DIFFERENTIAL CALCULUS

BY

TANUJIT CHAKRABORTY

Indian Statistical Institute

Mail : tanujitisi@gmail.com

LIMIT

Definition of limit:— $\underset{x\to a}{\text{Lt}} f(x) = l$ if for every $\epsilon > 0$, $\exists$ a 'δ'(>0) such that $|f(x) - l| < \epsilon$ whenever $|x - a| < \delta$.

Theorem:— $\underset{x\to a}{\text{Lt}} f(x) = l$ iff for every sequence $\{x_n\}$ converges to 'a', i.e. $\underset{n\to\infty}{\text{lt}} f(x_n) = l$.

~~If for two sequences {an} and {bn} converge to 'a', then~~

Remark:— Two sequences $\{a_n\}$ and $\{b_n\}$ are said to be equivalent if $\underset{n\to\infty}{\text{lt}} \frac{a_n}{b_n} = l \neq 0$.

Similarly, two functions $f(x)$ and $g(x)$ are said to be equivalent if for large x, $\underset{x\to\infty}{\text{Lt}} \frac{f(x)}{g(x)} = l \neq 0$.

Examples:—

(i) $\frac{\sqrt{n} + n}{n+1} \sim n\sqrt{n}$

(ii) $\sin\frac{1}{n} \sim \frac{1}{n}$.

(iii) $a^{1/n} - 1 \sim \frac{1}{n}$

(iv) $\sqrt{n+1} - \sqrt{n} \sim \frac{1}{\sqrt{n+1} + \sqrt{n}}$

(v) $\sqrt{n+1} - \sqrt{n} \sim \frac{1}{\sqrt{n}}$

(vi) $(n+1)^2 - n^2 \sim n$.

Note:— $\lim_{x\to a} f(x) = l \Leftrightarrow$ for given $\epsilon > 0$, $\exists$ a $\delta > 0 \ni$

$$0 < |x - a| < \delta \Rightarrow |f(x) - l| < \epsilon$$

$$\Leftrightarrow a - \delta < x < a + \delta \Rightarrow l - \epsilon < f(x) < l + \epsilon$$

$$\Leftrightarrow x \in (a - \delta, a + \delta) \Rightarrow f(x) \in (l - \epsilon, l + \epsilon).$$

(except possibly at $x = a$)

In other words, a real number l is a limit of the function f as x approaches to a if for every nbd of l $\exists$ a nbd of 'a' $\ni$ for every x in nbd of 'a', $f(x)$ is in nbd of l.

Ex. If f is given by $f(x) = \begin{cases} \frac{x^2 - a^2}{x^2 + a^2}, & x \neq a \\ 0, & \text{ow} \end{cases}$

then show that $\lim_{x\to a} f(x) = 2a$.

Sol.

$$|f(x) - 2a| < \epsilon$$

$$\Rightarrow \left|\frac{x^2 - a^2}{x - a} - 2a\right| < \epsilon$$

$$\Rightarrow |x - a| < \epsilon$$

Now if we choose a number $\delta \ni 0 \leq \delta \leq \epsilon$, then

$|f(x) - 2a| < \epsilon$ whenever $|x - a| < \delta \Rightarrow \lim_{x\to a} f(x) = 2a$.

Right-hand limit :- A real no. l is said to be a limit of a function f as x tends to 'a' from above (from the right) iff for every given $\epsilon > 0$, $\exists$ a +ve $\delta \ni$

$|f(x) - l| < \epsilon$ whenever $a < x < a + \delta$.

We write $\lim_{x \to a+0} f(x) = l = f(a+0)$

Here $a < x < a + \delta \Rightarrow l - \epsilon < f(x) < l + \epsilon$

Left hand limit :- A real number l is said to be a limit of a function f as x tends to 'a' from below (from the left) iff for every given $\epsilon > 0$, $\exists$ a +ve number $\delta \ni$

$|f(x) - l| < \epsilon$ whenever $a - \delta < x < a$.

We write $\lim_{x \to a-0} f(x) = l = f(a-0)$.

Here $a - \delta < x < a \Rightarrow l - \epsilon < f(x) < l + \epsilon$.

Ques:- A function $f(x)$ is said to have a limit l iff both the RHL and LHL exist and are equal to l.

Sol. By definion of limit,

$\lim_{x \to a} f(x) = l \Leftrightarrow$ for a given $\epsilon > 0$, $\exists$ a $\delta > 0 \ni$

$|f(x) - l| < \epsilon$ whenever $0 < |x - a| < \delta$

$|f(x) - l| < \epsilon$ whenever $a - \delta < x < a + \delta$

$\Leftrightarrow |f(x) - l| < \epsilon$ whenever $a - \delta < x < a$ and

$\Leftrightarrow |f(x) - l| < \epsilon$ whenever $a - \delta < x < a$ and $a < x < a + \delta$

$\Leftrightarrow \lim_{x \to a+} f(x) = l$ and $\lim_{x \to a-0} f(x) = l$.

Hence the result.

Ex. If a function f is defined as $f(x) = [1 - x]$ $\forall x \in \mathbb{R}$.

Then S.T. $\lim_{x \to 0+} f(x) \neq \lim_{x \to 0-} f(x)$.

sol. $x \in (0,1) \Rightarrow 0 < x < 1$

$\Rightarrow 0 < 1 - x < 1$

$\Rightarrow f(x) = 0 \ \forall x \in (0,1)$

So for any $\epsilon > 0$ and for any $\delta > 0$ but less than 1, we have

$|f(x) - 0| < \epsilon$ whenever $x \in (0, \delta)$

$\Rightarrow \lim_{x \to 0+0} f(x) = 0$

$x \in (-1, 0)$, $\Rightarrow -1 < x < 0$

$\Rightarrow 1 < 1 - x < 2$

$\Rightarrow f(x) = 1 \ \forall x \in (-1, 0)$

So for any $\epsilon > 0$ and for $\delta > 0$ but less than 1, we have

$|f(x) - 1| < \epsilon$ whenever $x \in (-\delta, 0)$

$\Rightarrow \lim_{x \to 0-0} f(x) = 1$.

$\therefore f(0+0) \neq f(0-0)$

Rule I:- To evaluate $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ divide the numerator and denominator by the highest power of x involved in $f(x)$ and $g(x)$.

Examples:- 1.

(i) $\lim_{x\to\infty} \{\sqrt{x+1} - \sqrt{x}\}$

Sol. $\lim_{x\to\infty} \{\sqrt{x+1} - \sqrt{x}\}$ Now, $\sqrt{x+1} - \sqrt{x} \sim \frac{1}{\sqrt{x}}$

$= \lim_{x\to\infty} \frac{1}{\sqrt{x}}$

$= 0.$

(ii) $\lim_{x\to\infty} \{\sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x}\}$

Sol. $\sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x}$

$$= \frac{x+\sqrt{x+\sqrt{x}} - x}{\sqrt{x+\sqrt{x+\sqrt{x}}} + \sqrt{x}}$$

$$= \frac{\sqrt{1+\frac{1}{\sqrt{x}}}}{\sqrt{1+\sqrt{\frac{1}{x}+\frac{1}{x^{3/2}}}} + 1}$$

$$\longrightarrow \frac{\sqrt{1+0}}{\sqrt{1+0}+1} = \frac{1}{2} \text{ as } x\to\infty.$$

(iii) $\lim_{x\to\infty} x^3 \{\sqrt{x^2+\sqrt{1+x^4}} - x\sqrt{2}\}$

Sol. $$\lim_{x\to\infty} \frac{x^3\{x^2+\sqrt{1+x^4} - 2x^2\}}{\{\sqrt{x^2+\sqrt{1+x^4}} + x\sqrt{2}\}}$$

$$= \lim_{x\to\infty} \frac{x^3\{\sqrt{1+x^4} - x^2\}}{\{\sqrt{x^2+\sqrt{1+x^4}} + x\sqrt{2}\}}$$

$$= \lim_{x\to\infty} \frac{x^3\{1+\cancel{x^4}-\cancel{x^4}\}}{\{\sqrt{x^2+\sqrt{1+x^4}} + x\sqrt{2}\}\{\sqrt{1+x^4}+x^2\}}$$

$$= \lim_{x\to\infty} \frac{1}{\{\sqrt{1+\sqrt{1+\frac{1}{x^4}}} + \sqrt{2}\}\{\sqrt{\frac{1}{x^4}+1} + 1\}}$$

$$= \frac{1}{\{\sqrt{1+\sqrt{1+0}} + \sqrt{2}\}\{\sqrt{1+0}+1\}}$$

$$= \frac{1}{4\sqrt{2}}.$$

2. (i) $$\lim_{x\to\infty} \frac{ax^{p}+bx^{p-2}+c}{dx^{p}+cx^{p-2}+b}, \quad p>0$$

Sol.

$$= \lim_{x\to\infty} \frac{a+\frac{b}{x^2}+\frac{c}{x^p}}{d+\frac{c}{x^2}+\frac{b}{x^p}}, \quad p>0$$

$$= \frac{a}{d}.$$

(ii) $$\lim_{x\to-\infty} \frac{x^4 \sin\frac{1}{x}+x^2}{1+|x|^3}$$

Sol.

$$= \lim_{x\to-\infty} \frac{x\sin\frac{1}{x}+\frac{1}{x}}{\frac{1}{x^3}-1} \quad \left[\text{Here } \lim_{x\to-\infty} x\sin\frac{1}{x}=1\right].$$

$$= \frac{1+0}{0-1}$$

$$= -1.$$

3. (i) If $\lim_{n\to\infty}\left(an-\frac{1+n^2}{1+n}\right)=b$, then find the values of a and b.

Solution:-

$$b = \lim_{n\to\infty}\left\{an-\frac{1+n^2}{1+n}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{an+an^2-1-n^2}{1+n}\right\}$$

$$= \lim_{n\to\infty}\frac{(a-1)n^2+(an-1)}{(1+n)}$$

If $(a-1)\neq 0$ then $\lim_{n\to\infty}\frac{(a-1)n^2+(an-1)}{(1+n)}=+\infty$ but it is given that the limitting value is b (finite).

$\therefore (a-1)=0 \Rightarrow a=1$. $\therefore$ $b=\lim_{n\to\infty}\frac{n-1}{n+1}=1$.

(ii) If $\lim_{x\to\infty}\left\{\sqrt{x^4+x^2+1}-ax^2-b\right\}=0$, then find a and b?

Solution:-

$$0 = \lim_{x\to\infty}\left\{\sqrt{x^4\mp x^2+1}-ax^2-b\right\}$$

$$= \lim_{x\to\infty}\frac{(x^4\mp x^2+1)-(ax^2+b)^2}{\sqrt{x^4\mp x^2+1}+ax^2+b}$$

$$= \lim_{x\to\infty}\frac{(1-a^2)x^4-(1+2ab)x^2+(1-b^2)}{\sqrt{x^4\mp x^2+1}+ax^2+b}$$

Hence $1-a^2=0 \Rightarrow a=\pm 1$.

If $a=1$, then $-(1+2b)=0$
$\Rightarrow b=-1/2$

If $a=-1$, then $(1-2b)=0$
$\Rightarrow b=1/2$.

■ **Rule II :-** $\lim\limits_{x\to\infty} \dfrac{x^n}{e^x} = 0,\ n>0.$

since, e^x increases more rapidly than x^n, $n>0$, for large x.

(OR) $\lim\limits_{x\to\infty} \dfrac{e^x}{x^n} = \infty,\ n>0.$

Alter:- $\lim\limits_{y\to\infty} \dfrac{(\log_e y)^n}{y} = 0$; where $y = e^x$ & $\log y = x$.

Example:- 1.

(i) $\lim\limits_{x\to\infty} (\log_e x - x)$

sol. $\log_e x - x$

$= \log_e\left(\dfrac{x}{e^x}\right) \longrightarrow -\infty$ as $n\to\infty$. $\left[\because \dfrac{x}{e^x} \to 0 \text{ as } x\to\infty\right]$

(ii) $\lim\limits_{x\to\infty} \dfrac{\log[x]}{x}$

sol. $\log[x] \simeq \log x$ for large x.

$\lim\limits_{x\to\infty} \dfrac{\log[x]}{x} = \lim\limits_{x\to\infty} \dfrac{\log x}{x} = 0.$

■ **Rule III :-** $\lim\limits_{n\to\infty} r^n = 0,\ |r|<1$

If $r>1$, $\lim\limits_{n\to\infty} r^n = +\infty$.

Example:- 1. If $x \in [0,1]$, then $f(x) = \lim\limits_{m\to\infty} \lim\limits_{n\to\infty} \{1+\cos^{2m}(\lfloor n . \pi x)\}$ is

(i) 1 or 2 according as x is rational or irrational.

(ii) 1 or 2 according as x is irrational or rational.

(iii) 1 for all x. [ISI]

(iv) 2 for all x.

Solution:- If x is rational, then $x = \frac{p}{q}$; p and q being integers and $\lfloor n . x$ = an integer, for sufficiently large n.

[If $n = q$, then $\lfloor n . x = \lfloor q . \frac{p}{q}$ = an integer]

Now, $\cos^2(\pi . \lfloor n\, x) = \cos^2(k\pi) = 1.$

$L = \lim\limits_{m\to\infty} \lim\limits_{n\to\infty} \{1+\cos^{2m}(\lfloor n . \pi x)\}$

$= 1+1$

$= 2.$

If x is irrational, then $\lfloor n . x \neq$ an integer.

$\Rightarrow r = \cos^2(\lfloor n . \pi x) < 1.$

& $\cos^{2m}(\lfloor n . \pi x) = r^m \longrightarrow 0$ as $m\to\infty$

& $L = 1+0 = 1.$

Ans : (ii)

Example 2.

(i) $\lim_{n\to\infty} \frac{2^{n+1}+3^{n+1}}{3^n - 2^n}$

Solution:- $\lim_{n\to\infty} \frac{2^{n+1}+3^{n+1}}{3^n-2^n}$

$= \lim_{n\to\infty} 3\left\{\frac{(2/3)^{n+1}+1}{1-(2/3)^n}\right\}$

$= 3\left\{\frac{0+1}{1-0}\right\}$

$= 3.$

(ii) $\lim_{n\to\infty} \frac{x^{2n}-1}{x^{2n}+1}, x\in\mathbb{R}$

Solution:-

$L = \lim_{n\to\infty} \frac{x^{2n}-1}{x^{2n}+1}, x\in\mathbb{R}$

If $|x|<1$, $x^{2n}\to 0$ as $n\to\infty$

$\therefore L = \frac{0-1}{1+0} = -1.$

If $|x|>1$, then $x^2>1$, $\frac{1}{x^{2n}} \to 0$ as $n\to\infty$.

$L = \lim_{n\to\infty} \frac{\left(1-\frac{1}{x^{2n}}\right)}{\left(1+\frac{1}{x^{2n}}\right)} = \frac{1-0}{1+0} = 1.$

If $|x|=1$, $x^{2n}=1$.

$\therefore L = \frac{1-1}{1+1} = 0.$

(iii) $\lim_{n\to\infty} \frac{4^n+3^n}{4^n-3^n}$

Solution:- $\lim_{n\to\infty} \frac{4^n+3^n}{4^n-3^n}$

$= \lim_{n\to\infty} \frac{1+\left(\frac{3}{4}\right)^n}{1-\left(\frac{3}{4}\right)^n}$

$= \frac{1+0}{1-0}$

$= 1.$

Rule IV :-

(i) $\lim_{x\to 0} (1+x)^{1/x} = e$

$\Leftrightarrow \lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

$\Leftrightarrow \lim_{x\to\infty} \left(1+\frac{k}{x}\right)^x = e^k$

(ii) $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$

$\Leftrightarrow \lim_{x\to 0} (1+x)^{1/x} = e \Leftrightarrow \lim_{x\to 0} (1+kx)^{1/x} = e^k$

$\Leftrightarrow \lim_{x\to 0} \frac{\log(1+x)}{x} = 1.$

(iii) $\lim_{x\to\infty} \left(1+\frac{a}{x}+\frac{b}{x^2}\right)^x = e^a \Leftrightarrow \lim_{x\to 0} \left(1+ax+bx^2\right)^{\frac{1}{x}} = e^a.$

Proof:- $\ln L = \lim_{x\to\infty} x \ln\left(1+\frac{a}{x}+\frac{b}{x^2}\right)$

$= \lim_{x\to\infty} x\left\{\left(\frac{a}{x}+\frac{b}{x^2}\right) - \frac{1}{2}\left(\frac{a}{x}+\frac{b}{x^2}\right)^2 + \cdots\cdots\right\}$

$= \lim_{x\to\infty} \left\{a + (\quad)\frac{1}{x} + (\quad)\frac{1}{x^2} + \cdots\cdots\right\}$

$= a$

$\therefore L = e^a.$

Examples:- 1. If α, β be the roots $ax^2+bx+c=0$, then $\lim_{x\to\alpha} \left(1+ax^2+bx+c\right)^{\frac{1}{x-\alpha}}$ is

(i) $a(\alpha-\beta)$ (ii) $\ln|a(\alpha-\beta)|$ (iii) $e^{a(\alpha-\beta)}$ (iv) $e^{a|\alpha-\beta|}$.

Solution:- $\lim_{x\to\alpha} \left\{1+ax^2+bx+c\right\}^{\frac{1}{(x-\alpha)}}$

$= \lim_{x\to\alpha} \left\{1+a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)\right\}^{\frac{1}{(x-\alpha)}}$

$= \lim_{x\to\alpha} \left\{1+a(x-\alpha)(x-\beta)\right\}^{\frac{1}{(x-\alpha)}}$

$= \left\{\lim_{u\to 0} (1+u)^{1/u}\right\}^{\lim_{x\to\alpha} a(x-\beta)}$

$= e^{a(\alpha-\beta)}.$

2 (i) $\lim_{x\to\infty}\left(\frac{x+6}{x+1}\right)^{x+4}$

Sol. $\lim_{x\to\infty}\left(\frac{x+6}{x+1}\right)^{(x+4)}$

$= \lim_{x\to\infty} \frac{\left(1+\frac{6}{x}\right)^x}{\left(1+\frac{1}{x}\right)^x} \cdot \frac{\left(1+\frac{6}{x}\right)^4}{\left(1+\frac{1}{x}\right)^4}$

$= \frac{e^6}{e} \cdot \frac{(1+0)^4}{(1+0)^4}$

$= e^5.$

(ii) $\lim_{x\to\infty}\left(\frac{3x-4}{3x+2}\right)^{\frac{x+1}{3}}$

Sol. $\lim_{x\to\infty}\left(\frac{3x-4}{3x+2}\right)^{\frac{x+1}{3}}$

$= \lim_{x\to\infty}\left\{\left(\frac{3-\frac{4}{x}}{3+\frac{2}{x}}\right)^{1/3}\right\}^{x} \cdot \lim_{x\to\infty}\left\{\frac{3-4/x}{3+2/x}\right\}^{1/3}$

$= \left(\frac{e^{-4}}{e^2}\right)^{1/3} \cdot 1.$

$= e^{-2/3}.$

(iii) $\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+5x+2}\right)^x$

Sol. $\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+5x+2}\right)^x$

$= \lim_{x\to\infty} \frac{\left(1+\frac{5}{x}+\frac{3}{x^2}\right)^x}{\left(1+\frac{1}{x}+\frac{2}{x^2}\right)^x}$

$= \frac{e^5}{e^1}$

$= e^4.$

(iv) $\operatorname{lt}_{x\to1} x^{\frac{1}{x-1}}$

Sol. $x-1 = y \to 0$

$\therefore \operatorname{lt}_{x\to1} x^{\frac{1}{x-1}} = \operatorname{lt}_{y\to0} (1+y)^{1/y}$

$= e.$

(v) $\operatorname{lt}_{x\to0}\left(\frac{1+5x^2}{1+3x^2}\right)^{1/x^2}$

Sol. $= \operatorname{lt}_{x^2\to0}\left(\frac{1+5x^2}{1+3x^2}\right)^{1/x^2}$

$= \frac{\operatorname{lt}_{x^2\to0}(1+5x^2)^{1/x^2}}{\operatorname{lt}_{x^2\to0}(1+3x^2)^{1/x^2}}$

$= \frac{e^5}{e^3}$

$= e^2.$

Rule V :→ Limits by using expansion:—

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\dots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots\dots$$

[Use these expansions only when variable in the expansion tends to zero, i.e. $x \to 0$]

Examples:—

1. Find the values of a and b, c such that

(a) $\lim_{x\to 0} \dfrac{x(1+a\cos x) - b\sin x}{x^3} = 1.$

Sol.

$$1 = \lim_{x\to 0} \frac{x\left\{1+a\left(1-\frac{x^2}{2!}+\dots\right)\right\} - b\left\{x - \frac{x^3}{3!}+\dots\right\}}{x^3}$$

$$= \lim_{x\to 0} \frac{x(1+a-b) + \left\{-\frac{a}{2!}+\frac{b}{3!}\right\}x^3 + \dots\dots}{x^3}$$

Here $1+a-b=0$ and $-\frac{a}{2!}+\frac{b}{3!} = 1$

$\Rightarrow a-b=-1,$ $\Rightarrow b-3a=6$

$\therefore a = -5/2,\ b = -3/2.$

(b) $\lim_{x\to 0} \dfrac{axe^x - b\log(1+x) + cxe^{-x}}{x^2\sin x} = 2.$

Sol.

$$2 = \lim_{x\to 0} \frac{ax\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\dots\infty\right) - b\left\{x - \frac{x^2}{2}+\dots\right\} + cx\left\{1-\frac{x}{1!}+\frac{x^2}{2!}-\dots\right\}}{x^2\left\{x - \frac{x^3}{3!}+\dots\right\}}$$

$$= \lim \frac{(a-b+c)x + \left(a+\frac{b}{2}-c\right)x^2 + \left(\frac{a}{2}-\frac{b}{3}+\frac{c}{2}\right)x^3}{x^3\left\{1-\frac{x^2}{3!}+\dots\right\}}$$

$\therefore (a-b+c)=0,\ a+\frac{b}{2}-c=0,\ \frac{a}{2}-\frac{b}{3}+\frac{c}{2} = 2$

$\therefore a=3,\ b=12,\ c=9.$

(c) $\lim_{x\to 0} \dfrac{a\cos x + bx\sin x - 5}{x^4} = c$, finite.

Sol.

$$c = \lim_{x\to 0} \frac{a\left\{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots\right\} + bx\left\{x - \frac{x^3}{3!}+\dots\right\} - 5}{x^4}$$

$$= \lim_{x\to 0} \frac{(a-5) + \left(-\frac{a}{2!}+b\right)x^2 - \left(\frac{a}{4!}-\frac{b}{3!}\right)x^4 + \dots\dots}{x^4}$$

$\therefore a-5=0$, $\therefore -\frac{a}{2}+b=0$, $\therefore \frac{a}{4!}-\frac{b}{3!} = c$

$\Rightarrow a=5$ $\Rightarrow b=\frac{5}{2}$ $\Rightarrow \frac{5}{24} - \frac{5}{12} = c$

$\Rightarrow -\frac{5}{24} = c.$

2. (i) $\lim_{x\to 0} \dfrac{\sin x - x + \frac{x^3}{6}}{x^5}$

(v) $\lim_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{1/x^2}$

$= \lim_{x\to 0}\left(\dfrac{x - \frac{x^3}{3!}}{x}\right)^{1/x^2}$

$= \lim_{x\to 0}\left(1 - \dfrac{x^2}{3!}\right)^{1/x^2}$

$= e^{-\frac{1}{3!}}$

$= e^{-\frac{1}{6}}.$

Sol. $\lim_{x\to 0} \dfrac{\sin x - x + x^3/6}{x^5}$

$= \lim_{x\to 0} \dfrac{\frac{x^5}{5!} + \cdots\cdots}{x^5}$

$= \dfrac{1}{5!}$

$= \dfrac{1}{120}$

(ii) $\lim_{x\to 0}\left(\dfrac{\tan x}{x}\right)^{1/x^2}$

Sol. $\left(\dfrac{\tan x}{x}\right)^{1/x^2}$

$= \left(1 + \dfrac{x^2}{3}\right)^{1/x^2}$

$\to e^{1/3}$ as $x \to 0$.

(iii) $\lim_{x\to a}\left(\dfrac{\tan x}{\tan a}\right)^{\frac{1}{(x-a)}}$

Sol. $\lim_{x\to a} \ln L = \lim_{x\to a} \dfrac{\log\tan x - \log\tan a}{(x-a)}$

$= \lim_{x\to a} \dfrac{f(x) - f(a)}{x-a}$

$= f'(a)$

$= \dfrac{d}{da}(\log\tan a)$

$= \dfrac{1}{\tan a}\cdot\sec^2 a$

$= 2\operatorname{cosec} 2a.$

$\lim_{x\to a} L = e^{2\operatorname{cosec}2a}.$

(iv) $\lim_{x\to\infty}\left(\sin\frac{1}{x} + \cos\frac{1}{x}\right)^x$ $= \lim_{x\to\infty}\left(\frac{1}{K}+1\right)^x = e.$

Sol. $\lim_{x\to\infty}\left(\sin\frac{1}{x} + \cos\frac{1}{x}\right)^x$

$= \lim_{x\to\infty}\left(1 + \sin\frac{2}{x}\right)^{\frac{x}{2}}$ $\dfrac{1}{\sin 2/x}\cdot\dfrac{\sin\frac{2}{x}}{\frac{2}{x}}$

$= \lim_{x\to\infty}\left(1 + \sin\frac{2}{x}\right)$

$= \left\{\lim_{u\to 0}(1+u)^{u}\right\}^{\lim_{v\to 0}\frac{\sin v}{v}}$; $u = \sin\frac{2}{x}$, $v = \frac{2}{x}$

$= (e^1)^1$

$= e.$

Rule VI :→ L'Hospital Rule :—

$$\underset{x\to a}{\mathrm{Lt}} \frac{f(x)}{g(x)} \quad \left[\text{form } \frac{0}{0} \text{ or } \frac{\infty}{\infty} \right]$$

$$= \underset{x\to a}{\mathrm{lt}} \frac{f'(x)}{g'(x)}.$$

Ex.1. (i) $\lim\limits_{x\to\infty} \dfrac{x^n}{e^x}, n \in \mathbb{N}$.

Sol. $\underset{x\to\infty}{\mathrm{lt}} \dfrac{x^n}{e^x}$

$$= \underset{x\to\infty}{\mathrm{lt}} \frac{nx^{n-1}}{e^x} \quad \left[\left(\frac{\infty}{\infty}\right) \text{form} \right]$$

$$\vdots$$

$$= \underset{x\to\infty}{\mathrm{lt}} \frac{n!}{e^x}$$

$$= 0.$$

(ii) $\lim\limits_{x\to 1+} \dfrac{\int_1^x |t-1|\,dt}{\sin(x-1)}$

Sol. $\lim\limits_{x\to 1+} \dfrac{\int_1^x |t-1|\,dt}{\sin(x-1)} \quad \left[\text{form} \left(\frac{0}{0}\right) \right].$

$$= \lim_{x\to 1+} \frac{|x-1|}{\cos(x-1)}$$

$$= \frac{0}{\cos 0} = \frac{0}{1} = 0.$$

(iii) $\lim\limits_{x\to 1} \displaystyle\int_{f(1)}^{f(x)} \frac{2t}{x-1}\,dt$

$= \lim\limits_{x\to 1}$

$=$

$=$

$= 2f(1)f'(1).$

(iv) $\lim\limits_{x\to 0} \dfrac{\int_0^x e^{t^2}(e^t-1)^2\,dt}{x(1-\cos x)}$

Sol. $\lim\limits_{x\to 0} \dfrac{\int_0^x e^{t^2}(e^t-1)^2\,dt}{x(1-\cos x)} \quad \left[\left(\frac{0}{0}\right) \text{form} \right]$

$$= \lim_{x\to 0} \frac{e^{x^2}(e^x-1)^2}{x\sin x + 1 - \cos x}$$

$$= \left(\lim_{x\to 0} e^{x^2}\right) \cdot \frac{\lim\limits_{x\to 0} \left(\frac{e^x-1}{x}\right)^2}{\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right) + \lim\limits_{x\to 0} \frac{1}{2}\left(\frac{\sin x/2}{x/2}\right)^2}$$

$$= 1 \cdot \frac{1}{1 + \frac{1}{2}}$$

$$= \frac{2}{3}.$$

Theorem:– If $\lim_{x\to a} f(x)$ exists, then it is unique.

Non-existance of Limit:–

Ex. 1. Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Solution:–
$$f(x) = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$$

For given $\epsilon(>0)$
Now $|f(x)-1| < \epsilon$ when $0<x<\delta_1$ for any $\delta_1(>0)$.

$\therefore f(0+0) = 1$.
Again $|f(x)+1| < \epsilon$ when $-\delta_2 < x < 0$ for any $\delta_2(>0)$.

$\therefore f(0-0) = -1$
Hence $f(0+0) \neq f(0-0)$
$\therefore$ limit does not exist here.

Ex. 2. Show that $\lim_{x\to 0} \frac{1}{x}$ does not exist.

Sol. Here ~~[illegible]~~ $f(x) = \frac{1}{x}$ is defined in $\mathbb{R} - \{0\}$ but unbounded in every nbd of 0.

$\therefore \lim_{x\to 0} f(x)$ does not exist.

Ex. 3. S.T. $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

Sol. Do yourself.

Ex. 4. $\lim_{x\to 0} x\tan\frac{1}{x}$ does not exist.

Sol. $\lim_{x\to 0} \frac{\left(\tan\frac{1}{x}\right)}{1/x}$

$= \lim_{z\to\infty} \frac{\tan z}{z}$ does not exist.

Alt. $\lim_{x\to 0} \tan\frac{1}{x}$ does not exist.

Theorem:– Let $D \subset \mathbb{R}$ and f and g be functions on D to $\mathbb{R}$. Let $c \in D'$. If f is bounded on $N'(c) \cap D$ for some deleted neighbourhood $N'(c)$ of c and $\lim_{x\to c} g(x) = 0$ then $\lim_{x\to c} (f\cdot g)(x) = 0$.

Examples:– 1. Prove that $\lim_{x\to 0} x\sin\frac{1}{x^2} = 0$.

Sol. Here $\lim_{x\to 0} x = 0$ and $\sin\frac{1}{x^2}$ is bounded in some deleted neighbourhood of 0. Therefore $\lim_{x\to 0} x\sin\frac{1}{x^2} = 0$.

2. Prove that $\lim_{x\to 0} \sqrt{x}\sin\frac{1}{x} = 0$.

Sol. $\lim_{x\to 0} \sqrt{x} = 0$ and $\sin\frac{1}{x}$ is bounded in some deleted neighbourhood of 0.
Therefore $\lim_{x\to 0} \sqrt{x}\sin\frac{1}{x} = 0$.

Sandwich theorem:– Let $D \subset \mathbb{R}$ and f, g, h be functions on D to $\mathbb{R}$. Let $c \in D'$. If $f(x) \le g(x) \le h(x)\ \forall x \in D - \{c\}$ and if $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = l$, then $\lim_{x\to c} g(x) = l$.

Ex. S.T. $\lim_{x\to 0} x\cos\frac{1}{x} = 0$.

Sol. $-1 \le \cos\frac{1}{x} \le 1\ \forall x>0$
$-x \le x\cos\frac{1}{x} \le x\ \forall x>0$

$x \le x\cos\frac{1}{x} \le -x\ \forall x<0$
$\therefore -|x| \le x\cos\frac{1}{x} \le |x|\ \forall x \neq 0$.
$\therefore$ As $\lim_{x\to 0} |x| = 0$, so $\lim_{x\to 0} x\cos\frac{1}{x} = 0$.

Cauchy's criterion for finite limit :-

Statement:- A real valued function f tends to a finite limit as $x \to c$ iff for every $\epsilon\ (>0)$, $\exists$ a deleted neighbourhood $N(c)$ of c $\ni$

$$|f(x_1) - f(x_2)| < \epsilon \quad \forall\ x_1, x_2 \in N(c).$$

Ex(1):- Use cauchy's criterion to show that $\lim_{x\to 0} \sin\frac{1}{x}$ or $\lim_{x\to 0}\cos\frac{1}{x}$ does not exist.

Solution:- Here $\sin\frac{1}{x}$ is defined for all $x \in \mathbb{R}$ and $x \neq 0$.

$\therefore$ 0 is a limit point of the domain of definition of the function.

Now, in any deleted nbd of 0 we take two points $x_1 = \frac{1}{(4n+1)\frac{\pi}{2}}$ and

$x_2 = \frac{1}{(4n-1)\frac{\pi}{2}}$ for sufficiently large value of n and

choose $\epsilon = 0.5$.

Now, $f(x_1) = \sin\frac{1}{x_1} = \sin(4n+1)\frac{\pi}{2} = 1$

and $f(x_2) = \sin\frac{1}{x_2} = \sin(4n-1)\frac{\pi}{2} = -1$.

$\therefore |f(x_1) - f(x_2)| = 2 \nless 0.5$

$\therefore$ By cauchy's criterion, $\lim_{x\to 0}\sin\frac{1}{x}$ does not exist.

In case of finiding $\lim_{x\to 0}\cos\frac{1}{x}$, we choose $x_1 = \frac{1}{2n\pi}$ and $x_2 = \frac{1}{(2n+1)\pi}$ in the neighbourhood of $x=0$, for sufficiently large value of n and then $|f(x_1) - f(x_2)| = 2 \nless \epsilon\ (\epsilon > 0)$

$\therefore \lim_{x\to 0}\cos\frac{1}{x}$ does not exist.

Ex.(2):- A function $f: \mathbb{R} \to \mathbb{R}$ is defined as follows

$$f(x) = \begin{cases} 1 & \text{when } x \in \mathbb{Q} \\ -1 & \text{when } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Show that $\lim_{x\to a} f(x)$ does not exist.

Solution:- a is a limit point of the domain of definition of f.

We choose $\epsilon = 0.5$ and take two points x_1 and x_2 in any deleted nbd $N'_\delta(a)$ of a, $\ni$ x_1 is rational and x_2 is irrational.

Then $f(x_1) = 1$ and $f(x_2) = -1$.

$\therefore |f(x_1) - f(x_2)| = 2 \nless 0.5$

$\therefore$ By cauchy's condition $\lim_{x\to a} f(x)$ does not exist.

By $|f(x)| = 1 \ \forall x \in \mathbb{R}$

$\therefore \lim_{x\to a}|f(x)| = 1$.

Conclusion:- $\lim_{x\to c} f(x) = l$, then $\lim_{x\to c}|f(x)| = |l|$. But the converse is not true.

SOLVED EXAMPLES:-

1. Examine if $\lim_{x\to 0}\left(\sin\frac{1}{x}+x\sin\frac{1}{x}\right)$ exists or not.

Solution:- We have that $\lim_{x\to 0}\sin\frac{1}{x}$ does not exist.

We know if $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, then

$\lim_{x\to c}\{f(x)+g(x)\}$ exists.

Hence $\lim_{x\to 0}\left(\sin\frac{1}{x}+x\sin\frac{1}{x}\right)$ does not exist.

2. Show that $\lim_{x\to\infty}\frac{[x]}{x}=1$, where $[x]$ denotes the greatest integer contained in x not greater than x.

Sol. We see that $|[x]-x|<1$.

choose $\epsilon>0$

Now $\left|\frac{[x]}{x}-1\right|=\left|\frac{[x]-x}{x}\right|<\frac{1}{x}<\epsilon$ if $x>\frac{1}{\epsilon}$.

$\therefore \lim_{x\to\infty}\frac{[x]}{x}=1.$

3. If $f(x)=x$, when x is rational

$=-x$, when x is irrational

Show that $\lim_{x\to a} f(x)$ exists only when $a=0$.

Sol. The function is defined for all real numbers of which a is a limit point.

For $a>0$, $\delta>0$ let $x_1\in\mathbb{Q}$ and $x_2\in\mathbb{R}-\mathbb{Q}$ in $(a, a+\delta)$.

Then $|f(x_1)-f(x_2)|=|x_1+x_2|>2a$

Hence by cauchy's condition the following limit does not exist for $a>0$.

Similarly, for $a<0$, $\lim_{x\to a} f(x)$ does not exist.

When $a=0$, we see that $|f(x)-0|<\epsilon$

$\Rightarrow |x|<\epsilon$ when $0<|x-0|<\epsilon=\delta$.

$\therefore$ The limit exists only when $a=0$.

4. If $\lim_{x\to a} g(x)=b$ and $\lim_{y\to b} f(y)=f(b)$, then

$$\lim_{x\to a} f\{g(x)\}=f\left(\lim_{x\to a} g(x)\right).$$

Sol. Use Cauchy's criterion.

This above result is known as limit of a function of a function.

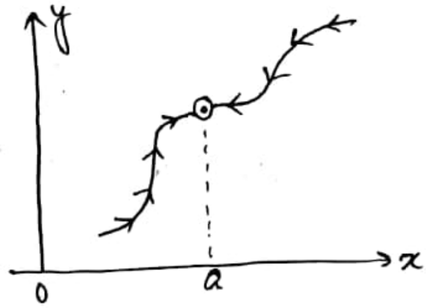
CONTINUITY AND DIFFERENTIABILITY

Continuity:- A function $f(x)$ is continuous at $x=a$ if for every $\epsilon>0$, $\exists$ a $\delta(>0)$ such that

$$|f(x)-f(a)|<\epsilon \Rightarrow |x-a|<\delta.$$

Alter:- A function $f(x)$ is continuous at $x=a$ if

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) = f(a)$$



Note:- 1. The discontinuity is removable by defining a new function $g(x)$ $\ni$

$$g(x)=\begin{cases} f(x) & \text{if } x\neq a \\ l=\lim_{x\to a} f(x) & \text{if } x=a \end{cases}$$

$f(x)$ is discontinuous at $x=a$. This discontinuity is not removable as $\lim_{x\to a} f(x)$ does not exist.

2. If for two sequences $\{x_n\}$ and $\{y_n\}$ converge to 'a', but $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$, then $\lim_{x\to a} f(x)$ does not exist.

Theorem:- $\lim_{x\to a} f(x)=l$ iff for every sequence $\{x_n\}$ converges to 'a', i.e. $\lim_{n\to\infty} f(x_n)=l$.

Examples:-

1. Let $f(x)=\begin{cases} 1 & \text{if } x \text{ is rational} \\ 2 & \text{if } x \text{ is irrational} \end{cases}$

Then show that $f(x)$ is nowhere continuous.

Solution:- $f(x)=\begin{cases} 1 & , x \text{ is rational} \\ 2 & , x \text{ is irrational} \end{cases}$

Let $a\in\mathbb{R}$ is a fixed real number.
Then consider a sequence $\{x_n\}$ of rational numbers converges to 'a'.
Then $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 1 = 1$.

Now consider a sequence $\{y_n\}$ of irrational nos. converges to 'a'.
Then $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} 2 = 2$.

$$\therefore \lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$$

$\therefore$ Hence $\lim_{x\to a} f(x)$ does not exist $\forall\, a\in\mathbb{R}$.

$\Rightarrow f(x)$ is discontinuous everywhere in $\mathbb{R}$.

2. Let $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$

then show that $f(x)$ is continuous at $x=0$.

Solution:- Let $a \in \mathbb{R}$ is a fixed real no.
Then consider a sequence $\{x_n\}$ of rational nos. converges to 'a'.

$$\text{Then } \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0$$

Let $\{y_n\}$ be another sequence of irrational nos. converges to 'a'.

$$\text{Then } \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} y_n^2 = a^2.$$

Now, $f(x)$ is continuous when $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$

$$\Rightarrow a^2 = 0 \Leftrightarrow a = 0.$$

$\therefore f(x)$ is continuous at $x=0$.

~~More similar problems:-~~

Differentiability:- A function $f(x)$ is differentiable at $x=a$,

if $\displaystyle\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists,

iff $\displaystyle\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} = \text{LHD} = \text{RHD} = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$

$= \text{finite}.$

Note that, $\displaystyle f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$

Remark:- Every differentiable function is continuous but the converse is not true.

Note:- By Taylor's series,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots\cdots$$
$$f(a-h) = f(a) - hf'(a) + \frac{h^2}{2!} f''(a) \cdots\cdots$$

Note that, $\displaystyle\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \left\{ f'(a) + \frac{h^2}{2!} f''(a) + \cdots\cdots \right\}$

$= f'(a).$

$$\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{h} = \lim_{h\to 0} \left\{ f'(a) + \frac{h^2}{3!} f'''(a) + \cdots \right\}$$

$= f'(a).$

and $\displaystyle\lim_{h\to 0} \frac{f(a+h)+f(a-h)-2f(a)}{h^2} = \lim_{h\to 0} \left\{ f''(a) + \frac{h^2}{12} f^{iv}(a) + \cdots\cdots \right\}$

$= f''(a).$

Ex. 1. If $f(x) = \begin{cases} x^n \sin\frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

Show that $f(x)$ is continuous but not differentiable at $x=0$ if $0<n\leq 1$.

Solution:- $\left|f(x) - f(0)\right| = \left|x^n \sin\frac{1}{x}\right| \leq |x|^n < \epsilon$ whenever $|x-0| < \epsilon^{1/n} = \delta$ if $n>0$

Hence $n>0$, $f(x)$ is continuous at $x=0$.

For differentiability, $\left|\frac{f(x)-f(0)}{x-0}\right| = \left|x^{n-1} \cdot \sin\frac{1}{x}\right| \leq |x|^{n-1} < \epsilon$

whenever $|x-0| < \epsilon^{\frac{1}{n-1}} = \delta$ if $n>1$

$\therefore$ for $0 \leq n \leq 1$, $f(x)$ is not differentiable.

Ex. 2. $f(x) = \begin{cases} ax^2 - b & , |x|<1 \\ -\frac{1}{|x|} & , |x| \geq 1 \end{cases}$

The above function is continuous & differentiable. Find a and b.

Solution:-

$$f(x) = \begin{cases} \frac{1}{x} & , x \leq -1 \\ ax^2 - b & , -1<x<1 \\ -\frac{1}{x} & , x \geq 1 \end{cases}$$

$f(x)$ is continuous at $x = \pm 1$.

Now, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (ax^2 - b) = a-b$

and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \left(-\frac{1}{x}\right) = -1$.

$\therefore a-b = -1$.

$$f'(x) = \begin{cases} -\frac{1}{x^2} & , x<-1 \\ 2ax & , -1<x<1 \\ \frac{1}{x^2} & , x \geq 1 \end{cases}$$

$LHD = f'(1^-) = \lim_{x\to 1^-} f'(x) = \lim_{x\to 1^-} (2ax) = 2a$

$RHD = f'(1^+) = \lim_{x\to 1^+} f'(x) = \lim_{x\to 1^+} \left(\frac{1}{x^2}\right) = 1$

$\therefore 2a = 1$

$\Rightarrow a = \frac{1}{2}$

$\therefore a-b = -1$

$\Rightarrow b = 3/2$.

Ex.(3):- S.T. $f(x) = x|x|$ is differentiable at $x \in \mathbb{R}$ but not twice differentiable at $x = 0$.

Sol.:→

$$f(x) = \begin{cases} x^2 & ; x \geq 0 \\ -2x & ; x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x & , x \geq 0 \\ -2x & , x < 0 \end{cases} \quad \therefore f'(0) \text{ exists.}$$

$$f''(x) = \begin{cases} 2 & ; x > 0 \\ -2 & ; x < 0 \end{cases}$$

$f''(0^-) = -2$, $f''(0^+) = 2$

$\therefore$ It is not twice differentiable at $x = 0$.

Ex.(4):- Show that $f(x) = |x|$ is continuous but not differentiable at $x=0$

Sol. Here $f(x) = |x| = \begin{cases} x , & x \geq 0 \\ -x , & x < 0 \end{cases}$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x)$
$= 0$.

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x) = 0$. and $f(0) = 0$.

Hence $f(x)$ is continuous at $x = 0$.

Note that $f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

$f'(0^+) = 1$, $f'(0^-) = -1$.

LHD $\neq$ RHD

Hence $f'(0)$ does not exist.

Ex.5. S.T. $|\ln x|$ is not differentiable at $x = 1$ but continuous at $x = 1$.

Sol. $f(x) = |\ln x| = |u|$ is not differentiable at $u = 0$, i.e. $\ln x = 0$
$\Rightarrow x = 1$.

Ex.6. $|\sin x|$ is not differentiable at $x =$ ____.

Sol. $\sin x = 0$
$\Rightarrow x = n\pi$, $n \in I$.

Ex.7. At which values of x. $f(x) = |x^2 - 3x + 2| + \cos|x|$ is not differentiable.

Sol. $|x^2 - 3x + 2| = |(x-1)(x-2)|$ is not differentiable at $x = 1, 2$.

$$\cos|x| = \begin{cases} \cos x , & \text{if } x \geq 0 \\ \cos(-x) & \text{if } x < 0 \end{cases} = \cos x \;\forall x.$$

$\therefore$ At $x = 1, 2$; $f(x)$ is not differentiable.

Ex.8. At which values of x, $f(x) = (x-1)|x^2 - 1| + |x - 2|$ is not differentiable

Sol.

$$f(x) = \begin{cases} -(x-1)^2 |x+1| + |x-2| & , x < 1 \\ (x-1)^2 |x+1| + |x-2| & , x > 1 \end{cases}$$

is not differentiable at $x = 1, 2$.

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Ex. 9. If $f(x) = a|\sin x| + b|x| + c|x|^3$ is differentiable at $x=0$, then $a=b=0$; $c \in \mathbb{R}$.

Solution:- $f(x) = a|\sin x| + b|x| + c|x^3|$

$|\sin x|$ is not differentible at $\sin x = 0$ at $x = n\pi$,

$|x|$ is not differentiable at $x=0$

but $|x|^3$ is differentiable at $x=0$.

$\therefore a=b=0$, $c \in \mathbb{R}$.

Ex. 10. $f(x) = |x|^p$ is differentible iff $p>1$.

Sol.

$$f(x) = \begin{cases} x^p, & x \geq 0 \\ (-x)^p, & x<0 \end{cases}$$

$$f'(x) = \begin{cases} p x^{p-1} & \text{if } x>0 \\ p(-x)^{p-1}(-1) & \text{if } x<0 \end{cases}$$

$f'(0^-) = \lim_{x \to 0^-} f'(x) = \lim_{x \to 0^-} (-p)(-x)^{p-1} = 0$ if $p>1$

$f'(0^+) = 0$ if $p>1$.

Ex. 11. If the derivative of $f(x) = \begin{cases} bx^2 + ax + 4, & x \geq -1 \\ ax^2 + b, & x<-1 \end{cases}$

is everywhere continuous then

(i) $a=2, b=3$, (ii) $b=2, a=3$ (iii) $a=-2, b=-3$.

Solution:- $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$

$\Rightarrow a(-1)^2 + b = b(-1)^2 + a(-1) + 4$

$\Rightarrow a+b = b-a+4$

$\Rightarrow a=2$

$$f'(x) = \begin{cases} 2bx + a & \text{if } x>-1 \\ 2ax & \text{if } x<-1 \end{cases}$$

$f'(-1^-) = 2a(-1)$

$f'(-1^+) = 2b(-1) + a$

$\therefore 2a = 2b - a$

$\Rightarrow b=3$

Ans (i)

Rolle's Theorem:– If $f(x)$ is

(i) continuous on $[a,b]$
(ii) differentiable on (a,b)
(iii) $f(a) = f(b)$

then $\exists$ a $\xi \in (a,b)$ such that $f'(\xi) = 0$.

Ex. Show that for the function $f(x) = \sqrt{x} - x,\ 0 \le x \le 1$, Rolle's theorem is not applicable but $\exists$ a $\xi \in (0,1) \ni f'(\xi) = 0$.

Sol. $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} (\sqrt{x} - x)$ does not exist.

Hence, Rolle's theorem is not applicable.

Now, $f'(x) = \frac{1}{2\sqrt{x}} - 1 = 0$

$\Rightarrow x = \frac{1}{4} \in (0,1)$.

Lagrange's mean value theorem:– If $f(x)$ is

(i) continuous on $[a,b]$
(ii) differentiable on (a,b)

then $\exists$ a 'c' $\in (a,b) \ni$

$$\frac{f(b) - f(a)}{b-a} = f'(c)$$

$$\Rightarrow f(b) = f(a) + f'(c)(b-a).$$

Ex. If $|f(x) - f(y)| \le |x-y|^2\ \forall x, y$ then s.t. $f(x)$ is constant.

Sol.
$$\left|\frac{f(x) - f(y)}{x-y}\right| \le |x-y|$$

$$0 \le \left|\lim_{x \to y} \frac{f(x) - f(y)}{x-y}\right| \le \lim_{x \to y} |x-y| = 0\ \forall y$$

$$\Rightarrow 0 \le |f'(y)| \le 0$$

$$\Rightarrow f'(y) = 0\ \forall y$$

$\Rightarrow f(x)$ is constant.

EXAPLES OF ROLLE'S THEOREM & LAGRANGE'S M.V.T :-

1. Show that $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$.

Sol. Let $f(x) = \tan^{-1}x$

& from Mean-value theorem,

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\Rightarrow \frac{\tan^{-1}b - \tan^{-1}a}{b-a} = \frac{1}{1+c^2}, \quad a<c<b$$

$$\Rightarrow \tan^{-1}b - \tan^{-1}a = \frac{b-a}{1+c^2}$$

i.e. $\frac{b-a}{1+b^2} < \frac{\tan^{-1}b - \tan^{-1}a}{1} < \frac{b-a}{1+a^2}$.

2. Let $f(x) = \sum_{k=0}^{n} a_k x^k$, where a_k's satisfy $\sum_{k=0}^{n} \frac{a_k}{k+1} = 0$, s.t. $\exists$ a root of $f(x)=0$ in the interval $(0,1)$.

Sol. Define $f(x) = \int_0^x f(t)\,dt$

$$= \int_0^x \frac{a_k}{k+1} x^{k+1}$$

Apply Rolle's theorem,

Monotonic Function

A function is monotonically increasing on D if for every $x_1 < x_2$ from D, $f(x_1) \le f(x_2)$ $f(x)$

Theorem:- A differentiable function $f(x)$ is increasing on D if $f'(x) > 0 \; \forall x \in D$.

Proof:- By Taylor's theorem,

$$f(x+h) = f(x) + hf'(x^*) \; ; \quad x < x^* < x+h.$$

let $h > 0$

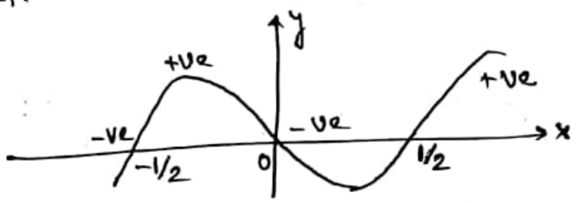
Then $f(x+h) - f(x) = h \cdot f'(x^*) > 0$

$\Rightarrow f(x+h) > f(x)$

$\Rightarrow f(x)$ is increasing.

Ex.(1):- Find the intervals of monotonicity of the function

$$y = 2x^2 - \log|x|.$$



Sol. $y = 2x^2 - \log|x|$, $x \neq 0$

$\frac{dy}{dx} = 4x - \frac{1}{x}$

$\frac{dy}{dx} > 0 \Rightarrow \frac{x(4x^2-1)}{x^2} > 0 \Rightarrow \left(x+\frac{1}{2}\right)x\left(x-\frac{1}{2}\right) > 0$

$\therefore \frac{dy}{dx} > 0$ if $x \in \left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$.

Ex.(2):- If $g(x) = f(x) + f(1-x)$ and $f''(x) < 0$, $0 \le x \le 1$, then find the intervals of monotonicity of $g(x)$.

Sol. $g(x) = f(x) + f(1-x)$ $\qquad f''(x) < 0$, $0 < x < 1$

For $g(x)$ to be increasing $(\uparrow)$ $\qquad \Rightarrow f'(x)$ is $\downarrow$

$g'(x) = f'(x) + (-1)f'(1-x) > 0$

$\Rightarrow f'(x) > f'(1-x)$

$\Rightarrow x < 1-x$

$\Rightarrow 0 < x < \frac{1}{2}$.

Ex. 3. Let $f(x) = (x-1)e^x + 1$. Then S.T. $f(x) \geq 0 \; \forall x$.

Sol. $f(x) = (x-1)e^x + 1 \quad f'(x) = (x-1)e^x + e^x$

$$= xe^x \begin{cases} > 0 & \text{if } x > 0 \\ < 0 & \text{if } x < 0 \end{cases}$$

For $x > 0$, $f(x)$ is $\uparrow$

then $f(x) > f(0) = 0$

For $x < 0$ $f(x)$ is $\downarrow$

then $f(x) > f(0) = 0$

Hence $f(x) \geq 0 \; \forall x \in \mathbb{R}$.

Ex.(4):- Show that $1+x\log\{x+\sqrt{x^2+1}\} \geq \sqrt{1+x^2}\ \forall x \geq 0$.

Solution:-

$$f(x) = 1+x\log\{x+\sqrt{x^2+1}\} - \sqrt{1+x^2}$$

$$f'(x) = \frac{x\left\{1+\frac{1}{2\sqrt{1+x^2}}\cdot 2x\right\}}{x+\sqrt{x^2+1}} + \log\left(x+\sqrt{x^2+1}\right)\cdot 1 - \frac{1}{2\sqrt{1+x^2}}$$

$\therefore f'(x) = \log_e\left(x+\sqrt{x^2+1}\right) > 0$ if $x>0$

$\Rightarrow f'(x)$ is increasing for $x \geq 0$

$\therefore f(x) > f(0)$

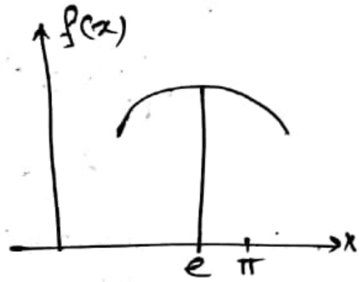
$\therefore 1+x\log\{x+\sqrt{x^2+1}\} > \sqrt{1+x^2}\ \forall x \geq 0$.

Ex.(5):- Which one is larger: π^e and e^π?

Solution:-

$\pi^e \gtrless e^\pi$

$\Rightarrow \pi^{1/\pi} \gtrless e^{1/e}$



Let $f(x) = x^{1/x}$

$\ln f(x) = \frac{1}{x}\ln x$

$\Rightarrow \frac{1}{f(x)}\cdot f'(x) = \frac{1}{x^2} - \ln x\cdot\frac{1}{x^2} = \frac{1-\ln x}{x^2}$

$\therefore f'(x) > 0$ if $1-\ln x > 0$ if $x > e$.

As, $\pi > e$, $f(\pi) < f(e)$

$\Rightarrow \pi^{1/\pi} < e^{1/e}$

$\Rightarrow \pi^e < e^\pi$.

Alter:- $x > 1+\log x\ \forall x \neq 1$

Take $x = \log\pi$, we have

$\log\pi > 1+\log(\log\pi)$

$\Leftrightarrow e^{\log\pi} > e^{1+\log(\log\pi)}$

$\Leftrightarrow \pi > e\cdot e^{\log(\log\pi)}$

$\Leftrightarrow \pi\log e > e\log\pi$

$\Leftrightarrow e^\pi > \pi^e$.

Taylor's Theorem:—

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(x-a)^n}{n!} f^n(c)$$, c lies between a & x.

Ex.(1):— If $f'(a) = f''(a) = f'''(a) = 0$ but $f^{(iv)}(a) > 0$ & $f^{(iv)}(x)$ is continuous at $x = a$, then show that $f(a)$ is local minimum.

Solution:— By Taylor's theorem,

$$f(x) = f(a) + \frac{(x-a)^4}{4!} f^{(iv)}(c)$$

$$\Rightarrow f(x) - f(a) = \frac{(x-a)^4}{4!} f^{(iv)}(c) \geq 0 \quad \forall\ a \in (a-\epsilon, a+\epsilon)$$

i.e. $f(x) \geq f(a) \quad \forall\ x \in (a-\epsilon, a+\epsilon)$

Ex.(2):— Let $f: B \to R$ be a function $\ni |f(x) - f(y)| \leq |x-y| \ \forall x, y \in B = [a,b]$.

Prove that for any function f satisfying the above conditions also satisfies $\left| \int_a^b f(x)\,dx - (b-a) f(a) \right| \leq \frac{1}{2}(b-a)^2$, provided f is integrable on $[a,b]$.

Sol.

$$\left| \frac{f(x) - f(y)}{x-y} \right| \leq 1 \quad \forall\ x, y \in B$$

$$\Rightarrow \left| \lim_{x \to y} \frac{f(x) - f(y)}{x-y} \right| \leq 1 \quad \forall\ y \in B$$

$$\Rightarrow |f'(y)| \leq 1 \quad \forall\ y \in B$$

Define $F(x) = \int_a^x f(t)\,dt$

$$\therefore \int_a^b f(x)\,dx = F(b) = F(a) + (b-a) F'(a) + \frac{(b-a)^2}{2!} F''(c)$$

$$= 0 + (b-a) f(a) + \frac{(b-a)^2}{2} f(c)$$

i.e. $$\left| \int_a^b f(x)\,dx - (b-a) f(a) \right| \leq \frac{(b-a)^2}{2} |f'(c)|$$

$$\leq \frac{(b-a)^2}{2}.$$

Ex.(3):- Let $f(0)=0$, show that $\lim_{h\to 0} \frac{f(h)+f(-h)}{h^2} = f''(0)$.

Solution:- $f(h) = f(0) + hf'(0) + \frac{h^2}{2!}f''(0) + \frac{h^3}{3!}f'''(c)$ [$\because f(0)=0$]

$\therefore f(h) = hf'(0) + \frac{h^2}{2!}f''(0) + \frac{h^3}{3!}f'''(c)$

similarly, $f(-h) = -hf'(0) + \frac{h^2}{2!}f''(0) - \frac{h^3}{3!}f'''(c)$

$\therefore f(h) + f(-h) = h^2 f''(0)$

$\therefore \lim_{h\to 0} \frac{f(h)+f(-h)}{h^2} = f''(0)$.

Ex.(4):- Let g be a function (continuous) with $g(1)=1$ $\ni$ $g(x+y) = 5g(x)g(y)$. Find $g(x)$.

Sol.

$g(x+y) = 5g(x)g(y)$

$5g(x+y) = 5g(x)\,5g(y)$

$\Rightarrow \log[5g(x+y)] = \log[5g(x)] + \log[5g(y)]$

$\Rightarrow f(x+y) = f(x) + f(y)$

$\Rightarrow f(x) = xf(1)$

$\therefore \log[5g(x)] = x\log[5g(1)] = x\log 5$
$= \log 5^x$

$\Rightarrow 5g(x) = 5^x$

$\Rightarrow g(x) = 5^{x-1}$.

Aliter:- Take $x=1$,

$g(y+1) = 5g(1)g(y)$

$g(y+1) = 5g(y)$

$\vdots$

$g(2) = 5g(1) = 5$

$\therefore g(y+1) = 5^y$

$\therefore g(y) = 5^{y-1}$

$\therefore g(x) = 5^{x-1}$.

FUNCTION OF SEVERAL VARIABLES

Functions: $z = f(x,y)$ be a function defined on $D \subseteq \mathbb{R}^2$.

Bolzano-Weierstrass Theorem in $\mathbb{R}^2$:— Every bounded infinite set $S \subset \mathbb{R}^2$ has at least one limit point in $\mathbb{R}^2$.

The limit of a function:—

Let $\lim_{(x,y)\to(a,b)} f(x,y) = l$ means for every $\epsilon > 0$, there corresponds a $\delta(>0)$ such that $|f(x,y) - l| < \epsilon$ whenever $(x-a)^2 + (y-b)^2 < \delta^2$.
i.e. $|x-a| < \delta$, $|y-b| < \delta$.

Examples:— Using definition show that

(i) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$

(ii) $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0$.

(iii) $\lim_{(x,y)\to(0,0)} xy\left(\frac{x^2-y^2}{x^2+y^2}\right) = 0$.

(iv) $\lim_{(x,y)\to(0,0)} e^{-(x^2+y^2)} = 1$.

(v) $\lim_{(x,y)\to(1,2)} (x^2+2y) = 5$

(vi) $\lim_{(x,y)\to(0,0)} \frac{\sqrt{1+x^2y^2}-1}{x^2+y^2} = 0$

(vii) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{1+x^2+y^2} = 0$.

(viii) $\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2} = 0$.

Solution:—

(i) Let $\epsilon > 0$ be given, then

Let, $x = r\cos\theta$, $y = r\sin\theta$

$|f(x,y) - l|$

$$= \left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| = \left|\frac{r^2\cos\theta\sin\theta}{r}\right| = \left|\frac{r}{2}\sin 2\theta\right|$$

$$\leq \frac{r}{2} = \frac{\sqrt{x^2+y^2}}{2} < \epsilon$$

whenever $x^2+y^2 < 4\epsilon^2 = \delta^2$

Then by definition, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Alternatively:—

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| = \frac{|x|\,|y|}{\sqrt{x^2+y^2}} \leq \frac{\sqrt{x^2+y^2}\cdot\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$$

$$= \sqrt{x^2+y^2} < \epsilon$$

if $x^2+y^2 < \delta^2 (= \epsilon^2)$

$\therefore$ By definition $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

(ii) $$\left|\frac{x^3-y^3}{x^2+y^2}-0\right| = \left|\frac{r^3(\cos^3\theta-\sin^3\theta)}{r^2}\right|$$

$$= |r(\cos^3\theta-\sin^3\theta)|$$

$$\leq r\{|\cos^3\theta|+|\sin^3\theta|\}$$

$$\leq 2r = 2\sqrt{x^2+y^2} < \epsilon$$

whenever $x^2+y^2 < \delta^2 = \frac{\epsilon^2}{4}$.

$\therefore$ By definition, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

(iii) Let us put $x = r\cos\theta, y = r\sin\theta$.

$x^2+y^2 = r^2$ implies $r\to 0$ as $(x,y)\to(0,0)$

$\therefore f(x,y) = r^2\sin\theta\cos\theta\cos 2\theta$

Now, $\left|xy\frac{x^2-y^2}{x^2+y^2}\right| = \left|\frac{r^2}{4}\sin 4\theta\right| \leq \frac{r^2}{4} < \epsilon$ if

$r^2 < 4\epsilon$ or, $x^2+y^2 < \delta^2 (=4\epsilon)$

$\therefore$ By definition, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Aliter:- We see $|x| \leq \sqrt{x^2+y^2}$ and $|y| \leq \sqrt{x^2+y^2}$

Also, $\left|\frac{x^2-y^2}{x^2+y^2}\right| \leq 1$.

$$\therefore \left|f(x,y)-0\right| = \left|xy\cdot\frac{x^2-y^2}{x^2+y^2}\right|$$

$$\leq x^2+y^2 < \epsilon$$

if $x^2+y^2 < \delta^2 (=\epsilon)$.

$\therefore$ By definition, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

(iv) $$\left|e^{-(x^2+y^2)}-1\right| \simeq x^2+y^2 < \epsilon \text{ if } x^2+y^2 < \delta^2 (=\epsilon)$$

$$\therefore \lim_{(x,y)\to(0,0)} e^{-(x^2+y^2)} = 1.$$

(v)

When $|x-1| < \delta$ and $|y-2| < \delta$, we get,

$$-4\delta+\delta^2 < x^2+2y-5 < \delta^2+4\delta$$

choosing $0<\delta\leq 1$, we get $-5\delta \leq x^2+2y-5 < 5\delta$

or, $|x^2+2y-5| < 5\delta (=\epsilon)$

Thus $|x^2+2y-5| < \epsilon$ when $|x-1| < \delta$ and $|y-2| < \delta \left(=\frac{\epsilon}{5}\right)$

(vi) $\left|\dfrac{\sqrt{1+x^2y^2}-1}{x^2+y^2}\right| \simeq \left|\dfrac{1}{2}\cdot\dfrac{x^2y^2}{x^2+y^2}\right| = \dfrac{r^2}{8}\left|\sin^2 2\theta\right| \le \dfrac{r^2}{8} < \epsilon$

$\lim\limits_{(x,y)\to(0,0)} \dfrac{\sqrt{1+x^2y^2}-1}{x^2+y^2} = 0$ if $r^2 < 8\epsilon$ or $x^2+y^2 < \delta^2 (=8\epsilon)$

(vii) We put $x = r\cos\theta,\ y = r\sin\theta$

$\therefore \left|\dfrac{x^2-y^2}{1+x^2+y^2}\right| = \left|\dfrac{r^2\cos 2\theta}{1+r^2}\right| = \dfrac{r^2}{1+r^2}\left|\cos 2\theta\right| \le r^2 < \epsilon \quad \left[\because \dfrac{1}{1+r^2} \le 1\right]$

if $x^2+y^2 < \delta^2 (=\epsilon)$

$\therefore \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{1+x^2+y^2} = 0.$

(viii) $\left|\dfrac{x^4+y^4}{x^2+y^2} - 0\right| = \left|\dfrac{x^4+y^4}{x^2+y^2}\right| \le \dfrac{|x|^4+|y|^4}{x^2+y^2} \le 2(x^2+y^2) < \epsilon$

$\left[\because x^2 \le x^2+y^2,\ x^4 \le (x^2+y^2)^2 \text{ and } y^4 \le (x^2+y^2)^2\right]$

if $x^2+y^2 < \delta^2 \left(=\dfrac{\epsilon}{2}\right)$

Non-Existance of Limit:— Let $\lim\limits_{(x,y)\to(a,b)} f(x,y) = l$.

Consider a path $y=\phi(x)$ of approaching to the limit (a,b),
i.e. $\phi(x) = y \longrightarrow b$ when $x \to a$.
Then for any path $y=\phi(x)$ of approaching to (a,b),

$$\lim_{x\to a} f(x,\phi(x)) = l.$$

If for two paths (different) $y=\phi_1(x)$ and $y=\phi_2(x)$ of approaching to (a,b),

$$\lim_{x\to a} f(x,\phi_1(x)) \neq \lim_{x\to a} f(x,\phi_2(x)) ;$$

then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Examples:- Show that the following limits do not exist:

(i) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ (ii) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$

(iii) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^3}{x^2+y^6}$ (iv) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x-y}{x+y}$

(v) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x-y}$ (vi) $\lim\limits_{(x,y)\to(0,0)} (x+y)\cdot\dfrac{y+(x+y)^2}{y-(x+y)^2}$

(vii) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2y^2+(x^2-y^2)^2}$.

In each case we shall show that for two different modes of approach of $(x,y)\to(0,0)$ the limits are different and hence the limit will cease to exist.

Solution:— (i) Let $y=mx$
As $x\to 0$, $y=mx\to 0$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2m}{x^2(1+m^2)} = \frac{m}{(1+m^2)}, \text{ depends on } m.$$

Hence $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist.

(ii) Let $y=mx^2$
$(x,y)\to(0,0)$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{mx^4}{x^4(1+m^2)} = \frac{m}{(1+m^2)}, \text{ depends on } m.$$

Hence the limit does not exist.

(iii) Let $x=my^3$,

$$\therefore \lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} = \lim_{x\to 0} \frac{my^6}{y^6(1+m^2)} = \frac{m}{(1+m^2)}, \text{ depends on } m.$$

$\therefore$ limit does not exist.

(iv) Letting $(x,y)\to(0,0)$ along $y=mx$, we see that the limit depends on m. Hence the limit does not exist.

(v) Let $(x,y)\to(0,0)$ along $y=x-mx^3$

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x-y} = \lim_{x\to 0} \frac{x^3+(x-mx^3)^3}{x^3}$$

$$= \lim_{x\to 0} \frac{1+(1-mx^2)^3}{m}$$

$= \frac{2}{m}$, which has different values for different m.

$\therefore$ limit does not exist.

(vi) $(x,y)\to(0,0)$ along $y=0$ and $y=x^2$ seperately.

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,0) = \lim_{x\to 0} x \cdot \frac{x^2}{-x^2} = 0.$$

Also $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,x^2)$

$$= \lim_{x\to 0} (x+x^2)\frac{x^2+x^2(1+x)^2}{x^2-x^2(1+x)^2}$$

$$= \lim_{x\to 0} \frac{(1+x)(2+2x+x^2)}{-(2+x)}$$

$$= -1.$$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(vii) Let $(x,y)\to(0,0)$ along $y=0$ and $y=x$ seperately.

$\therefore \lim_{x\to 0} f(x,0)=0$ and $\lim_{x\to 0} f(x,x)=1$.

$\therefore \lim_{x\to 0} f(x,0) \neq \lim_{x\to 0} f(x,x)$

$\therefore$ limit does not exist.

Iterated or Repeated Limits:— If $f(x,y)$ is defined in some neighbourhood of (a,b), then $\lim_{y\to b} f(x,y)$, if it exists, is a function of x, say, $\phi(x)$.

If $\lim_{x\to a} \phi(x)$ exists, then we write $\lim_{x\to a} \phi(x) = \lim_{x\to a}\left\{\lim_{y\to b} f(x,y)\right\} = l_1$, say and we say that l_1 is a repeated limit of f as $y\to b$, $x\to a$.

Similarly, the other repeated limit is

$$\lim_{y\to b}\left\{\lim_{x\to a} f(x,y)\right\} = l_2 \text{ (say)}.$$

Note:- If the simultaneous limits exist then two repeated limit, if they exists, are necessarily be equal, but the converse is not true. However, if the repeated limits are not equal, then simultaneous limit may not exist.

Ex.(1):- Give an example to show that repeated limits of a function $f(x,y)$ exist and are equal but the double limit does not exist as $(x,y)\to(a,b)$.

Let $f(x,y) = \dfrac{xy}{x^2+y^2}$

Find the two repeated limit of $f(x,y)$ at $(0,0)$. Also comment on the existance of $\lim_{(x,y)\to(0,0)} f(x,y)$.

Solution:— $\lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\}$ & $\lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\}$

$$= \lim_{y\to 0}\left\{\lim_{x\to 0}\frac{xy}{x^2+y^2}\right\} \qquad = \lim_{x\to 0}\left\{\lim_{y\to 0}\frac{xy}{x^2+y^2}\right\}$$

$$= \lim_{y\to 0}\{0\} \qquad = \lim_{x\to 0}\{0\}$$

$$= 0. \qquad = 0.$$

So, both repeated limits are equal.

Now, let $y = mx$, as $x\to 0$, $mx\to 0$.

Now, $\lim_{x\to 0} f(x,mx) = \lim_{x\to 0}\dfrac{mx^2}{(m^2+1)x^2} = \dfrac{m}{m^2+1}$, depends on m.

Hence, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

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Ex.(2):— Show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. But the repeated limits do not exists, where $f(x,y)$ is defined as

$$f(x,y) = \begin{cases} x\sin\frac{1}{y} + y\sin\frac{1}{x} & \text{when } xy \neq 0 \\ 0 & \text{when } xy = 0. \end{cases}$$

Solution:— $\lim_{x\to 0} f(x,y)$

$$= \left(\sin\frac{1}{y}\right)\left(\lim_{x\to 0} x\right) + y\underbrace{\left(\lim_{x\to 0}\sin\frac{1}{x}\right)}_{\text{does not exist}}$$

$\therefore \lim_{x\to 0} f(x,y)$ does not exist.

Similarly $\lim_{y\to 0} f(x,y)$ does not exist.

Hence $\lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\}$ and $\lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\}$ do not exist.

Now, for $\epsilon > 0$, $|f(x,y) - 0| = \left|x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right)\right| \leq |x| + |y| < \epsilon$

whenever $|x| < \delta = \frac{\epsilon}{2}$ and $|y| < \delta = \frac{\epsilon}{2}$.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = 0.$

Ex.(3):— Show that for $f(x,y) = \begin{cases} xy\cdot\frac{x^2-y^2}{x^2+y^2} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$

double limit & both repeated limits exist at $(0,0)$.

Solution:— We have already shown that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Now, $\lim_{x\to 0}\left\{\lim_{y\to 0} xy\cdot\frac{x^2-y^2}{x^2+y^2}\right\} = \lim_{x\to 0}\{0\} = 0.$

Also, $\lim_{y\to 0}\left\{\lim_{x\to 0} xy\cdot\frac{x^2-y^2}{x^2+y^2}\right\} = 0.$

Ex.(4):– Evaluate the following limits :—

(i) $\lim\limits_{(x,y)\to(0,0)} xy \cdot \dfrac{\sin(x^2+y^2)}{x^2+y^2}$

Sol. $L = \left\{\lim\limits_{(x,y)\to(0,0)} xy\right\}\left\{\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2}\right\}$

$= 0$.

(ii) $\lim\limits_{(x,y)\to(2,1)} \dfrac{\sin^{-1}(xy-2)}{\tan^{-1}(3xy-6)}$

Sol. $L = \lim\limits_{u=(xy-2)\to 0} \dfrac{\sin^{-1}u}{\tan^{-1}3u}$

(iii) $\lim\limits_{(x,y)\to(0,1)} \dfrac{x+y-1}{\sqrt{x}+\sqrt{1-y}}$

Sol. $L = \lim\limits_{(x,y)\to(0,1)} \dfrac{(\sqrt{x}+\sqrt{1-y})(\sqrt{x}-\sqrt{1-y})}{(\sqrt{x}+\sqrt{1-y})}$

$= \lim\limits_{(x,y)\to(0,1)} \{\sqrt{x}-\sqrt{1-y}\}$

Ex.(5):– Let $f(x,y) = \begin{cases} x\sin\frac{1}{y} + \dfrac{x^2-y^2}{x^2+y^2} & \text{if } y\neq 0 \\ 0 & \text{if } y=0 \end{cases}$

show that $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$ exists, but neither $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ nor $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$ exists.

Solution:– We first show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

$\therefore \lim\limits_{(x,y)\to(0,0)} \left\{x\sin\frac{1}{y} + \dfrac{x^2-y^2}{x^2+y^2}\right\}$

$= \lim\limits_{(x,y)\to(0,0)} x\sin\frac{1}{y} + \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$.

Now, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ $y=mx \longrightarrow 0$

$$= \lim_{x\to 0} \frac{x^2(1-m^2)}{x^2(1+m^2)} = \frac{1-m^2}{1+m^2}, \text{ depends on } m.$$

$\therefore \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ does not exist.

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

$$\lim_{x\to 0} \lim_{y\to 0} f(x,y) = \lim_{x\to 0} \left\{ \lim_{y\to 0} x\sin\frac{1}{y} \right\}$$

Now, as $\lim\limits_{y\to 0} \sin\frac{1}{y}$ does not exist, so $\lim\limits_{x\to 0} \lim\limits_{y\to 0} f(x,y)$ does not exist.

Now, $\lim\limits_{y\to 0} \lim\limits_{x\to 0} \left\{ x\sin\frac{1}{y} + \dfrac{x^2-y^2}{x^2+y^2} \right\}$

$$= \lim_{y\to 0} \left\{ 0 + \frac{-y^2}{y^2} \right\}$$

$$= -1.$$

Continuity of $f(x,y)$:—

A function $f(x,y)$ is continuous at (a,b) if for every $\epsilon > 0$, there corresponds a $\delta (>0)$ $\ni$ $|f(x,y) - f(a,b)| < \epsilon$ whenever $(x-a)^2 + (y-b)^2 < \delta^2$.

Note:— If $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, the question of continuity of f at (a,b) does not arise.

Examples:— Examine the continuity of the following functions :—

(i) $f(x,y) = \begin{cases} \dfrac{1}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Solution:—

$$|f(x,y) - f(0,0)| = \frac{1}{x^2+y^2} < \epsilon$$

whenever $x^2+y^2 > \frac{1}{\epsilon} = \delta^2$

Hence the limit does not exist.

$\therefore$ $f(x,y)$ is not continuous at $(0,0)$.

(ii) $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Solution:— Let $(x,y) \to (0,0)$ along $x - y = mx^3$ as $x \to 0$,

$y = x - mx^3 \to 0$.

$$\lim_{x\to 0} f\{x, x-mx^3\} = \lim_{x\to 0} \frac{1+(1-mx^2)^3}{m} = \frac{2}{m}, \text{ depends on } m.$$

$\therefore$ The limit does not exist and hence the function is not continuous at $(0,0)$.

(iii) $f(x,y) = \begin{cases} \dfrac{x^4+y^4}{x-y} & \text{when } x \neq y \\ 0 & \text{when } x = y \end{cases}$

S.T. f is not continuous at $(0,0)$.

Solution:—

$$x - y = mx^4$$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = \frac{2}{m}.$$

$\therefore$ $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Hence $f(x,y)$ is not continuous at $(0,0)$.

(iv) $f(x,y) = \begin{cases} \dfrac{e^{|x-2y|}}{x^2-4xy+4y^2}, & x \neq 2y \\ 0, & x = 2y \end{cases}$

Sol. Let $u = x-2y$

As, $(x,y) \to (0,0)$ so $u \to 0$.

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{u\to 0} \dfrac{e^{|u|}}{u^2} = +\infty.$

$\therefore$ The limit does not exist.

Hence $f(x,y)$ is not continuous at $(0,0)$.

(v) $f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Sol. $|f(x,y) - f(0,0)| = \left| xy\dfrac{x^2-y^2}{x^2+y^2} \right| = |x|\,|y| \left| \dfrac{x^2-y^2}{x^2+y^2} \right| \leq |x|\,|y|$

$\leq x^2+y^2$

$< \epsilon$

if $x^2+y^2 < \delta^2 (= \epsilon)$.

$\therefore f(x,y)$ is continuous at $(0,0)$.

(vi) $f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} & \text{where } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Sol. We know $x^2 \leq x^2+y^2 \quad \therefore x^3 \leq (x^2+y^2)^{3/2}$

similarly, $y^3 \leq (x^2+y^2)^{3/2}$

Now, $|f(x,y) - f(0,0)| = \left| \dfrac{x^3-y^3}{x^2+y^2} \right| \leq \dfrac{2(x^2+y^2)^{3/2}}{(x^2+y^2)}$

$= 2(x^2+y^2)^{1/2} < \epsilon$

if $x^2+y^2 < \delta^2 \left(= \dfrac{\epsilon^2}{4}\right)$

$\therefore f(x,y)$ is continuous at $(0,0)$.

(vii) $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$

Sol. Here $|f(x,y) - f(0,0)| = \left| \dfrac{xy}{\sqrt{x^2+y^2}} \right| \leq \dfrac{x^2+y^2}{\sqrt{x^2+y^2}}$

$= \sqrt{x^2+y^2} < \epsilon$

whenever $x^2+y^2 < \delta^2 (\delta = \epsilon)$

$\therefore f(x,y)$ is continuous at $(0,0)$.

(viii) $f(x,y) = \begin{cases} x\sin\frac{1}{y} + y\sin\frac{1}{x} & \text{when } xy \neq 0 \\ 0 & \text{when } xy = 0 \end{cases}$

S.T. f is continuous at $(0,0)$.

Sol. $|f(x,y) - f(0,0)| = \left|x\sin\frac{1}{y} + y\sin\frac{1}{x}\right| \leq |x| + |y| < \epsilon$

if $|x| < \delta$ and $|y| < \delta$ $\left(\delta = \frac{\epsilon}{2}\right)$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$.

$\therefore f(x,y)$ is continuous at $(0,0)$.

(ix) $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{when } x^2+y^2 \neq 0 \\ 0 & \text{when } x^2+y^2 = 0 \end{cases}$

S.T. f is not continuous at $(0,0)$.

Sol. $y = mx$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x, mx) = \lim_{x\to 0} \frac{x^2 m}{(1+m^2)x^2}$

$= \frac{m}{(1+m^2)}$, depends on m.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

$\therefore f(x,y)$ is not continuous at $(0,0)$.

(x) $f(x,y) = \begin{cases} (px+qy)\sin\frac{x}{y} & , y \neq 0 \\ 0 & , y = 0 \end{cases}$

Is f continuous at $(0,0)$?

Sol. $f(0,0) = 0$.

$|f(x,y) - f(0,0)| = \left|(px+qy)\sin\frac{x}{y} - 0\right| = |(px+qy)|\left|\sin\frac{x}{y}\right|$

$\leq |px+qy| \leq |p||x| + |q||y| < |p|\frac{\epsilon}{2|p|} + |q|\frac{\epsilon}{2|p|}$

when $|x| < \frac{\epsilon}{2|p|} = \delta_1$ and $|y| < \frac{\epsilon}{2|p|} = \delta_2$

$\therefore |f(x,y) - f(0,0)| < \epsilon$ when $|x| < \delta$ and $|y| < \delta$, where $\delta = \min\{\delta_1, \delta_2\}$

$\therefore f(x,y)$ is continuous at $(0,0)$.

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Partial Derivaties:-

The partial Derivatives of a function $f(x,y)$ at (a,b) with respect to x and y are respectively,

$$\frac{\partial f}{\partial x} \text{ or } f_x = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}, \text{ if it exists.}$$

$$\frac{\partial f}{\partial y} \text{ or } f_y = \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k}, \text{ if it exists.}$$

Note:- If the partial derivates f_x and f_y exist and are bounded in a region $R \subset \mathbb{R}^2$, then f is continuous in R.

Remark:- Unlike the situation for functions of one variable, the existance of 1^{st} order partial derivative does not imply the continuity at the point.

Example:- 1. If $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$

show that both the partial derivatives exist at $(0,0)$ but the function is not continuous there.

Solution:- Now, $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(0+h,0)-f(0,0)}{h}$

$$= \lim_{h\to 0} \frac{0}{h} = 0.$$

$\therefore f_x(0,0)$ exist and is equal to zero.

Similarly, $f_y(0,0) = \lim\limits_{k\to 0} \dfrac{f(0,0+k)-f(0,0)}{k}$

$$= \lim_{k\to 0} \frac{0}{k} = 0.$$

Also, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist.

Hence $f(x,y)$ is not continuous at $(0,0)$.

Ex. 2. Show that for the function $f(x,y) = |x|+|y|$, partial derivatives f_x and f_y do not exist at $(0,0)$ but $f(x,y)$ is continuous at $(0,0)$.

Sol. $\lim\limits_{x\to 0} \dfrac{f(x,0)-f(0,0)}{x} = \lim\limits_{x\to 0} \dfrac{|x|}{x} = \pm 1$, which does not exist.

$\therefore f_x(0,0)$ does not exist.

Similarly, $f_y(0,0)$ does not exist.

Here $|f(x,y)-f(0,0)| = \big||x|+|y|-0\big| = |x|+|y| < \epsilon$

if $|x| < \frac{\epsilon}{2}$ and $|y| < \frac{\epsilon}{2}$

choosing $\delta = \frac{\epsilon}{2}$, then

$|f(x,y)-f(0,0)| < \epsilon$ when $|x| < \delta$ and $|y| < \delta$.

$\therefore f(x,y)$ is continuous at $(0,0)$.

Ex. 3. Let $f(x,y) = \begin{cases} x^2\sin\frac{1}{x} + y^2\sin\frac{1}{y} & \text{when } x\neq 0, y\neq 0 \\ x^2\sin\frac{1}{x} & \text{when } y=0, x\neq 0 \\ y^2\sin\frac{1}{y} & \text{when } y\neq 0, x=0 \\ 0 & \text{when } x=0, y=0 \end{cases}$

Find $f_x(0,y)$ and $f_y(x,0)$.

Solution:- for any fixed y,

$$\lim_{x\to 0} \frac{f(x,y)-f(0,y)}{x}$$

$$= \lim_{x\to 0} \frac{x^2\sin\frac{1}{x} + y^2\sin\frac{1}{y} - y^2\sin\frac{1}{y}}{x}$$

$$= \lim_{x\to 0} x\sin\frac{1}{x}$$

$$= 0.$$

Also, for any fixed x,

$$\lim_{y\to 0} \frac{f(x,y)-f(x,0)}{y}$$

$$= \lim_{y\to 0} \frac{x^2\sin\frac{1}{x} + y^2\sin\frac{1}{y} - x^2\sin\frac{1}{x}}{y}$$

$$= \lim_{y\to 0} y\sin\frac{1}{y}$$

$$= 0.$$

$\therefore f_x(0,y) = f_y(x,0)$.

Ex.(4) :- For the function $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Examine existence of partial derivatives f_x and f_y at $(0,0)$.

Solution:- Here, $\displaystyle\lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x}$

$$= \lim_{x \to 0} \frac{\frac{x \cdot 0}{x^2} - 0}{x}$$

$$= \lim_{x \to 0} \frac{0}{x}$$

$$= 0.$$

Also, $\displaystyle\lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y} = \lim_{y \to 0} \frac{\frac{0 \cdot y^2}{0+y^4} - 0}{y} = 0.$

$\therefore f_x(0,0) = 0 = f_y(0,0).$

Ex.(5) :- If $f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} & \text{when } x^2+y^2 \neq 0 \\ 0 & \text{when } x^2+y^2 = 0 \end{cases}$

Show that $f_x(0,0) = 1$ and $f_y(0,0) = -1$.

Solution:- Here $\displaystyle\lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x}$

$$= \lim_{x \to 0} \frac{\frac{x^3 - 0}{x + 0}}{x}$$

$$= \lim_{x \to 0} \frac{x^3}{x^3}$$

$$= \lim_{x \to 0} 1$$

$$= 1.$$

Also, $\displaystyle\lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y}$

$$= \lim_{y \to 0} \frac{\frac{0 - y^3}{0 + y^2}}{y}$$

$$= -1.$$

$\therefore f_x(0,0) = 1$ and $f_y(0,0) = -1$.

DIFFERENTIABILITY of a function $f(x,y)$:—

Theorem 1:— If $f(x,y)$ is differentiable at (a,b), partial derivative $f_x(a,b)$ and $f_y(a,b)$ exist.

Theorem 2:— If $f(x,y)$ is differentiable at (a,b), it is continuous there.

Theorem 3:— (Sufficient condition for differentiability of $f(x,y)$ at (a,b))

Let $(a,b) \in D$, the domain of the function of f. If $f_x(a,b)$ exists and $f_y(x,y)$ is continuous at (a,b), then $f(x,y)$ is differentiable at (a,b).

Examples:- (1). The function $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{when } x^2+y^2 \neq 0 \\ 0 & \text{when } x^2+y^2=0 \end{cases}$

is not differentiable at $(0,0)$.

Solution:- We see that $f(x,y)$ is not continuous at $(0,0)$.

For, let $(x,y) \to (0,0)$ along $y=mx$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{x^2 m}{x^2(1+m^2)} = \frac{m}{(1+m^2)},$$

which is being a function of m. Hence $f(x,y)$ is not continuous at $(0,0)$. Therefore $f(x,y)$ is not differentiable at $(0,0)$ because differentiability would imply continuity.

Examples:-(2):- Show that $f(x,y) = |x|+|y|$ is not differentiable at $(0,0)$.

Solution:- We are to show that $f(x,y)$ is not continuous at $(0,0)$.

Here, $\displaystyle\lim_{x\to 0} \frac{f(0+x,0)-f(0,0)}{x} = \lim_{x\to 0} \frac{|x|}{x}$ does not exist.

$\therefore f(x,y)$ has no partial derivative w.r.t. x at $(0,0)$.

Similarly, $f_y(0,0)$ does not exist.

$\therefore f(x,y)$ is not differentiable at $(0,0)$, for, if the function were differentiable at $(0,0)$, then partial derivatives would exist there.

Ex. (3):- Let $f(x,y) = |x|(1+y)$. Examine whether f is differentiable at $(0,0)$.

Solution:- $\displaystyle f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{|h|-0}{h}$ does not exist.

and $\displaystyle f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k)-f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0.$

Now, we know the necessary condition for the differentiability of a function at a point is the existence of partial derivative at that point.

So, here f is not differentiable at $(0,0)$.

Ex. (4). Let $f(x,y)=\begin{cases} x, & \text{when } |y|<|x| \\ -x, & \text{when } |y|\geq |x| \end{cases}$

Examine whether f is differentiable at $(0,0)$.

Sol. Here $f_x(0,0)=\lim_{h\to 0}\dfrac{f(0+h,0)-f(0,0)}{h}$

$=\lim_{h\to 0}\dfrac{h-0}{h}=1$

and $f_y(0,0)=\lim_{k\to 0}\dfrac{f(0,0+k)-f(0,0)}{k}$

$=\lim_{k\to 0}\dfrac{-k-0}{k}=-1.$

If $f(x,y)$ is to be differentiable at $(0,0)$, we must have
$f(0+h,0+k)-f(0,0)=hf_x(0,0)+kf_y(0,0)+h\phi+k\psi$, where $\psi,\phi\to 0$ as $(h,k)\to(0,0)$.

So, $f(h,k)-0=h-k+h\phi+k\psi$

In particular if $k=-h$, we get

$-k=-2k-k\phi+k\psi$

$\Rightarrow 1=\psi-\phi$

As $k\to 0$, RHS $\to 0$ but LHS $\not\to 0$. So, f is not differentiable at $(0,0)$.

Ex. (5):- Let $f(x,y)=xy^2,\ y>0$

$=-xy^2,\ y\leq 0$

S.T. $f_{xy}(0,0)=f_{yx}(0,0)$.

Sol. $f_x(0,0)=\lim_{h\to 0}\dfrac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\dfrac{0-0}{h}=0.$

$f_x(0,k)=\lim_{h\to 0}\dfrac{f(h,k)-f(0,k)}{h}$

$=\begin{cases}\lim_{h\to 0}\dfrac{hk^2-0}{h} & \text{if } k>0 \\ \lim_{h\to 0}\dfrac{-hk^2-0}{h} & \text{if } k<0\end{cases}=\begin{cases}k^2, & \text{if } k>0\\ -k^2 & \text{if } k<0\end{cases}$

Therefore, $\lim_{k\to 0+}\dfrac{f_x(0,k)-f_x(0,0)}{k}=\lim_{k\to 0+}\dfrac{k^2-0}{k}=0.$

& $\lim_{k\to 0-}\dfrac{f_x(0,k)-f_x(0,0)}{k}=\lim_{k\to 0-}\dfrac{-k^2-0}{k}=0$

$\therefore f_{yx}(0,0)=0.$

Again, $f_y(0,0)=\lim_{k\to 0}\dfrac{f(0,k)-f(0,0)}{k}=0.$

Also, $\lim_{k\to 0+}\dfrac{f(h,k)-f(h,0)}{k}=\lim_{k\to 0+}\dfrac{hk^2-0}{k}=\lim_{h\to 0+}(hk)=0$

and $\lim_{k\to 0-}\dfrac{f(h,k)-f(h,0)}{k}=\lim_{k\to 0-}\dfrac{-hk^2-0}{k}=0.$

So, $f_y(h,0)=0$

So, $f_{xy}(0,0)=\lim_{h\to 0}\dfrac{f_y(h,0)-f_y(0,0)}{h}=\lim_{h\to 0}\dfrac{0-0}{h}=0.$

Differentiability :— Let $f(x,y)$ be a function defined on $D \subseteq \mathbb{R}^2$.

Consider the increment

$$\delta f = f(x+h, y+k) - f(x,y)$$
$$= A.h + B.k + h\phi(h,k) + k\psi(h,k),$$

where $A = f_x$, $B = f_y$ and ϕ and ψ are functions of (h,k) tend to zero as $(h,k) \to (0,0)$.

Then f is differentiable at (x,y).

Then $df = f_x dx + f_y dy$ is the differential at (x,y).

Note:-
$$\lim_{(h,k)\to(0,0)} \{ f(x+h, y+k) - f(x,y) \}$$
$$= \lim_{(h,k)\to(0,0)} \{ Ah + Bk + h\phi(h,k) + k\psi(h,k) \}$$
$$= 0.$$

Hence differentiability of $f(x,y)$ implies the continuity and also the existance of f_x and f_y. If a function is not continuous, then it is not differentiable.

Ex.(1):- Examine the differentiability of

$$f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

at $(x,y) = (0,0)$.

Solution:- The differentiability of f at $(0,0)$, we must have
$$\partial f = f(h,k) - f(0,0)$$
$$= Ah + Bk + h\phi + k\psi$$

where, $A = f_x(0,0)$, $B = f_y(0,0)$ and $\phi, \psi \to 0$ as $(h,k) \to (0,0)$.

Now, $A = f_x(0,0)$
$$= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\left(\frac{h^3 - 0^3}{h^2+0^2}\right) - 0}{h} = 1.$$

and $B = f_y(0,0)$
$$= \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{\frac{0^3 - k^3}{0^2+k^2} - 0}{k} = -1.$$

Now,
$$\frac{h^3 - k^3}{h^2+k^2} = \frac{h(h^2+k^2) - k(h^2+k^2)}{h^2+k^2} + \frac{-hk^2 + kh^2}{h^2+k^2}$$
$$= h(1) + (-1)k + h\left(\frac{-k^2}{h^2+k^2}\right) + k\left(\frac{h^2}{h^2+k^2}\right)$$
$$= h.(1) + k.(-1) + h\phi + k\psi.$$

but $\lim\limits_{(h,k)\to(0,0)} \phi = \lim\limits_{(h,k)\to(0,0)} \dfrac{-k^2}{h^2+k^2}$ &

$\lim\limits_{(h,k)\to(0,0)} \psi = \lim\limits_{(h,k)\to(0,0)} \dfrac{-h^2}{h^2+k^2}$ do not exist.

Hence, $f(x,y)$ is not differentiable at $(0,0)$.

Ex.(2):- Show that the function $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0 \\ 0, & (x,y)=(0,0) \end{cases}$

is continuous and possess partial derivatives but not differentiable at $(0,0)$.

Sol. Let us first show that f is continuous at $(0,0)$.

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| \leq \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} < \epsilon$$

whenever $x^2+y^2 < \delta^2 (= \epsilon^2)$.

$\therefore f(x,y)$ is continuous at $(0,0)$.

Now, $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = 0$

similarly, $f_y(0,0) = \lim\limits_{k\to 0} \dfrac{f(0,k)-f(0,0)}{k} = 0$.

Now, we are to show that $f(x,y)$ is not differentiable at $(0,0)$.

If possible let, $f(x,y)$ is differentiable at $(0,0)$.

$\therefore f(x,y) - f(0,0) = Ax + By + x\epsilon_1 + y\epsilon_2$

where, $\epsilon_1, \epsilon_2 \to 0$ as $x, y \to 0$.

$f(h,k) - f(0,0) = Ah + Bk + h\epsilon_1 + k\epsilon_2$ [Here $A=0, B=0$]

$\therefore \dfrac{hk}{\sqrt{h^2+k^2}} = h.\epsilon_1 + k\epsilon_2 = \epsilon\sqrt{h^2+k^2}$

$\cancel{\Rightarrow \dfrac{k/h}{\sqrt{1+k^2/h^2}} = \epsilon_1 + \left(\dfrac{k}{h}\right)\epsilon_2}$

$\Rightarrow \epsilon = \dfrac{hk}{h^2+k^2}$ which does not tend to a unique limit as $(h,k)\to(0,0)$.

In fact, if $(h,k)\to(0,0)$ along $k = mh$, we see that

$$\lim_{(h,k)\to(0,0)} \epsilon = \lim_{(h,k)\to(0,0)} \frac{hk}{h^2+k^2} = \lim_{h\to 0} \frac{mh^2}{h^2(1+m^2)}$$

$$= \frac{m}{1+m^2}, \text{ depends on } m.$$

$\therefore \lim\limits_{(h,k)\to(0,0)} \epsilon$ does not exist. Hence $f(x,y)$ is not differentiable at $(0,0)$.

Ex.(3):- If $f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Show that, $f(x,y)$ is differentiable at $(0,0)$.

Solution:- Partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist.

We see that $\lim\limits_{h \to 0} \dfrac{f(0+h,0) - f(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{f(h,0) - 0}{h}$

$$= \lim_{h \to 0} \frac{0-0}{h}$$

$$= 0.$$

$\therefore f_x(0,0) = 0.$

Similarly, $\lim\limits_{k \to 0} \dfrac{f(0,0+k) - f(0,0)}{k} = \lim\limits_{k \to 0} \dfrac{f(0,k) - 0}{k}$

$$= \lim_{k \to 0} \frac{0-0}{k}$$

$$= 0.$$

$\therefore f_y(0,0) = 0.$

Now, $f(0+h, 0+k) - f(0,0) = f(h,k) - f(0,0) = f(h,k) = \dfrac{hk(h^2-k^2)}{h^2+k^2}$

If we write $\dfrac{hk(h^2-k^2)}{h^2+k^2} = 0.h + 0.k + \epsilon\sqrt{h^2+k^2} = \epsilon\sqrt{h^2+k^2}$

$$\therefore |\epsilon| = \left| \frac{hk(h^2-k^2)}{(h^2+k^2)\sqrt{h^2+k^2}} \right| \leq \left| \frac{|h||k|}{\sqrt{h^2+k^2}} \right| \leq \sqrt{h^2+k^2} < \eta$$

whenever $h^2+k^2 < \delta^2 = \eta^2$.

$\therefore \lim\limits_{(h,k) \to (0,0)} \epsilon = 0.$

$\therefore f(x,y)$ is differentiable at $(0,0)$.

LEIBNITZ'S RULE

Theorem:— If f is a continuous function on $[a,b]$ and $u(x)$ and $v(x)$ are differentiable functions on x whose values lies in $[a,b]$ then

$$\frac{d}{dx}\left\{\int_{u(x)}^{v(x)} f(t)\,dt\right\} = f(v(x))\frac{dv(x)}{dx} - f(u(x))\frac{du(x)}{dx}$$

Proof:— Let $\frac{d}{dx}F(x) = f(x)$

$$\Rightarrow \int_{u(x)}^{v(x)} f(t)\,dt = F(v(x)) - F(u(x))$$

$$\Rightarrow \frac{d}{dx}\left\{\int_{u(x)}^{v(x)} f(t)\,dt\right\} = \frac{d}{dx}\left(F(v(x)) - F(u(x))\right)$$

$$\Rightarrow \frac{d}{dx}\left\{\int_{u(x)}^{v(x)} f(t)\,dt\right\} = F'(v(x))\frac{d(v(x))}{dx} - F'(u(x))\frac{d(u(x))}{dx}$$

$$\Rightarrow \frac{d}{dx}\left\{\int_{u(x)}^{v(x)} f(t)\,dt\right\} = f(v(x))\frac{d(v(x))}{dx} - f(u(x))\frac{d(u(x))}{dx}.$$

Ex.1. If $y = \int_{x^2}^{x^3} \frac{1}{\log t}\,dt$ (where $x>0$), then find $\frac{dy}{dx}$.

Solution:—

$$y = \int_{x^2}^{x^3} \frac{1}{\log t}\,dt$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^3)\frac{1}{\log x^3} - \frac{d}{dx}(x^2)\frac{1}{\log x^2}$$

$$= \frac{3x^2}{3\log x} - \frac{2x}{2\log x}$$

$$= x(x-1)(\log x)^{-1}.$$

Ex.2. If $\int_{\sin x}^{1} t^2 f(t)\,dt = 1 - \sin x$, where $x \in \left(0, \frac{\pi}{2}\right)$, then find the value of $f\left(\frac{1}{\sqrt{3}}\right)$.

Sol.

$$\int_{\sin x}^{1} t^2 f(t)\,dt = 1 - \sin x$$

Differentiating both sides w.r.t. x, we get

$$1^2 \times f(1)\,0 - \sin^2 x\, f(\sin x)\cos x = -\cos x$$

$$\Rightarrow f(\sin x) = \operatorname{cosec}^2 x = \frac{1}{\sin^2 x}$$

$$\Rightarrow f(z) = \frac{1}{z^2}$$

$$\Rightarrow f\left(\frac{1}{\sqrt{3}}\right) = 3.$$

Ex. (3):- Evaluate $\lim_{x\to\infty} \dfrac{\left(\int_0^x e^{x^2}dx\right)^2}{\int_0^x e^{2x^2}dx}$.

Sol. Since $e^{x^2} > 0$, $e^{2x^2} > 0$ in $[0,x]$, where $x > 0$.

$\int_0^x e^{x^2}dx$ & $\int_0^x e^{2x^2}dx \to \infty$ as $x\to\infty$

$L = \lim_{x\to\infty} \dfrac{\left(\int_0^x e^{x^2}\right)^2}{\int_0^x e^{2x^2}}$ is of the form $\frac{\infty}{\infty}$

$\therefore$ Using L'Hospital's Rule

$$L = \lim_{x\to\infty} \frac{2e^{x^2}\int_0^x e^{x^2}dx}{e^{2x^2}}$$

$$= 2\lim_{x\to\infty} \frac{\int_0^x e^{x^2}dx}{e^{x^2}} \quad \left(\tfrac{\infty}{\infty} \text{ form}\right)$$

$$= 2\lim_{x\to\infty} \frac{e^{x^2}}{2xe^{x^2}}$$

$$= \lim_{x\to\infty} \frac{1}{x} = 0.$$

Ex. (4). Let $f:(0,\infty)\to(0,\infty)$ be a differentiable function satisfying
$x\int_0^x (1-t)f(t)dt = \int_0^x tf(t)dt \;\forall\; x\in\mathbb{R}^+$ and $f(1)=1$.
Determine $f(x)$.

Sol. $$x\int_0^x (1-t)f(t)dt = \int_0^x tf(t)dt$$

Diff. both sides w.r.t. x, we get,

$$x(1-x)f(x) + \int_0^x (1-t)f(t)dt = xf(x)$$

$$\Rightarrow x^2 f(x) = \int_0^x (1-t)f(t)dt$$

Diff. both sides w.r.t. x again, we get

$$x^2 f'(x) + 2xf(x) = (1-x)f(x)$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \frac{1-3x}{x^2} \Rightarrow \int \frac{f'(x)}{f(x)}dx = \int \frac{1-3x}{x^2}dx$$

$$\Rightarrow \log\left[\frac{f(x)}{c}\right] = -\frac{1}{x} - 3\log x$$

Given $f(1)=1 \Rightarrow \log\left(\frac{1}{c}\right) = -1$, $c=e$.

$$\Rightarrow \log\left[\frac{f(x)\cdot x^3}{e}\right] = -\frac{1}{x} \Rightarrow f(x) = \frac{1}{x^3}e^{\left(1-\frac{1}{x}\right)}.$$

LEIBNITZ'S RULE

Theorem:- If the function $\phi(x)$ and $\psi(x)$ are defined on $[a,b]$ and differentiable functions at a point $x \in (a,b)$ and $f(x,t)$ is continuous then,

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(x,t)\,dt\right] = \int_{\phi(x)}^{\psi(x)} \frac{d}{dx} f(x,t)\,dt$$

$$+ \left\{\frac{d\psi(x)}{dx}\right\} f(x,\psi(x))$$

$$- \left\{\frac{d\phi(x)}{dx}\right\} f(x,\phi(x))$$

Corollary:- If f is continuous on $R = [a\ b\ c\ d]$, then

$$\int_a^b \left\{\int_c^d f(x,y)\,dy\right\} dx = \int_c^d \left\{\int_a^b f(x,y)\,dx\right\} dy.$$

i.e. the two repeated (iterated) integrals are equal.

WORKED EXAMPLES:—

1. If $f(x) = \int_a^x t^3 e^t\,dt$, then find $f'(x)$.

Solution:-

$$\frac{d}{dx} f(x) = \frac{d}{dx}\int_a^x t^3 e^t\,dt$$

$$= 1 \cdot x^3 \cdot e^x = x^3 e^x.$$

2. If $\phi(x) = \int_{1/x}^{\sqrt{x}} \sin(t^2)\,dt$, then $\phi'(1) = ?$

Sol.

$$\phi'(x) = \frac{1}{2\sqrt{x}} \sin x + \frac{1}{x^2} \sin\frac{1}{x^2}$$

$$\phi'(1) = \frac{1}{2}\sin 1 + 1\sin 1$$

$$= \frac{3}{2}\sin 1.$$

3. If $f(x) = \int_{x^2}^{x^4} \sin\sqrt{t}\,dt$ then $f'(x) = ?$

Sol.

$$f'(x) = 4x^3 \cdot \sin x^2 - 2x\sin x.$$

4. If $f(t) = \int_{-t}^{t} \frac{dx}{1+x^2}$ then $f'(1)$ is $= ?$

Sol.

$$f(t) = \left[\tan^{-1}x\right]_{-t}^{t} = \tan^{-1}t + \tan^{-1}t = 2\tan^{-1}t$$

$$f'(t) = 2\cdot\frac{1}{1+t^2}$$

$$f'(1) = 1.$$

5. If $F(x) = \frac{1}{x^2} \int_1^x (4t^2 - 2F'(t))\,dt$, then $F'(4)$ equals ?

Sol. $F'(x) = \frac{1}{x^2}(4x^2 - 2F'(x)) - \frac{2}{x^3}\int_1^x (4t^3 - 2F'(t))\,dt$

$\Rightarrow F'(4) = \frac{1}{16}[64 - 2F'(4)] - 0$

$\Rightarrow F'(4) = \frac{32}{9}$.

6. If $\int_0^{\pi} x f(\sin x)\,dx = A\int_0^{\pi/2} f(\sin x)\,dx$, then $A = ?$

Sol. $I = \int_0^{\pi} x f(\sin x)\,dx$

$2I = 2\pi\int_0^{\pi/2} f(\sin x)\,dx$

$\therefore A = \pi$.

7. The points of intersection of $F_1(x) = \int_2^x (2t-5)\,dt$ and $F_2(x) = \int_0^x 2t\,dt$, are

(A) $\left(\frac{6}{5}, \frac{36}{25}\right)$, (B) $= \left(\frac{2}{3}, \frac{4}{9}\right)$ (C) $\left(\frac{1}{3}, \frac{1}{9}\right)$ (D) $\left(\frac{1}{5}, \frac{1}{25}\right)$

Sol. (A) $y_1 = \int_2^x (2t-5)\,dt$, $y_2 = \int_0^x 2t\,dt$

Now point of intersection means those points at which

$y_2 = y_1 = y$

$\Rightarrow y_1 = x^2 - 5x + 6$

and $y_2 = x^2$

Now, solving, we have $x = \frac{6}{5}$, $x^2 = \frac{36}{25}$.

$\therefore$ Poin of intersection is $\left(\frac{6}{5}, \frac{36}{25}\right)$.

8. If $f(x) = \int_{x^2}^{x^2+1} e^{-t^2}\,dt$ then find the interval in which the function $f(x)$ is increasing.

Sol. $f(x)$ is said to be increasing if $f'(x) > 0$.

$f'(x) = e^{-(x^2+1)^2}\cdot 2x - e^{-(x^2)^2}\cdot 2x = 2xe^{-(x^4+1-2x^2)}\left(1 - e^{2x^2+1}\right)$

$\therefore f'(x) > 0 \Rightarrow x \in (-\infty, 0)$.

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9) If the value of the definite integral $\int^1 \frac{x\,dx}{x^3+16}$ lies in the interval $[a,b]$ then find the smallest interval.

Sol. The function $f(x) = \frac{x}{x^3+16}$ is an increasing function.

so, $\min f(x) = f(0) = 0$ and $\max f(x) = f(1) = \frac{1}{17}$.

Therefore by the law $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

$$\Rightarrow 0 \le \int_0^1 \frac{x}{x^3+16}\,dx \le \frac{1}{17}.$$

10) Find the points of extremum of the following integral

$$\int_0^{x^2} \frac{t^2-5t+4}{2+e^t}\,dt.$$

Sol. Let $F(x) = \int_0^{x^2} \frac{t^2-5t+4}{2+e^t}\,dt$

$$\Rightarrow F'(x) = \frac{x^4-5x^2+4}{2+e^{x^2}} \cdot 2x \quad \text{(By Leibnitz's rule)}$$

from $F'(0) = 0$, we get $x=0$ or $x^2 = \frac{5 \pm \sqrt{25-16}}{2} = 4, 1.$

Hence $x = 0, \pm 1, \pm 2.$

11) Let $f: R \to R$ and $g: R \to R$ be continuous functions, then the value of the integral $\int_{-\pi/2}^{\pi/2} [f(x)+f(-x)][g(x)-g(-x)]\,dx = ?$

Sol. Let $h(x) = \{f(x)+f(-x)\}\{g(x)-g(-x)\}$

$h(-x) = -h(x)$

$\therefore h(x)$ is an odd function, $\therefore \int_{-\pi/2}^{\pi/2} h(x)\,dx = 0.$

12) Let $f(x)$ be a function satisfying $f'(x) = f(x)$ with $f(0) = 1$ and $g(x)$ be the function satisfying $f(x) + g(x) = x^2$. Then the value of the integral $\int_0^1 f(x)g(x)\,dx$ is ?

Sol.

$f'(x) = f(x)$

$\frac{f'(x)}{f(x)} = 1$

$\Rightarrow \log f(x) = x + \log c$ since $f(0) = 1 \Rightarrow c = 1.$

$\Rightarrow f(x) = ce^x.$ $\therefore f(x) = e^x$, $g(x) = x^2 - e^x.$

$$\therefore \int_0^1 f(x)g(x)\,dx = \int_0^1 e^x(x^2-e^x)\,dx$$

13) Let the function $f(x)$ be defined as $\int_1^x \sqrt{2-t^2}\,dt$ then find the no. of real roots of the equation $x^2 - f'(x) = 0$.

Sol. $f'(x) = \sqrt{2-x^2}$

$\Rightarrow x^2 - \sqrt{2-x^2} = 0.$

or, $x^4 + x^2 - 2 = 0$

$\Rightarrow (x^2+2)(x^2-1) = 0$

$\Rightarrow x^2 = 1$ $\therefore$ there are 2 roots.

$\Rightarrow x = \pm 1.$

14) Let $\frac{d}{dx}F(x) = \left(\frac{e^{\sin x}}{x}\right)$, $x > 0$, if $\int_1^4 \frac{3}{x} e^{\sin x^3} dx = F(k) - F(1)$ then find one of the possible value of k.

Sol.

$$\frac{d}{dx}F(x) = \frac{e^{\sin x}}{x}$$

$$\Rightarrow \int_1^4 \frac{3}{x} e^{\sin x^3} dx = \int_1^4 \frac{3x^2}{x^3} e^{\sin x^3} dx$$

Put $x^3 = t \Rightarrow 3x^2 dx = dt$

$$F(t) = \int_1^{64} \frac{e^{\sin t}}{t} dt = \int_1^{64} F(t)\,dt = F(64) - F(1)$$

$\therefore k = 64.$

15) Let g and f be continuous function on R $\ni$ $f(x) = \int_0^x g(t)\,dt$ and $g(x) = \int_x^0 f(t)\,dt + 1$. Prove that $(f(x))^2 + (g(x))^2 \overset{?}{=} 1$.

Sol. Applying leibnitz rule on $f(x)$ and $g(x)$ we have

$$f'(x) = \frac{d}{dx}\int_0^x g(t)\,dt$$

$$= g(x) + 0 - 0 = g(x)$$

$$g'(x) = \frac{d}{dx}\left[\int_1^0 f(t)\,dt + 1\right]$$

$$= -f(x).$$

$f''(x) = g'(x) = -f(x)$

$\Rightarrow f''(x) = -f(x)$

$\Rightarrow f''(x)\cdot f'(x) = -f(x)f'(x)$

$\Rightarrow f''(x)\cdot f'(x) + f'(x)\,f(x) = 0$

integrating $[f'(x)]^2 + [f(x)]^2 = c \Rightarrow [f(x)]^2 + [g(x)]^2 = 1.$

PROBLEMS ON SERIES & SEQUENCE OF REAL NOS.

1. Prove that $\lim_{n\to\infty} \frac{n^{2/3} \cos \lfloor n}{n+1} = 0.$

Solution:-

$$-\frac{n^{2/3}}{n+1} \le \frac{n^{2/3}\cos \lfloor n}{n+1} \le \frac{n^{2/3}}{n+1} \;;\; n \in \mathbb{N} \quad \text{as} -1 \le \cos\lfloor n \le 1.$$

Now $\lim_{n\to\infty} \frac{n^{2/3}}{n+1} = \lim_{n\to\infty} \frac{\frac{1}{n^{1/3}}}{1+\frac{1}{n}} = 0.$

$\therefore$ By Squeeze's theorem, $\lim_{n\to\infty} \frac{n^{2/3}\cos\lfloor n}{n+1} = 0.$

2. If $a_n = \left(1+\frac{2}{n}\right)^n$, then find the value of $\lim_{n\to\infty}\left(1-\frac{a_n}{n}\right)^n.$

Solution:- $a_n = \left(1+\frac{2}{n}\right)^n$

let, $L = \lim_{n\to\infty}\left(1+\frac{a_n}{n}\right)^n$

$$\log L = \lim_{n\to\infty}\left\{\log_e\left(1-\frac{a_n}{n}\right)^n\right\}$$

$$= \lim_{n\to\infty}\left\{n\log_e\left(1-\frac{a_n}{n}\right)\right\}$$

$$= \lim_{n\to\infty}\left[-n\left\{\frac{a_n}{n}+\frac{a_n^2}{2n^2}+\cdots\right\}\right]$$

$$= -\lim(a_n) + 0$$

$$= -e^2$$

$\therefore L = e^{-e^2}.$

3. Evaluate: $\lim_{n\to\infty} \frac{1-2+3-4+5-6+\cdots+2n}{\sqrt{n^2+1}+\sqrt{n^2-1}}$

Solution:-

$$\lim_{n\to\infty} \frac{1-2+3-4+5-6+\cdots+2n}{\sqrt{n^2+1}+\sqrt{n^2-1}}$$

$$= \lim_{n\to\infty} \frac{-n}{\sqrt{n^2+1}+\sqrt{n^2-1}}$$

$$= \lim_{n\to\infty} \frac{-1}{\sqrt{1+\frac{1}{n^2}}+\sqrt{1-\frac{1}{n^2}}}$$

$$= -\frac{1}{2}.$$

4. Define $a_n = \frac{1000^n}{n!}$. Then $\{a_n\}$ has the maximum value at ______ point(s).

Sol. $\frac{a_{n+1}}{a_n} = \frac{1000}{n+1} \gtreqless 1$

$$\ldots < a_{998} < \underbrace{a_{999} = a_{1000}}_{2\,\text{max. values}} > a_{1001} > \ldots\ldots$$

5. Find the maximum among $1, 2^{1/2}, 3^{1/3}, \ldots\ldots, x^{1/x}, \ldots\ldots$

Sol. $\{n^{1/n}; n \in \mathbb{N}\}$

$\therefore f(x) = x^{1/x}$ is maximum at $x = e$.

and we know $2 < e < 3$.

$\therefore$ max. of $n^{1/n}$ = one of the quantities among $\max\{2^{1/2}, 3^{1/3}\}$

$= 3^{1/3}$.

6. Let f be a differentiable function on $(0,1)$ $\ni$ $|f'(x)| < 1 \; \forall \; 0 < x < 1$. Show that $\{a_n\}$, where $a_n = f(\frac{1}{n})$ is convergent.

Sol. $|f'(x)| < 1$, $0 < x < 1$

$\Rightarrow -1 < f'(x) < 1$

$\Rightarrow -x < f(x) < x$, on integration

$\therefore -\frac{1}{x} < a_n = f(\frac{1}{x}) < \frac{1}{x}$.

By squeeze theorem, $\lim_{n\to\infty}(a_n) = 0$.

7. Find $\lim_{n\to\infty} \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right) \cdots\cdots \left(1-\frac{1}{n^2}\right)$.

Sol. $\lim_{n\to\infty} \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right) \cdots\cdots \left(1-\frac{1}{n^2}\right)$

$= \lim_{n\to\infty} \frac{(2-1)(2+1)}{2\cdot 2} \cdot \frac{(3-1)(3+1)}{3\cdot 3} \cdot \frac{(4-1)(4+1)}{4\cdot 4} \cdots\cdots \frac{(n-1)(n+1)}{n\cdot n}$

$= \lim_{n\to\infty} \frac{1}{2} \cdot \frac{n+1}{n}$

$= \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)$

$= \frac{1}{2}$.

8. Find $\lim_{n\to\infty} \frac{1}{n^4} \sum_{k=1}^{n} k(k+1)(k+4)$

Sol.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k}{n}\left(\frac{k}{n}+\frac{1}{n}\right)\left(\frac{k}{n}+\frac{4}{n}\right) \simeq \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^3$$

$$\simeq \int_0^1 x^3 dx$$

$$= \frac{1}{4}.$$

9. Evaluate $\lim_{n\to\infty} \frac{(n!)^{1/n}}{(n+1)}$.

Sol. $L = \lim_{n\to\infty} \left\{\frac{\lfloor n}{(n+1)^n}\right\}^{1/n} = \lim_{n\to\infty} \{a_n^{1/n}\}$

Using Cauchy's 2nd Limit theorem :-

$$\therefore L = \lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right)$$

$$= \lim_{n\to\infty} \left\{\frac{\lfloor n+1}{(n+2)^{n+1}} \cdot \frac{(n+1)^n}{\lfloor n}\right\}$$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+2)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}} = \frac{1}{e}.$$

10. Show that $\left\{1+\frac{1}{2}+\cdots+\frac{1}{n}\right\}$ is not a cauchy sequence.

Solution:-

$$x_n = \sum_{k=1}^{n} \frac{1}{k}$$

$$|x_{2n} - x_n| = \left(\frac{1}{n+1} + \cdots + \frac{1}{2n}\right)$$

$> n \cdot \frac{1}{2n} = \frac{1}{2}$, which is not less than ϵ,

where $\epsilon < \frac{1}{2}$.

$\therefore \left\{1+\frac{1}{2}+\cdots+\frac{1}{n}\right\}$ is not a cauchy sequence.

— x —