MATRIX ALGEBRA

BY

TANUJIT CHAKRABORTY

Indian Statistical Institute

Mail : tanujitisi@gmail.com

MATRICES

Definition:- A collection of numbers arranged in rows and columns is said to be an array, A matrix is a rectangular array of numbers closed in addition, substraction, multiplication, and division.

We represent a matrix by $A=[a_{ij}]_{m\times n}$, $i=1(1)m$, $j=1(1)n$.

i.e. $\therefore A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$

Diagonal matrix:-

$A=[a_{ij}]$ is such that $a_{ij}=0 \ \forall i\neq j$

i.e. $A = \text{diag}[d_1, d_2, \ldots, d_n]$

Eg. $A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$ is a 3X3 diagonal mtx.

Scalar matrix:- $A=[a_{ij}] \ni a_{ij}=0 \ \forall i\neq j$

$a_{ij}=k \ \forall i=j$, $k\in\mathbb{N}$.

Eg $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$

~~Upper~~ **Triangular mtx:-** If every element above or below the leading diagonal of a square matrix is zero, then the matrix is called a triangular matrix.

Upper triangular mtx:- $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

Lower Triangular mtx:- $\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Equality of matrices:- Two matrices A and B of the same order are said to be equal, if and only if the corresponding elements are equal.

Multiplication of matrix by a scalar:- Matrix multiplication is associative and distributive, but not commutative.

$A = [a_{ij}]_{m\times n}$

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & & & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix} = [ka_{ij}]_{m\times n}$$

Addition of Matrices:-

$A = [a_{ij}]_{m\times n} \quad B = [b_{ij}]_{m\times n}$

$A + B = [a_{ij} + b_{ij}]$

Matrix addition is commutative, associative and distributive.

Note:- Only a square matrix can have a determinant.

Transpose of a matrix:- $A = [a_{ij}]_{m\times n}$

$A^T = [a_{ji}]_{n\times m}$

(1) $(A')' = A$.

(2) $(A+B)' = A' + B'$.

(3) $(AB)' = B'A'$.

symmetric and skew-symmetric matrices:-

For a symmetric mtx. A, $a_{ij} = a_{ji} \; \forall \; i, j$.

For a skew-symmetric mtx A, $a_{ij} = -a_{ji} \; \forall \; i \neq j$

$= 0 \quad \forall \; i = j$.

Ex.1. If A be any matrix, S.T. AA' and $A'A$ are symmetric.

Sol. $(AA')' = (A')'(A)' = AA'$

$(A'A)' = (A)'(A')' = A'A$

Hence AA' and $A'A$ both are symmetric.

Ex.2. If A and B are both symmetric, then AB is symmetric iff A and B commute.

Sol.

$A' = A, \; B' = B$

$(AB)' = B'A'$

$= BA$

$= AB$ iff A and B commute.

This shows (AB) is symmetric.

Ex.3. S.T. A^2 is symmetric, if A is either symmetric or skew-symmetric.

Sol. A^2 exists only if A is square mtx.

Let $A = [a_{ij}]$, $i, j = 1, 2, \ldots, n$. Then

$A^2 = [c_{ij}]$, $i, j = 1, 2, \ldots, n$, where,

$$c_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj}$$

Case I:- When A is symmetric, then $a_{kj} = a_{jk}$.

$$\therefore c_{ij} = \sum_{k=1}^{n} a_{ik} a_{jk}$$

$\therefore c_{ji} = \sum a_{jk} a_{ik}$; on interchanging i and j.

$\therefore$ Clearly, $c_{ij} = c_{ji}$, $\therefore A^2$ is symmetric.

Case II :- When A is skew-symmetric, then $a_{kj} = -a_{jk}$.

$$\therefore c_{ij} = \sum_{k=1}^{n} a_{ik}(-a_{jk}) = -\sum_{k=1}^{n} a_{ik} a_{jk}$$

so that $c_{ji} = \sum a_{jk} a_{ik}$, on interchanging i and j.

Clearly, $c_{ij} = c_{ji}$

$\therefore$ Hence A^2 is again symmetric.

$\therefore A^2$ is symmetric if A is either symmetric or skew-symmetric.

Ex.4. S.T. all positive integral powers of a symmetric matrix are symmetric.

Sol.

$$A' = A$$

$(A^n)' = (A\,A \ldots n \text{ times})'$, n be a positive integer

$$= A'A' \ldots n \text{ times}$$

$= A\,A \ldots n$ times as $A' = A$

$= A^n$; hence A^n is symmetric.

Ex.5. S.T. all positive odd (even) integral powers of a skew-symmetric matrix are skew-symmetric (symmetric).

Sol.

$$A' = -A$$

$$(A^n)' = (A\,A \ldots n \text{ times})'$$

$$= A'A' \ldots n \text{ times}$$

$= (-A)(-A) \ldots n$ times as $A' = -A$.

$$= (-1)^n A^n$$

$$= \begin{cases} A^n, & n = \text{even} \quad \therefore A^n \text{ is symmetric.} \\ -A^n, & n = \text{odd} \quad \therefore A^n \text{ is skew-symmetric.} \end{cases}$$

Ex.6. If A is symmetric (skew-symmetric), show that $B'AB$ is symmetric (skew-symmetric).

Sol. Case I:- $A^T = A$ $\quad (B'AB)' = (B)'A'(B')'$

$$= B'AB$$

Case II:- $A^T = -A$,

$$(B'AB)' = -B'AB$$

Ex.7. If A and B are symmetric (skew-symmetric), S.T. A+B is symmetric (skew-symmetric).

Ex.8. If A be any sq. matrix, S.T. $A+A'$ is symmetric and $A-A'$ is skew-symmetric.

Sol.

$$(A+A')' = (A')+(A')' = A'+A = (A+A')$$

$$(A-A')' = (A') - (A')' = A' - A = -(A-A').$$

Ex.9. If A, B are symmetric, S.T. $AB+BA$ is symmetric and $AB-BA$ is skew-symmetric.

Ex.10. Show that every square matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrix.

Sol.

$$A = \frac{1}{2}(A+A') + \frac{1}{2}(A-A')$$

$$(A+A')' = A' + (A')' = A'+A$$

$$(A-A')' = A' - A = -(A-A')$$

$\therefore$ $A+A'$ is symmetric and $A-A'$ is skew-symmetric.

Conjugate and Triangulated of matrix:-

A matrix obtained by replacing each element of a mtx A by its complex conjugate is called the conjugate mtx of A and is denoted by $\overline{A}$.

A matrix is said to be real iff $\overline{A} = A$.

$(\overline{A})'$ is called triangulated matrix of A.

Hermitian and Skew-Hermitian matrices:-

$A^* = [a_{ij}]$ is Hermitian iff $a_{ij} = \overline{a_{ij}} \ \forall\ i, j$

$= \overline{a}_{ii} \ \forall\ i = j$

i.e. every diagonal element of a Hermitian mtx is real.

e.g. $\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$, $\begin{bmatrix} 4 & 1-i \\ 1+i & 2 \end{bmatrix}$ are the examples of Hermitian mtx.

$A = [a_{ij}]$ is skew-Hermitian iff $a_{ij} = -\overline{a}_{ji} \ \forall\ i, j$.

i.e. Every diagonal element of a skew-hermitian mtx is either purely imaginary or zero.

e.g. $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1-i \\ -1-i & 2i \end{bmatrix}$

Q. Show that $A = \begin{bmatrix} 3 & 1+2i \\ 1-2i & 2 \end{bmatrix}$ is Hermitian.

Sol. $\overline{A}$ = conjugate of $A = \begin{bmatrix} 3 & 1-2i \\ 1+2i & 2 \end{bmatrix}$

and $A^* = (\overline{A})'$ transpose of $\overline{A} = \begin{bmatrix} 3 & 1+2i \\ 1-2i & 2 \end{bmatrix}$

Clearly, $A^* = A$; hence A is Hermitian.

Note:- The above all results of symmetric and skew-symmetric matrices are true if we replace symmetric mtx by

Hermitian matrix and skew-symmetric mtx by skew-Hermitian mtx.

Note:– 1. Every square mtx. can be uniquely represented as the sum of a hermitian and a skew-hermitian matrix.

2. Every square mtx. can be uniquely expressed as $P + iQ$, where P and Q are hermitian.

3. If A is Hermitian (skew-hermitian) mtx, then iA is a skew-hermitian (Hermitian) mtx.

Polynomials in square matrix with scalar coefficients:—

The algebra of polynomials in one square mtx A with scalar coefficients is the same as the algebra of ordinary polynomials. For example, two parallel identities are:

$$x^2 - (\alpha+\beta)x + \alpha\beta \equiv (x-\alpha)(x-\beta)$$

$$A^2 - (\alpha+\beta)A + \alpha\beta I \equiv (A-\alpha I)(A-\beta I)$$

More generally, if

$$x^n + p_1 x^{n-1} + \cdots + p_{n-1}x + p_n \equiv (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$$

then $A^n + p_1 A^{n-1} + \cdots + p_{n-1}A + p_n I \equiv (A-\alpha_1 I)(A-\alpha_2 I)\cdots(A-\alpha_n I)$

Ex.1. Find the scalar solutions of the mtx equation $A^2 - 5A + 7I = 0$, and show that $\begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$ is a non-scalar solution.

Sol. Consider the algebric equation

$$x^2 - 5x + 7 = 0;$$

$$x = \frac{5 \pm i\sqrt{3}}{2}$$

Hence the scalar solution are $A = \frac{1}{2}(5 \pm i\sqrt{3})I$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

i.e. $A = \begin{bmatrix} \frac{1}{2}(5+i\sqrt{3}) & 0 \\ 0 & \frac{1}{2}(5+i\sqrt{3}) \end{bmatrix}$

2nd part:–

If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

$$A^2 - 5A = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= -7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= -7I$$

Hence $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ is a non-scalar solution.

Idempotent Matrix:-

Definition:- A square matrix A such that $A^2 = A$ is called idempotent. If $A^T = A$ and $A^2 = A$, then the square matrix A is called symmetric idempotent.

Theorem:- Every non-singular idempotent mtx is an identity mtx.

Proof:- A matrix A is said to be non-singular or regular if $|A| \neq 0$.

As A is idempotent so $A^2 = A$ or $AA = A$.

Now, a square mtx B of the same order as A, such that $AB = BA = I$, is called the inverse mtx of A and is denoted by A^{-1} and exists as A is non-singular.

$$A^{-1}(AA) = A^{-1}A$$
$$\Rightarrow IA = I$$
$$\Rightarrow A = I.$$

Theorem:- If A and B are idempotent matrices, then AB is idempotent if A and B commute.

Proof:-

$$A^2 = A \;,\; B^2 = B$$

$$\begin{aligned}(AB)^2 &= (AB)(AB) \\ &= A(BA)B \\ &= A(AB)B \\ &= (AA)(BB) \\ &= A^2B^2 = AB \text{ as A and B commute. i.e. } AB = BA.\end{aligned}$$

$\therefore$ AB is idempotent.

Theorem:- If A is idempotent and $A+B = I$, then B is idempotent and $AB = BA = 0$.

Proof:- since $A+B = I$, $\therefore B = (I-A)$

$$\begin{aligned}B^2 &= (I-A)(I-A) \\ &= I - A - A + A^2 \\ &= I - A \qquad [\because A^2 = A] \\ &= B.\end{aligned}$$

$\therefore$ B is an idempotent matrix.

2nd Part:-

$$A+B = I$$
$$A(A+B) = AI$$
$$A^2 + AB = A$$
$$A + AB = A$$
$$\therefore AB = 0$$

Similarly $BA = 0$.

Theorem:- Show that the mtx A defined as $A = I_n - X(X'X)^{-1}X'$ is a symmetric and idempotent mtx.

Sol.

$$\begin{aligned}A &= I_n - X\left(X^{-1}(X')^{-1}\right)X' \\ &= I_n - \{(XX^{-1})\}\{(X')^{-1}X'\} \\ &= I_n - (I_n)(I_n) \\ &= I_n - I_n = 0\end{aligned}$$

Therefore $A' = A$ and $A^2 = A$.

Nilpotent Matrix:—

Definition:— If A be a nilpotent mtx such that $A^m = 0$, where m is positive integer, then A is called a nilpotent matrix.

If m be the least positive integer for which $A^m = 0$, then A is said to be a nilpotent matrix of index m. Thus a square mtx $A \ni A^m = 0$, but $A^{m-1} \neq 0$ is a nilpotent mtx of order m, m being a positive integer.

Involuntary Matrix:—

Definition:— A square mtx A is such that

$$A^2 = I \text{ or, } (I+A)(I-A) = 0$$

is called involutory. Clearly I is involutary.

Ex. Show that $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ is nilpotent.

Sol.

$$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Orthogonal matrix:—

Definition:— A square mtx A is said to be orthogonal if

$$A'A = I = AA'.$$

Note:— When A is orthogonal $|A|^2 = 1$, $|A| = \pm 1$. If $|A|$ is equal to 1, then A is called a proper mtx.

Theorem:— If A and B are n-square orthogonal matrices, then AB and BA are orthogonal matrices.

Proof:— Since A and B are orthogonal matrices, we have

$$AA' = I \quad \text{and } BB' = I$$

$$\begin{aligned}(AB)(AB') &= (AB)(B'A') \\ &= A(BB')A' \\ &= AA' \\ &= I\end{aligned}$$

$\therefore AB$ is orthogonal, similarly BA is also orthogonal.

Ex.1. S.T. the mtx $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Sol.

$AA' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; hence A is orthogonal.

Ex.2. If A is real skew-symmetric matrix such that $A^2 + I = 0$. Show that A is orthogonal.

Sol.

$$A = -A'$$

$$AA = -AA'$$

$$A^2 = -AA'$$

$$-I = -AA'$$

$$\therefore AA' = I$$

$\therefore A$ is orthogonal.

Scanned by CamScanner

Unitary matrix:—

Definition:— A square matrix A is called unitary if

$$A^* A = I = AA^*.$$

If A is real then $A^* = A'$, so that A is unitary if

$$A'A = AA' = I.$$

Ex. Show that $A = \begin{bmatrix} \frac{1}{2}(1+i) & -\frac{1}{2}(1-i) \\ \frac{1}{2}(1+i) & \frac{1}{2}(1-i) \end{bmatrix}$ is unitary mtx.

Sol. $A = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1+i & 1-i \end{bmatrix}$

$$A^* = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

$$AA^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is unitary.

Trace (or spur) of a square matrix:—

Definition:— The sum of the elements of the principal diagonal of a square matrix is called the trace (or, spur, a german word) of the matrix.

Thus if $A = [a_{ij}]$ be a square matrix of order n, then

$$\text{trace } A = a_{11} + a_{22} + a_{33} + \dots + a_{nn} = \sum_{i=1}^{n} a_{ii}.$$

Remark:— For the identity mtx I_n, trace $I_n = n$.

Ex. 1. Show that for any sq. mtx A, B:—

(i) Trace $(KA) = k$ trace (A), k being a scalar.

(ii) Trace $(KA+B) = k$ trace (A) + trace (B).

Sol. (i) $A = [a_{ij}]_{n\times n}$. Then $KA = [Ka_{ij}]_{n\times n}$

$$\text{Trace}(KA) = \sum_{i=1}^{n} ka_{ii}$$
$$= k\,\text{Trace}(A).$$

(ii) $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$

$$KA + B = [Ka_{ij} + b_{ij}]_{n\times n}$$

$$\text{Trace}(KA+B) = \sum_{i=1}^{n}(ka_{ii} + b_{ii})$$
$$= k\sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii}$$
$$= k\,\text{trace}(A) + \text{trace}(B)$$

e.g. If trace$(A) = 20$, trace$(B) = -8$, then trace$(A+B) = 12$.

Ex. 2. Show that trace A' = trace A.

Sol. Let $A = [a_{ij}]_{n\times n}$, $A' = [a'_{ij}]_{n\times n} = [a_{ji}]_{n\times n}$

therefore when $i=j$, we have trace(A) = trace(A').

Ex. 3. Show that trace (AB) = trace (BA), if AB and BA co-exist.

Sol.

$$A = [a_{ij}]_{m\times n} \quad B = [b_{ij}]_{n\times m}$$

$$AB = [c_{ij}]_{m\times m} \quad ; \quad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\therefore \text{trace}(AB) = \sum_{i=1}^{m} c_{ii}$$

$$= \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ik} b_{ki}$$

$$BA = [d_{ij}]_{n\times n}$$

$$d_{ij} = \sum_{k=1}^{m} b_{ik} a_{kj}$$

$$\therefore \text{trace}(BA) = \sum_{i=1}^{n} d_{ii}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{m} b_{ik} a_{ki}$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{m} b_{ki} a_{ik} \text{, interchanging } i \text{ and } k.$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki}$$

$$\therefore \text{trace}(AB) = \text{trace}(BA).$$

Ex. 4. S.T. trace $(AA') \geq 0$.

Sol.

$$A = [a_{ij}]_{m\times n}, \quad A' = [a'_{ij}]_{n\times m}$$

$$AA' = [c_{ij}]_{m\times m}, \quad c_{ij} = \sum_{k=1}^{n} a_{ik} a'_{kj}$$

$$\therefore \text{trace}(AA') = \sum_{i=1}^{m} c_{ii}$$

$$= \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ik} a'_{ki}$$

$$= \sum_{i=1}^{m} \sum_{k=1}^{n} (a_{ik})^2 \text{ as } a_{ik} = a'_{ki}$$

$$\geq 0$$

Ex. 5. S.T. trace $(C'AC)$ = trace (A), if C is an orthogonal mtx.

Sol.

$$\text{trace}(C'AC) = \text{trace}\{C'(AC)\}$$

$$= \text{trace}\{(AC)C'\} \text{, as trace } AB = \text{trace } BA.$$

$$= \text{trace}\{A(CC')\} \text{, as } C \text{ is orthogonal mtx.}$$

$$= \text{trace}(A).$$

The adjoint of a square matrix:-

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \text{Adj } A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

Theorem:- If A is a square matrix, then

$$A(\text{adj}A) = (\text{adj}A).A = |A|.I.$$

Sol. We have

The (i,j) the element of the product $A.(\text{Adj}A)$

= Product of the ith row of A and jth column of adj$A$

$$= a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn}$$

$$= \begin{cases} 0 & \text{if } i \neq j \\ |A| & \text{if } i = j \end{cases}$$

Thus in the product, only the diagonal elements exist and each is equal to $|A|$ while all other element are zero, so that

$$A.(\text{adj}A) = \begin{bmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ 0 & 0 & |A| & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & |A| \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= |A|.I.$$

Similarly, $(\text{adj}A).A = |A|.I.$

Corollary 1:- If $|A| \neq 0$, we have

$$|A|.|\text{Adj}A| = \begin{vmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & |A| \end{vmatrix} = |A|^n$$

$$\therefore |\text{Adj}A| = |A|^{n-1}.$$

Corollary 2:- If $|A| \neq 0$, then

$$A\left(\frac{1}{|A|}\text{Adj}A\right) = I = \left(\frac{1}{|A|}\text{adj}A\right)A.$$

Scanned by CamScanner

Singular matrix:– A sq. matrix is said to be singular if its determinant is zero.

Theorem:- If A and B are n-square matrices, then

$$\mathrm{Adj}\,AB = \mathrm{Adj}\,A \cdot \mathrm{Adj}\,B.$$

Sol. We have $AB.(\mathrm{adj}\,AB) = |AB|.I = (\mathrm{adj}\,AB).AB$

Now, $AB(\mathrm{adj}\,B)(\mathrm{Adj}\,A) = A(B.\mathrm{adj}\,B)(\mathrm{Adj}\,A)$

$= A.(|B|.I)(\mathrm{Adj}\,A)$

$= |B|.(A.\mathrm{adj}\,A)$

$= |B|\,|A|.I$

$= |AB|.I$

Also, $(\mathrm{adj}\,B)(\mathrm{adj}\,A)\,AB = (\mathrm{adj}\,B)[\mathrm{adj}(A).A]B$

$= (\mathrm{adj}\,B)|A|.IB$

$= |A|[(\mathrm{adj}\,B).B]$

$= |A|.|B|.I$

$= |AB|.I.$

$\therefore \mathrm{Adj}(AB) = \mathrm{Adj}(A)\,\mathrm{Adj}(B).$

Ex.1. Show that $\mathrm{Adj}\,A' = (\mathrm{Adj}\,A)'$

Sol. Obviously, the matrices $\mathrm{adj}\,A'$ and $(\mathrm{adj}\,A)'$ both will be of the same order as A. Now,

(i,j)th element of $\mathrm{Adj}\,A'$ = the co-factor of (j,i)th element of A' in the determinant $|A'|$.

= the co-factor of (i,j)th element of A in the determinant $|A|$

= (j,i)th element of Adj A.

$\therefore \mathrm{Adj}\,A' = (\mathrm{adj}\,A)'.$

Ex.2. Show that every skew-symmetric mtx of odd order is singular.

Solution:– Let A be a skew-symmetric matrix of order n, where n is odd.

Since A is skew-symmetric, we have

$A' = -A.$

$|A'| = |-A|$

$= (-1)^n |A|$

$= -|A|$, where n is odd and $|A'| = |A|$

$\therefore |A| = -|A|$

$\therefore 2|A| = 0$

$\Rightarrow |A| = 0.$

Hence a skew-symmetric mtx of odd order is singular.

WORKED EXAMPLES :-

1. Let A be a non-singular sq. mtx of order 3. If B is the mtx obtained from A by adding 3-multiple of its first row to its second row, then the value of $\det(2A^{-1}B)$ is
(A) 8 (B) 3 (C) 6 (D) 2

Sol. (D) $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$|A| \neq 0$,

$A \sim B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{11}+a_{21} & 3a_{12}+a_{22} & 3a_{13}+a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Since A and B are equivalent so $|A| = |B|$

$\therefore \text{Det}(2A^{-1}B) = 2|A^{-1}||B|$
$= 2 \cdot \frac{1}{|A|} \cdot |B|$
$= 2.$

2. Let $\underset{\sim}{u}$ be a unit column vector and $A = I - 2\underset{\sim}{u}\underset{\sim}{u}^T$. Then A^{-1} is
(A) $I - 2\underset{\sim}{u}\underset{\sim}{u}^T$ (B) $I + 2\underset{\sim}{u}\underset{\sim}{u}^T$ (C) $2\underset{\sim}{u}\underset{\sim}{u}^T - I$ (D) $4\underset{\sim}{u}\underset{\sim}{u}^T$.

Sol. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2.[1 \ 1 \ 1] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = I - 2\underset{\sim}{u}\underset{\sim}{u}^T$.

3. Let P, Q, R be matrices of order 3×5, 5×7, 7×3 respectively. The number of scalar additions required to compute $P(QR)$ is
(A) 114 (B) 126 (C) 128 (D) 138.

Sol. We have to calculate $P(QR)$.
Q and R is of order 5×7 and 7×3 respectively.
When we multiply it we have to make 6 scalar addition for each entry so 5 column in Q and 3 rows in R when we multiply Q and R.
we do $6 \times 5 \times 3 = 90$ addition further QR is of 5×3 matrix.
Similarly when we multiply P to QR we perform $4 \times 3 \times 3$ addition
$\therefore$ Total no. of additions $= 90 + 36 = 126$.

4. If A is a square matrix and $A-\frac{1}{2}I$ and $A+\frac{1}{2}I$ are orthogonal, prove that A is skew-symmetric & $A^2=-3/4\,I$.

Sol. Since $A-\frac{1}{2}I$ is orthogonal,

So, $(A-\frac{1}{2}I)(A-\frac{1}{2}I)^T = I$

or, $(A-\frac{1}{2}I)(A'-\frac{1}{2}I) = I$

or, $AA' - \frac{1}{2}IA' - \frac{1}{2}AI + \frac{1}{4}I^2 = I$

or, $AA' - \frac{1}{2}A' - \frac{1}{2}A = \frac{3}{4}I$ —— (1)

Similarly since $A+\frac{1}{2}I$ is orthogonal, we have

$AA' + \frac{1}{2}A' + \frac{1}{2}A = \frac{3}{4}I$. —— (2)

$\therefore$ (1) & (2) gives $A'+A=0$

(1) − (2)

$\Rightarrow A' = -A$

$\therefore$ A is skew-symmetric.

(1) + (2) gives $2AA' = \frac{3}{2}I$

$\therefore 2A(-A) = \frac{3}{2}I$

$\therefore A^2 = -\frac{3}{4}I$.

5. If $A = \begin{bmatrix} 1+i & 2-3i & 2 \\ 3-4i & 4+5i & 1 \\ 5 & 3 & 3-i \end{bmatrix}$. Find the conjugate of matrix A?

Sol. $\overline{A} = \begin{bmatrix} 1-i & 2+3i & 2 \\ 3+4i & 4-5i & 1 \\ 5 & 3 & 3+i \end{bmatrix}$

6. Let P and Q be two $n\times n$ nonzero matrices $\ni P+Q=0$, then show that, (i) P is non-singular

(ii) $P = Q^T$

(iii) $P = Q^{-1}$

(iv) Rank (P) = Rank (Q).

Sol. Given that P and Q are two $n\times n$ matrices such that $P+Q=0$.

$\therefore P+Q=0 \Rightarrow P=-Q$

$\Rightarrow |P| = |Q|$

$\Rightarrow |P| \neq 0$ iff $|Q| \neq 0$

i.e. P is non-singular iff Q is non-singular.

If Q is non-singular then Q^{-1} exists. $|Q^{-1}| = |Q^T| = |Q|$

$\therefore P = Q^T = Q^{-1}$.

But if $P+Q=0$ then Rank (P) = Rank (Q).

Example:- $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Then $P+Q=0$ and Rank (P) = Rank (Q).

7. Show that all $n\times n$ symmetric matrices over F form a vector space of dimension $\frac{n(n+1)}{2}$ over F. What is the dimension of the space of skew-symmetric matrices?

Sol. Let $A = (a_{ij})$ be an $n\times n$ symmetric mtx.
Then $a_{ij} = a_{ji}$ $\forall\, i \neq j$. Thus the number of independent entries are a_{ij} $(i<j)$ and a_{ij} $(1 \le i \le n)$ and these are

$$\frac{n(n-1)}{2} + n = \frac{n(n+1)}{2} \text{ in number.}$$

Hence the space has dimension $\frac{n(n+1)}{2}$.

For a skew-symmetric mtx, $a_{ij} = 0$, $1 \le i \le n$, $\forall\, i=j$ and $a_{ij} = -a_{ji}$
thus the number of LIN entries are $\frac{n(n-1)}{2}$ which is also the dimension of the space of skew-symmetric matrices.

8. (a) If A is a real skew-symmetric mtx and $A^2 + I = 0$, then S.T. A is orthogonal.
(b) If H is Hermitian mtx, what kind of mtx is e^{iH}?

Sol. (a) Let A be the real skew-symmetric mtx.
then $A^T = -A$.
Also we have $A^2 + I = 0$.
We have $A^T A = (-A)(A) = I$.

$\therefore$ A is orthogonal.

(b) Let H is Hermitian mtx. $H^* = H$.

For any mtx M, $\left(e^{M}\right)^* = e^{M^*}$

Let $e^{iH} = A$

Then $A^* A = \left(e^{iH}\right)^* \cdot e^{iH}$

$= I$.

Hence e^{iH} is unitary matrix.

Inverse or Reciprocal of a matrix:-

Let A be a square matrix of order n. Then the mtx B of order n, if it exists, such that

$$AB = BA = I_n,$$

is called the inverse or reciprocal of A and is denoted by A^{-1}.

$$A^{-1} = \left(\frac{\text{Adj}A}{|A|}\right), \text{ provided } |A| \neq 0.$$

Corollary:- We have $|A||A^{-1}| = |AA^{-1}| = |I| = 1$

Hence $|A^{-1}| = |A|^{-1}$.

Theorem:- The inverse of a matrix is unique.

Sol. If possible let B and C be two inverses of the same mtx A, then by definition

$$AB = BA = I$$
$$AC = CA = I$$

$\therefore C(AB) = CI = C$

$\therefore (CA)B = IB = B$

$\therefore C = B$, $\therefore$ the inverse is unique.

Theorem:- A square matrix A has an inverse if and only if $|A| \neq 0$, i.e. only a non-singular matrix has an inverse.

Proof:- The condition is necessary. Let B be the inverse of the mtx A.

Then $AB = I$

so, that $|A||B| = |I| = 1$

$\therefore |A| \neq 0$.

$\therefore$ The condition is sufficient.

Let $|A| \neq 0$, we assume that $B = \frac{\text{Adj}A}{|A|}$

$$\therefore AB = A\left(\frac{\text{Adj}A}{|A|}\right)$$
$$= \frac{1}{|A|} \cdot (A.\text{Adj}A)$$
$$= \frac{|A|. I}{|A|} = I$$

Similarly $BA = I$.

$\therefore AB = BA = I$. $\therefore$ Hence A has an inverse.

Ex. 1 Find A^{-1} when $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & 6 & -7 \end{bmatrix}$.

Sol. $|A| = 20$; $\text{Adj}\,A = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$

Hence $A^{-1} = \dfrac{\text{Adj}\,A}{|A|} = \begin{bmatrix} \frac{2}{20} & \frac{6}{20} & \frac{4}{20} \\ \frac{21}{20} & -\frac{7}{20} & -\frac{8}{20} \\ -\frac{18}{20} & \frac{6}{20} & \frac{4}{20} \end{bmatrix}$

Ex. 2. Show that the inverse of a regular symmetric (hermitian) matrix is symmetric (hermitian).

Sol. Let A be symmetric, so that $A' = A$

& $AA^{-1} = I = A^{-1}A$

Since $I = I'$, so, $A^{-1}A = (A^{-1}A)'$

$AA^{-1} = (A^{-1}A)'$

$= A'.(A^{-1})'$

$= A(A^{-1})'$

$\therefore A^{-1} = (A^{-1})'$

$\therefore A^{-1}$ is symmetric.

2nd Part :- Let A be hermitian, so that

$A^* = A$

$A^{-1}A = AA^{-1} = I$

$\therefore (A^{-1}A)^* = I^* = I = (AA^{-1})$

& $A^*(A^{-1})^* = AA^{-1}$

$\therefore A(A^{-1})^* = AA^{-1}$

$\therefore (A^{-1})^* = A^{-1}$

$\therefore A^{-1}$ is hermitian.

Ex. 3. Find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$.

Sol.

$\boxed{A = I \times A}$

$$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 1 & 3 \\ 1 & 3 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} A \quad \begin{bmatrix} C_1' \leftrightarrow C_2 - C_1 \\ C_3' \leftrightarrow C_4 - C_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \quad \begin{bmatrix} R_2' \leftrightarrow R_3 - R_2 \\ R_4' \leftrightarrow R_4 - R_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 2 & 2 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} A \quad [R_4' \leftrightarrow R_4 - R_1]$$

$[R_3' \leftrightarrow R_3 - 2R_2]$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 & 0 \\ -1 & -1 & 1 & 0 \\ 2 & 2 & -1 & 0 \\ 1 & 2 & -1 & 1 \end{bmatrix} A \quad [R_4' = R_4 + R_3]$$

$[R_1' = R_1 - R_2]$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 1 & -2 \\ -1 & -1 & 1 & 0 \\ -1 & -4 & 2 & -3 \\ 1 & 2 & -1 & 1 \end{bmatrix} A \quad [R_1' = R_1 - 2R_4]$$

$[R_3' = R_3 - 3R_4]$

$$I = A^{-1} A$$

$$\therefore A^{-1} = \begin{bmatrix} 0 & -3 & 1 & -2 \\ -1 & -1 & 1 & 0 \\ -1 & -4 & 2 & -3 \\ 1 & 2 & -1 & 1 \end{bmatrix}$$

Rank and Nullity of a matrix:—

The maximum order of the non-singular square sub-matrix of A is called the rank of A. , The mtx A is said to be of rank r, if and only if it has at least one nonsingular square sub-mtx of order r and all square submatrices of (r+1) and higher orders are singular. The rank of a mtx A is denoted by rank(A).

If A is a square mtx of order n, then n − rank(A) is called the nullity of the mtx A and denoted by N(A).

Remarks:—

1. If I is a unit mtx of order n, rank (I) = n.
2. If A is of order m×n, then rank (A) ≤ m and ≤ n.
3. If A′ is the transpose of A, rank (A′) = rank(A).

Ex.1. S.T. the rank of a skew-symmetric mtx can't be 1.

Sol. For a skew-symmetric mtx, the elements of the leading diagonal are all zero. If one element is non-zero, say x, then we have a minor of the form $\begin{vmatrix} 0 & x \\ -x & 0 \end{vmatrix} \neq 0$

Therefore rank of the mtx is ≥ 2. if all entries are zero then rank is 0.

Scanned by CamScanner

Ex.2. Find the rank of the mtx A, where $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}$.

Sol.

$|A| = 2(-9+8) + 2(-3+4) = 0$

But $\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} \neq 0$, so rank $(A) = 2$.

Normal form of a matrix :-

Theorem:- By means of elementary transformations every mtx A of order $m \times n$ and rank $r (>0)$ can be reduced to the following forms:

(i) $\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$ (ii) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (iii) $[I_r | 0]$ (iv) $[I_r]$

Ex. Reduce the mtx A to its normal form, where $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \end{bmatrix}$ and hence determine its rank.

Solution:-

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix} \qquad \begin{array}{l} R_2' \leftrightarrow R_2 - R_1 \\ R_3' \leftrightarrow R_3 - 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_3' \leftrightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & -6 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_1' \leftrightarrow R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} C_3' \leftrightarrow C_3 + C_1 \\ C_3' \leftrightarrow C_3 - 2C_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} C_4' \leftrightarrow C_4 + 6C_1 \\ C_4' \leftrightarrow C_4 - 5C_2 \end{array}$$

Hence the normal form of A is $\left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & 0 \end{array}\right]$

and rank$(A) = 2$.

WORKED EXAMPLES:-

1. If $r(A)$ denotes rank of a mtx A, then $r(AB)$ is
(a) $r(A)$ (b) $r(B)$ (c) $\leq \min[r(A), r(B)]$ (d) $> \min[r(A), r(B)]$

Sol. rank $(AB) \leq \min[r(A), r(B)]$

e.g. $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$AB = \begin{bmatrix} 7 & 10 \\ 7 & 10 \end{bmatrix}$ rank $(A) = 1$, $r(B) = 2$.

rank $(AB) = 1 \leq \min[r(A), r(B)]$.

2. Show that rank $(A') =$ rank $(A^*) =$ rank (A).

Sol. The transpose of a mtx is obtained by interchanging rows into columns into rows, and clearly this change does not alter the values of the determinants of minors. Hence

rank $(A') =$ rank (A).

Now assume that the value of a minor of A with complex elements be $a+ib$. Then the value of the corresponding minor of the conjugate mtx of A be $a-ib$.

incase $a+ib = 0$ then $a-ib = 0$.
in case $a+ib \neq 0$ then $a-ib \neq 0$.

Hence rank$(A^*) =$ rank $(A) =$ rank(A').

3. If A is an $(n \times 1)$ non-zero mtx and B is a $(1 \times n)$ non-zero mtx, then show that rank$(AB) = 1$.

Sol. Let $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}$ and $B = [b_{11} \ b_{12} \ \cdots \ b_{1n}]$.

Then $AB = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1n} \\ a_{21}b_{11} & a_{21}b_{12} & \cdots & a_{21}b_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1}b_{11} & a_{n2}b_{12} & \cdots & a_{n1}b_{1n} \end{bmatrix}$

AB is non-zero and there will be at least one non-zero minor of order 1.

Hence, rank$(AB) = 1$ because all 2nd and higher order minors are all zero here.

4. Find the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix}$

Solution:-

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 3 & -2 & 0 \\ 2 & 0 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{bmatrix} \qquad C_4' \leftrightarrow C_4 - C_1$$

$\therefore$ rank (A) = No. of LIN columns in A

$= 3.$

5. Reduce the mtx A to its normal form, where $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ and hence find the rank of the mtx.

Solution:-

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_4' \leftrightarrow R_4 - (R_1 + R_2)$$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C_1' \leftrightarrow C_1 - (C_2 + C_4)$$

$$\sim \begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C_2' = C_1 + C_2$$

5. Reduce the matrix A to its normal form, where

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \end{bmatrix}$$

and hence determine its rank.

Sol. To reduce a mtx into its normal form, let us use congruent row and column operations.

$$A \sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & 2 & 5 \\ 3 & 4 & 5 & 2 \end{bmatrix} \quad \text{By } R_{21}(-1)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix} \quad \text{By } R_{31}(-3)$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix} \quad \text{By } C_{21}(-1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix} \quad \text{By } C_{31}(-1).$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix} \quad \text{By } C_{41}(1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{By } R_{32}(-1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{By } C_{32}(-1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{By } C_{42}(-5)$$

Hence the normal form of A is $\left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & 0 \end{array}\right]$ and rank $(A) = 2$.

6. Reduce the matrix A to its normal form, where

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

and hence find the rank of the matrix.

Solution:- By performing the operation R_{12}, we have

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

By $R_{31}(-3)$, $R_{41}(-1)$ $\quad A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$

By $C_{31}(-1)$, $C_{41}(-1)$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$

By $R_{32}(-1)$, $R_{42}(-1)$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

By $C_{32}(3)$, $C_{42}(1)$, $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

which is of the form $\left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & 0 \end{array}\right]$; hence rank (A) = 2.

— o —

System of Linear Equations

1. The equations
$$x-y+2z=4$$
$$3x+y+4z=6$$
$$x+y+z=1 \quad \text{have}$$

(A) Unique solution (B) Infinite solutions (C) No solution (D) None.

Sol. → (B) $[A|b] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 3 & 1 & 4 & 6 \\ 1 & 1 & 1 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 4 & -2 & -6 \\ 0 & 2 & -1 & -3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 4 & -2 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ rank $[A|b]$ = rank $(A) = 2 < 3$

$\therefore$ it has infinite solutions.

2. The system of linear equations $x+y+z=2$, $2x+y-z=3$, $3x+2y+kz=4$ has a unique solution if

(A) $k=0$ (B) $k \neq 1$ (C) $k=1$ (D) $k \neq 0$.

Sol. $[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 3 \\ 3 & 2 & k & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & -1 \\ 0 & -1 & k-3 & -2 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & -1 \\ 0 & 0 & k & -1 \end{array}\right]$$

$\therefore k \neq 0$

Alt. method:—

Given system has a unique solution if

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} \neq 0$$

$\Rightarrow k \neq 0$

3. The system of equations

$$2x + y = 5$$
$$x - 3y = -1$$
$$3x + 4y = k$$

is consistent, when k is

(A) 1 (B) 2 (C) 5 (D) 10

Sol. (D) $[A|B] = \left[\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & -3 & -1 \\ 3 & 4 & k \end{array}\right]$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ 3 & 4 \end{bmatrix}$$

$$\text{rank}(A) = 2$$

$$\text{Det}[A|B] = -7k + 70$$

$\therefore$ The system is consistent if $\text{Det}[A|B] = 0$

$$\Rightarrow 7k = 70$$
$$\Rightarrow k = 10.$$

4. S.T. the S.O.E.s

$$\left.\begin{array}{l} x_1 - 2x_2 + x_3 - x_4 + 1 = 0 \\ 3x_1 - 2x_3 + 3x_4 + 4 = 0 \\ 5x_1 - 4x_2 + x_4 + 3 = 0 \end{array}\right\} \text{ is inconsistent.}$$

Sol. Here, we have

$$A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 3 & 0 & -2 & 3 \\ 5 & -4 & 0 & 1 \end{bmatrix}, \quad [A \,\vdots\, b] = \begin{bmatrix} 1 & -2 & 1 & -1 & -1 \\ 3 & 0 & -2 & 2 & -4 \\ 5 & -4 & 0 & 1 & -3 \end{bmatrix}$$

By elementary transformations, A and $[A \,\vdots\, b]$ can be reduced in the following forms:

$$A \sim \begin{bmatrix} 1 & 0 & \frac{1}{6} & 1 \\ 0 & 1 & -\frac{5}{6} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad [A \,\vdots\, b] \sim \begin{bmatrix} 1 & 0 & 1/6 & 1 & -4/3 \\ 0 & 1 & -5/6 & 1 & -1/6 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$\therefore$ rank$(A) = 2$ but rank$(A \,\vdots\, b) = 3$

So, the equations are inconsistent.

5. Find the rank of the matrix $A = \begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 1 & a & b & 0 \\ 0 & c & d & 1 \end{bmatrix}$.

Sol.

$$A \sim \begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ rank$(A) = 2$.

6. Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

Sol. $A = I_3 A$

Applying E-rocw tramsformation to the matrix applying $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - R_1$.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

$R_2 \to -\frac{1}{4} R_2$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - 2R_2$, $R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & \frac{1}{2} & 0 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/4 & 3/4 & -1 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{bmatrix} A$$

$\therefore I_3 = BA$

$\therefore A^{-1} = B$.

$$\therefore A^{-1} = B = \begin{bmatrix} -1/4 & 3/4 & -1 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{bmatrix}$$

7. Solve : $x_1 - x_2 + x_3 = 2$

$$3x_1 - x_2 + 2x_3 = -6$$

$$3x_1 + x_2 + x_3 = -18$$

Sol. Here, we have $A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 3 & -1 & 2 & -6 \\ 3 & 1 & 1 & -18 \end{bmatrix}$

Now, $|A| = 1(-1-2) + 1(3-6) + 1(3+3) = 0$.

Also, $\begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix} \neq 0$; hence rank $(A) = 2$.

We now proceed to reduce matrix B to its Echelon form by $R_{21}(-3)$, $R_{31}(-3)$,

$$B \sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & -1 & -12 \\ 0 & 4 & -2 & -24 \end{bmatrix}$$

By $R_{32}(-2)$,

$$B \sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & -1 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By $R_1(1/2)$

$$B \sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & -1/2 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\therefore$ rank $(B) = 2 =$ rank (A).

Hence the given equations are consistent.

Also the mtx equation becomes

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 0 \end{bmatrix}$$

$$\therefore \left.\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ x_2 - \tfrac{1}{2}x_3 &= -6 \end{aligned}\right\} \quad \therefore x_1 = -\tfrac{1}{2}x_3 - 4, \quad x_2 = \tfrac{1}{2}x_3 - 6.$$

and x_1, x_2 canbe expressed in terms of x_3 which is arbitrary. Since the rank is 2, two unknowns x_1, x_2 are expressed in terms of x_3, the system has ~~finite~~ infinite number of solutions.

8. Find two non-singular matrices P and Q such that PAQ is in the normal form where

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$. Also find the rank of the matrix A.

Solution:—

$$A = I_3 A I_3$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

E-operations on the matrix A (left hand equation) until it is reduced to the normal form.
Every E-row operation will also be applied to the pre factor I_3 of the product on the right hand member of the above equation and every column operation to the post-factor I_3

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing $R_2 \to -\frac{1}{2} R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing $R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing $C_3 \to C_3 - C_2$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & 0 \end{array}\right] = PAQ$$, where,

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -2 & -1 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ Rank(A) = 2.

EIGEN VALUES & VECTORS

Eigen Values :—

Let $A = [a_{ij}]_{n\times n}$ be any square matrix of order n and λ be an indeterminate.

$[A - \lambda I]$ is called characteristic matrix and $|A - \lambda I| = 0$ is called characteristic equation and roots of this characteristic equation is called the characteristic roots or characteristic values or eigen values or latent roots or proper values of the mtx A.

Note:- The set of the eigen values of A is called the spectrum of A.

Eigen Vectors :— If λ is a characteristic root of an $n\times n$ matrix A, then a non-zero vector $\underset{\sim}{x}$ such that

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

is called a characteristic vector or eigen vector of A corresponding to the characteristic root λ.

Remark:— 1. λ is a characteristic root of a matrix A if and only if there exists a non-zero vector $\underset{\sim}{x}$ such that $A\underset{\sim}{x} = \lambda\underset{\sim}{x}$.

2. If $\underset{\sim}{x}$ is a characteristic vector of a matrix A corresponding to the characteristic value λ. Here k is any non-zero scalar, then $k\underset{\sim}{x}$ is also a characteristic vector of A corresponding to the same characteristic value λ.

3. The characteristic vectors corresponding to distinct characteristic roots of a matrix are linearly independent.

Nature of an eigen value of the special types of matrices :—

1. The eigen value of a Hermitian matrix are all real.
2. The eigen value of a real symmetric mtx are all real.
3. The eigen value of a skew-Hermitian mtx are either pure imaginary or zero.
4. The eigen value of a skew symmetric mtx are either pure imaginary or zero.
5. The eigen values of a unitary matrices & an orthogonal matrix are of unit modulus.

Ex. S.T. the eigen value of a triangular matrix are just the diagonal elements of the matrix.

Sol.
Let, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$ be a triangular mtx of order 3.

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda) = 0$$

$$\Rightarrow \lambda = a_{11}, a_{22}, a_{33}.$$

Ex. 2. Determine the eigen values and eigen vectors of the matrix
$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}.$$

Sol.
$$|A - \lambda I| = 0$$
$$\Rightarrow \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$
$$\Rightarrow (5-\lambda)(2-\lambda) - 4 = 0$$
$$\Rightarrow (\lambda - 6)(\lambda - 1) = 0$$
$$\therefore \lambda_1 = 6,\ \lambda_2 = 1$$

Eigen vectors $\underset{\sim}{X_1} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of A corresponding to the eigen value 6.

$$(A - 6I)\underset{\sim}{x} = \underset{\sim}{0}$$
$$\Rightarrow \begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ by } R_2 \leftrightarrow R_2 + R_1$$

rank of the coefficient matrix $= 1$, there are $(2-1) = 1$ L.I. solution.

$$4x_2 = x_1$$
$$\therefore x_2 = 4,\ x_1 = 1$$

$\therefore \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value 6.

The eigen vectors $\underset{\sim}{X_2} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ corresponding to the eigen value 1.

$$(A - 1I)\underset{\sim}{x} = \underset{\sim}{0}$$
$$\Rightarrow \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \begin{cases} 4x_1 + 4x_2 = 0 \\ x_1 + x_2 = 0 \end{cases}$$
$$\therefore x_1 = -x_2$$
if $x_2 = 1$, $x_1 = -1$.

$\therefore \underset{\sim}{X_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the eigen vector of A corresponding to the eigen value 1.

So, $\underset{\sim}{X_1} = k\begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\underset{\sim}{X_2} = h\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are the eigenvectors corresponding to the eigen values 6, 1, respectively, where k, h are any non-zero scalars.

The Cayley - Hamilton theorem:—

Every square matrix satisfies ~~the~~ its characteristic equation.
i.e. if for a square matrix A of order n,

$$|A-\lambda I| = (-1)^n \left[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots\cdots + a_n \right]$$

then the matrix equation

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n I = 0$$

is satisfied by $x = A$,

i.e. $A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n I = 0$.

Corollary 1:— If A be a non-singular matrix, $|A| \neq 0$.

Premultiplying by A^{-1}

$$A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \cdots + a_{n-1} I + a_n A^{-1} = 0$$

or, $A^{-1} = -\left(\frac{1}{a_n}\right)\left(A^{n-1} + a_1 A^{n-2} + \cdots + a_{n-1} I\right)$

Corollary 2:— If m be a positive integer such that $m \geq n$, then multiplying the results by A^{m-n}.

$$A^m + a_1 A^{m-1} + \cdots + a_n A^{m-n} = 0.$$

Eigen values and Eigen vectors:—

If V is a vector space over the field F and T is a linear operator on V. An eigen value of T is a scalar c in F such that there is a non-zero vector $\alpha \in V$ with $T\alpha = c\alpha$.

If c is an eigen value of T, then

(a) Any α such that $T\alpha = c\alpha$ is called eigen vector of T associated with the eigen value c;

(b) the collection of all α such that $T\alpha = c\alpha$ is called the eigen space associated with c.

Eigen value of matrix A over F:— If A is an $n \times n$ matrix over the field F, an eigen value of A over F is a scalar c in F such that the matrix $(A - cI)$ is singular (not invertible).

Diagonalisable:— If T is a linear operator on the finite dimensional space V. Then T is diagonalizable if there is a basis for V each vector of which is an eigen vector of T.

Eigen polynomial:—

$$f(c) = |A - cI|.$$

Scanned by CamScanner

Some Important Theorem:—

1. If T is a linear operation on a finite dimensional space V and c is any scalar. then followings are equivalent:

(a) c is an eigen value of T

(b) The operator $(T-cI)$ is singular (not invertible)

(c) $\det(T-cI)=0$.

2. Similar matrices have the same eigen ~~...~~ polynomial.

3. If $T\alpha = c\alpha$ and F is any polynomial, then $F(T)\alpha = F(c)\alpha$

4. Suppose T is a linear operator on the finite dimensional space V; $c_1, \dots, c_k$ are k-distinct eigen values of T and W is the space of the eigen vectors associated with the eigen value c_i. If $W = W_1 + W_2 + \dots + W_k$, then

$$\dim(W) = \dim(W_1) + \dim(W_2) + \dots + \dim(W_k).$$

In fact, if B_i is an ordered basis for W_i, then $B = (B_1, \dots, B_k)$ is an ordered basis for W.

5. If T is a linear operator on a finite dimensional values of T and W_i is a null space of $(T-c_iI)$. The followings are equivalent:

(i) T is diagonalizable

(ii) The eigen polynomial for T is $F = (x-c_1)^{d_1}(x-c_2)^{d_2}\dots(x-c_k)^{d_k}$; with $\dim(W_i) = d_i$, $i = 1(1)k$.

(iii) $\dim V = \dim(W_1) + \dim(W_2) + \dots + \dim(W_k)$.

Theorem:— If $\underset{\sim}{\alpha}$ is a characteristic vector of T corresponding to the eigen value λ, then $k\underset{\sim}{\alpha}$ is also a ch. vector of T corresponding to the same ch. value λ. Here k is any non-zero scalar.

Proof:- Since $\underset{\sim}{\alpha}$ is a characteristic vector of T corresponding to the ch. value λ, therefore, $\underset{\sim}{\alpha} \neq \underset{\sim}{0}$ and

$$T(\underset{\sim}{\alpha}) = \lambda\underset{\sim}{\alpha} \quad ; \text{ let, } k \text{ be any non-zero scalar.}$$

$$T(k\underset{\sim}{\alpha}) = kT(\underset{\sim}{\alpha}) = k(\lambda\underset{\sim}{\alpha})$$
$$= \lambda(k\underset{\sim}{\alpha}).$$

$\therefore$ $k\underset{\sim}{\alpha}$ is a characteristic vector of T corresponding to the characteristic value λ.

Thus, corresponding to a ch. value λ, there may correspond more than one characteristic vectors.

Theorem:— If $\underset{\sim}{\alpha}$ is a ch. vector of T, then $\underset{\sim}{\alpha}$ can't correspond to more than one ch. values of T.

Proof:-

$$T\underset{\sim}{\alpha} = c_1\underset{\sim}{\alpha} \quad \text{and} \quad T\underset{\sim}{\alpha} = c_2\underset{\sim}{\alpha}\ ; \text{ where } c_1, c_2 \text{ are two distinct ch. values of } T.$$

$$c_1\underset{\sim}{\alpha} = c_2\underset{\sim}{\alpha}$$
$$\Rightarrow (c_1-c_2)\underset{\sim}{\alpha} = \underset{\sim}{0} \quad [\because \underset{\sim}{\alpha} \neq \underset{\sim}{0}]$$
$$\Rightarrow c_1 - c_2 = 0$$
$$\Rightarrow c_1 = c_2.$$

So, a ch. vector of a matrix, can't correspond to more than one characteristic value of that mtx.

Theorem:— Let T be a linear operator on an n-dimensional vector space V and A be the matrix of T relative to any ordered basis B. Then a vector $\underset{\sim}{\alpha}$ in V is an eigenvector of T corresponding to its eigenvalue c if and only if its coordinate vector $\underset{\sim}{X}$ relative to the basis B is an eigen-vector of A corresponding to its eigenvalue c.

Proof:— We have $[T-cI]_B = [T]_B - c[I]_B$

$$= A - cI.$$

If $\alpha \neq 0$, then the coordinate vector $\underset{\sim}{X}$ of $\underset{\sim}{\alpha}$ is also non-zero.

Now, $[(T-cI)(\alpha)]_B = [T-cI]_B[\alpha]_B$

$$= (A-cI)\underset{\sim}{X}$$

$\therefore (T-cI)(\alpha) = \underset{\sim}{0}$ iff $(A-cI)\underset{\sim}{X} = 0$

or, $T(\alpha) = c\underset{\sim}{\alpha}$ iff $A\underset{\sim}{X} = c\underset{\sim}{X}$.

or, α is an eigenvector of T iff $\underset{\sim}{X}$ is an eigen vector of A.

Ex. 1. Let V be an n-dimensional vector space over F. What is the ch. polynomial of (i) the identity operator on V, (ii) the zero operator on V.

Sol. Let B be any ordered basis for V.

(i) If I is the identity operator on V, then $[I]_B = I$. The characteristic polynomial of I, is $\det(I - xI)$

$$= \begin{vmatrix} 1-x & 0 & \cdots & 0 \\ 0 & 1-x & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1-x \end{vmatrix} = (1-x)^n.$$

(ii) If $\hat{0}$ is the zero operator on V, then $[\hat{0}]_B = \underset{\sim}{0}$, i.e. the null matrix of order n, then the ch. polynomial of $\hat{0}$ is

$$\det(\underset{\sim}{0} - xI) = \begin{vmatrix} -x & 0 & \cdots & 0 \\ 0 & -x & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -x \end{vmatrix} = (-1)^n x^n.$$

Ex. 2. If $c \in F$ is a ch. value of a linear operator T on a vector space $V(F)$, then for any polynomial $P(X)$ over F, $P(c)$ is a ch. value of $P(T)$.

Sol. Since c is a ch. value of T, therefore $\exists$ a non-zero vector $\underset{\sim}{\alpha}$ in V $\ni$

$$T\underset{\sim}{\alpha} = c\underset{\sim}{\alpha}$$

$$\Rightarrow T(T\underset{\sim}{\alpha}) = T(c\underset{\sim}{\alpha})$$

$$\Rightarrow T^2\underset{\sim}{\alpha} = cT(\underset{\sim}{\alpha}) = c^2\underset{\sim}{\alpha}$$

$\therefore c^2$ is a ch. value of T^2.

Repeating this process K times, we get

$$T^K\underset{\sim}{\alpha} = c^K\underset{\sim}{\alpha}.$$

$\therefore c^K$ is a ch. value of T^K, where K be any positive integer

Let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$, where $a_i \in F$.

Then $p(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_m T^m$.

We have $[p(T)](\alpha) = (a_0 I + a_1 T + \cdots + a_m T^m)(\alpha)$

$$= a_0 I \underset{\sim}{\alpha} + a_1 T(\underset{\sim}{\alpha}) + \cdots + a_m T^m(\underset{\sim}{\alpha})$$

$$= a_0 \underset{\sim}{\alpha} + a_1 (c\underset{\sim}{\alpha}) + \cdots + a_m (c^m \underset{\sim}{\alpha})$$

$$= (a_0 + c a_1 + \cdots + c^m a_m)\, \underset{\sim}{\alpha}.$$

$\therefore$ $P(c) = a_0 + c a_1 + \cdots + a_m c^m$ is a ch. value of $P(T)$.

Ex. 3. Find all (complex) ch. values and ch. vectors of the following matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Sol.

$$|A - xI| = \begin{vmatrix} -x & 1 \\ 0 & -x \end{vmatrix} = 0$$

$$\Rightarrow x^2 = 0 \Rightarrow x = 0$$

$\therefore$ 0 is the only ch. value of A.

Let x_1, x_2 be the components of the ch. vector corresponding to this ch. value 0.

Let $\underset{\sim}{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Now, $[A - 0.I]\, \underset{\sim}{X} = \underset{\sim}{0}$

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e. $x_2 = 0$

let $x_1 = k$; where k is any non-zero complex number.

$\therefore$ $\underset{\sim}{X} = \begin{bmatrix} k \\ 0 \end{bmatrix}$ is the ch. vector corresponding to the eigen value $\lambda = 0$, where k is any non-zero complex no.

WORKED EXAMPLES:–

1. The eigen values of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ is

(A) 2, 3, 6 (B) 2, 6, 7 (C) –2, 3, 6 (D) None.

Sol. (C) $|A-\lambda I| = 0$

$$\Leftrightarrow \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 5-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 5-\lambda \\ 3 & 1 \end{vmatrix}$$

$$= (\lambda+2)(\lambda-3)(\lambda-6) = 0$$

$$\Rightarrow \lambda = -2, 3, 6.$$

2. Find the eigen values of $(A^4+3A-2I)$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ are

(A) 2, 3, 20 (B) 2, 2, 2 (C) 2, 2, 20 (D) 20, 20, 2

Sol. (D)

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 15 & 15 \\ 0 & 16 & 15 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, $B = A^4+3A-2I$

$$= \begin{bmatrix} 1 & 15 & 15 \\ 0 & 16 & 15 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 3 & 3 \\ 0 & 6 & 3 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 18 & 18 \\ 0 & 20 & 18 \\ 0 & 0 & 2 \end{bmatrix}$$

Then the eigen values of $|B-\lambda I| = 0$

$$\Leftrightarrow \begin{vmatrix} 2-\lambda & 18 & 18 \\ 0 & 20-\lambda & 18 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 2, 20, 2.$$

3. Find the eigen values of A^4, where $A = \begin{pmatrix} 1 & 0 & -1 \\ 9 & 4 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

Sol.
$$A = \begin{pmatrix} 1 & 0 & -1 \\ 9 & 4 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Then the eigen values of the matrix can be determined from the ch. equation

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 9 & 4-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(\lambda-2)(\lambda-3) = 0$$

$$\Rightarrow \lambda = 1, 2, 3.$$

$\therefore$ The eigen values of A are: 1, 2, 3.

So, the eigen values of A^4 are: $1^4, 2^4, 3^4$, i.e. 1, 16, 81.

4. Use Cayley Hamilton theorem, find the inverse of $A = \begin{bmatrix} -1 & 1 & 1 \\ 3 & 1 & -1 \\ 2 & 2 & 1 \end{bmatrix}$

Sol.
$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 1 & 1 \\ 3 & 1-\lambda & -1 \\ 2 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda^3 + \lambda^2 - 4\lambda - 4 = 0$$

$$\Rightarrow A^3 - A^2 + 4A + 4 = 0$$

$$\Rightarrow A^{-1} = \frac{-A^2 + A + 4I}{4}$$

$$= \frac{1}{4}\left[-\begin{bmatrix} 6 & 2 & -1 \\ -2 & 2 & 1 \\ 6 & 6 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 & 1 \\ 3 & 1 & -1 \\ 2 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \right]$$

$$= \frac{1}{4}\begin{bmatrix} -3 & -1 & 2 \\ 5 & 3 & -2 \\ 4 & -4 & 4 \end{bmatrix}$$

5. If λ_1 and λ_2 are the values of λ for which $\begin{vmatrix} 1 & \lambda & 0 \\ \lambda & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0$, then $\lambda_1 + \lambda_2$ equals (A) -1 (B) 0 (C) 1 (D) 2.

Sol. (B) $\begin{vmatrix} 1 & \lambda & 0 \\ \lambda & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0$

$$\Rightarrow 1(2-1) - \lambda(\lambda) = 0$$

$\Rightarrow \lambda^2 = 1$ i.e. $\lambda_1 + \lambda_2 = 1 - 1 = 0$.

$$\Rightarrow \lambda = \pm 1$$

6. Find the characteristic roots and corresponding characteristic vectors for each of the following matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Sol. The characteristic equation is

$$|A - \lambda I| = \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = -\lambda^3 + 18\lambda^2 - 45\lambda = 0$$

$$\Rightarrow \lambda(\lambda-3)(\lambda-15) = 0$$

$\Rightarrow \lambda = 0, 3, 15$ are the 3 ch. roots of the mtx A.

If $\underset{\sim}{X}$ is a characteristic vector corresponding to the ch. root 0, then we have

$$A\underset{\sim}{X} = \lambda \underset{\sim}{X}$$

$$\Rightarrow [A - \lambda I]\underset{\sim}{X} = \underset{\sim}{0}$$

$$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 8x_1 - 6x_2 + 2x_3 = 0, \quad -6x_1 + 7x_2 - 4x_3 = 0, \quad 2x_1 - 4x_2 + 3x_3 = 0$$

let $x_1 = 1$, then $x_2 = 2$, $x_3 = 2$.

$\therefore \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is a ch. vector corresponding to the

It may similarly be shown by considering the equation

$(A - 3I)\underset{\sim}{X} = \underset{\sim}{0}$,

$(A - 15I)\underset{\sim}{X} = \underset{\sim}{0}$,

that the ch. vectors corresponding to the ch. roots 3 and 15 are arbitrary non-zero multiples of the vectors

$$\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

7. Find the eigen value of the mtx $\begin{bmatrix} 1 & 0 & 0 & -a/2 \\ 0 & 1 & 0 & -a/2 \\ 0 & 0 & 1 & -a/2 \\ 0 & 0 & 0 & a \end{bmatrix}$.

Sol.

$$\det[A - \lambda I] = 0$$

$$\Rightarrow (1-\lambda)(1-\lambda)(1-\lambda)(a-\lambda) = 0$$

$\Rightarrow \lambda = 1, 1, 1, a$ are the eigen values.

8. Let $M = \begin{pmatrix} 1 & 1+i & 2-i \\ 1-i & 2 & 3+i \\ 2+i & 3-i & 3 \end{pmatrix}$. If $B = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$, where $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ and $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$ are LIN eigen vectors of M, then the main diagonal of the matrix $B^{-1}MB$ has

(A) exactly one real entry (B) exactly two real entry

(C) exactly 3 real entry (C) no real entry.

Sol.

$M^T = M^*$

$\Rightarrow$ M is a Hermitian matrix

Since B is invertible mtx.

$\Rightarrow B^{-1}MB$ is a diagonal mtx whose diagonal entries are eigenvalues of M.

We know the eigen values of Hermitian mtx are real.

$\Rightarrow$ all ~~three~~ eigen values are real.

9. Let P be a 3X3 mtx $\ni$ for some c, the linear system $P=c$ has infinite number of solutions. Which one of the following is TRUE?

(A) The linear system $Px=b$ has infinite no. of solutions $\forall b$.

(B) Rank (P) = 3 (C) Rank (P) $\neq 1$ (D) Rank (P) ≤ 2.

Sol. (D) rank (P) $< n \Rightarrow Px=b$ has infinite no. of solutions

$\therefore$ Rank (P) ≤ 2.

10. Let P be a 2X2 mtx $\ni P^{102} = 0$. Then

(A) $P^2=0$ (B) $(1-P)^2=0$ (C) $(1+P)^2=0$ (D) $P=0$

Sol. (A) Since P is mtx of order 2 so its ch. equation is of order 2.

so P^{102} is equal to 0 iff $P^2=0$.

11. Let A be an nxn matrix $\ni P^{-1}AP > 0$ for every non-zero invertible mtx P where P is also an nxn mtx. Which of the following is TRUE?

(A) All eigen values of A are negative (B) All eigen values of A are positive.

Sol. (B) Since given that for mtx A, $P^{-1}AP > 0$.

Here $P^{-1}AP$ is a diagonal mtx whose diagonal elements are eigen values of matrix A. But $P^{-1}AP > 0$ shows the eigen values of A are all positive.

12) Let $P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, The eigenvectors corresponding to the eigenvalues i and $-i$ are respectively

(A) $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\begin{pmatrix} -1 \\ i \end{pmatrix}$ (B) $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ (C) $\begin{pmatrix} -1 \\ i \end{pmatrix}\begin{pmatrix} i \\ -i \end{pmatrix}$ (D) $\begin{pmatrix} i \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ i \end{pmatrix}$

Sol.

$\lambda_1 = i, \ \lambda_2 = -i$

$(P - \lambda_1 I)\underset{\sim}{X} = 0$

$\Rightarrow \begin{bmatrix} 0-i & 1 \\ -1 & -i \end{bmatrix}\underset{\sim}{X} = \underset{\sim}{0}$

$\Rightarrow \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{cases} -x_1 i + x_2 = 0 \\ -x_1 - ix_2 = 0 \end{cases}$

$x_1 = 1, \ x_2 = i$

$\therefore \begin{bmatrix} 1 \\ i \end{bmatrix}$ is the eigen vector corresponding to the eigen value i.

$(P - \lambda_2 I)\underset{\sim}{X}' = \underset{\sim}{0}$

$\Rightarrow \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{cases} ix_1 + x_2 = 0 \\ -x_1 + ix_2 = 0 \end{cases}$

$\therefore x_1 = -1, \ x_2 = i$

$\therefore \begin{bmatrix} -1 \\ i \end{bmatrix}$ is the eigenvector corresponding to the eigen value $-i$.

13) Let P, M, B be $n \times m$ matrices $\ni$ M and N are non-singular. If $\underset{\sim}{x}$ is an eigen vector P corresponding to the eigen value λ, then an eigen vector of $N^{-1}MPM^{-1}N$ corresponding to the eigenvalue λ is

(A) $MN^{-1}\underset{\sim}{x}$ (B) $M^{-1}N\underset{\sim}{x}$ (C) $NM^{-1}\underset{\sim}{x}$ (D) $N^{-1}M\underset{\sim}{x}$

Sol. (C) Since λ is eigenvalue of P and $\underset{\sim}{x}$ be the eigen vector corresponding to it.

$\Rightarrow P\underset{\sim}{x} = \lambda\underset{\sim}{x}$

$$N^{-1}MPM^{-1}N(N^{-1}M\underset{\sim}{x}) = N^{-1}MPM^{-1}(NN^{-1})M\underset{\sim}{x}$$
$$= N^{-1}MPM^{-1}M\underset{\sim}{x}$$
$$= N^{-1}MP(M^{-1}M)\underset{\sim}{x}$$
$$= N^{-1}MP\underset{\sim}{x}$$
$$= N^{-1}M\lambda\underset{\sim}{x}$$
$$= \lambda N^{-1}M\underset{\sim}{x}$$

$\therefore N^{-1}M\underset{\sim}{x}$ is eigenvector corresponding to λ.

14) Let $P = \begin{bmatrix} 1 & 1 \\ i & -1 \end{bmatrix}$. then

(A) P has two linearly independent eigenvectors (B) P has an eigen vectors
(C) P is non-singular (D) $\nexists$ a non-singular $S \ni S^{-1}PS$ is a diagonal mtx

sol.

(A)

---

14) Let P be an $n \times n$ idempotent mtx, that is $P^2 = P$, which of the following is FALSE?

(A) P^T is idempotent
(B) The possible eigenvalues of P are 0 and 1.
(C) The non-diagonal entries of P can be zero.
(D) There are infinite no. of $n \times n$ non-singular matrices that are idempotent

Solution:—

(D) Since P is idempotent matrix, i.e. $P^2 = P$, then P^T is also idempotent, as $(P^2)^T = P^T \Rightarrow (P^T)^2 = P^T$.

$P^2 = P \Rightarrow P(P - I) = 0 \Rightarrow$ eigen values of P are 0 and 1.

The non-diagonal entries of an idempotent mtx can be zero,

e.g. $P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $P^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = P$.

So, (D) is FALSE.

15) Let A and B are any arbitrary square matrices of order 2. then Show that AB and BA have same eigen values but may have different eigen vectors.

Solution:— Let us ~~take~~ an example.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$|AB - \lambda I| = 0 \Rightarrow (1-\lambda)(2-\lambda) = 2$$
$$\Rightarrow \lambda^2 - 3\lambda = 0$$
$$\Rightarrow \lambda(\lambda - 3) = 0$$
$$\Rightarrow \lambda = 0, \lambda = 3.$$

For $\lambda = 0$, $[AB - 0.I]\underset{\sim}{X} = \underset{\sim}{0}$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore x_1 + x_2 = 0$

$\therefore x_1 = 1, x_2 = -1$

$\therefore \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ be an eigen vector corresponding to $\lambda = 0$.

For $\lambda = 3$, $[AB - 3.I]\underset{\sim}{X} = \underset{\sim}{0}$

$$\Rightarrow \begin{bmatrix} 1-3 & 1 \\ 2 & 2-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -2x_1 + x_2 = 0, \; 2x_1 - x_2 = 0$

$\therefore x_2 = 2, x_1 = 1$

$\therefore \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is the eigen vector.

Here $BA = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

$\therefore |BA - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda = 0, 3.$

for $\lambda = 0$, $[BA - 0.I]\underset{\sim}{X} = \underset{\sim}{0}$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 + 2x_2 = 0$

$\Rightarrow x_1 = -2x_2$

$\therefore x_2 = -1/2$, $x_1 = 1$

$\therefore \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$ be the eigenvector corresponding to $\lambda = 0$.

for $\lambda = 3$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ be the eigenvector corresponding to $\lambda = 3$.

$\therefore$ AB & BA have same eigen values but not same eigen vectors.

16) A real 3×3 mtx M has eigen values ±1 and 2. S.T. (i) M is invertible (ii) $M^3 - 2M^2$ is singular (iii) M is diagonalisable.

Sol. We have a mtx M whose 3 eigenvalues are $+1, -1$ and 2. It's ch. equation can be given as

$(M+I)(M-I)(M-2I) = 0$

$\Rightarrow (M^2 - I)(M-2I) = 0$

$\Rightarrow M^3 - 2M^2 = M - 2I$

(i) By the properties of eigenvalues:
determinant of mtx = multiplication of eigenvalues

$|M| = (-1)\times(1)\times(2)$

$|M| = -2$ $\Rightarrow$ M is invertible.

here $|M| \neq 0$

(ii) $|M^3 - 2M^2| = |M - 2I|$

since $|M| = -2$

$|M - 2I| = 0$

$\Rightarrow |M^3 - 2M^2| = 0$

$\Rightarrow M^3 - 2M^2$ is singular.

(iii) Since M having 3 distinct eigenvalues, then the ch. vectors corresponding to distinct characteristics roots of a mtx are LIN.

And an $n\times n$ mtx is diagonalisable if and only if it possesses n linearly independent eigenvectors.

$\Rightarrow$ M is diagonalisable.