LINEAR ALGEBRA

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VECTORS AND VECTOR SPACES

FIELD :→ Suppose there is a set F of objects $x, y, z, \ldots$ and two operations on the elements of F as follows. The first operation, called addition, associates with each pair of elements x, y in F, an element $(x+y)$, the second operation called multiplication, associates with each pair x, y, an element xy, and these two operations satisfy the following properties :

i) **PROPERTIES OF ADDITION** :→

(a) Closure : $x \in F,\ y \in F \Rightarrow x+y \in F$.

(b) Commutative : $x+y = y+x \Rightarrow x, y \in F$.

(c) Associative : $x+(y+z) = (x+y)+z,\ \forall\ x, y, z \in F$.

(d) Nutral element : There is an unique element zero (0) in $F \ni$
$$x+0 = x \ \forall\ x \in F.$$

(e) Inverse : To each $x \in F$, there corresponds an unique element $(-x)$ in $F \ni x+(-x) = 0$.

ii) **PROPERTIES OF MULTIPLICATION** :→

(a) Closure : $x \in F,\ y \in F, \Rightarrow xy \in F$.

(b) Commutative : $xy = yx \Rightarrow x, y \in F$.

(c) Associative : $x(yz) = (xy)z \Rightarrow x, y, z \in F$

(d) Nutral element : There is an unique non-zero element 1 in F
$$\ni x \times 1 = x \ \forall\ x \in F.$$

(e) Inverse : To each non-zero $x \in F$, there corresponds an unique element x^{-1} (or, $\frac{1}{x}$) in $F \ni xx^{-1} = 1$.

iii) **PROPERTIES OF ADDITION & MULTIPLICATION** :→
(DISTRIBUTIVITY)

Multiplication distributes over addition, i.e.,
$$x \times (y+z) = xy + xz \ \forall\ x, y, z \in F$$

The set F together with these two operations is called a *field*.

Ex :→ $F = \{0, 1\}$.

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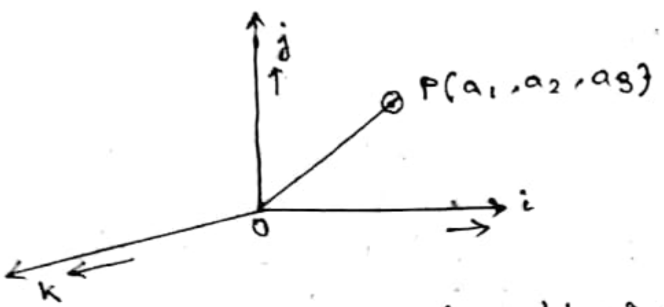


A vector in elementary physics, is a physical quantity having both magnitude and direction.

$$\overrightarrow{OP} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}.$$

Instead of characterising a vector by magnitude and direction, an equally satisfactory description could be achieved by the terminal point of vector originated from the origin.

Hence, we write $\rightarrow$ $\underset{\sim}{a} = (a_1, a_2, a_3)$, where a_i's the ith component.

An ordered array of numbers :→ An array of numbers $(a_1, a_2, a_3, \dots, a_n)$ is said to be an ordered array of numbers if $(a_{i_1}, a_{i_2}, a_{i_3}, \dots, a_{i_n})$ is not the same or equivalent to $(a_{j_1}, a_{j_2}, a_{j_3}, \dots, a_{j_n})$; where $(i_1, i_2, \dots, i_n)$ and $(j_1, j_2, \dots, j_n)$ are two different permutation of $(1, 2, \dots, n)$.

An ordered array of n-numbers $(a_1, a_2, \dots, a_n)$ will be called an ordered n-tuple.

Definition of Vector :→

1) An n-component vector $\underset{\sim}{a}$ is an ordered n-tuple written as a row $(\underset{\sim}{a}_1, \underset{\sim}{a}_2, \dots, \underset{\sim}{a}_n)$ or written as a column

$$\begin{pmatrix} \underset{\sim}{a}_1 \\ \underset{\sim}{a}_2 \\ \vdots \\ \underset{\sim}{a}_n \end{pmatrix}.$$

2)

** An ordered n-tuple of real numbers specifies a point in an n-dimensional space is called an n-component vector.

3) An ordered set of elements of a field is called a vector; the elements are called components. A vector of n components is called an 'n-component vector' or simply an 'n-vector'. An n-vector can be expressed in a horizontal or vertical line and in accordance, a row or column vector will appear.

TYPES :→

(a) Null Vector :

$$\underset{\sim}{0} = (0, 0, \ldots\ 0)'$$

all of those elements are zero.

(b) Unit Vector :

$$\underset{\sim}{e}_1 = (1, 0, 0, \ldots\ldots, 0)$$
$$\underset{\sim}{e}_2 = (0, 1, 0, \ldots\ldots, 0)$$
$$\vdots$$
(ith element)
$$\underset{\sim}{e}_i = (0, 0, 0, \ldots, 1, 0, \ldots, 0) \qquad [\bullet\ a_i' \underset{\sim}{e}_i = a_i \quad \forall\ i = 1(1)n.]$$
$$\vdots$$
$$e_n = (0, 0, 0, \ldots\ldots, 0, 1)$$

— are called the unit vectors.

(c) Sum Vector :

$$\underset{\sim}{1} = (1, 1, \ldots\ldots 1)'$$

all of whose components are unity. $[\bullet\ \underset{\sim}{1}.\underset{\sim}{x} = \sum_{i=1}^{n} x_i]$

VECTOR OPERATIONS :→

Let $\underset{\sim}{a} = (a_1, a_2, \ldots\ldots, a_n)'$ and $\underset{\sim}{b} = (b_1, b_2, \ldots, b_n)'$ be two n-component vectors.

(a) Equality : Then $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are said to be equal iff.

$\underset{\sim}{a}_i = \underset{\sim}{b}_i \quad \forall\ i = 1(1)n$. Then we can say, $\underset{\sim}{a} = \underset{\sim}{b}$.

NOTE : The vectors $(1, 2)$ and $(1, 2, 0)$ are not equal. Two vectors can't be equal unless they have the same number of components.

(b) Addition : The sum of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is defined as

$$\underset{\sim}{a} + \underset{\sim}{b} = (a_1 + b_1, a_2 + b_2, \ldots\ldots, a_n + b_n)'$$

NOTE : This definition is appied only to the vectors which have equal number of components.

(c) Scalar Multiplication : The product of a scalar λ (constant real no.) and a vector $\underset{\sim}{a}$ is defined as

$$\lambda \underset{\sim}{a} = (\lambda a_1, \lambda a_2, \ldots\ldots, \lambda a_n)'$$

(d) Subtraction :

$$\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{a} + (-1)\underset{\sim}{b}$$
$$= (a_1, a_2, \ldots\ldots, a_n)' + (-b_1, -b_2, \ldots, -b_n)'$$
$$= (a_1 - b_1, a_2 - b_2, \ldots\ldots, a_n - b_n)'$$

Some Geometrical Concepts :

(a) Scalar Product : The scalar product of two vectors $\underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ is defined to be scalar if

$$\underset{\sim}{a}'\underset{\sim}{b} = \sum_{i=1}^{n} a_i b_i \,.$$

PROPERTIES :

1) $\underset{\sim}{a}'\underset{\sim}{b} = \underset{\sim}{b}'\underset{\sim}{a}$

2) $(\lambda \underset{\sim}{a}) \cdot \underset{\sim}{b} = \lambda(\underset{\sim}{a} \cdot \underset{\sim}{b})$

3) $\underset{\sim}{a}(\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$

(b) Distance : The distance of a vector (or a point) $\underset{\sim}{a}$ to the vector (or the point) $\underset{\sim}{b}$ is defined as the scalar,

$$|\underset{\sim}{a} - \underset{\sim}{b}| = \sqrt{(a-b)'(a-b)}$$
$$= \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}$$

PROPERTIES :

1) $|\underset{\sim}{a} - \underset{\sim}{b}| = |\underset{\sim}{b} - \underset{\sim}{a}|$

2) $|\underset{\sim}{a} - \underset{\sim}{b}| \geq 0$

3) $|\underset{\sim}{a} - \underset{\sim}{b}| + |\underset{\sim}{b} - \underset{\sim}{c}| \geq |\underset{\sim}{c} - \underset{\sim}{a}|$

RESULT :→ Prove that for any two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$,

$$(\underset{\sim}{a} \cdot \underset{\sim}{b})^2 \leq |\underset{\sim}{a}|^2 \cdot |\underset{\sim}{b}|^2 .$$ [Cauchy - Schwartz Inequality]

Proof :→ For any scalar λ,

$$|\lambda \underset{\sim}{a} + \underset{\sim}{b}| \geq 0$$

$$\Leftrightarrow |\lambda \underset{\sim}{a} + \underset{\sim}{b}|^2 \geq 0$$

$$\Leftrightarrow (\lambda \underset{\sim}{a} + \underset{\sim}{b})(\lambda \underset{\sim}{a} + \underset{\sim}{b}) \geq 0$$

$\left[\because \underset{\sim}{a}'\underset{\sim}{a} = \sum_{i=1}^{n} a_i^2 = |\underset{\sim}{a}|^2\right]$

$$\Leftrightarrow \lambda^2 |\underset{\sim}{a}|^2 + 2\lambda(\underset{\sim}{a} \cdot \underset{\sim}{b}) + |\underset{\sim}{b}|^2 \geq 0$$

$$\Leftrightarrow |\underset{\sim}{a}|^2 \left\{\lambda^2 + 2\lambda \cdot \frac{(\underset{\sim}{a} \cdot \underset{\sim}{b})}{|\underset{\sim}{a}|^2}\right\} + |\underset{\sim}{b}|^2 \geq 0$$

$$\Leftrightarrow |\underset{\sim}{a}|^2 \left\{\lambda + \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|^2}\right\}^2 + |\underset{\sim}{b}|^2 - \frac{(\underset{\sim}{a} \cdot \underset{\sim}{b})^2}{|\underset{\sim}{a}|^2} \geq 0$$

$$\Leftrightarrow |\underset{\sim}{a}|^2 \left\{\lambda + \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|^2}\right\}^2 + \frac{|\underset{\sim}{a}|^2 |\underset{\sim}{b}|^2 - (\underset{\sim}{a} \cdot \underset{\sim}{b})^2}{|\underset{\sim}{a}|^2} \geq 0$$

For, $\lambda = -\dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|^2}$, then $\dfrac{|\underset{\sim}{a}|^2 |\underset{\sim}{b}|^2 - (\underset{\sim}{a} \cdot \underset{\sim}{b})^2}{|\underset{\sim}{a}|^2} \geq 0$

$$\Rightarrow \boxed{(\underset{\sim}{a} \cdot \underset{\sim}{b})^2 \leq |\underset{\sim}{a}|^2 \cdot |\underset{\sim}{b}|^2}$$, provided $\underset{\sim}{a}$ & $\underset{\sim}{b}$ have finite length.

'=' holds iff $|\lambda \underset{\sim}{a} + \underset{\sim}{b}| = 0$ for some λ.
iff $\lambda \underset{\sim}{a} + \underset{\sim}{b} = 0$.
iff $\underset{\sim}{b} = -\lambda \underset{\sim}{a}$, for some λ.
iff $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are collinear.

Remark → C-S inequality :–

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i\right)^2 \left(\sum_{i=1}^{n} b_i\right)^2$$

'=' holds iff $b_i = -\lambda a_i \ \forall\ i = 1(1)n$.
iff $b_i \propto a_i \ \forall\ i = 1(1)n$.

RESULT :→ For any three vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$, $|\underset{\sim}{a} - \underset{\sim}{b}| + |\underset{\sim}{b} - \underset{\sim}{c}| \geq |\underset{\sim}{c} - \underset{\sim}{a}|$
[Triangle Inequality]

Proof :→ $|\underset{\sim}{c} - \underset{\sim}{a}|^2$

$= |(\underset{\sim}{a} - \underset{\sim}{b}) + (\underset{\sim}{b} - \underset{\sim}{c})|^2$

$= (\underset{\sim}{a} - \underset{\sim}{b} + \underset{\sim}{b} - \underset{\sim}{c}) \cdot (\underset{\sim}{a} - \underset{\sim}{b} + \underset{\sim}{b} - \underset{\sim}{c})$

$= (\underset{\sim}{a} - \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b}) + 2(\underset{\sim}{a} - \underset{\sim}{b}) \cdot (\underset{\sim}{b} - \underset{\sim}{c}) + (\underset{\sim}{b} - \underset{\sim}{c})(\underset{\sim}{b} - \underset{\sim}{c})$

$= |\underset{\sim}{a} - \underset{\sim}{b}|^2 + 2 \cdot (\underset{\sim}{a} - \underset{\sim}{b})(\underset{\sim}{b} - \underset{\sim}{c}) + |\underset{\sim}{b} - \underset{\sim}{c}|^2$

$\leq |\underset{\sim}{a} - \underset{\sim}{b}|^2 + 2|\underset{\sim}{a} - \underset{\sim}{b}||\underset{\sim}{b} - \underset{\sim}{c}| + |\underset{\sim}{b} - \underset{\sim}{c}|^2$ [Applying C-S inequality]

[$\because \underset{\sim}{a} \cdot \underset{\sim}{b} \leq |\underset{\sim}{a} \cdot \underset{\sim}{b}| \leq |\underset{\sim}{a}| \cdot |\underset{\sim}{b}|$]

$\therefore |\underset{\sim}{c} - \underset{\sim}{a}|^2 \leq \{|\underset{\sim}{a} - \underset{\sim}{b}| + |\underset{\sim}{b} - \underset{\sim}{c}|\}^2$

$\Rightarrow |\underset{\sim}{a} - \underset{\sim}{b}| + |\underset{\sim}{b} - \underset{\sim}{c}| \geq |\underset{\sim}{c} - \underset{\sim}{a}|$

'=' holds iff $\underset{\sim}{b} - \underset{\sim}{c} = \lambda(\underset{\sim}{a} - \underset{\sim}{b})$ for some λ.
iff $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ are collinear.

(c) Length (Norm) :– The length of a vector $\underset{\sim}{a}$ is the distance between $\underset{\sim}{a}$ and the origin $\underset{\sim}{O}$.

$\therefore |\underset{\sim}{a}| = |\underset{\sim}{a} - \underset{\sim}{O}|$

$= \sqrt{\underset{\sim}{a}'\underset{\sim}{a}}$

$= \sqrt{\sum_i a_i^2}$

$\therefore |\underset{\sim}{a}|^2 = \underset{\sim}{a}' \cdot \underset{\sim}{a}$.

C.U.

(d) **Angle** :– The angle (θ) between two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, where $\underset{\sim}{a}, \underset{\sim}{b} \neq 0$, is given by —

$$\cos\theta = \frac{\underset{\sim}{a}' \underset{\sim}{b}}{|\underset{\sim}{a}| \cdot |\underset{\sim}{b}|} = \frac{\sum_{i=1}^{n} a_i b_i}{\sqrt{\sum_{i=1}^{n} a_i^2}\,\sqrt{\sum_{i=1}^{n} b_i^2}}$$

REMARK: The angle (θ) between two non-null vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is given by. $\cos\theta = \frac{\underset{\sim}{a}' \underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|}$.

Here for any θ, $|\cos\theta| \leq 1$

$$\Rightarrow \cos^2\theta \leq 1$$

$$\Rightarrow \left(\frac{\underset{\sim}{a}' \underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|}\right)^2 \leq 1$$

$$\Rightarrow (\underset{\sim}{a}' \underset{\sim}{b})^2 \leq |\underset{\sim}{a}|^2 |\underset{\sim}{b}|^2$$

Provided $\underset{\sim}{a}$ and $\underset{\sim}{b}$ have finite length.

It is C-S inequality, here '=' holds if $\underset{\sim}{a} = \lambda \underset{\sim}{b}$ for some scalar λ.

Some interpretations by vector operations :–

Let $x_1, x_2, \ldots, x_n$ be n-values of a variable x. Then,

$$\bar{x} = \sum_{i=1}^{n} x_i / n.$$

$$= \frac{1}{n} \cdot \underset{\sim}{1}' \underset{\sim}{x} \quad [\text{where } \underset{\sim}{x} = (x_1, x_2, \ldots, x_n)]$$

The deviations $\Rightarrow x_1 - \bar{x}, x_2 - \bar{x}, x_3 - \bar{x}, \ldots\ldots, x_n - \bar{x}$.

The deviation can be represented as

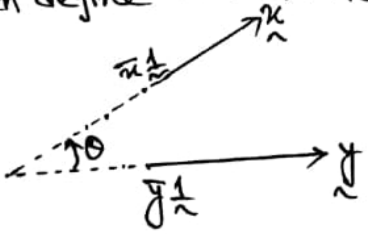
$$\underset{\sim}{x} - \bar{x}.\underset{\sim}{1} = (x_1, \ldots, x_n) - (\bar{x}, \bar{x}, \ldots, \bar{x})$$

$$= (x_1 - \bar{x}, \ldots\ldots, x_n - \bar{x})$$

As a measure of dispersion, we take the distance between $\underset{\sim}{x}$ and $\bar{x}\underset{\sim}{1}$, i.e. the length of deviations vector

$$|\underset{\sim}{x} - \bar{x}\underset{\sim}{1}| = \sqrt{\sum_i (x_i - \bar{x})^2}$$

Let, $(x_1, y_1), \ldots\ldots, (x_n, y_n)$ be n pairs of values on x & y. Then define the deviation vectors for $\underset{\sim}{x}$ and $\underset{\sim}{y}$.



$$\underset{\sim}{x} - \bar{x}.\underset{\sim}{1} = \underset{\sim}{d_1} \text{ and}$$

$$\underset{\sim}{y} - \bar{y}\underset{\sim}{1} = \underset{\sim}{d_2},$$

If θ, the angle between $\underset{\sim}{d_1}$ and $\underset{\sim}{d_2}$, is 0 then x and y are on a line. The smaller the angle, the more the vectors $\underset{\sim}{d_1}$ and $\underset{\sim}{d_2}$ closer to a line.

As a measure of linear relationship, we define $\cos\theta$ as

$$\cos\theta = \frac{\underset{\sim}{d_1} \cdot \underset{\sim}{d_2}}{|\underset{\sim}{d_1}||\underset{\sim}{d_2}|}$$

$$= \frac{(\underset{\sim}{x} - \bar{x}\underset{\sim}{1})(\underset{\sim}{y} - \bar{y}\underset{\sim}{1})}{|\underset{\sim}{x} - \bar{x}\underset{\sim}{1}||\underset{\sim}{y} - \bar{y}\underset{\sim}{1}|}$$

$$= \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_i (x_i - \bar{x})^2}\sqrt{\sum_i (y_i - \bar{y})^2}}$$

$= r_{xy}$, which is the correlation coefficient between x and y.

PROPERTY: $-1 \le r_{xy} \le 1$.

Interpret the cases:

1) $r_{xy} = 0$ where $\theta = 90°$

2) $r_{xy} = 1$ where $\theta = 0$

3) $r_{xy} = -1$ where $\theta = 180°$.

Linear Combination :→

An n-vector $\underset{\sim}{a}$ is said to be a linear combination of the n-vectors $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_n$ if for some scalars λ_i, $(i = 1, 2, \ldots, n)$, $\underset{\sim}{a}$ can be written as

$$\underset{\sim}{a} = \lambda_1 \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_2 + \cdots + \lambda_n \underset{\sim}{a}_n.$$

Euclidean Space :→

An n-dimensional Euclidean space is the collection of all n-component vectors such that:

i) for any $\underset{\sim}{a}, \underset{\sim}{b} \in E^n$; $\underset{\sim}{a} + \underset{\sim}{b} \in E^n$. → Closure property w.r.t. Addition.

ii) for any $\underset{\sim}{a} \in E^n$ and for any scalar λ, $\lambda \underset{\sim}{a} \in E^n$. → Closure property w.r.t. scalar multiplication.

iii) There is a non-negative quantity associated with any two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ called distance between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

In E^n, a vector $\underset{\sim}{a} \in E^n$ is a point in n-dimension Geometry.

C.U.

LINEAR DEPENDENCE :→

A set of vectors $\{a_1, a_2, \ldots, a_n\}$ from E^n is said to be linearly dependent if $\exists$ scalars λ_i, $(i = 1(1)n)$, not all zero, such that

$$\lambda_1 \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_2 + \cdots + \lambda_n \underset{\sim}{a}_n = 0$$

C.U.

LINEAR INDEPENDENCE :→

A set of vectors $\{b_1, b_2, \ldots, b_n\}$ is said to be linearly independent if $\lambda_i = 0$, $(i = 1(1)n)$, is the only solution of

$$\lambda_1 \underset{\sim}{b}_1 + \lambda_2 \underset{\sim}{b}_2 + \cdots + \lambda_n \underset{\sim}{b}_n = \underset{\sim}{0}$$

Examples :→

1) Show that $(0,0,1)$, $(0,1,1)$ and $(1,1,1)$ from E^3 are linearly independent.

Soln. →

Let, $\lambda_1(0,0,1) + \lambda_2(0,1,1) + \lambda_3(1,1,1) = \underset{\sim}{0}$ ——— (*)

$\Rightarrow (0,0,\lambda_1) + (0,\lambda_2,\lambda_2) + (\lambda_3,\lambda_3,\lambda_3) = (0,0,0)$

$\Rightarrow (\lambda_3, \lambda_2+\lambda_3, \lambda_1+\lambda_2+\lambda_3) = (0,0,0)$

$$\Rightarrow \begin{cases} \lambda_3 = 0 \\ \lambda_2 + \lambda_3 = 0 \\ \lambda_1 + \lambda_2 + \lambda_3 = 0 \end{cases} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0.$$

Hence, $\lambda_1 = \lambda_2 = \lambda_3 = 0$ is the only solution of (*).

Therefore, the vectors are linearly independent.

2) Show that the unit vectors from E^n are linearly independent.

Soln. → Let $\{e_1, e_2, \ldots, e_n\}$ be a set of unit vectors from E^n.

If $\exists$ scalars λ_i, $i = 1(1)n$, then we can write

$$\lambda_1 \underset{\sim}{e_1} + \lambda_2 \underset{\sim}{e_2} + \cdots + \lambda_n \underset{\sim}{e_n} = \underset{\sim}{0} \quad \text{———} (*)$$

$\Rightarrow \lambda_1(1, 0, 0, \ldots 0) + \lambda_2(0, 1, 0, \ldots, 0) + \cdots + \lambda_n(0, \ldots, 0, 1) = \underset{\sim}{0}$

$\Rightarrow (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) = (0, 0, 0, \ldots, 0)$

$\therefore \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \ldots, \lambda_n = 0.$

$\Rightarrow \lambda_i = 0 \;\forall\; i = 1(1)n$, is the only solution of $(*)$.

$\therefore \{e_1, e_2, \ldots, e_n\}$ is linearly independent.

3) Show that $(1, -1, 0, 0)$, $(1, 0, -1, 0)$, $(1, 0, 0, -1)$, $(3, -1, -1, -1)$ are linearly dependent.

Soln. →

Let, $\lambda_1(1, -1, 0, 0) + \lambda_2(1, 0, -1, 0) + \lambda_3(1, 0, 0, -1) + \lambda_4(3, -1, -1, -1) = \underset{\sim}{0}$

$\Rightarrow (\lambda_1 + \lambda_2 + \lambda_3 + 3\lambda_4, -\lambda_1 - \lambda_4, -\lambda_2 - \lambda_4, -\lambda_3 - \lambda_4) = (0, 0, 0, 0)$

$\Rightarrow \begin{cases} \lambda_1 = \lambda_2 = \lambda_3 = -\lambda_4 \\ \text{In particular, } \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1, \lambda_4 = -1. \end{cases}$

C.U Hence the vectors are linearly dependent.

4) Examine whether the following vectors are linearly dependent or not ?

i) $\underset{\sim}{\alpha_1} = (4, 3, 5)$; $\underset{\sim}{\alpha_2} = (1, 0, 4)$; $\underset{\sim}{\alpha_3} = (3, 6, 1)$

ii) $\underset{\sim}{\alpha_4} = (2, 0, 5)$; $\underset{\sim}{\alpha_5} = (1, 3, 2)$; $\underset{\sim}{\alpha_6} = (2, -12, 7)$

Soln. →

Let, $\lambda_1(4, 3, 5) + \lambda_2(1, 0, 4) + \lambda_3(3, 6, 1) + \lambda_4(2, 0, 5) + \lambda_5(1, 3, 2) + \lambda_6(2, -12, 7) = \underset{\sim}{0}$

$\Rightarrow (4\lambda_1 + \lambda_2 + 3\lambda_3 + 2\lambda_4 + \lambda_5 + 2\lambda_6,\; 3\lambda_1 + 6\lambda_3 + 3\lambda_5 - 12\lambda_6,\; 5\lambda_1 + 4\lambda_2 + \lambda_3 + 5\lambda_4 + 2\lambda_5 + 7\lambda_6) = (0, 0, 0)$

$$\Rightarrow \lambda_1 \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + \lambda_3 \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix} + \lambda_4 \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + \lambda_5 \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda_6 \begin{pmatrix} 2 \\ -12 \\ 7 \end{pmatrix} = 0$$

$$\text{or, } \begin{pmatrix} 4 & 1 & 3 & 2 & 1 & 2 \\ 3 & 0 & 6 & 0 & 3 & -12 \\ 5 & 4 & 1 & 5 & 2 & 7 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 1 & 3 & 2 & 1 & 2 \\ 3 & 0 & 6 & 0 & 3 & -12 \\ 5 & 4 & 1 & 5 & 2 & 7 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -3 & 2 & -2 & 14 \\ 1 & 0 & 2 & 0 & 1 & -4 \\ 0 & 4 & -9 & 5 & -3 & 27 \end{pmatrix}$$

① $R_2' = R_2/3$
② $R_1' = R_1 - R_2$
③ $R_3' = R_3 - 5R_2'$

$$\sim \begin{pmatrix} 1 & 1 & -3 & 2 & -2 & 14 \\ 0 & 1 & -5 & 2 & -3 & 18 \\ 0 & 4 & -9 & 5 & -3 & 27 \end{pmatrix}$$

$R_2' = R_1 - R_2$

$$\sim \begin{pmatrix} 1 & 0 & 2 & 0 & 1 & -4 \\ 0 & 1 & -5 & 2 & -3 & 18 \\ 0 & 0 & 11 & -3 & 9 & -45 \end{pmatrix}$$

$R_1' = R_1 - R_2$
$R_3' = R_3 - 4R_2$

$$\sim \begin{pmatrix} 1 & 0 & 2 & 0 & 1 & -4 \\ 0 & 1 & -4/3 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3/11 & 9/11 & -45/11 \end{pmatrix}$$

$R_2' = R_2 + \frac{1}{3} R_3$
$R_3' = R_3/11$

$= H$, an echelon matrix

$H\underset{\wedge}{x} = \underset{\wedge}{0}$ gives $\rightarrow \begin{cases} \lambda_1 + 2\lambda_3 + \lambda_5 - 4\lambda_6 = 0 \\ \lambda_2 - \frac{4}{3}\lambda_3 + \lambda_4 + 3\lambda_6 = 0 \\ \lambda_3 - \frac{3}{11}\lambda_4 + \frac{9}{11}\lambda_5 - \frac{45}{11}\lambda_6 = 0 \end{cases}$ U.O

$\Rightarrow \begin{cases} \lambda_1 + 2\lambda_3 + \lambda_5 - 4\lambda_6 = 0 \\ 3\lambda_2 - 4\lambda_3 + 3\lambda_4 + 9\lambda_6 = 0 \\ 11\lambda_3 - 3\lambda_4 + 9\lambda_5 - 45\lambda_6 = 0 \end{cases}$

---

2) $\lambda_1(4, 3, 5) + \lambda_2(1, 0, 4) + \lambda_3(3, 6, 1) = 0$ ------ (*)

$\Rightarrow (4\lambda_1 + \lambda_2 + 3\lambda_3, 3\lambda_1 + 6\lambda_3, 5\lambda_1 + 4\lambda_2 + \lambda_3) = (0, 0, 0)$

$\Rightarrow 4\lambda_1 + \lambda_2 + 3\lambda_3 = 0 \Rightarrow \lambda_2 - 5\lambda_3 = 0 \Rightarrow 4\lambda_2 - 20\lambda_3 = 0$
$\lambda_1 + 2\lambda_3 = 0 \Rightarrow \lambda_1 = -2\lambda_3$
$5\lambda_1 + 4\lambda_2 + \lambda_3 = 0 \Rightarrow 4\lambda_2 - 9\lambda_3 = 0$

$\Rightarrow \lambda_3 = 0, \lambda_1 = 0, \lambda_2 = 0$ is the only soln. of (*).

$\therefore$ the vectors are LIN.

Theorem 1. A set of vectors $\{a_1, a_2, \ldots, a_k\}$ from E^n is linearly dependent if and only if one of these vectors can be written as a linear combination of the others.

Proof:

- **Only if part:** Let $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_k$ are linearly dependent vectors from E^n. Then $\exists$ scalars λ_i's, not all zero, such that

$$\lambda_1 \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_2 + \cdots + \lambda_k \underset{\sim}{a}_k = \underset{\sim}{0} \qquad (*)$$

Let, non-zero scalar be λ_i, so $(*)$ implies —

$$-\lambda_i \underset{\sim}{a}_i = \lambda_1 \underset{\sim}{a}_1 + \cdots + \lambda_{i-1} \underset{\sim}{a}_{i-1} + \lambda_{i+1} \underset{\sim}{a}_{i+1} + \cdots + \lambda_k a_k$$

$$\Rightarrow \underset{\sim}{a}_i = \left(-\frac{\lambda_1}{\lambda_i}\right)\underset{\sim}{a}_1 + \left(-\frac{\lambda_2}{\lambda_i}\right)\underset{\sim}{a}_2 + \cdots + \left(-\frac{\lambda_{i-1}}{\lambda_i}\right)\underset{\sim}{a}_{i-1} + \left(-\frac{\lambda_{i+1}}{\lambda_i}\right)\underset{\sim}{a}_{i+1} + \cdots + \left(-\frac{\lambda_k}{\lambda_i}\right)a_k$$

$$= \sum_{j(\neq i)=1}^{k} \left(-\frac{\lambda_j}{\lambda_i}\right)\underset{\sim}{a}_j$$, a linear combination of the vectors $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_{i-1}, \underset{\sim}{a}_{i+1}, \ldots, a_k$.

Hence, one vector $\underset{\sim}{a}_i$ has been written as a linear combination of the others.

- **If part:** Let $\underset{\sim}{a}_i$ can be written as a linear combination of the other vectors, giving —

$$\underset{\sim}{a}_i = \lambda_1 \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_2 + \cdots + \lambda_{i-1} \underset{\sim}{a}_{i-1} + \lambda_{i+1} \underset{\sim}{a}_{i+1} + \cdots + \lambda_k \underset{\sim}{a}_k$$

for some scalars λ_i's.

$$\Rightarrow \lambda_1 \underset{\sim}{a}_1 + \cdots + \lambda_{i-1} \underset{\sim}{a}_{i-1} + (-1)\underset{\sim}{a}_i + \lambda_{i+1} \underset{\sim}{a}_{i+1} + \cdots + \lambda_k \underset{\sim}{a}_k = \underset{\sim}{0}$$

$\Rightarrow \lambda_1 \underset{\sim}{a}_1 + \cdots + \lambda_k \underset{\sim}{a}_k = \underset{\sim}{0}$ is satisfied for $\lambda_i = -1 \ (\neq 0)$, a non-zero scalar.

So, $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_k$ are linearly dependent.

Hence, the set of vectors is linearly dependent.

C.U.

Theorem 2. If $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_m\}$ is linearly independent and $\{\underset{\sim}{a}, \underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_m\}$ is linearly dependent, then $\underset{\sim}{a}$ is a linear combination of $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_m$.

Proof:→ Since $\{\underset{\sim}{a}, \underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_m\}$ is linearly dependent. Then $\exists$ scalars $\lambda, \lambda_1, \lambda_2, \ldots, \lambda_m$, not all zero, such that

$$\lambda \underset{\sim}{a} + \lambda_1 \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_2 + \cdots + \lambda_m \underset{\sim}{a}_m = \underset{\sim}{0} \qquad (1)$$

If $\lambda = 0$, then (1) implies that —

$$\lambda_1 \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_2 + \cdots + \lambda_m \underset{\sim}{a}_m = \underset{\sim}{0} \text{ for at least one } \lambda_i \neq 0.$$

which shows that $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_m\}$ is linearly dependent. But this contradicts our giving condition, so λ is always non zero.

As $\lambda \neq 0$, then (1) gives $\underset{\sim}{a} = -\frac{\lambda_1}{\lambda}\underset{\sim}{a}_1 - \frac{\lambda_2}{\lambda}\underset{\sim}{a}_2 - \cdots - \frac{\lambda_m}{\lambda}\underset{\sim}{a}_m$

i.e. $\underset{\sim}{a}$ is a linear combination of $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_m$.

Hence, the proof.

Example:→ 1) Show that the set of vectors:

$$\underset{\sim}{x}_1 = (1, -1, 0, 0, \ldots, 0, 0)$$
$$\underset{\sim}{x}_2 = (1, 0, -1, 0, \ldots, 0, 0)$$
$$\underset{\sim}{x}_3 = (1, 0, 0, -1, \ldots, 0, 0)$$
$$\vdots$$
$$\underset{\sim}{x}_{n-1} = (1, 0, 0, 0, \ldots, 0, -1)$$
$$\underset{\sim}{x}_n = (n-1, -1, -1, \ldots, -1, -1)$$

is linearly dependent. Also find a linearly independent set of vectors and determine the maximum number of linearly independent vectors in the set.

Soln.→ Note that — $\underset{\sim}{x}_n = \underset{\sim}{x}_1 + \underset{\sim}{x}_2 + \cdots + \underset{\sim}{x}_{n-1}$

$\Rightarrow \{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_n\}$ are linearly dependent.

Now, we consider the equation $\sum_{i=1}^{n-1} \lambda_i \underset{\sim}{x}_i = \underset{\sim}{0}$

$$\Rightarrow \left(\sum_{i=1}^{n-1} \lambda_i, -\lambda_1, -\lambda_2, \ldots, -\lambda_{n-1}\right) = (0, 0, \ldots, 0)$$

$$\Rightarrow \begin{cases} \sum_{i=1}^{n-1} \lambda_i = 0 \\ -\lambda_1 = 0 \\ \vdots \\ -\lambda_{n-1} = 0 \end{cases} \Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0$$

Hence, $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_{n-1}\}$ is linearly independent.
Here, the collection of n vectors $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_{n-1}, \underset{\sim}{x}_n\}$ is linearly dependent but the collection of $(n-1)$ vectors $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_{n-1}\}$ is linearly independent.

2) If $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_n$ are linearly independent and $\sum_{i=1}^{n} \alpha_i \underset{\sim}{x}_i = \sum_{i=1}^{n} \beta_i \underset{\sim}{x}_i$, then show that $\alpha_i = \beta_i \;\forall\, i = 1(1)n$.

<u>Soln.</u> →
$$\sum_{i=1}^{n} \alpha_i \underset{\sim}{x}_i = \sum_{i=1}^{n} \beta_i \underset{\sim}{x}_i$$

$$\Rightarrow \sum_{i=1}^{n} (\alpha_i - \beta_i) \underset{\sim}{x}_i = 0$$

$$\Rightarrow \alpha_i - \beta_i = 0 \;\forall\, i = 1(1)n.$$

since x_i's are linearly independent.

$$\Rightarrow \alpha_i = \beta_i \;\forall\, i = 1(1)n.$$

3) If $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are linearly independent. Then show that $\underset{\sim}{x} + \alpha \underset{\sim}{y}$ and $\underset{\sim}{x} + \beta \underset{\sim}{y}$ are also linearly independent if $\alpha \neq \beta$.

<u>Soln.</u> → Let, $L_1(\underset{\sim}{x} + \alpha \underset{\sim}{y}) + L_2(\underset{\sim}{x} + \beta \underset{\sim}{y}) = \underset{\sim}{0}$

$$\Rightarrow (L_1 + L_2)\underset{\sim}{x} + (L_2\beta + L_1\alpha)\underset{\sim}{y} = \underset{\sim}{0}$$

$\Rightarrow L_1 + L_2 = 0$ & $L_1\alpha + L_2\beta = 0$, since $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are linearly independent.

$$\Rightarrow L_1 = -L_2 \;\&\; L_1(\alpha - \beta) = 0.$$

$$\Rightarrow L_1 = 0,\; L_2 = 0 \text{ since } \alpha \neq \beta.$$

Hence $(\underset{\sim}{x} + \alpha \underset{\sim}{y})$ and $(\underset{\sim}{x} + \beta \underset{\sim}{y})$ are linearly independent if $\alpha \neq \beta$.

C.U

Theorem 3. Show that a set of vectors containing a null vector can't be linearly independent.

Proof:→ Let $\{\underset{\sim}{0}, \underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$ be a set containing a null vector.

For, $\lambda = 5, \lambda_1 = 0, \lambda_2 = 0, \ldots, \lambda_n = 0$, the equation

$$5\underset{\sim}{0} + \lambda_1 \underset{\sim}{a_1} + \lambda_2 \underset{\sim}{a_2} + \cdots + \lambda_n \underset{\sim}{a_n} = \underset{\sim}{0}$$ is satisfied.

Hence, $\exists$ a nonzero λ, so the set of vectors is linearly dependent.

$\therefore$ A set of vectors containing the null vector can't be linearly independent.

Theorem 4. Any n-vector must be linearly dependent on the unit vectors $\underset{\sim}{e_1}, \underset{\sim}{e_2}, \ldots, \underset{\sim}{e_n}$, which by themselves form an idependent set.

Proof:→ Let $\underset{\sim}{a}$ be an n-component vector, given by

$$\underset{\sim}{a} = (a_1, a_2, \ldots, a_n)$$

Since $\underset{\sim}{e_1} = (1, 0, \ldots 0)$, $\underset{\sim}{e_2} = (0\ 1\ 0 \ldots 0), \ldots$

$$\ldots \underset{\sim}{e_n} = (0, 0, \ldots, 0, 1).$$

We can write $\underset{\sim}{a} = a_1 \underset{\sim}{e_1} + a_2 \underset{\sim}{e_2} + \cdots + a_n \underset{\sim}{e_n}$.

This shows that $\underset{\sim}{a}$ is, indeed, linearly dependent on e_i $(i = 1(1)n)$.

Since $\underset{\sim}{a}$ is arbitrarily chosen, the first part of the theorem is established.

~~Also suppose~~ second part has been proved earlier, i.e. Unit vectors are linearly independent.

C.U.

Theorem 5. A null vector is linearly dependent on any other set of non-null vectors.

Proof:→

C.U. Theorem 6.

Any subset of a linearly independent set of vectors is also linearly independent.

<u>Proof</u>:→ Let $\{a_1, a_2, \ldots, a_n\}$ be linearly independent set of vectors. and $\{a_{i_1}, a_{i_2}, a_{i_3}, \ldots, a_{i_k}\}$ $(k < n)$ is a subset of it.
If possible, let this subset be linearly dependent, so $\exists$ some scalars $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_k}$, not all zero, such that

$$\lambda_{i_1} \underset{\sim}{a}_{i_1} + \lambda_{i_2} \underset{\sim}{a}_{i_2} + \cdots + \lambda_{i_k} \underset{\sim}{a}_{i_k} = \underset{\sim}{0}$$

$$\Rightarrow \lambda_1 \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_2 + \cdots + \lambda_n \underset{\sim}{a}_n = \underset{\sim}{0} \text{ for at least one } \lambda_{i_j}\ (j = 1(1)k)$$

non-zero and other λ_i's are zero.

Therefore, $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_n$ are linearly dependent which contradicts our original assumption.

Hence, any subset of a linearly independent set of vectors is linearly independent.

<u>Theorem 7.</u> Every superset of a linearly dependent set of vectors is also linearly dependent.

<u>Proof</u>:→ Let $\{a_1, a_2, \ldots, a_k\}$ be a linearly dependent set of vectors from $V_n(F)$ and $\{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_n\}$ be a superset of $\{a_1, a_2, \ldots, a_k\}$. As $\{a_1, a_2, \ldots, a_k\}$ is linearly dependent,
$\exists$ scalar λ_i, not all zero, satisfying —

$$\lambda_1 \underset{\sim}{a}_1 + \cdots + \lambda_k \underset{\sim}{a}_k = \underset{\sim}{0}.$$

$$\Rightarrow \lambda_1 \underset{\sim}{a}_1 + \cdots + \lambda_k \underset{\sim}{a}_k + 0 \cdot \underset{\sim}{b}_1 + \cdots + 0 \cdot \underset{\sim}{b}_n = \underset{\sim}{0} \text{ for at least one } \lambda_i \neq 0.$$

Hence, $\{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_n\}$ is linearly dependent.

So, every superset of linearly dependent set of vectors is linearly dependent.

<u>Result</u>:→ Suppose $k (< m)$ is the maximum number of linearly independent vectors in a set of m vectors. Then given any linearly independent subset of k vectors from this set, every other vector in the set can be written as a linear combination of these k vectors.

VECTOR SPACE :

C.U.

Definition :→ A vector space V_n is a collection of n-component vectors which is closed under the operations of addition and scalar multiplication.

i.e. i) If $\underset{\sim}{a}, \underset{\sim}{b} \in V_n$ then $\underset{\sim}{a}+\underset{\sim}{b} \in V_n$ [Closed under Addition]

ii) If $\underset{\sim}{a} \in V_n$, for any arbitrary scalar, $\lambda \underset{\sim}{a} \in V_n$ [Closed under Scalar multiplication]

Ex. → $V_2(R) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| x_1, x_2 \in R \right\}$ is a vector space.

REMARK :→ The concept of distance, scalar product and angle are not defined in vector spaces. Clearly, E^n satisfies the properties of a vector space, so E^n is a vector space. If we define length in V_n as in E^n, then V_n is identical with E^n. Although, E^n is a vector space, it does not follow that every V_n is E_n.

Example :→ Consider the collection of vectors of the form $(x_1, x_2, 0)$ from E^3 or V_3. Show that → the collection is a vector space.

Soln. Define $S_3 = \{(x_1, x_2, 0) : x_1, x_2 \in R\}$

Let $x, y \in S_3$.

Then $\underset{\sim}{x} = (x_1, x_2, 0)$ and $\underset{\sim}{y} = (y_1, y_2, 0)$

Now, $\underset{\sim}{x} + \underset{\sim}{y} = (x_1 + y_1, x_2 + y_2, 0) \in S_3$

and $\alpha \underset{\sim}{x} = (\alpha x_1, \alpha x_2, 0) \in S_3$

Hence, S_3 is closed under addition and scalar multiplication.

NOTE :– It is important to note that there are several subsets of E^n or V^n, which are itself a vector space.

VECTOR SUBSPACE :

C.U.

Definition :→

1) A vector subspace S_n of an n-dimensional vector space V_n is a subset of V_n which is itself a vector space.

2) A non empty subset S_n of vectors from n-dimensional vector space V_n is called a subspace of V_n if S_n is closed under —

i) Addition, i.e. $\underset{\sim}{a} \in S_n, \underset{\sim}{b} \in S_n \Rightarrow \underset{\sim}{a}+\underset{\sim}{b} \in S_n$, and

ii) Scalar multiplication, i.e. $\lambda \in R, \underset{\sim}{a} \in S_n \Rightarrow \lambda \underset{\sim}{a} \in S_n$.

Ex :→ $S_2 = \{(x_1, x_2, x_3) : x_1, x_2 \in R\}$ is a vector subspace.

REMARK :→

(a) For a vector space V_n if $\underset{\sim}{x} \in V_n$, then $\alpha \underset{\sim}{x} \in V_n$ for any scalar $\lambda \in R$. For $\alpha = 0$, we get $\underset{\sim}{0} \in V_n$. Hence, the null vector is a member of a vector space.

(b) Any subspace of E^3 is either E^3 itself, a plane through the origin or just the origin itself.

C.U.

Example :→ Show that S_1 is a vector subspace, where

$$S_1 = \{(x_1, x_2, x_3) : x_1 + x_2 = x_3\}.$$

Soln. → Clearly, $S_1 \in E^3$.

Consider two vectors $\underset{\sim}{x}$ and $y \in S$.

$\underset{\sim}{x} = (x_1, x_2, x_3)$, where $x_1 + x_2 = x_3$

$\underset{\sim}{y} = (y_1, y_2, y_3)$, where $y_1 + y_2 = y_3$.

Now, $\underset{\sim}{x} + \underset{\sim}{y} = (x_1+y_1, x_2+y_2, x_3+y_3)$ where $x_1+x_2=x_3$, $y_1+y_2=y_3$.

$\therefore \underset{\sim}{x} + \underset{\sim}{y} = (u_1, u_2, u_3)$, where $u_i = x_i + y_i$ and $u_1+u_2=u_3$.

$\Rightarrow \underset{\sim}{x} + \underset{\sim}{y} \in S$, and for any scalar λ,

$\therefore \lambda \underset{\sim}{x} = (\lambda x_1, \lambda x_2, \lambda x_3)$

$= (\alpha_1, \alpha_2, \alpha_3)$, where $\alpha_i = \lambda x_i$ and

$\alpha_1 + \alpha_2 = \lambda(x_1+x_2) = \lambda x_3$

$= \alpha_3$.

i.e. $\lambda \underset{\sim}{x} \in S_1$.

Hence S_1 is a vector subspace of E^3.

■ Example: 1) In each of the following find out whether the subsets given, form subspaces of the vector space V_2:

i) $S = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$

ii) $T = \{(x_1, x_2) : x_1 x_2 \geq 0\}$

Soln. →

i) For $\alpha < 0$, $\alpha \underset{\sim}{x} = (\alpha x_1, \alpha x_2)$, where $\alpha x_1 \leq 0, \alpha x_2 \leq 0$.
$\notin S$.

So, it's not a subspace.

ii) Consider the vectors $(3, 2)$ and $(-2, -4)$ from T.
Note that $(3, 2) + (-2, -4) = (1, -2)$ for which $x_1 x_2 < 0 \notin T$.

So, T is not a subspace.

e.g. 2) Consider a subset S of $V_2(R)$:
$S = \{(x, y) \mid 0 < x, y < \infty\} \subset V_2(R)$. Is S a subspace of $V_2(R)$?

Ans: → Consider any two vectors $\underset{\sim}{a}_1 = (a_{11}, a_{12})$ and $\underset{\sim}{a}_2 = (a_{21}, a_{22})$ from S.

Now, $\underset{\sim}{a}_1 + \underset{\sim}{a}_2 = (a_{11} + a_{21}, a_{12} + a_{22}) \in S$ because $a_{11} + a_{21}$ and $a_{21} + a_{22}$ are both positive, giving S is closed under addition.

But S is not closed under the operations of multiplication by a scalar λ, because for any vector $\underset{\sim}{a}_1 \in S$ and $\lambda \leq 0$, $\lambda \underset{\sim}{a}_1 = (\lambda a_{11}, \lambda a_{12}) \notin S$.

$\therefore$ S is not a subspace of $V_2(R)$.

e.g. 3) The subset $S = \{(x, y) \mid -\infty < x, y < \infty, x \neq 0, y \neq 0\}$ is not a subspace.

Ans: → For any vector, $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in S$ and $\lambda = 0$, $\lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin S$,

So, S is not closed under multiplication by a scalar, giving that S is not a subspace.

e.g. 4) The subset $S=\{(x,y) \mid y=a+bx, -\infty<x, y<\infty; a\neq 0, b\neq 0\}$ is not a subspace.

ANS:→ Consider any two vectors $\underset{\sim}{a_1}=(a_{11}, a_{12})$ and $\underset{\sim}{a_2}=(a_{21}, a_{22})$ from S.

Now, $\underset{\sim}{a_1}+\underset{\sim}{a_2}=(a_{11}+a_{21}, a_{12}+a_{22})$

$= (a_{11}+a_{21}, \underbrace{2a+b(a_{11}+a_{21})})$

$y=a+bx$
$a_{12}=a+ba_{11}$
$a_{22}=a+ba_{21}$

$\Rightarrow \underset{\sim}{a_1}+\underset{\sim}{a_2}\notin S.$

[this does not satisfy $y=a+bx$]

{Here, $x=(a_{11}+a_{21})$}

so, giving S is not closed under addition.
So, S is not a subspace.

Construction of a vector Subspace :

RESULT :→ Consider a set $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$ of vectors from V_n. Then show that the collection of all possible linear combinations of $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$ is a vector subspace of V_n.

Proof :→ The collection of all possible linear combinations of $\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}$ is

$$S=\left\{\sum_{i=1}^{n} l_i a_i : l_i \in R, i=1(1)n\right\}$$

Let, $\underset{\sim}{x}, \underset{\sim}{y}\in S$.

$\therefore \underset{\sim}{x}=\sum_{i=1}^{n} l_i \underset{\sim}{a_i}$ and $\underset{\sim}{y}=\sum_{i=1}^{n}\lambda_i \underset{\sim}{a_i}$, for some l_i's and λ_i's.

$$\therefore \underset{\sim}{x}+\underset{\sim}{y}=\sum_{i=1}^{n}(l_i+\lambda_i)\underset{\sim}{a_i}$$

$$=\sum_{i=1}^{n} l_i \underset{\sim}{a_i} \in S,$$

and $$\alpha\underset{\sim}{x}=\alpha\left(\sum_{i=1}^{n} l_i \underset{\sim}{a_i}\right)$$

$$=\sum_{i=1}^{n}(\alpha l_i)\underset{\sim}{a_i}$$

$$=\sum_{i=1}^{n} l_i \underset{\sim}{a_i} \in S.$$

Clearly, $S\subseteq V_n$.

Hence, S is a vector subspace of V_n.

Theorem 8. The intersection $S \cap T$ of any two subspaces of V is itself a subspace of V, although $S \cup T$ may not be the same.

Proof: Since S and T are both subspaces, each contains the zero vector. Hence the zero vector is in $S \cap T$, so that $S \cap T$ is non-empty. Again, let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be vectors in $S \cap T$ and let k be a scalar (in F).

By defn. of $S \cap T$, both $\underset{\sim}{a}$ and $\underset{\sim}{b}$ belongs to S as well as T, and because S as well as T is a subspace, $k\underset{\sim}{a} + \underset{\sim}{b} \in S$ and $k\underset{\sim}{a} + \underset{\sim}{b} \in T$. Hence, $k\underset{\sim}{a} + \underset{\sim}{b} \in S \cap T$. —(1)

Lemma: A non-empty subset S of a vector space V is a subspace of V iff for each pair of vectors $\underset{\sim}{a}, \underset{\sim}{b} \in S$ and each scalar $k \in F$, the vector $k\underset{\sim}{a} + \underset{\sim}{b} \in S$.

Proof: Suppose the condition is true. Then since S is non-empty, $\exists$ a vector $\underset{\sim}{c} \in S$ and hence $(-1)\underset{\sim}{c} + \underset{\sim}{c} = \underset{\sim}{0} \in S$. Then if $\underset{\sim}{a} \in S$ and the scalar $k \in F$, then the vector $k\underset{\sim}{a} = k\underset{\sim}{a} + \underset{\sim}{0} \in S$. In particular $(-1)\underset{\sim}{a} = -\underset{\sim}{a} \in S$. Finally, if $\underset{\sim}{a} \in S$, $b \in S$, then $\underset{\sim}{a} + \underset{\sim}{b} \in S$. Thus S is seen to be a subspace of V.

Conversely, if S is a subspace of V, $\underset{\sim}{a} \in S$ and $\underset{\sim}{b} \in S$ and the scalar $k \in F$, then $k\underset{\sim}{a} \in S$ and so, $k\underset{\sim}{a} + \underset{\sim}{b} \in S$. □

By this lemma (1) gives $S \cap T$ is a subspace.

It should be also be noted that the union of two subspaces S and T, denoted by $S \cup T$, and defined as the set of all vectors that belong to either S or T, may not be a subspace.

Spanning Set :

Definition : A set $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_r\}$ of vectors from a vector space V_n is said to be span or generate V_n if every vector in V_n can be written as a linear combination of $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_r$.

Then we say that $\{a_1, a_2, \ldots, a_r\}$ is a spanning set or generating set of the vector space V_n.

REMARK : The collection of all possible linear combinations of $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_r\}$ is $S = \left\{ \sum_{i=1}^{r} l_i \underset{\sim}{a}_i : l_i \in R \right\}$ which is a vector space.

The spanning set of S is $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_r\}$ or the span of $\{a_1, a_2, \ldots, a_r\}$ is $S = \left\{ \sum_{i=1}^{r} l_i \underset{\sim}{a}_i : l_i \in R \right\}$.

Example : For V_2, the unit vectors $\underset{\sim}{e}_1, \underset{\sim}{e}_2$ form a spanning set. Show that any two linearly independent vectors in V_2 form a spanning set of V_2. Also, show that — $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, (1,1)\}$ forms a spanning set of V_2.

Soln. → Let $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2\}$ be two linearly independent vectors from V_2.

If possible, let $\underset{\sim}{x} = l_1 \underset{\sim}{a}_1 + l_2 \underset{\sim}{a}_2$, where $\underset{\sim}{x} \in V_2$.

$$\underset{\sim}{x} = l_1 (a_{11}, a_{12}) + l_2 (a_{21}, a_{22})$$

$$\therefore (x_1, x_2) = (l_1 a_{11} + l_2 a_{21}, \; l_1 a_{12} + l_2 a_{22})$$

$$\Rightarrow \begin{cases} x_1 = l_1 a_{11} + l_2 a_{21} \\ x_2 = l_1 a_{12} + l_2 a_{22} \end{cases}$$

$$\Rightarrow \begin{cases} l_1 = \dfrac{x_1 a_{12} - x_2 a_{21}}{a_{11} a_{22} - a_{12} a_{21}} \\ l_2 = \dfrac{x_1 a_{12} - x_2 a_{11}}{a_{12} a_{21} - a_{11} a_{22}} \end{cases}$$

as $\underset{\sim}{a}_1 = (a_{11}, a_{12})$ and $\underset{\sim}{a}_2 = (a_{21}, a_{22})$ are linearly independent.

$$\Rightarrow \underset{\sim}{a}_2 \neq \lambda \underset{\sim}{a}_1$$

$$\Rightarrow \frac{a_{11}}{a_{21}} \neq \frac{a_{12}}{a_{22}}$$

$$\Rightarrow a_{11} a_{22} - a_{12} a_{21} \neq 0.$$

Hence, $\exists$ scalars l_1 and $l_2 \ni \underset{\sim}{x} = l_1 \underset{\sim}{a}_1 + l_2 \underset{\sim}{a}_2$

Hence, $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2\}$ spans V_2.

Again, for $\underset{\sim}{x} \in V_2$, $\underset{\sim}{x} = x_1 \underset{\sim}{e}_1 + x_2 \underset{\sim}{e}_2 + 0(1,1)$

$\Rightarrow \{e_1, e_2, (1,1)\}$ spans V_2.

Theorem 9. Any set of vectors which span V, a vector space and containing the smallest possible number of vectors must be linearly independent.

Proof :→ Let $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$ be a smallest spanning set of V.
If possible let $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$ is linearly dependent.
then, someone of them can be written as a linear combination of the others.
Let this one vector is $\underset{\sim}{a_n}$.

Then, $\underset{\sim}{a_n} = \sum_{i=1}^{n-1} \lambda_i \underset{\sim}{a_i}$ for some scalar λ_i.

Then, for any $\underset{\sim}{x} \in V$,

$$\underset{\sim}{x} = \sum_{i=1}^{n} l_i \underset{\sim}{a_i}$$

$$= \sum_{i=1}^{n-1} l_i \underset{\sim}{a_i} + l_n \underset{\sim}{a_n}$$

$$= \sum_{i=1}^{n-1} l_i \underset{\sim}{a_i} + l_n \left(\sum_{i=1}^{n-1} \lambda_i \underset{\sim}{a_i}\right)$$

$$= \sum_{i=1}^{n-1} (l_i + l_n \lambda_i) \underset{\sim}{a_i}$$

This implies $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_{n-1}}\}$ is a spanning set, which is a contradiction to the fact that $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$ is the smallest spanning set of V.

Hence, our assumption is not correct, i.e. $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$ must be linearly independent.

Remark :→ Converse of the theorem, a spanning set of vectors of V which is linearly independent, must be the minimal spanning.

Proof :→ Let $\{\underset{\sim}{a_1}, \ldots, \underset{\sim}{a_n}\}$ be a spanning and linearly independent set of vectors of V.
If possible, let the spanning set is not minimal.
Then, it is possible to have a set of vectors, say, $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_{n-1}}\}$ which is a spanning set of V.
Then $\underset{\sim}{a_n}$ can be written as a linear combination of $\{\underset{\sim}{a_1}, \ldots, \underset{\sim}{a_{n-1}}\}$.
$\Rightarrow \{\underset{\sim}{a_1}, \ldots, \underset{\sim}{a_{n-1}}, \underset{\sim}{a_n}\}$ is linearly dependent which is a contradiction, since $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$ is linearly independent.
Hence, a linearly independent spanning set of vectors is minimal spanning.

BASIS :-

(C.U) **Definition :-**

1) A linearly independent set of vectors that generates a subspace, is called a basis of the subspace.

2) A set of vectors $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_n\}$ is said to constitute a basis of a subspace S if —

i) $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_n\}$ spans S ;

ii) $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_n\}$ is a set of linearly independent vectors.

Ex:→ The unit vectors $\underset{\sim}{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \underset{\sim}{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \underset{\sim}{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ form a basis of $V_3(R)$ because these vectors are linearly independent and spans $V_3(R)$.

PROBLEMS:→

1) Do the vectors $\underset{\sim}{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \underset{\sim}{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \underset{\sim}{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ form a basis of

$$S = \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \middle| z=0, -\infty < x, y < \infty \right\} ?$$

Ans:→ No, because in $\underset{\sim}{e}_3$, $z \neq 0$, so, that defies the definition of spanning set.

2) Does $S = \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \middle| z=0, -\infty < x, y < \infty \right\}$ form a subspace of $V_3(R)$?

Ans:→ Considering any two vectors $\underset{\sim}{a}_1$ and $\underset{\sim}{a}_2$ from S.

$\underset{\sim}{a}_1 = (a_{11}, a_{12}, 0)$

$\underset{\sim}{a}_2 = (a_{21}, a_{22}, 0)$

Now, $\underset{\sim}{a}_1 + \underset{\sim}{a}_2 = (a_{11}+a_{21}, a_{12}+a_{22}, 0) \in S$.

because $a_{11}+a_{21} > -\infty$ and $a_{12}+a_{22} < \infty$.

So, S is closed under addition.

Now, for any scalar λ,

$$\lambda \underset{\sim}{a}_1 = (\lambda a_{11}, \lambda a_{12}, 0) \in S.$$

So, S is closed under multiplication by a scalar.

$\therefore$ S is a subspace of $V_3(R)$.

3) Show that $S = \{(x, y, z) : x+y+z=0 ;\ x, y, z \in \mathbb{R}\}$ is a subspace.

<u>Soln</u> → Consider a vector $(x, y, z) \in S$.

Here, $x+y+z=0$

$\Rightarrow \lambda(x+y+z)=0$

$\Rightarrow (\lambda x+\lambda y+\lambda z)=0$

$\therefore (\lambda x, \lambda y, \lambda z) \in S$.

i.e. S is closed under scalar multiplication.

Consider

$(x, y, z) \in S$ & $(u, v, w) \in S$

$\Rightarrow x+y+z=0$, $\Rightarrow u+v+w=0$

$\Rightarrow (x+y+z)=\underset{\sim}{0}$ & $\Rightarrow (u+v+w)=\underset{\sim}{0}$

i.e. $(x+u)+(y+v)+(z+w)=0$

$\Rightarrow (x+u, y+v, z+w) \in S$

$\therefore$ S is closed under vector addition.

Therefore S is a vector space. (<u>Ans</u>)

$\boxed{\text{C.U}}$ 4) Show that all the vectors $(\underset{\sim}{x_1}, \underset{\sim}{x_2}, \underset{\sim}{x_3})$ in a vector space V_3 which obey $x_1 - x_2 = 0$ form a subspace V and find a basis of this subspace.

OR

Show that, the set of vectors $S = \{(x_1, x_2, x_3) \mid x_1 - x_2 = 0,\ x_1, x_2, x_3 \in \mathbb{R}\}$ forms a subspace and find it's basis.

<u>Soln</u>.

i) Considering any two vectors $\underset{\sim}{a_1}$ and $\underset{\sim}{a_2}$ from S.

$\underset{\sim}{a_1} = (a_{11}, a_{12}, a_{13})$ &

$\underset{\sim}{a_2} = (a_{21}, a_{22}, a_{23})$

As $x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2$.

Here, $a_{11} = a_{12}$

$a_{21} = a_{22}$

Now, $\underset{\sim}{a}_1 + \underset{\sim}{a}_2 = (a_{11}+a_{21}, a_{12}+a_{22}, a_{13}+a_{23}) \in S$

$= (a_{11}+a_{21}, a_{11}+a_{21}, a_{13}+a_{23}) \in S.$

because $a_{11}+a_{21} \in R$ and $a_{13}+a_{23} \in R$.

So, S is closed under addition.

Now, for any scalar λ,

$\lambda \underset{\sim}{a}_1 = (\lambda a_{11}, \lambda a_{12}, \lambda a_{13}) \in S$.

So, S is closed under multiplication by a scalar.

$\therefore$ S forms a subspace.

ii)

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| x_1 - x_2 = 0, x_1, x_2, x_3 \in R \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_1 \\ x_3 \end{pmatrix} \middle| x_1, x_3 \in R \right\}$$

$$= \left\{ x_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \middle| x_1, x_3 \in R \right\}$$

$\Rightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ span S.

So, more over these two vectors are linearly independent.

So, they are the basis of S.

[NOTE:→ Here $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are not spanning set because e_1, e_2, e_3 are not in the subspace.]

5) The vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ form a basis of $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| x = y; x, y, z \in R \right\}$.

Does another basis of S exists?

<u>Ans</u>:→ $\left\{ \begin{pmatrix} a_{11} \\ a_{11} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ b_{31} \end{pmatrix} ; a_{11} \text{ and } b_{31} \in R \right\} \longrightarrow$ a basis.

If we put any another set then for this particular subspace the new set would be a linearly dependent set, which contradicts our definition of basis.

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C.U

Theorem 10. The set of unit vectors $\{\underset{\sim}{e_1}, \underset{\sim}{e_2}, \dots, \underset{\sim}{e_n}\}$ is a basis of $V_n(F)$.

Proof:→ The vectors $\underset{\sim}{e_1}, \underset{\sim}{e_2}, \dots, \underset{\sim}{e_n}$ are in $V_n(F)$ and any vector $\underset{\sim}{a} = (a_1, a_2, \dots, a_n)$ can be written as

$$\underset{\sim}{a} = a_1\underset{\sim}{e_1} + a_2\underset{\sim}{e_2} + \dots + a_n\underset{\sim}{e_n}$$

, a linear combination of n-unit vectors $\{\underset{\sim}{e_1}, \underset{\sim}{e_2}, \dots, \underset{\sim}{e_n}\}$. So, $\underset{\sim}{e_1}, \underset{\sim}{e_2}, \dots, \underset{\sim}{e_n}$ span or generate $V_n(F)$.

Further, these unit vectors $\underset{\sim}{e_1}, \dots, \underset{\sim}{e_n}$ are linearly independent because for some scalars λ_i,

$$\lambda_1\underset{\sim}{e_1} + \lambda_2\underset{\sim}{e_2} + \dots + \lambda_n\underset{\sim}{e_n} = \underset{\sim}{0}$$

$$\Rightarrow (\lambda_1, \lambda_2, \dots, \lambda_n) = (0, \dots, 0)$$

$$\Rightarrow \lambda_i = 0 \text{ for all } i = 1(1)n.$$

$\Rightarrow \underset{\sim}{e_1}, \underset{\sim}{e_2}, \dots, \underset{\sim}{e_n}$ are linearly independent.

So, $\{\underset{\sim}{e_1}, \underset{\sim}{e_2}, \dots, \underset{\sim}{e_n}\}$ being linearly independent generating set of $V_n(F)$, form a basis of $V_n(F)$.

C.U.

Theorem 11. A vector subspace has more than one basis.

Proof:→ Let $\underset{\sim}{a_1}, \underset{\sim}{a_2}, \dots, \underset{\sim}{a_k}$ are basis vectors of a subspace S. These implies for any scalar $\gamma \neq 0$,

To show, $\gamma\underset{\sim}{a_1}, \underset{\sim}{a_2}, \dots, \underset{\sim}{a_k}$ is also a basis of S, consider

$$\lambda_1\gamma\underset{\sim}{a_1} + \lambda_2\underset{\sim}{a_2} + \dots + \lambda_k\underset{\sim}{a_k} = \underset{\sim}{0}$$ where λ_i's are scalar, $i = 1(1)k$.

$$\Rightarrow \lambda_1\gamma = 0, \quad \lambda_i = 0 \text{ for all } i = 2(1)k.$$

As $\underset{\sim}{a_1}, \underset{\sim}{a_2}, \dots, \underset{\sim}{a_k}$ are linearly independent,

$$\Rightarrow \lambda_1 = 0, \quad \lambda_i = 0 \ \forall\ i = 2(1)k.$$

$\Rightarrow \gamma\underset{\sim}{a_1}, \underset{\sim}{a_2}, \dots, \underset{\sim}{a_k}$ are linearly independent. —— (1)

Further, let $\underset{\sim}{a}$ be any vector from S and hence, $\exists$ scalars α_i, such that

$\underset{\sim}{a} = \alpha_1\underset{\sim}{a_1} + \alpha_2\underset{\sim}{a_2} + \dots + \alpha_k\underset{\sim}{a_k}$, because $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \dots, \underset{\sim}{a_k}\}$ is a basis of S.

$$\Rightarrow \underset{\sim}{a} = \frac{\alpha_1}{\gamma}(\gamma\underset{\sim}{a_1}) + \alpha_2\underset{\sim}{a_2} + \dots + \alpha_k\underset{\sim}{a_k},$$

$\Rightarrow \underset{\sim}{a}$ is also a linear combination of $\{\gamma\underset{\sim}{a_1}, \underset{\sim}{a_2}, \dots, \underset{\sim}{a_k}\}$ —— (2)

Hence, From (1) & (2), we get —

$\{\gamma\underset{\sim}{a_1}, \underset{\sim}{a_2}, \dots, \underset{\sim}{a_k}\}$ is also a basis of S.

C.U

Theorem 12. Representation of a vector belonging to a subspace S, in terms of basis vectors of its, is unique.

Proof:→ Let $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_k\}$ be a given basis of a subspace S and $\underset{\sim}{a}$ be a vector belonging to S. Suppose $\underset{\sim}{a}$ can be written as a linear combination of $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_k\}$ in two ways as follows :—

$$\underset{\sim}{a} = \lambda_1 \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_2 + \cdots + \lambda_k \underset{\sim}{a}_k \quad \text{——— (1)}$$

and $$\underset{\sim}{a} = \mu_1 \underset{\sim}{a}_1 + \mu_2 \underset{\sim}{a}_2 + \cdots + \mu_k \underset{\sim}{a}_k \quad \text{——— (2)}$$

$$(1)-(2) \Rightarrow (\lambda_1 - \mu_1)\underset{\sim}{a}_1 + (\lambda_2 - \mu_2)\underset{\sim}{a}_2 + \cdots + (\lambda_k - \mu_k)\underset{\sim}{a}_k = \underset{\sim}{0}$$

$\Rightarrow \lambda_i - \mu_i = 0 \ \forall\ i = 1(1)k$, because $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_k$ being basis vectors are linearly independent.

$\Rightarrow \lambda_i = \mu_i \ \forall\ i = 1(1)k.$

Hence, the representation of any vector in terms of basis is unique.

Theorem 13. The representation of any vector in terms of an arbitrary set of spanning vectors is not unique.

Proof:→ Consider a spanning set $\{\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n\}$ of vectors of a vector space V_n. If it is linearly independent, then it will form a basis of V_n. Then the representation will be unique.

But if it is linearly dependent, the someone of them can be written as a linear combination of the others. Let, this one vector be $\underset{\sim}{a}_n$.

Then, $\underset{\sim}{a}_n = \sum_{i=1}^{n-1} \lambda_i \underset{\sim}{a}_i$, for some λ_i.

Again, for any $\underset{\sim}{x} \in V$,

$$\underset{\sim}{x} = \sum_{i=1}^{n} l_i \underset{\sim}{a}_i \text{, for some scalars } l_i. \quad \text{——— } (*)$$

and $$\underset{\sim}{x} = \sum_{i=1}^{n-1} l_i \underset{\sim}{a}_i + l_n \underset{\sim}{a}_n = \sum_{i=1}^{n-1} l_i \underset{\sim}{a}_i + l_n \left(\sum_{i=1}^{n-1} \lambda_i \underset{\sim}{a}_i \right)$$

$$\therefore \underset{\sim}{x} = \sum_{i=1}^{n-1} (l_i + l_n \lambda_i)\underset{\sim}{a}_i + 0.\underset{\sim}{a}_n \quad \text{——— } (**)$$

$(*)$ & $(**)$ shows $\Rightarrow$ the representation is not unique.

NOTE:→ How many choices are there of forming basis?

ANS. Infinite number of choices for the representation of any vector in terms of an arbitrary set of spanning vectors.

Theorem 14. (Change of Basis Technique)

If $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_p\}$ be a basis of V_n and $\underset{\sim}{y} \in V_n$ be $\ni$ $\underset{\sim}{y} = \sum_{i=1}^{p} \lambda_i \underset{\sim}{x}_i$, $\lambda_i \neq 0$. Show that $\{\underset{\sim}{y}, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_p\}$ will also be a basis of V_n.

Proof:→ Consider a vector $\underset{\sim}{z} \in V_n$. As $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_p\}$ is a basis of V_n, $\underset{\sim}{z}$ can be written as a linear combination of the basis vectors.

$\therefore \exists$ scalars θ_i, $i = 1(1)p$ such that

$$\underset{\sim}{z} = \sum_{i=1}^{p} \theta_i \underset{\sim}{x}_i \quad \cdots\cdots (1)$$

Given that, $\underset{\sim}{y} = \sum_{i=1}^{p} \lambda_i \underset{\sim}{x}_i$; $\lambda_i \neq 0$

$$\Rightarrow \underset{\sim}{x}_1 = \frac{1}{\lambda_1} \underset{\sim}{y} - \sum_{i=2}^{p} \frac{\lambda_i}{\lambda_1} x_i \quad [\lambda_1 \neq 0] \quad \cdots\cdots (2)$$

Combining (1) & (2) $\quad \underset{\sim}{z} = \frac{\theta_1}{\lambda_1} \underset{\sim}{y} + \sum_{i=2}^{p} \left(\theta_i - \frac{\theta_1}{\lambda_1} \lambda_i\right) \underset{\sim}{x}_i$

$\therefore \{\underset{\sim}{y}, \underset{\sim}{x}_2, \underset{\sim}{x}_3, \ldots\ldots, \underset{\sim}{x}_p\}$ spans V_n $\cdots\cdots (3)$

For the scalars $\alpha_1, \alpha_2, \ldots\ldots, \alpha_p$ consider

$$\alpha_1 y + \sum_{i=2}^{p} \alpha_i \underset{\sim}{x}_i = \underset{\sim}{0}$$

$$\text{i.e. } \alpha_1 \sum_{i=1}^{p} \lambda_i \underset{\sim}{x}_i + \sum_{i=2}^{p} \alpha_i \underset{\sim}{x}_i = \underset{\sim}{0}$$

$$\Rightarrow \alpha_1 \lambda_1 x_1 + \sum_{i=2}^{p} (\alpha_i + \alpha_1 \lambda_i) \underset{\sim}{x}_i = 0$$

$\Rightarrow \alpha_1 \lambda_1 = 0$, $\quad \alpha_i + \alpha_1 \lambda_i = 0 \quad \forall\, i = 2(1)p$.

as $x_1, x_2, \ldots\ldots, x_p$ are LIN vectors.

Now, $\lambda_1 \neq 0$, $\lambda_1 \alpha_1 = 0 \Rightarrow \alpha_1 = 0$

$\alpha_i + \alpha_1 \lambda_i = 0 \quad \forall\, i = 2(1)p$

$\Rightarrow \alpha_2 = \alpha_3 = \cdots\cdots = \alpha_p = 0$

$\therefore \quad \alpha_1 y + \sum_{i=2}^{p} \alpha_i x_i = \underset{\sim}{0} \Rightarrow \alpha_i = 0 \ \forall\, i$

Hence, $\{\underset{\sim}{y}, \underset{\sim}{x}_2, \underset{\sim}{x}_3, \ldots\ldots, \underset{\sim}{x}_p\}$ is a set of LIN vectors. $\cdots\cdots (4)$

(3) & (4) $\Rightarrow$ hence we get the result.

Remark: It is important to realize that "the choice of basis for a vector space is not unique".

Theorem 15. A set of n linearly independent n component vectors form a basis of E^n.

Proof:→ Let $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_n\}$ be a set of LIN n component vectors.

Note that $\underset{\sim}{x}_1 \neq \underset{\sim}{0} \in E^n$ [as it belongs to the set of LIN vectors]

Further assume that

$$\underset{\sim}{x}_1 = \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{pmatrix}, \quad x_{1i} \neq 0 \text{ for at least one } i, \ i = 1(1)n.$$

Clearly, $\underset{\sim}{x}_1 = x_{11}\underset{\sim}{e}_1 + x_{12}\underset{\sim}{e}_2 + \dots + x_{1n}\underset{\sim}{e}_n$

WLG, let $x_{11} \neq 0$, then $\{\underset{\sim}{x}_1, \underset{\sim}{e}_2, \dots, \underset{\sim}{e}_n\}$ will be a basis of E^n, as $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \dots, \underset{\sim}{e}_n\}$ is a basis of E^n.

Again, $\underset{\sim}{x}_2 \neq \underset{\sim}{0} \in E^n$, $\{\underset{\sim}{x}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3, \dots, \underset{\sim}{e}_n\}$ is a basis of E^n then $\exists$ scalars $\theta_1, \theta_2, \dots, \theta_n$, not all zero, such that

$$\underset{\sim}{x}_2 = \theta_1 \underset{\sim}{x}_1 + \sum_{i=2}^{n} \theta_i \underset{\sim}{e}_i .$$

if possible let, $\theta_1 \neq 0$ and $\theta_2 = \theta_3 = \dots = \theta_n = 0$

then $\underset{\sim}{x}_2 = \theta_1 \underset{\sim}{x}_1$ which is not possible as both of the vectors belongs to the set of LIN vectors.

So, there must be some non-zero θ_i's among $\theta_2, \theta_3, \dots, \theta_n$.

WLG, let $\theta_2 \neq 0$.

Hence, $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \underset{\sim}{e}_3, \underset{\sim}{e}_4, \dots, \underset{\sim}{e}_n\}$ will be a basis of E^n.

Proceeding in this way it can be shown that $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_n\}$ is a basis of E^n.

Corollary:—

If $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_n\}$ be a set of linearly independent n-component vectors then for any n-component vector, $\underset{\sim}{y} \neq$ $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_n, \underset{\sim}{y}\}$ must be a set of LD vectors.

C.U

Theorem 17. Number of vectors in a basis of a vector space is unique in the sense of that any two bases have the same number of vectors.

Proof:→ Consider a vector space V_n, if possible let $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_p\}$ and $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \dots, \underset{\sim}{y}_q\}$ be two bases of V_n. We are to show $p=q$.

Now, let us assume that $p \neq q$.

Let $p<q$.

Consider $\underset{\sim}{y}_1 (\neq 0) \in V_n$ as $\underset{\sim}{y}_1$ is a basis vector, must be non-null.

Now $\exists$ scalars $\theta_1, \theta_2, \dots, \theta_p$ with at least one $\theta_i \neq 0 \ni$ $(i=1(1)p)$.

$$\underset{\sim}{y}_1 = \sum_{i=1}^{p} \theta_i \underset{\sim}{x}_i \quad [\because \{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_p\} \text{ is a basis of } V_n]$$

WLG let $\theta_1 \neq 0$, $\{\underset{\sim}{y}_1, \underset{\sim}{x}_2, \underset{\sim}{x}_3, \dots, \underset{\sim}{x}_p\}$ will be a basis of V_n.

Consider $\underset{\sim}{y}_2 (\neq 0) \in V_n$.

Thus $\exists$ scalars $\lambda_1, \lambda_2, \dots, \lambda_p$ with at least one $\lambda_i \neq 0 \ \forall \ i=1(1)p$ such that

$$\underset{\sim}{y}_2 = \lambda_1 \underset{\sim}{y}_1 + \sum_{i=2}^{p} \lambda_i \underset{\sim}{x}_i \quad [\because \{\underset{\sim}{y}_1, \underset{\sim}{x}_2, \underset{\sim}{x}_3, \dots, \underset{\sim}{x}_p\} \text{ is a basis of } V_n]$$

If possible, let, $\lambda_1 \neq 0$, $\lambda_2 = \lambda_3 = \dots = \lambda_p = 0$, then

$\underset{\sim}{y}_2 = \lambda_1 \underset{\sim}{y}_1$, which is not true as both the vectors belong to the same basis.

There must be some non-zero λ_i among $\lambda_2, \lambda_3, \dots, \lambda_p$

WLG, let $\lambda_2 \neq 0$, then $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \underset{\sim}{x}_3, \dots, \underset{\sim}{x}_p\}$ will be a basis of V_n.

Proceeding in this way it can be shown that $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \dots, \underset{\sim}{y}_p\}$ will be a basis of V_n.

Now $\exists$ $\underset{\sim}{y}_{p+1} \in V_n$.

Hence it can be written as a linear combination of the vectors in the basis $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \dots, \underset{\sim}{y}_p\}$, i.e. $\underset{\sim}{y}_{p+1}$ is linearly dependent on $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \dots, \underset{\sim}{y}_p\}$ which is not possible as $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \dots, \underset{\sim}{y}_p, \underset{\sim}{y}_{p+1}\}$ belongs to the basis $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \dots, \underset{\sim}{y}_q\}$.

$$\therefore p \not< q.$$

Similar arguments lead to $p \not> q$, so,

$$p = q.$$

- **Corollary:**

(1) Every basis of $V_n(F)$ contains exactly n vectors.

Proof:→ We know that $\{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\}$ is a basis for V_n or E^n. As any two bases for a vector space have the same number of vectors, hence ~~a ba~~ every basis of $V_n(F)$ contains exactly n vectors.

(2) Any set of $(n+1)$ vectors from V_n or E^n is linearly dependent.

Proof:– Let $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_{n+1}\}$ be a set of $n+1$ vectors from E^n.
If possible, suppose $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_{n+1}\}$ is linearly independent. Then any subset of $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_{n+1}\}$, namely $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n\}$ containing n vectors is linearly independent.
Now, $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n\}$ forms a basis for E^n.
Then $\hat{a}_{n+1}$ can be expressed as a linear combination of $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n\}$ which contradicts with the fact that $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_{n+1}\}$ is linearly independent.
By contradiction, hence the proof.

(3) S.T. it is not possible to have $(n+1)$ mutually orthogonal vectors (non-zero) with n components.

Proof:→ Let $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_{n+1}\}$ is a set of $(n+1)$ vectors.
If possible, suppose that the vectors are mutually orthogonal.
But as we know mutually orthogonal vectors are necessarily LIN, so any subset of it containing n vectors will be linearly independent.

OR

* Any set of mutually orthogonal non-null vectors from E^n is LIN.
If possible, let the $(n+1)$ vectors are mutually orthogonal, and they will be LIN, which contradicts the fact that "any set containing the $(n+1)$ vectors from E^n is necessarily LD".
Hence the conclusion. *

Dimension :

Definition : 1) The number of vectors in a basis of a subspace is called the dimension of the subspace.

2) Dimension of a vector space V_n, denoted by $d(V_n)$ is defined as number of linearly independent vectors required to span V_n, i.e. number of vectors in any basis of V_n. clearly —

i) $d(E^n) = n$.

ii) $d(V_n) \leq n$.

iii) $d(V_n) \geq p$ if we have an arbitrary collection of p linearly independent vectors in V_n.

Example:

1) $\dim(V_2(R)) = 2$
because $\underset{\sim}{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}$ & $\underset{\sim}{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$ form a basis of $V_2(R)$.

2) Can you find the $\dim(S)$ where $S = \left\{\begin{pmatrix}x\\1\end{pmatrix} \middle| x \in \mathbb{R}\right\}$

ANS: $\begin{pmatrix}a_{11}\\1\end{pmatrix}, \begin{pmatrix}a_{12}\\1\end{pmatrix}$ As, $\begin{pmatrix}a_{11}\\1\end{pmatrix} + \begin{pmatrix}a_{12}\\1\end{pmatrix} = \begin{pmatrix}a_{11}+a_{12}\\2\end{pmatrix} \notin S$

Here S is not closed under addition and thus S is not a subspace, so $\dim(S)$ is not defined in this case.

3) $S = \left\{\begin{pmatrix}x\\x\end{pmatrix} \middle| x \in R\right\}$. Calculate $\dim(S)$?

Soln. → $\underset{\sim}{a} = \begin{pmatrix}a_1\\a_2\end{pmatrix}, \underset{\sim}{b} = \begin{pmatrix}b_1\\b_2\end{pmatrix} \in S$; $\underset{\sim}{a} + \underset{\sim}{b} = \begin{pmatrix}a_1+b_1\\a_2+b_2\end{pmatrix} \in S$,

$\lambda \underset{\sim}{a} = \begin{pmatrix}\lambda a_1\\\lambda a_2\end{pmatrix} \in S \ \forall \lambda \in R.$

So, S is a subspace.

$$S = \left\{\begin{pmatrix}x\\x\end{pmatrix} \middle| x \in R\right\} = \left\{x\begin{pmatrix}1\\1\end{pmatrix} \middle| x \in R\right\}.$$

So, $\begin{pmatrix}1\\1\end{pmatrix}$ forms a basis of S, giving $\dim(S) = 1$.

4) $S_1 = \left\{\begin{pmatrix}x\\0\end{pmatrix} \middle| x \in R\right\}$ and $S_2 = \left\{\begin{pmatrix}0\\y\end{pmatrix} \middle| y \in R\right\}$ are two subspace of $V_2(R)$. Find $\dim(S_1+S_2)$?

Soln. → $S_1 + S_2 = \left\{\begin{pmatrix}x\\0\end{pmatrix} + \begin{pmatrix}0\\y\end{pmatrix} \middle| x, y \in R\right\}$

$= \left\{\begin{pmatrix}x\\y\end{pmatrix} \middle| x, y \in R\right\} = V_2(R)$

Subspace	Dimension
S_1 =	1
S_2 =	1
S_1+S_2 =	2

5) $S_1 = \left\{ \begin{pmatrix} x \\ x \end{pmatrix} \middle| x \in R \right\}$ and $S_2 = \left\{ \begin{pmatrix} y \\ -y \end{pmatrix} \middle| y \in R \right\}$ are too subspaces of $V_2(R)$. Find $\dim(S_1+S_2)$?

Soln. →

$$S_1+S_2 = \left\{ \begin{pmatrix} x \\ x \end{pmatrix} + \begin{pmatrix} y \\ -y \end{pmatrix} \middle| x, y \in R \right\}$$

$$= \left\{ \begin{pmatrix} x+y \\ x-y \end{pmatrix} \middle| x, y \in R \right\}$$

$$= \left\{ (x+y)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (x-y)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \middle| x, y \in R \right\}$$

$\Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ form a basis of S_1+S_2.

$\boxed{\text{C.U.}}$ $\therefore \dim(S_1+S_2) = 2.$

$\underset{\wedge}{a} = \begin{pmatrix} a_1+a_2 \\ a_1-a_2 \end{pmatrix}, \underset{\wedge}{b} = \begin{pmatrix} b_1+b_2 \\ b_1-b_2 \end{pmatrix}$

$\underset{\wedge}{a} + \underset{\wedge}{b} = \begin{pmatrix} a_1+b_1 + b_1+b_2 \\ a_1+b_1-(a_2+b_2) \end{pmatrix}$

$\in (S_1+S_2)$

$\therefore \lambda \underset{\wedge}{a} = \lambda \begin{pmatrix} a_1+a_2 \\ a_1-a_2 \end{pmatrix} \in S_1$

$\therefore S_1+S_1$ is also a subspace.

6) Consider the vectors $\underset{\wedge}{x_1} = (1, 3, 2)$ and $x_2 = (-2, 4, 3)$ in V_3 or F^3. S.T. span of $\{\underset{\wedge}{x_1}, \underset{\wedge}{x_2}\}$ is given by $S = \{(\xi_1, \xi_2, \xi_3) : \xi_1 - 7\xi_2 + 10\xi_3 = 0\}$

Soln. → span of $\{\underset{\sim}{x_1}, \underset{\sim}{x_2}\}$ is

$$S = \{ l_1 x_1, l_2 x_2 : l_1, l_2 \in R \}$$

$$= \{ l_1(1,3,2), l_2(-2,4,3) : l_1, l_2 \in R \}$$

$$= \{ (l_1 - 2l_2, 3l_1 + 4l_2, 2l_1 + 3l_2) : l_1, l_2 \in R \}$$

Let, $\xi \in S$, then $S = \{ (\xi_1, \xi_2, \xi_3) : \xi_1, \xi_2, \xi_3 \in R \}$

$\xi_1 = l_1 - 2l_2$, $\xi_2 = 3l_1 + 4l_2$, $\xi_3 = 2l_1 + 3l_2$

$$\xi_1 - 7\xi_2 + 10\xi_3 = l_1 - 2l_2 - 21l_1 - 28l_2 + 20l_1 + 30l_2$$

$$\therefore \xi_1 - 7\xi_2 + 10\xi_3 = 0.$$

$$S = \{ (\xi_1, \xi_2, \xi_3) : \xi_1 - 7\xi_2 + 10\xi_3 = 0 \}$$

Theorem 18. If S_n and V_n be two disjoint vector space, i.e. $S_n \cap V_n = \{\underset{\sim}{0}\}$, then $\dim(S_n + V_n) = \dim(S_n) + \dim(V_n)$.

Proof: Let $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_N\}$ be a basis of S_n and $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots, \underset{\sim}{y}_s\}$ be a basis of V_n. In this case it is enough to show that $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_N, \underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots, \underset{\sim}{y}_s\}$ is a basis of $S_n + V_n$, in that case we have

$$\dim(S_n + V_n) = \dim(S_n) + \dim(V_n) = N + S.$$

Consider a vector $\underset{\sim}{z} \in (S_n + V_n)$.

$$\underset{\sim}{z} = \underset{\sim}{z}_1 + \underset{\sim}{z}_2$$

where $\underset{\sim}{z}_1 \in S_n$, $\underset{\sim}{z}_2 \in V_n$

Clearly, $\underset{\sim}{z}_1$ & $\underset{\sim}{z}_2$ be respectively written as,

$\underset{\sim}{z}_1 = \sum_{i=1}^{N} \theta_i \underset{\sim}{x}_i$ since $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_N\}$ spans S_n,

&

$\underset{\sim}{z}_2 = \sum_{i=1}^{S} \lambda_i \underset{\sim}{y}_i$ $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots, \underset{\sim}{y}_s\}$ spans V_n.

Thus, $\underset{\sim}{z} = \sum_{i=1}^{N} \theta_i \underset{\sim}{x}_i + \sum_{i=1}^{S} \lambda_i \underset{\sim}{y}_i$

Hence, $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_N, \underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots, \underset{\sim}{y}_s\}$ spans $(S_n + V_n)$. ——— (1)

Now for scalars α_i's and β_i's consider

$$\sum_{i=1}^{N} \alpha_i \underset{\sim}{x}_i + \sum_{i=1}^{S} \beta_i \underset{\sim}{y}_i = \underset{\sim}{0}$$

$$\Rightarrow \sum_{i=1}^{N} \alpha_i \underset{\sim}{x}_i = -\sum_{i=1}^{S} \beta_i \underset{\sim}{y}_i \quad \text{——— (2)}$$

Note that $\rightarrow \sum_{i=1}^{S} \beta_i \underset{\sim}{y}_i \in V_n$.

$\Rightarrow$ R.H.S of (2) belongs to V_n

$\Rightarrow$ L.H.S. of (2) belongs to V_n

Again L.H.S of (2) belongs to S_n

So, L.H.S. of (2) belongs to $V_n \cap S_n$.

Now, we have, $V_n \cap S_n = \{\underset{\sim}{0}\}$

$\therefore \sum_{i=1}^{N} \alpha_i \underset{\sim}{x}_i = \underset{\sim}{0}$ ------ (3)

$\Rightarrow \alpha_i = 0 \ \forall \ i = 1(1)N$ as x_i's are LIN.

(2) & (3) implies $\Rightarrow \sum_{i=1}^{S} \beta_i \underset{\sim}{y}_i = \underset{\sim}{0} \Rightarrow \beta_i = 0 \ \forall \ i = 1(1)s$ as y_i's are LIN.

$\therefore \sum_{i=1}^{N} \alpha_i \underset{\sim}{x}_i + \sum_{i=1}^{S} \beta_i \underset{\sim}{y}_i = 0 \Rightarrow \alpha_i = 0$ & $\beta_i = 0 \ \forall \ i$

$\therefore \{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_N, \underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots, \underset{\sim}{y}_s\}$ is a set of LIN vectors. ——— (4)

(1) & (4) $\Rightarrow \dim(S_n + V_n) = \dim(S_n) + \dim(V_n)$.

Result: For two subspaces W_1 & W_2 of V_n, S.T.

i) $W_1 \cap W_2$ is also a subspace of V_n,

ii) $\dim(W_1 \cap W_2) \leq \min\{\dim(W_1), \dim(W_2)\}$.

Proof:

i) Let, $\underset{\sim}{a}, \underset{\sim}{b} \in W_1 \cap W_2$

$\Rightarrow \underset{\sim}{a}, \underset{\sim}{b} \in W_i$ for $i=1,2$.

$\Rightarrow \underset{\sim}{a}, \underset{\sim}{b} \in W_1$.

So, $\underset{\sim}{a}+\underset{\sim}{b} \in W_1$ and $\underset{\sim}{a}+\underset{\sim}{b} \in W_2$ as W_1 & W_2 being two subspaces closed under addition. — ①

$\Rightarrow \underset{\sim}{a}+\underset{\sim}{b} \in W_1 \cap W_2$.

For any scalar $\lambda \in F$,

$\lambda \underset{\sim}{a} \in W_1$ and $\lambda \underset{\sim}{b} \in W_2$.

$\therefore \lambda \underset{\sim}{a} \in W_1 \cap W_2$.

$\therefore W_1 \cap W_2$ is also closed under the property of multiplication. — ②

① & ② $\Rightarrow$ So, $W_1 \cap W_2$ is also a subspace of V_n.

Example:

$$W_1 = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \middle| x, y \in R \right\},$$

$$W_2 = \left\{ \begin{pmatrix} x \\ x \\ z \end{pmatrix} \middle| x, z \in R \right\}$$

$$\therefore W_1 \cap W_2 = \left\{ \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} \middle| x \in R \right\}$$

is also a subspace.

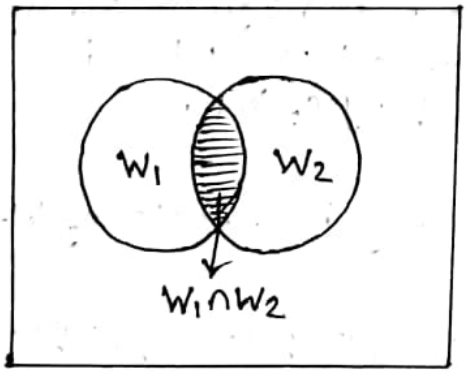
$\therefore \dim(W_1) = 2$,

$\therefore \dim(W_2) = 2$,

$\dim(W_1 \cap W_2) = 1$

$\therefore \dim(W_1 \cap W_2) \leq \min\{\dim(W_1), \dim(W_2)\}$

ii)



$V_n(F)$

Clearly, the maximum no. of LIN vectors in $(W_1 \cap W_2)$ $\leq$ the maximum no. of LIN vectors in W_1 $[\because W_1 \cap W_2 \subseteq W_1]$

$\Rightarrow \dim(W_1 \cap W_2) \leq \dim(W_1)$

Similarly, $W_1 \cap W_2 \subseteq W_2$

$\Rightarrow \dim(W_1 \cap W_2) \leq \dim(W_2)$

$\therefore \dim(W_1 \cap W_2) \leq \min\{\dim(W_1), \dim(W_2)\}$

Remark: 1. $\dim(W_1 \cap W_2) \leq \sqrt{\dim(W_1)\dim(W_2)}$.

2. Let V be a vector space over a field F and W be a subspace of V. Then $\dim(V/W) = \dim V - \dim W$.

Result: Prove that $S_n' \cap S_n''$ is also a vector subspace of V_n.

Let $S_2' = \{\lambda \underset{\sim}{e_1}\}$ and $S_2'' = V_2$. What is $S_2' \cap S_2''$?

Proof: Let $\hat{x}, \hat{y} \in S_n' \cap S_n''$

Then $\underset{\sim}{x} \in S_n'$ and $\hat{x} \in S_n''$; $\hat{y} \in S_n'$ and $y \in S_n''$

$\therefore \hat{x} + \hat{y} \in S_n'$ as S_n' is closed under vector addition.

$\therefore \hat{x} + y \in S_n' \cap S_n''$

For any λ, $\lambda \in R$, $\lambda \hat{x} \in S_n'$ and $\lambda y \in S_n''$ as S_n' and S_n'' are closed under scalar multiplication.

$\therefore \lambda \hat{x} \in S_n' \cap S_n'' \quad \forall \lambda$.

Hence, $S_n' \cap S_n''$ is a vector subspace of V_n.

$$V_2 = \{\alpha \hat{e}_1 + \lambda \underset{\sim}{e_2} : \alpha, \lambda \in R\} = S_2''$$

$$S_2' = \{\alpha \hat{e}_1\}$$

$$\therefore S_2' \cap S_2'' = S_2' \cap V_2 = \{\alpha \hat{e}_1\} = S_2'.$$

Theorem 19. For two subspaces W_1 and W_2 of $V_n(F)$

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

Proof: Let $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_p\}$ be a basis of $W_1 \cap W_2$.

As this set is LIN & contains vectors from W_1 and W_2, It can be extended so as to constitute a basis

$S_1 = \{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_p, \hat{b}_1, \hat{b}_2, \ldots, \hat{b}_q\}$ of W_1.

For similar reason

$S_2 = \{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_p, \hat{c}_1, \hat{c}_2, \ldots, \hat{c}_n\}$ is a basis of W_2, contains vectors a_i's $\forall i = 1(1)p$.

We now prove that $S = \{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_p, \hat{b}_1, \ldots, \hat{b}_q, \hat{c}_1, \ldots, \hat{c}_n\}$ is a basis of $W_1 + W_2$.

Let $\underset{\sim}{d} \in W_1 + W_2$

$\Rightarrow \hat{d} = \hat{d}_1 + \hat{d}_2$ where $\hat{d}_1 \in W_1$ & $\underset{\sim}{d}_2 \in W_2$

$$\Rightarrow \underset{\sim}{d} = \left[\sum_{i=1}^{p} \lambda_{1i} \hat{a}_i + \sum_{i=1}^{q} \lambda_{2i} \hat{b}_i\right] + \left[\sum_{i=1}^{p} \gamma_{1i} \hat{a}_i + \sum_{i=1}^{n} \gamma_{2i} c_i\right]$$

(for some scalar λ and γ)

$$= \sum_{i=1}^{p} (\lambda_{1i} + \gamma_{1i}) \underset{\sim}{a_i} + \sum_{i=1}^{q} \lambda_{2i} \hat{b}_i + \sum_{i=1}^{n} \gamma_{2i} \underset{\sim}{c_i}$$

$\Rightarrow$ S is a spanning set of $(W_1 + W_2)$.

To show that S is LIN, we consider the equation $\longrightarrow$

$$\sum_{i=1}^{p} k_{1i}\underset{\sim}{a_i} + \sum_{i=1}^{q} k_{2i}\underset{\sim}{b_i} + \sum_{i=1}^{r} k_{3i}\underset{\sim}{c_i} = 0 \qquad (\text{k's are scalars})$$

$$\Rightarrow \sum_{i=1}^{p} k_{1i}\underset{\sim}{a_i} + \sum_{i=1}^{q} k_{2i}\underset{\sim}{b_i} = -\sum_{i=1}^{r} k_{3i}\underset{\sim}{c_i} \quad \text{——} (1)$$

(1) shows that $-\sum_{i=1}^{r} k_{3i}\underset{\sim}{c_i} \in W_1$,

as it is a Linear combination of the basis vector of W_1, more over being a linear combination of the basis vectors of W_2,

$$\sum_{i=1}^{r} k_{3i}\underset{\sim}{c_i} \in W_2 .$$

$\therefore \sum_{i=1}^{r} k_{3i}\underset{\sim}{c_i} \in (W_1 \cap W_2)$, and hence

$-\sum_{i=1}^{r} k_{3i}\underset{\sim}{c_i} = \sum_{i=1}^{p} \delta_i \underset{\sim}{a_i}$ for some scalars δ_i, as $\{a_1, a_2, \ldots, a_p\}$ is a basis of W_1.

$$\Rightarrow \sum_{i=1}^{p} \delta_i \underset{\sim}{a_i} + \sum_{i=1}^{r} k_{3i}\underset{\sim}{c_i} = 0$$

$\Rightarrow \delta_i = 0$ and $k_{3i} = 0$ $\forall i$, because the set $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_p}, \underset{\sim}{c_1}, \ldots, \underset{\sim}{c_r}\}$ is LIN.

Thus, $\sum_{i=1}^{p} k_{1i}\underset{\sim}{a_i} + \sum_{i=1}^{q} k_{2i}\underset{\sim}{b_i} = 0$

$\Rightarrow \begin{cases} k_{1i} = 0 \ \forall\ i = 1(1)p, \\ k_{2i} = 0 \ \forall\ i = 1(1)q, \end{cases}$ because $\{\underset{\sim}{a_1}, a_2, \ldots, a_p, \underset{\sim}{b_1}, \ldots, \underset{\sim}{b_q}\}$ is also LIN set of vectors.

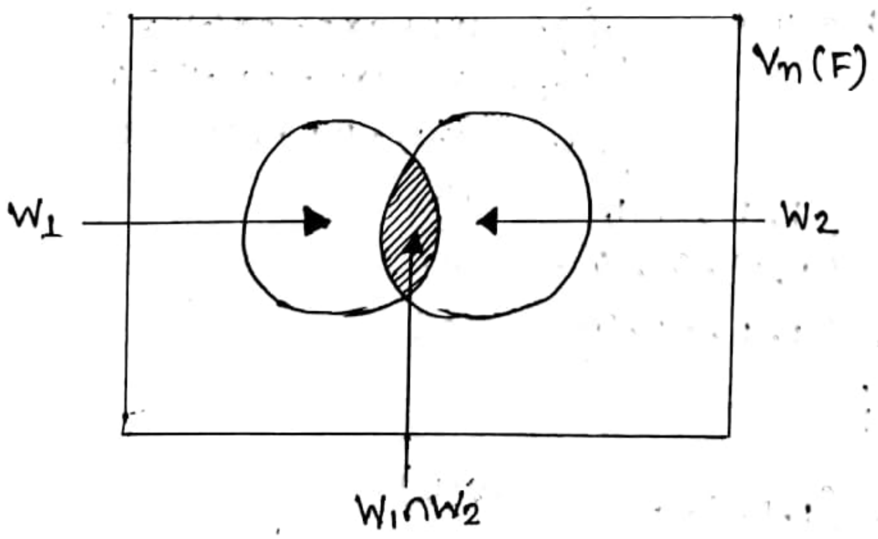
So, the set S containing all the $(p+q+r)$ vectors is LIN.

$\therefore$ S is a basis of $W_1 + W_2$.

Now, $\dim(W_1 + W_2) = p + q + r$

$$= (p+q) + (p+r) - p$$

$$= \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$



Orthogonal Vectors :

Definition :—

1) Two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are said to be orthogonal if $\underset{\sim}{a}'\underset{\sim}{b} = 0$. We generally use the notation $a \perp b$ to mean that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are orthogonal. Clearly, the null vector $\underset{\sim}{0}$ is orthogonal to every vector.

2) A set of vectors $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_n\}$ from E^n is said to be mutually orthogonal if $\underset{\sim}{a}_i{}'\underset{\sim}{a}_j = 0 \; \forall \; i \neq j$.

NOTE: It is noted that the unit vectors are orthogonal since $\underset{\sim}{e}_i{}'\underset{\sim}{e}_j = 0 \; \forall \; i \neq j$.

Let θ be the angle between the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, then

$$\cos\theta = \frac{\underset{\sim}{a}'\underset{\sim}{b}}{\|\underset{\sim}{a}\|\|\underset{\sim}{b}\|}, \text{ where, } \|\underset{\sim}{a}\| = \sqrt{\sum_{i=1}^{n} a_i^2}, \quad \|\underset{\sim}{b}\| = \sqrt{\sum_{i=1}^{n} b_i^2}$$

Now, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are said to be orthogonal then, $\underset{\sim}{a}'\underset{\sim}{b} = 0$

i.e. $\cos\theta = 0$

i.e. $\theta = \frac{\pi}{2}$

i.e. vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.

Example:—

(a) $\underset{\sim}{a} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ are orthogonal vectors.

(b) The set containing $\underset{\sim}{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\underset{\sim}{a}_2 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ and $\underset{\sim}{a}_3 = \begin{pmatrix} -5 \\ 1 \\ 2 \end{pmatrix}$ is orthogonal because

$$\underset{\sim}{a}_1{}'\underset{\sim}{a}_2 = 1\times 0 + 1\times 2 + 2\times(-1) = 0$$
$$\underset{\sim}{a}_2{}'\underset{\sim}{a}_3 = 0 + 1\times 2 + 2\times(-1) = 0$$
$$\underset{\sim}{a}_1{}'\underset{\sim}{a}_3 = 1\times(-5) + 1\times 1 + 2\times 2 = 0$$

(c)
$$\underset{\sim}{a}_1 = (1, 1, 1, 1, \ldots, 1)$$
$$\underset{\sim}{a}_2 = (1, -1, 0, 0, \ldots, 0)$$
$$\underset{\sim}{a}_3 = (1, 1, -2, 0, \ldots, 0)$$
$$\vdots$$
$$\underset{\sim}{a}_n = (1, 1, 1, \ldots \; -(n-1))$$

is a set of orthogonal vectors.

(d) Let W_1 be the subspace generated by the vectors $\begin{pmatrix}1\\1\\2\end{pmatrix}$ and $\begin{pmatrix}0\\2\\-1\end{pmatrix}$ and W_2, the subspace generated by $\begin{pmatrix}-5\\1\\2\end{pmatrix}$ are orthogonal.

$$W_1 = \left\{ \lambda_1 \begin{pmatrix}1\\1\\2\end{pmatrix} + \lambda_2 \begin{pmatrix}0\\2\\-1\end{pmatrix} : \lambda_1, \lambda_2 \in R \right\}$$

$$W_2 = \left\{ \lambda \begin{pmatrix}-5\\1\\2\end{pmatrix} : \lambda \in R \right\}$$

For, $\underset{\sim}{a} = \lambda_1 \begin{pmatrix}1\\1\\2\end{pmatrix} + \lambda_2 \begin{pmatrix}0\\2\\-1\end{pmatrix}$ and $\underset{\sim}{b} = \lambda \begin{pmatrix}-5\\1\\2\end{pmatrix}$,

$$\underset{\sim}{a}' \underset{\sim}{b} = 0.$$

Orthonormal Vectors :

Definition: — $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \dots, \underset{\sim}{a_n}\}$ is said to be a set of orthonormal vectors if

$$\underset{\sim}{a_i}' \underset{\sim}{a_j} = 0 \quad \forall \; i \neq j$$
$$\underset{\sim}{a_i}' \underset{\sim}{a_j} = 1 \quad \forall \; i = j \rightarrow \text{(with unit length)}$$

If we define a set of vectors $\{\underset{\sim}{b_1}, \underset{\sim}{b_2}, \dots, \underset{\sim}{b_n}\}$ and $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \dots, \underset{\sim}{a_n}\}$, form a set of mutually orthogonal vectors, such that

$$\underset{\sim}{b_i} = \frac{\underset{\sim}{a_i}}{\|\underset{\sim}{a_i}\|} \quad \forall \; i$$

then $\{\underset{\sim}{b_i}\}$ will be a set of orthonormal vectors.

Orthogonal Basis :

- **Definition:** A basis of a subspace of E^n is called orthogonal if the basis vectors are orthogonal. A basis is called orthonormal if the basis vectors are of unit length in addition to orthogonal.

Example: — The vectors $\begin{pmatrix}1\\1\end{pmatrix}$ & $\begin{pmatrix}2\\-2\end{pmatrix}$ form an orthogonal basis of E^2 because $\begin{pmatrix}1\\1\end{pmatrix}'\begin{pmatrix}2\\-2\end{pmatrix} = 0$.

The vectors $\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$ & $\begin{pmatrix}2/2\sqrt{2}\\-2/2\sqrt{2}\end{pmatrix}$ being of unit length form an orthonormal basis of E^2.

In particular, $\{\underset{\sim}{e_1}, \underset{\sim}{e_2}, \dots, \underset{\sim}{e_n}\}$ forms an orthonormal basis for E^n.

Theorem 20. Any set of orthogonal vectors not containing the null vector is linearly independent.

Proof:→ Let $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n\}$ be a set of mutually orthogonal non-null vectors from E^n. To show that these vectors are LIN, we consider the equation —

$$\lambda_1 \hat{a}_1 + \lambda_2 \hat{a}_2 + \cdots + \lambda_n \hat{a}_n = \underset{\sim}{0} \quad \text{——— (1)}$$

By considering the scalar products of $\underset{\sim}{a}_i$ and the equation (1), $i = 1(1)n$, we get —

$$\lambda_1 \hat{a}_1' \hat{a}_1 + \lambda_2 \underset{\sim}{a}_1' \underset{\sim}{a}_2 + \cdots + \lambda_n \underset{\sim}{a}_1' \underset{\sim}{a}_n = 0$$

$$\lambda_1 \hat{a}_2' \hat{a}_1 + \lambda_2 \hat{a}_2' \hat{a}_2 + \cdots + \lambda_n \underset{\sim}{a}_2' a_n = 0$$

$$\vdots$$

$$\lambda_1 \hat{a}_n' \hat{a}_1 + \lambda_2 \hat{a}_n' \hat{a}_2 + \cdots + \lambda_n \hat{a}_n' a_n = 0$$

$$\lambda_1 \hat{a}_1' \hat{a}_1 + \lambda_2 \times 0 + \cdots + \lambda_n \times 0 = 0$$

$$\Rightarrow \quad \lambda_1 \times 0 + \lambda_2 \underset{\sim}{a}_2' \underset{\sim}{a}_2 + \cdots + \lambda_n \times 0 = 0$$

$$\vdots$$

$$\lambda_1 \times 0 + \lambda_2 \times 0 + \cdots + \lambda_n \underset{\sim}{a}_n' \underset{\sim}{a}_n = 0$$

$\Rightarrow \quad \lambda_1 = 0, \lambda_2 = 0, \ldots, \lambda_n = 0$ as a_i's are non-null vectors, gives $\underset{\sim}{a}_i' \underset{\sim}{a}_i \neq 0$, $i = 1(1)n$.

$\Rightarrow$ The set of vectors $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n\}$ is LIN.

Theorem 21. Any set of n mutually orthogonal, non-null vectors from E^n forms a basis for E^n.

Proof:→ Let $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3, \ldots, \underset{\sim}{v}_n\}$ be a set of non-null orthogonal vectors from E^n.

Consider the equation

$$\lambda_1 \underset{\sim}{v}_1 + \lambda_2 \underset{\sim}{v}_2 + \cdots + \lambda_n \underset{\sim}{v}_n = \underset{\sim}{0}$$

Note that

$$\hat{v}_i (\lambda_1 \underset{\sim}{v}_1 + \lambda_2 \underset{\sim}{v}_2 + \cdots + \lambda_n \underset{\sim}{v}_n) = \underset{\sim}{v}_i \cdot \underset{\sim}{0}$$

$$\Rightarrow \lambda_1 (\underset{\sim}{v}_i \cdot \underset{\sim}{v}_1) + \cdots + \lambda_i (\underset{\sim}{v}_i' \underset{\sim}{v}_i) + \cdots + \lambda_n (\underset{\sim}{v}_i' \underset{\sim}{v}_n) = \underset{\sim}{0}$$

$$\Rightarrow \lambda_i |\underset{\sim}{v}_i|^2 = 0 \quad \forall\ i = 1(1)n$$

$\Rightarrow \lambda_i = 0$, since $|v_i| \neq 0\ \forall\ i = 1(1)n$, i.e. v_i's are non-null.

Hence the set $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \ldots, \underset{\sim}{v}_n\}$ is LIN.

We know any set of n LIN set of vectors forms a basis for E^n. Hence, $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \ldots, \underset{\sim}{v}_n\}$ forms a basis of E^n.

Result: — If $\{x_1, x_2, \dots, x_{n-1}\}$ be a set of mutually orthogonal n-component vectors & if $\exists$ two non-null vectors y and z such that each of y and z is orthogonal to any vector belonging to the given set. Then y must be a scalar multiple of z.

Soln → Note that $\{x_1, x_2, \dots, x_{n-1}, z\}$ is an orthogonal basis of E^n. Now, $y (\neq 0) \in E^n$

$\because \exists$ scalars $\theta_1, \theta_2, \dots, \theta_n$, at least one $\theta_i \neq 0 \ni$

$$y = \sum_{i=1}^{n-1} \theta_i x_i + \theta_n z$$

$$\Rightarrow y' x_1 = \sum_{i=1}^{n-1} \theta_i x_i' x_1 + \theta_n z' x_1$$

$\because \theta_1 x_1' x_1 = 0$

$\therefore \theta_1 = 0$ as $x_1' x_1 \neq 0$

Similarly it can be shown that $\theta_2 = \theta_3 = \dots = \theta_{n-1} = 0$.

Thus $y = \theta_n z$ $[\theta_n \neq 0]$.

Orthogonal Basis : —

Definition: — 1) If we consider a set of n mutually orthogonal non-null vectors from E^n then it forms a basis for E^n. Hence, a set of n mutually orthogonal non-null vectors from E^n is known as an Orthogonal Basis for E^n.

Orthonormal Basis : —

If a_i is a non-null vector, then

$$v_i = \frac{a_i}{|a_i|}$$ is a vector of unit length.

For a set of $\{a_1, a_2, \dots, a_n\}$ of mutually orthogonal non-null vectors from E^n, we obtain a set $u_i = \frac{a_i}{|a_i|}$, $i = 1(1)n$.

Then,
$$u_i \cdot v_j = \frac{a_i \cdot a_j}{|a_i||a_j|}$$

$$= \begin{cases} 0 & \forall\ i \neq j \\ 1 & \forall\ i = j \end{cases}$$

The set $\{a_1, \dots, a_n\}$ of mutually orthogonal vectors of unit length and forms a basis for E^n. This type of set of vectors from E^n is known as an orthonormal basis for E^n.

GRAM-SCHMIDT ORTHOGONALISATION PROCESS :

~~By this process a set of independent vectors from E^n can be converted to an~~

Any set of n given linearly independent vectors from E^n can be converted into an orthonormal basis by a procedure known as the Schmidt orthogonalization process. Let us suppose that $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \cdots, \underset{\sim}{a}_n$ are n LIN vectors from E^n. We select any vector from this set, for example, $\underset{\sim}{a}_1$. This vectors are non-null vectors i.e. $\underset{\sim}{a}_i \neq 0$, $i = 1(1)n$.

Let us define the vector of unit length $\underset{\sim}{u}_1$ as

$$\underset{\sim}{u}_1 = \frac{\underset{\sim}{a}_1}{|\underset{\sim}{a}_1|}.$$

To obtain a vector $\underset{\sim}{v}_2$ orthogonal to $\underset{\sim}{u}_1$, we subtract from $\underset{\sim}{a}_2$ a scalar multiple of $\underset{\sim}{u}_1$; i.e. $\underset{\sim}{v}_2$ is expressed as

$$\underset{\sim}{v}_2 = \underset{\sim}{a}_2 - \lambda_1 \underset{\sim}{u}_1,$$

where λ_1 is determined so that $\underset{\sim}{u}_1' \underset{\sim}{v}_2 = 0$.

$$\text{or, } \underset{\sim}{u}_1'(\underset{\sim}{a}_2 - \lambda_1 \underset{\sim}{u}_1) = 0$$

$$\text{or, } \underset{\sim}{u}_1' \underset{\sim}{a}_2 - \lambda_1 = 0$$

$$\text{or, } \lambda_1 = \underset{\sim}{u}_1' \underset{\sim}{a}_2$$

Therefore, $v_2 = a_2 - (\underset{\sim}{u}_1' \underset{\sim}{a}_2)\underset{\sim}{u}_1$.

A second unit length vector orthogonal to $\underset{\sim}{u}_1$ is defined by

$$\underset{\sim}{u}_2 = \frac{\underset{\sim}{v}_2}{|\underset{\sim}{v}_2|};$$

this can be done since $|v_2| \neq 0$ because

$v_2 = 0$ when, $\underset{\sim}{a}_2 = (\underset{\sim}{u}_1' \underset{\sim}{a}_2)\underset{\sim}{u}_1$

$$= (\underset{\sim}{u}_1' \underset{\sim}{a}_2) \cdot \frac{\underset{\sim}{a}_1}{|\underset{\sim}{a}_1|} = \left(\frac{\underset{\sim}{u}_1' \underset{\sim}{a}_2}{|\underset{\sim}{a}_1|}\right) \underset{\sim}{a}_1$$

It is possible only when $a_1 = 0$, but $\underset{\sim}{a}_1 \neq 0$ as $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \cdots, \underset{\sim}{a}_n$ are non-zero LIN vectors. so, $|v_2| \neq 0$

To obtain a vector $\underset{\sim}{v}_3$ orthogonal to both $\underset{\sim}{u}_1$ and $\underset{\sim}{u}_2$, we write —

$$\underset{\sim}{v}_3 = \underset{\sim}{a}_3 - (\underset{\sim}{u}_1' \underset{\sim}{a}_3)u_1 - (\underset{\sim}{u}_2' \underset{\sim}{a}_3)\underset{\sim}{u}_2$$

The third unit vector which is orthogonal to $\underset{\sim}{u}_1, \underset{\sim}{u}_2$ is

$$\underset{\sim}{u}_3 = \frac{\underset{\sim}{v}_3}{|\underset{\sim}{v}_3|}.$$

This is also valid as $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{a}_3$ are LIN; giving $|v_3| \neq 0$.

This procedure is continued until an orthonormal basis is obtained. In general,

$$\underset{\sim}{v}_n = \underset{\sim}{a}_n - \sum_{i=1}^{n-1} (\underset{\sim}{u}_i' \underset{\sim}{a}_n)\underset{\sim}{u}_i ,$$

$$\underset{\sim}{u}_n = \frac{\underset{\sim}{v}_n}{|\underset{\sim}{v}_n|}.$$

Orthonormal vectors are $u_1, u_2, \ldots$
Orthogonal vectors are $v_1, v_2, \ldots$

Example :— Using the Schmidt process, construct an orthonormal basis from $\underset{\sim}{a}_1 = [2,3,0]$, $\underset{\sim}{a}_2 = [6,1,0]$, $\underset{\sim}{a}_3 = [0,2,4]$.

Soln → Let scalars are $\lambda_1, \lambda_2, \lambda_3$ such that

$$\lambda_1 \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_2 + \lambda_3 \underset{\sim}{a}_3 = (0,0,0)$$

$$\Rightarrow \lambda_1(2,3,0) + \lambda_2(6,1,0) + \lambda_3(0,2,4) = (0,0,0)$$

$$\therefore 2\lambda_1 + 6\lambda_2 = 0 ; \quad 3\lambda_1 + \lambda_2 + 2\lambda_3 = 0 ; \quad 4\lambda_3 = 0$$

$$\therefore \lambda_3 = 0, \therefore 3\lambda_1 + \lambda_2 = 0, \quad \lambda_1 + 3\lambda_2 = 0$$

$$\therefore \lambda_2 = -3\lambda_1 ; \therefore \lambda_1 + 3(-3\lambda_1) = 0 \Rightarrow \lambda_1 - 9\lambda_1 = 0 \Rightarrow \lambda_1 = 0$$

$$\therefore \lambda_2 = 0$$

∴ The vectors are LIN.

$$\underset{\sim}{u}_1 = \frac{\underset{\sim}{a}_1}{|\underset{\sim}{a}_1|} = \frac{[2,3,0]}{\sqrt{13}} = [0.554, 0.831, 0] ;$$

$$\underset{\sim}{v}_2 = \underset{\sim}{a}_2 - (\underset{\sim}{u}_1' \underset{\sim}{a}_2)\underset{\sim}{u}_1 ;$$

$$(\underset{\sim}{u}_1' \underset{\sim}{a}_2) = (0.554, 0.831, 0)'(6,1,0)$$
$$= 4.16$$

$$(\underset{\sim}{u}_1' \underset{\sim}{a}_2)\underset{\sim}{u}_1 = [2.30, 3.45, 0] ;$$

$$\underset{\sim}{v}_2 = [3.70, -2.45, 0] ;$$

$$\underset{\sim}{u}_2 = \frac{\underset{\sim}{v}_2}{|\underset{\sim}{v}_2|} = [0.831, -0.554, 0] ;$$

$$\underset{\sim}{v}_3 = \underset{\sim}{a}_3 - (\underset{\sim}{u}_1' \underset{\sim}{a}_3)\underset{\sim}{u}_1 - (\underset{\sim}{u}_2' \underset{\sim}{a}_3)\underset{\sim}{u}_2 ;$$

$$\underset{\sim}{u}_1' \underset{\sim}{a}_3 = 1.664, \quad \underset{\sim}{u}_2' \underset{\sim}{a}_3 = -1.106 ;$$

$$(\underset{\sim}{u}_1' \underset{\sim}{a}_3)\underset{\sim}{u}_1 = [0.921, 1.386, 0], \quad (\underset{\sim}{u}_2' \underset{\sim}{a}_3)\underset{\sim}{u}_2 = [-0.921, 0.614, 0] ;$$

$$\underset{\sim}{v}_3 = [0,0,4] ;$$

$$\underset{\sim}{u}_3 = [0,0,1].$$

Example 2) Construct an orthogonal basis for E^3 with $(1,1,1)$ as the starting vector.

Soln →

$$\underset{\sim}{a}_1 = (1,1,1).$$

$$\underset{\sim}{u}_1 = \frac{(1,1,1)}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}}(1,1,1).$$

Note that $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ forms a basis for E^3.

Now, $(1,1,1) = 1.\underset{\sim}{e}_1 + 1.\underset{\sim}{e}_2 + 1.\underset{\sim}{e}_3$

$\therefore \{(1,1,1), \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ forms a basis for E^3, by replacement theorem.

Now, $\underset{\sim}{v}_1 = (1,1,1)$ and $\underset{\sim}{u}_1 = \frac{\underset{\sim}{v}_1}{|\underset{\sim}{v}_1|} = \frac{1}{\sqrt{3}}(1,1,1)$

$$\therefore \underset{\sim}{v}_2 = \underset{\sim}{a}_2 - (\underset{\sim}{a}_2' \underset{\sim}{u}_1).\underset{\sim}{u}_1$$

$$= \underset{\sim}{e}_2 - (\underset{\sim}{e}_2' \underset{\sim}{u}_1).\underset{\sim}{u}_1$$

$$= (0,1,0) - \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}(1,1,1)$$

$$= \left(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right).$$

$$\underset{\sim}{u}_2 = \frac{\left(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right)}{\sqrt{2/3}}.$$

$$= \frac{-1}{\sqrt{6}}(1,-2,1)$$

$$\underset{\sim}{v}_3 = \underset{\sim}{a}_3 - (\underset{\sim}{a}_3' \underset{\sim}{u}_1).\underset{\sim}{u}_1 - (\underset{\sim}{a}_3' \underset{\sim}{u}_2).\underset{\sim}{u}_2$$

$$= \underset{\sim}{e}_3 - (\underset{\sim}{e}_3' \underset{\sim}{u}_1).\underset{\sim}{u}_1 - (\underset{\sim}{e}_3' \underset{\sim}{u}_2).\underset{\sim}{u}_2$$

$$= (0,0,1) - \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}(1,1,1) + \frac{2}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}}(1,-2,1)$$

$$= (0,-1,1).$$

$$\underset{\sim}{u}_3 = \frac{\underset{\sim}{v}_3}{|\underset{\sim}{v}_3|} = \frac{1}{\sqrt{2}}(0,-1,1).$$

Hence, $\{(1,1,1), -\frac{1}{3}(1,-2,1), (0,-1,1)\}$ is an orthogonal basis for E^3.

Also, $\{\frac{1}{\sqrt{3}}(1,1,1), -\frac{1}{\sqrt{6}}(1,-2,1), \frac{1}{\sqrt{2}}(0,-1,1)\}$ is an orthonormal basis for E^3.

[C.U.]

Example 3> Construct an orthonormal basis for E^4 with $\frac{1}{2}(1,1,1,1)$ as the starting vector.

Soln.→ $\underset{\sim}{a}_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$$\underset{\sim}{u}_1 = \frac{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}{\sqrt{1/4+1/4+1/4+1/4}} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

Note that, $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3, \underset{\sim}{e}_4\}$ forms a basis for E^4.

Now, By replacement theorem $\left\{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \underset{\sim}{e}_2, \underset{\sim}{e}_3, \underset{\sim}{e}_4\right\}$ is also a basis for E^4.

Since $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}\underset{\sim}{e}_1 + \frac{1}{2}\underset{\sim}{e}_2 + \frac{1}{2}\underset{\sim}{e}_3 + \frac{1}{2}\underset{\sim}{e}_4$.

Now, we shall apply orthogonalization process to the set

$$\left\{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \underset{\sim}{e}_2, \underset{\sim}{e}_3, \underset{\sim}{e}_4\right\}, \quad \underset{\sim}{u}_1 = \frac{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}{\sqrt{1/4+1/4+1/4+1/4}}$$

$$\therefore \underset{\sim}{v}_2 = \underset{\sim}{a}_2 - (\underset{\sim}{a}_2 ' \underset{\sim}{u}_1)\cdot\underset{\sim}{u}_1$$

$$= \underset{\sim}{e}_2 - (\underset{\sim}{e}_2 ' \underset{\sim}{u}_1)\cdot\underset{\sim}{u}_1$$

$$= (0,1,0,0) - \frac{1}{2}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$= \left(-\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$$

$$\underset{\sim}{u}_2 = \frac{\left(-\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}\right)}{\sqrt{1/16 + 9/16 + 1/16 + 1/16}}$$

$$= -\frac{1}{4}\Big/\sqrt{\frac{12}{16}}\,(1,-3,1,1)$$

$$= -\frac{1}{2\sqrt{3}}(1,-3,1,1)$$

$$\underset{\sim}{v}_3 = \underset{\sim}{e}_3 - (\underset{\sim}{u}_1 ' \underset{\sim}{e}_3)\underset{\sim}{u}_1 - (\underset{\sim}{u}_2 ' \underset{\sim}{e}_3)\underset{\sim}{u}_2$$

$$= (0,0,1,0) - \frac{1}{2}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) - \frac{1}{(2\sqrt{3})^2}(1,-3,1,1)$$

$$= \left(-\frac{1}{3}, 0, \frac{2}{3}, -\frac{1}{3}\right)$$

$$\underset{\sim}{u}_3 = \frac{\left(-\frac{1}{3}, 0, \frac{2}{3}, -\frac{1}{3}\right)}{\sqrt{\frac{1}{9} + \frac{4}{9} + \frac{1}{9}}} = -\frac{1}{\sqrt{6}}(1,0,-2,1)$$

$$\underset{\sim}{v}_4 = \underset{\sim}{e}_4 - (\underset{\sim}{u}_1' \underset{\sim}{e}_4)\underset{\sim}{u}_1 - (\underset{\sim}{u}_2' \underset{\sim}{e}_4)\underset{\sim}{u}_2 - (\underset{\sim}{u}_3' \underset{\sim}{e}_4)\underset{\sim}{u}_3$$

$$= (0,0,0,1) - \frac{1}{2}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) - \frac{1}{(2\sqrt{3})^2}(1,-3,1,1) - \frac{1}{(\sqrt{6})^2}(1,0,-2,1)$$

$$= \left(-\frac{1}{2},0,0,\frac{1}{2}\right)$$

$$\underset{\sim}{u}_4 = \frac{\left(-\frac{1}{2},0,0,\frac{1}{2}\right)}{\sqrt{\frac{1}{4}+\frac{1}{4}}} = -\frac{1}{2}\cdot\sqrt{2}(1,0,0,-1) = -\frac{1}{\sqrt{2}}(1,0,0,-1)$$

Now, $\left\{\frac{1}{2}(1,1,1,1),\ -\frac{1}{2\sqrt{3}}(1,-3,1,1),\ -\frac{1}{\sqrt{6}}(1,0,-2,1),\ -\frac{1}{\sqrt{2}}(1,0,0,-1)\right\}$ is an orthonormal basis for E^3.

Result: → Let $\underset{\sim}{a}_1$ and $\underset{\sim}{a}_2$ be two LIN vectors and $\underset{\sim}{b}\ (\neq 0)$ is orthogonal to $\underset{\sim}{a}_1$ and $\underset{\sim}{a}_2$. Show that $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{b}$ are LIN.

Proof: → Let, $S(\underset{\sim}{a}_1, \underset{\sim}{a}_2)$ and $S(\underset{\sim}{b})$ be two subspaces generated by $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2\}$ and $\{\underset{\sim}{b}\}$ respectively. As $\underset{\sim}{b}(\neq 0)$ is orthogonal to $\underset{\sim}{a}_1$ and $\underset{\sim}{a}_2$ both, $S(\underset{\sim}{a}_1, \underset{\sim}{a}_2)$ and $S(\underset{\sim}{b})$ are mutually orthogonal subspaces, implying —

$$S(\underset{\sim}{a}_1, \underset{\sim}{a}_2) \cap S(\underset{\sim}{b}) = \{\underset{\sim}{0}\}$$

To show $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{b}$ are linearly independent, consider—

$$\lambda_1 \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_2 + \lambda \underset{\sim}{b} = \underset{\sim}{0}$$

$$\Rightarrow \lambda_1 \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_2 = -\lambda \underset{\sim}{b}$$

$$\Rightarrow \begin{cases} \lambda_1 \underset{\sim}{a}_1 + \lambda_2 \underset{\sim}{a}_2 = 0 & \text{as } S(\underset{\sim}{a}_1, \underset{\sim}{a}_2) \cap S(\underset{\sim}{b}) = \{\underset{\sim}{0}\} \\ \lambda \underset{\sim}{b} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \lambda_1 = 0,\ \lambda_2 = 0 & \text{as } \underset{\sim}{a}_1, \underset{\sim}{a}_2 \text{ are LIN.} \\ \lambda = 0 & \text{as } b \neq 0. \end{cases}$$

$$\Rightarrow \underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{b} \text{ are LIN.}$$

Problem: → Let S_1 and S_2 be two vector subspaces generated by $\{(2,3,0,-1),(-1,0,2,3)\}$ and $\{(1,3,2,2),(3,3,-2,-4)\}$, respectively. Show that S_1 and S_2 are identical.

Solution: Here the two sets are both LIN.

To show $S_1 = S_2$, we consider the following equations with unknown scalars $\lambda_1, \lambda_2, \mu_1, \mu_2$:

$$\lambda \begin{pmatrix} 2 \\ 3 \\ 0 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 0 \\ 2 \\ 3 \end{pmatrix} = \mu_1 \begin{pmatrix} 1 \\ 3 \\ 2 \\ 2 \end{pmatrix} + \mu_2 \begin{pmatrix} 3 \\ 3 \\ -2 \\ -4 \end{pmatrix}$$

or, $$\begin{pmatrix} 2 & -1 & -1 & -3 \\ 3 & 0 & -3 & -3 \\ 0 & 2 & -2 & 2 \\ 1 & 3 & -2 & 4 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \mu_1 \\ \mu_2 \end{pmatrix} = \underset{\sim}{0} \quad \text{——— (1)}$$

We reduce the co-efficient matrix A of (1) via elementary row operations:

$$A = \begin{pmatrix} 2 & -1 & -1 & -3 \\ 3 & 0 & -3 & -3 \\ 0 & 2 & -2 & 2 \\ 1 & 3 & -2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A_1 \quad \text{——— (2)}$$

$$\sim \begin{pmatrix} 1/2 & 1/2 & 0 & 1 \\ -1/2 & 1/2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A_2 \quad \text{——— (3)}$$

(2) $\Rightarrow$

$$\Rightarrow \left.\begin{array}{l} \lambda_1 = \mu_1 + \mu_2 \\ \lambda_2 = \mu_1 - \mu_2 \end{array}\right\} \to (4) \qquad (3) \Rightarrow \; \Rightarrow \left.\begin{array}{l} \mu_1 = (\lambda_1 + \lambda_2)/2 \\ \mu_2 = (\lambda_1 - \lambda_2)/2 \end{array}\right\} \text{——— (5)}$$

For $\underset{\sim}{a} \in S_1$, $\exists$ λ_1, λ_2 $\ni$ $\lambda_1 \underset{\sim}{a_1} + \lambda_2 \underset{\sim}{a_2} = \underset{\sim}{a}$. Putting these λ_1, λ_2 in (5), we get μ_1, μ_2 so that (1) is satisfied, and thus $\underset{\sim}{a}$ being a linear combination of $(1,3,2,2)$ and $(3,3,-2,-4)$ lies in S_2. Hence, $S_1 \subseteq S_2$. For the same reason, using (4), $S_2 \subseteq S_1$, and thus we have $S_1 = S_2$.

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C.U.

2) Obtain the maximum collection of linearly independent vectors using the following :

$\underset{\sim}{\alpha}_1 = (1, 0, 1, 1, 1, 0)'$, $\underset{\sim}{\alpha}_2 = (1, 0, 1, 1, 0, 1)'$,

$\underset{\sim}{\alpha}_3 = (1, 0, 1, 0, 1, 1)'$, $\underset{\sim}{\alpha}_4 = (1, 1, 0, 1, 1, 0)'$,

$\underset{\sim}{\alpha}_5 = (1, 1, 0, 1, 0, 1)'$, $\underset{\sim}{\alpha}_6 = (1, 1, 0, 0, 1, 1)'$

Also find a vector which is orthogonal to each of vectors belonging to the maximal collection.

Soln. → $\exists$ any $\lambda \ni$

$$\underset{\sim}{\alpha}_2 = \lambda \underset{\sim}{\alpha}_1$$

$$\Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda = 1 \\ \lambda = 1 \\ \lambda = 0 \end{cases}$$ which is not possible either all λ_i's are zero, then $\underset{\sim}{\alpha}_2$ and $\underset{\sim}{\alpha}_1$ are LIN.

If possible, let $\underset{\sim}{\alpha}_3 = \lambda_1 \underset{\sim}{\alpha}_1 + \lambda_2 \underset{\sim}{\alpha}_2$

$$\Rightarrow \left\{ \begin{array}{l} \lambda_1 + \lambda_2 = 1 \\ \lambda_1 + \lambda_2 = 0 \end{array} \right\}$$ which is not possible.

$\therefore \{\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \underset{\sim}{\alpha}_3\}$ is LIN.

If possible, let $\underset{\sim}{\alpha}_4 = \lambda_1 \underset{\sim}{\alpha}_1 + \lambda_2 \underset{\sim}{\alpha}_2 + \lambda_3 \underset{\sim}{\alpha}_3$

$$\Rightarrow \left\{ \begin{array}{l} \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ \lambda_1 + \lambda_2 = 1 \\ \lambda_1 + \lambda_3 = 1 \\ \lambda_2 + \lambda_3 = 0 \end{array} \right\}$$ which is not possible.

$\therefore \{\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \underset{\sim}{\alpha}_3, \underset{\sim}{\alpha}_4\}$ is LIN.

If possible, let $\underset{\sim}{\alpha}_5 = \lambda_1 \underset{\sim}{\alpha}_1 + \lambda_2 \underset{\sim}{\alpha}_2 + \lambda_3 \underset{\sim}{\alpha}_3 + \lambda_4 \underset{\sim}{\alpha}_4$

$$\Rightarrow \begin{cases} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\ \lambda_4 = 1 \\ \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ \lambda_1 + \lambda_3 + \lambda_4 = 0 \\ \lambda_2 + \lambda_3 = 1 \end{cases}$$

$\Rightarrow \lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 0, \lambda_4 = 1.$

i.e. $\underset{\sim}{\alpha}_5$ is LD of $\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \underset{\sim}{\alpha}_3, \underset{\sim}{\alpha}_4$.

If possible, let $\underset{\sim}{\alpha}_6 = \lambda_1 \underset{\sim}{\alpha}_1 + \lambda_2 \underset{\sim}{\alpha}_2 + \lambda_3 \underset{\sim}{\alpha}_3 + \lambda_4 \underset{\sim}{\alpha}_4$

$$\Rightarrow \begin{cases} \lambda_4 = 1 \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\ \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ \lambda_1 + \lambda_2 + \lambda_4 = 0 \\ \lambda_2 + \lambda_3 = 1 \end{cases} \Rightarrow \begin{cases} \lambda_1 = -1 \\ \lambda_2 = 0 \\ \lambda_3 = 1 \\ \lambda_4 = 1 \end{cases}$$

Hence, $\underset{\sim}{\alpha}_6$ is LD of $\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \underset{\sim}{\alpha}_3, \underset{\sim}{\alpha}_4$.

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Hence, a maximal collection of linearly independent vectors will be . $\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \underset{\sim}{\alpha}_3, \underset{\sim}{\alpha}_4$.

The choice of this 4 LIN vectors is not unique.

▨ Let the vector be $\underset{\sim}{x} = \{x_1, x_2, x_3, \ldots, x_6\}'$

We are to find x_i's $\ni$

$$\underset{\sim}{\alpha}_i' \underset{\sim}{x} = 0 \quad \forall\ i = 1(1)4.$$

$$\Rightarrow \begin{cases} x_1 + x_3 + x_4 + x_5 = 0 \\ x_1 + x_3 + x_4 + x_6 = 0 \\ x_1 + x_3 + x_5 + x_6 = 0 \\ x_1 + x_2 + x_4 + x_5 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_2 = x_3 \\ x_4 = x_5 = x_6 \end{cases}$$

$$\underset{\sim}{x} = \begin{pmatrix} -\beta - 2\alpha \\ \beta \\ \beta \\ \alpha \\ \alpha \\ \alpha \end{pmatrix}$$

choose, $\alpha = 1, \beta = 1$, then $\underset{\sim}{x} = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

w.r.t. this particular problem we have exactly 2 LIN vectors each of which is orthogonal to $\underset{\sim}{\alpha}_i, i = 1(1)4$.

A choice of such two vectors (LIN) are: $\begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

▣ **Orthogonal spaces:—** A vector $\underset{\sim}{a}$ is orthogonal to every vector in S_n, i.e. if $\underset{\sim}{a}'\underset{\sim}{b} = 0 \ \forall\ \underset{\sim}{b} \in S_n$.

So, two subspaces S_n' and S_n'' of E^n are said to be orthogonal if $\underset{\sim}{a}'\underset{\sim}{b} = 0 \ \forall\ \underset{\sim}{a} \in S_n'$ and $\forall\ \underset{\sim}{b} \in S_n''$;

~~i.e. if any vector, $\underset{\sim}{a} \in S_n'$, is orthogonal to S_n'', i.e. if~~

Orthocomplement space of vector space :

Defn: 1) Let S_n be a subspace of E^n. The set of vectors orthogonal to every vector in S_n, is called the orthogonal complement of S_n, and is denoted by $O(S_n)$.

2) Let S_n be a subspace of E^n, then orthocomplement of S_n is defined as $-$ $O(S_n) = \{\underset{\sim}{x} : \underset{\sim}{x}'y = 0 \;\forall\; y \in S_n\}$.
Clearly S_n and $O(S_n)$ are two distinct subspaces of E^n,
i.e. $S_n \cap O(S_n) = \{\underset{\sim}{0}\}$.

NOTE: $-$ E^n is partitioned into S_n and $O(S_n)$, i.e. S_n and $O(S_n)$ are disjoint. Therefore, if we have a basis of S_n and a basis of $O(S_n)$, combining these two, we get a basis of E^n.

Basis Combination : Consider a subspace S_n of V_n. Let the number of vectors in a basis of S_n be s and that a basis of V_n be v, then a basis of S_n can be extended to a basis of V_n by augmenting $v-s$ LIN vectors, not belonging to S_n.

Ex. Find the basis of the orthocomplement of span of $\{\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2\}$ where $\underset{\sim}{\alpha}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$ and $\underset{\sim}{\alpha}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$.

Soln. $\rightarrow$ It is enough to find two LIN vectors each of which is orthogonal to both α_1 and α_2. Let us denote those vectors by $\underset{\sim}{\beta}_1$ and $\underset{\sim}{\beta}_2$, then $\{\underset{\sim}{\beta}_1, \underset{\sim}{\beta}_2\}$ will be a basis of the required vector space.

Consider a non-null vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ which is orthogonal to $\underset{\sim}{\alpha}_i$, $i = 1, 2$.

$$x_1 + x_2 - x_3 - x_4 = 0$$
$$x_1 - x_2 + x_3 - x_4 = 0$$

$$x_1 = x_4$$
$$x_2 = x_3$$

$\therefore$ $\underset{\sim}{\beta}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\underset{\sim}{\beta}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

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Theorem 22. $O(S_n)$ is a vector subspace of E^n.

Proof:→ Let $\{\underset{\sim}{b_1}, \underset{\sim}{b_2}, \ldots, \underset{\sim}{b_r}\}$ be a basis of S_n.
Consider two vectors $\underset{\sim}{x}$ and $\underset{\sim}{y} \in O(S_n)$.
Then $\underset{\sim}{x}$ & $\underset{\sim}{y}$ are orthogonal to S_n.

$\Rightarrow \underset{\sim}{x} \perp \underset{\sim}{b_i}, \; i=1(1)r$
and $\underset{\sim}{y} \perp \underset{\sim}{b_i}, \; i=1(1)r$.

$\Rightarrow (\underset{\sim}{x}+\underset{\sim}{y}) \underset{\sim}{b_i} = \underset{\sim}{0} \quad \forall i=1(1)r$
$\Rightarrow (\underset{\sim}{x}+\underset{\sim}{y}) \perp \underset{\sim}{b_i} = \underset{\sim}{0} \quad \forall i=1(1)r$
$\Rightarrow (\underset{\sim}{x}+\underset{\sim}{y})$ is orthogonal to S_n.
$\Rightarrow (\underset{\sim}{x}+\underset{\sim}{y}) \in O(S_n)$.

Again, for any scalar λ,
$(\lambda \cdot \underset{\sim}{x}) \cdot \underset{\sim}{b_i} = \lambda (\underset{\sim}{x} \cdot \underset{\sim}{b_i}) = 0 \quad \forall i=1(1)r$
$\Rightarrow \lambda \underset{\sim}{x} \perp b_i, \; \forall i=1(1)r$.
$\Rightarrow \lambda \underset{\sim}{x}$ is orthogonal to S_n.
$\Rightarrow \lambda \underset{\sim}{x} \in O(S_n)$.

Hence, $O(S_n)$ is a vector space and $O(S_n) \subseteq E^n$.
Hence, $O(S_n)$ is a vector subspace of E^n.

Example:→ i) $S_3 = \{(x_1, 0, x_3) : x_1, x_3 \in R\}$
ii) $S_3 = \{(x_1, x_2, x_3) : x_1+x_2+x_3 = 0\}$

Soln.→ i) Let, $\underset{\sim}{x} \in S_3$.
Then, $\underset{\sim}{x} = (x_1, 0, x_3)$
$= x_1 (1,0,0) + x_3 (0,0,1)$

Clearly, $\{\underset{\sim}{e_1}, \underset{\sim}{e_3}\}$ forms a basis for S_3. Let $\underset{\sim}{y}$ is orthogonal to S_3.

$\therefore \underset{\sim}{y}$ is orthogonal to $\underset{\sim}{e_1}$ and $\underset{\sim}{e_3}$.

$\Rightarrow \underset{\sim}{y} \cdot \underset{\sim}{e_1} = 0 \quad \Rightarrow y_1 = 0 = y_3$
$\therefore \underset{\sim}{y} = (0, y_2, 0) \perp S_3$.
$\Rightarrow \underset{\sim}{y} \cdot \underset{\sim}{e_3} = 0$

$\therefore O(S_3) = \{(0, y_2, 0) : y_2 \in R\}$

Remark: $\dim(S_3) = 2$
$\dim(O(S_3)) = 1$
$\therefore \dim(S_3) + \dim O(S_3) = 3 = \dim(E^3)$.

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ii) $S_3 = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0\}$

$\therefore$ Let $\underset{\sim}{x} \in O(S_3)$

$\underset{\sim}{x} = (x_1, x_2, -x_1 - x_2)$ $\quad [\because x_1 + x_2 + x_3 = 0 \quad \therefore x_3 = -x_1 - x_2]$

$= x_1(1, 0, -1) + x_2(0, 1, -1)$

Clearly, $\{(1, 0, -1), (0, 1, -1)\}$ forms a basis for S_3.

Let $\underset{\sim}{y}$ is orthogonal to S_3.

$\therefore \underset{\sim}{y} \perp (1, 0, -1)$, $\underset{\sim}{y} \perp (0, 1, -1)$.

Theorem 23. Every vector space V has a basis.

Soln. → If $V = \{\underset{\sim}{0}\}$ then the basis of V can be formed by using the vector $\underset{\sim}{0}$ only.

Let us consider the case, where V is a non-trivial vector space. Then a basis of V can be formed by choosing sequencially non-null vectors $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots\}$ from V $\ni$ no $\underset{\sim}{a}_i$ is dependent on its predecessors.

At first we choose a non-null vector $\underset{\sim}{a}_1$ from V & see whether the other vectors in V can be expressed as a scalar multiple of it. If yes then we stop and the basis of V is formed by $\underset{\sim}{a}_1$ along. If no then we choose a second non-null vector $\underset{\sim}{a}_2 \in V$, which is not a scalar multiple of $\underset{\sim}{a}_1$ and check whether the remaining vectors can be expressed as a linear combination of $\underset{\sim}{a}_1$ and $\underset{\sim}{a}_2$. If yes then we stop but if no then we continue picking up another vector in V which is LIN of the previous too and so on.

In this process it may so happen that after the k^{th} stage no independent vector is left in V, in which case $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_k$ constitute a basis of V, and V is said to be finite (k) dimensional vector space.

Problem: → i) If S be a set of vectors and $O(S)$ is the set of vectors which is orthogonal to S. Then S.T. $S \subset O_{(O(S))}$. Discuss the situation where $S = O_{(O(S))}$. Discuss the decomposition is unique.

Soln. →

Hints: Consider a vector $\underset{\sim}{x} \in S$ then $\underset{\sim}{x}'\underset{\sim}{y} = 0 \; \forall \; \underset{\sim}{y} \in O(S)$.

Here, $S \subset O_{(O(S))}$

When S is a subspace then $x \in O_{(O(S))}$, then $x \in S$.

In that case, $O_{(O(S))} = S$.

☑ If possible suppose $\underset{\sim}{x} = \underset{\sim}{y}_1 + \underset{\sim}{z}_1 \; \forall \; \underset{\sim}{y}_1 \in S_n$ and $\underset{\sim}{z}_1 \in O(S_n)$

Now, $\underset{\sim}{y} \in S_n$, $\underset{\sim}{y}_1 \in S_n$ $\therefore \underset{\sim}{y} - \underset{\sim}{y}_1 \in S_n$,

$\underset{\sim}{z} \in O(S_n)$, $\underset{\sim}{z}_1 \in O(S_n)$, $\therefore \underset{\sim}{z} - \underset{\sim}{z}_1 \in O(S_n)$

Then, $\underset{\sim}{x} - \underset{\sim}{x} = \underset{\sim}{y} - \underset{\sim}{y}_1 + \underset{\sim}{z} - \underset{\sim}{z}_1$

$\Rightarrow \underset{\sim}{y} - \underset{\sim}{y}_1 = -(\underset{\sim}{z} - \underset{\sim}{z}_1)$

So, here, $(\underset{\sim}{y} - \underset{\sim}{y}_1)$ and $(\underset{\sim}{z} - \underset{\sim}{z}_1) \in S_n \cap O(S_n)$.

Since $\underset{\sim}{0}$ is the common vector between $S_n \cap O(S_n)$

We get, $\underset{\sim}{y} - \underset{\sim}{y}_1 = \underset{\sim}{0}$, $\underset{\sim}{z} - \underset{\sim}{z}_1 = \underset{\sim}{0}$

∴ The above decomposition is unique.

2) If S_r is a r-dimensional subspace of the n-dimensional vector space E^n, then S.T. $\dim[O(S_r)] = n - \dim(S_r)$.

Proof:→ Let $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_r}\}$ be an orthonormal basis for S_r then $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_r}\}$ can be extended to $A = \{\underset{\sim}{a_1}, \ldots, \underset{\sim}{a_{r+1}}, \ldots, \underset{\sim}{a_n}\}$ where $\underset{\sim}{a_{r+1}}, \ldots, \underset{\sim}{a_n} \in A \ni A$ forms an orthonormal basis for E^n. Consider a vector $\underset{\sim}{b} \in E^n \ni \underset{\sim}{b}'\underset{\sim}{a_i} = 0 \ \forall \ i = 1(1)r$.

Hence, $\underset{\sim}{b}$ is orthogonal to every vector in S_r, since any vector in S_r can be expressed as a linear combination of $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$

Now, $\underset{\sim}{b} \in E^n$ and $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$ forms a basis for V_n so $\exists$ scalars $\lambda_1, \lambda_2, \ldots, \lambda_n \ni \underset{\sim}{b} = \sum_{i=1}^{n} \lambda_i \underset{\sim}{a_i}$.

Now, $\underset{\sim}{a_i}\underset{\sim}{b} = 0$

$\Rightarrow \underset{\sim}{a_i}(\lambda_1 \underset{\sim}{a_1} + \lambda_2 \underset{\sim}{a_2} + \cdots + \lambda_n \underset{\sim}{a_n}) = 0$

$\Rightarrow \lambda_1 \underset{\sim}{a_1}'\underset{\sim}{a_1} = 0$ [since $\underset{\sim}{a_i}'\underset{\sim}{a_j} = 0 \ \forall \ i \neq j$]

$\Rightarrow \lambda_1 = 0$ [$\because \|\underset{\sim}{a_1}\| = 1$]

Similarly, $\underset{\sim}{a_2}'\underset{\sim}{b} = 0 \Rightarrow \lambda_2 = 0$

$\vdots$

$\underset{\sim}{a_r}'\underset{\sim}{b} = 0 \Rightarrow \lambda_r = 0$

Hence, $\underset{\sim}{b} = \sum_{i=r+1}^{n} \lambda_i \underset{\sim}{a_i}$

Hence, every vector in $O(S_r)$ can be expressed as a linear combination of $\{\underset{\sim}{a_{r+1}}, \underset{\sim}{a_{r+2}}, \ldots, \underset{\sim}{a_n}\}$, in other words $\{\underset{\sim}{a_{r+1}}, \underset{\sim}{a_{r+2}}, \ldots, \underset{\sim}{a_n}\}$ spans $O(S_r)$.

Since $\{\underset{\sim}{a_{r+1}}, \underset{\sim}{a_{r+2}}, \ldots, \underset{\sim}{a_n}\}$ is a subset of lineably independent set $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}\}$, $\therefore \{\underset{\sim}{a_{r+1}}, \underset{\sim}{a_{r+2}}, \ldots, \underset{\sim}{a_n}\}$ is LIN and it forms a basis for $O(S_r)$.

$$\therefore \dim[O(S_r)] = n - r = n - \dim(S_r).$$

C.U. Show that for a matrix $A^{n\times n}$, A is idempotent if and only if $R(A)+R(I_n-A)=n$.

Soln. → If Part: → $R(A)+R(I_n-A)=n$ —— i

$V_c(I_n) \subseteq V_c\{A\}+V_c\{I_n-A\}$ dim[

$\Rightarrow \dim[V_c(I_n)] \leq \dim[V_c\{A\}+V_c\{I_n-A\}]$

$V_c(A) \cap V_c(I_n-A) = \underset{\sim}{0}$

Considering another matrix $B \ni B=(I_n-A)A$

so, every column vector of $B \in V_c(I_n-A)$

Again, $B=A(I_n-A)$

so, every column vector of $B \in V_c(A)$

$\therefore B \in V_c(A) \cap V_c(I_n-A)$

$\therefore$ every column vectors of B is $\underset{\sim}{0}$.

$\therefore B=0$

$\therefore A(I_n-A)=0$

$\therefore A^2=A$, i.e. A is an idempotent matrix.

Only if part: →

A is an idempotent matrix of order n.

i.e. $A^2=A$

$\Rightarrow A(I_n-A)=0$

Now, the rank of sum of two matrices can't exceed the sum of their ranks, so we have

$\text{Rank}[A+I_n-A] \leq \text{Rank}(A)+\text{Rank}(I_n-A)$

$\Rightarrow \text{Rank}(A)+\text{Rank}(I_n-A) \geq n$ —— ① [As, $\text{Rank}(I_n)=n$]

From Sylvester's inequality,

$\text{Rank}[A(I_n-A)] \geq \text{Rank}(A)+\text{Rank}(I_n-A)-n$

$\Rightarrow \text{Rank}(A)+\text{Rank}(I_n-A) \leq n$ —— ② [$\because$ the product of two matrices is a zero matrix, then its rank must be zero]

Comparing ① & ②, we get

$\text{Rank}(A)+\text{Rank}(I_n-A)=n$.

C.U.

Show that the choice of basis is not unique by giving suitable example.

ANS:- Consider a vector subspace $V_2(F)$.

As we know that $\{\underset{\sim}{e_1}, \underset{\sim}{e_2}\}$ form a basis of $V_2(F)$.

Now, to show that the choice of basis is not unique we are to show that $\{\lambda \underset{\sim}{e_1}, \underset{\sim}{e_2}\}$ will form a basis of $V_2(F)$ for any $\lambda \neq 0$.

Considering $\underset{\sim}{a} = \begin{pmatrix} \underset{\sim}{a_1} \\ \underset{\sim}{a_2} \end{pmatrix} \in V_2(F)$

$$\underset{\sim}{a} = \frac{a_1}{\lambda}(\lambda \underset{\sim}{e_1}) + a_2 \underset{\sim}{e_2}$$

So, $\underset{\sim}{a}$ can be written as a linear combination of ~~basis~~ $\{\lambda \underset{\sim}{e_1}, \underset{\sim}{e_2}\}$.

To show $\{\lambda \underset{\sim}{e_1}, \underset{\sim}{e_2}\}$ linearly independent, considering the equation with scalars ν_1, ν_2

$$\nu_1 \lambda \underset{\sim}{e_1} + \nu_2 \underset{\sim}{e_2} = \underset{\sim}{0}$$

$\Rightarrow \nu_1 \lambda = 0, \ \nu_2 = 0$, as $\underset{\sim}{e_1}, \underset{\sim}{e_2}$ are LIN.

$\Rightarrow \nu_1 = 0$ as $\lambda \neq 0$, $\nu_2 = 0$

So, $\{\lambda \underset{\sim}{e_1}, \underset{\sim}{e_2}\}$ being LIN vectors span $V_2(F)$. So, $\{\lambda \underset{\sim}{e_1}, \underset{\sim}{e_2}\}$ form a basis of $V_2(F)$.

Hence, the result is proved by e.g.

— x —

WORKED EXAMPLES:-

1. If $U = L\{(1,2,1),(2,1,3)\}$, $W = L\{(1,0,0),(0,0,1)\}$, show that U and W are subspace of $\mathbb{R}^3$. Determine $\dim U$, $\dim W$, $\dim(U \cap W)$, Deduce that $\dim(U+W) = 3$.

Solution:- Let $\alpha = (1,2,1)$, $\beta = (2,1,3)$, $\gamma = (1,0,0)$, $\delta = (0,0,1)$

$\{\alpha, \beta\}$ is linearly independent and therefore U is a subspace of $\mathbb{R}^3$ of dimension 2.

$\{\gamma, \delta\}$ is linearly independent and therefore W is a subspace of $\mathbb{R}^3$ of dimension 2.

Let λ be a vector in $U \cap W$. Then $\lambda = a\alpha + b\beta$ for some real a & b. also $\lambda = c\gamma + d\delta$ for some real c & d.

therefore, $a(1,2,1) + b(2,1,3) = c(1,0,0) + d(0,0,1)$

$\Rightarrow a + 2b = c$, $2a + b = 0$, $a + 3b = d$.

so, $a = -b/2$, $c = \frac{3}{2}b$, $d = \frac{5}{2}b$,

$\therefore \lambda = b(3/2, 0, 5/2)$; $b \in \mathbb{R}$

Hence $U \cap W$ be a subspace of dimension 1.

So, $\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$

$= 2 + 2 - 1 = 3$.

2. Find a basis and the dimension of the subspace W of $\mathbb{R}^3$, where $W = \{(x,y,z) \in \mathbb{R}^3 : x+y+z = 0\}$.

Sol. Let $\xi = (a,b,c) \in W$. Then $a,b,c \in \mathbb{R}$ and $a+b+c = 0$.

therefore $\xi = (a, b, -a-b) = a(1,0,-1) + b(0,1,-1)$

Let $\alpha = (1,0,-1)$, $\beta = (0,1,-1)$.

We find α and β are linearly independent in $W = L\{\alpha, \beta\}$.

So, $\{\alpha, \beta\}$ is a basis of W and $\dim(W) = 2$.

3. Find a basis and the dimension of the subspace W of $\mathbb{R}^3$, where $W = \{(x,y,z) \in \mathbb{R}^3 : x + 2y + z = 0, 2x + y + 3z = 0\}$.

Sol. Let $\xi = (a,b,c) \in W$,

$a + 2b + c = 0$, $2a + b + 3c = 0$; $a,b,c \in \mathbb{R}$.

Solving, we have $\frac{a}{5} = \frac{b}{-1} = \frac{c}{-3} = k$

ξ takes the form $k\begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix}'$, where k is an arbitrary real no.

Therefore, $W = L\{\alpha\}$, where $\alpha = (5,-1,-3)$.

Since $\{\alpha\}$ is a linearly independent set, $\{\alpha\}$ is a basis of W and $\dim(W) = 1$.

4. Extend the set of vectors $\{(2,3,-1), (1,-2,-4)\}$ to an orthogonal basis of the euclidean space $\mathbb{R}^3$ with standard inner product and then find the associated orthonormal basis.

Solution:- Let $\underset{\sim}{\alpha}_1 = (2,3,-1)$, $\underset{\sim}{\alpha}_2 = (1,-2,-4)$ $\therefore$ $\underset{\sim}{\alpha}_1' \underset{\sim}{\alpha}_2 = \underset{\sim}{0}$.

$\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2$ are orthogonal vectors. Let $\underset{\sim}{\alpha}_3 = (0,0,1)$. Then $\{\alpha_1, \alpha_2, \alpha_3\}$ is linearly independent because $\begin{vmatrix} 2 & 3 & -1 \\ 1 & -2 & -4 \\ 0 & 0 & 1 \end{vmatrix} \neq 0$.

So, $\{\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \underset{\sim}{\alpha}_3\}$ is a basis of $\mathbb{R}^3$.

Let $\beta = \underset{\sim}{\alpha}_3 - c_1\underset{\sim}{\alpha}_1 - c_2\underset{\sim}{\alpha}_2$, where $c_1 = \dfrac{(\alpha_3, \alpha_1)}{(\alpha_1, \alpha_1)}$, $c_2 = \dfrac{(\alpha_3, \alpha_2)}{(\alpha_2, \alpha_2)}$.

Then β is orthogonal to $\underset{\sim}{\alpha}_1$ and $\underset{\sim}{\alpha}_2$ and $L\{\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \underset{\sim}{\alpha}_3\} = L\{\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \beta\}$

therefore, $\{\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \underset{\sim}{\beta}\}$ is an orthogonal basis of $\mathbb{R}^3$.

$$c_1 = -1/14, \; c_2 = -4/21 \text{ and therefore}$$

$$\beta = (0,0,1) + \frac{1}{14}(2,3,-1) + \frac{4}{21}(1,-2,-4) = \left(\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}\right).$$

Hence an extended orthogonal basis is $\left\{(2,3,-1), (1,-2,-4), \left(\frac{1}{3}, -\frac{1}{6}, +\frac{1}{6}\right)\right\}$

and the associated orthonormal basis is

$$\left\{\frac{1}{\sqrt{14}}(2,3,-1), \frac{1}{\sqrt{21}}(1,-2,-4), \frac{1}{\sqrt{6}}(2,-1,1)\right\}.$$

5. Use Gram-Schmidt process to obtain an orthogonal basis from the basis set $\{(1,0,1), (1,1,1), (1,3,4)\}$ of the Euclidean space $\mathbb{R}^3$ with standard inner product.

Solution:— Let $\underset{\sim}{\alpha}_1 = (1,0,1)$, $\underset{\sim}{\alpha}_2 = (1,1,1)$, $\underset{\sim}{\alpha}_3 = (1,3,4)$.

Let $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2 - c_1\beta_1$, where c_1 is the scalar component of α_2 along β_1.

Then β_2 is orthogonal to β_1 and

$$L\{\beta_1, \beta_2\} = L\{\beta_1, \alpha_2\} = L\{\alpha_1, \alpha_2\}$$

$c_1 = \dfrac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)} = 1.$ Therefore, $\beta_2 = \alpha_2 - \beta_1 = (0,1,0)$.

Let $\beta_3 = \alpha_3 - d_1\beta_1 - d_2\beta_2$; where d_1, d_2 are scalar component of α_3 along β_1, β_2, respectively.

Then β_3 is orthogonal to β_1, β_2 and $L\{\beta_1, \beta_2, \beta_3\} = L\{\alpha_1, \alpha_2, \alpha_3\}$.

$$d_1 = \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)} = \frac{(\alpha_3, \alpha_1)}{(\alpha_1, \alpha_1)} = \frac{5}{2}, \quad d_2 = \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)} = \frac{3}{1} = 3.$$

Therefore $\beta_2 = (1,3,4) - \frac{5}{2}(1,0,1) - 3(0,1,0) = \frac{3}{2}(-1,0,1)$.

Therefore, an orthogonal basis is

$$\left\{(1,0,1), (0,1,0), \frac{3}{2}(-1,0,1)\right\}.$$

END

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LINEAR TRANSFORMATION

Linear mapping or Linear Transformation:—

Definition:— Let V and W be vector spaces over the same field F. A mapping $T: V \to W$ is said to be a linear mapping if it satisfies the following conditions —

1. $T(\alpha+\beta) = T(\alpha)+T(\beta) \ \forall \ \alpha, \beta \in V$.
2. $T(c\alpha) = cT(\alpha) \ \forall \ c$ in F and $\forall \ \alpha \in V$.

These two conditions can be replaced by the single condition —

$$T(a\alpha+b\beta) = aT(\alpha)+bT(\beta) \ \forall \ a, b \text{ in } F \text{ and all } \alpha, \beta \text{ in } V.$$

Note:— 1. A linear mapping $T: V \to W$ is also a homomorphism of V to W.

2. Generally a linear mapping T is a transformation from one vector V to another vector space W, both over the same field of scalars. But the co-domain space W may be the space V itself. In this case T is said to be a linear mapping on V.

Examples:—

1. The identity mapping: The mapping $T: V \to V$ defined by $T(\alpha) = \alpha$ $\forall$ α in V, is a linear mapping. This is called the identity mapping on V and is denoted by I_V.

2. The zero mapping: The mapping $T: V \to W$ defined by $T(\alpha) = 0$ (zero vector of V) $\forall \ \alpha \in V$, is called the zero mapping.

3. Let P the vector space of all real polynomials. The mapping $D: P \to P$ defined by $Dp(x) = \frac{d}{dx} p(x)$, $p(x) \in P$ is a linear mapping.

4. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x_1, x_2, x_3) = (x_1, x_2, 0)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.

$\to$ Let $\alpha = (x_1, x_2, x_3)$, $\beta = (y_1, y_2, y_3) \in \mathbb{R}^3$.

Then $\alpha+\beta = (x_1+y_1, x_2+y_2, x_3+y_3)$

$$\begin{aligned} T(\alpha+\beta) &= (x_1+y_1, x_2+y_2, 0) \\ &= (x_1, x_2, 0) + (y_1, y_2, 0) \\ &= T(\alpha) + T(\beta). \end{aligned}$$

For $c \in \mathbb{R}$, $c\alpha = (cx_1, cx_2, cx_3)$,

$$\begin{aligned} T(c\alpha) &= (cx_1, cx_2, 0) \\ &= c(x_1, x_2, 0) \\ &= cT(\alpha). \end{aligned}$$

Therefore T is a linear mapping.

Remark:— $\underset{\sim}{Y}^{m\times 1} = A^{m\times n} \underset{\sim}{X}^{n\times 1}$ is a linear mapping from E^n into E^m as $A(\underset{\sim}{x}+\underset{\sim}{y}) = A\underset{\sim}{x}+A\underset{\sim}{y}$ & $A(k\underset{\sim}{x}) = kA(\underset{\sim}{x})$.

5. Let V be the vector space of all real valued functions continuous on the closed interval $[a,b]$ and let $T: V \to \mathbb{R}$ be defined by $T(f) = \int_a^b f(t)\,dt$, $f \in V$. Then T is a linear functional.

6. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x_1, x_2, x_3) = (x_1+1, x_2+1, x_3+1)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.

$\rightarrow$ Let $\alpha = (x_1, x_2, x_3)$, $\beta = (y_1, y_2, y_3) \in \mathbb{R}^3$

Then $\alpha+\beta = (x_1+y_1, x_2+y_2, x_3+y_3)$

$$T(\alpha+\beta) = (x_1+y_1+1, x_2+y_2+1, x_3+y_3+1)$$
$$= (x_1, x_2, x_3) + (y_1, y_2, y_3) + (1,1,1)$$
$$\neq T(\alpha) + T(\beta).$$

Therefore, T is not a linear mapping.

Properties of a linear Transformation:—

Theorem:— Let V and W be vector spaces over a field F and $T: V \to W$ be a linear mapping. Then
(i) $T(\underset{\sim}{0}) = \underset{\sim}{0}$, where $\underset{\sim}{0}$ is the null vector in V and W respectively.
(ii) $T(-\alpha) = -T(\alpha)\ \forall\ \alpha \in V$.
(iii) $T(\alpha - \beta) = T(\alpha) - T(\beta)\ \forall\ \alpha, \beta \in V$.
(iv) $T(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) = a_1T(\alpha_1) + a_2T(\alpha_2) + \cdots + a_nT(\alpha_n)$; where $\alpha_1, \alpha_2, \ldots, \alpha_n \in V$ and $a_1, a_2, \ldots, a_n \in F$.

Proof:— (i) Let $\alpha \in V$, Then $T(\alpha) \in W$, We have

$$T(\alpha) + \underset{\sim}{0} = T(\alpha) \quad [\because \underset{\sim}{0} \text{ is a zero vector of } V \text{ and } T(\alpha) \in W]$$
$$= T(\alpha + \underset{\sim}{0})$$

Now, in the vector space W, we have

$$T(\alpha) + \underset{\sim}{0} = T(\alpha) + T(\underset{\sim}{0})$$
$$\Rightarrow T(\underset{\sim}{0}) = \underset{\sim}{0}$$

(ii) $T[\alpha + (-\alpha)] = T(\alpha) + T(-\alpha)$

$$\Rightarrow T(0) = T(\alpha) + T(-\alpha)$$
$$\Rightarrow \underset{\sim}{0} = T(\alpha) + T(-\alpha)$$
$$\Rightarrow T(-\alpha) = -T(\alpha)\ \forall\ \alpha \in V.$$

(iii) $T(\alpha - \beta) = T[\alpha + (-\beta)]$

$$= T(\alpha) + T(-\beta)$$
$$= T(\alpha) - T(\beta)$$

(iv) By induction the proof can easily be done.

Kernel of T OR Null Space of T :—

Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear mapping. The set of all vectors $\alpha \in V \ni T(\alpha) = \underset{\sim}{0}$, $\underset{\sim}{0}$ being the null vector in W, is said to be the kernel of T or the null space of T, and denoted by KerT or N(T).

$$\ker T \; (N(T)) = \{\alpha \in V : T(\alpha) = \underset{\sim}{0}\}$$

*Theorem:— $T: V \to W$ is a linear mapping, then T is injective or one-to-one mapping if and only if $\ker T = \{\underset{\sim}{0}\}$.

Proof:— Only if part:— Let T is injective, $T(\underset{\sim}{0}) = \underset{\sim}{0}$ is the only pre imag

$\therefore \ker T = \{\underset{\sim}{0}\}$.

If part:— $\ker(T) = \{\underset{\sim}{0}\}$, and α, β be two elements of $V \ni$

$T(\alpha) = T(\beta)$ in W.

$\underset{\sim}{0} = T(\alpha) - T(\beta)$

$\Rightarrow T(\alpha - \beta) = \underset{\sim}{0}$, since T is linear.

$\Rightarrow \alpha - \beta \in \ker T$ and since $\ker(T) = \{0\}$, so $\alpha = \beta$.

Thus $T(\alpha) = T(\beta) \Rightarrow \alpha = \beta$ and therefore T is injective.

Note:— $A\underset{\sim}{x} = \underset{\sim}{0}$ has only trivial solution $\underset{\sim}{x} = \underset{\sim}{0} \Rightarrow \text{rank}(A^{m\times n}) = n$.

Theorem:— Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear mapping. Then KerT is a subspace of V.

Image of a linear mapping:— Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear mapping. The images of the elements of V under the mapping T form a subset of W. This subset is said to be the image of T and is denoted by ImT or, it is also called Range of T (R(T)).

$$\text{Im}(T) \text{ or } R(T) = \{T(\underset{\sim}{\alpha}) : \underset{\sim}{\alpha} \in V\}.$$

Theorem:— Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear mapping. Then ImT is a subspace of W.

Proof:— Obviously R(T) is a non-empty subset of W.

Let $\beta_1, \beta_2 \in R(T)$, then $\exists$ vectors $\underset{\sim}{\alpha_1}, \underset{\sim}{\alpha_2}$ in $V \ni T(\alpha_1) = \beta_1, T(\alpha_2) = \beta_2$

Let a, b be any element of the field F. We have

$$a\beta_1 + b\beta_2 = aT(\alpha_1) + bT(\alpha_2)$$

$$= T(a\alpha_1 + b\alpha_2) \quad [\because T \text{ is a linear mapping}]$$

Now, V is a vector space, therefore $\alpha_1, \alpha_2 \in V$ and

$$a, b \in F \Rightarrow a\alpha_1 + b\alpha_2 \in V.$$

Consequently, $T(a\alpha_1 + b\alpha_2) = a\beta_1 + b\beta_2 \in R(T)$.

Therefore R(T) is a subspace of W.

Theorem:– Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear mapping $\ni$ $\ker T = \{0\}$. Then the image of a LIN set of vectors $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ in V are LIN in W.

Proof:– To proof the Linear independence of the vectors $T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)$ in W. Let us consider the relation

$$c_1 T(\alpha_1) + c_2 T(\alpha_2) + \cdots + c_n T(\alpha_n) = \underset{\sim}{0}, \text{ where } c_1, \ldots, c_n \in F.$$

$$\Rightarrow T(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n) = \underset{\sim}{0}, \text{ since } T \text{ is linear.}$$

$$\Rightarrow c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = \underset{\sim}{0}, \text{ since } \operatorname{Ker}(T) = \{0\}.$$

$$\Rightarrow c_1 = c_2 = \cdots = c_n = 0, \text{ since } \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \text{ is}$$

linearly independent. This proves linear independence of the vectors $T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)$ in W.

~~**Theorem:–** Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear mapping $\ni$ $\ker T = \{\underset{\sim}{0}\}$. Then the image of a LIN set of vectors $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ in V are LIN in W.~~

~~**Proof:–**~~

Theorem:– Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear mapping and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of V, then the vectors $T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)$ generate $\operatorname{Im} T$ or $R(T)$.

Proof:– Let $\xi \in \operatorname{Im} T$. Then $\exists$ an element α in V $\ni$

$$T(\alpha) = \xi.$$

Let $\underset{\sim}{\alpha} = c_1\underset{\sim}{\alpha}_1 + c_2\underset{\sim}{\alpha}_2 + \cdots + c_n\underset{\sim}{\alpha}_n$ for some scalars $c_i \in F$.

Then
$$\xi = T(c_1\underset{\sim}{\alpha}_1 + c_2\underset{\sim}{\alpha}_2 + \cdots + c_n\underset{\sim}{\alpha}_n)$$
$$= c_1 T(\underset{\sim}{\alpha}_1) + c_2 T(\underset{\sim}{\alpha}_2) + \cdots + c_n T(\underset{\sim}{\alpha}_n), \text{ since } T \text{ is linear.}$$

Since each $T(\alpha_i) \in \operatorname{Im} T$, it follows that $\operatorname{Im} T$ is generated by the vectors $T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)$.

Note:– $T: V \to W$ is a linear transformation. If $\{\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n\}$ is a basis for V, then $\{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)\}$ is a spanning set of W. Further, if T is one-to-one, i.e. $\ker(T) = \{\underset{\sim}{0}\}$, then $\{T(\underset{\sim}{\alpha}_1), T(\underset{\sim}{\alpha}_2), \ldots, T(\underset{\sim}{\alpha}_n)\}$ is a basis for W.

Proof:– The above two theorem show it.

WORKED EXAMPLES:—

1. A mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x_1, x_2, x_3) = (x_1+x_2+x_3, 2x_1+x_2+2x_3, x_1+2x_2+x_3)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$. Show that T is a linear mapping. Find $\ker T$ and dimension of $\ker T$. Is it one-one? Find $\operatorname{Im} T$ and the dimension of $\operatorname{Im} T$.

Solution:—

1st Part:— Let $\underset{\sim}{\alpha} = (a_1, a_2, a_3) \in \mathbb{R}^3$, $\underset{\sim}{\beta} = (b_1, b_2, b_3) \in \mathbb{R}^3$.

Then $T(\alpha) = (a_1+a_2+a_3,\ 2a_1+a_2+2a_3,\ a_1+2a_2+a_3)$,

$T(\beta) = (b_1+b_2+b_3,\ 2b_1+b_2+2b_3,\ b_1+2b_2+b_3)$.

$\alpha+\beta = (a_1+b_1, a_2+b_2, a_3+b_3)$.

$$T(\alpha+\beta) = \big((a_1+b_1)+(a_2+b_2)+(a_3+b_3),\ 2(a_1+b_1)+(a_2+b_2)+2(a_3+b_3),\ (a_1+b_1)+2(a_2+b_2)+(a_3+b_3)\big)$$

$$= \big((a_1+a_2+a_3)+(b_1+b_2+b_3),\ (2a_1+a_2+2a_3)+(2b_1+b_2+2b_3),\ (a_1+2a_2+a_2)+(b_1+2b_2+b_3)\big)$$

$$= T(\alpha)+T(\beta)$$

Let $c \in \mathbb{R}$, $c\alpha = (ca_1, ca_2, ca_3)$

$$T(c\underset{\sim}{\alpha}) = (ca_1+ca_2+ca_3,\ 2ca_1+ca_2+2ca_3,\ ca_1+2ca_2+ca_3)$$

$$= c\,(a_1+a_2+a_3,\ 2a_1+a_2+2a_3, a_1+2a_2+a_3)$$

$$= cT(\underset{\sim}{\alpha})$$

$\therefore T$ is a linear mapping.

2nd Part:— $\ker(T) = \{\underset{\sim}{x} : T(\underset{\sim}{x}) = \underset{\sim}{0}\}$

$T(x_1, x_2, x_3) = (0,0,0)$

$\Rightarrow$ $x_1+x_3+x_3 = 0$, $2x_1+x_2+2x_3=0$, $x_1+2x_2+x_3=0$

From 1st two equations,

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1} = k$$

$\therefore x_1 = k, x_2 = 0, x_3 = -k$. the last equation is satisfied.

$\therefore (x_1, x_2, x_3) = k(1,0,-1);\ k \in \mathbb{R}$

$\therefore \ker(T) = \{k(1,0,-1) : k \in \mathbb{R}\}$

$\therefore \dim \ker T = 1$.

As $\ker T \neq \{0\}$, so the linear mapping T is <u>not</u> one-to-one.

3rd Part:-

If $\{\alpha_1, \alpha_2, \alpha_3\}$ be a basis of the domain space $\mathbb{R}^3$, $\operatorname{Im}T$ is the linear span of the vectors $T(\alpha_1), T(\alpha_2), T(\alpha_3)$.

$\{e_1=(1,0,0), e_2=(0,1,0), e_3=(0,0,1)\}$ is a basis of $\mathbb{R}^3$.

$T(e_1)=(1,2,1), T(e_2)=(1,1,2), T(e_3)=(1,2,1)$

$\Rightarrow \{T(e_1), T(e_2), T(e_3)\}$ is a spanning set of $\operatorname{Im}T$.

Since $T(e_1)=T(e_3)$, $\operatorname{Im}T$ is the linear span of the vectors $(1,2,1)$ and $(1,1,2)$.

Hence $\operatorname{Im}T = L\{(1,2,1),(1,1,2)\}$

$\therefore \dim(\operatorname{Im}T)=2$.

Alter:-

$T(x) = (x_1+x_2+x_3,\ 2x_1+x_2+2x_3,\ x_1+2x_2+x_3)$

$T(x) = x_1(1,2,1) + x_2(1,1,2) + x_3(1,2,1)$

$\therefore T(x)$ is a linear combination of the vectors $(1,2,1), (1,1,2)$

Hence $\operatorname{Im}T = \{l_1(1,2,1)+l_2(1,1,2) : l_1, l_2 \in \mathbb{R}\}$

$= L\{(1,2,1),(1,1,2)\}$

$\therefore \dim(\operatorname{Im}T)=2$.

2. A linear mapping $T:\mathbb{R}^3 \to \mathbb{R}^4$ is defined by $T(x_1,x_2,x_3) = (x_2+x_3,\ x_3+x_1,\ x_1+x_2,\ x_1+x_2+x_3)$, $(x_1,x_2,x_3)\in\mathbb{R}^3$. Find $\operatorname{Ker}T$. Verify that the set $\{T(e_1), T(e_2), T(e_3)\}$ is linearly independent in $\mathbb{R}^4$. Find $\operatorname{Im}T$ and the dimension of $\operatorname{Im}T$.

Solution:- 1st Part:-

$\operatorname{Ker}T = \{(x_1,x_2,x_3)\in\mathbb{R}^3 : T(x_1,x_2,x_3)=(0,0,0,0)\}$

Let $(x_1,x_2,x_3)\in \operatorname{Ker}T$.

Then $x_2+x_3=0,\ x_3+x_1=0,\ x_1+x_2=0,\ x_1+x_2+x_3=0$.

The solution is $x_1=x_2=x_3=0$.

Therefore $\operatorname{Ker}T=\{0\}$.

2nd Part:-

$T(e_1)=(0,1,1,1),\ T(e_2)=(1,0,1,1),\ T(e_3)=(1,1,0,1)$.

To examine linear independence of the set $\{T(e_1), T(e_2), T(e_3)\}$, let $c_1T(e_1)+c_2T(e_2)+c_3T(e_3)=0$, $c_i\in\mathbb{R}$.

Then $c_1(0,1,1,1)+c_2(1,0,1,1)+c_3(1,1,0,1)=(0,0,0,0)$

$\Rightarrow c_2+c_3=0,\ c_1+c_2=0,\ c_1+c_3=0,\ c_1+c_2+c_3=0$.

The solution is $c_1=c_2=c_3=0$.

This proves that $\{T(e_1), T(e_2), T(e_3)\}$ is linearly independent in $\mathbb{R}^4$.

3rd Part:— ImT is the linear span of the vectors $T(\alpha_1), T(\alpha_2), T(\alpha_3)$ where $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis of $\mathbb{R}^3$.

$\{\underset{\sim}{e_1} = (1,0,0), \underset{\sim}{e_2} = (0,1,0), \underset{\sim}{e_3} = (0,0,1)\}$ is a basis of $\mathbb{R}^3$.

$T(\underset{\sim}{e_1}) = (0,1,1,1)$, $T(\underset{\sim}{e_2}) = (1,0,1,1)$, $T(\underset{\sim}{e_3}) = (1,1,0,1)$.

Therefore, $ImT = L\{(0,1,1,1), (1,0,1,1), (1,1,0,1)\}$.

The set $\{(0,1,1,1), (1,0,1,1), (1,1,0,1)\}$ is linearly independent.

Therefore the dimension of ImT is 3.

3. Show that the transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (x+y, y+z, z+x)$ is a linear transformation. Also find the dimension of $\ker(T)$ and S.T. the transformation is one-to-one. Find ImT. Is it onto?

Solution:— 1st Part:— ~~Let, $\underset{\sim}{\alpha} = (a_1, a_2, a_3) \in \mathbb{R}^3$, $\underset{\sim}{\beta} = (b_1, b_2, b_3) \in \mathbb{R}^3$~~

~~$T(a\underset{\sim}{\alpha} + b\underset{\sim}{\beta}) = T(aa_1 + a$~~

Let $\underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{R}^3$, then

$$
\begin{aligned}
T(\alpha\underset{\sim}{x} + \beta\underset{\sim}{y}) &= T(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3) \\
&= (\alpha(x_1+x_2) + \beta(y_1+y_2), \alpha(x_2+x_3) + \beta(y_2+y_3), \alpha(x_3+x_1) + \beta(y_3+y_1)) \\
&= \alpha(x_1+x_2, x_3+x_2, x_1+x_3) + \beta(y_1+y_2, y_2+y_3, y_3+y_1) \\
&= \alpha T(\underset{\sim}{x}) + \beta T(\underset{\sim}{y}).
\end{aligned}
$$

$\therefore T$ is a linear transformation.

2nd Part:— $\ker T = \{\underset{\sim}{x} : T(\underset{\sim}{x}) = \underset{\sim}{0}\}$.

$T(\underset{\sim}{x}) = \underset{\sim}{0} \Rightarrow x+y=0,\ y+z=0,\ z+x=0$

$\Rightarrow x=0, y=0, z=0$

$\therefore \ker T = \{0\}$ and hence T is one-to-one.

$\therefore \dim(\ker T) = 0$.

3rd Part:— $\{\underset{\sim}{e_1}, \underset{\sim}{e_2}, \underset{\sim}{e_3}\}$ is a basis for $\mathbb{R}^3$.

$\Rightarrow \{T(\underset{\sim}{e_1}), T(\underset{\sim}{e_2}), T(\underset{\sim}{e_3})\}$ is a basis for $Im(T)$.

$\Rightarrow \{(1,0,1), (1,1,0), (0,1,1)\}$ is a basis for $Im(T)$.

$\therefore ImT = L\{(1,0,1), (1,1,0), (0,1,1)\}$

$ImT = \mathbb{R}^3$ and therefore T is an onto mapping.

4. Determine $T: \mathbb{R}^3 \to \mathbb{R}^2$ which maps the basis vectors $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ of $\mathbb{R}^3$ to $(1,1), (2,3), (-1,2)$ of $\mathbb{R}^2$. Find $T(\underset{\sim}{x})$.

Solution:— Let, $\underset{\sim}{x} \in \mathbb{R}^3$

Then, $\underset{\sim}{x} = x_1\underset{\sim}{e}_1 + x_2\underset{\sim}{e}_2 + x_3\underset{\sim}{e}_3$

$$T(\underset{\sim}{x}) = x_1T(\underset{\sim}{e}_1) + x_2T(\underset{\sim}{e}_2) + x_3T(\underset{\sim}{e}_3)$$
$$= x_1(1,1) + x_2(2,3) + x_3(-1,2)$$
$$= (x_1 + 2x_2 - x_3,\ x_1 + 3x_2 + 2x_3).$$

5. Determine the linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^2$ which maps the basis vectors $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ of $\mathbb{R}^3$ to the vectors $(1,1), (2,3), (3,2)$, respectively.

(i) Find $T(1,1,0)$, $T(6,0,-1)$. (ii) Find $\ker T$ and $\operatorname{Im} T$.

(iii) Prove that T is not one-to-one, but onto.

Solution:— Let $\xi = (x,y,z)$ be an arbitrary vector in $\mathbb{R}^3$.

$$\xi = x(1,0,0) + y(0,1,0) + z(0,0,1).$$

Since T is linear,

$$T(\xi) = xT(1,0,0) + yT(0,1,0) + zT(0,0,1)$$
$$= x(1,1) + y(2,3) + z(3,2)$$
$$= (x+2y+3z,\ x+3y+2z).$$

So, T is defined by $T(x,y,z) = (x+2y+3z,\ x+3y+2z)$; $(x,y,z) \in \mathbb{R}^3$.

(i) $T(1,1,0) = (3,4)$; $T(6,0,-1) = (3,4)$.

(ii) Let $(x,y,z) \in \ker T$. Then $T(x,y,z) = (0,0)$

$$\Rightarrow x+2y+3z=0,\ x+3y+2z=0$$

The solution is $\dfrac{x}{-5} = \dfrac{y}{1} = \dfrac{z}{1} = k$

$$\Rightarrow (x,y,z) = k(-5,1,1), \text{ where } k \in \mathbb{R}.$$

Consequently, $\ker T = L\{\underset{\sim}{\alpha}\}$, where $\underset{\sim}{\alpha} = (-5,1,1)$.

$\operatorname{Im} T$ is the linear span of the vectors $T(\alpha_1), T(\alpha_2), T(\alpha_3)$ where $\{\alpha_1, \alpha_2, \alpha_3\}$ is any basis of $\mathbb{R}^3$.

Since $\{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis of $\mathbb{R}^3$,

$$\operatorname{Im} T = L\{(1,1), (2,3), (3,2)\}.$$

(iii) Since $\ker T \neq \{\underset{\sim}{0}\}$, T is not one-to-one.

$\operatorname{Im} T = L\{(1,1), (2,3), (3,2)\}$.

These vectors are linearly dependent in $\mathbb{R}^2$.

The subset $\{(1,1), (2,3)\}$ is LIN in $\mathbb{R}^2$.

Therefore, $\dim \operatorname{Im} T = 2$.

Since $\operatorname{Im} T$ is a subspace of $\mathbb{R}^2$ and $\dim \operatorname{Im} T = 2$,

$\operatorname{Im} T = \mathbb{R}^2$. Therefore T is onto.

Nullity and Rank of a linear mapping:—

Let V and W be vector spaces over a field F and $T: V \to W$ be a linear mapping.
Then $\ker T$ is a subspace of V. The dimension of $\ker T$ is called the nullity of T. $\operatorname{Im} T$ is a subspace of W. The dimension of $\operatorname{Im} T$ is called the rank of T.

$$\text{Nullity}(T) = \dim(\ker T) = \dim N(T)$$
$$\text{Rank}(T) = \dim(\operatorname{Im} T) = \dim R(T)$$

Note:- If V be a finite dimensional vector space then $\ker T$, being a subspace of V, is finite dimensional. The number of vectors in a basis of V is finite. As $\operatorname{Im} T$ is generated by the images of the vectors in a basis of V, $\operatorname{Im} T$ is also finite dimensional.

Theorem:- Let V and W be vector spaces over a field F and V is finite dimensional. If $T: V \to W$ be a linear mapping then
$$\dim \ker T + \dim \operatorname{Im} T = \dim V.$$
OR
$$\text{Nullity of } T + \text{Rank of } T = \dim V.$$

Proof:-

Case I:- Let $\ker T = V$, then $\operatorname{Im} T$ consists of $\underset{\sim}{0}$, where $\underset{\sim}{0}$ is the null element in W. $\therefore \dim \operatorname{Im} T = 0$.
$\therefore \dim \ker T + \dim \operatorname{Im} T = \dim V + 0 = \dim V$.

Case-II:- Let $\ker T = \{\underset{\sim}{0}\}$, and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of V.
Then $\{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)\}$ is a basis of $\operatorname{Im} T$.
$\therefore \dim \ker T = 0$, $\dim V = n$, $\dim \operatorname{Im} T = n$.
$\therefore$ The theorem holds good.

Case III:- ~~Let $\ker T = \{\underset{\sim}{0}\}$ and let~~
Let $\ker T$ be a non-trivial proper subspace of V. Let $\{\underset{\sim}{\alpha_1}, \underset{\sim}{\alpha_2}, \ldots, \underset{\sim}{\alpha_k}\}$ be a basis of $\ker T$. This basis of $\ker T$ can be extended to a basis of V and let the extended basis of V be $\{\underset{\sim}{\alpha_1}, \ldots, \underset{\sim}{\alpha_k}, \underset{\sim}{\alpha_{k+1}}, \ldots, \underset{\sim}{\alpha_n}\}$.
$\operatorname{Im} T$ is generated by the vectors $T(\underset{\sim}{\alpha_1}), T(\underset{\sim}{\alpha_2}), \ldots, T(\underset{\sim}{\alpha_n})$,

As, $T(\alpha_1) = T(\alpha_2) = \cdots = T(\alpha_k) = \underset{\sim}{0}$, $\operatorname{Im} T$ is generated by the vectors $T(\alpha_{k+1}), T(\alpha_{k+2}), T(\alpha_{k+3}), \ldots, T(\alpha_n)$.
Considering the equation with c_{k+i} scalars with $i = 1(1)(n-k)$.
$= 1(1)\,\overline{n-k}$.

$$c_{k+1}T(\alpha_{k+1}) + c_{k+2}T(\alpha_{k+2}) + \cdots + c_nT(\alpha_n) = \underset{\sim}{0}, \text{ where } c_j \in F.$$
$$\Rightarrow T(c_{k+1}\alpha_{k+1} + c_{k+2}\alpha_{k+2} + \cdots + c_n\alpha_n) = \underset{\sim}{0}$$
$\Rightarrow c_{k+1}\alpha_{k+1} + \cdots + c_n\alpha_n$ is in $\ker T$.
Since $\{\alpha_1, \ldots, \alpha_k\}$ is a basis for $\ker T$.
$\therefore$ The linear independence of $\{\alpha_1, \ldots, \alpha_n\}$ requires $c_{k+1} = c_{k+2} = \cdots = c_n = 0$.
$\therefore \{T(\alpha_{k+1}), \ldots, T(\alpha_n)\}$ is LIN and is a basis for $\operatorname{Im} T$.
So, $\dim(\operatorname{Im} T) = n-k$ and $\dim \ker T + \dim \operatorname{Im} T = k + (n-k) = n$
$= \dim V$.

Theorem:– Let V and W be finite dimensional vector spaces of the same dimensional over a field F and $T: V \to W$ be a linear mapping. Then T is one-to-one $\Leftrightarrow$ T is onto.

Proof:– Let T be one-to-one.

$$\ker(T) = \{\underset{\sim}{0}\}$$

$$\dim \ker T = 0.$$

$$\therefore \dim \ker T + \dim \operatorname{Im} T = \dim V$$

$$\Rightarrow \dim \operatorname{Im} T = \dim V$$

$$\Rightarrow \operatorname{Im} T = \dim W.$$

But $\operatorname{Im} T$ is a subspace of W and so $\operatorname{Im} T = W$.

Hence T is onto.

Conversely, let T is onto. Then $\operatorname{Im} T = W$.

$$\therefore \dim \ker T + \dim \operatorname{Im} T = \dim V$$

$$\Rightarrow \dim \ker T + \dim W = \dim V$$

As, $\dim W = \dim V$, $\therefore \dim \ker T = 0$

$$\Rightarrow \ker T = \{\underset{\sim}{0}\}$$

$$\Rightarrow T \text{ is one-to-one.}$$

Ex. 1:– Let V be a vector space and T is a linear transformation from V into V. Prove that the following two statements about T are equivalent:

(i) $R(T) \cap N(T) = \{0\}$

(ii) $T[T(\underset{\sim}{\alpha})] = \underset{\sim}{0} \Rightarrow T(\underset{\sim}{\alpha}) = \underset{\sim}{0}$.

Solution:– To show (i) $\Rightarrow$ (ii)

$$R(T) \cap N(T) = \{\underset{\sim}{0}\}$$

$$\Rightarrow T(\underset{\sim}{\alpha}) = \underset{\sim}{0}$$

$$\Rightarrow T(\underset{\sim}{\alpha}) \in R(T) \cap N(T)$$

$$\Rightarrow T(\underset{\sim}{\alpha}) \in N(T)$$

$$\Rightarrow T(T(\underset{\sim}{\alpha})) = \underset{\sim}{0}.$$

To show (ii) $\Rightarrow$ (i)

$\alpha \in R(T) \cap N(T)$

$\therefore \alpha \in R(T)$, $\alpha \in N(T)$, $\therefore T(\underset{\sim}{\alpha}) = \underset{\sim}{0}$

Now let $T(\beta) = \alpha$

$$\Rightarrow T(T(\beta)) = T(\alpha)$$

$$= \underset{\sim}{0} \quad \text{but } T(\beta) = \alpha \neq \underset{\sim}{0}$$

This contradicts the hypothesis,

$\therefore \alpha \notin R(T) \cap N(T)$; $\therefore R(T) \cap N(T) = \{\underset{\sim}{0}\}$.

WORKED EXAMPLES:— (Continued)

1. Let T be a linear operator on V and let $\text{Rank}\,T^2 = \text{Rank}\,T$, then show that $\text{Range}\,T \cap \ker T = \{\underset{\sim}{0}\}$.

Solution:— $T: V \to V$, $T^2: V \to V$

$\text{Rank}(T) = \dim V - \dim \ker T$

$\text{Rank}(T^2) = \dim V - \dim \ker T^2$

$\Rightarrow \dim \ker T = \dim \ker T^2$.

we claim $\ker T = \ker T^2$

$\underset{\sim}{x} \in \ker T \Rightarrow T(\underset{\sim}{x}) = \underset{\sim}{0} \Rightarrow T^2(\underset{\sim}{x}) = T(\underset{\sim}{0}) = \underset{\sim}{0}$

$\underset{\sim}{x} \in \ker T^2 \Rightarrow \ker(T) \subseteq \ker(T^2)$

Now, $\underset{\sim}{x} \in \text{Range}\,T \cap \ker T$

$\Rightarrow \underset{\sim}{x} \in \text{Range}\,T$ and $\underset{\sim}{x} \in \ker T$

$\Rightarrow T(\underset{\sim}{x}) = \underset{\sim}{0}$, $\underset{\sim}{x} = T(\underset{\sim}{y})$, for some $\underset{\sim}{y} \in V$.

$\Rightarrow T(T(\underset{\sim}{y})) = \underset{\sim}{0}$

$\Rightarrow T^2(\underset{\sim}{y}) = \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{y} \in \ker T^2 = \ker T$

$\Rightarrow T(\underset{\sim}{y}) = \underset{\sim}{0} \Rightarrow \underset{\sim}{x} = \underset{\sim}{0}$

$\Rightarrow \ker T \cap \text{Range}\,T = \{\underset{\sim}{0}\}$.

2. Determine the linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ which maps the basis vectors $(0,1,1)$, $(1,0,1)$, $(1,1,0)$ of $\mathbb{R}^3$ to $(1,1,1)$, $(1,1,1)$, $(1,1,1)$ respectively. Verify that $\dim \ker T + \dim \text{Im}\,T = 3$.

Solution:— $\underset{\sim}{\xi} = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0)$; c_i are unique scalars.

$c_2 + c_3 = x$, $c_1 + c_3 = y$, $c_1 + c_2 = z$.

solving $c_1 = \dfrac{y+z-x}{2}$, $c_2 = \dfrac{z+x-y}{2}$, $c_3 = \dfrac{x+y-z}{2}$.

Since T is linear,

$$T(\underset{\sim}{\xi}) = c_1 T(0,1,1) + c_2 T(1,0,1) + c_3 T(1,1,0)$$

$$= c_1(1,1,1) + c_2(1,1,1) + c_3(1,1,1)$$

$$= (c_1+c_2+c_3,\ c_1+c_2+c_3,\ c_1+c_2+c_3)$$

$$= \left(\frac{x+y+z}{2}, \frac{x+y+z}{2}, \frac{x+y+z}{2}\right);$$

$$\therefore T(x,y,z) = \left(\frac{x+y+z}{2}, \frac{x+y+z}{2}, \frac{x+y+z}{2}\right); \ (x,y,z) \in \mathbb{R}^3.$$

To find $\ker T$, let $T(x,y,z) = (0,0,0)$;

$x+y+z = 0$, let, $y = c$, $z = d$, $x = -(c+d)$.

$\therefore (x,y,z) = (-c-d, c, d) = c(-1,1,0) + d(-1,0,1)$; $c, d \in \mathbb{R}$.

Hence $\ker T = L\{(-1,1,0), (-1,0,1)\}$ and since $(-1,1,0)$ and $(-1,0,1)$ are LIN. $\dim \ker T = 2$.

ImT is the linear span of the vectors $T(\alpha), T(\beta), T(\gamma)$, where $\{\alpha, \beta, \gamma\}$ is any basis of the domain space $\mathbb{R}^3$.

since $(0,1,1), (1,0,1), (1,1,0)$ is a basis of $\mathbb{R}^3$,

$ImT = L\{(1,1,1)\}$. Hence $\dim ImT = 1$.

$\therefore \dim KerT + \dim ImT = 2+1 = 3$.

3. Determine the linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ which maps the basis vectors $(0,1,1), (1,0,1), (1,1,0)$ of $\mathbb{R}^3$ to the vectors $(2,0,0), (0,2,0), (0,0,2)$ respectively. find $KerT$ and ImT. Verify that $\dim KerT + \dim ImT = 3$.

Solution:-

$(x,y,z) = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0)$; c_i are unique scalars.

$c_2 + c_3 = x, \quad c_3 + c_1 = y, \quad c_1 + c_2 = z$.

solving, we have $c_1 = \frac{y+z-x}{2}, \quad c_2 = \frac{z+x-y}{2}, \quad c_3 = \frac{x+y-z}{2}$.

since T is linear,

$$T(x,y,z) = c_1 T(0,1,1) + c_2 T(1,0,1) + c_3 T(1,1,0)$$

$$= \frac{y+z-x}{2}(2,0,0) + \frac{z+x-y}{2}(0,2,0) + \frac{x+y-z}{2}(0,0,2)$$

$$= (y+z-x, z+x-y, x+y-z) ; (x,y,z) \in \mathbb{R}^3.$$

$$T(x,y,z) = (0,0,0) = (y+z-x, z+x-y, x+y-z)$$

$$\therefore x = y = z = 0.$$

$KerT = \{\underset{\sim}{0}\}$, $\dim KerT = 0$.

ImT is the linear span of the vectors $T(\alpha_1), T(\alpha_2), T(\alpha_3)$, where $\{\alpha_1, \alpha_2, \alpha_3\}$ is any basis of the domain space $\mathbb{R}^3$.

since $\{(0,1,1), (1,0,1), (1,1,0)\}$ is a basis of $\mathbb{R}^3$,

$ImT = L\{(2,0,0), (0,2,0), (0,0,2)\}$.

& $\dim(ImT) = 3$.

Hence, $\dim(KerT) + \dim(ImT) = 0 + 3 = 3$.

4. Find a linear operator T on $\mathbb{R}^3 \ni KerT$ is the subspace $U = \{(x,y,z) \in \mathbb{R}^3 : x+y+z = 0\}$ of $\mathbb{R}^3$.

Solution:- Let $\underset{\sim}{\xi} = (a,b,c)$ be a vector in the subspace U.

Then $a+b+c = 0$.

$\underset{\sim}{\xi} = (a,b,-a-b) = a(1,0,-1) + b(0,1,-1)$; $a,b \in \mathbb{R}$.

This shows that $\underset{\sim}{\xi} \in L\{(1,0,-1), (0,1,-1)\}$

& $\{(1,0,-1), (0,1,-1)\}$ is a basis of U.

$U = KerT$, $T(1,0,-1) = (0,0,0)$, $T(0,1,-1) = (0,0,0)$

The basis $\{(1,0,-1),(0,1,-1)\}$ of U can be extended to a basis $\{(1,0,-1),(0,1,-1),(1,0,0)\}$ of $\mathbb{R}^3$.

Let T be the linear operator on $\mathbb{R}^3 \ni T(1,0,-1)=(0,0,0)$, $T(0,1,-1)=(0,0,0)$, $T(1,0,0)=(1,0,0)$, then $\ker T = U$.

Let $(x,y,z)\in\mathbb{R}^3$, $(x,y,z)=c_1(1,0,-1)+c_2(0,1,-1)+c_3(1,0,0)$

$$c_1=-y-z,\ c_2=y,\ c_3=x+y+z.$$

$$\begin{aligned}T(x,y,z) &= (-y-z)T(1,0,-1)+yT(0,1,-1)+(x+y+z)T(1,0,0)\\ &= (-y-z)(0,0,0)+y(0,0,0)+(x+y+z)(1,0,0)\\ &= (x+y+z,0,0);\ (x,y,z)\in\mathbb{R}^3.\end{aligned}$$

Note:- As the image of the basis vector $(1,0,0)$ can be chosen arbitrarily (other than $(0,0,0)$), T is not unique.

5. Find a linear mapping $T:\mathbb{R}^3\to\mathbb{R}^3 \ni \operatorname{Im}T$ is the subspace $U=\{(x,y,z)\in\mathbb{R}^3 : x+y-z=0\}$ of $\mathbb{R}^3$.

Solution:- Let $\xi=(a,b,c)$ be a vector in the subspace U, then $a+b-c=0$.

$$\xi=(a,b,a+b)=a(1,0,1)+b(0,1,1)$$

$\therefore \xi\in L\{(1,0,1),(0,1,1)\}$.

$\therefore \{(1,0,1),(0,1,1)\}$ is a basis of U as $(1,0,1)$ and $(0,1,1)$ are LIN.

$\operatorname{Im}T$ is generated by the images of the vectors of a basis. Let us take the standard basis $\{(1,0,0),(0,1,0),(0,0,1)\}$ of $\mathbb{R}^3$ and let

$$T(1,0,0)=(1,0,1),\ T(0,1,0)=(0,1,1),\ T(0,0,1)=(0,1,1).$$

Then $\operatorname{Im}T = L\{(1,0,1),(0,1,1)\}=U$.

Let $(x,y,z)\in\mathbb{R}^3$.

$$\begin{aligned}T(x,y,z) &= xT(1,0,0)+y(T(0,1,0))+zT(0,0,1),\ \text{since } T \text{ is linear.}\\ &= x(1,0,1)+y(0,1,1)+z(0,1,1)\\ &= (x,\ y+z,\ x+y+z),\ (x,y,z)\in\mathbb{R}^3.\end{aligned}$$

Note:- As the image of the basis vectors $(0,0,1)$ can be chosen arbitrarily (as a scalar multiple of $(0,1,1)$, or as a scalar multiple of $(1,0,1)$), T is not unique.

Linear Operator (Definition):- Let V be a vector space. A linear operator on V is a function T from V into $V \ni$

$$T(a\alpha+b\beta)=aT(\alpha)+bT(\beta)\quad \forall\ \alpha,\beta \text{ in } V \text{ and } \forall\ a,b\in F.$$

Thus T is a linear operator on V if T is a linear transformation from V into V itself.

Composition of linear mapping :—

Let V, W and U be vector spaces over a field F and let $T: V \to W$; $S: W \to U$ be linear mapping. The composite mapping $SoT: V \to U$ is defined by $SoT(\alpha) = S\{T(\alpha)\}$, $\alpha \in V$.

The composite SoT is generally denoted by ST and it is also said to be the product mapping ST.

Theorem:- Let V, W and U be vector spaces over a field F and $T: V \to W$, $S: W \to U$ be linear mapping. Then the composite mapping $ST: V \to U$ is linear.

Proof:- Let α, β in V and $c \in F$.

$$ST(\alpha+\beta) = S[T(\alpha+\beta)]$$

$= S[T\alpha + T\beta]$, since T is linear.

$= S(T\alpha) + S(T\beta)$, since S is linear.

$= ST(\alpha) + ST(\beta)$.

$$ST(c\alpha) = S[T(c\alpha)]$$

$= S[cT(\alpha)]$, since T is linear.

$= cS[T(\alpha)]$, since S is linear.

$= cST(\alpha)$.

This proves that ST is linear.

Note:- The unique inverse of the mapping T is denoted by T^{-1}.

Theorem:- Let V and W be vector spaces over a field F. A linear mapping $T: V \to W$ is invertible if and only if T is one-to-one and onto.

Theorem:- Let V and W be vector spaces over a field F. If a linear mapping $T: V \to W$ be invertible, then the inverse mapping $T^{-1}: W \to V$ is linear.

Proof:- Let $\alpha', \beta' \in W$ and $T^{-1}(\alpha') = \alpha$, $T^{-1}(\beta') = \beta$.

Then $\alpha, \beta \in V$ and $T(\alpha) = \alpha'$, $T(\beta) = \beta'$

Since T is linear, $T(\alpha+\beta) = T(\alpha) + T(\beta) = \alpha' + \beta'$,

therefore $T^{-1}(\alpha' + \beta') = \alpha + \beta = T^{-1}(\alpha') + T^{-1}(\beta')$

since T is linear, $T(c\alpha) = cT(\alpha)$, $c \in F$.

$= c\alpha'$.

Therefore, $T^{-1}(c\alpha') = c\alpha = cT^{-1}(\alpha')$ $\forall$ $c \in F$.

This proves T^{-1} is linear.

Definition:- A linear mapping $T: V \to W$ is said to be non-singular if T is invertible.

Isomorphism:- Let V and W be vector spaces over a field F. A linear mapping $T: V \to W$ is said to be an isomorphism if T is both one-to-one and onto.

Since T is both one-to-one and onto, T is invertible and $T^{-1}: W \to V$ is also a linear mapping which is both one-to-one and onto.

Thus the existance of an isomorphism $T: V \to W$ implies the existance of another isomorphism $T^{-1}: W \to V$. In this case the vector spaces V and W are said to be isomorphic.

Theorem:- Two finite dimensional vector spaces V and W over a field F are isomorphic if and only if $\dim V = \dim W$.

Isomorphisms from V to F^n.

Theorem:- Let V be a vector space of dimension n over a field F. Then V is isomorphism of F^n.

Proof:- An isomorphism between V and F^n can be establlshed in many ways.

Let $(\beta_1, \beta_2, \dots, \beta_n)$ be an ordered basis of V. Then any vector ξ of V can be expressed as

$\xi = c_1\beta_1 + c_2\beta_2 + \dots + c_n\beta_n$, where $c_1, c_2, \dots c_n$ are scalars in F.

Let us define a mapping $\phi: V \to F^n$ by

$$\phi(\xi) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \text{ where } \xi = (c_1\beta_1 + c_2\beta_2 + \dots + c_n\beta_n) \in V.$$

Let $\alpha = a_1\beta_1 + a_2\beta_2 + \dots + a_n\beta_n \in V$,

$\beta = b_1\beta_1 + b_2\beta_2 + \dots + b_n\beta_n \in V$.

Then $\alpha + \beta = (a_1+b_1)\beta_1 + (a_2+b_2)\beta_2 + \dots + (a_n+b_n)\beta_n \in V$.

$$\phi(\alpha) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad \phi(\beta) = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \text{ and}$$

$$\phi(\alpha+\beta) = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \phi(\alpha) + \phi(\beta) \quad \dots\dots\dots (i)$$

$$\phi(p\alpha) = \begin{pmatrix} pa_1 \\ pa_2 \\ \vdots \\ pa_n \end{pmatrix} = p \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = p\,\phi(\alpha) \quad \dots\dots (ii)$$

From (i) and (ii) ϕ is a ~~homomorphism~~ linear mapping.

To prove that ϕ is one-to-one, let $\alpha, \beta \in V \ni$.

$$\phi(\alpha) = \phi(\beta)$$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$\Rightarrow a_1 = b_1,\ a_2 = b_2, \ldots\ldots, a_n = b_n$

$\Rightarrow \underset{\sim}{\alpha} = \underset{\sim}{\beta}$

so, ϕ is one-to-one.

To prove that ϕ is onto, let $\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$ be an element in F^n.

Then $r_1\beta_1 + r_2\beta_2 + \cdots\cdots + r_n\beta_n \in V$.

and $\phi(r_1\beta_1 + r_2\beta_2 + \cdots + r_n\beta_n) = \begin{pmatrix} b_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$

so, ϕ is onto.

Since ϕ is both one-to-one and onto, so ϕ is an isomorphism.

Since ϕ is isomorphism, V is isomorphic to F^n.

__Ex(1)__:- A linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x,y,z) = (x-y, x+2y, y+3z)$, $(x,y,z) \in \mathbb{R}^3$. Show that T is non-singular and determine T^{-1}.

Solution:— T is a linear mapping. Let us find $\ker T$.

Let $(a,b,c) \in \ker T$. Then $(a-b, a+2b, b+3c) = (0,0,0)$

Therefore $a-b=0,\ a+2b=0,\ b+3c=0$.

This gives $a=b=c=0$.

$\ker T = \{\underset{\sim}{0}\}$. $\therefore$ Therefore T is one-to-one.

Here $V = \mathbb{R}^3$, $W = \mathbb{R}^3$ and therefore $\dim V = \dim W$.

Since $T: V \to W$ is one-to-one, T is onto. Since T is one-to-one and onto, so it is non-singular.

Let $T^{-1}(x,y,z) = (a,b,c)$

Then $(x,y,z) = T(a,b,c) = (a-b, a+2b, b+3c)$

therefore, $a-b = x,\ a+2b = y,\ b+3c = z$.

$$\Rightarrow a = \frac{1}{3}(2x+y),\ b = \frac{1}{3}(-x+y),\ c = \frac{1}{9}(x-y+3z)$$

therefore $T^{-1}: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$T^{-1}(x,y,z) = \left(\frac{2}{3}x + \frac{1}{3}y,\ -\frac{1}{3}x + \frac{1}{3}y,\ \frac{1}{9}x - \frac{1}{9}y + \frac{1}{3}z\right);$$

$(x,y,z) \in \mathbb{R}^3$.

Ex.(2):→ A linear mapping $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ maps the vectors $(1,2,3)$, $(2,3,1)$, $(3,1,2)$ to the vectors $(0,1,1)$, $(1,0,1)$, $(1,1,0)$ respectively. Show that ϕ is an isomorphism.

Solution:— The set of vectors $\{(1,2,3), (2,3,1), (3,1,2)\}$ is LIN set in $\mathbb{R}^3$, a vector space of dimension 3. Therefore it is a basis of $\mathbb{R}^3$.

The set of vectors $\{(0,1,1), (1,0,1), (1,1,0)\}$ is LIN set in $\mathbb{R}^3$, a vector space of dimension 3. Therefore it is a basis of $\mathbb{R}^3$.

The domain space and the co-domain space of the linear mapping ϕ are of the same dimension and ϕ maps a basis of the domain space to a basis of the co-domain space. So, ϕ is an isomorphism.

Ex.(3):→ A linear mapping $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ maps the vectors $(0,1,1)$, $(1,0,1)$, $(1,1,0)$ to the vectors $(1,1,-1)$, $(1,-1,1)$, $(1,0,0)$ respectively. Show that ϕ is not an isomorphism.

Solution:— The set of vectors $\{(0,1,1), (1,0,1), (1,1,0)\}$ is a LIN set in $\mathbb{R}^3$. Therefore it is a basis of $\mathbb{R}^3$. The images of this basis vectors are the vectors $(1,1,-1)$, $(1,-1,1)$, $(1,0,0)$ respectively. The set $\{(1,1,-1), (1,-1,1), (1,0,0)\}$ is an LIN set in $\mathbb{R}^3$ and ~~the~~ therefore it is not a basis of $\mathbb{R}^3$.

The linear mapping ϕ maps a basis of $\mathbb{R}^3$ to a set which is not a basis of $\mathbb{R}^3$. So ϕ is not an isomorphism.

Ex.(4):→ Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation defined by $T(x,y,z) = (x+y, x-z)$; then the dimension of the null space of T is

(A) 0 (B) 1 (C) 2 (D) 3.

Solution:—

(B) Null space of $T = \{(x,y,z) : T(x,y,z) = (0,0)\}$

$\therefore (x+y, x-z) = (0,0)$

$\Rightarrow x+y=0, \quad x-z=0$

$\Rightarrow x = -y = z$

Let, $x = t$, $\therefore (x,y,z) = t(1,-1,1)$

$\therefore$ Null space of T is $= \{t(1,-1,1) : t \in \mathbb{R}\}$

$\therefore \dim N(T) = 1$.

Matrix representation of a linear mapping :-

T is completely determined by the images $T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)$. Each $T(\alpha_i)$ in W is a linear combination of the vectors $\beta_1, \beta_2, \ldots, \beta_m$.

Let
$$T(\alpha_1) = a_{11}\beta_1 + a_{21}\beta_2 + \cdots + a_{m1}\beta_m$$
$$T(\alpha_2) = a_{12}\beta_1 + a_{22}\beta_2 + \cdots + a_{m2}\beta_m$$
$$\vdots$$
$$T(\alpha_n) = a_{1n}\beta_1 + a_{2n}\beta_2 + \cdots + a_{mn}\beta_m,$$
where

[similar as $Y = AX$, where A is the associated matrix.]

a_{ij} are unique scalars in F determined by the ordered basis $(\beta_1, \beta_2, \ldots, \beta_m)$.

The co-ordinate vector of $T(\alpha_j)$ relative to the ordered basis $(\beta_1, \beta_2, \ldots, \beta_m)$ is given by the j^{th} column of A.

The matrix $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ is said to be the matrix associated with the linear mapping T relative to the chosen ordered bases of V and W. A is also called the matrix of T relative to the chosen ordered bases.

WORKED EXAMPLES :-

1. A linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3, x_1 - 3x_2 - 2x_3)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.
Find the matrix of T relative to the ordered bases
(i) $\{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$ and $\{(1,0), (0,1)\}$ of $\mathbb{R}^2$;
(ii) $\{(0,1,0), (1,0,0), (0,0,1)\}$ of $\mathbb{R}^3$ and $\{(0,1), (1,0)\}$ of $\mathbb{R}^2$;
(iii) $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ and $\{(1,0), (0,1)\}$ of $\mathbb{R}^2$;

Solution:— (i)
$$T(1,0,0) = (3,1) = 3(1,0) + 1(0,1);$$
$$T(0,1,0) = (-2,-3) = -2(1,0) - 3(0,1)$$
$$T(0,0,1) = (1,-2) = 1(1,0) - 2(0,1)$$

$\therefore$ The matrix of $T = \begin{pmatrix} 3 & -2 & 1 \\ 1 & -3 & -2 \end{pmatrix}$.

(ii)
$$T(0,1,0) = (-2,-3) = -3(0,1) - 2(1,0)$$
$$T(1,0,0) = (3,1) = 1(0,1) + 3(1,0)$$
$$T(0,0,1) = (1,-2) = -2(0,1) + 1(1,0)$$

$\therefore$ The mtx of $T = \begin{pmatrix} -3 & 1 & -2 \\ -2 & 3 & 1 \end{pmatrix}$.

(iii)
$$T(0,1,1) = (-1,-5) = -1(1,0) - 5(0,1)$$
$$T(1,0,1) = (4,-1) = 4(1,0) - 1(0,1)$$
$$T(1,1,0) = (1,-2) = 1(1,0) - 2(0,1)$$

$\therefore$ The mtx of $T = \begin{pmatrix} -1 & 4 & 1 \\ -5 & -1 & -2 \end{pmatrix}$.

2. Let $(\alpha_1, \alpha_2, \alpha_3)$, (β_1, β_2) be ordered bases of the real vector spaces V and W respectively. A linear mapping $T: V \to W$ maps the basis vectors as $T(\alpha_1) = \beta_1 + \beta_2$, $T(\alpha_2) = 3\beta_1 - \beta_2$, $T(\alpha_3) = \beta_1 + 3\beta_2$.

Find the matrix of T relative to the ordered bases

(i) $(\alpha_1, \alpha_2, \alpha_3)$ of V and (β_1, β_2) of W;

(ii) $(\alpha_1 + \alpha_2, \alpha_2, \alpha_3)$ of V and $(\beta_1, \beta_1 + \beta_2)$ of W;.

Solution:- (i) $T(\alpha_1) = \beta_1 + \beta_2$; $T(\alpha_2) = 3\beta_1 - \beta_2$; $T(\alpha_3) = \beta_1 + 3\beta_2$

$\therefore$ the matrix of $T = \begin{pmatrix} 1 & 3 & 1 \\ 1 & -1 & 3 \end{pmatrix}$.

(ii) $T(\alpha_1 + \alpha_2) = T(\alpha_1) + T(\alpha_2) = 4\beta_1 = 4\beta_1 + 0(\beta_1 + \beta_2)$;

$T(\alpha_2) = (3\beta_1 - \beta_2) = 4\beta_1 - 1(\beta_1 + \beta_2)$;

$T(\alpha_3) = \beta_1 + 3\beta_2 = -2\beta_1 + 3(\beta + \beta_2)$

$\therefore$ the mtx of $T = \begin{pmatrix} 4 & 4 & -2 \\ 0 & -1 & 3 \end{pmatrix}$.

3. The matrix of a linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^2$ relative to the ordered bases $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ and $\{(1,0), (1,1)\}$ of $\mathbb{R}^2$ is $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$. Find T. Also find the matrix of T relative to the ~~ordered bases $\{(0,1,0), (1,0,1), (0,1,0)\}$ of $\mathbb{R}^3$ and $\{(1,1), (0,1)\}$ of $\mathbb{R}^2$~~

~~Solution:- $T(0,1,1) = 1(1,0) + 2(1,1)$~~

ordered bases $\{(1,1,0), (1,0,1), (0,1,1)\}$ of $\mathbb{R}^3$ and $\{(1,1), (0,1)\}$ of $\mathbb{R}^2$.

Solution:- $T(0,1,1) = 1(1,0) + 2(1,1) = (3,2)$

$T(1,0,1) = 2(1,0) + 1(1,1) = (3,1)$

$T(1,1,0) = 4(1,0) + 0(1,1) = (4,0)$

Let $(x,y,z) \in \mathbb{R}^3$ and let $(x,y,z) = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0)$; where c_i are scalars in $\mathbb{R}$.

Then $c_2 + c_3 = x$, $c_1 + c_3 = y$, $c_1 + c_2 = z$

$\therefore$ $c_1 = \frac{1}{3}(y+z-x)$, $c_2 = \frac{1}{2}(z+x-y)$, $c_3 = \frac{1}{2}(x+y-z)$.

$T(x,y,z) = c_1 T(0,1,1) + c_2 T(1,0,1) + c_3 T(1,1,0)$

$= c_1(3,2) + c_2(3,1) + c_3(4,0)$

$= (3c_1 + 3c_2 + 4c_3, \; 2c_1 + c_2)$

$= \left(2x + 2y + z, \; \frac{1}{2}(-x + y + 3z)\right)$, $(x,y,z) \in \mathbb{R}^3$,

2nd Part:-

$T(1,1,0) = (4,0) = 4(1,1) - 4(0,1)$

$T(1,0,1) = (3,1) = 3(1,1) - 2(0,1)$

$T(0,1,1) = (3,2) = 3(1,1) - 1(0,1)$

The matrix of $T = \begin{pmatrix} 4 & 3 & 3 \\ -4 & -2 & -1 \end{pmatrix}$.

Theorem:– Let $T: V \to V'$ be a linear transformation $\ni$ $\dim V = \dim V'$, then the followings are equivalent:

(a) T is non-singular;
(b) T is invertible;
(c) T is one-to-one;
(d) T is onto;

PROBLEMS:–

1. The matrix of a linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ w.r.t. the ordered basis $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ is given by

$$\begin{pmatrix} 0 & 3 & 0 \\ 2 & 3 & -2 \\ 2 & -1 & 2 \end{pmatrix}.$$

Find T. Find the matrix of T relative to the ordered basis $\{(2,1,1), (1,2,1), (1,1,2)\}$ of $\mathbb{R}^3$.

Solution: $\to$

$$T(0,1,1) = 0(0,1,1) + 2(1,0,1) + 2(1,1,0) = (4,2,2)$$
$$T(1,0,1) = 3(0,1,1) + 3(1,0,1) - 1(1,1,0) = (2,2,6)$$
$$T(1,1,0) = 0(0,1,1) - 2(1,0,1) + 2(1,1,0) = (0,2,-2)$$

Let $(x,y,z) \in \mathbb{R}^3$ and let $(x,y,z) = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0)$ where $c_i \in \mathbb{R}$. Then

$$c_1 = \frac{-x+y+z}{2}, \quad c_2 = \frac{x-y+z}{2}, \quad c_3 = \frac{x+y-z}{2}.$$

$$T(x,y,z) = c_1 T(0,1,1) + c_2 T(1,0,1) + c_3 T(1,1,0)$$
$$= c_1(4,2,2) + c_2(2,2,6) + c_3(0,2,-2)$$
$$= (-x+y+3z,\ x+y+z,\ x-3y+5z).$$

$$T(2,1,1) = (2,4,4); \quad T(1,2,1) = (4,4,0); \quad T(1,1,2) = (6,4,8);$$

Let $(2,4,4) = c_1(2,1,1) + c_2(1,2,1) + c_3(1,1,2)$.
Then $c_1 + c_2 + c_3 = 5/2$ and therefore $c_1 = -1/2$, $c_2 = 3/2$, $c_3 = \frac{3}{2}$.

Let $(4,4,0) = c_1(2,1,1) + c_2(1,2,1) + c_3(1,1,2)$
Then $c_1 + c_2 + c_3 = 2$ and therefore $c_1 = 2$, $c_2 = 2$, $c_3 = -2$.

Let $(6,4,8) = c_1(2,1,1) + c_2(1,2,1) + c_3(1,1,2)$
then $c_1 + c_2 + c_3 = 9/2$ and therefore $c_1 = 3/2$, $c_2 = -1/2$, $c_3 = \frac{7}{2}$.

$$\therefore\ m(T) = \begin{pmatrix} -1/2 & 2 & 3/2 \\ 3/2 & 2 & -1/2 \\ -3/2 & -2 & 7/2 \end{pmatrix}.$$

The Transpose of a Linear Transformation:-

1. Let V and W be vector space over the field F. For each linear transformation T from V into W, there is a unique linear transformation T^t from W^* into V^* such that $(T^t g)(\alpha) = g(T\alpha)$ for every $g \in W^*$ and $\alpha \in V$.

2. Let V and W be vector spaces over the field F, and let T be a linear transformation from V into W. The null space of T^t is the annihilator of the range of T. Then V and W are finite dimensional, then

(i) $\text{Rank}(T^t) = \text{rank}(T)$

(ii) the range of T^t is annihilator of the null space of T.

WORKED EXAMPLES:-

1. Let $(\alpha_1, \alpha_2, \alpha_3)$ be an ordered basis of a real vector space V and a linear mapping $T: V \to V$ is defined by

$$T(\alpha_1) = \alpha_1 + \alpha_2 + \alpha_3, \; T(\alpha_2) = \alpha_1 + \alpha_2, \; T(\alpha_3) = \alpha_1.$$

Show that T is non-singular. Find the matrix of T^{-1} relative to the ordered basis $(\alpha_1, \alpha_2, \alpha_3)$.

Solution:- Let $m(T)$ be the matrix of T relative to the ordered basis $(\alpha_1, \alpha_2, \alpha_3)$.

$$\text{Then } m(T) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$m(T)$ is non-singular and therefore T is non-singular.

$$T^{-1}(\alpha_1 + \alpha_2 + \alpha_3) = \alpha_1 \Rightarrow T^{-1}(\alpha_1) + T^{-1}(\alpha_2) + T^{-1}(\alpha_3) = \alpha_1$$

$$T^{-1}(\alpha_1 + \alpha_2) = \alpha_2 \Rightarrow T^{-1}(\alpha_1) + T^{-1}(\alpha_2) = \alpha_2$$

$$T^{-1}(\alpha_1) = \alpha_3 \Rightarrow T^{-1}(\alpha_1) = \alpha_3$$

; since T^{-1} is also linear.

∴ We have $T^{-1}(\alpha_1) = \alpha_3$, $T^{-1}(\alpha_2) = \alpha_2 - \alpha_3$,

$$T^{-1}(\alpha_3) = \alpha_1 - \alpha_2.$$

$$\text{Therefore, } m(T^{-1}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

2) Let T be the linear transformation from $\mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$.
If $\underset{\sim}{\beta} = \{(1,0,-1), (1,1,1), (1,0,0)\}$, $\underset{\sim}{\beta}' = \{(0,1), (1,0)\}$ be ordered bases of $\mathbb{R}^3, \mathbb{R}^2$, respectively. Then find the matrix of T relative to $\underset{\sim}{\beta}, \underset{\sim}{\beta}'$, also find rank (T) and nullity (T).

Solution:- (i)
$$T(1,0,-1) = (1,-3) = -3(0,1) + 1(1,0)$$
$$T(1,1,1) = (2,1) = 1(0,1) + 2(1,0)$$
$$T(1,0,0) = (1,-1) = -1(1,0) + 1(0,1)$$

$\therefore$ the matrix of T is $\begin{bmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

(ii) Let $(x_1, x_2, x_3) \in \ker T$.

$T(x_1, x_2, x_3) = (0,0) = (x_1 + x_2, 2x_3 - x_1)$

$\therefore x_1 + x_2 = 0, \; 2x_3 - x_1 = 0$

$\Rightarrow x_1 = -x_2, \; x_1 = 2x_3$

$\therefore x_1 = -x_2 = 2x_3$, let $x_3 = t$,

$\therefore \ker(T) = \{t(2,-2,1) : t \in \mathbb{R}\}$

$\therefore \dim \ker T = 1$.

i.e. nullity$(T) = 1$.

We know, $\text{rank}(T) + \text{nullity}(T) = \dim \mathbb{R}^3 = 3$

$\Rightarrow \text{rank}(T) = 3 - 1 = 2$.

3) Show that $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x,y,z) = (y,x)$ is a linear transformation. For the basis sets $S_1 = (\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ of $\mathbb{R}^1$ and $S_2 = \{(1,0), (2,3)\}$ of $\mathbb{R}^2$. Find the mtx of T.

Solution:-
$$T(\alpha \underset{\sim}{u} + \beta \underset{\sim}{v}) = (\alpha y_1 + \beta y_2, \alpha x_1 + \alpha x_2); \quad \underset{\sim}{u} = (x_1, y_1, z_1), \; \underset{\sim}{v} = (x_2, y_2, z_2) \in \mathbb{R}^3.$$
$$= \alpha(y_1, x_1) + \beta(y_2, x_2)$$
$$= \alpha T(\underset{\sim}{u}) + \beta T(\underset{\sim}{v}).$$

Hence T is a linear transformation.

$T(\underset{\sim}{e}_1) = (0,1)$, $T(\underset{\sim}{e}_2) = (1,0)$, $T(\underset{\sim}{e}_3) = (0,0)$

$$T(\underset{\sim}{e}_1) = (0,1) = -\tfrac{2}{3}(1,0) + \tfrac{1}{3}(2,3)$$
$$T(\underset{\sim}{e}_2) = (1,0) = 1(1,0) + 0(2,3)$$
$$T(\underset{\sim}{e}_3) = (0,0) = 0(1,0) + 0(2,3)$$

$\because (0,1) = x\alpha_1 + y\alpha_2$
$= x(1,0) + y(2,3)$
$= (x + 2y, 3y)$
$\therefore x = -2/3, \; y = 1/3$

The matrix representation of T is $\begin{bmatrix} -2/3 & 1 & 0 \\ 1/3 & 0 & 0 \end{bmatrix}$.

PROBLEMS:—

1. Let T be an invertible linear operator on $V_3(\mathbb{R})$ defined by $T(x,y,z) = (3x,\ x-y,\ 2x+y+z)\ \forall\ (x,y,z) \in V_3(\mathbb{R})$. Then $T^{-1}(3,0,4)$ is equal to

(A) $(4,0,3)$ (B) $(1,1,1)$ (C) $(1,2,3)$ (D) $(0,0,0)$

Sol. (B) If $T(x,y,z) = (3,0,4)$ then $T^{-1}(3,0,4) = (x,y,z)$

Now, $T(x,y,z) = (3,0,4)$

$\Rightarrow (3x,\ x-y,\ 2x+y+z) = (3,0,4)$

$\Rightarrow 3x = 3,\ x-y=0,\ 2x+y+z=4$

$\Rightarrow x=1,\ y=1,\ z=1$

$\therefore T^{-1}(3,0,4) = (1,1,1)$

2. The linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined as $T(x_1,x_2,x_3) = (x_1-x_2+2x_3,\ 2x_1+x_2-x_3,\ -x_1-2x_2)$. Then the null space of T is

(A) $\{(0,0,-2), (-1,0,2)\}$ (B) $\{(0,0,0)\}$ (C) $\{(1,0,0),(0,0,0)\}$ (D) None

Sol. $T(x_1,x_2,x_3) = (0,0,0) = (x_1-x_2+2x_3,\ 2x_1+x_2-x_3,\ -x_1-2x_2)$

$\Rightarrow x_1-x_2+2x_3=0,\quad 2x_1+x_2-x_3=0,\quad -x_1-2x_2=0$

Coefficient mtx is $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 1 & -2 & 0 \end{bmatrix} = A$

$|A| = -9 \neq 0$

$\therefore$ Rank$(A) = 3$

Hence the equations have the only trivial solution $x_1 = x_2 = x_3 = 0$.

$\therefore (0,0,0)$ is the only vector which belongs to the null space of T.

3. The linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined as $T(x,y,z) = (3x-2y+z,\ x-3y-2z)$, then the range of T is

(A) $\{(3,-2,1), (1,-3,-2)\}$ (B) $\{(3,1), (-2,-3), (1,-2)\}$
(C) $\{(1,0,0), (0,1,0), (0,0,1)\}$ (D) None of the above.

Solution:- (B) $\{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis for $\mathbb{R}^3$.

$\therefore T(1,0,0) = (3,1)$

$T(0,1,0) = (-2,-3)$

$T(0,0,1) = (1,-2)$

4. Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(x,y,z) = (x+2y-z,\ y+z,\ x+y-2z)$. Then which of the following is not a null space of T

(A) $\{(3,1,-1)\}$ (B) $\{(6,-4,2)\}$ (C) $\{(2,-2/3,2/3)\}$, (D) $\{(6,-2,2)\}$

Sol. → (D)

$$T(3,1,-1) = (0,0,0)$$
$$T(6,-4,2) = (0,0,0)$$
$$T(2,-2/3,2/3) = (0,0,0)$$
But $T(6,-2,2) \neq (0,0,0)$.

5. With respect to the standard basis vectors a linear transformation $T:\mathbb{R}^4 \to \mathbb{R}^3$ is given by the matrix $\begin{bmatrix} 3 & -1 & -1 & 1 \\ -2 & 2 & -2 & -2 \\ -1 & 1 & 3 & -1 \end{bmatrix}$

Then dimension of $\ker(T)$ is

(A) 1 (B) 2 (C) 3 (D) 4

Sol. (A) $\quad \text{dimension}(\ker T) = \dim(\mathbb{R}^4) - \text{rank}(T)$
$$= 4-3$$
$$= 1.$$

6. The coordinate of the vector $(2,1,-6)$ relative to the basis $\{\alpha_1, \alpha_2, \alpha_3\}$, where $\alpha_1 = (1,1,2)$, $\alpha_2 = (3,-1,0)$, $\alpha_3 = (2,0,-1)$ is

(A) $(-7/8, -15/8, 17/4)$ (B) $(-9/4, 13/8, 17/4)$ (C) $(-7/2, -9/4, -15/8)$
(D) None of these.

Sol. Let a_i, $i=1(1)3$ are scalars in $\mathbb{R}$.
$$(2,1,-6) = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$$
$$\Rightarrow (2,1,-6) = a_1(1,1,2) + a_2(3,-1,0) + a_3(2,0,-1)$$
$$\Rightarrow (2,1,-6) = (a_1+3a_2+2a_3,\ a_1-a_2,\ 2a_1-a_3)$$
$$\Rightarrow a_1+3a_2+2a_3 = 2,\quad a_1-a_2 = 1,\quad 2a_1-a_3 = -6$$
Solving these equations, we get
$$\Rightarrow a_1 = -7/8,\quad a_2 = -15/8,\quad a_3 = 17/4.$$

7. A linear transformation $T: V \to W$ is defined as $T(x,y,z) = (2x, 4y, 5z)$. Then find the mtx of T w.r.t. the basis $(2/3,0,0)$, $(0,1/2,0)$, $(0,0,1/4)$.

Sol.
$$T(2/3,0,0) = (4/3,0,0) = 2(2/3,0,0)$$
$$T(0,1/2,0) = (0,2,0) = 4(0,1/2,0)$$
$$T(0,0,1/4) = (0,0,5/4) = 5(0,0,1/4)$$
Hence the required mtx is $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

8. Let T be a linear operator on $\mathbb{R}^2$, the mtx of which in the standard ordered basis is $A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$

Prove that the only subspaces of $\mathbb{R}^2$ invariant under T are $\mathbb{R}^2$ and zero subspaces.

Sol. Ch. polynomial of A (or T) is $\det \begin{bmatrix} x-1 & 1 \\ -2 & x-2 \end{bmatrix} = x^2 - 3x + 4 = 0$, whose roots are not real.

& Eigen values of A (or T) donot exist in $\mathbb{R}$. If W is an invariant subspace of $\mathbb{R}^2$ s.t. $W \neq 0$, $\mathbb{R}^2$ with $\dim W = 1$.

Let W be spanned by v. Then $T(v) \in W \Rightarrow Tv = \alpha v$, $v \neq 0$

$\Rightarrow \alpha$ is an eigen value of T ($\alpha \in \mathbb{R}$), a contradiction.

$\therefore$ 0 and $\mathbb{R}^2$ are the only invariant subspaces of $\mathbb{R}^2$.

9. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T(x,y,z) = (x+y+z,\ y+z,\ z)$. Then $T^n(x,y,z)$ is ? $(n \geq 1)$.

Sol.
$$T^2(x,y,z) = (x+2y+3z,\ y+2z,\ z)$$
$$T^3(x,y,z) = (x+3y+6z,\ y+3z,\ z)$$
$$\vdots$$
$$T^n(x,y,z) = \left(x+ny+\frac{n(n+1)}{2}z,\ y+nz,\ z\right)$$

10. Let $V = \mathbb{R}^3$ and $T: V \to V$ is a linear map $\ni$ the mtx of T w.r.t. the standard basis is $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Show that T is an isomorphism of V' onto V'. Find the mtx of T^{-1} w.r.t. the same basis and verify that it is the inverse of A.

Sol.
$$T(x,y,z) = \underset{\sim}{0},$$
Then, $(x+z,\ y+z,\ x+y) = (0,0,0)$
$$\Rightarrow x = y = z = 0.$$

Hence T is one-to-one, so it is an isomorphism.

Let $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ be the standard basis.

Since $T(\underset{\sim}{e}_1) = \underset{\sim}{e}_1 + \underset{\sim}{e}_3$, $T(\underset{\sim}{e}_2) = \underset{\sim}{e}_2 + \underset{\sim}{e}_3$, $T(\underset{\sim}{e}_3) = \underset{\sim}{e}_1 + \underset{\sim}{e}_2$

$$\Rightarrow \underset{\sim}{e}_1 = \frac{1}{2}\left[T(\underset{\sim}{e}_1) + T(\underset{\sim}{e}_3) - T(\underset{\sim}{e}_2)\right] = \frac{1}{2}T(\underset{\sim}{e}_1 + \underset{\sim}{e}_3 - \underset{\sim}{e}_2) \quad [\text{As } T \text{ is linear}]$$

$$\& \ T^{-1}(\underset{\sim}{e}_1) = \frac{1}{2}(\underset{\sim}{e}_1 + \underset{\sim}{e}_3 - \underset{\sim}{e}_2)$$

Similarly, $T^{-1}(\underset{\sim}{e}_2) = \frac{1}{2}(\underset{\sim}{e}_2 + \underset{\sim}{e}_3 - \underset{\sim}{e}_1)$

and $T^{-1}(\underset{\sim}{e}_3) = \frac{1}{2}(\underset{\sim}{e}_1 + \underset{\sim}{e}_2 - \underset{\sim}{e}_3)$

Hence the mtx of T^{-1} is $\frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} = M$

Now, $AM = I_3$, so we get that it is also the inverse of A.

SOME PROBLEMS ON LINEAR ALGEBRA

1. $A^{n\times n}$; $a_{ij} = 1, \forall i=j$
$= \rho \ \forall i \neq j$

Find the determinant of A.

Solution:-

$$\begin{pmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho & 1 & \rho & \cdots & \rho \\ \rho & \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \vdots & & \vdots \\ \rho & \rho & \rho & \cdots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+\overline{n-1}\rho & \rho & \cdots & \rho \\ 1+\overline{n-1}\rho & \cancel{\rho}1 & \cdots & \rho \\ \rho & \rho & \cdots & \rho \\ \vdots & \vdots & & \vdots \\ 1+\overline{n-1}\rho & \rho & \cdots & 1 \end{pmatrix}$$

$$= (1+\overline{n-1}\rho) \begin{pmatrix} 1 & \rho \\ 1 & \\ \vdots & \\ 1 & \end{pmatrix}$$

PROBLEMS ON LINEAR ALGEBRA [FROM CU & ISI PAPERS]

Ex. 1. Let A and B be two invertible $n\times n$ matrices. Assume that $A+B$ is invertible. Then show that $A^{-1}+B^{-1}$ is invertible.

Sol.

$$|A(A^{-1}+B^{-1})B| = |A+B|$$

$$\Rightarrow A^{-1}+B^{-1} = \frac{|A+B|}{|A||B|}$$

Ex. 2. Let A and B be $n\times n$ real matrices $\ni A^2=A$, $B^2=B$. Let $I-(A+B)$ is invertible. Show that $\text{rank}(A)=\text{rank}(B)$.

Sol.

$$A[I-A-B] = A-A^2-AB$$
$$= A^2-A^2-AB$$
$$= -AB$$

$$\& \ [I-A-B]B = B-AB-B^2$$
$$= -AB$$

$\therefore$ rank (A) = rank $[A(I-A-B)]$ = rank $(-AB)$ = rank (B).

$$\therefore r(A) = r(B).$$

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Cayley Hamilton Theorem:— Let $A^{n\times n}$ be a matrix satisfies the matrix equation $f(A)=0$ of degree n, the ch. equation is $f(\lambda)=0$.

Ex. 3. $A^{2\times 2} \ni A^2=0$

$$\Rightarrow \lambda^2=0$$
$$\Rightarrow \lambda=0,0$$

Now, $(I+A)\underset{\sim}{x} = \underset{\sim}{x} + A\underset{\sim}{x}$
$= \underset{\sim}{x} + \lambda \underset{\sim}{x}$
$= \underset{\sim}{x} + 0.\underset{\sim}{x}$
$= 1.\underset{\sim}{x}$, $\underset{\sim}{x} \neq 0$

Now, $\lambda = 1, 1$, then $|I+A| = 1.1 = 1$.

Ex. 4 Let $A^{n\times n}$ be an orthogonal matrix and n is even. Let $|A|=-1$. Find $|I-A|$.

Sol. A is a mtx with charcteristic root λ.

$A' = A^{-1}$, as A is orthogonal.

$\Rightarrow \lambda = \frac{1}{\lambda}$ $\qquad |A| = -1 = \lambda_1 \lambda_2 \cdots \lambda_n$

$\Rightarrow \lambda = \pm 1$; $\qquad \Rightarrow$ at least one λ_i's must be 1.

ch. equation:— $|\lambda I - A| = 0$, set $\lambda = 1$

$\Rightarrow |I-A| = 0$.

Ex. 5. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$. Find $A^{100} + A^5$.

Sol. $0 = |A - \lambda I_2| = \begin{vmatrix} 1-\lambda & -1 \\ 2 & -2-\lambda \end{vmatrix} = \lambda^2 + \lambda$

Using C-H Theorem, $A^2 + A = 0$

$A^5 = (A^2)^2 . A = (-A)^2 . A = A^2 . A = -A . A = A$

$A^{100} = (A^5)^{20} = (A^5)^4 = (A^2)^2 = (-A)^2 = A^2 = -A$

$\therefore A^{100} + A^5 = 0$.

Ex. 6 Find $\begin{vmatrix} 1+x_1y_1 & 1+x_1y_2 & \cdots & 1+x_1y_n \\ 1+x_2y_1 & 1+x_2y_2 & \cdots & 1+x_2y_n \\ \vdots & \vdots & & \vdots \\ 1+x_ny_1 & 1+x_ny_2 & \cdots & 1+x_ny_n \end{vmatrix}$

Sol. $\Delta = \begin{vmatrix} 1 & x_1 & 0 & \cdots & 0 \\ 1 & x_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & 0 & \cdots & 0 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & \cdots & 1 \\ y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \end{vmatrix} = 0 \times 0 = 0$.

Ex. Let P be a matrix of order > 1 and entries are +ve integers. Suppose P^{-1} exists and has integer entries, then what are the set of possible values of $|P|$?

(ISI)

Sol. P has integer entries.

$$\Rightarrow \lambda_1 + \lambda_2 + \cdots + \lambda_n = \text{trace}(P) = \text{integer}$$

$$\Rightarrow \sum_{i<j} \lambda_i \lambda_j = \text{Sum of minors of order}$$

$$\vdots$$

$$\prod_{i=1}^{n} \lambda_i = |P| = \text{integer}.$$

Then the eigen values of P^{-1} are $\frac{1}{\lambda_i}$ and they are also integers.

$$\Rightarrow \lambda_i = \frac{1}{\lambda_i}$$

$$\Rightarrow \lambda_i = \pm 1$$

$$\Rightarrow |P| = \prod_{i=1}^{n} \lambda_i = \pm 1.$$

Ex. Let XY be a bivariate normal vector $\ni$ $E(X) = E(Y) = 0$ & $V(X) = V(Y) = 1$. Let S be a subset of $\mathbb{R}^2$ and defined by

$$S = \{(a,b) : (ax+by) \text{ is independent of } y\}$$

(i) S.T. S be a subspace (ii) Find of its dimension.

Sol. $S = \{(a,b) : (ax+by) \text{ is independent of } Y\}$ (ISI)

(i) $(a_1, b_1), (a_2, b_2) \in S$.

Then $a_1 x + b_1 y$ is independent of Y.

$a_2 x + b_2 y$ is independent of Y.

$\Rightarrow (\alpha a_1 + \beta a_2) x + (\alpha b_1 + \beta b_2) y$ is independent of Y.

$\Rightarrow (\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2) \in S$

$\Rightarrow \alpha(a_1, b_1) + \beta(a_2, b_2) \in S \ \forall (\alpha, \beta) \in \mathbb{R}$

Hence, S is a subspace.

(ii) $(a,b) \in S$

$\Rightarrow ax + by$ is indep. of Y.

$\Rightarrow \text{cov}[ax+by, y] = 0$

$\Rightarrow a\,\text{cov}[x,y] + b\,\text{cov}[y,y] = 0$

$\Rightarrow a\rho + b.1 = 0$ since $\text{cov}[x,y] = \rho$

$\Rightarrow b = -a\rho$ & $\text{cov}[y,y] = \text{Var}(Y) = 1$, as $E(X) = 0 = E(Y)$ & $V(X) = 1 = V(Y)$

$\therefore (a,b) = a(1, -\rho), \ a \in \mathbb{R}$

$\therefore S = \{(a,b) : (a,b) = a(1,-\rho), \ a \in \mathbb{R}\}$

$\therefore \dim(S) = 1.$

Characteristic Roots & Ch. vectors:-

In mathematics and social sciences we may be requested in the solution of $\underset{\sim}{A x} = \lambda \underset{\sim}{x}$. We are looking for non-trivial solutions, i.e. for vectors $\underset{\sim}{x} \neq \underset{\sim}{0}$ ∋

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

$$\Rightarrow (A - \lambda I_n)\underset{\sim}{x} = \underset{\sim}{0} \quad \cdots\cdots\cdots (1)$$

There will be a non-trivial solution $\underset{\sim}{x} \neq \underset{\sim}{0}$,

iff $r(A - \lambda I_n) < n$

iff $|A - \lambda I_n| = 0$.

iff
$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \quad \cdots\cdots\cdots (2)$$

Note that, $f(\lambda) = |A - \lambda I_n|$

$= (-\lambda)^n + b_{n-1}(-\lambda)^{n-1} + \cdots + b_0$; a polynomial in λ,

is called the characteristic polynomial of the matrix $A^{n\times n}$.

The equation (2), i.e. $|A - \lambda I_n| = 0$ or $f(\lambda) = 0$ is called the <u>characteristic equation</u> of the mtx A.

The n^{th} degree polymial equation $f(\lambda) = 0$ has n roots, roots may be real or imaginary and all roots need not to be different.

The roots, say $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the equation (2), i.e.,

$|A - \lambda I_n| = 0$ are called the characteristic root or eigen value or latent roots of the matrix A.

When $\lambda \neq \lambda_i$, $i = 1(1)n$, then $|A - \lambda I_n| \neq 0$ and $(A - \lambda I_n)\underset{\sim}{x} = \underset{\sim}{0}$ has the only solution $\underset{\sim}{x} = \underset{\sim}{0}$.

When $\lambda = \lambda_i$, $i = 1(1)n$, then $|A - \lambda_i I_n| = 0$ and $(A - \lambda_i I_n)\underset{\sim}{x} = \underset{\sim}{0}$ has at least one non-null solution $\underset{\sim}{x}$.

The maximum number of LIN vectors $\underset{\sim}{x} \neq \underset{\sim}{0}$ which satisfy $[A - \lambda_i I_n]\underset{\sim}{x} = \underset{\sim}{0}$ is $\dim\{N(A - \lambda_i I_n)\}$.

The vectors $\underset{\sim}{x} \neq \underset{\sim}{0}$ which satisfy $(A - \lambda_i I_n)\underset{\sim}{x} = \underset{\sim}{0}$ are called the characteristic vector or eigen vectors corresponding to the eigen values λ_i of A.

Note that $f(\lambda) = |A - \lambda I_n|$

$$= (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda).$$

Ex.9. Let A be an upper triangular matrix with $d_1, \dots, d_n$ as the diagonal elements, show that the eigen values of A are $d_1, d_2, \dots, d_n$.

Solution:- Characteristic equation gives

$$|A - \lambda I_n| = 0 = \begin{vmatrix} d_1-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & d_2-\lambda & a_{23} & \cdots & a_{2n} \\ 0 & 0 & d_3-\lambda & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n-\lambda \end{vmatrix}$$

$$= \prod_{i=1}^{n} (d_i - \lambda)$$

$\Rightarrow \lambda = d_1, \dots, d_n$

Hence $d_1, d_2, \dots, d_2$ are the eigen values.

Ex.10. Find the characteristic roots of $A = \alpha\alpha^T$, $\underset{\sim}{\alpha} \neq \underset{\sim}{0}$.

Solution:- Characteristic equation is

$$0 = |\underset{\sim}{\alpha}\underset{\sim}{\alpha}^T - \lambda I_n|$$

$$= (-\lambda)^n \left|I_n - \frac{1}{\lambda} \cdot \underset{\sim}{\alpha}\underset{\sim}{\alpha}^T\right|$$

$$= (-\lambda)^n \left(1 - \frac{1}{\lambda} \cdot \underset{\sim}{\alpha}'\underset{\sim}{\alpha}\right)$$

$$= (-1)^n \lambda^{n-1} (\lambda - \underset{\sim}{\alpha}'\underset{\sim}{\alpha})$$

Hence the ch. roots are $\lambda_1 = \underset{\sim}{\alpha}'\underset{\sim}{\alpha}$ and $\lambda_2 = 0$ with multiplicity $(n-1)$

Ex.11. Find the ch. roots of $A = (I_n - \underset{\sim}{\alpha}\underset{\sim}{\alpha}^T)$.

Solution:- Characteristic equation is

$$0 = |I_n - \underset{\sim}{\alpha}\underset{\sim}{\alpha}^T - \lambda I_n|$$

$$\Rightarrow |(1-\lambda)I_n - \underset{\sim}{\alpha}\underset{\sim}{\alpha}^T| = 0$$

$$\Rightarrow (1-\lambda)^n \left|I_n - \frac{1}{1-\lambda}\underset{\sim}{\alpha}\underset{\sim}{\alpha}^T\right| = 0$$

$$\Rightarrow (1-\lambda)^n \left(1 - \frac{1}{1-\lambda}\underset{\sim}{\alpha}^T\underset{\sim}{\alpha}\right) = 0$$

$$\Rightarrow (1-\lambda)^{n-1} \{(1-\lambda) - \underset{\sim}{\alpha}^T\underset{\sim}{\alpha}\} = 0$$

Rank $(I_n - \underset{\sim}{\alpha}\underset{\sim}{\alpha}')$
$= \text{trace}(I_n) - \text{trace}(\underset{\sim}{\alpha}\underset{\sim}{\alpha}')$
$= n - 1$.

$\Rightarrow \lambda = 1 - \underset{\sim}{\alpha}^T\underset{\sim}{\alpha}$ or. 1 with multiplicity $(n-1)$.

Ex.12 (a) If $AA' = I_n$, then find the eigen value of A.

Sol. $A^T\underset{\sim}{x} = \lambda\underset{\sim}{x}$, $\underset{\sim}{x} \neq \underset{\sim}{0}$ [$\because |A - \lambda I_n| = 0 \Rightarrow |A^T - \lambda I_n| = 0$
$\Rightarrow$ A and A^T have the same set of eigen values]

$\therefore A\underset{\sim}{x} = \lambda\underset{\sim}{x}$, $\underset{\sim}{x} \neq \underset{\sim}{0}$

$\therefore A^T\underset{\sim}{x} = \lambda\underset{\sim}{x}$

$\Rightarrow AA^T\underset{\sim}{x} = \lambda A\underset{\sim}{x} = \lambda^2\underset{\sim}{x}$

$\Rightarrow I_n\underset{\sim}{x} = \lambda^2\underset{\sim}{x}$

$\Rightarrow (1-\lambda^2)\underset{\sim}{x} = \underset{\sim}{0}$, $\underset{\sim}{x} \neq \underset{\sim}{0}$

$\Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$.

(b) If $AA^T = n.I_n$ then find the eigen values of A.

Sol. $AA^T\underset{\sim}{x} = \lambda^2\underset{\sim}{x}$, $\underset{\sim}{x} \neq \underset{\sim}{0}$

$\Rightarrow n.I_n\underset{\sim}{x} = \lambda^2\underset{\sim}{x}$, $\underset{\sim}{x} \neq \underset{\sim}{0}$

$\Rightarrow \lambda^2 = n$

$\Rightarrow \lambda = \pm\sqrt{n}$.

Ex.13. Find the eigen value of $A = \begin{bmatrix} 1 & p & p \cdots p \\ p & 1 & p \cdots p \\ \vdots & \vdots & \vdots \\ p & p & p \cdots 1 \end{bmatrix} = (1-p)I_n + p\underset{\sim}{1}.\underset{\sim}{1}^T$

Sol.

$A = (1-p)I_n + p\underset{\sim}{1}.\underset{\sim}{1}^T$

characteristic equation is:

$|A - \lambda I_n| = |(1-p)I_n + p\underset{\sim}{1}.\underset{\sim}{1}^T - \lambda I_n| = 0$

$\Rightarrow |(1-p-\lambda)I_n + p\underset{\sim}{1}.\underset{\sim}{1}^T| = 0$

$\Rightarrow (1-p-\lambda)^n \left| I_n + \frac{p\underset{\sim}{1}.\underset{\sim}{1}^T}{(1-p-\lambda)} \right| = 0$

$\Rightarrow (1-p-\lambda)^n \left\{ 1 + \frac{p}{1-p-\lambda} \underset{\sim}{1}^T.\underset{\sim}{1} \right\} = 0$

$\Rightarrow (1-p-\lambda)^{n-1} \left\{ 1-p-\lambda + p\underset{\sim}{1}^T\underset{\sim}{1} \right\} = 0$

$\Rightarrow \lambda = 1-p+p\underset{\sim}{1}^T\underset{\sim}{1}$, $(1-p)$ with multiplicity $(n-1)$.

Ex.14. If A is an idempotent matrix, then find the eigen values of A and hence show that rank(A) = trace(A).

Sol. Let λ be an eigen value of A.

By definition, $A\underset{\sim}{x} = \lambda\underset{\sim}{x}$, $\underset{\sim}{x} \neq \underset{\sim}{0}$

$\Rightarrow A^2\underset{\sim}{x} = \lambda\underset{\sim}{x}$

$\Rightarrow A(A\underset{\sim}{x}) = \lambda\underset{\sim}{x}$

$\Rightarrow \lambda(A\underset{\sim}{x}) = \lambda\underset{\sim}{x}$

$\Rightarrow \lambda(\lambda\underset{\sim}{x}) = \lambda\underset{\sim}{x}$

$\Rightarrow \lambda^2\underset{\sim}{x} = \lambda\underset{\sim}{x}$

$\Rightarrow \lambda(\lambda-1)\underset{\sim}{x} = \underset{\sim}{0}$ $\forall \underset{\sim}{x} \neq \underset{\sim}{0}$

$\Rightarrow \lambda = 0, 1$.

Now, we can find an orthogonal mtx Q such that

$Q'AQ = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$

$\therefore$ Rank(A) = rank $\{Q'AQ\}$, since $|Q| = \pm 1$, i.e. Q is non-singular

$= r\{\text{diag}(\lambda_1, \ldots, \lambda_n)\}$

= No. of non-zero λ_i's

$= \sum_{i=1}^{n} \lambda_i$, since $\lambda_i = 0, 1$.

$= \text{trace}\{\text{diag}(\lambda_1, \ldots, \lambda_n)\}$

$= \text{trace}(Q'AQ)$ [$A = Q^T$, $B = AQ$

$= \text{trace}(AQQ')$ $\Rightarrow \text{trace}(AB) = \text{tr}(BA)$]

$= \text{trace}(A)$.

Ex. 15. Let λ be an eigen value of an n.s. mtx A. Show that $\frac{1}{\lambda}$ is an eigen value of A^{-1}. Further show that A and A^{-1} have the same set of eigen values. Can A have zero as eigen value?

Sol. Let $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$, $\underset{\sim}{x} \neq \underset{\sim}{0}$

$\Rightarrow A^{-1} A \underset{\sim}{x} = \lambda A^{-1} \underset{\sim}{x}$ [multiplying by A^{-1}]

$\Rightarrow \frac{1}{\lambda} A^{-1} A \underset{\sim}{x} = A^{-1} \underset{\sim}{x}$

$\Rightarrow \frac{1}{\lambda} \underset{\sim}{x} = A^{-1} \underset{\sim}{x}$ [$\because$ as $A^{-1}A = 1$]

$\Rightarrow \frac{1}{\lambda}$ is an eigen value of A^{-1}. (Proved)

If possible let $\lambda = 0$,

Then $A\underset{\sim}{x} = \underset{\sim}{0}$ $\forall \underset{\sim}{x} \neq \underset{\sim}{0}$

$\Rightarrow$ rank(A) < n, if $A^{n \times n}$ mtx.

$\Rightarrow |A| = 0$ which is a contradiction to the fact that A is non-singular.

Hence $\lambda \neq \underset{\sim}{0}$.

Ex. 16. Show that eigen values of real square mtx A (not necessarily symmetric) are same as the eigen values of A^T.

Are the eigen vectors of A same as the A^T?

Sol. Characteristic equation of A is

$$|A - \lambda I_n| = 0 = |A - \lambda I_n|' = |(A - \lambda I_n)^T|$$

i.e. $0 = |A^T - \lambda I_n|$ is the ch. equation of A^T.

$(A - \lambda I_n)\underset{\sim}{x} = \underset{\sim}{0}$, $\underset{\sim}{x} \neq \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{x} \in N(A - \lambda I_n) - \{\underset{\sim}{0}\}$

and $(A^T - \lambda I_n)\underset{\sim}{y} = \underset{\sim}{0}$, $\underset{\sim}{y} \neq \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{y} \in N(A^T - \lambda I_n) - \{\underset{\sim}{0}\}$

If A is symmetric then A and A^T will have the same set of eigen vectors.

If A is not symmetric, then A and A^T will have different set of eigen vectors.

Ex. 17. If A and B are row-equivalent $m \times n$ matrices, then the homogeneous system of linear equations $AX = 0$ and $BX = 0$ have exactly the same solution.

Proof:- Suppose, we pass from A to B by a finite sequence of elementary row operations:

$$A = A_0 \to A_1 \to A_2 \to \cdots \to A_k = B.$$

It is enough to prove that the systems $A_jX = 0$ and $A_{j+1}X = 0$ have the same solution, i.e. that one elementary row operation does not disturb the set of solutions.

So, suppose that B is obtained from A by a single elementary row operation. No matter which of the types of operations are used, each equation in the system $BX = 0$ will be a linear combination of the equations in the system $AX = 0$. Since the inverse of an elementary row operation is an elementary row operation, each equation in $AX = 0$ will be a linear combination of the equations in $BX = 0$. Hence, there are two equivalent systems and have the same solutions.

Ex. 18. Let A and B be two matrices such that B is obtained by interchanging its 1st and 2nd rows. Find an n.s. mtx P ∋ $PA = B$

Sol. Let $A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$ then $B = \begin{pmatrix} a_2 \\ a_1 \\ \vdots \\ a_m \end{pmatrix}$

$B = E_{12}A$, E_{12} is the elementary mtx obtained from I_n by interchanging the first and second row.

$\therefore B = PA$, (given)

$\therefore E_{12} = P$

P being an elementary mtx and non-singular because $|P| = -|I_n| = -1 \neq 0$.

Ex. 19. Let A and B be two matrices of order $p \times q$ and $q \times p$, respectively. Show that $|AB| = 0$ if $p > q$.

Sol. AB is matrix of ~~rank~~ $p \times p$.

$$\text{Rank}(AB) \leq \min[\text{Rank}(A), \text{Rank}(B)] \quad \text{——— (1)}$$

For $q < p$, $\text{rank}(A) \leq q$ and $\text{rank}(B) \leq q$.

giving that $\text{Rank}(AB) \leq q$.

$\Rightarrow$ Columns of $(AB)_{p \times p}$ are linearly dependent.

$\Rightarrow |AB| = 0$.

Ex. 20. For two subspaces W_1 and W_2 of $V_n(F)$,
$$\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

Proof:-

Let $\{\alpha_1, \ldots, \alpha_k\}$ be a basis of $W_1 \cap W_2$.
As this set is linearly independent and contains vectors from W_1, it can be extended to form a basis
$$\{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m\} \text{ of } W_1.$$
and for the similar reason
$$\{\alpha_1, \ldots, \alpha_k, \gamma_1, \ldots, \gamma_n\} \text{ of } W_2.$$

The subspace W_1+W_2 is spanned by the vectors
$$\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_n$$
and these vectors form an independent set:

For suppose $\sum x_i\alpha_i + \sum y_j\beta_j + \sum z_r\gamma_r = 0.$

Then, $-\sum z_r\gamma_r = \sum x_i\alpha_i + \sum y_j\beta_j$ which shows that $\sum z_r\gamma_r$ belongs to W_1. As $\sum z_r\gamma_r$ also belong to W_2 it follows that
$$\sum z_r\gamma_r = \sum c_i\alpha_i,$$
for certain scalars $c_1, \ldots, c_k$. Because the set
$$\{\alpha_1, \ldots, \alpha_k, \gamma_1, \ldots, \gamma_n\}$$
is independent, each of the scalars $z_r = 0$. Thus
$$\sum x_i\alpha_i + \sum y_j\beta_j = 0$$
and since $\{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m\}$ is also an independent set, each $x_i = 0$ and each $y_j = 0$. Thus,
$$\{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_n\}$$
is a basis for W_1+W_2. Finally
$$\begin{aligned}\dim(W_1) + \dim(W_2) &= (k+m) + (k+n) \\ &= k + (k+m+n) \\ &= \dim(W_1 \cap W_2) + \dim(W_1+W_2)\end{aligned}$$
$$\therefore \dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

EX.21. Let A be a square matrix of order $p \ni$ for some non-null vector $\underset{\sim}{\alpha}$, $A\underset{\sim}{\alpha}=0$. Show that for any square matrix B of order p, $|AB|=0$.

Sol. $A\underset{\sim}{\alpha}=\underset{\sim}{0}$ for $\underset{\sim}{\alpha}\neq\underset{\sim}{0}$

Nullity of $A \geq 1$

$R(A) \leq n-1$

$\therefore$ A is non-singular.

$\therefore$ $|A|=0$

$\therefore |AB|=|A||B|=0$.

EX. 22. Find the dimension of the subspace of E^3:

i) $S_1=\{(x_1, x_2, 0): x_1, x_2 \in R\}$

ii) $S_2=\{(x_1, x_2, x_3): x_1-x_2=x_3\}$

iii) $S_3=\{\lambda(1,1,1): \lambda \in R\}$

Sol. (i) Let $\underset{\sim}{x} \in S_3$ be an arbitrary vector.

Then $\underset{\sim}{x}=(x_1, x_2, 0)$, $x_1, x_2 \in R$

$= x_1\underset{\sim}{e}_1 + x_2\underset{\sim}{e}_2$.

Clearly, $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2\}$ spans S_3 since it is LIN set of vectors.

$\therefore \{e_1, e_2\}$ forms a basis for S_3;

Hence, $\dim(S_3)$ = No. of vectors in a basis = 2.

(ii) Let $\underset{\sim}{x} \in S_3$ be an arbitrary vector.

Then $\underset{\sim}{x}=(x_1, x_2, x_3)$: $x_1-x_2=x_3$

$=(x_1, x_2, x_1-x_2)$

$=x_1(1,0,1)+x_2(0,1,-1)$

Clearly $\{(1,0,1),(0,1,-1)\}$ spans S_3 and they are LIN, i.e. they form a basis for S_3. Hence, $\dim(S_3)$ = No. of vectors in a basis = 2.

Remark:- It is also intuitively appealing to takes the dimension of a vector space as the number of independent components in a vector of the vector space.

Clearly, $S_3=\{(x_1, x_2, x_3)): x_1, x_2 \in R\}$ has 2 independent components in each vector, $\dim(S_3)=2$.

Ex.23. For any square mtx, the determinant of the transposed matrix is the same as the determinant of the matrix itself.

Solution:- Let the square matrix be of order n. Let $A=(a_{ij})$ then the determinant of A can A be written as

$$|A| = \sum \mp a_{1k_1} a_{2k_2} a_{3k_3} \ldots\ldots a_{nk_n} \quad \cdots\cdots (1)$$

where, the sum is taken over all permutations $(k_1, \ldots, k_n)$ of first n natural numbers and the sign attached to every term of the sum is minus or plus if the corresponding permutation is odd or even.

Let us re-arrange the factors of each term so as to bring the secondary suffixes in the natural order $(1, 2, \ldots, n)$ and modifying the primary suffixes accordingly. The term $a_{1k_1} a_{2k_2} a_{3k_3} \ldots a_{nk_n}$ then taken as the form $a_{l_1 1} a_{l_2 2} a_{l_3 3} \ldots\ldots a_{l_n n}$ and the permutation $(l_1, \ldots, l_n)$ is odd or even according to the permutation $(k_1, \ldots, k_n)$ is odd or even.

Thus, $$|A| = \sum \mp a_{l_1 1} a_{l_2 2} \ldots\ldots a_{l_n n} \quad \cdots\cdots (2)$$

If A' be the transpose of A, a'_{ij} be the (i,j)th term of A' then $a'_{ij} = a_{ji} \ \forall i, j$.

So, $$|A'| = \sum \mp a'_{1l_1} a'_{2l_2} \ldots\ldots a'_{nl_n}$$

or $$|A'| = \sum \mp a_{l_1 1} a_{l_2 2} \ldots\ldots a_{l_n n} \quad \cdots\cdots (3)$$

From (2) & (3), $|A| = |A'|$.

Ex.24. If the two rows of a determinant $|A|$ are equal, then $|A| = 0$.

Solution:-

Since the two rows of the determinant $|A|$ are identical, if we interchange the identical rows, then its value remains $|A|$. Further, due to the interchange of two rows, the sign of the determinant gets changed. Thus, we get,

$$|A| = -|A|$$
$$\therefore |A| = 0.$$

Ex. 25. $A\underset{\sim}{x} = \underset{\sim}{0} \ \forall \ \underset{\sim}{x} \neq \underset{\sim}{0}$, What can you say about A?

Sol. $A\underset{\sim}{x} = \underset{\sim}{0} \ \forall \ \underset{\sim}{x} \neq \underset{\sim}{0}$

$\Rightarrow$ The rows of A are linearly dependent.

$\therefore$ A is singular.

Ex. 26. Suppose $AA' = pI_p$, A $p \times p$ matrix, then What can you say about $A'A$?

Sol. As, $AA' = pI_p$

$$\Rightarrow \frac{A}{\sqrt{p}} \cdot \frac{A'}{\sqrt{p}} = I_p$$

$$\Rightarrow BB' = I_p, \text{ where } B = \frac{A}{\sqrt{p}}, \ B' = \frac{A'}{\sqrt{p}}$$

$\therefore$ B is an orthogonal matrix.

$$\Rightarrow B'B = I_p$$

$$\Rightarrow \frac{A'}{\sqrt{p}} \cdot \frac{A}{\sqrt{p}} = I_p$$

$$\Rightarrow A'A = pI_p$$

And $\det(A'A) = \det(pI_p) = p^p \neq 0$ for $p \neq 0$.

$\therefore (A'A)^{-1}$ exists and $A'A$ is non-singular.

Ex. 27. Give an example of an idempotent matrix whose rank is 1.

Sol.
$$A = \begin{pmatrix} 1/n & 1/n & \cdots & 1/n \\ 1/n & 1/n & \cdots & 1/n \\ \vdots & \vdots & & \vdots \\ 1/n & 1/n & \cdots & 1/n \end{pmatrix}$$

e.g. $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$

$A^2 = A$ .. Rank of A mtx = trace of A mtx.

$\text{Rank}(A) = 1$.

Note:- If
$$A = \begin{pmatrix} K/n & 1/n & \cdots & 1/n \\ 1/n & K/n & \cdots & 1/n \\ \vdots & \vdots & & \vdots \\ 1/n & 1/n & \cdots & K/n \end{pmatrix}$$

Here, $\text{Rank}(A) = K$.

Ex. 28. Consider two subspaces S_n' and S_n'' of V_n. Define

$$S_n' + S_n'' = \{ \underset{\sim}{a} + \underset{\sim}{b} : \underset{\sim}{a} \in S_n', \underset{\sim}{b} \in S_n'' \}.$$

Prove that $S_n' + S_n''$ is a vector subspace of V_n. If $S_2' = \{ \lambda(2,1) : \lambda \in R \}$ and $S_2'' = \{ \alpha(1,3) : \alpha \in R \}$, show that $S_2' + S_2'' = V_2$. Illustrate geometrically.

Solution:- Let $\underset{\sim}{x}, \underset{\sim}{y} \in S_n' + S_n''$

Then $\underset{\sim}{x} = \underset{\sim}{a_1} + \underset{\sim}{b_1}$

$\underset{\sim}{y} = \underset{\sim}{a_2} + \underset{\sim}{b_2}$ where $\underset{\sim}{a_1}, \underset{\sim}{a_2} \in S_n'$

$\underset{\sim}{b_1}, \underset{\sim}{b_2} \in S_n''$

Now $\underset{\sim}{x} + \underset{\sim}{y} = (\underset{\sim}{a_1} + \underset{\sim}{a_2}) + (\underset{\sim}{b_1} + \underset{\sim}{b_2})$

$= \underset{\sim}{a}^* + \underset{\sim}{b}^*$; $\underset{\sim}{a}^*, \underset{\sim}{b}^* \in S_n', S_n''$.

As S_n' and S_n'' are closed under vector addition,

$\underset{\sim}{x} + \underset{\sim}{y} \in S_n' + S_n''$.

Again, for any $\alpha \in R$, $\alpha \underset{\sim}{x} = \alpha \underset{\sim}{a_1} + \alpha \underset{\sim}{b_1}$

$= \underset{\sim}{a}' + \underset{\sim}{b}'$

$\therefore \alpha \underset{\sim}{x} \in S_n' + S_n''$ for any $\alpha \in R$.

Hence, $S_n' + S_n''$ is a subspace of V_n.

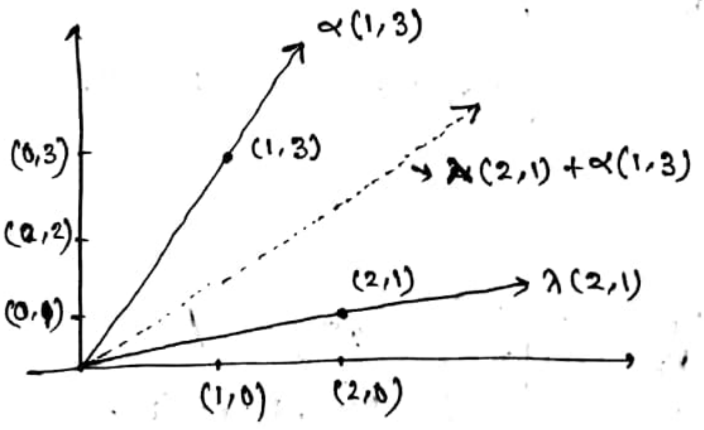
Here, $S_2' + S_2'' = \{ \lambda(2,1) + \alpha(1,3) : \lambda, \alpha \in R \}$

Note that $(2,1)$ and $(1,3)$ are LIN from V_2, because $(2,1)$ can't be written as a linear combination of $(1,3)$.

Hence every vector in V_2 can be written as a linear combination of $(2,1)$ and $(1,3)$.

i.e. $V_2 = \{ \lambda(2,1) + \alpha(1,3) : \lambda, \alpha \in R \}$

$\therefore S_2' + S_2'' = V_2$



For different (λ, α), we get different vector in V_2. Geometrically, every vector in V_2 can be obtained as $\lambda(2,1) + \alpha(1,3)$.

MATRIX ALGEBRA

Elementary Row Operation :—

i) Multiplication of one row of A by a non-zero scalar 'c'.
ii) A replacement of r^{th} row of A by row 'r' plus c times row 's', c be any scalar and $r \neq s$.
iii) Interchange of two rows of A.

Row-equivalent Matrix :—

Definition:– For two matrix A and B of order $m \times n$ over the field F, we say that B is row-equivalent to A if B can be obtained from A by a finite sequence of elementary row operations.

Result:– If A and B are row-equivalent, then the homogeneous system of equations $A\underset{\sim}{x} = 0$ & $B\underset{\sim}{x} = 0$ have exactly the same solution.

Row-reduced Matrix :—

Definition:– An $m \times n$ matrix R is called Row-reduced matrix if
i) the first non-zero entry in each non-zero row of R is 1, &
ii) each column that contains the leading non-zero entry of some non-zero row has all its other entries 0.

e.g. A row reduced matrix is : $\begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$

Result:– Every $m \times n$ matrix is row-equivalent to a row-reduced matrix.

Echelon Matrix :—

Definition:– An $m \times n$ matrix A is called an echelon matrix if
i) the 1st k (≥ 0) rows are non-zero, and the remaining $(m-k)$ rows are zero.
ii) in the i^{th} row $i = 1, 2, \ldots, k$ (if $k \geq 1$), the first non-zero element is 1, and
iii) the arrangement of the rows is such that $c_1 < c_2 < \cdots < c_k$, where c_i is the column in which the leading non-zero element 1 of row i occurs.

Ex:– $H = \begin{pmatrix} 0 & 1 & h_{13} & h_{14} & h_{15} & h_{16} \\ 0 & 0 & 0 & 1 & h_{25} & h_{26} \\ 0 & 0 & 0 & 0 & 1 & h_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ is an echelon matrix.

e.g. $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}$

Result:- i) If $(A+B)$ is defined then $(A+B)' = A'+B'$.
ii) If AB is defined then $(AB)' = B'A'$.

Proof:- i) (i,j)th element of $A'+B'$
$= (i,j)$th element of $A' + (i,j)$th element of B'
$= (j,i)$th element of $A + (j,i)$th element of B.
$= (j,i)$th element of $(A+B)$
$= (i,j)$th element of $(A+B)'$ $\forall\, i,j$.

ii) (i,j)th element of $B'A'$
$=$ scalar product of ith row of B' and jth column of A'
$=$ scalar product of ith column of B and jth ~~column~~ row of A
$= (j,i)$th element of AB
$= (i,j)$th element of $(AB)'$.

Elementary Matrices:- A square matrix $E_{m\times m}$ is said to be an elementary matrix if it can be obtained from the identity matrix I_m by means of a single elementary row operations. So, elementary matrices are of three distinct type corresponding to the three types of elementary row operation.

(i) $(E_{ij})_{m\times m}$ is the elementary matrix obtained by interchanging the ith and jth rows of the identity matrix I_m.

Example: $E_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ obtained from $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(ii) $E_i(\lambda)$ is the matrix obtained by multiplying the ith row of I_m, by $\lambda (\neq 0)$.

$$E_2(3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(iii) $E_i(\lambda|j)$ is the matrix obtained from I_m by adding the ith row $\lambda(\neq 0)$ times the jth row.

$$E_2(5|3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}.$$

Result:- Let A and B are $m\times n$ matrices over the field F. then B is row equivalent to A if and only if $B = PA$, where P is a product of some $m\times m$ elementary matrices.

Rank of a Matrix:– Let A be a matrix of order $m \times n$. We may look upon A as an ordered set of row vectors as

$$A = \begin{pmatrix} a_1' \\ a_2' \\ \vdots \\ a_m' \end{pmatrix}, \text{ where } a_i'\text{s being the } i^{th} \text{ row of } A.$$

The set spanned by these m rows, a_i' is a sub-space of $V_n(F)$ and is called the row-space of the matrix A. Again, each of the n columns of $A = (b_1, \dots, b_n)$ consist of m elements and is an m-vector belonging to $V_m(F)$, the set spanned by the n-columns $b_1, \dots, b_n$ is a subspace of $V_m(F)$ and is called the column space of A.

The dimension of the row-space (column space) of A is is called row rank (column rank) of the mtx A. Since row rank and column rank are the same, so it simply called rank of A.

Result:– For two matrices A and B of the same order, show that

$$\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$$

Proof:– Let A and B be of the same order $m \times n$ and $C = A+B$, considering these matrices as sequences of column vectors as $A = (a_1, \dots, a_n)$, $B = (b_1, \dots, b_n)$ and $C = (c_1, \dots, c_n)$, where $c_j = a_j + b_j \ \forall\, j = 1(1)n$.

Denoting by $S(A), S(B)$ and $S(C)$ as the column spaces of A, B, C respectively, we get —

$$S(C) \subseteq S(A) + S(B) \quad \text{—— (1)}$$

Further, we know that for two subspaces W_1 and W_2,

$$\dim(W_1 + W_2) \leq \dim(W_1) + \dim(W_2) \quad \text{—— (2)}$$

$$\dim(S(C)) \leq \dim(S(A) + S(B)), \text{ due to (1)}$$

$$\leq \dim(S(A)) + \dim(S(B)), \text{ due to (2)}.$$

$$\Rightarrow \text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B).$$

Result:– $\text{rank}(AB) \leq \min[\text{rank}(A), \text{rank}(B)]$

Proof:– $A_{m\times p}$, $B_{p\times n}$ mtxs and $C = AB_{m\times n}$.

a_j, b_j, c_j be the columns of A, B and C respectively

Then $\underset{m\times 1}{c_j} = j^{th}$ column of $C = A_{m\times p}\, b_{j\,p\times 1}$, where $b_j = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{pmatrix}$

$$= b_{1j}a_1 + b_{2j}a_2 + \dots + b_{pj}a_j, \text{ where}$$

$\Rightarrow$ Each of the m columns of C is a linear combination of the p columns of A. Consequently, Column space $(AB) \subseteq$ Column space (A),

$$\Rightarrow \text{rank}(AB) \leq \text{rank}(A)$$

Now, $\text{rank}(AB) = \text{rank}[(AB)'] = \text{rank}(B'A') \leq \text{rank}(B') = \text{rank}(B)$

$\therefore$ We have $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$

Theorem:- For any matrix $A_{m\times n} = (a_{ij})$, show that
Row rank (A) = Column rank (A).

Proof:- Let the row space of A has dimension k and the row-vectors $v_1, v_2, \dots, v_k$ form a basis of the row-space of A. Then each of the rows of A, denoted by $R_1, R_2, \dots$, etc. can be expressed as a linear combination of $v_1, v_2, \dots, v_k$, implying that $\exists$ scalars c_{ij} $\ni$

$$\left.\begin{aligned} R_1 &= c_{11}v_1 + c_{12}v_2 + \cdots + c_{1k}v_k \\ R_2 &= c_{21}v_1 + c_{22}v_2 + \cdots + c_{2k}v_k \\ &\vdots \\ R_m &= c_{m1}v_1 + c_{m2}v_2 + \cdots + c_{mk}v_k \end{aligned}\right\} \text{——— (1)}$$

By collecting the j^{th} component from both sides of each equation in (1), with the notation $v_i = (v_{i1}, v_{i2}, \dots, v_{in})$, we get $\longrightarrow$

$$\left.\begin{aligned} a_{1j} &= c_{11}v_{1j} + c_{12}v_{2j} + \cdots + c_{1k}v_{kj} \\ a_{2j} &= c_{21}v_{1j} + c_{22}v_{2j} + \cdots + c_{2k}v_{kj} \\ &\vdots \\ a_{mj} &= c_{m1}v_{1j} + c_{m2}v_{2j} + \cdots + c_{mk}v_{kj} \end{aligned}\right\}$$

or,

$$\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} = v_{1j}\begin{pmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{pmatrix} + v_{2j}\begin{pmatrix} c_{12} \\ c_{22} \\ \vdots \\ c_{m2} \end{pmatrix} + \cdots + v_{kj}\begin{pmatrix} c_{1k} \\ c_{2k} \\ \vdots \\ c_{mk} \end{pmatrix} \text{——— (2)}$$

The L.H.S. of (2) is the j^{th} column of A, and thus each column of A lies in the subspace spanned by the k vectors on the RHS of (2). So, the dimension of the column space of A $\leq$ k.

As, k = dimension of the row space of A.

dim (column space of A) $\leq$ dim (row space of A) ——— (3)

Further, as the matrix A is arbitrary, inequality (3) is true for A', i.e.,

dim(column space of A') $\leq$ dim (row space of A')

or, dim(row space of A) $\leq$ dim (column space of A) ——— (4)

Inequalities (3) and (4) give ———

dim(columnspace of A) = dim(row space of A).

i.e. $\boxed{\text{Column rank (A) = Row rank (A)}}$

Theorem:- Let $H_{m\times n}$ be a non-zero echelon matrix having k non-zero rows. Then the non-zero rows form a basis of the row-space of H, giving $\text{rank}(H) = k$.

Proof:- Let $h_1', h_2', h_3', \ldots, h_k'$ be k non-zero rows of H with $h_i' = (h_{i1}, h_{i2}, \ldots, h_{in})$ $\forall i = 1, 2, 3, \ldots, k$.

Certainly, these k non-zero rows span the row-space of H. So to show that h_i', $i = 1, 2, \ldots, k$, form a basis of the row space of H, we need to show that these k-vectors are LIN. For this, we consider the following equations with unknown scalars λ_i $(i = 1, 2, \ldots, k)$.

$$\lambda_1 h_1' + \lambda_2 h_2' + \cdots + \lambda_k h_k' = \underset{\sim}{0}' \quad \text{——— (1)}$$

As, H is an echelon matrix, row h_i' has the element leading 1 in column c_i (say), $i = 1, 2, \ldots, k$. The elements of column c_i below row i, all are zero. Hence, equation (1) implies —

$$\lambda_1 = 0$$
$$\lambda_1 h_{1c_2} + \lambda_2 = 0$$
$$\lambda_1 h_{1c_3} + \lambda_2 h_{2c_3} + \lambda_3 = 0$$
etc.....

Solving these equations successively, we have then
$$\lambda_1 = \lambda_2 = \cdots\cdots = \lambda_k = 0.$$

Thus, the non-zero rows of H spans the row-space and are linearly independent implying the rows $h_1', h_2', \ldots, h_k'$ form a basis of the row-space of H.

Cor:- Let $H_{m\times n}$ be an echelon mtx row-equivalent to $A_{m\times n}$, then the non-zero rows of H form a basis of the row-space of A.

Proof:- Let $a_1, a_2, \ldots, a_m$ be the m-rows of the matrix A and $h_1, h_2, \ldots, h_k$ $(k \le m)$ be the non-zero rows of H.

We can write the row space of A as —

$$\text{Row space}(A) = \{\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m \mid \lambda_1, \ldots, \lambda_m \in F\}$$

As H is row-equivalent to A, then A is also row-equivalent to H. Then a_i is a linear combination of $h_1, h_2, \ldots, h_k$ $\forall i = 1(1)m$.

Therefore, Row space of $A = \{\gamma_1 h_1 + \gamma_2 h_2 + \cdots + \gamma_k h_k \mid \gamma_1, \ldots, \gamma_k \in F\}$

$\Rightarrow h_1, \ldots, h_k$ generate the row-space of A.

Further, we know that the non-zero rows $h_1, \ldots, h_k$ of H are linearly independent. Hence the result.

Result:- If H is an echelon matrix row-equivalent to A, the $\text{rank}(A) = \text{rank}(H)$.

Rank – Factorization theorem :–

Statement :– Let A be an $m \times n$ matrix of rank r. Then $\exists$ two matrices $B_{m\times r}$ and $C_{r\times n}$ such that $A = BC$ and $\text{rank}(B) = r$ and $\text{rank}(C) = r$.

Proof :– $\text{Rank}(A) = r$

Let us suppose that $\{\underset{\sim}{b_1}, \underset{\sim}{b_2}, \ldots, \underset{\sim}{b_r}\}$ forms a basis of the column space of A.

Let $\underset{\sim}{\beta_j}$ denote the j^{th} column vector of A. Then $\exists$ real scalars $\ni$ $\underset{\sim}{\beta_j}$ can be expressed in terms of $\underset{\sim}{b_1}, \underset{\sim}{b_2}, \ldots, \underset{\sim}{b_r}$ as follows

$$\underset{\sim}{\beta_j} = c_{1j}\underset{\sim}{b_1} + c_{2j}\underset{\sim}{b_2} + \cdots + c_{rj}\underset{\sim}{b_r}$$

$$\text{or, } \beta_{ij} = (\underset{\sim}{b_1} \; \underset{\sim}{b_2} \cdots \underset{\sim}{b_r}) \begin{pmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{rj} \end{pmatrix} = B_{m\times r}\, C_{j\, r\times 1} \quad \forall j = 1(1)n.$$

Now, $A = (\underset{\sim}{\beta_1} \cdots \underset{\sim}{\beta_n})$

$= (B\underset{\sim}{c_1} \; Bc_2 \cdots Bc_n)$

$= B_{m\times r} C_{r\times n}$ [Proved]

Now, we also have

$\text{rank}(BC) \le \min\{r(B), r(C)\}$

i.e. $\text{rank}(BC) \le \text{rank}(B)$

$\le \text{rank}(C)$

i.e. $\text{rank}(A) \le \text{rank}(B)$ & $\text{rank}(A) \le \text{rank}(C)$

$\therefore r \le \text{rank}(B)$ & $r \le \text{rank}(C)$

But B has r column vectors and C has r row vectors.

& $\text{rank}(B) \le r$ and $\text{rank}(C) \le r$

$\therefore$ Finally we get $\text{rank}(B) = \text{rank}(C) = r$.

RANK RELATED PROBLEMS :–

1. Let A be an $m \times n$ mtx with rank m and S be a $r \times m$ mtx with rank r, then show that $\text{rank}(SA) = r$.

Solution :– $\text{rank}(A) + \text{rank}(S) - m \le \text{rank}(SA) \le \min\{\text{rank}(S), \text{rank}(A)\}$

$\Rightarrow r + m - m \le \text{rank}(SA) \le \min\{r, m\} = r$

i.e. $\text{rank}(SA) = r \le m$.

2. If $\text{rank}(A - I) = p$ and $\text{rank}(B - I) = q$, then S.T. $\text{rank}(AB - I) \le p + q$

Solution :– $AB - I = A(B - I) + (A - I)$

$\text{Rank}(AB - I) \le \text{Rank}[A(B - I)] + \text{Rank}(A - I)$

$\le \text{Rank}(B - I) + \text{Rank}(A - I)$

$\le p + q$.

3) If A be an $m\times n$ mtx and B be an $s\times n$ mtx $\ni AB'=0$, then $\text{rank}(A'A+B'B) = r(A)+r(B)$.

Sol. Let us define, $C = \left(\underset{n\times m}{A'} \;\vdots\; \underset{n\times s}{B'} \right)$

Then $CC' = \left(A' \;\vdots\; B' \right) \begin{pmatrix} A \\ \hdashline B \end{pmatrix}$

$= A'A + B'B$.

and $C'C = \begin{pmatrix} A \\ \hdashline B \end{pmatrix} \left(A' \;\vdots\; B' \right) = \left(\begin{array}{c:c} AA' & AB' \\ \hdashline BA' & BB' \end{array} \right)$

$= \left(\begin{array}{c:c} AA' & 0 \\ \hdashline 0 & BB' \end{array} \right)$

Since $\text{rank}(CC') = \text{rank}(C'C)$

$\therefore \text{rank}(A'A+B'B) = \text{rank}\left(\begin{array}{c:c} AA' & 0 \\ \hdashline 0 & BB' \end{array} \right)$

$= \text{rank}(AA') + \text{rank}(BB')$

$= \text{rank}(A) + \text{rank}(B)$.

PROBLEM:- Determine all the idempotent diagonal matrices of order n.

Solution:- Let $A = \text{diagonal}[d_1, \dots, d_n]$ be an idempotent mtx.

Then $A^2 = A$.

$A = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$, $A^2 = \begin{bmatrix} d_1^2 & 0 & \cdots & 0 \\ 0 & d_2^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^2 \end{bmatrix}$

$A^2 = A$ gives $d_1^2 = d_1,\ d_2^2 = d_2,\ \dots,\ d_n^2 = d_n$

$\therefore d_1 = 0,1;\ d_2 = 0,1;\ \dots;\ d_n = 0,1$.

Hence the required idempotent diagonal mtx is

$\text{diag}[d_1, \dots, d_n]$ with $d_i = 0,1 \ \forall\ i = 1(1)n$.

SOME BASIC THEOREMS :-

Theorem:-

1. If $AB = A$ and $BA = B$. S.T. A and B are idempotent.

Solution:- We have $AB = A$

$A(AB) = A$, $[\because BA = A]$

$\Rightarrow (AB)A = A$

or, $A^2 = A \Rightarrow A$ is idempotent.

Similar for the later.

2. If B is an idempotent matrix, S.T. $A = I - B$ is also idempotent and that $AB = BA = 0$.

Solution:- As B is idempotent, $\therefore B^2 = B$.

$A^2 = (I-B)^2 = (I-B)(I-B)$

$= I - IB - BI + B^2$

$= I - B - B + B$

$= I - B$

$= A$

$\therefore A$ is idempotent.

$\therefore AB = (I-B)B = B - B^2 = 0$.

$\therefore BA = B(I-B) = B - B^2 = 0$.

3. A matrix is said to be involutory if $A^2 = I$. Show that A is involutory iff $(I+A)(I-A) = 0$.

Solution:- Let A be an involutory matrix of order n.

Then $A^2 = I$.

$\therefore I - A^2 = 0$

$\therefore (I+A)(I-A) = 0$.

Conversely, if $(I+A)(I-A) = 0$

then $I^2 - IA + AI - A^2 = 0$

or, $I - A^2 + AI - AI = 0$

or, $I - A^2 = 0$

or, $A^2 = I$.

$\therefore A$ is involutory.

IDEA ABOUT INVERSE :—

Definition 1:- An $m \times n$ matrix A is said to be of full row rank if its rank is m, i.e., if its rows are linearly independent. Similarly A is said to be of full column rank if its columns are linearly independent.

Definition 2:- A left inverse of a matrix A is any matrix B such that $BA = I$. A right inverse of A is any matrix C such that $AC = I$. A mtx B is said to be an inverse of A if it is both a left inverse and a right inverse of A.

PROPERTIES OF INVERSE:- A square matrix A is said to be non-singular if it has an inverse. A square matrix which does not possess an inverse is said to be singular.

Theorem 1:- If A is non-singular, then A^{-1} and A^T are also non-singular

$$(A^{-1})^{-1} = A \quad \text{and} \quad (A^T)^{-1} = (A^{-1})^T.$$

Proof:- Since $(A^{-1})A = I$, it follows that A is a right inverse and so the inverse of A^{-1} $\Rightarrow A = (A^{-1})^{-1}$

Similarly, $A^T(A^{-1})^T = (A^{-1}A)^T = I$

$$\Rightarrow (A^T)^{-1} = (A^{-1})^T.$$

Theorem 2:- Let A and B be square matrices of the same order. Then AB is non-singular iff both A and B are non-singular.

Also then $(AB)^{-1} = B^{-1}A^{-1}$.

Proof:- If AB is non-singular and $C = (AB)^{-1}$, then $ABC = I$, so BC is an inverse of A and A is non-singular. Also, $CAB = I$, so, B is non-singular.

Conversely, Let A and B be both non-singular. Then

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I.$$

So, $B^{-1}A^{-1}$ is the inverse of AB and AB is non-singular.

By repeated application of the preceeding theorem, it can be shown that if $A_1, A_2, \ldots, A_K$ are non-singular matrices of the same order, then $(A_1A_2\cdots A_K)^{-1} = A_K^{-1}A_{K-1}^{-1}\cdots A_1^{-1}$.

Note:- (1). If $A = I$, the identity matrix, then $A^{-1} = I$.

(2). If $A = \text{diag}(d_1, \ldots, d_k)$, a diagonal mtx, then

$$A^{-1} = \text{diag}\left(\frac{1}{d_1}, \ldots, \frac{1}{d_k}\right).$$

Theorems on the inverse of Elementary Matrices :-

Theorem:- We have the followings:

(a) $E_{ij}^{-1} = E_{ij}$ (b) $E_i^{-1}(\lambda|j) = E_i(-\lambda|j)$ (c) $E_i^{-1}(\lambda) = E_i(1/\lambda)$

Proof:-

(a) E_{ij} is obtained from the identity matrix I of the same order by interchanging the i^{th} & the j^{th} rows (or, columns) of I. As such, if the i^{th} and j^{th} rows (or, columns) of E_{ij} itself are interchanged, which will mean post-multiplication or pre-multiplication of E_{ij} by E_{ij} then one gets back the identity matrix.

Hence, $E_{ij}E_{ij} = I$ implying that $E_{ij}^{-1} = E_{ij}$.

(b) $E_i(\lambda|j)$ is obtained from the identity matrix of the same order by adding to the i^{th} row (j^{th} column) λ times the j^{th} row (i^{th} column) of the latter.

Hence, if we subtract from the i^{th} row (j^{th} column) of $E_i(\lambda|j)$ λ times the j^{th} row (i^{th} column), we get back the identity matrix. Since this means premultiplication (post multiplication) of $E_i(\lambda|j)$ by $E_i(-\lambda|j)$, we ~~have~~ get the identity matrix back.

$$E_i(-\lambda|j)\,E_i(\lambda|j) = I \Rightarrow E_i^{-1}(\lambda|j) = E_i(-\lambda|j).$$

(c) $E_i(\lambda)$ is obtained from the identity matrix I of the same order by multiplying the i^{th} row or column by the non-zero scalar λ. Hence, if we post multiply $E_i(\lambda)$ by $E_i(1/\lambda)$ or premultiply $E_i(\lambda)$ by $E_i(1/\lambda)$, this will have the effect of multiplying of the i^{th} row or column of $E_i(\lambda)$ by $1/\lambda$. Thus,

$$E_i(1/\lambda)\,E_i(\lambda) = E_i(\lambda)\,E_i(1/\lambda) = I,$$

implying that $E_i^{-1}(\lambda) = E_i(1/\lambda)$

NORMAL FORM OF A MATRIX :-

<u>Definition</u>:- A matrix is said to be in normal form if it is $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ for some r. We use the conversion that it denotes a null mtx if $r=0$. Clearly the rank of this mtx is r.

<u>Theorem</u>:- Every matrix can be reduced to a matrix in normal form (by elementary (row and column) operation.

<u>Proof</u>:- If $A=0$, then A itself is in normal form. So let $A \neq 0$. Then by interchanging the first row with another row and first column with another column, if necessary, we make $a_{11} \neq 0$. Then we sweep out the first column and the first row using a_{11} as the pivot. At this stage the matrix is of the form $A = \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix}$. If $B=0$, the current matrix is in normal form. If $B \neq 0$, by interchanging the second row with some later row and the second column with some later column, if necessary, we make $a_{22} \neq 0$ & sweep out the 2nd column & the 2nd row with a_{22} as the pivot. Note that these operations do not disturb the first row and the first column. Now the matrix is of the form $A = \begin{bmatrix} I_2 & 0 \\ 0 & C \end{bmatrix}$

If $C=0$, then the work is done; otherwise we proceed as before until we arrive at a matrix in normal form.

<u>NOTE</u>:- Since row operations amount to premultiplication and column operations amount to past multiplication by non-singular matrices, we have the following theorem :-

<u>Theorem</u>:- Let A be an $m \times n$ matrix of rank r. Then $\exists$ n.s. matrices P of order m and Q of order n $\ni$

$$\longrightarrow \qquad PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

By taking $R = P^{-1}$ and $S = Q^{-1}$, we can rewrite the above as

$$A = R \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} S$$

This representation is often more useful than earlier. For instance, we can readily get a rank-factorization of A from the later. Partition

$$R = [R_1 \vdots R_2]$$

$$S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$$

where, R_1 is of order $m \times r$ and S_1 is of order $r \times n$. Then $A = R_1 S_1$, So, (R_1, S_1) is a rank factorization of A.

Problem:- Let A be a skew-symmetric matrix and $(I+A)$ is non-singular matrix. Show that $B = (I-A)(I+A)^{-1}$ is orthogonal.

Solution:- Since A is skew-symmetric, $A' = -A$ and $(I+A)$ is a non-singular matrix, i.e., $|I+A| \neq 0$.

$$B = (I-A)(I+A)^{-1}$$

$$B^T = \{(I-A)(I+A)^{-1}\}^T$$

$$= \{(I+A)^{-1}\}^T (I-A)^T$$

$$= (I+A^T)^{-1}(I+A)$$

$$= (I-A)^{-1}(I+A) .$$

Thus, $B^T B = (I-A)^{-1}(I+A)(I-A)(I+A)^{-1}$

$$= I .$$

$$BB^T = I$$

So, it follows that B is orthogonal.

* ———————————————— *

DETERMINANTS

DEFINITION:- Consider any permutation, say $p = (k_1, k_2, \ldots\ldots, k_n)$; of the first n natural numbers. Such a permutation has some or no inversions, an inversion being a derivation from the natural order of two positive integers, the bigger integer preceeding the smaller. A permutation will be said to be an even or odd permutation according as the number of inversions in it is even or odd.

Consider now a square matrix of order n, say the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots\cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots\cdots & a_{nn} \end{pmatrix} \quad \ldots\ldots\ldots\ldots(i)$$

By definition, the determinant of A, denoted by $|A|$ is given by,

$$|A| = \sum \mp a_{1k_1} a_{2k_2} \ldots\ldots a_{nk_n} \ldots\ldots\ldots\ldots(ii)$$

where, the sum is taken over all the $n!$ permutations of the first n natural numbers and the sign attached to any given term is a plus $(+)$ or a minus $(-)$ sign according as the permutation is even or odd. This rule is to be followed irrespective of whether the value of the product $a_{1k_1} a_{2k_2} \ldots\ldots a_{nk_n}$ itself is +ve or −ve.

Example:- Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Here $n = 3$ & we are to consider all the $3! = 6$ permutations of the first 3 natural numbers 1, 2 & 3. These are listed below:-

(1, 2, 3) :- Even permutation, the # of inversions being 0
(1, 3, 2) :- Odd " , " " " " 1
(2, 1, 3) :- Odd " , " " " " 1
(2, 3, 1) :- Even " , " " " " 2
(3, 1, 2) :- " " , " " " " 2
(3, 2, 1) :- Odd " , " " " " 3

As such, we have $|A| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$

Thus, $|A| = |a_{ij}| = ((a_{ij}))$

Adjoint of a matrix:- Given any square matrix $A = (a_{ij})$, not necessary non-singular, the adjoint of A, denoted by $\mathrm{Adj}\,A$, is the matrix of the same order where $(i,j)^{th}$ element is A_{ij}, the cofactor of a_{ji} in $|A|$.

Thus, $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ while $\mathrm{Adj}\,A = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$

Theorem:- If A be any non-singular matrix then $A^{-1} = \dfrac{\mathrm{Adj}\,A}{|A|}$.

Proof:- We have, [redacted] $\sum_{j=1}^{n} a_{ij}a_{i'j} = |A|$ if $i = i'$
$= 0$ if $i \neq i'$

Again, considering the columns of A, we have,

$$\sum_{i=1}^{n} a_{ij}A_{ij'} = |A| \text{ if } j = j'$$
$$= 0 \text{ if } j \neq j'.$$

As such,

$$(\mathrm{adj}\,A)\,A = \begin{pmatrix} \sum_{i=1}^{n} a_{i1}A_{i1} & \sum_{i=1}^{n} a_{i2}A_{i1} & \cdots & \sum_{i=1}^{n} a_{i1}A_{i1} \\ \vdots & \vdots & & \vdots \\ \sum_{i=1}^{n} a_{i1}A_{in} & \sum_{i=1}^{n} a_{i2}A_{in} & \cdots & \sum_{i=1}^{n} a_{in}A_{in} \end{pmatrix}$$

$$= \begin{pmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & |A| \end{pmatrix} = |A|\cdot I \text{ ---- } (1)$$

$= |A|$, being a scalar.

Again,

$$A\,(\mathrm{Adj}\,A) = \begin{pmatrix} \sum_{j=1}^{n} a_{1j}A_{1j} & \sum_{j=1}^{n} a_{1j}A_{2j} & \cdots & \sum_{j=1}^{n} a_{1j}A_{nj} \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^{n} a_{nj}A_{1j} & \sum_{j=1}^{n} a_{nj}A_{2j} & \cdots & \sum_{j=1}^{n} a_{nj}A_{nj} \end{pmatrix}$$

$$= \begin{pmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & |A| \end{pmatrix}$$

$= |A|$ ------ (2)

From (1) & (2),

$$\left(\frac{\mathrm{Adj}\,A}{|A|}\right)A = A\left(\frac{\mathrm{Adj}\,A}{|A|}\right) = I.$$

consequently, the reciprocal matrix of A is obtained as

$$A^{-1} = \frac{1}{|A|}(\mathrm{Adj}A)$$

$$= \begin{pmatrix} \frac{A_{11}}{|A|} & \frac{A_{21}}{|A|} & \cdots\cdots & \frac{A_{n1}}{|A|} \\ \frac{A_{12}}{|A|} & \frac{A_{22}}{|A|} & \cdots\cdots & \frac{A_{n2}}{|A|} \\ \vdots & \vdots & & \vdots \\ \frac{A_{1n}}{|A|} & \frac{A_{2n}}{|A|} & \cdots\cdots & \frac{A_{nn}}{|A|} \end{pmatrix}$$

In other words, if we write $A^{-1} = (a^{ij})$, then $a^{ij} = \frac{A_{ji}}{|A|} \forall j, i.$

Theorem:- Prove that any square matrix $A_{n\times n}$ possesses an inverse iff $|A_{n\times n}| \neq 0$.

Proof:- Only if Part:- Let the matrix $A_{n\times n}$ be invertible and B be the inverse of A then $AB = I_n$

$\therefore |AB| = 1$

or, $|A||B| = 1$

since the product of two determinants is one, $|A_{n\times n}|$ must be non-zero, so, $|A_{n\times n}| \neq 0$ is the necessary condition.

If Part:- Suppose that the condition $|A_{n\times n}| \neq 0$ holds. Then let us define a matrix B by the relation $B = \frac{\mathrm{Adj}A}{|A|}$

Then $AB = \frac{1}{|A|} A\,\mathrm{Adj}A$

$BA = \frac{1}{|A|}\mathrm{Adj}A.A$

But we proved that $A(\mathrm{Adj}A) = (\mathrm{Adj}A)A = |A|.I_n$

Here $AB = BA = I_n$.

so, A is an invertible matrix and B is the inverse of A.

Problem:- If A be an $n\times n$ matrix, prove that $|\mathrm{Adj}A| = |A|^{n-1}$.

Solution:- We have $A.(\mathrm{Adj}A) = |A|.I_n$

$\therefore |A(\mathrm{Adj}A)| = |A|^n|I_n|$ [$\because |AB| = |A||B|$ & $|kA| = k^n|A|$]

$\Rightarrow |A||\mathrm{Adj}A| = |A|^n$

$\Rightarrow |\mathrm{Adj}A| = |A|^{n-1}$ [If $|A| \neq 0$]

If $|A| = 0$, then $|\mathrm{Adj}A| = 0$.

Hence the result.

Problem:- If A be an $n \times n$ matrix, show that the rank of $\text{Adj} A$ is n, 1 or 0 according as the rank of A is $n, n-1$ or less than $n-1$.

Solutions:- (i) Let A be an $n \times n$ matrix. Then

$$A(\text{Adj} A) = |A| \cdot I_n$$

$$\therefore |A|\,|\text{Adj} A| = |A|^n |I_n|$$

$$\therefore |A|\,|\text{Adj} A| = |A|^n$$

Now, since there is $\text{rank}(A) = n$

therefore $|A| \neq 0$

$\therefore |A|\,|\text{Adj} A| = |A|$ gives $|\text{Adj} A| = |A|^{n-1} \neq 0$.

$\therefore$ The matrix $|\text{Adj} A| \neq 0$ hence it is of full rank, i.e. rank n.

(ii) If the rank of A is $n-1$, then at least one minor of order $(n-1)$ of the matrix is not equal to zero, therefore the matrix $\text{Adj} A$ will be a non-zero matrix and the rank of the matrix $\text{Adj} A$ will be greater than zero.

Again, the rank of A is $n-1$, therefore, $|A| = 0$.

$\therefore A(\text{Adj} A)$ is a zero matrix and hence is of rank zero.

Hence, by sylvester's inequality,

$$\text{rank}(A\,\text{adj} A) \geq \text{rank}(A) + \text{rank}(\text{adj} A) - n$$

or, $\text{rank}(A) + \text{rank}(\text{adj} A) - n \leq 0$

or, $\text{rank}(\text{adj} A) \leq 1$.

But we have shown that $\text{rank}(\text{adj} A) > 0$

Hence, $\text{rank}(\text{adj} A) = 1$.

(iii) If the rank of A is less than $n-1$, then all the minors order $n-1$ of the matrix A will be zero. Therefore, the matrix $\text{adj} A$ will be a zero matrix & hence $\text{rank}(\text{adj} A) = 0$.

INVERSE OF A PARTITIONED MATRIX :-

Let A be a square matrix of order n, written in the partitioned form

$$A_{n \times n} = \begin{pmatrix} \underset{m \times m}{A_{11}} & \underset{m \times (n-m)}{A_{12}} \\ \underset{(n-m) \times m}{A_{21}} & \underset{(n-m) \times (n-m)}{A_{22}} \end{pmatrix}$$

Theorem:- If A be of the above structure and A_{11} is non-singular then $|A| = |A_{11}||A_{22} - A_{21}A_{11}^{-1}A_{12}|$

Proof:- Consider together with A, the following $(n\times n)$ matrices,

$$B_{n\times n} = \begin{pmatrix} I_m & O_{m\times(n-m)} \\ -A_{21}A_{11}^{-1}{}_{(n-m)\times m} & I_{(n-m)} \end{pmatrix}$$

and $$C_{n\times n} = \begin{pmatrix} I_m & -A_{11}^{-1}A_{12}{}_{m\times(n-m)} \\ O_{(n-m)\times m} & I_{(n-m)} \end{pmatrix}$$

Then $|B| = |I_m||I_{(n-m)}| = 1$ and $|C| = |I_m||I_{n-m}| = 1$.

Again, $$BAC = \begin{pmatrix} I_m & 0 \\ -A_{21}A_{11}^{-1} & I_{n-m} \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} I_m & -A_{11}^{-1}A_{12} \\ 0 & I_{n-m} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}$$

Hence taking determinants on both sides, we get,

$$|BAC| = |A| = |A_{11}||A_{22} - A_{21}A_{11}^{-1}A_{12}|.$$

Theorem:- If A has the same structure as above and is non-singular Then

$$A^{-1} = \begin{pmatrix} A_{11}^{-1}(I_m + A_{12}F^{-1}A_{21}A_{11}^{-1}) & -A_{11}^{-1}A_{12}F^{-1} \\ -F^{-1}A_{21}A_{11}^{-1} & F^{-1} \end{pmatrix}$$

where, $F = A_{22} - A_{21}A_{11}^{-1}A_{12}$.

Proof:- Since A is non-singular, A_{11} is also non-singular so that A_{11}^{-1} exists. Now, let $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ and $AB = I$.

i.e. $$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} I_m & O_{m\times(n-m)} \\ O_{(n-m)\times m} & I_{(n-m)} \end{pmatrix}$$

We have the following equations $\longrightarrow$

$$A_{11}B_{11} + A_{12}B_{21} = I_m \quad \text{---(i)}$$
$$A_{11}B_{12} + A_{12}B_{22} = O_{m\times(n-m)} \quad \text{---(ii)}$$
$$A_{21}B_{11} + A_{22}B_{21} = O_{(n-m)\times m} \quad \text{---(iii)}$$
$$A_{21}B_{12} + A_{22}B_{22} = I_{(n-m)} \quad \text{---(iv)}$$

Equation (i) gives $\quad B_{11} + A_{11}^{-1} A_{12} B_{21} = A_{11}^{-1}$ ——— (v)

Premultiplying by A_{21} and subtracting from (iii)

$$A_{22} B_{21} - A_{21} A_{11}^{-1} A_{12} B_{21} = - A_{21} A_{11}^{-1}$$

or, $(A_{22} - A_{21} A_{11}^{-1} A_{12}) B_{21} = - A_{21} A_{11}^{-1}$

or, $F B_{21} = - A_{21} A_{11}^{-1}$

or, $B_{21} = -F^{-1} A_{21} A_{11}^{-1}$

So, from (v),

$$B_{11} - A_{11}^{-1} A_{12} F^{-1} A_{21} A_{11}^{-1} = A_{11}^{-1}$$

$$B_{11} = A_{11}^{-1} (I_m + A_{12} F^{-1} A_{21} A_{11}^{-1})$$

Pre multiplying equation (ii) by $A_{21} A_{11}^{-1}$ & subtracting from (iv)

$$A_{21} B_{12} + A_{22} B_{22} - A_{21} B_{12} - A_{21} A_{11}^{-1} A_{12} B_{22} = I_{(n-m)}.$$

or, $(A_{22} - A_{21} A_{11}^{-1} A_{12}) B_{22} = I_{(n-m)}$.

or, $F B_{22} = I_{(n-m)}$

or, $B_{22} = F^{-1}$.

~~Subtracting from equation~~

Substituting in equation (ii)

$$B_{12} = - A_{11}^{-1} A_{12} F^{-1}$$

Hence the result.

Result:- Let A be an n.s. mtx of order n and $\underset{\sim}{u}$ and $\underset{\sim}{v}$ be two n-component column vectors. Then show that $|A + \underset{\sim}{u}\underset{\sim}{v}'| \neq 0$ iff $1 + \underset{\sim}{v}' A^{-1} \underset{\sim}{u} \neq 0$. Then also show that

$$(A + \underset{\sim}{u}\underset{\sim}{v}')^{-1} = A^{-1} - \frac{(A^{-1}\underset{\sim}{u})(\underset{\sim}{v}' A^{-1})}{1 + \underset{\sim}{v}' A^{-1} \underset{\sim}{u}}.$$

ANSWER:- $\quad 1 + \underset{\sim}{v}' A^{-1} \underset{\sim}{u} \neq 0$

Let us define, $M = \begin{pmatrix} A & -\underset{\sim}{u} \\ \underset{\sim}{v}' & 1 \end{pmatrix}$

$\therefore |M| = |A + \underset{\sim}{u}\underset{\sim}{v}'|$

Since A is non-singular, A^{-1} exists.

Now, again, $\quad |M| = |A| (1 + \underset{\sim}{v}' A^{-1} \underset{\sim}{u})$

So, $\quad 1 + \underset{\sim}{v}' A^{-1} \underset{\sim}{u} \neq 0 \Leftrightarrow |M| \neq 0$

$\Leftrightarrow |A + \underset{\sim}{u}\underset{\sim}{v}'| \neq 0$

Thus, $\quad |A + \underset{\sim}{u}\underset{\sim}{v}'| \neq 0$ iff $1 + \underset{\sim}{v}' A^{-1} \underset{\sim}{u} \neq 0$

Now, $$(A+\underset{\sim}{u}\underset{\sim}{v}')\left(A^{-1} - \frac{(A^{-1}\underset{\sim}{u})(\underset{\sim}{v}'A^{-1})}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}\right)$$

$$= AA^{-1} + \underset{\sim}{u}\underset{\sim}{v}'A^{-1} - \frac{A(A^{-1}\underset{\sim}{u})(\underset{\sim}{v}'A^{-1})}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}} - \frac{\underset{\sim}{u}\underset{\sim}{v}'(A^{-1}\underset{\sim}{u})(\underset{\sim}{v}'A^{-1})}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}$$

$$= I + \underset{\sim}{u}\underset{\sim}{v}'A^{-1} - \frac{(AA^{-1})(\underset{\sim}{u}\underset{\sim}{v}'A^{-1}}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}} - \frac{\underset{\sim}{u}(\underset{\sim}{v}'A^{-1}\underset{\sim}{u})\underset{\sim}{v}'A^{-1}}{1+\underset{\sim}{v}A^{-1}\underset{\sim}{u}}$$

$$= I + \underset{\sim}{u}\underset{\sim}{v}'A^{-1} - \frac{\underset{\sim}{u}\underset{\sim}{v}'A^{-1}}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}} - \frac{(\underset{\sim}{v}'A^{-1}\underset{\sim}{u})\underset{\sim}{u}\underset{\sim}{v}'A^{-1}}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}$$

$$= \underline{I} + u\underset{\sim}{v}'A^{-1} - \frac{(1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u})\underset{\sim}{u}\underset{\sim}{v}'A^{-1}}{(1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u})}$$

$$= I.$$

Further, $$\left(A^{-1} - \frac{(A^{-1}\underset{\sim}{u})(\underset{\sim}{v}'A^{-1})}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}\right)(A+\underset{\sim}{u}\underset{\sim}{v}')$$

$$= A^{-1}A - \frac{(A^{-1}\underset{\sim}{u})(\underset{\sim}{v}'A^{-1})A}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}} + A^{-1}\underset{\sim}{u}\underset{\sim}{v}' - \frac{(A^{-1}\underset{\sim}{u})(\underset{\sim}{v}'A^{-1})\underset{\sim}{u}\underset{\sim}{v}'}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}$$

$$= I - \frac{(A^{-1}\underset{\sim}{u}\underset{\sim}{v}')(A^{-1}A)}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}} + A^{-1}\underset{\sim}{u}\underset{\sim}{v}' - \frac{(A^{-1}\underset{\sim}{u}\underset{\sim}{v}')(\underset{\sim}{v}'A^{-1}\underset{\sim}{u})}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}$$

$$= I - \frac{A^{-1}\underset{\sim}{u}\underset{\sim}{v}'}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}} + A^{-1}\underset{\sim}{u}\underset{\sim}{v}' - \frac{(\underset{\sim}{v}A^{-1}\underset{\sim}{u})(A^{-1}\underset{\sim}{u}\underset{\sim}{v}')}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}$$

$$= I + A^{-1}\underset{\sim}{u}\underset{\sim}{v}' - \frac{(A^{-1}\underset{\sim}{u}\underset{\sim}{v}')(1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u})}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}$$

$$= I.$$

Hence, $$(A+\underset{\sim}{u}\underset{\sim}{v}')^{-1} = A^{-1} - \frac{(A^{-1}\underset{\sim}{u})(\underset{\sim}{v}'A^{-1})}{1+\underset{\sim}{v}'A^{-1}\underset{\sim}{u}}.$$

Problem:- Find the inverse of the matrix $A = \begin{pmatrix} a & b & b \cdots & b \\ b & a & b \cdots & b \\ \vdots & \vdots & \vdots & \vdots \\ b & b & b \cdots & a \end{pmatrix}_{n \times n}$.

ANSWER:-

The given matrix $A = \begin{pmatrix} a & b & b \cdots & b \\ b & a & b \cdots & b \\ \vdots & \vdots & \vdots & \\ b & b & b \cdots & a \end{pmatrix}$

$$= \begin{pmatrix} a-b & 0 & 0 \cdots & 0 \\ 0 & a-b & 0 \cdots & 0 \\ \vdots & \vdots & \vdots & \\ 0 & 0 & 0 \cdots & a-b \end{pmatrix} + \begin{pmatrix} b & b \cdots & b \\ b & b \cdots & b \\ \vdots & \vdots & \vdots \\ b & b \cdots & b \end{pmatrix}$$

$= (a-b) I_n + b J_n$ I_n : Identity mtx of order n.
J_n : Sum mtx of order n.

$= (a-b) I_n + b \underset{\sim}{1}\underset{\sim}{1}'$

$= (a-b) \left[I_n + \frac{b}{a-b} \underset{\sim}{1}\underset{\sim}{1}' \right]$; $\underset{\sim}{1}_{n \times 1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$= (a-b)(C + \underset{\sim}{u}\underset{\sim}{v}')$, $C = I_n$, $\underset{\sim}{u} = \frac{b}{a-b} \underset{\sim}{1}$
$v' = \underset{\sim}{1}'$

Here $|C + \underset{\sim}{u}\underset{\sim}{v}'| \neq 0$.

$$\therefore A^{-1} = \frac{1}{(a-b)} \left[C^{-1} - \frac{(C^{-1}\underset{\sim}{u})(\underset{\sim}{v}'C^{-1})}{1 + \underset{\sim}{v}'C^{-1}\underset{\sim}{u}} \right]$$

$$= \frac{1}{(a-b)} \left[I_n - \frac{\left\{ \left(\frac{b}{a-b} \right) \underset{\sim}{1} \right\} (\underset{\sim}{1}')}{1 + \underset{\sim}{1}'\underset{\sim}{1}} \right]$$

$$= \frac{1}{(a-b)} \left[I_n - \frac{\frac{b}{a-b} J_n}{n+1} \right] \quad \text{(Ans)}$$

Problem:- Find the inverse of the following matrix

$$A = \begin{pmatrix} np_1(1-p_1) & -np_1p_2 & \cdots & -np_1p_{k-1} \\ -np_1p_2 & np_2(1-p_2) & \cdots & -np_2p_{k-1} \\ \vdots & \vdots & & \vdots \\ -np_1p_{k-1} & -np_2p_{k-1} & \cdots & np_{k-1}(1-p_{k-1}) \end{pmatrix}_{(k-1)\times(k-1)}$$

where $\sum_{i=1}^{k-1} p_i < 1$.

ANSWER:-

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$$A = n\begin{pmatrix} p_1(1-p_1) & -p_1p_2 & \cdots & -p_1p_{k-1} \\ -p_1p_2 & p_2(1-p_2) & \cdots & -p_2p_{k-1} \\ \vdots & \vdots & & \vdots \\ -p_1p_{k-1} & -p_2p_{k-1} & \cdots & p_{k-1}(1-p_{k-1}) \end{pmatrix}.$$

$$= n\left[\begin{pmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & p_{k-1} \end{pmatrix} - \begin{pmatrix} p_1^2 & p_1p_2 & p_1p_3 & \cdots & p_1p_{k-1} \\ p_1p_2 & p_2^2 & p_2p_3 & \cdots & p_2p_{k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ p_1p_{k-1} & p_2p_{k-1} & p_3p_{k-1} & \cdots & p_{k-1}^2 \end{pmatrix}\right]$$

$$= n\left[\mathrm{diag}(p_1, \cdots, p_{k-1}) + \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_{k-1} \end{pmatrix}(-p_1, -p_2, \cdots, -p_{k-1})\right]$$

$$= n\left[C + \underset{\sim}{u}\underset{\sim}{v}'\right], \quad C = \mathrm{diag}(p_1, \cdots, p_{k-1}), \ \underset{\sim}{u} = \begin{pmatrix} p_1 \\ \vdots \\ p_{k-1} \end{pmatrix},$$

$$\underset{\sim}{v} = -\underset{\sim}{u}.$$

Now, $A^{-1} = \frac{1}{n}\left(C + \underset{\sim}{u}\underset{\sim}{v}'\right)^{-1}$

$$= \frac{1}{n}\left[C^{-1} - \frac{(C^{-1}\underset{\sim}{u})(\underset{\sim}{v}'C^{-1})}{1 + \underset{\sim}{v}'C^{-1}\underset{\sim}{u}}\right].$$

Now, $$C^{-1}\underset{\sim}{u} = \begin{pmatrix} 1/p_1 & 0 & \cdots & 0 \\ 0 & 1/p_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/p_{k-1} \end{pmatrix}\begin{pmatrix} p_1 \\ \vdots \\ p_{k-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \underset{\sim}{1}_{(k-1)}.$$

and $$\underset{\sim}{v}'C^{-1} = \begin{pmatrix} -p_1 \\ -p_2 \\ \vdots \\ -p_{k-1} \end{pmatrix}\begin{pmatrix} 1/p_1 & 0 & \cdots & 0 \\ 0 & 1/p_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/p_{k-1} \end{pmatrix} = (-1 \ -1 \ \cdots \ -1)$$
$$= -\underset{\sim}{1}'_{(k-1)}.$$

and, $$\underset{\sim}{v}'C^{-1}\underset{\sim}{u} = -\underset{\sim}{1}'_{k-1}\begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_{k-1} \end{pmatrix} = -\sum_{i=1}^{k-1} p_i$$

$$1 + \underset{\sim}{v}'C^{-1}\underset{\sim}{u} = 1 - \sum_{i=1}^{k-1} p_i$$

$$\therefore A^{-1} = \frac{1}{n}\left[\mathrm{diag}(1/p_1, 1/p_2, \cdots, 1/p_{k-1}) + \frac{\underset{\sim}{1}_{k-1}\underset{\sim}{1}'_{k-1}}{1 - \sum_{i=1}^{k-1} p_i}\right]$$

$$= \frac{1}{n}\left[\mathrm{diag}(1/p_1, \cdots, 1/p_{k-1}) + \frac{J_{k-1}}{1 - \sum_1^{k-1} p_i}\right], \text{ define, } p_k = 1 - \sum_1^{k-1} p_i$$

$$A^{-1} = \frac{1}{n}\left[\mathrm{diag}(1/p_1, \cdots, 1/p_{k-1}) + \frac{J_{k-1}}{p_k}\right]$$

$$= \frac{1}{n}\begin{pmatrix} 1/p_1 + 1/p_k & 1/p_k & 1/p_k & \cdots & 1/p_k \\ 1/p_k & 1/p_2 + 1/p_k & 1/p_k & \cdots & 1/p_k \\ \vdots & \vdots & \vdots & & \vdots \\ 1/p_k & 1/p_k & 1/p_k & \cdots & 1/p_{k-1} + 1/p_k \end{pmatrix}_{(k-1)\times(k-1)}.$$

————

- Evaluate the value of the determinant:

$$\begin{vmatrix} a & b & b & b & \cdots & b \\ b & a & c & c & \cdots & c \\ b & c & a & c & \cdots & c \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ b & c & c & c & \cdots & a \end{vmatrix}$$

<u>Solution:-</u>

$$\begin{vmatrix} a & b & b & b & \cdots & b \\ b & a & c & c & \cdots & c \\ b & c & a & c & \cdots & c \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ b & c & c & c & \cdots & a \end{vmatrix}$$

$$= \begin{vmatrix} a & b & b & b & \cdots & b \\ 0 & a' & c' & c' & \cdots & c' \\ 0 & c' & a' & c' & \cdots & c' \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & c' & c' & c' & \cdots & a' \end{vmatrix} \qquad \begin{aligned} R_i' &= R_i - \frac{b}{a} R_1 \ \forall i > 1 \\ a' &= a - \frac{b^2}{a} \\ c' &= c - \frac{b^2}{a} \end{aligned}$$

$$= a \begin{vmatrix} a' & c' & c' & \cdots & c' \\ c' & a' & c' & \cdots & c' \\ \vdots & \vdots & \vdots & & \vdots \\ c' & c' & c' & \cdots & a' \end{vmatrix}$$

$$= a \left[a' + (n-2) c' \right] (a' - c')^{n-2}.$$

MATRIX AND DETERMINANTS

A **Matrix** of order $m \times n$ is rectangular array of mn elements in m rows and n columns. It is usually denoted by an upper case letter.

If a_{ij} be the element in the i^{th} row and j^{th} column of the matrix A, we write

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$= (\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n), \quad \underset{\sim}{\alpha}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}_{m\times 1}$$

$$= \begin{pmatrix} \underset{\sim}{\beta}_1 \\ \underset{\sim}{\beta}_2 \\ \vdots \\ \underset{\sim}{\beta}_m \end{pmatrix}, \quad \underset{\sim}{\beta}_i = (a_{i1}, a_{i2}, \ldots, a_{in})_{1\times n}$$

$$= ((a_{ij}))\; \begin{matrix} i = 1(1)m \\ j = 1(1)n \end{matrix}$$

Square Matrix:- A matrix is said to be a square matrix if no of rows of the matrix is equal to the no of columns.

A square matrix A of $m \times n$ is denoted by A_n, as $m=n$, so $A_n = ((a_{ij}))$

then a_{ii}'s are called diagonal elements of matrix A_n and a_{ij}'s are called off-diagonal elements of A_n.

Diagonal Matrix:- If all the off diagonal ~~matrix~~ elements of a square matrix vanish then it reduces to a diagonal matrix.

A diagonal matrix of order $n \times n$ consisting of elements $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is denoted by $diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Scalar Matrix:- If all diagonal elements of a diagonal matrix became equal then it reduces to a scalar matrix.

$$A = diag(\lambda, \lambda, \ldots, \lambda)$$

if $\lambda = 1$, then A is said to be an identity matrix.

$$I_n = ((\delta_{ij})), \quad \delta_{ij} = \text{kronecker's - '}\delta\text{'}.$$

Triangular matrix:- Let A be a square matrix of order n and $A = ((a_{ij}))$

Then A is said to be upper triangular matrix if $a_{ij} = 0$, $\forall i > j$.

And A is " " " lower " " " " " $\forall i < j$.

Null Matrix:- A matrix is said to be null mtx if all of its elements vanish, denoted by $O_{m\times n}$.

Sum Matrix:- A matrix is said to be a sum matrix if all of its elements become unity.

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Matrix Addition:- Two matrices are said to be conformable for addition if they are of same order.

Let, $A = ((a_{ij}))_{m\times n}$ and $B = ((b_{ij}))_{m\times n}$

then $A + B = ((a_{ij} + b_{ij}))_{m\times n}$.

Matrix addition is commutative and associative.

$$A+B = B+A.$$

$$(A+B)+C = A+(B+C).$$

Multiplication by a Scalar:- Let $A = ((a_{ij}))_{m\times n}$, λ = Scalar

$$\lambda A = ((\lambda a_{ij}))_{m\times n}$$

Combining the above two operations, we have

$$\alpha A + \beta B = ((\alpha a_{ij} + \beta b_{ij}))$$

$$A = ((a_{ij})),\ B = ((b_{ij}))$$

If we choose $\alpha = 1$ and $\beta = -1$, we get matrix subtraction.

Matrix Multiplication:- Two matrices A and B are said to be conformable for matrix multiplication in the given order if no. of columns of A = no. of columns of B.

Let $A = \begin{pmatrix} \underset{\sim}{\alpha}_1 \\ \underset{\sim}{\alpha}_2 \\ \vdots \\ \underset{\sim}{\alpha}_m \end{pmatrix}_{m\times n}$, $B = (\underset{\sim}{\beta}_1, \underset{\sim}{\beta}_2, \ldots, \underset{\sim}{\beta}_n)_{n\times n}$

then AB is defined as $AB = ((\alpha_i \beta_j))_{m\times n}$

$$= \left(\left(\sum_{k=1}^{n} a_{ik} b_{kj}\right)\right) \text{ if}$$

$$A = ((a_{ij})),\ B = ((b_{ij}))$$

If AB is defined well, BA may not be so and if AB and BA are both defined then they may not be equal.

Idempotent matrix:- A is said to be an idempotent matrix if

$$A^2 = A.$$

Now, $\left(I_n - \frac{1_n 1_n'}{n}\right)\left(I_n - \frac{1_n 1_n'}{n}\right)$

$$= I_n - \frac{1_n 1_n'}{n} - \frac{1_n 1_n'}{n} + \frac{1_n (1_n' 1_n) 1_n'}{n^2}$$

$$= I_n - \frac{1_n 1_n'}{n}.$$

Trace over matrix:- Trace over a matrix defined as the sum of the diagonal elements of A and denoted by $tr(A)$.

Properties:- $tr(A+B) = tr(A) + tr(B)$

$$tr(AB) = tr(BA)$$

$$tr\left(I_n - \frac{1_n 1_n'}{n}\right) = tr(I_n) - \frac{1}{n} tr(\underset{\sim}{1}_n \underset{\sim}{1}_n')$$

$$= n - \frac{1}{n} \times n$$

$$= n - 1.$$

Transpose of a Matrix:- Transpose of a matrix A is denoted by A'/A^T and it is defined as a matrix obtained by replacing rows of A by columns of A or columns by rows.

Properties:- $(A')'=A$, $(AB)'=B'A'$, $(A+B)'=A'+B'$.

Symmetric and Skew-symmetric Matrix:- A square matrix A is said to be a symmetric matrix if $A'=A$ and will be skew-symmetric if $A'=-A$.

$$A'=A \Leftrightarrow a_{ij}=a_{ji} \ \forall\ (i,j)$$

$$A'=-A \Leftrightarrow a_{ij}=-a_{ji} \ \forall\ (i,j) \text{ and } a_{ii}=0 \ \forall\ i.$$

Here, AA', $A'A$ and $(A\pm A')$ are also symmetric matrices. (Check)

Note:- Any square matrix can uniquely be written as a sum of symmetric and skew-symmetric matrix.

$$A=\frac{1}{2}(A+A')+\frac{1}{2}(A-A').$$

DETERMINANTS:-

$$A=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33}-a_{23}a_{32}) - a_{12}(a_{21}a_{33}-a_{23}a_{31}) + a_{13}(a_{21}a_{32}-a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{23}a_{31}$$

$$= \sum_{1\le i_1\ne i_2\ne i_3\le 3} (-1)^{N(i_1,i_2,i_3)} a_{1i_1}a_{2i_2}a_{3i_3}$$

$$= \sum_{1\le i_1\ne i_2\ne i_3\le 3} (-1)^{N(i_1,i_2,i_3)} a_{i_1 1}a_{i_2 2}a_{i_3 3}$$

$$= ((a_{ij}))_{n\times n}.$$

$$\therefore\ |A| = \det(A) = \sum_{1\le i_1\ne i_2\ne \cdots \ne i_n\le n} (-1)^{N(i_1,i_2,i_3,\ldots,i_n)} \prod_{j'=1}^{n} a_{i_{j'} j},$$

where $N(i_1,\ldots,i_n)$ = No. of inversions in $(i_1,\ldots,i_n)$.

Summation is taken over all possible permutations of $(1,2,\ldots,n)$, i.e. no. of terms under the summation is $n!$.

Properties: 1. $\operatorname{Det}(A) = \operatorname{Det}(A')$

Sol. Let $A = ((a_{ij}))_{n\times n}$, where $B = ((b_{ij}))_{n\times n}$, $B = A'$.

clearly, $a_{ji} = b_{ij}\ \forall (i,j)$

$$|A'| = |B| = \sum_{1\le i_1\ne i_2\ne\cdots\ne i_n\le n} (-1)^{N(i_1,\ldots,i_n)} a_{i_1 1} a_{i_2 2}\cdots a_{i_n n}$$

$$= |A|.$$

P: 2. $\det(\lambda A) = \lambda^n \det(A)$, where A is of order n.

Sol. Let $A = ((a_{ij}))$, $B = ((b_{ij}))$

$B = \lambda A$, clearly, $b_{ij} = \lambda a_{ij}\ \forall (i,j)$

$$|B| = \sum_{1\le i_1\ne i_2\ne\cdots\ne i_n\le n} (-1)^{N(i_1,\ldots,i_n)} b_{1i_1} b_{2i_2}\cdots b_{ni_n}$$

$$= \sum_{1\le i_1\ne i_2\ne\cdots\ne i_n\le n} (-1)^{N(i_1,\ldots,i_n)} \lambda a_{1i_1}\lambda a_{2i_2}\cdots \lambda a_{ni_n}.$$

$$= \lambda^n |A|.$$

P.3. Let $A = \begin{pmatrix} \underset{\sim}{\alpha}_1' \\ \vdots \\ \underset{\sim}{\alpha}_m' \end{pmatrix} = ((a_{ij}))$ and $B = \begin{pmatrix} \underset{\sim}{\alpha}_1' + \sum_{j\ne 1} \lambda_j \underset{\sim}{\alpha}_i' \\ \underset{\sim}{\alpha}_2' \\ \vdots \\ \underset{\sim}{\alpha}_m' \end{pmatrix}$ then $|B| = |A|$

Sol. Let $B = ((b_{ij}))_{m\times m}$

$$b_{1j} = a_{1j} + \sum_{k\ne 1} \lambda_k a_{kj}, \quad b_{ij} = a_{ij} + \sum_{k\ne 1} \lambda_k a_{kj}$$

$$|B| = \sum_{1\le i_1\ne i_2\ne\cdots\ne i_m\le m} (-1)^{N(i_1,\ldots,i_m)} b_{1i_1} b_{2i_2}\cdots b_{mi_m}$$

$$= \sum_{1\le i_1\ne i_2\ne\cdots\ne i_m\le m} (-1)^{N(i_1,\ldots,i_m)} \left(a_{1i_1} + \sum_{k=2}^{m} \lambda_k a_{ki_1}\right) a_{2i_2}\cdots a_{mi_m}$$

$$= \sum_{1\le i_1\ne i_2\ne\cdots\ne i_m\le m} (-1)^{N(i_1,\ldots,i_m)} a_{1i_1} a_{2i_2}\cdots a_{mi_m} + 0 = |A|.$$

P.4:- $|AB| =$ [redacted] $|A||B|$

Let $A = ((a_{ij}))_{n\times n}$ $\qquad AB = \left(\left(\sum a_{ik} b_{kj}\right)\right)_{n\times n}$

$B = ((b_{ij}))_{n\times n}$

$$|AB| = \sum_{1\le i_1\ne i_2\ne\cdots\ne i_n\le n} (-1)^{N(i_1,\ldots,i_n)} \sum_{k=1}^{n} a_{1k} b_{ki_1} \sum_{k=1} a_{2k} b_{ki_2} \cdots \sum_{k=1}^{n} a_{nk} b_{ki_n}.$$

$$= \sum_{1\le k_1, k_2,\ldots,k_n\le n} a_{1k_1} a_{2k_2}\cdots a_{nk_n} \sum_{1\le i_1\ne i_2\ne\cdots\ne i_n\le n} (-1)^{N(i_1,\ldots,i_n)} b_{k_1 i_1} b_{k_2 i_2}\cdots b_{k_n i_n}$$

Rewrite B as $B = B(1,2,\ldots,n)$

$B(k_1,\ldots,k_n)$ = matrix obtained from B replacing the [redacted] 1st row by k_1th row and 2nd row by k_2th row and so on, $\forall i = 1,\ldots,n\ \forall i$.

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$$|AB| = \sum_{1 \le k_1, k_2, \dots, k_n \le n} a_{1k_1} a_{2k_2} \cdots a_{nk_n} |B(k_1, \dots, k_n)|$$

$$= \sum_{1 \le k_1 \ne k_2 \ne \cdots \ne k_n \le n} (-1)^{N(k_1,\dots,k_n)} |B(1,2,\dots,n)| a_{1k_1} a_{2k_2} \cdots a_{nk_n}$$, if at least two k_i's are equal then $|B(k_1,\dots,k_n)|$ vanishes.

$$= |B| \sum_{1 \le k_1 \ne \cdots \ne k_n \le n} (-1)^{N(k_1,\dots,k_n)} a_{1k_1} a_{2k_2} \cdots a_{nk_n}$$

$$= |A||B| .$$

Minor • Cofactor • Inverse :-

Let A be a square matrix of order $n \times n$, $A = ((a_{ij}))$, then minor of a_{ij} is determinant of the matrix obtained from A omitting the ith row and jth column.

Cofactor of $a_{ij} = A_{ij}$, say

$= (-1)^{i+j} \times$ minor of a_{ij}.

Adjoint of $A = A^*$ or $Adj(A)$

$= ((A_{ij}))^T$

Now, $A = ((a_{ij}))_{n \times n}$; A_{ij} = Cofactor of a_{ij}

$$\sum a_{ij} A_{i'j} = \begin{cases} |A| & \text{if } i = i' \\ 0 & \text{if } i \ne i', \forall j . \end{cases}$$

or, $$\sum a_{ij} A_{ij'} = \begin{cases} |A| & \text{if } j = j' \\ 0 & \text{if } j \ne j' \ \forall i . \end{cases}$$

Result:- $A = ((a_{ij}))_{n \times n}$, A_{ij} = Co-factor of a_{ij}, $A^* = Adj(A)$.

Then $AA^* = A^*A = |A| I_n$.

Proof:-

$$AA^* = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{n1} \\ A_{21} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \sum a_{1k} A_{1k} & \sum a_{2k} A_{2k} & \cdots & \sum a_{1k} A_{nk} \\ \sum a_{2k} A_{1k} & \sum a_{2k} A_{2k} & \cdots & \sum a_{2k} A_{nk} \\ \vdots & \vdots & & \vdots \\ \sum a_{nk} A_{1k} & \sum a_{nk} A_{2k} & \cdots & \sum a_{nk} A_{nk} \end{pmatrix}$$

$$= \mathrm{diag}(|A|, |A|, \dots, |A|)$$

$$= |A| \cdot I_n .$$

Singular and Non-Singular Matrices :—

A matrix (square) is said to be a non-singular matrix if $|A| \neq 0$ otherwise it is said to be singular.

Let us assume that A is non-singular, i.e. $|A| \neq 0$

$$A\left(\frac{1}{|A|}\cdot A^*\right) = \left(\frac{1}{|A|}\cdot A^*\right)A = I_n \qquad \text{[SEE P.T.O.]}$$

$$[\because |A| \neq 0]$$

if $B = \frac{1}{|A|}\cdot A^*$, then $AB = BA = I_n$.

[Analogous to the feature of the real numbers (non-zero), $x \times \frac{1}{x} = \frac{1}{x} \times x = 1$.]

Here B is the inverse matrix of A and usually denoted by A^{-1},

then $AA^{-1} = A^{-1}A = I$.

Result:- Inverse of a square matrix A exists if and only if A is non-singular.

Proof:- **If Part:-** Let A be a non-singular matrix, $|A| \neq 0$.

Then $AA^* = A^*A = |A|I$

$$\Rightarrow A\left(\frac{1}{|A|}A^*\right) = \left(\frac{1}{|A|}A^*\right)A = I$$

$$\Rightarrow A^{-1} = \frac{1}{|A|}A^* \text{ exists if } |A| \neq 0.$$

Only if Part:- Let A^{-1} exists.

$$AA^{-1} = A^{-1}A = I$$

$$\Rightarrow |AA^{-1}| = 1$$

$$\Rightarrow |A||A^{-1}| = 1$$

$$\Rightarrow |A| \neq 0.$$

Properties 1:- Inverse of a square matrix is unique.

$\Rightarrow$ If possible let B and C be the inverse of A. Then

$AB = BA = I$ ——— ① $\times C$ (Pre-multiply)

$AC = CA = I$ ——— ② $\times B$ (Post-multiply)

$CAB = CBA = C$

$ACB = CAB = B$

$\therefore B = C$.

P-2:- $(AB)^{-1} = B^{-1}A^{-1}$.

Sol.

$$(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = AA^{-1} = I$$

$$(AB)^{-1}ABB^{-1}A^{-1} = (AB)^{-1}I = (AB)^{-1}$$

$$\Rightarrow IB^{-1}A^{-1} = (AB)^{-1}$$

$$\therefore (AB)^{-1} = B^{-1}A^{-1}.$$

Result:- Let $A_{p\times p}$ be a non-singular matrix and $B_{p\times q}$ and $C_{q\times p}$ such that $A+BC$ is also non-singular then,

$$[A_{p\times p} + B_{p\times q}C_{q\times p}]^{-1} = A^{-1} - A^{-1}B(I_p + CA^{-1}B)CA^{-1},$$

when the inverses exist.

Proof:-

$$D = A+BC$$

$$DA^{-1} = (A+BC)A^{-1}$$
$$= AA^{-1} + BCA^{-1}$$
$$= I_p + BCA^{-1}$$

$$DA^{-1}B = (I_p + BCA^{-1})B$$
$$= B + BCA^{-1}B$$
$$= B(I_q + CA^{-1}B)$$

$$\Rightarrow DA^{-1}B(I_q + CA^{-1}B)^{-1} = B.$$

$$\Rightarrow DA^{-1}B(I_q + CA^{-1}B^{-1})C = BC$$

$$\Rightarrow A + DA^{-1}B(I_q + CA^{-1}B^{-1})C = A + BC = D$$

$$\Rightarrow D(I_p - A^{-1}B(I_q + CA^{-1}B)^{-1}C) = A$$

$$\Rightarrow D(I_p - A^{-1}B(I_q + CA^{-1}B)^{-1}C)A^{-1} = I_p$$

$$\Rightarrow D^{-1}D(A^{-1} - A^{-1}B(I_q + CA^{-1}B)^{-1}CA^{-1}) = D^{-1}I_p$$

$$D^{-1} = (A+BC)^{-1} = A^{-1} - A^{-1}B(I_q + CA^{-1}B)^{-1}CA^{-1}.$$

General Case:- $A_{p\times p}$, $C_{q\times q}$ are non-singular matrices
$B_{p\times q}$, $D_{q\times p}$ are non-singular matrices

Then $\ni$ $(A+BCD)$, a non-singular matrix then

$$(A+BCD)^{-1} = A^{-1} - A^{-1}B(I_q + CDA^{-1}B)^{-1}CDA^{-1}$$

$$= A^{-1} - A^{-1}B(C^{-1}(I_q + CDA^{-1}B))^{-1}DA^{-1}$$

Particular Case:- $(A + \underset{\sim}{\alpha}\underset{\sim}{\beta}')^{-1} = A^{-1} - A'\underset{\sim}{\alpha}(1 + \underset{\sim}{\beta}'A^{-1}\underset{\sim}{\alpha})^{-1}\underset{\sim}{\beta}'A^{-1}$

$$= A^{-1} - \frac{1}{1+\underset{\sim}{\beta}'A^{-1}\underset{\sim}{\alpha}} A^{-1}\underset{\sim}{\alpha}\underset{\sim}{\beta}'A^{-1}.$$

Partitioning and Augmenting :-

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1s} & a_{1\,\overline{s+1}} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2s} & a_{2\,\overline{s+1}} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{ms} & a_{m\,\overline{s+1}} & \cdots & a_{mn} \\ a_{\overline{m+1}\,1} & a_{\overline{m+1}\,2} & \cdots & a_{\overline{m+1}\,s} & a_{\overline{m+1}\,\overline{s+1}} & \cdots & a_{\overline{m+1}\,n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ns} & a_{n\,\overline{s+1}} & \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11\,m\times s} & A_{12\,m\times \overline{n-s}} \\ A_{21\,\overline{n-m}\times s} & A_{22\,\overline{n-m}\times\overline{n-s}} \end{bmatrix}$$

Here A is partitioned into sub-matrices $A_{11}, A_{12}, A_{21}, A_{22}$. In particular, the submatrices may reduce to vectors or scalars.

Example:-

$$\left(\begin{array}{c|ccc} 2 & 3 & 4 & 6 \\ \hline 7 & 9 & 1 & 2 \\ 4 & 5 & 6 & 3 \end{array}\right) = \begin{pmatrix} 2 & \underset{\sim}{\alpha}'_{1\times 3} \\ \underset{\sim}{\beta}'_{2\times 1} & C_{2\times 3} \end{pmatrix}$$

Again $A_{m\times n}$ is augmented by $B_{m\times n}$ may be of the form $(A/B)_{m\times 2n}$ or $\left(\frac{A}{B}\right)_{2m\times n}$.

Problem:- 1. $\begin{vmatrix} A_{n\times n} & O_{n\times m} \\ O_{m\times n} & I_m \end{vmatrix} = |A|$.

Proof:-

$$\begin{vmatrix} A_{n\times n} & O_{n\times m} \\ O_{m\times n} & I_m \end{vmatrix} = 1 \begin{vmatrix} A_{n\times n} & O_{n\times\overline{m-1}} \\ O_{\overline{m-1}\times n} & I_{m-1} \end{vmatrix}$$

$$= 1 \begin{vmatrix} A_{n\times n} & O_{n\times\overline{m-2}} \\ O_{\overline{m-2}\times n} & I_{m-2} \end{vmatrix}$$

$$= 1 \begin{vmatrix} A_{n\times n} & O_{n\times 1} \\ O_{1\times n} & 1 \end{vmatrix}$$

$$= |A|.$$

Problem:- 2. $\begin{vmatrix} A_{m\times m} & B_{m\times n} \\ O_{n\times m} & C_{n\times n} \end{vmatrix} = |A|\,|C|$

Proof:-

$$= \begin{vmatrix} I & O \\ O & C \end{vmatrix} \times \begin{vmatrix} A & B \\ O & I \end{vmatrix}$$

$$= |C| \times |A|$$

$$= |A|\,|C|.$$

Problem:-3:- $\begin{vmatrix} A_{m\times n} & O_{m\times n} \\ O_{m\times n} & B_{n\times n} \end{vmatrix} = |A|\,|B|$

Proof:-

$$\begin{pmatrix} A_{m\times m} & O_{m\times n} \\ O_{n\times m} & I_n \end{pmatrix}\begin{pmatrix} I_n & O_{m\times n} \\ O_{n\times m} & B_{n\times m} \end{pmatrix}$$

$$= \begin{pmatrix} A_{m\times n} & O_{m\times n} \\ O_{n\times n} & B_{n\times n} \end{pmatrix} = |A|\,|B| .$$

Result:- Show that —

$$\det \begin{bmatrix} \underset{m\times m}{A_{11}} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}_{n\times n} = |A_{11}|\,|A_{22} - A_{21}A_{11}^{-1}A_{12}| \cdots\cdots (1)$$

$$= |A_{22}|\,|A_{11} - A_{12}A_{22}^{-1}A_{21}| \cdots\cdots (2)$$

if A_{11} is an non-singular matrix.

Proof:-

$$\begin{bmatrix} I_m & O_{m\times \overline{n-m}} \\ -A_{21}A_{11}^{-1} & I_{\overline{n-m}} \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

$$\therefore \det \begin{bmatrix} I_m & O_{m\times \overline{n-m}} \\ -A_{21}A_{11}^{-1} & I_{\overline{n-m}} \end{bmatrix} \det \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$= \det \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

$$= |A_{11}|\,|A_{22} - A_{21}A_{11}^{-1}A_{12}| \cdots\cdots (1)$$

$$\begin{bmatrix} I_m & -A_{12}A_{22}^{-1} \\ 0_{\overline{n-m}\times m} & I_{n-m} \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

$$\therefore \det \begin{bmatrix} I_m & -A_{12}A_{22}^{-1} \\ O_{\overline{n-m}\times m} & I_{n-m} \end{bmatrix} \det \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$= \det \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

$$= |A_{22}|\,|A_{11} - A_{12}A_{22}^{-1}A_{21}| \cdots\cdots (2)$$

Rank of a Matrix :-

$V_C(A)$: Column space of A (Vector space generated by the columns of A).

$V_R(A)$: Row-space of A.

Column Rank of A is defined as $\dim\{V_C(A)\}$ or number of LIN columns of A.

Row rank of A = Column rank of A = $R(A) \leq \min(m,n)$, where $A_{m\times n} = A$.

Definition:- Rank of the matrix A is the order of highest order non-vanishing minor of A.

Some Useful Results:-

1. $\text{Rank}(AB) \leq \min[\text{Rank}(A), \text{Rank}(B)]$

Proof:- Let A be a matrix of order $m\times n$ and B be a matrix of order $n\times p$.

suppose $A = ((a_{ij}))$

$B = ((b_{ij}))$

$$A = (\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \dots, \underset{\sim}{\alpha}_n)$$

$$AB = (\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \dots, \underset{\sim}{\alpha}_n)\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{pmatrix}$$

$$= \left(\sum_{k=1}^{n} b_{k1}\underset{\sim}{\alpha}_k, \sum_{k=1}^{n} b_{k2}\underset{\sim}{\alpha}_k, \dots, \sum_{k=1}^{n} b_{kp}\underset{\sim}{\alpha}_k\right)$$

Columns of AB are linear combination of columns of A.

$$V_C(AB) \subseteq V_C(A)$$

$$\Rightarrow \dim V_C(AB) \leq \dim V_C(A)$$

$$\Rightarrow R(AB) \leq R(A) \quad \dots\dots (i)$$

Similarly we can show that $R(AB) \leq R(B) \quad \dots\dots (ii)$

Combining (i) & (ii) we have, $R(AB) \leq \min[R(A), R(B)]$.

2. If A be an idempotent matrix then Rank (A) = Trace(A).

Proof:- Let A be a matrix of order $n\times n$, such that $A^2 = A$ and rank (A) = r.

By rank-factorisation theorem, we have

$$A = B_{n\times r} C_{r\times n} \text{ where } \operatorname{rank}(B) = \operatorname{rank}(C) = r.$$

$$A^2 = A$$

$$\Rightarrow BCBC = BC$$

$$\Rightarrow B'BCBCC' = B'BCC'$$

We know, $R(B'B) = \operatorname{Rank}(B) = r$, where BB' is of order r.

i.e. BB' is non-singular $\Rightarrow (BB')^{-1}$ exists.

Similarly, $(CC')^{-1}$ exists.

$$(B'B)^{-1} B'BCBCC'(CC')^{-1}$$

$$= (B'B)^{-1} B'BCC'(CC')^{-1}$$

$$\Rightarrow CB = I_r$$

$\therefore$ Trace (CB) = trace(BC) = trace(A) = trace(I_r) = r = rank(A)

3. Rank of a matrix remain unaltered if it is premultiplied or post multiplied by a non-singular matrix,

$$R(PA) = R(AQ) = R(PAQ) = R(A), \text{ where}$$

$P_{m\times n}$, $Q_{n\times n}$ are non-singular matrices.

Proof:-

$$R(PA) \le R(A) \cdots\cdots (1)$$

Again, $R(A) = R(I_m A)$

$= R(P^{-1}PA)$ [P^{-1} exists, since P is n.s.]

$\le R(PA) \quad \cdots (2)$

(1) & (2) gives, $R(A) = R(PA)$.

Null Space & Nullity :-

Let A be a matrix of order $m \times n$.

Define, $N(A) = \{ \underset{\sim}{x} : A\underset{\sim}{x} = \underset{\sim}{0} \}$

$N(A)$ constitutes a vector space.

Suppose, $A = \begin{pmatrix} \underset{\sim}{\alpha}_1' \\ \underset{\sim}{\alpha}_2' \\ \vdots \\ \underset{\sim}{\alpha}_m' \end{pmatrix}$

$$A\underset{\sim}{x} = \underset{\sim}{0} \Rightarrow \begin{pmatrix} \underset{\sim}{\alpha}_1'\underset{\sim}{x} \\ \underset{\sim}{\alpha}_2'\underset{\sim}{x} \\ \vdots \\ \underset{\sim}{\alpha}_m'\underset{\sim}{x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\Rightarrow$ i.e. $\underset{\sim}{x}$ is orthogonal to any row of A.

$N(A)$ and $V_R(A)$ are orthogonal spaces.

i.e. $N(A) \perp V_R(A)$.

Thus, if $R(A) = r$,

i.e. if $\dim V_R(A) = r$

then $\dim N(A) = n - r$

$$\left[N(A) + V_R(A) = E_n \;\; \& \;\; N(A) \cap V_R(A) = \{\underset{\sim}{0}\} \right]$$

Here $N(A)$ is termed as Null space of the matrix A and $\dim(N(A))$ is the Nullity of A.

Result:- S.T. $\dim \left[\underset{m\times n}{N(A)} \right] = n - R(A)$.

Proof:- Let $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_s\}$ be a basis of $V_R(A)$

and $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \dots, \underset{\sim}{y}_t\}$ be a basis of $N(A)$.

$\therefore$ By definition of $N(A)$, we have,

$$\underset{\sim}{x}_i' \underset{\sim}{y}_j = 0 \;\; \forall\, i, j.$$

$\therefore$ $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_s, \underset{\sim}{y}_1, \underset{\sim}{y}_2, \dots, \underset{\sim}{y}_t\}$ be a set of LIN vectors
n-component vectors can't contain more than n vectors.

$$s + t \le n$$

if possible, let $s + t < n$.

$\exists$ at least one vector which is orthogonal to any one of

$$\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_s, \underset{\sim}{y}_1, \ldots, \underset{\sim}{y}_t$$

Let $\underset{\sim}{\gamma}$ be such a vector $\ni$ $\underset{\sim}{x}_i'\underset{\sim}{\gamma} = \underset{\sim}{0}$ $\forall i = 1, 2, \ldots, s$.

$$\Rightarrow \underset{\sim}{\gamma} \in N(A)$$

$$\Rightarrow \underset{\sim}{\gamma} = \sum_{i=1}^{t} \theta_i \underset{\sim}{y}_i \text{ for some } \theta_i\text{'s}.$$

$\underset{\sim}{y}_i'\underset{\sim}{\gamma} = \underset{\sim}{0}$ $\forall i = 1(1)t$

$s+t \nless n$.

$\therefore \dim(\gamma_R(A)) + \dim(N(A)) = n$

$\therefore \dim(N(A)) = n - R(A)$.

Result:- $R(AA') = R(A) = R(A') = R(AA')$.

Proof:- Let $x \in N(A'A)$

$$\Rightarrow A'A\underset{\sim}{x} = \underset{\sim}{0}$$

$$\Rightarrow \underset{\sim}{x}'A'A\underset{\sim}{x} = \underset{\sim}{0}$$

i.e. $\underset{\sim}{y}'\underset{\sim}{y} = \sum y_i^2 = 0$, $\underset{\sim}{y} = A\underset{\sim}{x}$

$$\Rightarrow y_i = 0 \ \forall i.$$

i.e. $\underset{\sim}{y} = \underset{\sim}{0}$

$A\underset{\sim}{x} = \underset{\sim}{0}$, $\underset{\sim}{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$, assuming A has order $m \times n$.

$$\Rightarrow \underset{\sim}{x} \in N(A)$$

$$N(A'A) \subseteq N(A) \text{ ------- } (1)$$

Let $\underset{\sim}{x} \in N(A)$

$$\Rightarrow A\underset{\sim}{x} = \underset{\sim}{0}$$

$$\Rightarrow A'A\underset{\sim}{x} = \underset{\sim}{0}$$

$\Rightarrow \underset{\sim}{x} \in N(A'A)$; $N(A) \subseteq N(A'A) \Rightarrow R(A) = R(A'A)$.

Now, let $A' = B$, so for the matrix B, rank $(B'B) = r(B)$.

i.e. $R(AA') = R(A') = R(A) = R(A'A)$.

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Reducing a matrix into Normal form:—

Let A be a matrix of order $m\times n$, suppose $R(A) = r < \min(m,n)$. Then $\exists$ non-singular matrices P and Q such that

$$PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof:- Since $R(A) = r < \min(m,n)$

$\exists$ a non-singular matrix P_1 such that $P_1A = E$, an echelon matrix with r non-null rows.

Clearly, $E = \begin{pmatrix} E_1 \\ 0 \end{pmatrix}_{m\times n}$, E_1 is also an echelon matrix.

Now $\text{Rank}(E_1) = r =$ Column rank of E_1.

E_1 has $n-r$ columns each of which is LD on the rest r LIN columns. Hence through column operations those dependent columns can be reduced to null columns.

i.e. $E_1 \sim \left(\underset{r\times r}{E_2} \quad \underset{r\times n}{0} \right)$, where E_2 has full rank.

$\therefore \exists$ a non-singular matrix $P_2 \ni$

$$P_1AP_2 = \begin{pmatrix} E_2 & 0 \\ 0 & 0 \end{pmatrix}$$

As E_2 is a square matrix of full rank through row column operation it can be reduced to I_r.

Hence $\exists$ non-singular matrices P_3 and P_4

$$P_3P_1AP_2P_4 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{m\times n}$$

Let $P_3P_1 = P$ and $P_2P_4 = Q$

Clearly, P and Q are non-singular matrices as they are product of non-singular matrices.

A square matrix of full rank is always non-singular.

Let A be a square matrix of order n possesing full rank then $\exists$ non-singular matrices P and Q $\ni$

$$PAQ = I_n.$$

$$|P||A||Q| = 1$$

$$\therefore |A| \neq 0.$$

- **Sylvester's Inequality :-**

$$R(A_{n\times n} B_{n\times n}) \geq R(A) + R(B) - n,$$

Proof:- Let $R(A) = r < n$

$\exists$ n.s. matrices P and Q such that

$$PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

Define a square matrix C of the order $n\times n$ such that

$$PCQ = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix}$$

Clearly, $P(A+C)Q = I_n$

Therefore, $|P(A+C)Q| = |I_n| = 1$

$\Rightarrow |P|\,|A+C|\,|Q| = 1.$

$\Rightarrow A+C$ is non singular.

Hence, $R(B) = R((A+C)B)$

$= R(AB+CB)$

$< R(AB) + R(CB)$

$\leq R(AB) + R(C)$

$= R(AB) + n - r$

$= R(AB) + n - R(A)$

$\therefore R(AB) \geq R(A) + R(B) - n.$

Orthogonal Transformation:- A matrix transformation $\underset{\sim}{Y} = A_{n\times n}\underset{\sim}{x}$ is said to be orthogonal iff A is orthogonal.

Here, $AA^T = I_n$

$\Rightarrow |A| = \pm 1$.

$\Rightarrow$ A is singular.

Now, A matrix transformation is said to be non-singular iff A is non-singular.

i.e. $\underset{\sim}{y} = A\underset{\sim}{x} \Rightarrow \underset{\sim}{x} = A^{-1}\underset{\sim}{y}$, i.e. N.s. transformation is one-one

An orthogonal transformation is an n.s. transformation & hence it is one-to-one.

Note that, $\underset{\sim}{y}^T\underset{\sim}{y} = (A\underset{\sim}{x})^T(A\underset{\sim}{x})$

$\Rightarrow \underset{\sim}{x}^T A^T A\underset{\sim}{x} = \underset{\sim}{x}^T\underset{\sim}{x}$

i.e. $\sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} x_i^2$

i.e. the length of the vector is preserved under orthogonal transformation.

Helmert's Transformation :—

$(x_1, x_2, \ldots, x_n) \longrightarrow (y_1, y_2, \ldots, y_n) \ni$

$$y_1 = \frac{x_1}{\sqrt{n}} + \frac{x_2}{\sqrt{n}} + \cdots + \frac{x_n}{\sqrt{n}}$$

$$y_2 = \frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}}$$

$$y_3 = \frac{x_1}{\sqrt{6}} + \frac{x_2}{\sqrt{6}} - \frac{2x_3}{\sqrt{6}}$$

$$\vdots$$

$$y_n = \frac{x_1}{\sqrt{n(n-1)}} + \frac{x_2}{\sqrt{n(n-1)}} + \cdots + \frac{x_{n-1}}{\sqrt{n(n-1)}} - \frac{(n-1)x_n}{\sqrt{n(n-1)}}$$

i.e. $\underset{\sim}{y} = P\underset{\sim}{x}$, where P is an orthogonal matrix.

$\Rightarrow \underset{\sim}{y}'\underset{\sim}{y} = \underset{\sim}{x}'P'P\underset{\sim}{x} = \underset{\sim}{x}' I_n \underset{\sim}{x} = \underset{\sim}{x}'\underset{\sim}{x}$

$\Rightarrow \sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} x_i^2$.

Example:-

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

$$ns^2 = \sum_{i=1}^{n}(x_i-\bar{x})^2 = \sum_{i=2}^{n} y_i^2$$

Note that, $y_1 = \frac{\sum x_i}{\sqrt{n}} = \frac{n\bar{x}}{\sqrt{n}} = \sqrt{n}\,\bar{x}$

Now, $ns^2 = \sum_{i=1}^{n}(x_i-\bar{x})^2 = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2$

$$= \sum_{i=1}^{n} y_i^2 - y_1^2$$

$$= \sum_{i=2}^{n} y_i^2$$

$$ns^2 = \sum_{i=2}^{n}\frac{i}{i-1}(x_i-\bar{x}_i)^2 \ ; \ \bar{x}_i = \sum_{j=1}^{i} x_j/i \ , \ i=1(1)n.$$

$$\Rightarrow ns^2 = \sum_{i=2}^{n}\left\{\sqrt{\frac{i}{i-1}}\left(\frac{i-1}{i}x_i - \frac{x_1}{i} - \frac{x_2}{i} - \cdots - \frac{x_i}{i}\right)\right\}^2$$

$$= \sum_{i=2}^{n}\left\{\frac{x_1}{\sqrt{i(i-1)}} + \frac{x_2}{\sqrt{i(i-1)}} + \cdots + \frac{x_{i-1}}{\sqrt{i(i-1)}} - \frac{(i-1)x_i}{\sqrt{i(i-1)}}\right\}^2$$

$$= \sum_{i=2}^{n} y_i^2$$

[CU]

- Problem:- Show that the SD s of a set of observation $x_1, x_2, \ldots, x_n$ is given by $n.s^2 = \sum_{i=2}^{n}\frac{i}{i-1}(x_i-\bar{x}_i)^2$, where $\bar{x}_i = \sum_{j=1}^{i} x_j/i$, $i=2(1)n$.

Solution:- Note that, $\sqrt{\frac{i}{i-1}}(\bar{x}_i - x_i)$

$$= \sqrt{\frac{i}{i-1}}\left(\frac{x_1 + \cdots + x_{i-1} + x_i - i.x_i}{i}\right)$$

$$= \frac{x_1 + \cdots + x_{i-1} - (i-1)x_i}{\sqrt{i(i-1)}} \ , \ i=2(1)n.$$

Let, $y_1 = \dfrac{x_1+x_2+\cdots+x_n}{\sqrt{n}}$,

$y_i = \dfrac{x_1+\cdots+x_{i-1}-(i-1)x_i}{\sqrt{i(i-1)}}$, $i=2(1)n$.

$$y_1 = \frac{x_1}{\sqrt{n}} + \frac{x_2}{\sqrt{n}} + \cdots + \frac{x_n}{\sqrt{n}}$$

$$y_2 = \frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}} = \frac{x_1-x_2}{\sqrt{2}}$$

$$y_3 = \frac{x_1}{\sqrt{6}} + \frac{x_2}{\sqrt{6}} - \frac{2x_3}{\sqrt{6}} = \frac{x_1+x_2-2x_3}{\sqrt{2\cdot 3}}$$

$$y_4 = \frac{x_1}{\sqrt{12}} + \frac{x_2}{\sqrt{12}} + \frac{x_3}{\sqrt{12}} - \frac{3x_4}{\sqrt{12}} = \frac{x_1+x_2+x_3-3x_4}{\sqrt{3\cdot 4}}$$

$$\vdots$$

$$y_{n-1} = \frac{x_1+\cdots+x_{n-1}-(n-2)x_{n-1}}{\sqrt{(n-2)(n-1)}}$$

$$y_n = \frac{x_1+\cdots+x_{n-1}-(n-1)x_n}{\sqrt{(n-1)n}}$$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1/\sqrt{n} & 1/\sqrt{n} & 1/\sqrt{n} & \cdots & 1/\sqrt{n} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & \cdots & 0 \\ 1/\sqrt{6} & +1/\sqrt{6} & -2/\sqrt{6} & 0 \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1/\sqrt{n(n-1)} & \frac{1}{\sqrt{(n-1)n}} & \cdots & \cdots & -\frac{n-1}{\sqrt{(n-1)n}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$\Rightarrow \underset{\sim}{y} = A\underset{\sim}{x}$, where A is an orthogonal matrix.

$$\underset{\sim}{y}'\underset{\sim}{y} = (A\underset{\sim}{x})'(A\underset{\sim}{x})$$
$$= \underset{\sim}{x}'A'A\underset{\sim}{x}$$
$$= \underset{\sim}{x}'\underset{\sim}{x}$$

$$\Leftrightarrow \sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} x_i^2$$

$$\Rightarrow \sum_{i=1}^{n} x_i^2 - y_1^2 = \sum_{i=2}^{n} y_i^2$$

$$\Rightarrow \sum_{i=1}^{n} x_i^2 - n\bar{x}^2 = \sum_{i=2}^{n} y_i^2$$

$$\Rightarrow \sum_{i=1}^{n} (x_i-\bar{x})^2 = \sum_{i=2}^{n} \frac{i}{i-1}(x_i-\bar{x}_i)^2$$

$$\Rightarrow ns^2 = \sum_{i=2}^{n} \frac{i}{i-1}(x_i-\bar{x})^2 .$$

• [CU] For $n \geq 2$, suppose $x_1, x_2, \ldots, x_n$ are not all equal. Consider the orthogonal transformation of $(y_1, y_2, \ldots, y_n)$ to $(z_1, z_2, \ldots, z_n)$ such that

$$\underset{\sim}{z}_{n\times 1} = \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots\cdots & \frac{1}{\sqrt{n}} \\ \frac{x_1-\bar{x}}{\sqrt{\sum_i (x_i-\bar{x})^2}} & \frac{x_2-\bar{x}}{\sqrt{\sum_i (x_i-\bar{x})^2}} & \cdots\cdots & \frac{x_n-\bar{x}}{\sqrt{\sum_i (x_i-\bar{x})^2}} \\ a_{31} & a_{32} & \cdots\cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{pmatrix}_{n\times n} \cdot \underset{\sim}{y}_{n\times 1}$$

show that $\sum_{i=3}^{n} z_i^2 = \left\{\sum_{i=1}^{n}(y_i-\bar{y})^2\right\}\left\{1 - r_{xy}^2\right\}$,

where r_{xy} is the correlation coefficient between x and y.

Solution:- Here, $\underset{\sim}{z}^{n\times 1} = A^{n\times n} \cdot \underset{\sim}{y}^{n\times 1}$

where, A is the given orthogonal matrix.

By property, $\underset{\sim}{z}'\underset{\sim}{z} = \underset{\sim}{y}'\underset{\sim}{y}$

$$\Rightarrow \sum_{i=1}^{n} z_i^2 = \sum_{i=1}^{n} y_i^2$$

$$\Rightarrow \sum_{i=3}^{n} z_i^2 = \sum_{i=1}^{n} y_i^2 - z_1^2 - z_2^2$$

Here, $z_1 = \dfrac{y_1 + y_2 + \cdots + y_n}{\sqrt{n}} = \sqrt{n}\,\bar{y}$.

$$z_2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})y_i}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2}}$$

$$= \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2}}$$

$$= r_{xy} \cdot \sqrt{\sum_{i=1}^{n}(y_i-\bar{y})^2}$$

$$\therefore \sum_{i=3}^{n} z_i^2 = \left(\sum_{i=1}^{n} y_i^2 - n\bar{y}^2\right) - r_{xy}^2 \cdot \sum_{i=1}^{n}(y_i-\bar{y})^2$$

$$= \sum_{i=1}^{n}(y_i-\bar{y})^2 - r_{xy}^2 \cdot \sum_{i=1}^{n}(y_i-\bar{y})^2$$

$$= \left\{\sum_{i=1}^{n}(y_i-\bar{y})^2\right\}\left(1 - r_{xy}^2\right)$$

Gaussian Reduction or Gaussian Elimination for n.s. matrix:

Consider a system $A\underset{\sim}{x} = \underset{\sim}{b}$; where $A^{n\times n}$ is n.s. Let P be the product of elementary row matrices which take $(A, \underset{\sim}{b})$ into $(H, \underset{\sim}{d})$, where H is an echelon matrix.

$$\therefore A\underset{\sim}{x} = \underset{\sim}{b}$$

$$\Rightarrow PA\underset{\sim}{x} = P\underset{\sim}{b}$$

$$\Rightarrow H\underset{\sim}{x} = \underset{\sim}{d}$$

$$\Rightarrow \begin{pmatrix} 1 & h_{12} & h_{13} & \cdots & h_{1\,n-1} & h_{1n} \\ 0 & 1 & h_{23} & \cdots & h_{2\,n-1} & h_{2n} \\ 0 & 0 & 1 & \cdots & h_{3\,n-1} & h_{3n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & h_{n-1\,n} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{pmatrix}$$

$$\therefore x_n = d_n$$

$$\therefore x_{n-1} + h_{n-1\,n}\, x_n = d_{n-1}$$

$$\vdots$$

$$\therefore x_2 + h_{23}x_3 + \cdots + h_{2n}x_n = d_2$$

$$\therefore x_1 + h_{12}x_2 + \cdots + h_{1n}x_n = d_1$$

SYSTEM OF LINEAR EQUATION

Theorem:- A consistent system $\underset{m\times n}{A}\underset{\sim}{x} = \underset{\sim}{b}$ has a unique solution iff A is of full column rank [i.e. $r(A) = n$] & at least two solution iff $r(A) = r(A \,\vdots\, b) < n$.

Proof:- [**Lemma:** The solution set S of $A\underset{\sim}{x} = \underset{\sim}{b}$ be a consistent system, can be written as $\{\underset{\sim}{u} + \underset{\sim}{v} : \underset{\sim}{v} \in N(A)\}$, where $\underset{\sim}{u}$ is a particular solution of $A\underset{\sim}{x} = \underset{\sim}{b}$.

Proof:- Let $\underset{\sim}{\omega} \in S$. Then $A\underset{\sim}{u} = \underset{\sim}{b}$ and $A\underset{\sim}{\omega} = \underset{\sim}{b}$.

$$\Rightarrow A(\underset{\sim}{\omega} - \underset{\sim}{u}) = 0$$

$$\Rightarrow \underset{\sim}{\omega}, \underset{\sim}{u} \in N(A).$$

$$\therefore \underset{\sim}{\omega} = \underset{\sim}{u} + \underset{\sim}{\omega} - \underset{\sim}{u} = \underset{\sim}{u} + \underset{\sim}{v}, \text{ where } \underset{\sim}{v} \in N(A).$$

$$\therefore \underset{\sim}{\omega} \in \{\underset{\sim}{u} + \underset{\sim}{v} : \underset{\sim}{v} \in N(A)\}$$

Let $\underset{\sim}{\omega} \in \{\underset{\sim}{u} + \underset{\sim}{v} : \underset{\sim}{u} \in N(A)\}$

Then $A\underset{\sim}{\omega} = A\underset{\sim}{u} + A\underset{\sim}{v}$

$$= \underset{\sim}{b} + \underset{\sim}{0} = \underset{\sim}{b}.$$

$$\Rightarrow \underset{\sim}{\omega} \in S.$$

Hence, $S = \{\underset{\sim}{u} + \underset{\sim}{v} : \underset{\sim}{v} \in N(A)\}$.]

The system has a unique solution iff no. of solution in S is one.

iff $\dim\{N(A)\} = 0$ [S has only one solution

iff $n - r(A) = 0$ $\Rightarrow \underset{\sim}{u} + \underset{\sim}{0}$ is the only one solution.

iff $r(A) = n$. $\Rightarrow \underset{\sim}{v} = \underset{\sim}{0}$ is the only member of $N(A)$.

i.e. $N(A) = \{\underset{\sim}{0}\}$]

The system has at least two solution iff the no. of solutions in S is ≥ 2

iff $\underset{\sim}{u} + \underset{\sim}{0}$ is a solution and there is a non-null $\underset{\sim}{v} \in N(A) \ni$ $\underset{\sim}{v} + \underset{\sim}{u}$ is a solution.

iff $\dim(N(A)) \geq 1$

iff $n - r(A) \geq 1$

iff $r(A) \leq n-1 < n$

$\Rightarrow r(A) = r(A \mid b) < n$.

Theorem:- Let A be a matrix of order $m\times n$, then the system $\underset{m\times n}{A}\underset{\sim}{x} = \underset{\sim}{0}_{m\times 1}$ will have a non-trivial solution iff $R(A) < n$.

Proof:- If Part:- Suppose $R(A) < n$

We have to show that $A\underset{\sim}{x} = \underset{\sim}{0}$ has a trivial solution.

Let $A = (\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n)$

since $R(A) < n$

$\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n$ are linearly dependent.

so $\exists$ scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ at least one in non-zero $\ni$

$$\lambda_1\underset{\sim}{\alpha}_1 + \lambda_1\underset{\sim}{\alpha}_2 + \cdots + \lambda_n\underset{\sim}{\alpha}_n = \underset{\sim}{0}$$

$$\Rightarrow (\underset{\sim}{\alpha}_1 \ \underset{\sim}{\alpha}_2 \cdots\cdots \underset{\sim}{\alpha}_n)\begin{pmatrix}\lambda_1\\ \lambda_2\\ \vdots\\ \lambda_n\end{pmatrix} = \underset{\sim}{0}$$

$\Rightarrow A.\underset{\sim}{\lambda} = \underset{\sim}{0}$, where $\underset{\sim}{\lambda} = (\lambda_1, \lambda_2 \cdots \lambda_n)' \neq 0$

Hence $\underset{\sim}{x} = \underset{\sim}{\lambda}\ (\neq 0)$ is a solution of the system $A\underset{\sim}{x} = \underset{\sim}{0}$.

Hence, we have a non-trivial solution of the system $A\underset{\sim}{x} = \underset{\sim}{0}$.

Only If Part:- Suppose $\exists$ a non-trivial solution, say.

$\underset{\sim}{x} = \underset{\sim}{h}\ (\neq \underset{\sim}{0})$ of the system $A\underset{\sim}{x} = \underset{\sim}{0}$.

$\therefore A\underset{\sim}{h} = \underset{\sim}{0}$

$$\Rightarrow (\underset{\sim}{\alpha}_1 \ \underset{\sim}{\alpha}_2 \cdots \underset{\sim}{\alpha}_n)\begin{pmatrix}h_1\\ h_2\\ \vdots\\ h_n\end{pmatrix} = \underset{\sim}{0}, \text{ where } h_i \neq 0 \text{ for at least one } i.$$

$$\Rightarrow \sum_{i=1}^{n} h_i\alpha_i = \underset{\sim}{0}$$

$\Rightarrow \underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n$ are linearly dependent.

$\Rightarrow R(A) < n$.

Theorem:- For a square matrix A, suppose $A\underset{\sim}{x} = \underset{\sim}{0}$ for some zero vector $\underset{\sim}{x}$. Explain why A is singular.

Proof:- $A = (\underset{\sim}{\alpha}_1\ \underset{\sim}{\alpha}_2 \cdots \underset{\sim}{\alpha}_n)_{n\times n}$

$\underset{\sim}{x} = (x_1\ x_2 \cdots x_n)'$ $[x_i \neq 0$ for atleast one $i]$

Now, $A\underset{\sim}{x} = \underset{\sim}{0}$

$\Rightarrow \sum_i \alpha_i x_i = \underset{\sim}{0}$

$\underset{\sim}{\alpha}_i$'s are linearly dependent.

$\Rightarrow R(A) < n$.

$\Rightarrow \det(A) = 0$.

Theorem:- If $R(A) = r\ (<n)$ then the system of equations $A\underset{\sim}{x} = \underset{\sim}{0}$ has exactly $(n-r)$ independent non-trivial solution.

Proof:- Let $N(A) = \{\underset{\sim}{x} : A\underset{\sim}{x} = \underset{\sim}{0}\}$

The set of all solutions of the system $A\underset{\sim}{x} = \underset{\sim}{0}$.

Since $\text{Rank}(A) = r$

$\Rightarrow \dim[N(A)] = n-r$.

Hence, $N(A)$ has $n-r$ linearly independent non-trivial vectors of $\underset{\sim}{x}$.

$\therefore A\underset{\sim}{x} = \underset{\sim}{0}$ has $(n-r)$ non-trivial solution.

Theorem:- S.T. if a system of linear equation has two distinct solution then $\exists$ an infinite no. of solutions.

Proof:- Suppose $\underset{\sim}{x}_1$ & $\underset{\sim}{x}_2$ are two distinct solution of the system $A\underset{\sim}{x} = \underset{\sim}{b}$.

Then $A\underset{\sim}{x}_1 = \underset{\sim}{b}$ & $A\underset{\sim}{x}_2 = \underset{\sim}{b}$

Let us define, $\underset{\sim}{x}^* = \lambda\underset{\sim}{x}_1 + (1-\lambda)\underset{\sim}{x}_2$ for some scalar λ.

$$\therefore A\underset{\sim}{x}^* = \lambda A\underset{\sim}{x}_1 + (1-\lambda)A\underset{\sim}{x}_2$$
$$= \lambda\underset{\sim}{b} + (1-\lambda)\underset{\sim}{b}$$
$$= \underset{\sim}{b}, \quad \lambda \in \mathbb{R}.$$

$\underset{\sim}{x} = \underset{\sim}{x}^*$ is also a solution $A\underset{\sim}{x} = \underset{\sim}{b}$ and as λ can be chosen in an infinite number of ways. Hence the system has an infinite no. of solutions.

PROBLEMS ON MATRICES & DETERMINANTS

1. Diagonalise the following matrix using non-singular transformation and find the non-singular matrices which diagonalise

$$A = \begin{pmatrix} 1 & 1 & 2 \\ -2 & 2 & -1 \\ 1 & 2 & -1 \end{pmatrix}.$$

Solution:-

$$A \sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 4 & 3 \\ 0 & 1 & -3 \end{pmatrix} \quad \begin{matrix} R_2' \leftrightarrow R_2 + 2R_1 \\ R_3' \leftrightarrow R_3 - R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & -15/4 \end{pmatrix} \quad R_3' \leftrightarrow R_3 - \frac{R_2}{4}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & -15/4 \end{pmatrix} \quad \begin{matrix} C_2' \leftrightarrow C_2 - C_1 \\ C_3' \leftrightarrow C_3 - 2C_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -15/4 \end{pmatrix} \quad C_3' \leftrightarrow C_3 - \frac{3}{4}C_2$$

$$= \Lambda$$

$$E_2 E_1 A E_3 E_4 = \Lambda$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/4 & 1 \end{pmatrix}$$

$$P = E_2 \times E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3/2 & -1/4 & 1 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3/4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q = E_3 \times E_4 = \begin{pmatrix} 1 & -1 & -5/4 \\ 0 & 1 & -3/4 \\ 0 & 0 & 1 \end{pmatrix}$$

$\therefore PAQ = \Lambda$

2) Find the rank of the matrix $\begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{pmatrix}$.

<u>Solution:-</u> $A \sim \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$ $\quad R_2' = R_2 - R_1$, $R_3' = R_3 - 2R_1$, $R_4' = R_4 - 3R_1$

$\sim \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad R_4' = R_4 - R_2$

$\text{Rank}(A) = 3$.

3) Find $\det \begin{pmatrix} \sigma_1^2 & p\sigma_1\sigma_2 & \cdots & p\sigma_1\sigma_n \\ p\sigma_1\sigma_2 & \sigma_2^2 & \cdots & p\sigma_2\sigma_n \\ \cdots & \cdots & \cdots & \cdots \\ p\sigma_n\sigma_1 & p\sigma_n\sigma_2 & \cdots & \sigma_n^2 \end{pmatrix}$; $\sigma_i \neq 0 \ \forall i = 1(1)n$.

<u>Sol.</u> $\Delta = \sigma_1^2\sigma_2^2 \cdots \sigma_n^2 \begin{vmatrix} 1 & p & \cdots & p \\ p & 1 & \cdots & p \\ \vdots & \vdots & & \vdots \\ p & p & \cdots & 1 \end{vmatrix}$ $\quad R_i' = \frac{R_i}{\sigma_i} \ \forall i$, $\quad C_i' = \frac{C_i}{\sigma_i} \ \forall i$

$$= \prod_{i=1}^{n} \sigma_i^2 \left\{ (1-p)^{n-1} \{1 + (n-1)p\} \right\}.$$

<u>(Ans)</u>

<u>Cor.</u>

$\begin{vmatrix} 1+x & 1 & \cdots & 1 \\ 1 & 1+x & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1+x \end{vmatrix} = (n+x) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1+x & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1+x \end{vmatrix}$ $\quad R_1' = \sum R_i$, $\quad R_1'' = \frac{R_1}{(n+x)}$

$= (n+x) \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 1 & x & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & x \end{vmatrix}$ $\quad C_i = C_i - C_1$

$= (n+x) \cdot x^{n-1}$

$= x^n + nx^{n-1}$.

4. Find $\begin{vmatrix} 1 & p & p & \cdots & p \\ p & 1 & p & \cdots & p \\ \vdots & \vdots & \vdots & & \vdots \\ p & p & p & \cdots & 1 \end{vmatrix}$. (OR). $A^{n\times n}$, where $a_{ij} = \begin{cases} 1 & \text{if } i=j \\ p & \text{if } i\neq j \end{cases}$

Find $|A|$. [ISI M.STAT]

Sol.

$$\Delta = \begin{vmatrix} 1+(n-1)p & 1+(n-1)p & \cdots & 1+(n-1)p \\ p & 1 & \cdots & p \\ \vdots & \vdots & & \vdots \\ p & p & \cdots & p \end{vmatrix} \qquad R_1' = \sum_i R_i$$

$$= [1+(n-1)p] \begin{vmatrix} 1 & 1 & \cdots & 1 \\ p & 1 & \cdots & p \\ \vdots & \vdots & & \vdots \\ p & p & \cdots & 1 \end{vmatrix}$$

$$= [1+(n-1)p] \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1-p & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1-p \end{vmatrix} \qquad R_i' = R_i - pR_1 \ \forall i>1$$

$$= 1+(n-1)p\,(1-p)^{n-1}.$$ [expanding by its first column]

5.

Find $\begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ \vdots & \vdots & \vdots & & \vdots \\ b & b & b & \cdots & a \end{vmatrix}$.

Sol.

$$\Delta = [a+(n-1)b] \begin{vmatrix} 1 & 1 & \cdots & 1 \\ b & a & \cdots & b \\ \vdots & \vdots & & \vdots \\ b & b & & a \end{vmatrix}$$

$$= [a+(n-1)b] \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & a-b & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a-b \end{vmatrix} \qquad R_i' = R_i - bR_1 \ \forall i>1.$$

[expanding by the first column]

$$= [(a+(n-1)b] (a-b)^{n-1}$$
$$= (a-b)^n + nb(a-b)^{n-1}.$$

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6. <u>Vandermonde's Determinant :-</u>

Find
$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \end{vmatrix}$$

<u>Sol.</u>

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & (a_2-a_1) & (a_3-a_1) & \cdots & (a_n-a_1) \\ 0 & a_2(a_2-a_1) & a_3(a_3-a_1) & \cdots & a_n(a_n-a_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a_2^{n-2}(a_2-a_1) & a_3^{n-2}(a_3-a_1) & \cdots & a_n^{n-2}(a_n-a_1) \end{vmatrix} \qquad R_i' = R_i - a_1 R_{i-1} \quad \forall i > 1.$$

$$= \begin{vmatrix} a_2-a_1 & a_3-a_1 & \cdots & a_n-a_1 \\ a_2(a_2-a_1) & a_3(a_3-a_1) & \cdots & a_n(a_n-a_1) \\ \vdots & \vdots & & \vdots \\ a_2^{n-2}(a_2-a_1) & a_3^{n-2}(a_3-a_1) & \cdots & a_n^{n-2}(a_n-a_1) \end{vmatrix} \qquad \text{[expanding by the first column]}$$

$$= \prod_{i=2}^{n} (a_i - a_1) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_2 & a_3 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_2^{n-2} & a_3^{n-2} & \cdots & a_n^{n-2} \end{vmatrix} \qquad C_i' = \frac{C_i}{a_{i+1} - a_1}$$

$$= \prod_{i=1}^{n} (a_i - a_1) \prod_{i=3}^{n} (a_i - a_2) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_3 & a_4 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_3^{n-3} & a_4^{n-3} & \cdots & a_n^{n-3} \end{vmatrix}$$

$$= \prod_{i=1}^{n} (a_i - a_1) \prod_{i=3}^{n} (a_i - a_2) \cdots \prod_{i=1}^{n} (a_i - a_{n-2}) \begin{vmatrix} 1 & 1 \\ a_{n-1} & a_n \end{vmatrix}$$

$$= \prod_{i>j} \prod (a_i - a_j)$$

$$= (-1)^{\frac{n(n-1)}{2}} \prod_{i<j} \prod (a_i - a_j) .$$

7. Find $\begin{vmatrix} 1+a_1^2 & a_1a_2 & a_1a_3 & \cdots & a_1a_n \\ a_2a_1 & 1+a_2^2 & a_2a_3 & \cdots & a_2a_n \\ a_3a_1 & a_3a_2 & 1+a_3^2 & \cdots & a_3a_n \\ \vdots & \vdots & \vdots & & \vdots \\ a_na_1 & a_na_2 & a_na_3 & \cdots & 1+a_n^2 \end{vmatrix}$

Sol.

$$\Delta = \begin{vmatrix} 1+a_1^2 & a_1a_2 & a_1a_3 & \cdots & a_1a_n \\ a_2a_1 & 1+a_2^2 & a_2a_3 & \cdots & a_2a_n \\ a_3a_1 & a_3a_2 & 1+a_3^2 & \cdots & a_3a_n \\ \vdots & \vdots & \vdots & & \vdots \\ a_na_1 & a_na_2 & a_na_3 & & 1+a_n^2 \end{vmatrix}$$

$$= a_1^2a_2^2\ldots a_n^2 \begin{vmatrix} \left(\frac{1}{a_1^2}+1\right) & 1 & 1 & \cdots & 1 \\ 1 & \left(\frac{1}{a_2^2}+1\right) & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & \left(1+\frac{1}{a_n^2}\right) \end{vmatrix} \quad R_i' = \frac{R_i}{a_i},\ C_i' = \frac{C_i}{a_i}\ \forall i$$

$$= \begin{vmatrix} 1+a_1^2 & a_1^2 & a_1^2 & \cdots & a_1^2 \\ a_2^2 & 1+a_2^2 & a_2^2 & \cdots & a_2^2 \\ \vdots & \vdots & \vdots & & \vdots \\ a_n^2 & a_n^2 & a_n^2 & \cdots & 1+a_n^2 \end{vmatrix} \quad R_i' = R_i a_i^2\ \forall i$$

$$= \begin{vmatrix} 1+\sum a_i^2 & 1+\sum a_i^2 & \cdots & 1+\sum a_i^2 \\ a_2^2 & 1+a_2^2 & \cdots & a_2^2 \\ \vdots & \vdots & & \vdots \\ a_n^2 & a_n^2 & \cdots & 1+a_n^2 \end{vmatrix} \quad R_1' = \sum_i R_i$$

$$= \left(1+\sum_i a_i^2\right) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_2^2 & 1+a_2^2 & \cdots & a_2^2 \\ \vdots & \vdots & & \vdots \\ a_n^2 & a_n^2 & \cdots & 1+a_n^2 \end{vmatrix}$$

$$= \left(1+\sum_i a_i^2\right) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} \quad R_i' = R_i - a_i^2 R_1\ \forall i > 1.$$

$$= 1 + \sum_{i=1}^{n} a_i^2.$$

LINEAR ALGEBRA

SYSTEM OF LINEAR EQUATION

Suppose that there are m equations and n unknowns, say $x_1, x_2, \ldots, x_n$. We may put the equations in the form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

We may write the equations in the matrix form —

$$A\underset{\sim}{x} = \underset{\sim}{b} \quad \text{———} (*)$$

where, $A = ((a_{ij}))_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n}$, and

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = (x_1, x_2, \ldots, x_n)' \quad \& \quad \underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1} = (b_1, b_2, \ldots, b_m)'$$

Definition: — $\boxed{\text{C.U.}}$

Consistent system and Inconsistent System: —

By a solution of the system $(*)$, we mean any set of values $x_1, x_2, \ldots, x_n$ that satisfies the m equations simultaneously. The system is said to be consistent if it has at least one solution, otherwise it is said to be inconsistent.

* **Ex.1.** The system of equations

$$x_1 + 2x_2 = 5$$
$$2x_1 + 4x_2 = 3$$

— it is an inconsistent system since it does not have any solution. The term 'inconsistent' is appropriate to this situation, for the first equation implies $2x_1 + 4x_2 = 10$, which is incompatible with the second equation.

* **Ex. 2.** The system

$$x_1 + 2x_2 = 5$$
$$3x_1 - x_2 = 1$$

– it is a consistent system having a unique solution with $x_1 = 1, x_2 = 2$.

* **Ex. 3.** The equation

$$x_1 + 2x_2 = 5$$

– it is a consistent system having infinite number of solutions.

Now we shall go for finding conditions under which a system will be consistent.

Case of a homogeneous System :– If in a system each term in the R.H.S is zero or in the other word, $\underset{\sim}{b} = \underset{\sim}{0}$, then the system is said to be Homogeneous System.

$$A\underset{\sim}{x} = \underset{\sim}{0} \quad \text{———} (**)$$

– A system of homogeneous equation is necessarily consistent as it posses a trivial solution $\underset{\sim}{x} = \underset{\sim}{0}$.

i.e. $\underset{\sim}{x} = \underset{\sim}{0}$ is always a soln. of $(**)$, ~~whether~~ whatever the matrix A may be, we call the solution a trivial solution.

Thus it may be of interest to know whether the system has any non-trivial soln. (non-null) or, it possess the trivial solution as the only solution. If there is atleast one non-trivial soln. then it is of interest to find a set of maximum number of linearly independent non-null soln.

Theorem :→ The necessary and sufficient condition for a system $\underset{m\times n}{A}\,\underset{n\times 1}{\underset{\sim}{x}} = \underset{m\times 1}{\underset{\sim}{0}}$ to have a non trivial soln. is rank $(A) < n$ ***

Proof :– Let rank$(A) < n$.

$\Leftrightarrow$ the columns of A, say $\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}$ are linearly dependent.

$\Leftrightarrow$ the equation $x_1\underset{\sim}{a_1} + x_2\underset{\sim}{a_2} + \cdots + x_n\underset{\sim}{a_n} = \underset{\sim}{0}$ has at least one non-trivial solution.

$\Leftrightarrow$ $A\underset{\sim}{x} = \underset{\sim}{0}$ has at least one non trivial solution.

$\Leftrightarrow$ System is consistent.

Equivalent matrices and Sweep-out method :—

We know that the rank of a matrix is obtained by reducing A to an echelon matrix through elementary row (column) operations, i.e., by pre-multiplying (post multiplying) A by elementary matrices.

Defn.→ Any matrix B obtained from a given matrix A by performing a succession of elementary (row or column) operations is said to be equivalent to A. Thus we can say that B is equivalent to A if and only if $B = PAQ$ where P and Q are the products of elementary matrices.

Since P and Q are non-singular then we may write P^{-1} and Q^{-1} exists so, we have $A = P^{-1}BQ^{-1}$. Clearly A is equivalent to B in case B is also equivalent to A.

We may say that A and B are mutually equivalent if $B = PAQ$ holds.

Result:→ If A and B are equivalent matrices then $\text{rank}(A) = \text{rank}(B)$

Proof:→ One should note that an echelon matrix H is obtained from a given matrix A by elementary row (or column) operations so that $H = AQ$ or, $H = PA$; where A and H are equivalent matrices. i.e. in our case, $B = PA$ or, $B = AQ$, where P and Q are non-singular matrices. We know — rank of a matrix is unchanged by pre or post multiplication of non-singular matrix.

Hence the result.

Result:→ If A and B are equivalent matrices then $A\underset{\sim}{x} = \underset{\sim}{0}$ and $B\underset{\sim}{x} = 0$ has the same solution space.

Definition of equivalent system:— Two systems of linear equations are equivalent if each equation in each system is a linear combination of the equations in the other system.

Remark: Equivalent system of linear equations have exactly the same solutions.

Result:→ Each matrix A is equivalent to some matrix B in which, for some integer $r(\geq 0)$, the first r elements on the diagonal are unity and all other elements are zero. A and B have the same rank, r.

Proof:→ We know that matrix A can be reduced to an echelon matrix H some of whose columns, say columns $c_1, c_2, \ldots, c_r$, are unit vectors, all but the first r rows of H being non-null vectors. Now, the columns $c_1, c_2, \ldots, c_r$, can by column ~~operations~~ interchanges be brought into the first r positions. Thus we arrive a matrix of the form

$$PAQ_1 = \begin{pmatrix} I_r & D \\ 0 & 0 \end{pmatrix},$$

Now D can be converted into a null matrix by performing suitable column operations. Thus, we have —

$$PAQ_1Q_2 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

or, taking $Q = Q_1Q_2$,

$$PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

Taking $B = PAQ$, we may say that A has been reduced to the desired form, again, B has rank r and so has A, which is equivalent to B.

**

Theorem:→ For a matrix A prove that $\dim(S) = n - \dim$ (row space of A) where S is the solution space of $A\underset{\sim}{x} = \underset{\sim}{0}$.

Proof:→ Let R be the row-reduced echelon matrix which is row-equivalent to A then the solution space of $A\underset{\sim}{x} = \underset{\sim}{0}$ will be same as the solution space of $R\underset{\sim}{x} = \underset{\sim}{0}$.

Let R has r non-zero rows. Then the system $R\underset{\sim}{x} = \underset{\sim}{0}$ simply express r of the unknowns $x_1, x_2, \ldots, x_n$ in terms of the remaining $(n-r)$ unknowns. Now suppose that the leading non-zero entries of R occur in the columns $k_1, k_2, \ldots, k_r$ respectively. Define J be the set of all indices different from $k_1, k_2, \ldots, k_r$, i.e., $J = \{1, 2, 3, \ldots, n\} - \{k_1, k_2, \ldots, k_r\}$. Then the system $R\underset{\sim}{x} = \underset{\sim}{0}$ can be written in the form —

$$\left.\begin{array}{l} x_{k_1} + \sum_{j\in J} c_{1j} x_j = 0 \\ x_{k_2} + \sum_{j\in J} c_{2j} x_j = 0 \\ \vdots \\ x_{k_r} + \sum_{j\in J} c_{rj} x_j = 0 \end{array}\right\} \longrightarrow (*)$$

, where c_{ij}'s, $i = 1(1)r$ are certain scalars.

Here each solution of $(*)$ is obtained by assigning values to those x_j's with $j \in J$ and computing the different values of $x_{k_1}, x_{k_2}, \ldots, x_{k_r}$.

For each j in J, let ε_j be the solution obtained by assigning $x_j = 1$ and $x_{j'} = 0$, $j \neq j' \in J$.

As ε_j has a 1 in the jth row and 0 in the other rows indexed by the entries in J. then ε_j's are linearly independent.

Now if $\underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ is in the solution space then $\underset{\sim}{\alpha} = \sum_{j\in J} a_j \varepsilon_j$ is also in the solution space and is a solution space such that $x_j = a_j \ \forall j \in J$. the solution with this property is unique and therefore $\underset{\sim}{\alpha} = \underset{\sim}{a}$ and $\underset{\sim}{a}$ is in the span of $\{\varepsilon_j : j \in J\}$. Thus the set of $(n-r)$ vectors $\underset{\sim}{\varepsilon_j}$, $j \in J$ forms a basis of the solution space S.

$$\therefore \dim(S) = n - r$$
$$= n - \dim(\text{rowspace of } A).$$

Homogeneous Linear Equations:—

If $b_1 = b_2 = \cdots = b_m = 0$, the set of equations is said to be homogeneous.

For a set of homogeneous equations, the condition $\text{rank}(A \,\vdots\, b) = \text{rank}(A)$ is automatically satisfied.

If $r = n$, the only solution is $x_1 = 0 = x_2 = \cdots = x_n$ (trivial solution).

If $r < n$, the $(n-r)$ unknowns may have arbitrary values. Consequently the necessary and sufficient condition for non-zero solutions is $r < n$, i.e., the rank is less than the no. of unknowns.

When there are n equations and n unknowns, this condition becomes $|A| = 0$, and when there are $m\,(<n)$ equations and n unknowns, there are always non-trivial solutions.

Theorem: Let rank$(A) = r\ (< n)$, then the system $A\underset{\sim}{x} = \underset{\sim}{0}$ has exactly $(n-r)$ independent solutions.

Proof: Let S denotes the solution space of $A\underset{\sim}{x} = \underset{\sim}{0}$ and R denotes the row-space of A.
Obviously, S will be an orthocomplement of R.
Let, $\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_r$ be an orthogonal basis of R. this can be extended to an orthogonal basis of E^n, say, $\{\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_r, \underset{\sim}{\beta}_1, \underset{\sim}{\beta}_2, \ldots, \underset{\sim}{\beta}_{n-r}\}$. Here in contradiction we prove that $\underset{\sim}{\beta}_1, \underset{\sim}{\beta}_2, \ldots, \underset{\sim}{\beta}_{n-r}$ is an orthogonal basis of S. $\therefore \dim(S) = n-r$.
Consider a vector $\underset{\sim}{\beta} \in S$, which can't be written as a linear combination of $\underset{\sim}{\beta}_1, \underset{\sim}{\beta}_2, \ldots, \underset{\sim}{\beta}_{n-r}$. Then the set $\underset{\sim}{\beta}_1, \underset{\sim}{\beta}_2, \ldots, \underset{\sim}{\beta}_{n-r}, \underset{\sim}{\beta}$ will be linearly independent and generate the subspace $S_2 \subseteq S$.
Now the set $\{\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n, \underset{\sim}{\beta}_1, \ldots, \underset{\sim}{\beta}_{n-r}, \underset{\sim}{\beta}\}$ is a set of non-zero orthogonal vectors and this is a linearly independent set of $(n+1)$ vectors in E^n, which is impossible.
Therefore $\{\underset{\sim}{\beta}_1, \underset{\sim}{\beta}_2, \ldots, \underset{\sim}{\beta}_{n-r}\}$ forms a basis of S. So, $\dim(S) = n-r$, i.e. the system has exactly $(n-r)$ independent solutions.

Solution Space: The vector space generated by the soln. of $A\underset{\sim}{x} = \underset{\sim}{0}$ is the solution space of the system of equation.
i.e. $\{x : A\underset{\sim}{x} = \underset{\sim}{0}\}$ which is the null space of A $[N(A)]$ → Notation
We know, $\dim(N(A)) = (n-r)$, where $r = R(A)$.
Clearly $\dim(N(A)) = 0$ if A is of full column rank, in that case we have the trivial soln as the only soln.

If $\dim(N(A)) > 0$ we may search for a basis of the soln. space.
[basis is the set of maximum no. of LIN soln.]

Suppose $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_{n-r}\}$ be a basis of the soln. space. Then any soln. of $A\underset{\sim}{x} = \underset{\sim}{0}$ can be written as a linear combination of the basis vectors, i.e., the general soln. to the system of eqn. will be of the form $\sum_{i=1}^{n-r} \lambda_i \underset{\sim}{x}_i$.

NOTE: — If there exists a non-null solution then it must be orthogonal to the non-null rows of A.

Rewrite A as

$$A = (\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots\ldots, \underset{\sim}{\alpha}_n)$$

and $\underset{\sim}{x} = (x_1, x_2, \ldots\ldots, x_n)$

Then $A\underset{\sim}{x} = \underset{\sim}{0}$ reduces to

$$\sum_{i=1}^{n} x_i \underset{\sim}{\alpha}_i = \underset{\sim}{0} \quad \text{———} (***)$$

$\therefore$ to search for a non-trivial solution is equivalent to check whether (***) holds good for at least one $x_i \neq 0$.

In other words, the system of equation defined by $A\underset{\sim}{x} = \underset{\sim}{0}$ possesses a non-trivial solution if column of A are linearly dependent.

Ex.1. Obtain the general soln. to the system of equations:

$$x_1 + x_2 + x_3 + x_4 = 0$$
$$x_1 - 2x_2 + 2x_4 = 0$$
$$x_1 + 7x_2 + 3x_3 - x_4 = 0$$

Soln. → $A\underset{\sim}{x} = \underset{\sim}{0}$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 2 \\ 1 & 7 & 3 & -1 \end{pmatrix} = (\underset{\sim}{\alpha}_1\ \underset{\sim}{\alpha}_2\ \underset{\sim}{\alpha}_3\ \underset{\sim}{\alpha}_4)$$

$\underset{\sim}{\alpha}_1$ and $\underset{\sim}{\alpha}_2$ are linearly independent.

If possible, let $\underset{\sim}{\alpha}_3 = \lambda_1 \underset{\sim}{\alpha}_1 + \lambda_2 \underset{\sim}{\alpha}_2$

$$\Rightarrow \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \lambda_2 \\ \lambda_1 - 2\lambda_2 \\ \lambda_1 + 7\lambda_2 \end{pmatrix}$$

$$\Rightarrow \lambda_1 = \frac{2}{3}, \ \lambda_2 = \frac{1}{3}$$

i.e. $\frac{2}{3}\underset{\sim}{\alpha}_1 + \frac{1}{3}\underset{\sim}{\alpha}_2 - \underset{\sim}{\alpha}_3 = 0$

$$\Rightarrow (\underset{\sim}{\alpha}_1\ \underset{\sim}{\alpha}_2\ \underset{\sim}{\alpha}_3\ \underset{\sim}{\alpha}_4) \begin{pmatrix} 2/3 \\ 1/3 \\ -1 \\ 0 \end{pmatrix} = \underset{\sim}{0}$$

i.e. $\begin{pmatrix} 2/3 \\ 1/3 \\ -1 \\ 0 \end{pmatrix}$ is a non trivial solution.

If possible, let $\Rightarrow \underset{\sim}{\alpha}_4 = \lambda_1 \underset{\sim}{\alpha}_1 + \lambda_2 \underset{\sim}{\alpha}_2$

$$\Rightarrow \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \lambda_2 \\ \lambda_1 - 2\lambda_2 \\ \lambda_1 + 7\lambda_2 \end{pmatrix}$$

$\Rightarrow \lambda_2 = -1/3, \ \lambda_1 = 4/3$

$\Rightarrow$ another non-trivial independent solution is $\begin{pmatrix} 4/3 \\ -1/3 \\ 0 \\ -1 \end{pmatrix}$.

Here, $R(A) = 2 \Rightarrow \dim(N(A)) = 2$.

A basis of the solution space is $\left\{ \begin{pmatrix} 2/3 \\ 1/3 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 4/3 \\ -1/3 \\ 0 \\ -1 \end{pmatrix} \right\}$

A basis of the soln. space = $\begin{pmatrix} 2\alpha + 4\beta \\ \alpha - \beta \\ -3\alpha \\ -3\beta \end{pmatrix}$

$\boxed{\text{Or}}$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 2 \\ 1 & 7 & 3 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 1 & -1 \\ 0 & -6 & -2 & 2 \end{pmatrix} \begin{matrix} ① R_2' = R_1 - R_2 \\ ② R_3' = R_1 - R_3 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} ① R_2' = R_2/3 \\ ② R_3' = 2R_2 + R_3 \end{matrix}$$

$= H$, an echelon matrix.

Here, rank$(A) = 2$.

$$H\underset{\sim}{x} = 0 \Rightarrow \left.\begin{matrix} x_1 + x_2 + x_3 + x_4 = 0 \\ \text{and, } x_2 + \frac{x_3}{3} - \frac{x_4}{3} = 0 \end{matrix}\right\} \Rightarrow$$

x_1	x_2	x_3	x_4
-2	0	1	1
$2/3$	$-2/3$	1	-1

$\Rightarrow$ General soln. is $\begin{pmatrix} -2\gamma + \frac{2}{3}\delta \\ -2/3\,\delta \\ \gamma + \delta \\ \gamma - \delta \end{pmatrix}$

<u>NOTE</u>:- If m exceeds n, then at least $m-n$ equation become redem, thus we consider $m \leq n$.

Ex.2. Obtain the general soln. to the system of equations :—

$$x_1 + 3x_2 + 2x_3 + 3x_4 = 0$$
$$x_2 + 2x_3 + x_4 - x_1 = 0$$
$$3x_1 + 4x_2 + x_3 + 4x_4 = 0$$

Soln.→ $A\underset{\sim}{x} = \underset{\sim}{0}$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 2 & 3 \\ -1 & 1 & 2 & 1 \\ 3 & 4 & 1 & 4 \end{bmatrix} \underset{\sim}{x} = \underset{\sim}{0}$$

or, $\left(\underset{\sim}{\alpha_1}\ \underset{\sim}{\alpha_2}\ \underset{\sim}{\alpha_3}\ \underset{\sim}{\alpha_4}\right)\underset{\sim}{x} = \underset{\sim}{0}$

Here, $R(A) = 2$.

$\underset{\sim}{\alpha_1}$ and $\underset{\sim}{\alpha_2}$ are linearly independent,

Note that — $\underset{\sim}{\alpha_3} = \underset{\sim}{\alpha_2} - \underset{\sim}{\alpha_1}$

$$\Rightarrow \left(\underset{\sim}{\alpha_1}\ \underset{\sim}{\alpha_2}\ \underset{\sim}{\alpha_3}\ \underset{\sim}{\alpha_4}\right)\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} = \underset{\sim}{0}$$

i.e. $\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$ is a solution.

Again, $\alpha_2 - \alpha_4 = 0$

$$\Rightarrow \left(\underset{\sim}{\alpha_1}\ \underset{\sim}{\alpha_2}\ \underset{\sim}{\alpha_3}\ \underset{\sim}{\alpha_4}\right)\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \underset{\sim}{0}$$

$\Rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$ is another lineatily independent solution.

A basis of the solution space is $\rightarrow \left\{\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}\right\}$

A basis of the solution space $\rightarrow \begin{pmatrix} \alpha \\ \beta - \alpha \\ \alpha \\ -\beta \end{pmatrix}$

Ex. 3. Consider the system of equations:

$$x_1+x_2+x_3+x_4=0$$
$$x_1+2x_2+3x_3+4x_4=0$$
$$2x_1+x_3-x_4=0$$

Soln.

$$A=\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 2 & 3 & 4\\ 2 & 1 & 0 & -1\end{pmatrix}, \quad x=\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}$$

The sweep-out method method gives —

$$A\sim\begin{pmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 2 & 3\\ 0 & -1 & -2 & -3\end{pmatrix}\sim\begin{pmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 2 & 3\\ 0 & 0 & 0 & 0\end{pmatrix}\sim\begin{pmatrix}1 & 0 & -1 & -2\\ 0 & 1 & 2 & 3\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$= R$, a row-reduced echelon matrix.

So, Rank$(A)=2$.

$$R\underset{\sim}{x}=\underset{\sim}{0} \Rightarrow x_1-x_3-2x_4=0 \;\&\; x_2+2x_3+3x_4=0$$

x_1	x_2	x_3	x_4
1	-2	1	0
3	-5	1	1

$\Rightarrow$ General form is $\rightarrow$ $\begin{pmatrix}\alpha+3\beta\\ -2\alpha-5\beta\\ \alpha+\beta\\ \beta\end{pmatrix}$

EX. 4. The system of equations ,
$x-y+z=0$, $x+2y-z=0$, $2x+y+3z=0$ have
(A) infinite solutions (B) Trivial solution (C) No-solution.

Sol. (B)

$$A=\begin{pmatrix}1 & -1 & 1\\ 1 & 2 & -1\\ 2 & 1 & 3\end{pmatrix}\sim\begin{pmatrix}1 & -1 & 1\\ 1 & 2 & -1\\ 0 & 0 & 3\end{pmatrix}\sim\begin{pmatrix}1 & -1 & 1\\ 0 & 3 & -2\\ 0 & 0 & 3\end{pmatrix}$$

$$\begin{pmatrix}1 & -1 & 1\\ 0 & 3 & -2\\ 0 & 0 & 3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$$\Rightarrow \begin{cases}x-y+z=0\\ 3y-2z=0\\ 3z=0\end{cases}$$

$\therefore$ $z=0$, $y=0$, $x=0$ is the only solution of the given system of equations.

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Case of non-homogeneous system :→

If $\underset{\sim}{b} \neq \underset{\sim}{0}$ for the system of equations $A\underset{\sim}{x} = \underset{\sim}{b}$ then $\underset{\sim}{x} = 0$ can't be a solution. It means that if there is at all a solution then it must be a non-trivial solution. We will now establish a necessary and sufficient condition for a non-homogeneous system to be consistent.

$$A\underset{\sim}{x} = \underset{\sim}{b} \quad , \underset{\sim}{b} \neq 0.$$

This can be written as — $\sum_{i=1}^{n} x_i \underset{\sim}{\alpha_i} = \underset{\sim}{b}$.

Augmented Matrix :→

By the augmented matrix for the system we mean the matrix $(A : b)$ of order $m \times \overline{n+1}$ obtained by adjoining one more column, viz. $\underset{\sim}{b}$. thus

$$(A : b) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & & a_{mn} & b_m \end{pmatrix}_{m \times \overline{n+1}}$$

$\boxed{\text{C.U.}}$ ***

Theorem :—

The system of equation $A\underset{\sim}{x} = \underset{\sim}{b}$ is compartible (consistent) or has at least one soln. if and only if rank $(A : \underset{\sim}{b})$ = rank (A).

[Due to Kronecker – Capelli]

Proof :→

If Part :→ Suppose rank $(A : \underset{\sim}{b})$ = rank (A)

$\therefore$ $\underset{\sim}{b}$ can be written as a linear combination of the columns of A,

$$\Rightarrow \underset{\sim}{b} \in V_c(A)$$

Suppose $A = (\underset{\sim}{\alpha_1}, \underset{\sim}{\alpha_2}, \ldots, \underset{\sim}{\alpha_n})$

Therefore there exists scalars $\lambda_1, \lambda_2, \ldots, \lambda_n \ni \underset{\sim}{b} = \sum_{i=1}^{n} \lambda_i \underset{\sim}{\alpha_i}$

$$\Rightarrow (\underset{\sim}{\alpha_1}, \underset{\sim}{\alpha_2}, \ldots, \underset{\sim}{\alpha_n}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \underset{\sim}{b}.$$

Hence $\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}$ is a solution to the system of equation $A\underset{\sim}{x} = \underset{\sim}{b}$.

Therefore, the system is consistent (compartable)

Only if part :— Suppose the system $A\underset{\sim}{x} = \underset{\sim}{b} \, (\neq 0)$ is consistent and let $\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}$ is a solution of the system.

$$A \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \underset{\sim}{b}$$

$$\Rightarrow \sum_{i=1}^{n} \lambda_i \hat{\alpha}_i = \underset{\sim}{b}$$

i.e. $\underset{\sim}{b}$ is linearly dependent on the columns of A.

$$\Rightarrow \underset{\sim}{b} \in V_c(A)$$

$\therefore$ column rank of A = column rank of $A \vdots b$

i.e. $\text{rank}(A) = \text{rank}(A \vdots b)$.

Theorem :— A non-homogeneous system of n-linear equations in n unknowns $\underset{n\times n}{A}\underset{n\times 1}{\underset{\sim}{x}} = \underset{n\times 1}{\underset{\sim}{b}}$ has the unique solution $\underset{\sim}{x} = A^{-1}b$ if and only if $\text{rank}(A) = n$ or A is non-singular.

Proof :—

If part :— Let $\text{rank}(A) = n$.

Then A^{-1} exists. Putting $x = A^{-1}b$ in the L.H.S of the system —

$$AX = A(A^{-1}b) = (AA^{-1}).b = Ib = b = \text{R.H.S}$$

$\therefore x = A^{-1}b$ is a solution of the system.

To show that this is the unique solution, let $\underset{\sim}{x} = \hat{\alpha}$ be another solution, we can write $\underset{\sim}{\alpha} = A^{-1}b + \hat{\beta}$, where $\hat{\beta} = \underset{\sim}{\alpha} - A^{-1}b$.

As α is a solution of $A\underset{\sim}{\alpha} = \underset{\sim}{b}$ $\Rightarrow A(A^{-1}b + \hat{\beta}) = \underset{\sim}{b}$

$$\Rightarrow (AA^{-1}b) + A\hat{\beta} = \underset{\sim}{b}$$

$$\Rightarrow Ib + A\hat{\beta} = \underset{\sim}{b}$$

$$\therefore A\underset{\sim}{\beta} = \underset{\sim}{0}$$

If $\hat{\beta} = \underset{\sim}{0}$ then this is obviously satisfied, on the other hand if $\hat{\beta} \neq 0$, then this means that the columns of A ~~will~~ will form a linearly dependent set. This means $\text{rank}(A) < n$ which is a contradiction. So, $x = A^{-1}\hat{b}$ is the unique solution.

Only if part :- Suppose $\underset{\sim}{x} = A^{-1}\underset{\sim}{b}$ is a unique solution of $A\underset{\sim}{x} = \underset{\sim}{b}$.
$\Rightarrow$ $\underset{\sim}{x}$ is the unique solution of $A\underset{\sim}{x} = \underset{\sim}{0}$.
so, $\underset{\sim}{x} = \underset{\sim}{0}$ is the only solution $x_i = 0$,
$\Rightarrow \sum_{i=1}^{n} \underset{\sim}{\alpha_i} x_i = 0$ has the only solution $x_i = 0$,
so, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent, so, rank$(A) = n$,
where $\underset{\sim}{\alpha_i}$: ith column vectors of A.

Note :-
$$A\underset{\sim}{x} = \underset{\sim}{b} \quad (\underset{\sim}{b} \neq 0)$$
$$\Leftrightarrow \sum_i x_i \underset{\sim}{\alpha_i} = \underset{\sim}{b}$$
If α_i's are linearly independent then the collection constitutes the basis of the column space, in that case the representation of $\underset{\sim}{b}$ in terms of columns of A will be unique, i.e. the choice of $(x_1, x_2, \ldots, x_n)'$ is unique. Hence the system possesses a unique solution.
On the other hand, if the columns of A are linearly independent then representation of $\underset{\sim}{b}$ will not be unique.

General solution to a system of non-homogeneous equation :-

Let $A\underset{\sim}{x} = \underset{\sim}{b}\ (\neq 0)$ be a system of non-homogeneous equation where y is a specific solution to the system of eqn. and $\underset{\sim}{z}$ is the general solution to $A\underset{\sim}{x} = \underset{\sim}{0}$, then $(\underset{\sim}{y} + \underset{\sim}{z})$ will be the general solution to $A\underset{\sim}{x} = \underset{\sim}{b}$.

Ex.1. Obtain the general soln. to the following system of eqn. :
$$x_1 + x_2 + x_3 + x_4 = 2$$
$$x_1 - 2x_2 + 2x_4 = -1$$
$$x_1 + 7x_2 + 3x_3 - x_4 = 8$$

Soln. $\rightarrow$
$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 2 \\ 1 & 7 & 3 & -1 \end{pmatrix}, \quad \underset{\sim}{b} = \begin{pmatrix} 2 \\ -1 \\ 8 \end{pmatrix}$$
$$A\underset{\sim}{x} = \underset{\sim}{b}$$
$\Leftrightarrow PA\underset{\sim}{x} = P\underset{\sim}{b}$, where P is an non-singular matrix $\ni$ $PA = H$, an echelon matrix.
i.e. $H\underset{\sim}{x} = P\underset{\sim}{b}$

Reducing, $(A \mid \underset{\sim}{b}) \underset{\text{trans}}{\overset{\text{r.s.}}{\sim}} (H \mid P\underset{\sim}{b})$.

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & 7 & 3 & -1 & 8 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 3 & 1 & -1 & 3 \\ 0 & -6 & -2 & 2 & -6 \end{array}\right) \quad \begin{array}{l} (1)\ R_2' = R_1 - R_2 \\ (2)\ R_3' = R_1 - R_3 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1/3 & -1/3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} (1)\ R_3' = R_3 + 2R_2 \\ (2)\ R_2' = R_2/3 \end{array}$$

$\Rightarrow R(A \vdots b) = R(A) = 2$

Hence it is consistent.

$\boxed{\text{Or}}$.

Note that, $\underset{\sim}{b} = \underset{\sim}{\alpha}_1 + \underset{\sim}{\alpha}_2$,

$A = (\underset{\sim}{\alpha}_1\ \underset{\sim}{\alpha}_2\ \underset{\sim}{\alpha}_3\ \underset{\sim}{\alpha}_4)$

So, here $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

The general soln. of $A\underset{\sim}{x} = \underset{\sim}{0}$ is $\begin{pmatrix} 2\alpha + 4\beta \\ \alpha - \beta \\ -3\alpha \\ -3\beta \end{pmatrix}$ [Obtained before]

Thus the general solution of $A\underset{\sim}{x} = \underset{\sim}{b}$ is $\begin{pmatrix} 2\alpha + 4\beta + 1 \\ \alpha - \beta + 1 \\ -3\alpha \\ -3\beta \end{pmatrix}$

NOTE: $\rightarrow$ In particular if A is non-singular then the unique solution is $A^{-1}\underset{\sim}{b}$.

Ex. 2. Obtain the general solution to the following system of equation:

$$x_1 + 3x_2 + 2x_3 + 3x_4 = -1$$
$$x_2 + 2x_3 + x_4 - x_1 = -3$$
$$3x_1 + 4x_2 + x_3 + 4x_4 = 2$$

Soln. → Let it be $A\underset{\sim}{x} = \underset{\sim}{b}$

First of all we are to show that the system is consistent

i.e. $R(A \vdots b) = R(A)$.

Here, $\begin{pmatrix} 1 & 3 & 2 & 3 \\ -1 & 1 & 2 & 1 \\ 3 & 4 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}$

It is to be noted that — $\underset{\sim}{b} = \underset{\sim}{\alpha}_1 - \underset{\sim}{\alpha}_3$.

Hence a specific form is $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$.

Again the general solution of $A\underset{\sim}{x} = \underset{\sim}{0}$ is $\begin{pmatrix} \alpha \\ \beta-\alpha \\ \alpha \\ -\beta \end{pmatrix}$ [Obtained earlier]

Thus the general solution to this system will be $\begin{pmatrix} \alpha+1 \\ \beta-\alpha \\ \alpha-1 \\ -\beta \end{pmatrix}$.

Ex. 3. Show whether the system is consistent or not:

$$x_1 + x_2 + x_3 = 4$$
$$2x_1 + 5x_2 + 2x_3 = 3$$
$$x_1 + 7x_2 - 7x_3 = 5$$

Soln. → $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{pmatrix}$ and the method of sweep-out gives → $A \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -4/3 \\ 0 & 0 & 0 \end{pmatrix}$

As such, $R(A) = 2$.

Again, $(A \vdots b) = \begin{pmatrix} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & -4/3 & -5/3 \\ 0 & 0 & 0 & 11 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & -4/3 & -5/3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Hence, $R(A \vdots b) = 3$.

Since, $R(A \vdots b) > R(A)$, the system of equations is inconsistent, i.e. does not have a solution.

Ex. 4.

Check the consistency of this equations :

$$x_1 + x_2 + x_3 = 4$$
$$2x_1 + 5x_2 - 2x_3 = 3.$$

Soln.→ We have $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & -4 \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -4/3 \end{pmatrix}$$

Hence, $R(A) = 2$.

$$(A : b) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -4/3 & -5/3 \end{array}\right)$$

$R(A : b) = 2$

Since $r(A : b) \geq 2$, $(A : b)$ having one more column than A, and also, $r(A : b) \leq 2$, for $(A : b)$ has just two ~~row~~ rows, we have also $r(A : b) = 2$

As such, the pair of equations reduces to

$$x_1 + x_2 + x_3 = 4$$
$$x_2 - 4/3\, x_3 = -5/3$$

$x_1 + x_3 = 4 - \alpha$

$-\frac{4}{3} x_3 = -\frac{5}{3} - \alpha$

$\Rightarrow x_3 = \frac{5}{4} + \frac{3}{4}\alpha = \frac{5+3\alpha}{4}$

$x_1 = (4 - \alpha) - \left(\frac{5+3\alpha}{4}\right) = \frac{11-7\alpha}{4}$

x_1	x_2	x_3
$10/3$	$-1/3$	1
$17/3$	$-5/3$	0

Thus the general solution of the system will be —

$$= \begin{pmatrix} \frac{11-7\alpha}{4} \\ \alpha \\ \frac{5+3\alpha}{4} \end{pmatrix}$$

Questions from C.U. Question Papers

1) Examine the consistency or otherwise of the system $A\underset{\sim}{x} = \underset{\sim}{b}$ by citing examples of both. (4)

2) Examine the consistency of the following system of linear equations:

(i) $x_1 = 1$ $x_2 = -1$

(ii) $x_3 = 1$ $x_4 = 0$

$$2x_1 + 5x_2 + 3x_3 + 6x_4 = 7$$
$$3x_1 - x_2 + 2x_3 + 8x_4 = 9$$
$$5x_1 + 2x_2 - 9x_3 - x_4 = 17$$

3) State a necessary and sufficient condition for the existence of a solution $A^{n\times n}\underset{\sim}{x}^{n\times 1} = b^{n\times 1}$. Show that if a system of linear equations has two distinct solutions then there exists an infinite number of solutions. (5)

4) Find the value of c for which the following equations admit a solution:

$$4x_1 + 6x_3 = 1$$
$$4x_3 - 2x_2 = 7 + c \qquad (4)$$
$$2x_1 - x_2 + 5x_3 = 4$$

5) Discuss when the system of non-homogeneous linear equations $A\underset{\sim}{x} = \underset{\sim}{b}$ has (i) a unique solution, (ii) no solution. (4)

Linear Equations – Regular case; Cramer's solution:

Consider the n-linear equation in n unknowns $x_1, x_2, \ldots, x_n$,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots &\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}\right\}$$

Let
$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \text{Determinant of the coefficients.}$$

If $|A| \neq 0$, the set of equations is said to be regular. then have a unique solution given by

$$\frac{x_1}{|A_1|} = \frac{x_2}{|A_2|} = \cdots\cdots = \frac{x_n}{|A_n|} = \frac{1}{|A|};$$

This is known as Cramer's solution.

3) State a necessary and sufficient condition for the existance of a solution to $\underset{n\times n}{A}\ \underset{\sim}{x}_{n\times 1} = \underset{\sim}{b}_{n\times 1}$. Show that if a system of linear equations has two distinct solutions then $\exists$ an infinite number of solutions. (5)

<u>Ans:</u>–

$A = ((a_{ij}))_{m\times n}$

$$(A:b) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & & a_{mn} & b_m \end{pmatrix}_{m\times \overline{n+1}}, \text{ where } \underset{\sim}{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}_{n\times 1}$$

A necessary and sufficient condition for the existance of a solution to $\underset{n\times n}{A}\ \underset{n\times 1}{x} = \underset{n\times 1}{b}$ is rank$(A \vdots b)$ = rank(A).

Let $\underset{m\times n}{A}\ \underset{n\times 1}{x} = \underset{m\times 1}{\underset{\sim}{b}}$ be a system of linear equations having two distinct solution $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$.

$\Rightarrow A\underset{\sim}{x}_1 = \underset{\sim}{b}$ and $A\underset{\sim}{x}_2 = \underset{\sim}{b}$.

$\Rightarrow A(\underset{\sim}{x}_1 - \underset{\sim}{x}_2) = 0$

$\Rightarrow (\underset{\sim}{x} - \underset{\sim}{x}_2)\ (\neq 0)$ is a solution of the corresponding homogeneous system $A\underset{\sim}{x} = \underset{\sim}{0}$.

$\Rightarrow \lambda(x_1 - x_2)$ is a solution of $A\underset{\sim}{x} = \underset{\sim}{0}$ for any scalar λ.

$\Rightarrow x_1 + \lambda(x_1 - x_2)$ is a solution of $A\underset{\sim}{x} = \underset{\sim}{b}$ because

$$\begin{aligned} & A\{x_1 + \lambda(x_1 - x_2)\} \\ &= Ax_1 + A\lambda(x_1 - x_2) \\ &= \underset{\sim}{b} + 0 \\ &= \underset{\sim}{b} \end{aligned}$$

$\Rightarrow$ infinitely many solutions :— $x_1 + \lambda(x_1 - x_2)$ for varying λ — of $A\underset{\sim}{x} = \underset{\sim}{b}$ exists.

4) Find the value of c for which the following equations admit a solution : (4)

$$4x_1 + 6x_3 = 1$$
$$4x_3 - 2x_2 = 7 + c$$
$$2x_1 - x_2 + 5x_3 = 4$$

Ans :- Given that the system admits a solution for a value of c.

$\Rightarrow$ rank $(A \vdots b)$ = rank (A)

Here $A = \begin{pmatrix} 4 & 0 & 6 \\ 0 & -2 & 4 \\ 2 & -1 & 5 \end{pmatrix}$ and the method of sweep-out gives $\rightarrow$ $A \sim \begin{pmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

As such, rank$(A) = 2$

Again, $(A \vdots b) = \left(\begin{array}{ccc:c} 4 & 0 & 6 & 1 \\ 0 & -2 & 4 & 7+c \\ 2 & -1 & 5 & 4 \end{array}\right) \sim \left(\begin{array}{ccc:c} 1 & 0 & 3/2 & 1/4 \\ 2 & -1 & 5 & 4 \\ 0 & -2 & 4 & 7+c \end{array}\right)$

$\sim \left(\begin{array}{ccc:c} 1 & 0 & 3/2 & 1/4 \\ 0 & -1 & 2 & 7/2 \\ 0 & -1 & 2 & \frac{7+c}{2} \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 3/2 & 1/4 \\ 0 & 1 & -2 & -7/2 \\ 0 & 0 & 0 & c/2 \end{array}\right)$

So, we know from the given assumption that rank $(A \vdots b) = r(A) = 2$

$\Rightarrow c/2 = 0$

i.e. $c = 0$.

So, for $c = 0$ the equations admit a solution.

6) Investigate for what values of λ and μ, the system of equations

$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu$$

has i) no solution.

ii) a unique solution.

iii) an infinite no. of solution.

Soln →

$$(A \,\vdots\, \underset{\sim}{b}) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right) \quad \begin{array}{l} R_2' = R_2 - R_1 \\ R_3' = R_3 - R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right) \quad R_3' = R_3 - R_2$$

i) No solution $\Rightarrow r(A) < r(A \,\vdots\, \underset{\sim}{b})$

$\Rightarrow \lambda = 3$ but $\mu \neq 10$.

$[\, r(A) = 2 < 3 = r(A \,\vdots\, b) \,]$

ii) Unique solution $\Rightarrow r(A) = 3$

$\Rightarrow \lambda \neq 3$.

iii) An infinite no. of solution

$\Rightarrow r(A) = r(A \,\vdots\, b) < 3$

$\Rightarrow \lambda = 3, \ \mu = 10$.

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Theorem:- If A is a square matrix of order n, then A is row-equivalent to the identity matrix I_n if and only if the system of ~~linear~~ equations $A\underset{\sim}{x}=\underset{\sim}{0}$ has the only solution $x_1=0, x_2=0, \ldots, x_n=0$.

[

$$I_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A_{n\times n} \sim A_1 \sim \cdots\cdots \sim \text{Identity mtx.}$$

$\vdots$

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

]

Proof:-

Only If Part:- Let $A_{n\times n}$ be row-equivalent to I_n. Then $A\underset{\sim}{x}=0$ and $I_n\underset{\sim}{x}=0$ have exactly the same solution(s).

As $I_n\underset{\sim}{x}=0$ has the only solution $x_1=0, x_2=0, \ldots, x_n=0$, then the system $A\underset{\sim}{x}=0$ has the only solution $x_1=0, x_2=0, \ldots\ldots, x_n=0$.

If Part:- Let $A\underset{\sim}{x}=\underset{\sim}{0}$ has the only solution $x_1=0, \ldots\ldots, x_n=0$.

Further, let R be an $n\times n$ row-reduced echelon matrix which is row equivalent to A and r be the number of non-zero rows of R. Now $R\underset{\sim}{x}=0$ has no non-zero solution implies $r \geq n$ [because any homogeneous system of m equation in n variables has a non-zero solution if $n>m$]. But since R has n rows, $r=n$ this means that R actually has a leading non-zero entry of 1 in each of the n rows, and as these 1's occur each in a different one of the n columns, R must be the identity matrix I_n of order $n\times n$.

- **What is elementary Congruent Operation?**

A pair of elementary operations — one row operation and the other column operation — is said to constitute an elementary congruent operation if the corresponding elementary matrices are such that one is transpose of the other.

It may be noted that each matrix obtained from a given matrix A by subjecting to a series of congruent elementary operation is congruent to A.

Problems:-

1. Show that the three equations $-2x+y+z=a$, $x-2y+z=b$, $x+y-2z=c$

Have no solution, unless $a+b+c=0$.

Solution:- $[A:b] = \left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 1 & 1 & -2 & c \end{array}\right]$

$= \left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 0 & 0 & 0 & a+b+c \end{array}\right]$

If $a+b+c=0$, then $r(A) = r(A:b) = 2$.
Then the system of equation has infinitely many solutions.

If $a+b+c \neq 0$, then $r(A:b) = 3 \neq r(A) = 2$.
Then these 3 equations has no solution.

2. Use matrices to find the solution set of

$$2x+y+z=1$$
$$x-2y-3z=1$$
$$3x+2y+4z=5$$

Solution:-

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & -3 \\ 3 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$$

i.e. $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & -3 \\ 3 & 2 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{9}\begin{pmatrix} -2 & -2 & -1 \\ -13 & 5 & 7 \\ 8 & -1 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$ $\therefore$ Solution set is $\{(1,-3,2)\}$.

3. For which value of μ, the following system of equations is inconsistent:
$3x+2y+z=10$, $2x+3y+2z=10$, $x+2y+\mu z=10$.

Solution:- The given system of equations is inconsistent if

$$\begin{vmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & \mu \end{vmatrix} = 0 \Rightarrow 5\mu - 7 = 0$$
$$\Rightarrow \mu = 7/5.$$

END

Quadratic Forms

Linear Form :— An expression of the type $\sum_{i=1}^{n} a_i x_i$, where a_i are constant coefficients and x_i are variables, is called a linear form w.r.t. the variables. It may be denoted by $L(x)$, being looked upon as a function of the vector variable $\underset{\sim}{x} = [x_1, x_2, \ldots, x_n]'$. Putting $\underset{\sim}{a} = [a_1, a_2, \ldots, a_n]'$. We may write $L(x) = \underset{\sim}{a}'\underset{\sim}{x}$.

Quadratic Form :— An expression of the type $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$, where a_{ij} are constant coefficients while x_i and x_j are variables, is called a quadratic form w.r.t. the variables. It is denoted by $Q(x)$, being looked upon as a function of the vector variable $\underset{\sim}{x}$. Putting $A = (a_{ij})$, we may write $Q(\underset{\sim}{x}) = \underset{\sim}{x}' A \underset{\sim}{x}$.

Def^n^ → A quadratic form in $x_1, x_2, \ldots, x_n$ is a second degree homogeneous function in n variables $x_1, x_2, \ldots, x_n$, i.e.

$$Q(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{i \neq j} a_{ij} x_i x_j$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$$

Define, $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)'$, and

$$A = ((a_{ij}))_{n \times n}$$

$$\therefore Q(\underset{\sim}{x}) = \underset{\sim}{x}' A \underset{\sim}{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j = a_{11} x_1 x_1 + a_{12} x_1 x_2 + \cdots + a_{1n} x_1 x_n + a_{21} x_2 x_1 + \cdots + a_{2n} x_2 x_n + \cdots + a_{n1} x_n x_1 + \cdots + a_{nn} x_n x_n$$

Ex. 1. $n = 2$,

$$Q(x, y) = ax^2 + bxy + cxy + dy^2$$
$$= (x \;\; y)\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

For a quadratic form $\underset{\sim}{x}' A \underset{\sim}{x}$, A is said to be a matrix of the quadratic form. If the matrix of a quadratic form be not symmetric then it can reduced to a symmetric matrix.

$n = 2$,

$$Q(x, y) = ax^2 + bxy + cxy + dy^2$$
$$= ax^2 + \left(\frac{b+c}{2}\right)xy + \left(\frac{c+b}{2}\right)xy + dy^2$$
$$= (x \;\; y)\begin{pmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Note :→ The square matrix A in a quadratic form [redacted] Q(x) may, without any loss of generality, be supposed to be a symmetric matrix. For, in case A is not symmetric, we may take another matrix $B = (b_{ij})$ such that

$$b_{ij} = \frac{a_{ij} + a_{ji}}{2} \text{ for all } i, j.$$

which implies that $b_{ii} = a_{ii}$ for all i.

then, $b_{ij} = b_{ji}$ for all ij, so that

$B = B'$, i.e. B is a symmetric matrix.

Now,
$$x'Bx = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} x_i x_j$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\frac{a_{ij}+a_{ji}}{2}\right) x_i x_j$$
$$= \frac{1}{2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji} x_i x_j\right]$$
$$= \frac{1}{2}\left[x'Ax + x'A'x\right]$$
$$= x'Ax.$$

$(x'Ax)' = x'A'x$
But $x'Ax$ is a scalar,
so → $(x'Ax)' = x'Ax$
$\therefore x'Ax = x'A'x$.

In our discussion, then, we shall always assume A [redacted] is symmetric. We shall also assume that each element of $\underset{\sim}{x}$ can take only real values.

The determinant $|A|$ is said to be the discriminant of the quadratic form $\hat{x}'A\hat{x}$.

Ex.1. $x^2 - 2x_2^2 + 3x_3^2 + 4x_1x_2 - 6x_1x_3 + 5x_2x_3$
is a q.f., with
$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & -2 & 5/2 \\ -3 & 5/2 & 3 \end{pmatrix}$$

Ex.2. [redacted] $x_1^2 + 4x_2^2 + 9x_3^2 + 12x_1x_2 - 16x_2x_3 + 8x_1x_3$
is a q.f. with
$$A = \begin{pmatrix} 1 & 6 & 4 \\ 6 & 4 & -8 \\ 4 & -8 & 9 \end{pmatrix}$$

Classification of Quadratic forms / Classification of the matrix of the Quadratic forms :—

Every real quadratic form $Q(\underset{\sim}{x})$ can be put into one of the following broad categories, depending on the range of values that it may assume, i.e. depending on the nature of A:

i) Non-negative definite Quadratic form :— If $Q(\underset{\sim}{x}) \geq 0 \; \forall \; x \in E^n$ then it is said to be n.n.d. quadratic form.

i.e. $Q(\underset{\sim}{x}) = \underset{\sim}{x}' A \underset{\sim}{x}$ is said to be an n.n.d. quadratic form if

$$\underset{\sim}{x}' A \underset{\sim}{x} \geq 0 \; \forall \; \underset{\sim}{x}$$

A is then said to be Non-negative definite matrix.

An n.n.d. matrix will be either positive definite or positive-semi definite matrix.

An n.n.d. matrix is also called a matrix which is atleast p.s.d.q.f.

ii) **Positive definite Quadratic form :— A quadratic form $Q(\underset{\sim}{x}) = \underset{\sim}{x}' A \underset{\sim}{x}$ is said to be a p.d. quadratic form if

$$\underset{\sim}{x}' A \underset{\sim}{x} > 0 \; \forall \; \underset{\sim}{x} \neq \underset{\sim}{0}$$
$$= 0 \text{ iff } \underset{\sim}{x} = \underset{\sim}{0}$$

$Q(\underset{\sim}{x})$ is a p.d. quadratic form means A is a p.d. matrix.

iii) **Positive semi-definite Quadratic form :— A quadratic form $Q(\underset{\sim}{x}) = \underset{\sim}{x}' A \underset{\sim}{x}$ is said to be a p.s.d. quadratic form if

$$\underset{\sim}{x}' A \underset{\sim}{x} \geq 0 \; \forall \; \underset{\sim}{x}$$
$$= 0 \text{ for atleast one } \underset{\sim}{x} \neq \underset{\sim}{0}.$$

$Q(\underset{\sim}{x})$ is a p.s.d. quadratic form means A is a p.s.d. matrix.

Moreover, An n.n.d. quadratic form which is not p.d. is said to be a positive semi-definite (or p.s.d) quadratic form.

iv) Non-positive Definite Quadratic form :— A quadratic form $Q(\underset{\sim}{x}) = \underset{\sim}{x}' A \underset{\sim}{x}$ is said to be non-positive definite quadratic form if

$$\underset{\sim}{x}' A \underset{\sim}{x} \leq 0 \; \forall \; \underset{\sim}{x}.$$

i.e. if $Q(\underset{\sim}{x}) \leq 0 \; \forall \; x \in E^n$, then it is said to be a n.p.d. q.f.

- A n.p.d. $\Longleftrightarrow$ $-A$ n.n.d.

A non-positive definite quadratic form (at least negative semi definite) is either negative definite or negative semi definite quadratic form.

v) ** Negative definite Quadratic form : A quadratic form $Q(\underset{\sim}{x}) = \underset{\sim}{x}'A\underset{\sim}{x}$ is said to be a negative definite (or n.d.) quadratic form if

$$\underset{\sim}{x}'A\underset{\sim}{x} < 0 \quad \forall \underset{\sim}{x} \neq \underset{\sim}{0}$$
$$= 0 \text{ iff } \underset{\sim}{x} = \underset{\sim}{0}$$

- A : nd $\Longleftrightarrow$ $-A$: pd matrix.

vi) Negative Semi-definite Quadratic Form : A quadratic form $Q(\underset{\sim}{x}) = \underset{\sim}{x}'A\underset{\sim}{x}$ is said to be an n.s.d. quadratic form if

$$\underset{\sim}{x}'A\underset{\sim}{x} \leq 0 \quad \forall \underset{\sim}{x}$$
$$= 0 \text{ for at least one } \underset{\sim}{x} \neq 0.$$

- A : nsd $\Longleftrightarrow$ A : psd

An n.p.d. quadratic form which is not n.d. is said to be negative semi-definite (or n.s.d.).

vii) Indefinite Quadratic form : A quadratic form $Q(\underset{\sim}{x})\, \underset{\sim}{x}'A\underset{\sim}{x}$ is said to be indefinite if

$$Q(\underset{\sim}{x}) = \underset{\sim}{x}'A\underset{\sim}{x} \geq 0 \text{ for some } \underset{\sim}{x}$$
$$< 0 \text{ for some } \underset{\sim}{x}$$

A q.f. which is neither p.s.d. nor n.s.d. is called indefinite quadratic form.

Ex.1.
$$Q(x) = 5x_1^2 - 5x_1x_2 + 4x_2^2$$
$$= 5\left(x_1^2 - x_1x_2 + \frac{x_2^2}{4}\right) + \frac{11}{4}x_2^2$$
$$= 5\left(x_1 - \frac{x_2}{2}\right)^2 + \frac{11}{4}x_2^2$$

$\therefore Q(x) \geq 0 \quad \forall\ x_1, x_2 \neq 0$

$Q(x) = 0 \Rightarrow \left(x_1 - \frac{x_2}{2}\right) = 0$ and $x_2 = 0$

i.e. iff $x_1 = x_2 = 0$.

$\therefore Q(x)$ is a positive definite quadratic form.

Ex. 2. $Q(x) = 3x_1^2 - 6x_1x_2 + 3x_2^2$
$= 3(x_1 - x_2)^2$

$\therefore Q(x) \geq 0 \quad \forall x_1, x_2$

Here, $Q(x) = 0 \Rightarrow x_1 = x_2$

$Q(x) = 0$ for at least one $x \neq 0$.

$\therefore$ It is positive semidefinite quadratic form.

Ex. 3. $Q(x) = 4x_1^2 - 7x_1x_2 - 2x_2^2$

Here $Q(x)$ may assume values that may be positive, negative or zero. Thus, for instance, in case $x_1 = 1, x_2 = 0$,

$Q(x) = 4$;

when $x_1 = 0, x_2 = 1$,

$Q(x) = -2$;

and when $x_1 = 2, x_2 = 1$ or $x_1 = x_2 = 0$,

$Q(x) = 0$.

As such, we now have an indefinite quadratic form.

Theorem :→ If $x'Ax$ is positive definite (p.d.), then $x'(-A)x$ is negative definite (n.d.). Conversely, if $x'Ax$ is negative definite (n.d.) then $x'(-A)x$ is positive definite (p.d.).

Proof:→

We have, $x'(-A)x = \sum_i \sum_j (-a_{ij}) x_i x_j$

$= -\sum_i \sum_j a_{ij} x_i x_j$

$= -x'Ax$. ——— (1)

Case-I → Now, let $x'Ax$ is p.d. q.f.,

then for $x \neq 0$, we have $x'Ax > 0$,

$\Rightarrow -x'Ax < 0$ [Applying (1)]

$\Rightarrow x'(-A)x < 0$.

if $x = 0$, we have $x'Ax = 0$

$\Rightarrow x'(-A)x = 0$

Hence, $x'(-A)x$ must be n.d. q.f.

Case-II → Again, let $x'Ax$ is n.d. q.f.,

then for $x \neq 0$, we have $x'Ax < 0$,

$\Rightarrow x'(-A)x < 0$.

for $x = 0$, we have $x'Ax = 0$

$\Rightarrow x'(-A)x = 0$

also hence, $x'(-A)x$ must be p.d. q.f.

Congruence of matrix :→ A square matrix B of order n over a field F is said to be congruent to another square matrix A of the same order over the same field F if there exists a non-singular matrix P over F such that $B = P'AP$.

Result :→ The relation of 'congruence of matrices' is an equivalence relation in the set of all $n \times n$ matrices over a field F.

* **Result :→** Every matrix congruent to a symmetric matrix is also a symmetric matrix.

Proof :→ Let A be a symmetric matrix, so $A = A^T$.

Suppose B is congruent to A, then $B = P'AP$.

$$\therefore B' = (P'AP)'$$
$$= P'A'P$$
$$= P'AP \quad [\because A \text{ is symmetric}]$$
$$= B$$

$\therefore$ B' is also congruent and B is a symmetric matrix.

Congruence of Quadratic form or Equivalance of Quadratic Forms :-

Defn :→ Two quadratic forms $x'Ax$ and $y'By$ over a field F are said to be congruent or equivalent over F if these respective matrices A and B are congruent over F. Thus $x'Ax$ is congruent to $y'By$ if $\exists$ a matrix P $\ni$ $B = P'AP$. Since congruence of matrices is an equivalence relation, therefore, congruence of quadratic form is also an equivalence relation.

The linear transformation of a quadratic form :→ Consider a quadratic form $x'Ax$ and a non-singular linear transformation $x = Py$ so that P is a non-singular matrix.

Putting $x = Py$, we get → $x'Ax = (Py)'A(Py)$
$$= y'P'APy$$
$$= y'(P'AP)y = y'By \quad \left[\begin{array}{l}\text{where} \\ P'AP = B\end{array}\right]$$

Since B is congruent to a symmetric matrix A, So, B is also symmetric. Thus, $y'By$ is a quadratic form. It is called a linear transformation of the form $x'Ax$ by the non-singular matrix P. The matrix of the quadratic form $y'By$ is $B = P'AP$. Thus, the quadratic form $y'By$ is congruent to $x'Ax$.

Result :→ The ranges of values of two congruent quadratic form are the same.

Congruent reduction of a symmetric matrix :–

Theorem :– If A be any n rowed non-zero symmetric matrix of rank r over a field F then ∃ an n-rowed non-singular matrix P over F such that $P'AP = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$ where A_1 is a non-singular diagonal matrix of order r over F and O is the null matrix of suitable order. [For Practical]

Proof :→ We shall proof the theorem by induction on n, the order of the given matrix. If n=1, the theorem is obviously true. Let us suppose that the theorem is true for all symmetric matrices of order (n−1). Then we have to show that it is also true for an n×n symmetric matrix A.

Let $A = (a_{ij})_{n\times n}$ be a symmetric matrix of rank r over a field F. First we shall show that ∃ a matrix $B = (b_{ij})_{n\times n}$ over F congruent to A such that $b_{11} \neq 0$.

Case I → If $a_{11} \neq 0$, then we take B = A.

Case II → If $a_{11} = 0$, but some diagonal element of A, say, $a_{ii} \neq 0$, then applying the congruent operation $R_i \leftrightarrow R_1$, $C_i \leftrightarrow C_1$ to A, we obtain a matrix B congruent to A such that $b_{11} = a_{ii} \neq 0$.

Case III → Suppose that each diagonal element of A is 0. Since A is a non-zero matrix, let a_{ij} be a non-zero element of A. Then $a_{ij} = a_{ji} \neq 0$. Applying the congruent operation $R_i \leftrightarrow R_i + R_j$, $C_i \leftrightarrow C_i + C_j$ to A, we obtain a matrix $D = [d_{ij}]$ congruent to A ∋ $d_{ii} = a_{ij} + a_{ji} = 2a_{ij} \neq 0$.

Now applying the congruent operation $R_i \leftrightarrow R_1$, $C_i \leftrightarrow C_1$ to D, We obtain a matrix $B = (b_{ij})_{n\times n}$ congruent to D and therefore, also congruent to A ∋ $b_{11} = d_{ii} \neq 0$. Thus there always exists a matrix $B = [b_{ij}]_{n\times n}$ congruent to a symmetric matrix ∋ the leading element of B is not zero. Since B is congruent to a symmetric matrix, therefore B itself is a symmetric matrix. Since $b_{11} \neq 0$, therefore all elements in the first row and first column of B, except the leading element, can be made zero by suitable congruent operations. We thus have a matrix $C = \begin{bmatrix} a_{11} & 0 \cdots 0 \\ 0 & \\ \vdots & B_1 \\ 0 & \end{bmatrix}$ congruent to B and therefore, also congruent to $A \ni B_1$ is a square matrix of order $(n-1)$, since C is congruent to a symmetric matrix A, therefore C is also a symmetric matrix and consequently B_1 is also a symmetric matrix of order $(n-1)$. Therefore by our induction hypothesis, it can be reduced to a diagonal matrix by congruent operations. If the congruent operations applied to B_1 for this purpose applied to C, they will not affect row one and column one of C. So C can be reduced to a diagonal matrix by congruent operations. Thus A is congruent to a diagonal matrix, say, $\text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_k, 0, 0, \ldots 0]$. Thus $\exists$ a non-singular matrix $P \ni P'AP = \text{diag}[\lambda_1, \ldots, \lambda_k, 0, \ldots, 0]$.

Since rank $(A) = r$, and the rank of a matrix does not change on multiplication by a non-singular matrix, therefore, the rank of $\text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_k, 0, \ldots, 0]$ is also r. So, precisely r elements of $\text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_k, 0, \ldots, 0]$ are non-zero. $\therefore k = r$ and thus $P'AP = \text{diag}[\lambda_1, \ldots, \lambda_r, 0, \ldots, 0]$

Thus A can be reduced to diagonal form by Congruent operations.

Theorem:→ If A be any n-rowed non-zero symmetric matrix of rank r over the field of real numbers, then $\exists$ a real non-singular matrix $P \ni P'AP = \{1, \ldots 1, -1, \ldots, -1, 0, \ldots, 0\}$ where 1 appears $s\ (= 0, 1, \ldots, r)$ times and -1 appears $(r-s)$ times.

Proof :→ As A is a non-zero symmetric matrix of rank r over the field of real numbers. There exists a non-singular matrix Q over the real field $\ni$ Q'AQ is a diagonal matrix with first r diagonal entries nonzero and remaining $(n-r)$ entries zero, i.e. $Q'AQ = \text{diag}\{\lambda_1, \dots, \lambda_r, 0, \dots, 0\}$, now let s of the λ_j's are positive and remaining $(r-s)$ λ_j's are negative. Performing suitable elementary congruent operations on Q'AQ of the type; interchanging ith and jth row and ith and jth column we get a diagonal matrix whose 1st s diagonal entries are positive and remaining $(r-s)$ entries are negative. Without loss of generality let $\lambda_1, \dots, \lambda_s$ are positive and $\lambda_{s+1}, \lambda_{s+2}, \dots, \lambda_r$ are negative then $\exists$ real numbers $\beta_1, \beta_2, \dots, \beta_r \ni$

$\lambda_1 = \beta_1^2, \lambda_2 = \beta_2^2, \dots, \lambda_s = \beta_s^2, \beta_{s+1}^2 = -\lambda_{s+1}, \dots, \beta_r^2 = -\lambda_r$

Let $R = \text{diag}\{1/\beta_1, 1/\beta_2, \dots, 1/\beta_r, 1, \dots, 1\}$ and

$$P = QR$$

$$\Rightarrow P'AP = R'Q'AQR$$

$$= R'(Q'AQ)R$$

$$= \text{diag}\{1/\beta_1, \dots 1/\beta_r, 1, \dots, 1\}\, \text{diag}\{\lambda_1, \dots, \lambda_r, 0, \dots, 0\}\, \text{diag}\{1/\beta_1, \dots, 1/\beta_r, 1, \dots, 1\}$$

$$= \text{diag}\{\underbrace{1, 1, \dots, 1}_{s\text{ times}}, \underbrace{-1, -1, \dots, -1}_{(r-s)\text{ times}}, \underbrace{0, 0, \dots 0}_{(n-r)\text{ times}}\}$$

$$R'(Q'AQ)R = \begin{pmatrix} 1/\beta_1 & & & & 0 \\ & \ddots & & & \\ & & 1/\beta_r & & \\ & & & 1 & \\ 0 & & & & \ddots \; 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & & & & 0 \\ & \ddots & & & \\ & & \lambda_r & & \\ & & & 0 & \\ 0 & & & & \ddots \; 0 \end{pmatrix} \begin{pmatrix} 1/\beta_1 & & & & 0 \\ & \ddots & & & \\ & & 1/\beta_r & & \\ & & & 1 & \\ 0 & & & & \ddots \; 1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1/\beta_1^2 & & & & 0 \\ & \ddots & & & \\ & & \lambda_r/\beta_r^2 & & \\ & & & 0 & \\ 0 & & & & \ddots \; 0 \end{pmatrix}$$

$[\beta_1^2 = \lambda_1, \dots, \beta_s^2 = \lambda_s,$
$\beta_{s+1}^2 = -\lambda_{s+1}, \dots,$
$\beta_r^2 = -\lambda_r]$

$$= \begin{pmatrix} \underbrace{1 \;\; 1 \; \dots \; 1}_{s\text{ times}} & & & 0 \\ & \underbrace{-1 \; \dots \; -1}_{r-s\text{ times}} & & \\ 0 & & \underbrace{0 \; \dots \; 0}_{n-r\text{ times}} \end{pmatrix} = \text{diag}\{1, \dots, 1, -1, \dots, -1, 0, \dots, 0\}$$

SOME ELEMENTARY RESULTS :→

- **Result 1.** A positive definite (p.d.) quadratic form remains positive definite under a non-singular linear transformation.

Proof:→ Let $Q(x) = x'Ax$ is a p.d. quadratic form and $y = Bx$ be a ~~[illegible]~~ non-singular linear transformation, under this non-singular linear transformation $x = B^{-1}y$. Therefore,

$$Q(x) = (B^{-1}y)'A(B^{-1}y)$$
$$= y'[(B^{-1})'AB^{-1}]y$$
$$= y'Cy \quad [\text{where } C = (B^{-1})'AB^{-1}]$$
$$= Q^*(y), \text{ a quadratic form in } y_i\text{'s } (i=1(1)n) \quad \text{---(1)}$$

From (1) we see that — the range value of $Q^*(y)$ is the same as the range value $Q(x)$.

$$\therefore Q^*(y) \geq 0 \quad \forall y \quad [\text{since } Q(x) \text{ is p.d.}]$$

Now $y \neq 0 \Rightarrow x = B^{-1}y \neq 0$

Let $B^{-1} = (c_1, c_2, \ldots, c_n)$, since B^{-1} is non-singular. So, $c_1, c_2, \ldots, c_n$ are linearly independent.

So, $B^{-1}y = \underset{\sim}{c}_1 y_1 + \underset{\sim}{c}_2 y_2 + \cdots + \underset{\sim}{c}_n y_n = 0$ iff $\underset{\sim}{y} = \underset{\sim}{0}$.

$\therefore Q^*(y) = 0$ if $\underset{\sim}{y} = \underset{\sim}{0}$ and $Q^*(y) > 0$ if $\underset{\sim}{y} \neq 0$.

$\therefore Q^*(y)$ is positive definite (p.d.).

- **Result 2.** If $x'Ax$ is a real quadratic form of rank r, i.e. the rank of the ~~matrix~~ symmetric matrix A is r, then $\exists$ a non-singular linear transformation $x = Py$ which transforms $x'Ax$ to $y'(P'AP)y = y_1^2 + y_2^2 + \cdots + y_s^2 - y_{s+1}^2 - y_{s+2}^2 - \cdots - y_r^2$.

Proof :→

We know that for an n-rowed non zero matrix A of rank r $\exists$ a non-singular matrix P $\ni$

$$P'AP = \text{diag}\{\underbrace{1, 1, \ldots, 1}_{s\text{ times}}, \underbrace{-1, -1, \ldots, -1}_{r-s\text{ times}}, \underbrace{0, 0, \ldots, 0}_{n-r\text{ times}}\}$$

Let us consider now the non-singular transformation

$$x = Py$$

$$\therefore x'Ax = (Py)'A(Py)$$

$$= y'P'AP\,y$$

$$= y'(P'AP)y$$

$$= y'\left[\text{diag}\{\underbrace{1, 1, \ldots, 1}_{s\text{ times}}, \underbrace{-1, \ldots, -1}_{r-s\text{ times}}, \underbrace{0, \ldots, 0}_{n-r\text{ times}}\}\right] y$$

$$= y_1^2 + y_2^2 + \cdots + y_s^2 - y_{s+1}^2 - y_{s+2}^2 - \cdots - y_r^2 .$$

[C.U.]

Result 3. If $x'Ax$ is a real p.d. quadratic form in n variables $x_1, x_2, \ldots, x_n$ then it can be reduced by a non-singular linear transformation to the form

$$Q^*(y) = y'y = \sum_{i=1}^{n} y_i^2$$

Proof:→ Since $x'Ax$ is a real p.d. q.f. of rank r, then $\exists$ a non-singular matrix P which transform $x'Ax$ to the form

$$Q^*(y) = y_1^2 + \cdots + y_s^2 - y_{s+1}^2 - \cdots - y_r^2$$

Now, since $Q(x) > 0$ if $x \neq 0$
$= 0$ if $x = 0$,

then $Q^*(y) \geq 0 \ \forall\, y$

Now in order that $Q^*(y)$ to be p.d., there must be no-zero coefficient in the expression of $Q^*(y)$ since when $y_1 = 0, \ldots, y_r = 0, y_{r+1} \neq 0, \ldots, y_n \neq 0$.

i.e. $y \neq \underset{\sim}{0}$ then also $Q^*(\underset{\sim}{y}) = \underset{\sim}{0}$, i.e. the condition for a p.d. q.f. is violated. Hence $r = n$.

$$\therefore Q^*(y) = y_1^2 + \cdots + y_s^2 - y_{s+1}^2 - \cdots - y_n^2$$

Now if $(y_1^2 + \cdots + y_s)^2 < (y_{s+1}^2 + \cdots + y_n^2)$

Then $Q^*(y) < 0$, again the condition of a p.d.q.f. is violated. So, to satisfy the condition no negative co-efficient will be there. Hence $s = n$.

$$\therefore Q^*(y) = y'y = \sum_{i=1}^{n} y_i^2 .$$

[C.U.]

Result 4. The determinant of a p.d.q.f. is positive.

Proof:→ Let $x'Ax$ be a real p.d.q.f.. Since $x'Ax$ is p.d., then $\exists$ a non-singular transformation $y = Px \ni$

$$P'AP = I$$

$\therefore$ Taking determinant, $|P'||A||P| = 1$

or, $|A||P|^2 = 1 \quad [\because |P| = |P'|]$

or, $|A| = \dfrac{1}{|P|^2} > 0$

$\therefore$ The determinant of a real p.d.q.f. is positive.

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[C.U.]

Result 4.* Suppose A is a p.d. matrix then show that
(a) all diagonal elements of A are positive.
(b) all principal submatrices are p.d.
(c) A^{-1} is a positive-definite matrix.

Proof:→

(a) A : p.d. $\Rightarrow \underset{\sim}{x}' A \underset{\sim}{x} > 0 \quad \forall\, x \neq \underset{\sim}{0}$

$A = ((a_{ij}))_{n\times n}$ and $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)'$.

Choose $\underset{\sim}{x} = \underset{\sim}{e}_i = (0, 0, \ldots, 0, \underset{\substack{\uparrow \\ i\text{th component}}}{1}, 0, \ldots, 0, 0)$

$\therefore \underset{\sim}{x}' A \underset{\sim}{x} \Big|_{\underset{\sim}{x} = \underset{\sim}{e}_i} = a_{ii} > 0$

(b) A : p.d.
$\Rightarrow \underset{\sim}{x}' A \underset{\sim}{x} > 0 \;\forall\; \underset{\sim}{x} \neq \underset{\sim}{0}$.

Let $A = ((a_{ij}))_{n\times n}$

Partition $\underset{\sim}{x}$ and A as $\underset{\sim}{x} = \begin{pmatrix} \underset{\sim}{x}_{(1)m\times 1} \\ \underset{\sim}{x}_{(2)} \end{pmatrix}_{n\times 1}$

$$A = \begin{pmatrix} A_{11\,m\times m} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}_{n\times n}$$

Note that, $\underset{\sim}{x}' A \underset{\sim}{x}$

$$= \begin{pmatrix} \underset{\sim}{x}'_{(1)} & \underset{\sim}{x}'_{(2)} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \underset{\sim}{x}_{(1)} \\ \underset{\sim}{x}_{(2)} \end{pmatrix}$$

$$= \begin{pmatrix} \underset{\sim}{x}'_{(1)} A_{11} + \underset{\sim}{x}'_{(2)} A_{21} & \underset{\sim}{x}'_{(1)} A_{12} + \underset{\sim}{x}'_{(2)} A_{22} \end{pmatrix} \begin{pmatrix} \underset{\sim}{x}_{(1)} \\ \underset{\sim}{x}_{(2)} \end{pmatrix}$$

$$= \underset{\sim}{x}'_{(1)} A_{11} \underset{\sim}{x}_{(1)} + \underset{\sim}{x}'_{(2)} A_{21} \underset{\sim}{x}_{(1)} + \underset{\sim}{x}'_{(1)} A_{12} \underset{\sim}{x}_{(2)} + \underset{\sim}{x}'_{(2)} A_{22} \underset{\sim}{x}_{(2)}$$

Choose, $\underset{\sim}{x} \neq 0$, $\underset{\sim}{x}_{(1)} \neq \underset{\sim}{0}$ and $\underset{\sim}{x}_{(2)} = 0$.

$\underset{\sim}{x}' A \underset{\sim}{x} > 0 \quad \forall\; \underset{\sim}{x} \neq \underset{\sim}{0}$.

$\Rightarrow \underset{\sim}{x}' A \underset{\sim}{x} \Big|_{\underset{\sim}{x}_{(2)} = \underset{\sim}{0}} = \underset{\sim}{x}'_{(1)} A_{11} \underset{\sim}{x}_{(1)} > 0 \quad \forall\; \underset{\sim}{x}_{(1)} \neq \underset{\sim}{0}$.

$\Rightarrow A_{11}$ is p.d. of order $m \times m$, where A_{11} is principal submatrix of A. Hence the fact holds good for any 'm'.

(c)

A be a p.d. matrix.

Then $\exists$ a non-singular matrix P show that

$$P'AP = I_n$$

$$\Rightarrow \quad (P'AP)^{-1} = I_n$$

or, $\quad P^{-1}A^{-1}(P')^{-1} = I_n$

i.e. $\quad P^{-1}A^{-1}(P^{-1})' = I_n$.

$$\therefore \quad Q'A^{-1}Q = I_n$$

[Taking $(P^{-1})' = Q$, i.e. $P^{-1} = Q'$, since P is non-singular, Q is also non-singular]

So, A^{-1} is also congruent to I_n,

$\Rightarrow A^{-1}$ is p.d., i.e. $x'A^{-1}x$ is p.d.

Result 5. Suppose A be a p.d. matrix and P be a non-singular matrix, then show that $P'AP$ or PAP' will also be p.d. matrix.

Proof :> A be a p.d. matrix $\Rightarrow \underset{\sim}{x}'A\underset{\sim}{x} > 0 \;\forall\; \underset{\sim}{x} \neq \underset{\sim}{0}$.

Then $\exists$ a non-singular transformation $\underset{\sim}{x} = P\underset{\sim}{y}$,

As P is non-singular matrix

$$\underset{\sim}{x}'A\underset{\sim}{x} > 0$$

$$\Rightarrow \underset{\sim}{y}'(P'AP)\underset{\sim}{y} > 0$$

In order to show $P'AP$ is p.d. matrix it is enough to verify $\underset{\sim}{x} \neq \underset{\sim}{0}$

$$\Rightarrow \underset{\sim}{y} \neq \underset{\sim}{0}$$

Rewrite P and $\underset{\sim}{y}$ as —

$P = (\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n)$ and $\underset{\sim}{y} = (y_1, y_2, \ldots, y_n)$

Then $P\underset{\sim}{y} = \sum_{i=1} y_i \underset{\sim}{\alpha}_i$

$$\therefore \sum y_i \underset{\sim}{\alpha}_i = \underset{\sim}{x}$$

As P is non-singular matrix, then columns of P must be linearly independent i.e. α_i's are LIN vectors. Hence in order to get a non-null $\underset{\sim}{x}$ at least one y_i must be non-zero. i.e. $\underset{\sim}{x} \neq \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{y} \neq \underset{\sim}{0} \quad \therefore \underset{\sim}{y}'P'AP\underset{\sim}{y} > 0 \;\forall\; \underset{\sim}{y} \neq 0$

$\Rightarrow P'AP$ is p.d.

• **Result 6.** Suppose P be a non singular matrix, show that $P'P$ or PP' will be a p.d. matrix.

Proof:→ Let $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)' \neq \underset{\sim}{0}$

i.e. $x_i \neq 0$ for at least one i.

$\therefore \underset{\sim}{x}'\underset{\sim}{x} = \sum x_i^2 > 0$ for at least one $x_i \neq 0$.

Consider the non-singular transformation $\underset{\sim}{x} = P\underset{\sim}{y}$, where P is a non-singular matrix.

$$\therefore \underset{\sim}{x}'\underset{\sim}{x} > 0$$

$$\Rightarrow \underset{\sim}{y}'P'P\underset{\sim}{y} > 0$$

In order to show $P'P$ is a p.d. matrix it is enough to verify that

$$\underset{\sim}{x} \neq \underset{\sim}{0} \Rightarrow \underset{\sim}{y} \neq \underset{\sim}{0}.$$

Rewrite P as $P = (\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n)$

then $P\underset{\sim}{y} = \sum_{i=1}^{n} y_i \underset{\sim}{\alpha}_i$ [where $\underset{\sim}{y} = (y_1, \ldots, y_n)'$]

$$\therefore \sum y_i \underset{\sim}{\alpha}_i = \underset{\sim}{x}$$

As P is a non-singular matrix, then columns of P must be LIN, i.e. $\underset{\sim}{\alpha}_i$'s are LIN vectors. Hence in order to get a non-null $\underset{\sim}{x}$ at least one y_i must be non-zero.

i.e. $\underset{\sim}{x} \neq 0$,

$\Rightarrow \underset{\sim}{y} \neq 0$.

$\therefore \underset{\sim}{y}'P'P\underset{\sim}{y} > 0 \;\forall\; \underset{\sim}{y} \neq 0$.

$\Rightarrow P'P$ is p.d.

- **Result 7.** (Lagrange's method / Method of Completing the square)

If $A_{n\times n}$ be a p.d. matrix then there exists an n.s. matrix $P \ni P'AP$ or $PAP' = \Lambda$, Λ is a diagonal with positive elements.

Proof :→ (Practical)

$$A = ((a_{ij}))_{n\times n}$$

Partition A as $\begin{bmatrix} a_{11} & \underset{\sim}{a}_{(1)}' \\ \underset{\sim}{a}_{(1)} & A_{22} \end{bmatrix}$

Now A is a p.d. matrix $\Rightarrow a_{11} > 0$, hence through row and column operation A can be reduced to

$$P_1 A P_1' = \begin{bmatrix} a_{11} & \underset{\sim}{0}' \\ \underset{\sim}{0} & A_{22}^{(1)} \end{bmatrix}, \text{ where } A_{22}^{(1)} = ((a_{ij}^{(1)})), i,j = 2(1)n.$$

thus there exists a non-singular matrix $P_1 \ni$

$$P_1 A P_1' = \begin{bmatrix} a_{11} & \underset{\sim}{0}' \\ \underset{\sim}{0} & A_{22}^{(1)} \end{bmatrix}$$

Here
$$P_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\frac{a_{21}}{a_{11}} & 0 & 0 & \cdots & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -\frac{a_{n1}}{a_{11}} & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Partition $P_1 A P_1'$ as
$$= \begin{pmatrix} a_{11} & 0 & \underset{\sim}{0}' \\ 0 & a_{22}^{(1)} & \underset{\sim}{a}_{(2)}^{(1)\prime} \\ \underset{\sim}{0} & \underset{\sim}{a}_{(2)}^{(1)} & A_{33}^{(1)} \end{pmatrix}$$

as A is a p.d. matrix and P_1 be an n.s. matrix, so $P_1 A P_1'$ must be p.d. matrix.

Hence $a_{22}^{(1)} > 0$

Therefore there exists an n.s. matrix P_2 such that

$$P_2 P_1 A P_1' P_2' = \begin{pmatrix} a_{11} & 0 & \underset{\sim}{0}' \\ 0 & a_{22}^{(1)} & \underset{\sim}{0}' \\ \underset{\sim}{0} & \underset{\sim}{0} & A_{33}^{(2)} \end{pmatrix}$$

~~as P_1AP_1' is a p.d. matrix~~

$$A_{33}^{(2)} = ((a_{ij}^{(2)})) \quad , i,j = 3(1)n$$

where, $P_2 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & -\frac{a_{32}^{(1)}}{a_{22}^{(1)}} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & -\frac{a_{n2}^{(1)}}{a_{22}^{(1)}} & 0 & \cdots & 1 \end{bmatrix}$

as P_1AP_1' is a p.d. matrix and P_2 is an n.s. matrix, so $P_2P_1AP_1'P_2'$ must be a p.d. matrix, hence $a_{33}^{(2)} > 0$,

Proceeding in this way it can be shown that

$$P_{n-1}P_{n-2}\cdots P_2P_1AP_1'P_2'\cdots P_{n-1}'$$
$$= diag\left(a_{11}, a_{22}^{(1)}, a_{33}^{(2)}, \cdots, a_{nn}^{(n-1)}\right)$$

, a matrix with +ve elements.

where P_i's are n.s. matrices.

Define $P = P_{n-1}P_{n-2}\cdots P_2P_1$, as a product of n.s. matrices.

Clearly P is an n.s. matrix

$$\Lambda = diag(\lambda_1, \cdots, \lambda_n)$$

where, $\lambda_i = a_{ii}^{(i-1)}$, $i = 1, 2, \ldots, n$.

$\lambda_i > 0 \;\forall\, i$.

therefore $PAP' = \Lambda$, a diagonal matrix with positive elements.

• **Result 8.** If $A_{n\times n}$ be a p.s.d. matrix with rank$(A) = r < n$ then $\exists$ an n.s. matrix $P \ni PAP' = \text{diag}(\lambda_1, \dots, \lambda_r, 0, \dots, 0)$ when $\lambda_i > 0$, $i = 1, 2, \dots, r$.

Proof:→ Since rank$(A_{n\times n}) = r < n$, $\exists$ an n.s. matrix $P \ni$

$$P'AP = \begin{bmatrix} \Lambda_{r\times r} & 0_{r\times \overline{n-r}} \\ 0 & 0 \end{bmatrix}_{n\times n}$$, where Λ is a diagonal matrix with non-null elements.

now, A is p.s.d. matrix.

$\Rightarrow P'AP$ is also p.s.d. matrix as P is n.s.

Let $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_r)$

Note that, $\underset{\sim}{y}' P'AP \underset{\sim}{y}$

$$= \begin{pmatrix} \underset{\sim}{y}'_{(1)} & \underset{\sim}{y}'_{(2)} \end{pmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \underset{\sim}{y}_{(1)} \\ \underset{\sim}{y}_{(2)} \end{pmatrix}$$, where $\underset{\sim}{y} = \begin{pmatrix} \underset{\sim}{y}_{(1)r\times 1} \\ \underset{\sim}{y}_{(2)} \end{pmatrix}_{n\times 1}$

$$= \underset{\sim}{y}'_{(1)} \Lambda \underset{\sim}{y}_{(1)}$$

$$= \sum_{i=1}^{r} \lambda_i y_i^2$$, where $\underset{\sim}{y}_{(1)} = (y_1, y_2, \dots, y_r)'$

since $\lambda_i \neq 0$, $i = 1(1)r$.

and $\sum_{i=1}^{r} \lambda_i y_i^2 \geq 0$ as $P'AP$ is p.s.d., we have $\lambda_i > 0$, $i = 1(1)r$.

C.U

• **Result 9.** If A be a p.d. matrix then it can be written as $A = P'P$ or PP' where P is non-singular matrix.

Proof:→ As A be a p.d. matrix, $\exists$ an n.s. matrix $Q \ni$

$$QAQ' = \Lambda = \text{diag}(\lambda_1, \dots, \lambda_n),\ \lambda_i > 0\ \forall i.$$

$$= \Lambda^{1/2} \Lambda^{1/2}$$ [assume A is of order $n\times n$]

, when $\Lambda^{1/2} = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$

$$\Rightarrow A = Q^{-1} \Lambda^{1/2} \Lambda^{1/2} (Q')^{-1}$$ [Q^{-1} exists and is n.s. as Q is n.s.]

$$= Q^{-1} \Lambda^{1/2} \Lambda^{1/2} (Q^{-1})'$$ [Q is n.s.]

$$= PP'$$ [P is n.s. because it's a product of two non-singular matrices]

C.U.

- **Result 10.** If A be a p.d. matrix then A can be written as $B'B$ (or BB') when B is also p.d. matrix.

Proof:→ If A be a p.d. matrix then ∃ an orthogonal matrix $Q \ni QAQ' = \Lambda$

$= \text{diag}(\lambda_1, \cdots, \lambda_n)$, $\lambda_i > 0 \; \forall i$

$A = Q^{-1}\Lambda^{1/2}\Lambda^{1/2}(Q')^{-1}$ [Q^{-1} exists as it is equals to Q' as Q is an orthogonal matrix as, $\Lambda^{1/2} = \text{diag}(\sqrt{\lambda_1}, \cdots \sqrt{\lambda_n})$]

$= Q^{-1}\Lambda^{1/2} Q Q' \Lambda^{1/2}(Q^{-1})'$

$= (Q'\Lambda^{1/2}Q)(Q'\Lambda^{1/2}Q)$ [$\because Q^{-1} = Q'$]

$= BB'$

Q is an orthogonal matrix.

$\Rightarrow$ Q is n.s., Λ is a p.d. matrix,

$\Rightarrow Q'\Lambda^{1/2}Q$ is also a p.d. matrix,

i.e. B is a p.d. matrix.

- **Result 11.** If A be a p.s.d. matrix then A can be written as PP' or $P'P$ where P is a singular matrix.

Proof:→ Let A be a matrix of order $n \times n$ with $R(A) = r < n$.

then ∃ an n.s. matrix $Q \ni$

$QAQ' = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_r, 0, \cdots, 0)$

$= \Lambda$, say, where $\lambda_i > 0$, $i = 1(1)r$.

$\Rightarrow A = Q^{-1}\Lambda^{1/2}\Lambda^{1/2}(Q')^{-1}$, [Q^{-1} exists and is n.s. as Q is n.s.]

$\Lambda^{1/2} = \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \cdots, \sqrt{\lambda_r}, 0, \cdots, 0)$

Note that, $|P| = |Q^{-1}||\Lambda^{1/2}|$ [$\because$ P is a singular matrix]

$= 0$

$A = Q^{-1}\Lambda^{1/2}\Lambda^{1/2}(Q^{-1})'$

$= PP'$ where P is a singular matrix.

[redacted] $\boxed{\text{C.V.}}$

Result 12. If A be a real p.s.d. matrix with $\sum_i \sum_j a_{ij} = 0$, then show that $\sum_j a_{ij} = 0 \ \forall i$.

Proof :→ $\square$ $A = ((a_{ij}))_{n \times n}$

Given $\sum_i \sum_j a_{ij} = 0 = \underset{\sim}{1}_n' A \underset{\sim}{1}_n$

Since A is a p.s.d. matrix, $\exists$ a singular matrix $P \ni$

$$A = P'P$$

$\therefore \underset{\sim}{1}_n' A \underset{\sim}{1}_n = \underset{\sim}{1}_n' P'P \underset{\sim}{1}_n = \underset{\sim}{x}'\underset{\sim}{x}$, where $\underset{\sim}{x} = P\underset{\sim}{1}_n = (x_1, x_2, \ldots, x_n)'$

$$= \sum_{i=1}^{n} x_i^2$$

Now, $\sum_{i=1}^{n} x_i^2 = 0 \Rightarrow x_i = 0 \ \forall i$.

$\therefore \underset{\sim}{x} = \underset{\sim}{0}$

or, $P\underset{\sim}{1}_n = \underset{\sim}{0}$

or, $P'P\underset{\sim}{1}_n = \underset{\sim}{0}$

or, $A\underset{\sim}{1}_n = \underset{\sim}{0}$

or, $\sum_j a_{ij} = 0 \ \forall i$ ///

OR

$\sum_i \sum_j a_{ij} = 0 \Rightarrow \underset{\sim}{1}'A\underset{\sim}{1} = 0$ ——(1)

We know, $P'AP = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{n \times n}$

$$\therefore A = (P^{-1})' \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P^{-1}$$

$$= (P^{-1})' \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P^{-1}$$

$= B'B$, say, where $B = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P^{-1}$.

$\therefore A = B'B$.

From (1) → $\underset{\sim}{1}' B'B\underset{\sim}{1} = 0$

$\Rightarrow (B\underset{\sim}{1})' B\underset{\sim}{1} = 0$ [$\because \underset{\sim}{Y}'\underset{\sim}{Y} = 0$ only when $\underset{\sim}{Y} = \underset{\sim}{0}$]

$\therefore B\underset{\sim}{1} = \underset{\sim}{0}$

$\therefore B'(B\underset{\sim}{1}) = \underset{\sim}{0}$ (Premultiplying by B')

$\therefore A\underset{\sim}{1} = \underset{\sim}{0}$, i.e. $\sum_j a_{ij} = 0 \ \forall i$

[C.V.]

Result 13. If $A=((a_{ij}))_{n\times n}$ be a symmetric p.d. matrix then $|A| \le \prod_{i=1}^{n} a_{ii}$. When does the equality hold?

Proof: → Let $A=((a_{ij}))_{n\times n}$ be a p.d. matrix;

Partition A as
$$A=\begin{bmatrix} A_{n-1} & \underset{\sim}{a}_{(n)} \\ \underset{\sim}{a}'_{(n)} & a_{nn} \end{bmatrix}$$

$$|A| = |A_{n-1}|\,\underbrace{|a_{nn} - \underset{\sim}{a}'_{(n)} A^{-1}_{n-1} \underset{\sim}{a}_{(n)}|}_{\text{scalar}}$$

Now A_{n-1} is the principal submatrix of order $\overline{n-1}$ obtained from A. As A is p.d. matrix. $\Rightarrow$ A_{n-1} will be a p.d. matrix.

$\Rightarrow$ A^{-1}_{n-1} is p.d. matrix.

$\Rightarrow$ $\underset{\sim}{a}'_n A^{-1}_{n-1} \underset{\sim}{a}_n \geq 0$

$$\therefore |A| = |A_{n-1}|\,|a_{nn} - \underset{\sim}{a}'_{(n)} A^{-1}_{n-1} \underset{\sim}{a}_{(n)}|$$
$$\leq |A_{n-1}|\, a_{nn}$$

Again partition A_{n-1} as
$$A_{n-1} = \begin{bmatrix} A_{n-2} & \underset{\sim}{a}_{n-1} \\ \underset{\sim}{a}'_{n-1} & a_{\overline{n-1}\times\overline{n-1}} \end{bmatrix}$$

Similar arguments lead to

$$\therefore |A_{n-1}| \leq |A_{n-2}|\, a_{\overline{n-1}\times\overline{n-1}}$$

$$\therefore |A| \leq |A_{n-2}|\, a_{\overline{n-1}\times\overline{n-1}}\, a_{nn}$$

proceeding this way ⋮

$$\leq \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} a_{33}\cdots a_{nn}$$

$$\leq (a_{11}a_{22} - a_{12}^2)\, a_{33}\cdots a_{nn}$$
$$\leq a_{11}a_{22}a_{33}\cdots a_{nn}$$
$$\leq \prod_{i=1}^{n} a_{ii}$$

'=' holds when A be a p.d. diagonal matrix.

C.U

- **Result 14.** If $A = ((a_{ij}))_{n\times n}$ be a non-singular matrix then $|AA'| \le \prod_i \left(\sum_j a_{ij}^2\right)$

Proof:→ Since A is an n.s. matrix ⇒ AA' must be p.d. matrix.

Let, $B = AA' = ((b_{ij}))$ Clearly, $b_{ij} = \sum_k a_{ik} a_{jk}$

by previous result :→

$$|B| \le \prod_i b_{ii}$$
$$= \prod_i \sum_k a_{ik}^2$$

Hence the result.

~~Note that, $AA' = B$ is n.n.d.~~
~~By the~~

Problem:— Reduce the quadratic form $5x^2+y^2+10z^2-4yz-10zx$ to the normal form and show that it is positive definite.

Solution:— The associated symmetric mtx is

$$A = \begin{pmatrix} 5 & 0 & -5 \\ 0 & 1 & -2 \\ -5 & -2 & 10 \end{pmatrix}$$

Let us apply congruence operations on A to reduce it to the normal form

$$A \xrightarrow{R_3+R_1} \begin{pmatrix} 5 & 0 & -5 \\ 0 & 1 & -2 \\ 0 & -2 & 5 \end{pmatrix} \xrightarrow{C_3+C_1} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 5 \end{pmatrix}$$

$$\xrightarrow{R_3+2R_2} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_3+2C_2} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{\sqrt{5}}R_1} \begin{pmatrix} \sqrt{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\frac{1}{\sqrt{5}}C_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The normal form is $x^2+y^2+z^2$

The rank of the quadratic form is 3.

∴ the quadratic form is positive definite.

Canonical reduction of Quadratic Forms :→

Let A be a p.d. matrix then $\underset{\sim}{x}' A \underset{\sim}{x} = Q(\underset{\sim}{x}) > 0$ $\forall \underset{\sim}{x} \neq \underset{\sim}{0}$.

$$Q(\underset{\sim}{x}) \longrightarrow Q(\underset{\sim}{y}) = \underset{\sim}{y}' \Lambda \underset{\sim}{y} = \sum_{i=1}^{n} \lambda_i y_i^2 \text{, where } \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n), \lambda_i > 0 \, \forall i$$

and $\underset{\sim}{y} = (y_1, \ldots, y_n)'$

$$Q(\underset{\sim}{y}) \longrightarrow Q(\underset{\sim}{z}) = \sum_{i=1}^{n} z_i^2$$

$\exists$ an n.s. matrix $P \ni P'AP = \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n), \lambda_i > 0 \, \forall i$

Choose $\underset{\sim}{y} \ni \underset{\sim}{x} = P\underset{\sim}{y}$, then we get —

$$Q(\underset{\sim}{x})\Big|_{\underset{\sim}{x} = P\underset{\sim}{y}} = x'Ax = \underset{\sim}{y}'P'AP\underset{\sim}{y} = \underset{\sim}{y}'\Lambda\underset{\sim}{y} = \sum_{i=1}^{n} \lambda_i y_i^2$$

Choose $\underset{\sim}{z} \ni \underset{\sim}{y} = \Lambda^{-1/2}\underset{\sim}{z}$, where $\Lambda^{-1/2} = \text{diag}\left(\frac{1}{\sqrt{\lambda_1}}, \frac{1}{\sqrt{\lambda_2}}, \ldots, \frac{1}{\sqrt{\lambda_n}}\right)$

$$Q(\underset{\sim}{y})\Big|_{\underset{\sim}{y} = \Lambda^{-1/2}\underset{\sim}{z}} = y'\Lambda y = \underset{\sim}{z}' \underbrace{\Lambda^{-1/2}\Lambda^{1/2}}_{I} \underbrace{\Lambda^{1/2}\Lambda^{-1/2}}_{I} \underset{\sim}{z}$$

$$= \underset{\sim}{z}'\underset{\sim}{z}$$

$$= \sum_{i=1}^{n} z_i^2 \text{, where } \Lambda^{1/2} = \text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$$

Def^n. of canonical form :→ If by any real non-singular linear transformation a real quadratic form be expressed as a sum and difference of the squares of the new variables then this later expression is called the canonical ~~[illegible]~~ form of the given form.

C.U.

- **Result** * If A be a p.s.d. matrix of order $n\times n$ with $\text{rank}(A)=r<n$, then $Q(\underset{\sim}{x})=\underset{\sim}{x}'A\underset{\sim}{x} \longrightarrow Q(\underset{\sim}{y})=\sum_{i=1}^{r}\lambda_i y_i^2$ (or)

* Every quadratic form $\underset{\sim}{x}'A\underset{\sim}{x}$ can be reduced to a canonical form $\sum_{i=1}^{n}\lambda_i y_i^2=\underset{\sim}{y}'\Lambda\underset{\sim}{y}$, where $\Lambda=\text{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)$ by an n.s. transformation of variables.

Proof :→ Let, $\lambda_1,\lambda_2,\ldots,\lambda_n$ be the eigen values and $\underset{\sim}{u}_1,\underset{\sim}{u}_2,\ldots,\underset{\sim}{u}_n$ be the corresponding eigen vectors. We can take these vectors $\underset{\sim}{u}_i$'s as orthogonal.

Take $Q=(\underset{\sim}{u}_1,\underset{\sim}{u}_2,\ldots,\underset{\sim}{u}_n)$, obviously Q is non-singular.

Then $\underset{\sim}{x}=Q\underset{\sim}{y}$ is a non-singular transformation.

It reduces the q.f. to,
$$\begin{aligned}\underset{\sim}{x}'A\underset{\sim}{x} &= (Q\underset{\sim}{y})'A(Q\underset{\sim}{y})\\ &= \underset{\sim}{y}'(Q'AQ)\underset{\sim}{y}\\ &= \underset{\sim}{y}'\Lambda\underset{\sim}{y} \quad (\#)\\ &= \sum_{i=1}^{n}\lambda_i y_i^2\end{aligned}$$

Thus the theorem is established.

[(#) → As A be a square matrix of order n and $\lambda_1,\ldots,\lambda_n$ be the eigen values, then an real n.s. matrix Q exists, ∋ $Q'AQ=\text{diag}(\lambda_1,\ldots,\lambda_n)=\Lambda$ is true. Using this result there.]

Corollary: We have a connection between the nature of the eigen values and the nature of the matrix (or, q.f.).

If $\lambda_i>0$ for each i, then $\underset{\sim}{x}'A\underset{\sim}{x}$ is positive definite (p.d.)
$\lambda_i<0$ for each i, then $\underset{\sim}{x}'A\underset{\sim}{x}$ is negative definite (n.d.)
$\lambda_i\geq 0$ ∀ i and $\lambda_i=0$ for some i, then $\underset{\sim}{x}'A\underset{\sim}{x}$ is p.s.d.
$\lambda_i\leq 0$ ∀ i and $\lambda_i=0$ for some i, then $\underset{\sim}{x}'A\underset{\sim}{x}$ is n.s.d.
$\lambda_i>0$ for some i & $\lambda_i<0$ for some i, then $\underset{\sim}{x}'A\underset{\sim}{x}$ is indefinite.

[C.V.]

● **Result** The necessary and sufficient condition for a quadratic form $x'Ax$ to be positive definite is the leading principal minors of its matrix A are all positive. i.e.
Let A be a matrix of order $n \times n$, $A = ((a_{ij}))_{n \times n}$, then

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0, \ldots\ldots, |A| > 0.$$

Proof :→ Necessity part (Only if part) :–

Let $Q(\underset{\sim}{x}) = \underset{\sim}{x}'A\underset{\sim}{x}$ be a p.d.q.f. in n variables $x_1, x_2, \ldots, x_n$ and let $m (\le n)$ be a natural number, putting $x_{m+1} = x_{m+2} = \ldots\ldots = x_n = 0$ in the p.d.q.f. $\underset{\sim}{x}'A\underset{\sim}{x}$, we arrive to another p.d.q.f. in m variables $x_1, x_2, \ldots, x_m$, The determinant of Whose matrix is the leading principal minor of the matrix A.
Since, the determinant of every p.d.q.f. is positive.
∴ every leading principal minor of the matrix A is positive.

Sufficiency part (If part) :– Method of induction will be used to proof this part. For a single variable $\underset{\sim}{x}$, the q.f.
$Q(\underset{\sim}{x}) = ax_1^2 > 0$ for $x_1 \neq 0$ if $a > 0$.
Now, suppose that the theorem is true for m variables.

Consider any q.f. in (m+1) variables with the corresponding symmetric matrix $A \ni$ the leading pricipal minors of A is positive.
Let us partition of A as follows:

$$A_{m+1 \times m+1} = \begin{pmatrix} \underset{m \times m}{B} & \underset{m \times 1}{b_1} \\ \underset{1 \times m}{b_1'} & k \end{pmatrix}$$

the leading principal minors of A and B are all positive as the theorem is supposed to be true for m variables $\exists$ a n.s. matrix P over the real field $\ni$ $P'BP = I_m$
Let us now determine $C \ni$

$\begin{pmatrix} P' & 0 \\ C' & 1 \end{pmatrix} \begin{pmatrix} B & b_1 \\ b_1' & k \end{pmatrix} \begin{pmatrix} P & C \\ 0 & 1 \end{pmatrix}$ has its R.H. top corner element zero.

the product matrix equals to $\begin{pmatrix} P'B & P'b_1 \\ C'B + b_1' & C'b_1 + k \end{pmatrix} \begin{pmatrix} P & C \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} P'BP & P'BC + P'b_1 \\ C'BP + b_1'P & C'BC + b_1'C + C'b_1 + k \end{pmatrix}$$

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∴ We have $P'BC + P'\underset{\sim}{b_1} = 0$

$\Rightarrow P'BC = -P'\underset{\sim}{b_1}$

$\Rightarrow BC = -\underset{\sim}{b_1}$

$\therefore C = -(B^{-1}\underset{\sim}{b_1})$

Under this choice of C, we also have

$C'BP + b_1'P = 0$ and

$$C'BC + C'\underset{\sim}{b_1} = \underset{\sim}{b_1}'(B^{-1})'BB^{-1}\underset{\sim}{b} - \underset{\sim}{b_1}'(B^{-1})'\underset{\sim}{b_1}$$

$$= \underset{\sim}{b_1}'(B^{-1})'\underset{\sim}{b_1} - \underset{\sim}{b_1}'(B^{-1})'\underset{\sim}{b_1}$$

$$= 0$$

∴ Product matrix $= \begin{pmatrix} I_m & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{b_1}'C + k \end{pmatrix}$

Taking determinant in both sides,

$$|P'|\,|A|\,|P| = |I_m|\,(\underset{\sim}{b_1}'C + k)$$

or, $|A|\{|P|\}^2 = (\underset{\sim}{b_1}'C + k)$

Now by assumption, $|A| > 0$ and $|P| \neq 0$.

So, $(\underset{\sim}{b_1}'C + k) > 0$

Let $\underset{\sim}{b_1}'C + k = \beta^2$

$$\therefore \begin{pmatrix} P' & 0 \\ C' & 1 \end{pmatrix} A \begin{pmatrix} P & C \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I_m & \underset{\sim}{0} \\ \underset{\sim}{0} & \beta^2 \end{pmatrix}$$

Premultiplying and postmultiplying both sides by

$$\begin{pmatrix} I_m & \underset{\sim}{0} \\ \underset{\sim}{0} & \beta^{-1} \end{pmatrix}$$

$$\begin{pmatrix} I_m & \underset{\sim}{0} \\ \underset{\sim}{0} & \beta^{-1} \end{pmatrix}\begin{pmatrix} P' & 0 \\ C' & 1 \end{pmatrix} A \begin{pmatrix} P & C \\ 0 & 1 \end{pmatrix}\begin{pmatrix} I_m & \underset{\sim}{0} \\ \underset{\sim}{0} & \beta^{-1} \end{pmatrix} = I_{m+1}$$

or, $Q'AQ = I_{m+1}$, where $Q = \begin{pmatrix} P & C \\ 0 & 1 \end{pmatrix}\begin{pmatrix} I_m & \underset{\sim}{0} \\ \underset{\sim}{0} & \beta^{-1} \end{pmatrix}$

Since A is congruent to I_{m+1}, therefore A is p.d., i.e., the corresponding q.f. is p.d. Hence, by induction result follows.

$\boxed{\text{C.U.}}$

• Corollary : The necessary and sufficient condition for a q.f. $Q(\underset{\sim}{x}) = \underset{\sim}{x}'A\underset{\sim}{x}$ to be negative definite is that the leading principal minors of A starting from the first are alternatively negative and positive, i.e.

$$a_{11} < 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} < 0, \ldots\ldots, (-1)^n|A| \gtrless 0$$ according as n is even or odd.

Proof :→ $\underset{\sim}{x}'A\underset{\sim}{x}$ is negative definite if and only if $\underset{\sim}{x}'(-A)\underset{\sim}{x}$ is positive definite. But by virtue of the above theorem, $\underset{\sim}{x}'(-A)\underset{\sim}{x}$ is positive definite if and only if

$$-a_{11} > 0, \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0, \begin{vmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{vmatrix} > 0, \ldots\ldots$$

Thus the necessary and sufficient conditions for $x'(-A)\underset{\sim}{x}$ to be p.d. or, equivalently, $\underset{\sim}{x}'A\underset{\sim}{x}$ to be n.d. is the following:

$$-a_{11} > 0, \text{ i.e. } a_{11} < 0,$$

$$(-1)^2 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \text{ i.e. } \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0,$$

$$(-1)^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0, \text{ i.e. } \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} < 0, \ldots\ldots$$

$(-1)^n|A| \gtrless 0$ according as n is even or odd.

Hence the proof.

Ex.1. Consider the quadratic form: $4x_1^2 - 10x_1x_2 + 7x_2^2$.

Here, $A = \begin{pmatrix} 4 & -5 \\ -5 & 7 \end{pmatrix}$

As such, $a_{11} = 4 > 0$, $|A| = 3 > 0$.

Consequently, the quadratic form is positive definite (p.d.)

Ex.2. Consider now the quadratic form: $4x_1^2 - 10x_1x_2 + 3x_2^2$.

Here, $A = \begin{pmatrix} 4 & -5 \\ -5 & 3 \end{pmatrix}$

As such, $a_{11} = 4 > 0$, $|A| = -13 < 0$,

so, that this quadratic form is neither positive definite (p.d.) nor negative definite (n.d.).

Orthogonal Matrix :—

A matrix A is said to be an orthogonal matrix if

$$A^TA = AA^T = I$$

i.e. $A^T = A^{-1}$.

Rewrite $\underset{n\times n}{A} = (\underset{\sim}{\alpha_1}, \underset{\sim}{\alpha_2}, \dots, \underset{\sim}{\alpha_n})$,

$\therefore A^TA = I$

then, $\Rightarrow \begin{pmatrix} \underset{\sim}{\alpha_1}' \\ \underset{\sim}{\alpha_2}' \\ \vdots \\ \underset{\sim}{\alpha_n}' \end{pmatrix} (\underset{\sim}{\alpha_1}, \underset{\sim}{\alpha_2}, \dots, \underset{\sim}{\alpha_n}) = I$

i.e. $((\alpha_i'\alpha_j)) = I$

$\Leftrightarrow \underset{\sim}{\alpha_i}'\underset{\sim}{\alpha_j} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$

i.e. $(\underset{\sim}{\alpha_1}\, \underset{\sim}{\alpha_2}, \dots, \underset{\sim}{\alpha_n})'$ are mutually orthogonal vectors.

Again if $\underset{n\times n}{A} = \begin{pmatrix} \underset{\sim}{\beta_1}' \\ \underset{\sim}{\beta_2}' \\ \vdots \\ \underset{\sim}{\beta_n}' \end{pmatrix}$ then $A^T = (\underset{\sim}{\beta_1}, \underset{\sim}{\beta_2}, \dots, \underset{\sim}{\beta_n})$

$$AA^T = \begin{pmatrix} \underset{\sim}{\beta_1}' \\ \vdots \\ \underset{\sim}{\beta_n}' \end{pmatrix} (\underset{\sim}{\beta_1}, \underset{\sim}{\beta_2}, \dots, \underset{\sim}{\beta_n})$$

$= I$

i.e. $((\underset{\sim}{\beta_i}'\underset{\sim}{\beta_j})) = I$

$\Rightarrow \underset{\sim}{\beta_i}'\underset{\sim}{\beta_j} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$

i.e. $(\underset{\sim}{\beta_1}, \dots, \underset{\sim}{\beta_n})$ are mutually orthogonal vectors.

Ex:→ I, $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$, $\begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \dots & \dots & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \dots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \dots & \dots & \frac{-(n-1)}{\sqrt{n(n-1)}} \end{bmatrix}_{n\times n}$, etc.

C.U.

Result: → Show that the following matrix is orthogonal $(I-S)(I+S)^{-1}$ when S is a skew-symmetric matrix, and $(I-S)$ is non-singular.

We have to show –
i) $A'A = I$ or,
ii) $AA' = I$

Proof: →
$$\{(I-S)(I+S)^{-1}\}^T \{(I-S)(I+S)^{-1}\}$$
$$= \{(I+S)^{-1}\}^T (I-S)^T (I-S)(I+S)^{-1}$$
$$= \{(I+S)^T\}^{-1} (I-S')(I-S)(I+S)^{-1}$$
$$= (I+S')^{-1}(I+S)(I-S)(I+S)^{-1}$$
$$= (I-S)^{-1}(I+S)(I-S)(I+S)^{-1}$$

[Note that, $(I+S)(I-S) = I-S+S-S^2 = I-S^2 = (I-S)(I+S)$] — (1)

$$= \underbrace{(I-S)^{-1}(I-S)}\underbrace{(I+S)(I+S)^{-1}}$$
$$= I.I$$
$$= I$$

OR

$$(I-S)(I+S)^{-1}\{(I-S)(I+S)^{-1}\}'$$
$$= (I-S)(I+S)^{-1}\{(I+S)^{-1}\}'(I-S')$$
$$= (I-S)(I+S)^{-1}(I+S')^{-1}(I-S')$$
$$= (I-S)(I+S)^{-1}(I-S)^{-1}(I+S)$$
$$= (I-S)[(I+S)(I-S)]^{-1}(I+S)$$

[Using (1)]

$$= (I-S)(I-S)^{-1}(I+S)^{-1}(I+S)$$
$$= II$$
$$= I.$$

1) If $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

Show that the matrix $((\delta_{ij} + x_i x_j))$ is p.d., $i, j = 1(1)p$. (ISI)

Soln →

$$\text{Let } A = ((\delta_{ij} + x_i x_j))$$
$$= I_n + x x^T \quad , \text{ where, } x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Consider an associated Q.F.

$$u'Au = u'(I_n + xx^T)u$$
$$= u' I_n u + u' x x' u$$
$$= u'u + (x'u)'(x'u)$$

For $u \neq 0$, $\quad u'u = \sum_{i=1}^{n} u_i^2 > 0$

$$\Rightarrow \quad u'Au > 0$$

Hence, $u'Au$ is positive definite,

i.e. $A = ((\delta_{ij} + x_i x_j))$ is p.d.

Quadratic Form

- N.n.d.
  - p.d.
  - p.s.d.
- N.p.d.
  - n.d.
  - N.s.d.
- Indefinite

Remark: →

1) If $x'Ax$ is n.n.d./p.d./p.s.d. then $x'(-A)x$ is n.p.d./n.d./n.s.d.

2) If $x'Ax$ is n.n.d. or n.p.d. or Indefinite, then A is said to be n.n.d. or n.p.d. or indefinite. If a quadratic form has some definiteness, then the associated matrix has the same definiteness.

EXAMPLE:

1) $x_1^2 + 2x_2^2$ is p.d.
2) $x_1^2 + (x_1 - x_2)^2$ is p.d.
3) $Q(x_1, x_2) = (x_1 - x_2)^2$ is p.s.d.
4) $Q(x_1, x_2) = x_1^2 + x_1 x_2$ is indefinite.
5) $Q(x_1, x_2) = x_1^2 - x_2^2$ is indefinite.

Theorem:— A non-singular transformation of the variables changes a quadratic form into another q.f. with the same definiteness.

Proof:→

Consider a quadratic form $\underset{\sim}{x}' A \underset{\sim}{x}$ and an n.s. transformation

$\underset{\sim}{y} = P\underset{\sim}{x} \Rightarrow \underset{\sim}{x} = P^{-1}\underset{\sim}{y}$, P is n.s.

Note that $\underset{\sim}{x}' A \underset{\sim}{x} = \underset{\sim}{y}' \{(P^{-1})' A (P^{-1})\} \underset{\sim}{y} = \underset{\sim}{y}' B \underset{\sim}{y}$

Clearly, the range sets of $\underset{\sim}{x}' A \underset{\sim}{x}$ and $\underset{\sim}{y}' B \underset{\sim}{y}$ are identical.

Note that $\underset{\sim}{x} = \underset{\sim}{0}$

$\Leftrightarrow \underset{\sim}{y} = 0$, since P is n.s.

$[\because \underset{\sim}{y} = P\underset{\sim}{x} \Rightarrow \underset{\sim}{x} = P^{-1}\underset{\sim}{y}]$

Now, $\underset{\sim}{x}' A \underset{\sim}{x} = 0$

or, $\underset{\sim}{x} \neq 0 \Rightarrow \underset{\sim}{y} \neq 0$ as P is n.s.

[That is, $\underset{\sim}{x}' A \underset{\sim}{x} = 0$ iff $\underset{\sim}{x} = \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{y}' B \underset{\sim}{x} = 0$ iff $\underset{\sim}{y} = \underset{\sim}{0}$

or, $\underset{\sim}{x}' A \underset{\sim}{x} > 0$ for all $\underset{\sim}{x} \neq 0$

$\Rightarrow \underset{\sim}{y}' B \underset{\sim}{x} > 0 \quad \forall \quad \underset{\sim}{y} \neq 0$]

Problem:- Find the values of p for which $(1-p)I_n + p\underset{\sim}{1}\underset{\sim}{1}'$ is p.d.?

Soln.

$(1-p)I_n + p\underset{\sim}{1}\underset{\sim}{1}'$

$$= \begin{bmatrix} 1 & p & p & \cdots & p \\ p & 1 & p & \cdots & p \\ \vdots & \vdots & \vdots & & \vdots \\ p & p & p & \cdots & 1 \end{bmatrix}$$

The k^{th} order principal minor is

$$\Delta_k = \begin{bmatrix} 1 & p & p & \cdots & p \\ p & 1 & p & \cdots & p \\ \vdots & \vdots & \vdots & & \vdots \\ p & p & p & \cdots & 1 \end{bmatrix}_{k\times k} = (1+\overline{k-1}\,p)(1-p)^{k-1}$$

$(1-p)I_n + p\underset{\sim}{1}\underset{\sim}{1}'$ is p.d.

iff $\Delta_k > 0$, $\forall$ $k = 1(1)n$.

iff $\Delta_1 = 1 > 0$, $\Delta_2 = (1+p)(1-p) > 0$

$\Delta_k = (1+\overline{k-1}\,p)(1-p)^{k-1} > 0$ $\forall$ $k = 3(1)n$.

iff $-1 < p < 1$, $(1+\overline{k-1}\,p) > 0$ $\forall$ $k = 3(1)n$

iff $-\dfrac{1}{k-1} < p < 1$, $k = 2(1)n$.

iff $-\dfrac{1}{n-1} < p < 1$.

Theorem:→ A is n.n.d., iff $A = B'B$, for some B.

Proof:→

If part:— Let, $A = B'B$

Then $\underset{\sim}{x}' A \underset{\sim}{x} = \underset{\sim}{x}'(B'B)\underset{\sim}{x}$

$\Rightarrow (B\underset{\sim}{x})'(B\underset{\sim}{x}) = y'y \geq 0$, where $y = B\underset{\sim}{x}$

$\therefore \underset{\sim}{x}' A \underset{\sim}{x} \geq 0 \ \forall x$

$\Rightarrow$ A is n.n.d..

Only if part:— Let A be n.n.d. then $\exists$ an n.s. matrix C $\ni$ $C'AC = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, $r = \text{rank}(A)$.

$$\Rightarrow A = (C')^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} (C)^{-1}$$

$$= \left\{ \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} C^{-1} \right\}^T \left\{ \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} C^{-1} \right\}$$

$$= B^T B \text{, where } B = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} C^{-1},$$

with $\text{rank}(B) = r$.

Corollary:— A is p.d. iff $A = B'B$, for some non-singular B.

Proof:→ **If part:—** Let $A = B'B$, B is n.s.

Then, $\underset{\sim}{x}' A \underset{\sim}{x} = (B\underset{\sim}{x})^T (B\underset{\sim}{x}) = y^T y \geq 0 \ \forall \underset{\sim}{y}$

Now, $\underset{\sim}{x}' A \underset{\sim}{x} = 0$

$\Leftrightarrow \hat{y}'\hat{y} = 0$

$\Leftrightarrow \underset{\sim}{y} = 0 \Rightarrow B\underset{\sim}{x} = \underset{\sim}{0} \Rightarrow \underset{\sim}{x} = B^{-1}\underset{\sim}{0} = \underset{\sim}{0}$

Hence, A is p.d. [A is p.d. $\Rightarrow |A| > 0$, $\Rightarrow r(A_{n \times n}) = n$]

Only if part:— Let A is p.d.

Then $\exists$ a n.s. matrix C $\ni$ $C'AC = I_n$

$\Rightarrow A = (C^{-1})' C^{-1} = B'B$, where $B = C^{-1}$ is n.s.

— X —

Example:—

1) If $\underset{\sim}{x}' A \underset{\sim}{x}$ is p.s.d. then S.T. $|A| = 0$.

Soln. → A is p.s.d.

$\therefore \exists$ an n.s. mts $C \ni C^T A C = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, where

$r = \text{rank}(A) < n$.

$$A = \left\{ \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} C^{-1} \right\}^T \left\{ \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} C^{-1} \right\}$$

$\therefore A = B^T B$, where $B = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} C^{-1}$

with rank$(B) = r < n$

$\therefore |A| = |B^T||B| = |B|^2 = 0$ $\left[\because r(\underset{n \times n}{B}) < n \Rightarrow |B| = 0 \right]$

2) If A is p.d.. Then S.T. A^{-1} is also p.d.

Soln. → A is p.d.

$\Leftrightarrow \underset{\sim}{x}' A \underset{\sim}{x} > 0 \quad \forall \underset{\sim}{x} \neq \underset{\sim}{0}$.

Also, $|A| > 0$, i.e. A is n.s.

Consider the non-singular transformation:

$\underset{\sim}{y} = A \underset{\sim}{x}$

Now, $\underset{\sim}{x}' A \underset{\sim}{x} > 0 \quad \forall \underset{\sim}{x} \neq \underset{\sim}{0}$

$\Rightarrow (A^{-1}\underset{\sim}{y})' A (A^{-1}\underset{\sim}{y}) > 0 \quad \forall A^{-1}\underset{\sim}{y} \neq 0$

$\Rightarrow \underset{\sim}{y}' (A')^{-1} A A^{-1} \underset{\sim}{y} > 0 \quad \forall \underset{\sim}{y} \neq 0$ as A is n.s.

$\Rightarrow \underset{\sim}{y}' A^{-1} \underset{\sim}{y} > 0, \quad \forall \underset{\sim}{y} \neq \underset{\sim}{0}$ as A is symmetric(WL

$\therefore A^T = A$.

$\Rightarrow A^{-1}$ is p.d.

Problem:— 1. Reduce the equation $3x^2+5y^2+3z^2+2xy+2yz+2zx=1$ into canonical form.

Solution:-
$$A=\begin{pmatrix}3&1&1\\1&5&1\\1&1&3\end{pmatrix},\quad \underset{\sim}{X}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$\therefore \underset{\sim}{X}^T A \underset{\sim}{X}=1.$

The ch. equation of A is
$$\begin{vmatrix}3-x&1&1\\1&5-x&1\\1&1&3-x\end{vmatrix}=0$$

or, $(x-2)(x-3)(x-6)=0.$

$\therefore$ The eigen values of A are $2, 3, 6$.

The eigen vector corresponding to the eigen value 2 are $c\begin{pmatrix}1\\0\\-1\end{pmatrix};\ c\neq 0.$

The " " " " " 3 " $c\begin{pmatrix}1\\-1\\1\end{pmatrix},\ c\neq 0.$

The " " " " " " 6 " $c\begin{pmatrix}1\\2\\1\end{pmatrix},\ c\neq 0.$

Let $\underset{\sim}{\alpha}=\begin{pmatrix}1\\0\\-1\end{pmatrix},\ \underset{\sim}{\beta}=\begin{pmatrix}1\\-1\\1\end{pmatrix},\ \underset{\sim}{\gamma}=\begin{pmatrix}1\\2\\1\end{pmatrix}$. Then the set $\{\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}\}$ is an orthogonal set. The orthonormal set of eigen vectors is
$$\left\{\frac{1}{\sqrt2}\begin{pmatrix}1\\0\\-1\end{pmatrix},\ \frac{1}{\sqrt3}\begin{pmatrix}1\\-1\\1\end{pmatrix},\ \frac{1}{\sqrt6}\begin{pmatrix}1\\2\\1\end{pmatrix}\right\}.$$

Let $P=\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt3}&\frac{1}{\sqrt6}\\0&-\frac{1}{\sqrt3}&\frac{2}{\sqrt6}\\-\frac{1}{\sqrt2}&\frac{1}{\sqrt3}&\frac{1}{\sqrt6}\end{pmatrix}$. Then P is an orthogonal mtx.

Let us apply the orthogonal transformation $\underset{\sim}{X}=P\underset{\sim}{X}'$, where $\underset{\sim}{X}'=\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$.

Then the equation transforms to $(X')^T(P^TAP)X'=1.$

$P^TAP=(P^{-1}AP)$ is a diagonal matrix D which has the same eigen value as those of A.

$$AP=\begin{pmatrix}3&1&1\\1&5&1\\1&1&3\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt3}&\frac{1}{\sqrt6}\\0&-\frac{1}{\sqrt3}&\frac{2}{\sqrt6}\\-\frac{1}{\sqrt2}&\frac{1}{\sqrt3}&\frac{1}{\sqrt6}\end{pmatrix}=\begin{pmatrix}2\cdot\frac{1}{\sqrt2}&3\cdot\frac{1}{\sqrt3}&6\cdot\frac{1}{\sqrt6}\\2\cdot0&3\cdot-\frac{1}{\sqrt3}&6\cdot\frac{2}{\sqrt6}\\2\cdot-\frac{1}{\sqrt2}&3\cdot\frac{1}{\sqrt3}&6\cdot\frac{1}{\sqrt6}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt3}&\frac{1}{\sqrt6}\\0&-\frac{1}{\sqrt3}&\frac{2}{\sqrt6}\\-\frac{1}{\sqrt2}&\frac{1}{\sqrt3}&\frac{1}{\sqrt6}\end{pmatrix}\begin{pmatrix}2&0&0\\0&3&0\\0&0&6\end{pmatrix}$$

$$=PD,\ \text{where } D=\begin{pmatrix}2&0&0\\0&3&0\\0&0&6\end{pmatrix}.$$

So, $P^{-1}AP=D=\begin{pmatrix}2&0&0\\0&3&0\\0&0&6\end{pmatrix}$

The equation transforms to $(X')^TDX'=1$

i.e. to $2x'^2+3y'^2+6z'^2=1.$

2. Reduce the equation $7x^2 - 2xy + 7y^2 - 16x + 16y - 8 = 0$ into canonical form and determine the nature of the conic.

Solution:— Let $A = \begin{pmatrix} 7 & -1 \\ -1 & 7 \end{pmatrix}$, $B = \begin{pmatrix} -16 & 16 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

Then the equation takes the form $X^T A X + BX - 8I_1 = 0$

The eigen values of A are $8, 6$.

The eigen vectors corresponding the eigen values 8 and 6 are $c\begin{pmatrix} 1 \\ -1 \end{pmatrix}, c \neq 0$; $d\begin{pmatrix} 1 \\ 1 \end{pmatrix}, d \neq 0$, respectively.

The orthonormal set of eigen vectors is $\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$.

Let $P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Then P is an orthogonal matrix.

$P^T A P = (P^{-1}AP)$ is a diagonal matrix which has the same eigenvalues as those of A.

So, $P^T A P = \begin{pmatrix} 8 & 0 \\ 0 & 6 \end{pmatrix}$. $BP = \begin{pmatrix} -16\sqrt{2} & 0 \end{pmatrix}$.

By the orthogonal transformation $X = PX'$, where $X' = \begin{pmatrix} x' \\ y' \end{pmatrix}$, the equation transforms to $8x'^2 + 6y'^2 - 16\sqrt{2}x' - 8 = 0$

or, $8(x' - \sqrt{2})^2 + 6y'^2 = 24$.

Let us apply the translation $x'' = x' - \sqrt{2}$,
$y'' = y'$

$\therefore$ The equation transforms to $8x''^2 + 6y''^2 = 24$.

The canonical form is $8x^2 + 6y^2 = 24$.

The equation represents an ellipse.

EIGEN VALUES & EIGEN VECTORS

■ Differentiation with respect to a vector :→

Let $f \equiv f(x_1, x_2, \ldots, x_n)$ be a function of the n real variables $x_1, x_2, \ldots, x_n$.

Suppose further that the partial derivatives $\frac{\partial f}{\partial x_i}$ $(i = 1(1)n)$ exists for $x_1 = a_1, x_2 = a_2, \ldots, x_n = a_n$.

Then by the vector of partial derivatives, denoted by

$$\left[\frac{\partial f}{\partial \underset{\sim}{x}}\right]_{\underset{\sim}{x} = \underset{\sim}{a}} = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right]'_{\substack{x_1 = a_1 \\ x_2 = a_2 \\ \vdots \\ x_n = a_n}}$$

E.g. Suppose $f(\underset{\sim}{x}) = Q(\underset{\sim}{x}) = \underset{\sim}{x}' A \underset{\sim}{x}$, where A is a symmetric matrix, $A = ((a_{ij}))_{n \times n}$

$$\begin{aligned} f(\underset{\sim}{x}) = &\left(a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2\right) \\ &+ 2x_1(a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n) \\ &+ 2x_2(a_{23}x_3 + a_{24}x_4 + \cdots + a_{2n}x_n) \\ &+ 2x_3(a_{34}x_4 + a_{35}x_5 + \cdots + a_{3n}x_n) \\ &+ \\ &+ 2x_{n-1}(a_{n-1\,n}x_n) \end{aligned}$$

We have, $\frac{\partial f}{\partial x_1} = 2a_{11}x_1 + 2a_{12}x_2 + \cdots + 2a_{1n}x_n$

$$= 2\sum_{j=1}^{n} a_{1j}x_j = 2\underset{\sim}{a}_1' \underset{\sim}{x}$$

In the same way,

$$\frac{\partial f}{\partial x_2} = 2\sum_{j=1}^{n} a_{2j}x_j = 2\underset{\sim}{a}_2' \underset{\sim}{x}$$

$$\vdots$$

$$\frac{\partial f}{\partial x_n} = 2\sum_{j=1}^{n} a_{nj}x_j = 2\underset{\sim}{a}_n' \underset{\sim}{x}$$

$$\therefore \left[\frac{\partial f}{\partial \underset{\sim}{x}}\right]_{\underset{\sim}{x} = \underset{\sim}{a}} = 2\left(\underset{\sim}{a}_1'\underset{\sim}{x} + \underset{\sim}{a}_2'\underset{\sim}{x} \ldots\ldots \underset{\sim}{a}_n'\underset{\sim}{x}\right)'$$

$$= 2\begin{pmatrix} \underset{\sim}{a}_1' \\ \underset{\sim}{a}_2' \\ \vdots \\ \underset{\sim}{a}_n' \end{pmatrix} \underset{\sim}{x}$$

Maxima and minima of a real function : →

Suppose the function f has partial derivatives of both first and second orders in the neighbourhood of the point $\underset{\sim}{x} = a$. Let us write

$$g_i = \frac{\partial f}{\partial x_i}\Big|_{\underset{\sim}{x}=\underset{\sim}{a}} \quad , \quad g_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_{\underset{\sim}{x}=\underset{\sim}{a}}$$

Define,
$$g = [g_1, g_2, \cdots, g_n] = \frac{\partial f}{\partial \underset{\sim}{x}}\Big|_{\underset{\sim}{x}=\underset{\sim}{a}}$$

$$G = ((g_{ij}))$$

so that G is a symmetric matrix.

By Taylor's series for f at $x = a+h$, where $h = [h_1, h_2, \cdots h_n]$ we may, for sufficiently small h_i, write

$$f(a+h) = f(a) + h'g + \frac{1}{2!} h'Gh + R,$$

where R is a reminder term $\ni R \to 0$ as $h \to 0$.

A necessary condition for f to have a maximum or a minimum at $\underset{\sim}{a}$ is that $g = 0$, since this ensures that f is stationary at a. Whether this is a maximum or a minimum or neither a maximum nor a minimum depends on the quadratic form $h'Gh$. One may then assert:

i) If G is a positive definite matrix, then f is a minimum at $\underset{\sim}{a}$.

ii) If G is a negative definite, then f is a maximum at $\underset{\sim}{a}$.

iii) If G is indefinite, then f is neither a maximum nor a minimum at $\underset{\sim}{a}$.

Method of Lagrangian Multiplier →

$f(x)$
$g(\underset{\sim}{x}) = c$

$$Z = f(x_1, x_2, \cdots, x_n) + \lambda\left(g(x_1, x_2, \cdots, x_n) - c\right)$$

$$\frac{\partial Z}{\partial x_1} = 0, \cdots\cdots, \frac{\partial Z}{\partial x_n} = 0, \frac{\partial Z}{\partial \lambda} = 0$$

$\underset{\sim}{x}'A\underset{\sim}{x}$

$\underset{\sim}{x}'\underset{\sim}{x} = 1.$

$$g = \underset{\sim}{x}'A\underset{\sim}{x} - \lambda\underset{\sim}{x}'\underset{\sim}{x}$$

$$\therefore g = x'(A - \lambda I)\underset{\sim}{x}.$$

And, $$\frac{\partial g}{\partial \underset{\sim}{x}} = 2(A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$$

if $\underset{\sim}{x} \neq \underset{\sim}{0}$, then this implies rank $(A - \lambda I) < n$

$$\therefore |A - \lambda I| = 0.$$

An optimization Problem :→ Suppose $\underset{\sim}{x}'A\underset{\sim}{x}$ is a quadratic form in the n variables $x_1, x_2, \ldots, x_n$ which is to be maximized or minimized subject to the condition $\underset{\sim}{x}'\underset{\sim}{x} = 1$.

Introducing a Lagrange multiplier λ, we are then led to consider the quadratic form

$$\underset{\sim}{x}'A\underset{\sim}{x} - \lambda\underset{\sim}{x}'\underset{\sim}{x} = \underset{\sim}{x}'(A-\lambda I)\underset{\sim}{x} \quad \text{——— (1)}$$

To get a maximum or a minimum, we equate.

$$\frac{\partial}{\partial x_1}\left(\underset{\sim}{x}'(A-\lambda I)\underset{\sim}{x}\right) = 0, \ldots\ldots, \frac{\partial}{\partial x_n}\left(\underset{\sim}{x}'(A-\lambda I)\underset{\sim}{x}\right) = 0$$

This yields $(A-\lambda I)\underset{\sim}{x} = 0$ ——— (2)

One solution of the system of linear equations (2) is, of course, $\underset{\sim}{x} = 0$. But we are not interested in this trivial solution since it does not meet the condition $\underset{\sim}{x}'\underset{\sim}{x} = 1$. Here, a non-trivial solution exists iff rank $(A-\lambda I) < n$.

i.e. $|A-\lambda I| = 0$ ——— (3)

*** **Definitions :→** The values of the scalar parameter λ satisfying $|A-\lambda I_n| = 0$ are called the eigen values or latent roots or eigen roots or characteristic roots of the square matrix $(A_{n\times n})$.

Any non-null vector satisfying $(A-\lambda I)\underset{\sim}{x} = 0$ corresponding to an eigen value is called an eigen vector or latent vector or characteristic vector of A.

In other words, let λ_0 be an eigen value of A and x_0 be a non-null vector satisfying $(A-\lambda_0 I_n)\underset{\sim}{x} = 0$. We call this x_0 as an eigen ~~value~~ vector of A corresponding to λ_0.

☐ Let λ be a characteristic root of $A_{n\times n}$. If r be the rank of $(A-\lambda I)$, then there are $(n-r)$ linearly independent characteristic vectors of A corresponding to this λ.

* Characteristic Polynomial and Equations and Space :-

The determinant $|A-\lambda I|$, which has λ in each diagonal element, is a polynomial in λ, called the characteristic polynomial for A. The heighest power of λ, of course, comes from the product of the n diagonal elements, thus the term with the heighest power of λ is $(-\lambda)^n$, so that the characteristic polynomial of a square matrix of order n is a polynomial of degree n. Putting $|A-\lambda I| = f(\lambda)$, we may, therefore write

$$f(\lambda) = b_0 + b_1(-\lambda) + b_2(-\lambda)^2 + \cdots + b_{n-1}(-\lambda)^{n-1} + (-\lambda)^n \quad \text{---} (*)$$

From a fundamental theorem of algebra, we know that the equation $(*) = 0$,

$$\text{i.e. } |A-\lambda I| = 0$$

$$\text{or, } \quad f(\lambda) = 0,$$

has n roots, say $\lambda_1, \lambda_2, \ldots, \lambda_n$. These roots, i.e. eigen values, need not be all distinct, but if each distinct root is counted a number of times equal to its multiplicity, then we have in total n roots. Further, each distinct root may be either real or complex and the complex roots must occur in pairs. The equation is called the characteristic equation for A.

For any value of λ different from the roots of the characteristic equation, the only solution of $(A-\lambda I)\underset{\sim}{x} = \underset{\sim}{0}$ is $\underset{\sim}{x} = \underset{\sim}{0}$. On the other hand, if λ is put equal to any of the eigen values, say to λ_i, then $|A-\lambda_i I| = 0$ and so there is at least one $\underset{\sim}{x} \neq \underset{\sim}{0}$ which satisfies $(A-\lambda I)\underset{\sim}{x} = 0$ i.e. there exists at least one eigen vector. The maximum number of independent vectors satisfying

$$(A-\lambda_i I)\underset{\sim}{x} = 0$$

is the nullity of $(A-\lambda_i I)$. In other words, the highest number of independent eigenvectors corresponding to the eigen value λ_i is the same as the nullity of $(A-\lambda_i I)$, i.e. $n-r$, where r is the rank of the matrix A.

Every non-zero linear combination of this (n-r) independent characteristic vectors is also a characteristic vector of A corresponding to this λ. The set of all these linear combinations together with the null vector is a subspace. The subspace is called the characteristic space of A related to the characteristic root λ. It may be noted that the characteristic space of A corresponding to the characteristic root λ is actually the null space of $(A-\lambda I)$.

* What do you mean by a characteristic value problem?

A problem which arises frequently in application of linear algebra, is that of finding value of a scalar λ for which $\exists$ non-null vectors $\underset{\sim}{x}$ satisfying

$A\underset{\sim}{x} = \lambda \underset{\sim}{x}$, A be a square matrix, i.e., we verify wheather for a non-null vector $\underset{\sim}{x}$ the matrix A operating on it, produces a scalar multiple of $\underset{\sim}{x}$. Such a problem is known as a characteristic value problem. A well known example of an eigen value problem emarges from a problem of constrained optimization. Suppose we are to maximize the quadratic form

$Q(\underset{\sim}{x}) = \underset{\sim}{x}'A\underset{\sim}{x}$, with respect to the variables $x_1, x_2, \ldots, x_n$

Subject to the restriction that the length of the vector remain constant $\sum_i x_i^2 = c$

Then by the Lagranges method of optimisation, we maximize,

$$f(x) = \underset{\sim}{x}'A\underset{\sim}{x} + \lambda(c - \underset{\sim}{x}'\underset{\sim}{x})$$

w.r.t. $\underset{\sim}{x}$, when λ is a Lagranges multiplier.

Clearly $\frac{\partial}{\partial x} f(\underset{\sim}{x}) = 2A\underset{\sim}{x} - 2\lambda\underset{\sim}{x} = 0$

i.e. $A\underset{\sim}{x} = \lambda\underset{\sim}{x}$

Again for a given λ we search for a non-null vector $\underset{\sim}{x}$ satisfying $A\underset{\sim}{x} = \lambda\underset{\sim}{x}$

is equivalent to obtain a non-null vector from the null space of $(A-\lambda I)$.

We have a non-trivial solution to

$$(A-\lambda I)\underset{\sim}{x} = 0$$

if $R(A-\lambda I) < n$ [assuming A is of order $n \times n$]

i.e. there will be a non-null $\underset{\sim}{x}$ satisfying $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$

if $|A-\lambda I| = 0$

Let, $A = ((a_{ij}))_{n \times n}$.

Then $|A - \lambda I_n| = 0$

$$\equiv \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \quad \text{——— (1)}$$

Note that LHS of (1) is a polynomial in λ of degree n since the highest order term in λ comes out to be $(-\lambda)^n$. Thus solution to the equation $|A-\lambda I| = 0$ has n-roots on λ, namely $\lambda_1, \lambda_2, \ldots, \lambda_n$, these roots are known as characteristic ~~values~~ roots or, eigen values or, latent roots of matrix A, and a non-null vector satisfying $A\underset{\sim}{x} = \lambda_i \underset{\sim}{x}$ is ~~the~~ called the eigen vector, clearly the maximum number of LIN vectors $\underset{\sim}{x}$ satisfying $A\underset{\sim}{x} = \lambda_i \underset{\sim}{x}$ is the nullity of $(A - \lambda_i I)$, these roots need not to be distinct but if a root is counted as many time as its multiplicity, we have n-roots in all.

Again the eigen values may real or complexed. As $|A - \lambda I|$ is a polynomial in $(-\lambda)$ of degree n, it can be written as

$$|A - \lambda I| = b_0 + b_1(-\lambda) + \cdots + b_{n-1}(-\lambda)^{n-1} + (-\lambda)^n$$

$$= (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots\cdots (\lambda_n - \lambda)$$

$$\Rightarrow b_0 = \prod_i \lambda_i$$

$$b_1 = \sum_{i_1 > i_2 > \cdots > i_{n-1}} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_{n-1}}$$

$$\vdots$$

$$b_{n-2} = \sum_{i > j} \lambda_i \lambda_j$$

$$b_{n-1} = \sum_i \lambda_i$$

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Example :→ Consider the matrix

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$$

The characteristic equation of A is

$$\begin{vmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

or, $(4-\lambda)(1-\lambda) - 4 = 0$

or, $\lambda(\lambda - 5) = 0$.

As such, the eigen values are $\lambda_1 = 0, \lambda_2 = 5$.

To determine the eigenvectors $\underset{\sim}{x}$ corresponding to λ_1, we have to solve the linear homogeneous equations

$$(A - \lambda_1 I)\underset{\sim}{x} = \underset{\sim}{0},$$

i.e. the equations

$$4x_1 + 2x_2 = 0,$$

$$\Rightarrow 2x_1 + x_2 = 0.$$

these gives $x_2 = -2x_1$

Take $x_1 = \alpha \ (\neq 0)$, so, $x_2 = -2\alpha$

Any vector of the form $\begin{pmatrix} \alpha \\ -2\alpha \end{pmatrix}$, $\alpha \neq 0$ is an eigen vector corresponding to $\lambda_1 = 0$.

To determine the eigen vector corresponding to $\lambda_2 \ (=5)$, we have to solve the following linear homogeneous eqn.

$$(A - \lambda_2 I)\underset{\sim}{x} = \underset{\sim}{0}.$$

$$\Rightarrow \left\{ \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \right\} \underset{\sim}{x} = \underset{\sim}{0}$$

$$\Rightarrow -x_1 + 2x_2 = 0$$

i.e, $2x_1 - 4x_2 = 0$

or, $x_1 = 2x_2$

Take $x_1 = \beta$, $x_2 = \beta/2$.

Any vector of the form $\begin{pmatrix} \beta \\ \beta/2 \end{pmatrix}$, $\beta \neq 0$ is an eigen vector corresponding to an eigen value $\lambda_2 = 5$.

*(1.)

Problem :→ Obtain the eigen values and hence the eigen vectors of the matrix $A = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$.

Soln. →

$$\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} 2-\lambda & \sqrt{2} \\ \sqrt{2} & 1-\lambda \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underset{\sim}{0}$$

$$\therefore \begin{vmatrix} 2-\lambda & \sqrt{2} \\ \sqrt{2} & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(1-\lambda) - 2 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda = 0$$

$$\therefore \lambda(\lambda - 3) = 0$$

i.e. $\lambda = 0, 3$.

when $\lambda = 0$, $\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \sqrt{2}x_1 + x_2 = 0 \quad \text{—— (1)}$$

$$\& \quad x_1^2 + x_2^2 = 1 \quad \text{—— (2)}$$

[as eigen vectors are of unit length]

Let $x_2 = a$,

$$\therefore x_1 = -\frac{a}{\sqrt{2}} \quad [\text{from (1)}]$$

Putting these values in the eqn. (2),

$$\left(\frac{-a}{\sqrt{2}}\right)^2 + a^2 = 1$$

$$\Rightarrow a = \sqrt{\frac{2}{3}}$$

$$\therefore x_2 = \sqrt{\frac{2}{3}}, \quad x_1 = -\frac{1}{\sqrt{3}}$$

when $\lambda = 3$, $\begin{bmatrix} 2-3 & \sqrt{2} \\ \sqrt{2} & 1-3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow -x_1 + \sqrt{2}x_2 = 0 \quad \text{—— (3)}$$

$$\& \quad x_1^2 + x_2^2 = 1 \quad \text{—— (4)}$$

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Let $x_2 = b$, $\therefore x_1 = \sqrt{2}\cdot b$ [From (3)]

Putting these values into eqn.(4),

$$(\sqrt{2}b)^2 + b^2 = 1$$

$$\Rightarrow b = \sqrt{1/3}$$

$$\therefore x_2 = \sqrt{1/3}, \quad x_1 = \sqrt{2/3}$$

Hence, the eigen values are 0 and 3 and eigen vectors are

$$\begin{pmatrix} -1/\sqrt{3} \\ \sqrt{2/3} \end{pmatrix}, \begin{pmatrix} \sqrt{2/3} \\ \sqrt{1/3} \end{pmatrix}.$$

*Similar Matrices :-

Defn. → 1) Two square matrices A and B of the same order, say n are said to be similar if there exists a non-singular matrix P of order n, such that $B = PAP^{-1}$.

2) A and B be two square matrices, are said to be similar if they have same characteristics equation and hence same set of eigen values.

Result :→ 1) If A and B are similar matrices, then ~~they~~ they must have the same eigenvalues.

Proof :→ Since A and B are similar, we have.

$$B = PAP^{-1} \quad \text{for some ~~non-zero scalar~~ n.s. matrix } P.$$

Now,
$$B - \lambda I = PAP^{-1} - \lambda I$$
$$= P(A - \lambda I)P^{-1}$$

Taking determinants of both sides, we have

$$|B - \lambda I| = |P|\,|A - \lambda I|\,|P^{-1}|$$
$$= |A - \lambda I|\,|P|\,|P^{-1}|$$
$$= |A - \lambda I| \quad \text{, since } |P||P^{-1}| = |PP^{-1}| = |I| = 1.$$

Thus, the two matrices A and B have the same characteristic equation and hence the same eigen values.

(OR)

$$|A - \lambda I| = 0$$
$$\Rightarrow |PAP^{-1} - P\lambda IP^{-1}| = 0$$
$$\Rightarrow |PAP^{-1} - \lambda I| = 0$$
$$\Rightarrow |B - \lambda I| = 0 \quad [\because B = PAP^{-1}, \text{ as A and B are similar matrices}]$$

Therefore A and PAP^{-1} (or, B) give same characteristic equation.

IMPORTANT THEOREMS ON EIGEN VALUES & EIGEN VECTORS :-

Theorem :→ If λ is an eigenvalue of the square matrix A, then $\lambda - k$ is an eigen value of $(A - kI)$.

Proof :→ Since λ is an eigen value of A, there exists $\underset{\sim}{x} \neq \underset{\sim}{0}$, ∋

$$(A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$$

i.e. $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$

$$\Rightarrow A\underset{\sim}{x} - kI\underset{\sim}{x} = \lambda\underset{\sim}{x} - kI\underset{\sim}{x}$$

$$\therefore (A - kI)\underset{\sim}{x} = (\lambda - k)\underset{\sim}{x}$$

$\Rightarrow (\lambda - k)$ is an eigen value of $(A - kI)$.

Theorem :→ If λ is an eigenvalue of A, then λ^m is an eigenvalue of A^m, for any positive integer m.

Proof :→ Since λ is an eigen value of A, ∃ $\underset{\sim}{x} \neq \underset{\sim}{0}$ ∋ $A\underset{\sim}{x} = \lambda\underset{\sim}{x}$

$$\Rightarrow A(A\underset{\sim}{x}) = A(\lambda\underset{\sim}{x})$$

$$\Rightarrow A^2\underset{\sim}{x} = \lambda^2\underset{\sim}{x}$$

$\Rightarrow \lambda^2$ is an eigen value of A^2.

We have, $A^2\underset{\sim}{x} = \lambda^2\underset{\sim}{x}$

$$\Rightarrow A(A^2\underset{\sim}{x}) = A(\lambda^2\underset{\sim}{x}) = \lambda^2 A\underset{\sim}{x}$$

$$\Rightarrow A^3\underset{\sim}{x} = \lambda^3\underset{\sim}{x}$$

$\Rightarrow \lambda^3$ is an eigen value of A^3.

In the same way, we can prove that for any $m = 2, 3, 4, \ldots$; λ^m is an eigen value of A^m.

Theorem :→ If λ is a characteristic roots of $A_{n\times n}$, then $k\lambda$ is a ch. roots of KA.

Proof :→ As λ is a ch. root of $A_{n\times n}$, then ∃ some non-zero $\underset{\sim}{x}$ such that $A\underset{\sim}{x} = \lambda\underset{\sim}{x}$

$$\Rightarrow KA\underset{\sim}{x} = k\lambda\underset{\sim}{x}$$

$$\Rightarrow (KA)\underset{\sim}{x} = (k\lambda)\underset{\sim}{x}$$

$\Rightarrow k\lambda$ is an eigen value of KA.

Theorem: If λ is an eigen value of an n.s. matrix A, then $\frac{|A|}{\lambda}$ is an eigen value of $Adj(A)$.

Proof: Since λ is a ch. roots of A, then $\frac{1}{\lambda}$ is a ch. root A^{-1}. (By the next theorem)

We know that, $A^{-1} = \frac{Adj(A)}{|A|}$ and $k\lambda$ is a ch. root of kA, for any scalar k.

So, λ^{-1} is a characteristic root of $\frac{Adj(A)}{|A|}$, giving $\frac{|A|}{\lambda}$ is an eigen value of $Adj(A)$.

***

Theorem: Suppose A is n.s. and λ is an eigen value of A, then λ^{-1} is an eigen value of A^{-1}. Further A and A^{-1} has the same set of eigen vectors.

Proof: Let A be an n.s. matrix and λ is an eigen value of A, then there exists $\underset{\sim}{x} \neq \underset{\sim}{0}$ such that $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$ —— (1)

Premultiplying both sides by A^{-1}, we get

$$A^{-1}(A\underset{\sim}{x}) = A^{-1}\lambda x$$

$$\Rightarrow (A^{-1}A)\underset{\sim}{x} = \lambda(A^{-1}x)$$

$$\text{or, } \underset{\sim}{x} = \lambda(A^{-1}x)$$

$$\text{or, } \frac{1}{\lambda}\underset{\sim}{x} = A^{-1}x$$

$$\text{i.e. } \lambda^{-1}\underset{\sim}{x} = A^{-1}x \quad \text{—— (2)}$$

This implies that λ^{-1} is an eigen value of A^{-1}.

From (1) & (2), we see that $\underset{\sim}{x}$ is an eigen vector of A corresponding to λ iff $\underset{\sim}{x}$ is an eigen vector of A^{-1} corresponding to λ^{-1}. Thus A and A^{-1} has the same set of eigen vectors.

Theorem: If λ be an eigen value of $A_{n\times n}$ and $\underset{\sim}{x}$ be the corresponding eigen vector then for any n.s. matrix $P_{n\times n}$, λ is also an eigen value of $P^{-1}AP$ and $P^{-1}\underset{\sim}{x}$ is the corresponding eigen vector.

Proof: Since λ is an eigen value of $A_{n\times n}$ and $\underset{\sim}{x}$ be a corresponding eigen vector $\Rightarrow$ $A\underset{\sim}{x} = \lambda\underset{\sim}{x}$

Premultiplying both sides by P^{-1}, we get $\to$ $P^{-1}A\underset{\sim}{x} = P^{-1}\lambda\underset{\sim}{x}$

Let $x = Py$ or $y = P^{-1}x$, where P is an n.s. matrix, so P^{-1} exists.

$$\text{or, } P^{-1}APy = P^{-1}\lambda Py = \lambda P^{-1}Py$$

$$\text{or, } P^{-1}APy = \lambda y$$

$\therefore$ λ is an eigen value of $P^{-1}AP$ treating as the corresponding eigen vector where $y = P^{-1}x$.

Theorem: → Let $\lambda_1, \lambda_2, \dots, \lambda_K$ be distincted eigen values of A and $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_K$ be the corresponding eigen vectors. then show that $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_K$ are Linearly independent.

Proof: → Since $\underset{\sim}{x}_i$ is the characteristic vector corresponding to the eigen value λ_i of A, where $i = 1(1)K$, ∋

$$A\underset{\sim}{x}_i = \lambda_i \underset{\sim}{x}_i$$

To show that the ch. vectors $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_K$ are LIN, let us consider the equation

$$\delta_1 \underset{\sim}{x}_1 + \delta_2 \underset{\sim}{x}_2 + \dots + \delta_K \underset{\sim}{x}_K = \underset{\sim}{0} \quad \text{—— (1)}$$

Premultiplying both sides by A, we get ⟶

$$A.\delta_1 \underset{\sim}{x}_1 + A\delta_2 \underset{\sim}{x}_2 + \dots + A\delta_K \underset{\sim}{x}_K = \underset{\sim}{0}$$

$$\Rightarrow \delta_1 (A\underset{\sim}{x}_1) + \delta_2 (A\underset{\sim}{x}_2) + \dots + \delta_K (A\underset{\sim}{x}_K) = \underset{\sim}{0}$$

$$\text{or, } \delta_1 \lambda_1 \underset{\sim}{x}_1 + \delta_2 \lambda_2 \underset{\sim}{x}_2 + \dots + \delta_K \lambda_K \underset{\sim}{x}_K = \underset{\sim}{0} \quad \text{—— (2)} \quad [\because A\underset{\sim}{x}_i = \lambda_i \underset{\sim}{x}_i]$$

Premultiplying ~~both side~~ by A, once again

$$\delta_1 \lambda_1^2 \underset{\sim}{x}_1 + \delta_2 \lambda_2^2 \underset{\sim}{x}_2 + \dots + \delta_K \lambda_K^2 \underset{\sim}{x}_K = \underset{\sim}{0} \quad \text{—— (3)}$$

Proceeding similarly, we will get

$$\delta_1 \lambda_1^m \underset{\sim}{x}_1 + \delta_2 \lambda_2^m \underset{\sim}{x}_2 + \dots + \delta_K \lambda_K^m \underset{\sim}{x}_K = \underset{\sim}{0} \quad \text{—— (m)}$$

for $m = 3, 4, \dots, K-1$.

Now we write these K equations in matrix form

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_K \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \dots & \lambda_K^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda_1^{K-1} & \lambda_2^{K-1} & \lambda_3^{K-1} & \dots & \lambda_K^{K-1} \end{pmatrix} \begin{pmatrix} \delta_1 \underset{\sim}{x}_1 \\ \delta_2 \underset{\sim}{x}_2 \\ \delta_3 \underset{\sim}{x}_3 \\ \vdots \\ \delta_K \underset{\sim}{x}_K \end{pmatrix} = \underset{\sim}{0}$$

Since, the λ_i's are all distinct, the above matrix coefficients are all singular. Hence the equation has the solution $\delta_1 \underset{\sim}{x}_1 = \underset{\sim}{0}, \delta_2 \underset{\sim}{x}_2 = \underset{\sim}{0}, \dots, \delta_K \underset{\sim}{x}_K = \underset{\sim}{0}$.

But since $\underset{\sim}{x}_i$'s are eigen vectors $\underset{\sim}{x}_i \neq 0 \ \forall i = 1(1)K$.

So, $\delta_1 = \delta_2 = \dots = \delta_K = 0$.

Hence, $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_K$ are Linearly Independent.

Theorem: → The eigen vectors corresponding to different eigen values constitute a set of mutually orthogonal vector.

Proof: → The eigen values of $A_{n \times n}$ are all distinct. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the distinct eigen values of a symmetric matrix $A_{n\times n}$. Let us suppose, λ_i and λ_j are two distinct eigen values of A and further assume that $\underset{\sim}{x_i}$ and $\underset{\sim}{x_j}$ are the eigen vectors corresponding to λ_i and λ_j, respectively.

Then we have, $A\underset{\sim}{x_i} = \lambda_i \underset{\sim}{x_i}$

$$\text{implying}\quad \underset{\sim}{x_j}' A \underset{\sim}{x_i} = \lambda_i \underset{\sim}{x_j}' \underset{\sim}{x_i} \quad \text{——— (1)}$$

and $A\underset{\sim}{x_j} = \lambda_j \underset{\sim}{x_j}$

$$\text{implying}\quad \underset{\sim}{x_i}' A \underset{\sim}{x_j} = \lambda_j \underset{\sim}{x_i}' \underset{\sim}{x_j} \quad \text{——— (2)}$$

Noting that, $\underset{\sim}{x_i}'\underset{\sim}{x_j} = \underset{\sim}{x_j}'\underset{\sim}{x_i}$ and $\underset{\sim}{x_i}'A\underset{\sim}{x_j} = \underset{\sim}{x_j}'A\underset{\sim}{x_i}$ (since A is symmetric), Now, (1) − (2) implies

$$\underset{\sim}{x_i}'\underset{\sim}{x_j}(\lambda_i - \lambda_j) = 0$$

Now, $\lambda_i - \lambda_j \neq 0$ [as λ_i, λ_j are two distinct values]

$$\Rightarrow \underset{\sim}{x_i}'\underset{\sim}{x_j} = 0$$

Hence the eigen vectors are orthogonal.

NOTE: → If $\underset{\sim}{x}$ be an eigen vector corresponding to the eigen value λ of the matrix $A_{n\times n}$ then a multiple of $\underset{\sim}{x}$ will as well be an eigen vectors corresponding to λ. Thus it is usual to assume that the eigen vectors are of length unity.

If A be an matrix of order $n\times n$ having distinct eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the eigen vectors corresponding the eigen values constitute an orthonormal basis of E_n.

[Example of this is Problem No. (1).]

Theorem: → The value of the quadratic form $x'A\underset{\sim}{x}$ corresponding to an eigen vector x_i for any eigen value λ_i of A, which satisfies the condition $x_i'x_i = 1$, must be λ_i itself.

Proof: → Since λ_i is an eigen value of A and $\underset{\sim}{x_i}$ a corresponding eigen vector, we have $A\underset{\sim}{x_i} = \lambda_i\underset{\sim}{x_i}$

$$\Rightarrow \underset{\sim}{x_i}'A\underset{\sim}{x_i} = \lambda_i \underset{\sim}{x_i}'\underset{\sim}{x_i}$$
$$= \lambda_i,$$

since $\underset{\sim}{x_i}'\underset{\sim}{x_i} = 1$.

****

Theorem :→ If A is a real symmetric matrix, then the eigen values of A must be all real. / The characteristic roots of a real symmetric matrix are all real.

Proof :→

1st Method :– Let λ be the root of a real symmetric matrix A, then there exists a non-zero vector $\underset{\sim}{x}$ such that

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

Premultiplying both sides by $\underset{\sim}{x}'$ we obtain

$$\underset{\sim}{x}'A\underset{\sim}{x} = \lambda \underset{\sim}{x}'\underset{\sim}{x} \quad \text{———} (1)$$

Transposing (1) we get

$$(\underset{\sim}{x}'A\underset{\sim}{x})' = (\lambda\underset{\sim}{x}'\underset{\sim}{x})'$$

$$\Rightarrow \underset{\sim}{x}'A\underset{\sim}{x} = \underset{\sim}{x}'x\lambda \qquad [\text{since } A' = A, \text{ as } A \text{ is symmetric}]$$

Thus, $\lambda = \dfrac{\underset{\sim}{x}'A\underset{\sim}{x}}{\underset{\sim}{x}'\underset{\sim}{x}}$ is real since $\underset{\sim}{x}'A\underset{\sim}{x}$ and $\underset{\sim}{x}'\underset{\sim}{x}$ are both real matrices of single elements. ///.

Method 2 :– Suppose λ be a complex ~~eigen vector~~ eigen value of symmetric matrix A, in particular λ may reduce to a real eigenvalues, then $\exists$ at least one vector $\underset{\sim}{x}$ satisfying

$$A\underset{\sim}{x} = \lambda\underset{\sim}{x} \quad \text{———} (1)$$

Taking the complex conjugate, we have

$$A\underset{\sim}{x}^* = \lambda^* \underset{\sim}{x}^* \quad \text{———} (2)$$

where $\underset{\sim}{x}^*$ is conjugate of $\underset{\sim}{x}$ and λ^* is conjugate of $\underset{\sim}{\lambda}$.

If $\underset{\sim}{x} = (x_1\ x_2 \cdots x_n)'$ and $\underset{\sim}{x}^* = (x_1^*\ x_2^* \cdots x_n^*)'$, assuming A is of order $n\times n$, then $\underset{\sim}{x}_i^*$ is conjugate of $\underset{\sim}{x}_i\ \forall i$.

Premultiplying (1) by $(\underset{\sim}{x}^*)'$ and (2) by $\underset{\sim}{x}'$ respectively, we obtain

$$\underset{\sim}{x}^{*\prime}A\underset{\sim}{x} = \lambda\underset{\sim}{x}^{*\prime}\underset{\sim}{x} \quad \text{———} (3)$$

$$\underset{\sim}{x}'A\underset{\sim}{x}^* = \lambda^*\underset{\sim}{x}^{*\prime}\underset{\sim}{x} \quad \text{———} (4)$$

Subtracting (3) & (4), we get

$$\underset{\sim}{x}^{*\prime}A\underset{\sim}{x} - \underset{\sim}{x}'A\underset{\sim}{x}^* = (\lambda-\lambda^*)\underset{\sim}{x}^{*\prime}\underset{\sim}{x}$$

since A is a symmetric matrix,

$$[\underset{\sim}{x}^{*\prime}A\underset{\sim}{x}]' = \underset{\sim}{x}'A'\underset{\sim}{x}^* = \underset{\sim}{x}'A\underset{\sim}{x}^*$$

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We get, $(\lambda - \lambda^*) \underset{\sim}{x}^{*\prime} \underset{\sim}{x} = 0$,

Let, $\underset{\sim}{x} = \begin{bmatrix} x_{11} + i x_{12} \\ x_{21} + i x_{22} \\ \vdots \\ x_{n1} + i x_{n2} \end{bmatrix}$

$\Rightarrow \underset{\sim}{x}^* = \begin{bmatrix} x_{11} - i x_{12} \\ x_{21} - i x_{22} \\ \vdots \\ x_{n1} - i x_{n2} \end{bmatrix}$

Thus $\underset{\sim}{x}^{*\prime} \underset{\sim}{x} = \sum_{i=1}^{n} (x_{i1}^2 + x_{i2}^2) > 0$ [since x is non-null]

Therefore $\lambda_i = \lambda_i^*$

Thus if $\lambda = a + ib$, then $\lambda^* = a - ib$, and

$\lambda = \lambda^* \Rightarrow b = 0$.

$\therefore$ The roots are real.

☑ Another Method :- Let A be a real symmetric matrix and let $\underset{\sim}{x} + i\underset{\sim}{y}$ be the real/complex eigen vector corresponding to the eigen value λ, where $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are both vectors with real elements, at least one of them being non-null.

$\therefore$ $A(\underset{\sim}{x} + i\underset{\sim}{y}) = \lambda(\underset{\sim}{x} + i\underset{\sim}{y})$

Premultiplying both sides by the conjugate transpose $(\underset{\sim}{x} - i\underset{\sim}{y})'$, we get

$$(\underset{\sim}{x} - i\underset{\sim}{y})' A (\underset{\sim}{x} + i\underset{\sim}{y}) = \lambda (\underset{\sim}{x} - i\underset{\sim}{y})'(\underset{\sim}{x} + i\underset{\sim}{y})$$

or, $\left\{ (\underset{\sim}{x}' A \underset{\sim}{x} + \underset{\sim}{y}' A \underset{\sim}{y}) + i(\underset{\sim}{x}' A \underset{\sim}{y} - \underset{\sim}{y}' A \underset{\sim}{x}) \right\} = \lambda \left\{ (\underset{\sim}{x}'\underset{\sim}{x} + \underset{\sim}{y}'\underset{\sim}{y}) + i(\underset{\sim}{x}'\underset{\sim}{y} - \underset{\sim}{y}'\underset{\sim}{x}) \right\}$

Since A is symmetric, $\underset{\sim}{x}' A \underset{\sim}{y} - \underset{\sim}{y}' A \underset{\sim}{x} = 0$ and $\underset{\sim}{x}'\underset{\sim}{y} - \underset{\sim}{y}'\underset{\sim}{x} = 0$

$\therefore$ the imaginary part vanishes from both sides.

$\therefore$ $\underset{\sim}{x}' A \underset{\sim}{x} + \underset{\sim}{y}' A \underset{\sim}{y} = \lambda(\underset{\sim}{x}'\underset{\sim}{x} + \underset{\sim}{y}'\underset{\sim}{y})$

It gives $\lambda = \dfrac{\underset{\sim}{x}' A \underset{\sim}{x} + \underset{\sim}{y}' A \underset{\sim}{y}}{\underset{\sim}{x}'\underset{\sim}{x} + \underset{\sim}{y}'\underset{\sim}{y}}$

Since $\underset{\sim}{x} \neq 0$ or $\underset{\sim}{y} \neq 0$, then $\underset{\sim}{x}'\underset{\sim}{x} + \underset{\sim}{y}'\underset{\sim}{y} \neq 0$

$\therefore$ λ is real whatever be the nature of the eigen vector.

[*** → corollary later]

* Diagonalisation of a symmetric matrix using orthogonal transformation

Example :- Let us consider the symmetric matrix

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & \sqrt{3} \\ 0 & \sqrt{3} & 6 \end{pmatrix}$$

$$\therefore |A - \lambda I| = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 4-\lambda & \sqrt{3} \\ 0 & \sqrt{3} & 6-\lambda \end{vmatrix}$$

$$= (7-\lambda)(3-\lambda)(3-\lambda)$$

Hence the eigen values are $\lambda_1 = 7$, $\lambda_2 = \lambda_3 = 3$.

Corresponding to these, we shall find three eigenvectors that are orthonormal.

(i) An eigenvector corresponding to λ_1 is a solution of

$$A\underset{\sim}{x} = 7\underset{\sim}{x}$$

i.e. of the system [redacted] $3x_1 = 7x_1$

$$4x_2 + \sqrt{3}x_3 = 7x_2$$

$$\sqrt{3}x_2 + 6x_3 = 7x_3.$$

The first of the equations, gives

$$x_1 = 0,$$

and from second, and three we get, $x_2 = \dfrac{x_3}{\sqrt{3}}$.

As such, an eigenvector corresponding to λ_1 has the form $\left[0, \frac{1}{\sqrt{3}}x_3, x_3\right]$. In order to normalize it; i.e. to make its length equal to unity, we have to take

$$\frac{1}{3}x_3^2 + x_3^2 = 1$$

i.e. [redacted] $x_3 = \pm \dfrac{\sqrt{3}}{2}$.

Taking the positive value of x_3, we may then have as our eigenvector $\underset{\sim}{u}_1$, given by

$$\underset{\sim}{u}_1 = \left[0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right]$$

(ii) An eigen vector $\underset{\sim}{x} = [x_1, x_2, x_3]$ corresponding to λ_2 must be orthogonal to $\underset{\sim}{u}_1$ and it may also be of unit lenght, we must have

$$\frac{1}{2}x_2 + \frac{\sqrt{3}}{2}x_3 = 0$$

and $$x_1^2 + x_2^2 + x_3^2 = 1.$$

Taking $x_1 = 1$, then solving this we will get $x_2 = x_3 = 0$.

Hence, an eigenvector of unit ~~leg~~ length corresponding to λ_2 and orthogonal to $\underset{\sim}{u}_1$ is $\underset{\sim}{u}_2 = [1, 0, 0]$

(iii) To obtain an eigen vector corresponding to λ_3 which is of unit length and orthogonal to u_1 and u_2, we have first to solve the equations

$$\frac{1}{2}x_2 + \frac{\sqrt{3}}{2}x_3 = 0$$

$$x_1 = 0$$

and $$x_1^2 + x_2^2 + x_3^2 = 1.$$

so, $x_3 = \pm 1/2$, considering the positive value, we get $x_2 = -\frac{\sqrt{3}}{2}$

then $\underset{\sim}{u}_3 = [0, -\frac{\sqrt{3}}{2}, \frac{1}{2}]$.

Consider now a square matrix Q whose columns are the three eigenvectors $\underset{\sim}{u}_1, \underset{\sim}{u}_2, \underset{\sim}{u}_3$. thus

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 0 & 1/2 \end{pmatrix}$$

One should note that

$$Q'AQ = \begin{pmatrix} 0 & 1/2 & \sqrt{3}/2 \\ 1 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2} & 1/2 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & \sqrt{3} \\ 0 & \sqrt{3} & 6 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

Thus $Q'AQ$ is a diagonal matrix with the three eigen values as its diagonal elements.

Theorem:→ If A be a real symmetric matrix of order $n\times n$ then there exists a real non-singular orthogonal matrix Q such that $Q'AQ = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where the columns of Q are the eigen vectors of length unity corresponding to the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A, may not be all distinct.

Proof:→ Method 1.

Let $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_n$ be eigenvectors of A corresponding to eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively, which are so taken as to make them mutually orthogonal and of unit length.

Define, $Q = (\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_n)$.

Clearly, Q is an orthogonal matrix since $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_n\}$ is a set of mutually orthonormal vectors. We thus have–

i) $A\underset{\sim}{x}_i = \lambda_i \underset{\sim}{x}_i \quad (i = 1(1)n)$;

ii) $\underset{\sim}{x}_i' \underset{\sim}{x}_j = 0$ if $i \neq j \quad (i, j = 1(1)n)$; and

iii) $\underset{\sim}{x}_i' \underset{\sim}{x}_j = 1$ if $i = j = 1(1)n$.

Consequently,

$$Q'AQ = \begin{pmatrix} \underset{\sim}{x}_1' \\ \underset{\sim}{x}_2' \\ \vdots \\ \underset{\sim}{x}_n' \end{pmatrix} A (\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_n)$$

$$= \begin{pmatrix} \underset{\sim}{x}_1' \\ \vdots \\ \underset{\sim}{x}_n' \end{pmatrix} (A\underset{\sim}{x}_1, A\underset{\sim}{x}_2, \ldots, A\underset{\sim}{x}_n)$$

$$= ((\underset{\sim}{x}_i' A \underset{\sim}{x}_j))$$

$$= ((\underset{\sim}{x}_i' \lambda_j \underset{\sim}{x}_j))$$

$$= ((\lambda_j \underset{\sim}{x}_i' \underset{\sim}{x}_j))$$

$$= ((\lambda_j \delta_{ij})), \quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$= \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

<u>Method 2.</u> Let A be a real symmetric matrix and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of A, now corresponding to each λ_i, $\exists$ a characteristic vector or eigen vector $\underset{\sim}{u}_i$, now we have already proved that the set of eigen vectors are linearly independent.

Now, every LIN set of vectors can be converted to an orthonormal set by Gram-schmidt orthogonalization process. So, $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n$ can be taken as orthonormal, now we take $Q = (\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n)$

the ith diagonal element of $Q'AQ = \hat{u}_i' A \underset{\sim}{u}_i$

$$= \underset{\sim}{u}_i' \lambda_i \underset{\sim}{u}_i \quad [\because A\hat{u}_i = \lambda_i \underset{\sim}{u}_i \,\forall\, i = 1(1)n]$$

$$= \lambda_i (\hat{u}_i' \hat{u}_i)$$

$$= \lambda_i \quad [\because \hat{u}_i' \hat{u}_i = 1, \text{ since } \hat{u}_i\text{'s are orthonormal}]$$

the (i,j)th element of $Q'AQ = \hat{u}_i' A \hat{u}_j$
$(i \neq j)$

$$= \hat{u}_i' \lambda_j \underset{\sim}{u}_j \quad [\because A\hat{u}_j = \lambda_j \hat{u}_j]$$

$$= \lambda_j (\hat{u}_i' \underset{\sim}{u}_j)$$

$$= 0 \quad [\because \hat{u}_i' \underset{\sim}{u}_j = 0 \text{ for } i \neq j]$$

$$\therefore Q'AQ = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$

<u>NOTE</u> :→ Take $Q'AQ = \Lambda$

If λ_i's are all positive, $\lambda_i^{1/2}$ is defined,

$$\therefore \Lambda^{1/2} = (\lambda_1^{1/2}, \lambda_2^{1/2}, \ldots\ldots, \lambda_n^{1/2})$$

$$\therefore A = (Q')^{-1} \Lambda Q^{-1}$$

$$= (Q^{-1})' \Lambda^{1/2} \Lambda^{1/2} Q^{-1}$$

We call $Q^{-1} = P$.

$$= P' \Lambda^{1/2} \Lambda^{1/2} P$$

$$= (\Lambda^{1/2} P)' (\Lambda^{1/2} P)$$

$$= B'B, \quad [B = \Lambda^{1/2} P]$$

$\boxed{A = P'\Lambda P}$ ⟶ It is called singular value decomposition of a matrix. In terms of eigen values and eigen vectors, we can write a matrix A as

$$A = \lambda_1 \hat{u}_1 \hat{u}_1' + \lambda_2 \hat{u}_2 \hat{u}_2' + \ldots\ldots + \lambda_n \hat{u}_n \hat{u}_n'$$

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Cayley Hamilton Theorem :-

*** Every square matrix satisfies it's own characteristic equations ***

$\Rightarrow$ Let A be a square matrix of order n, I be the identity matrix of order n and O be the zero matrix of order n. The theorem states that if

$|A-\lambda I| = (-1)^n(\lambda^n + P_1\lambda^{n-1} + P_2\lambda^{n-2} + \cdots + P_n) = 0$

be the characteristic equation then $(A)^n + P_1(A)^{n-1} + \cdots + P_nI = 0$

Problem 1. $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$. Use C-H theorem to express $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ as a linear polynomial in A.

Proof :- Ch. eqn. $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = 0$$

or, $(1-\lambda)(3-\lambda) + 2 = 0$

or, $\lambda^2 - 4\lambda + 5 = 0$

$\therefore$ the matrix A satisfies $A^2 - 4A + 5I = 0$

$A^2 = 4A - 5I$

$A^3 = 4A^2 - 5A$

$\vdots$

$A^6 = 4A^5 - 5A^4$

$A^4 = 4A^3 - 5A^2$

$A^4 - 4A^3 = -5A^2$

$\therefore 3A^4 - 12A^3 = -15A^2$

$\therefore A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$

$= -5A^4 + 8A^4 - 12A^3 + 14A^2$

$= 3A^4 - 12A^3 + 14A^2$

$= -15A^2 + 14A^2$

$= -A^2$

$= 5I - 4A$

2. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, Find A^9 and A^{-1} by using C-H theorem?

Proof: $\rightarrow$ $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Ch. equn. s.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 2\lambda + 1 = 0$$

The matrix A satisfies $A^3 - 2A + I = 0$.

$$A^3 = 2A - I.$$

$$\therefore A^9 = (A^3)^3 = (2A - I)^3$$

$$= 8A^3 - 12A^2 - I + 6A$$

$$= 16A - 8I - 12A^2 - I + 6A$$

$$= 22A - 12A^2 - 9I$$

$$= \begin{bmatrix} 22 & 0 & 44 \\ 0 & -22 & 22 \\ 0 & 22 & 0 \end{bmatrix} - \begin{bmatrix} 12 & 24 & 24 \\ 0 & 24 & -12 \\ 0 & -12 & 12 \end{bmatrix} - \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -24 & 20 \\ 0 & -55 & 34 \\ 0 & 34 & -21 \end{bmatrix}$$

Here, $|A| = -1 \neq 0$, $\therefore A^{-1}$ exists.

$$A^3 - 2A + I = 0$$

$$\Rightarrow A^{-1} = 2I - A^2$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

[11] Questions from C.U. Previous Papers

1) What is characteristic equation of matrix? Discuss the nature and the number of roots of such equation. (4)

2) If A and B are two square matrices of the same order, then show that AB and BA have the same set of characteristic roots. (6)

3) Discuss the nature of the characteristic roots of a positive semi-definite quadratic form. (3)

4) If $\underset{\sim}{\alpha}$ is a non-null p-component column vector, then find the characteristic roots of $\underset{\sim}{\alpha}\underset{\sim}{\alpha}'$. (3)

5) If A and B are two non-singular matrices of the same order and $C = BAB^{-1}$ and λ is any scalar, prove that —

$$|C+\lambda I| = |A+\lambda I|. \quad (3)$$

(***)

Corollary: → If A is a real skew-symmetric, then all the eigen values are purely imaginary or zero.

Hints: → From (3) & (4) of method 2) of the original theorem, we have,

$$0 = (\lambda+\lambda^*)\,\underset{\sim}{x}'\underset{\sim}{x}^* \quad \text{as} \quad \underset{\sim}{x}'A\underset{\sim}{y} = (\underset{\sim}{x}'A\underset{\sim}{y})'$$

$$= \underset{\sim}{y}'A'\underset{\sim}{x}$$

$$\Rightarrow \lambda = -\lambda^*$$

$$\Rightarrow \lambda \text{ is purely imaginary.} \qquad = -\underset{\sim}{y}'A\underset{\sim}{x}$$

[1] **Result:** — Show that if λ is a ch. root of the mtx A then $K+\lambda$ is a ch. root of the mtx $A+KI$.

Sol. Let λ be a ch. root of the mtx A and $\underset{\sim}{x}$ be a corresponding vector. Then $\underset{\sim}{x}$ is a non-zero vector $\ni$ $A\underset{\sim}{x} = \lambda\underset{\sim}{x}$.

Now,
$$(A+KI)\underset{\sim}{x} = A\underset{\sim}{x} + KI\underset{\sim}{x}$$
$$= \lambda\underset{\sim}{x} + K\underset{\sim}{x}$$
$$= (\lambda+K)\underset{\sim}{x} \quad \text{——— (1)}$$

Since $\underset{\sim}{x} \neq 0$, therefore from the relation (1), we see that the scalar $\lambda+K$ is a characteristic value of the mtx $(A+KI)$ and $\underset{\sim}{x}$ is a corresponding ch. vector.

Result.

* If A be a symmetric matrix of order $n \times n$ with eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then $\det(A) = \prod_{i=1}^{n} \lambda_i$.

Proof :→

Method 1. Since $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the matrix A, the characteristic polynomial of the matrix A can be written in the factor form

$$|A - \lambda I| = f(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) \cdots (\lambda_n - \lambda)$$

Putting $\lambda = 0$ in both sides of the above equation, we get

$$|A| = \prod_{i=1}^{n} \lambda_i \quad ///.$$

Method 2. We know—

For a symmetric matrix $A_{n \times n}$ having distinct eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the orthogonal matrix P diagonalises A, where $P'AP = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ where the columns of P are the eigen vectors of length unity corresponding to the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Here, $P'AP = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ which implies

$$|P'AP| = \det[\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)]$$

$$= \prod_{i=1}^{n} \lambda_i$$

$$\Rightarrow |A||P'P| = \prod_{i=1}^{n} \lambda_i$$

$$\therefore |A| = \prod_{i=1}^{n} \lambda_i \quad , \text{since } P'P = I \text{ as P is orthogonal.}$$

31 Result!— Prove that the characteristic roots of Hermitian mtx are real.

Sol. Let A be a Hermitian matrix. λ be an ch. root of A and $\underset{\sim}{x}$ be the corresponding eigen value. then

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x}.$$

Premultiplying both sides by x^θ, we get

$$\underset{\sim}{x}^\theta A \underset{\sim}{x} = \underset{\sim}{x}^\theta \lambda \underset{\sim}{x}$$

Taking conjugate transpose of the both sides we get

$$(\underset{\sim}{x}^\theta A \underset{\sim}{x})^\theta = (\lambda \underset{\sim}{x}^\theta \underset{\sim}{x})^\theta$$

$$\Rightarrow \underset{\sim}{x}^\theta A \underset{\sim}{x} = \underset{\sim}{x}^\theta \underset{\sim}{x} \lambda$$

$$\Rightarrow \lambda = \frac{\underset{\sim}{x}^\theta A \underset{\sim}{x}}{\underset{\sim}{x}^\theta \underset{\sim}{x}}$$ is real since $\underset{\sim}{x}^\theta A \underset{\sim}{x}$ and $\underset{\sim}{x}^\theta \underset{\sim}{x}$ are real.

C.U.

Ex. Show that the eigen values of a p.d. / n.n.d. symmetric matrix are all positive / non-negative.

Soln. → By definition,

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x}, \quad \underset{\sim}{x} \neq \underset{\sim}{0}.$$

$$\Rightarrow \underset{\sim}{x}' A \underset{\sim}{x} = \lambda \underset{\sim}{x}' \underset{\sim}{x}$$

$$\Rightarrow \lambda = \frac{\underset{\sim}{x}' A \underset{\sim}{x}}{\underset{\sim}{x}' \underset{\sim}{x}}.$$

Note that, $\underset{\sim}{x} \neq 0$,

$$\underset{\sim}{x}' \underset{\sim}{x} > 0$$

If A is p.d. symmetric matrix, then

$$\underset{\sim}{x}' A \underset{\sim}{x} > 0 \quad \forall \underset{\sim}{x} \neq \underset{\sim}{0}$$

$$\text{and } \lambda = \frac{\underset{\sim}{x}' A \underset{\sim}{x}}{\underset{\sim}{x}' \underset{\sim}{x}} > 0$$

If A is n.n.d. symmetric matrix, then

$$\underset{\sim}{x}' A \underset{\sim}{x} \geq 0 \quad \forall \underset{\sim}{x} \neq \underset{\sim}{0}$$

$$\text{and } \lambda = \frac{\underset{\sim}{x}' A \underset{\sim}{x}}{\underset{\sim}{x}' \underset{\sim}{x}} \geq 0$$

Ex. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Find an orthogonal matrix Q such that $Q'AQ = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, where λ, λ_2 are the eigen values of A. Hence, find A^8?

ANS: →

$$|A - \lambda I_n| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = -1, 3.$$

$$\lambda = -1, \quad \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow x_2 = -x_1$$

$$\Rightarrow \underset{\sim}{x} = x_1 (1, -1)$$

$\underset{\sim}{u}_1 = \frac{1}{\sqrt{2}}(1, -1)$ is an orthonormal eigen vector corresponding to $\lambda = -1$.

when $\lambda = 3$, $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$\Rightarrow x_1 = x_2$

$\therefore \underset{\sim}{x} = x_2(1,1)$

$\therefore \underset{\sim}{u}_2 = \frac{1}{\sqrt{2}}(1,1)$ is an orthonormal eigen vector corresponding to $\lambda = 3$.

Hence $Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ is an orthogonal mtx

such that $Q'AQ = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$.

$$Q'A^8Q = Q'AQ\,Q'A^7Q \qquad [\because QQ' = I_n]$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} Q'A^7Q$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}^8 = \begin{pmatrix} (-1)^8 & 0 \\ 0 & 3^8 \end{pmatrix}$$

[Alternative way: → We know that, if A has an eigen value λ, then A^m has an eigen value λ^m, and A, A^m have the same set of eigen vectors.

$Q'AQ = \text{diag}(\lambda_1, \dots, \lambda_n)$

and $Q'A^mQ = \text{diag}(\lambda_1^m, \dots, \lambda_n^m)$]

$$A^8 = (Q')^{-1}\begin{pmatrix} (-1)^8 & 0 \\ 0 & 3^8 \end{pmatrix}$$

$$= Q\begin{pmatrix} 1 & 0 \\ 0 & 3^8 \end{pmatrix}Q'$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3^8 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3^8 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -3^8 \\ 1 & 3^8 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Ex. If A is a p.d. symmetric matrix, then S.T. $\exists$ a p.d. matrix $B \ni A = BB$.

Soln. $\exists$ an orthogonal matrix $Q \ni Q'AQ =$ diag$(\lambda_1, \ldots, \lambda_n)$, where $\lambda_i > 0 \; \forall i = 1(1)n$.

$$Q'AQ = \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \sqrt{\lambda_n} \end{pmatrix} QQ' \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \sqrt{\lambda_n} \end{pmatrix}$$

$$\Rightarrow A = Q \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \sqrt{\lambda_n} \end{pmatrix} Q'Q \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \sqrt{\lambda_n} \end{pmatrix} Q^T$$

$$= BB, \text{ where } B = Q \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \sqrt{\lambda_n} \end{pmatrix} Q^T$$

$\Rightarrow Q'BQ$ is p.d.

$\Rightarrow B$ is p.d.

Ex. If A and B are two square matrices, then show that AB and BA have the same characteristic roots.

Soln. $\rightarrow$ If rank$(A^{n \times n}) = r$, then $\exists$ two n.s. matrices P and $Q \ni PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$

We have, $PABP^{-1} = PAQQ^{-1}BP^{-1}$

$$= \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

$\left[\text{where, } Q^{-1} B^{n \times n} P^{-1} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \; C_{11} \text{ is of order } r \times r \right]$

$$= \begin{pmatrix} C_{11}^{r \times r} & C_{12} \\ 0 & 0 \end{pmatrix}$$

Now, $Q^{-1}BAQ = Q^{-1}BP^{-1}PAQ$

$$= \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} C_{11} & 0 \\ C_{21} & 0 \end{pmatrix}$$

Note that, the ch. equation of AB is

$$0 = |AB - \lambda I_n|$$

$$= |PABP^{-1} - \lambda I_n|$$

$$= \left| \begin{pmatrix} C_{11} & C_{12} \\ 0 & 0 \end{pmatrix} - \lambda \begin{pmatrix} I_r & 0 \\ 0 & I_{n-r} \end{pmatrix} \right|$$

$$\left[\because |AB - \lambda I_n| = |P^{-1}| \, |PABP^{-1} - \lambda PP^{-1}| \, |P| \right.$$

$$= \frac{1}{|P|} |PABP^{-1} - \lambda I_n| \, |P|$$

$$\left. = |PABP^{-1} - \lambda I_n| \right]$$

$$= \begin{vmatrix} C_{11} - \lambda I_r & C_{12} \\ 0 & -\lambda I_{n-r} \end{vmatrix}$$

$$= |C_{11} - \lambda I_r| \, |-\lambda I_{n-r}|$$

$$= (-\lambda)^{n-r} |C_{11} - \lambda I_r|$$

and the ch. eqn. of BA is

$$0 = |BA - \lambda I_n| = |Q^{-1}BAQ - \lambda I_n|$$

$$= \begin{vmatrix} C_{11} - \lambda I_r & 0 \\ 0 & -\lambda I_{n-r} \end{vmatrix}$$

$$= (-1)^{n-r} |C_{11} - \lambda I_r|$$

Since AB & BA have the same characteristic equation, they have the same set of eigen values.

Problem:- 1. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$. If

$A = P^{-1}DP$, then the matrix D is equal to

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (C) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Solution:-

A and P are non-singular matrices.

If $A = P^{-1}DP$

then D be a diagonal matrix having eigen values as diagonal entry.

then we find eigen values of A.

characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)\left[(1-\lambda)^2 - 1\right] = 0$$

$$\Rightarrow \lambda(3-\lambda)(\lambda-2) = 0$$

$$\therefore \lambda = 0, 2, 3.$$

So, D becomes $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

2. A real quadratic form $\underset{\sim}{X}'A\underset{\sim}{X}$ is positive definite if

(A) All eigen values of $A > 0$ (B) All eigen values of $A < 0$

(C) All eigen values of $A = 0$ (D) None.

Solution:- $Q(X) = X'AX$ is positive definite

$\Leftrightarrow Q(\underset{\sim}{X}) \geq 0$; $\underset{\sim}{X} \neq 0$

$\Leftrightarrow$ All eigen values of $A > 0$.

END