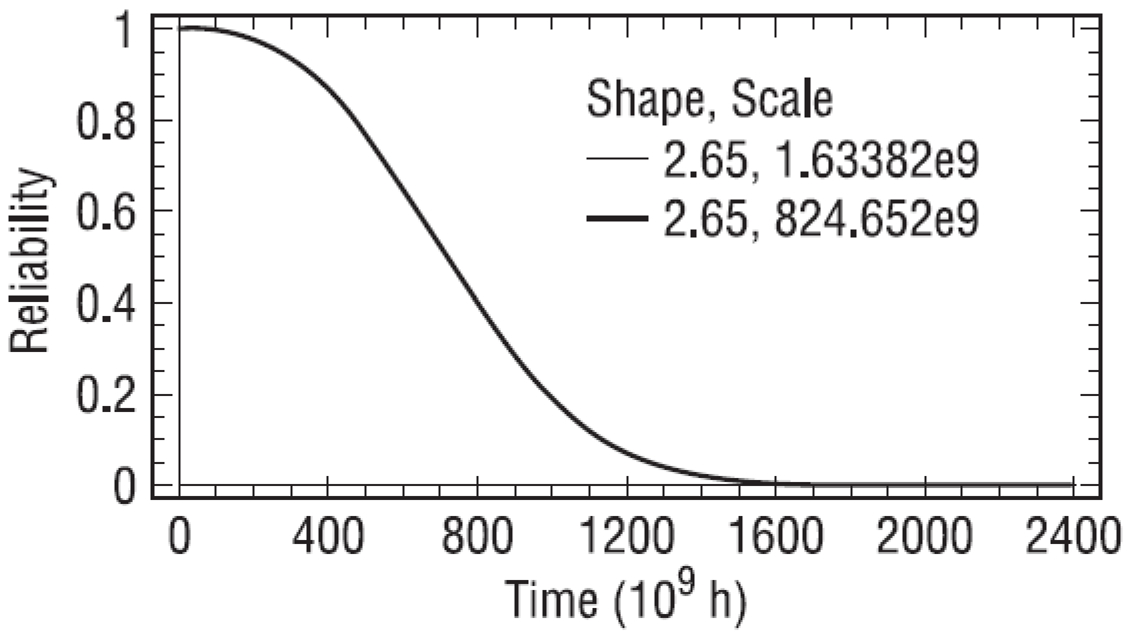


Figure 22.4. Reliability functions at 80 and 90 psi

The Arrhenius Model

Elevated temperature is the most commonly used environmental stress for accelerated life testing of microelectronic devices. The effect of temperature on the device is generally modeled using the Arrhenius reaction rate equation given by

$$r = Ae^{-\left(\frac{E_a}{kT}\right)} \qquad (22.14)$$

where r is the speed of reaction. A is an unknown non−thermal constant, $E_a(eV)$ is the activation energy (i.e., energy that a molecule must have before it can take part in the reaction), k is the Boltzmann constant (8.623$\times$ $10^{-5}eV\ K^{-1}$), and T(K) is the temperature.

Activation energy E_a is a factor that determines the slope of the reaction rate curve with temperature, i.e., it describes the acceleration effect that temperature has on the rate of a reaction and is expressed in electron volts (eV). For most applications, E_a is treated as a slope of a curve rather than a specific energy level. A low value of E_a indicates a small slope or a reaction that has a small dependence on temperature. On the other hand, a large value of E_a indicates a high degree of temperature dependence.

Assuming that device life is proportional to the inverse reaction rate of the process, then Equation 22.14 can be rewritten as

$$L_{30} = 719 \exp \frac{0.42}{4.2998 \times 10^{-5}} \times \left(\frac{1}{30 + 273} - \frac{1}{180 + 273}\right)$$
$$= 31.0918 \times 10^{6}$$

The median lives of the units at normal operating temperature L_O and accelerated temperature L_S are related by

$$\frac{L_O}{L_S} = \frac{Ae^{E_a/kT_o}}{Ae^{E_a/kT_s}}$$

Or

$$L_O = L_S \exp \frac{E_a}{k}\left(\frac{1}{T_o} - \frac{1}{T_s}\right) \qquad (22.15)$$

The thermal acceleration factor is

$$A_F = \exp \frac{E_a}{k}\left(\frac{1}{T_o} - \frac{1}{T_s}\right)$$

The calculation of the median life (or percentile of failed units) is dependent on the failure time distribution. When the sample size is small it becomes difficult to obtain accurate results. In this case, it is advisable to use different percentiles of failures and obtain a weighted average of the median lives. One of the drawbacks of this model is the inability to obtain a reliability function that relates the failure times under stress conditions to failure times under normal condition. We can only obtain a point estimate of life. We now illustrate the use of the Arrhenius model in predicting median life under normal operating conditions.

Table 22.5. Failure time data (hours) for oxide breakdown

Temperature 180°C	Temperature 150°C
112	162
260	188
298	288
327	350
379	392
487	681
593	969
658	1303

701	1527
720	2526
734	3074
736	3652
775	3723
915	3781
974	4182
1123	4450
1157	4831
1227	4907
1293	6321
1335	6368
1472	7489
1529	8312
1545	13778
2029	14020
1568	18640

Example 3. The gate oxide in MOS devices is often a source of device failure, especially for high−density device arrays that require thin gate oxides. The reliability of MOS devices on bulk silicon and the gate oxide integrity of these devices have been the subject of investigation over the years. A producer of MOS devices conducts an accelerated test to determine the expected life at 30℃. Two samples of 25 devices each are subjected to stress levels of 150℃ and 180℃. The oxide breakdown is determined when the potential across the oxide reaches a threshold value. The times of breakdown are recorded in Table 22.5. the activation energy of the device is 0.42 eV. Obtain the reliability function of these devices.

Solution:

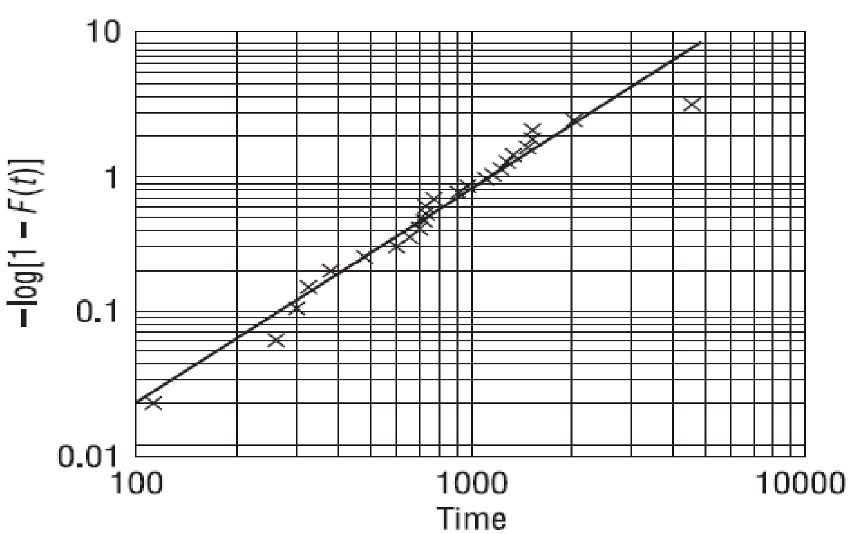


Figure 22.5. Weibull probability plot

Figure 22.5 shows that it is appropriate to fit the failure data using Weibull distributions with shape parameters approximately equal to unity. This means that they can be represented by exponential distributions with means of 1037 and 4787 h for the respective temperatures of 180 ℃ and 150℃ respectively. Therefore, we determine the 50th percentiles for these temperatures using Equation 22.8 as being 719 and 3318 respectively.

$$3318 = 719 \exp \frac{0.42}{k} \left(\frac{1}{150 + 273} - \frac{1}{180 + 273}\right)$$

Which results in $k = 4.2998 \times 10^{-5}$.

The median life under normal conditions of 30°C is

$$L_{30} = 719 \exp \frac{0.42}{4.2998 \times 10^{-5}} \times \left(\frac{1}{30 + 273} - \frac{1}{180 + 273}\right)$$
$$= 31.0918 \times 10^{6}$$

The mean life is 44.8561×10^{6} and the reliability function is

$$R(t) = \exp(- 44.8561 \times 10^{6})$$