Chapter 1

Introduction

> The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive.
>
> –Robert M. Solow (1956), p. 65.

1.1 Macroeconomics

1.1.1 The field

Macroeconomics has two defining characteristics. First, it is a study of the economic interactions in society as a whole. This could also be said of microeconomic general equilibrium theory, however. The second defining characteristic of macroeconomics is that it aims at understanding the empirical regularities in the behavior of aggregate economic variables such as aggregate production, investment, unemployment, the general price level for goods and services, the inflation rate, the interest rate, the level of real wages, the stock market level, the foreign exchange rate, productivity growth etc. Thus macroeconomics focuses on the major lines of the economics of a society. The aim is three-fold:

1. to explain the level of the aggregate variables as well as their movement over time in the short run and the long run;

2. to make well-founded forecasts possible;

3. to provide foundations for rational economic policy applicable to macroeconomic problems, be they short-run distress or problems of a more long-term, structural character.

To make our complex economic environment accessible for theoretical analysis we use economic models. What *is* an economic model? It is a way of organizing one's thoughts about the economic functioning of a society. A more specific answer is to define an economic model as a conceptual structure based on a set of mathematically formulated assumptions which have an economic interpretation and from which empirically testable predictions can be derived. In particular, a macroeconomic model is an economic model concerned with macroeconomic phenomena, i.e., the short-run fluctuations of aggregate variables as well as their long-run trend.

Any economic analysis is based upon a conceptual framework. Formulating this framework as a precisely stated economic model helps to break down the issue into assumptions about the concerns and constraints of households and firms and the character of the market environment within which these agents interact. The advantage of this approach is that it makes rigorous reasoning possible, lays bare where the possible underlying disagreements behind different interpretations of economic phenomena are, and makes sensitivity analysis of the conclusions amenable. Moreover, by being explicit about agents' concerns and the social structures (market forms, social conventions, and juridical institutions) conditioning their interactions, this approach allows analysis of policy interventions, using the well-established tools of welfare economics. And expressing our assumptions mathematically opens up for use of powerful mathematical theorems from the mathematical toolbox. Without these math tools it would in many cases be impossible to reach any conclusion whatsoever.

Undergraduate students of economics are often perplexed or even frustrated by macroeconomics being so preoccupied with composite theoretical models. Why not study the issues each at a time? The reason is that the issues, say housing prices and changes in unemployment, are not separate, but parts of a complex system of mutually dependent variables. This also suggests that macroeconomics must take advantage of theoretical and empirical knowledge from other branches of economics, including microeconomics, industrial organization, game theory, political economy, behavioral economics etc.

At the same time our models necessarily give a *simplified* picture of the economic reality. Ignoring secondary aspects and complications is indispensable to be able to focus on the essential features of a given problem. In particular macroeconomics deliberately simplifies the description of the individual actors so as to make the analysis of the interaction between different types of actors manageable.

The assessment of − and choice between − *competing* simplifying frameworks should be based on how well they perform in relation to the three-fold aim listed above, given the problem at hand. A key condition for good performance is the empirical tenability of the model's implications. A guiding principle in the development of useful models therefore lies in the confrontation with data. This can be based on a variety of methods ranging from sophisticated econometric techniques to more qualitative case studies.

The union of connected and non-contradictory economic models and the theorems derived from these constitute an *economic theory*. Being about the interaction of *human* beings in *societies*, the subject matter of economic theory is complex and at the same time history dependent. This explains why economic theory is far from the natural sciences with respect to precision and undisputable empirical foundation. In particular in macroeconomics one should be aware of the existence of differing conceptions and in some matters even conflicting theoretical schools.

1.1.2 The different "runs"

This text subdivides macroeconomics into "short-run", "medium-run", and "long-run" macroeconomics. The first concentrates on the behavior of the macroeconomic variables within a time horizon of a few years, whereas "long-run" macroeconomics deals with a considerably longer time horizon − indeed, long enough for changes in the capital stock, population, and technology to have a dominating influence on changes in the level of production. The "medium run" is then something in between.

To be more specific, *long-run macromodels* study the evolution of an economy's productive capacity over time. Typically a time span of at least 15 years is considered. The analytical framework is by and large *supply-dominated*. That is, variations in the employment rate for labor and capital due to demand fluctuations are abstracted away. This is of course a simplification, but can be justified by the fact that these variations, at least in advanced economies, tend to remain within a fairly narrow band, anyway. Therefore, under "normal" circumstances the economic outcome after, say, a 30 years' interval reflects primarily the change in supply side factors such as the labor force, the capital stock, and the technology. The fluctuations in demand and monetary factors are of limited quantitative importance within such a time horizon.

In contrast, when we speak of *short-run macromodels*, we think of models concentrating on mechanisms that determine how fully an economy uses its productive capacity at a given point in time. The focus is on the level of output and employment within a time horizon less than, say, three years. These models are typically *demand-dominated*. In this time perspective the demand side, monetary factors, and price rigidities matter significantly. Shifts in aggregate demand (induced by, e.g., changes in fiscal or monetary policy, exports, interest rates, the general state of confidence, etc.) tend to be accommodated by changes in the produced quantities rather than in the prices of manufactured goods and services. In contrast, variations in the supply of production factors and technology are of limited importance within this time span. With Keynes' words the aim of short-run analysis is to explain "what determines the actual employment of the available resources" (Keynes 1936, p. 4).

The short and the long run make up the traditional subdivision of macroeconomics. But it is convenient and fruitful to include also a *medium run*, referring to a time interval of, say, three-to-fifteen years.[1] We shall call models attempting to bridge the gap between the short run and the long run *medium-run macromodels*. These models deal with the regularities exhibited by *sequences* of short periods. However, in contrast to long-run models which focus on the trend of the economy, medium-run models attempt to understand the pattern characterizing the fluctuations around the trend. In this context, variations at both the demand and supply side are important. Indeed, at the centre of attention is the dynamic interaction between demand and supply factors, the correction of expectations, and the time-consuming adjustment of wages and prices. Such models are also called *business cycle models.* They have received substantial attention in recent years, both as attempts to explain the economic fluctuations we see and as frameworks suitable for reconciling and integrating short-run and long-run macroeconomics.

There is a further sense in which we deviate from the conventional textbook terminology relating to branches of macroeconomics. This relates to what the "long run" embraces. In this book we do not attempt to cover the modern theory of endogenous technical change. Since the path-breaking contributions by the American economist Paul Romer in the late 1980s, there has been a surge of "new growth theory" or "endogenous growth theory", focusing on endogenous technical change. This has developed growth theory into a more specialized discipline studying the factors and mechanisms that *determine* the evolution of technology and productivity.

In order not to have too many balls in the air at the same time, this text does not attempt to account for this expanding line of work within macroeconomics.[2] Thus, when we refer to "long-run macromodels", we do not think of growth models in this modern sense. Instead, we simply think of macromod-

[1] These number-of-years figures should not be understood as more than a rough indication. Their appropriateness will certainly depend on the specific problem and circumstances at hand.

[2] The reader is referred to the dedicated textbooks listed in *Bibliographic notes* at the end of this chapter.

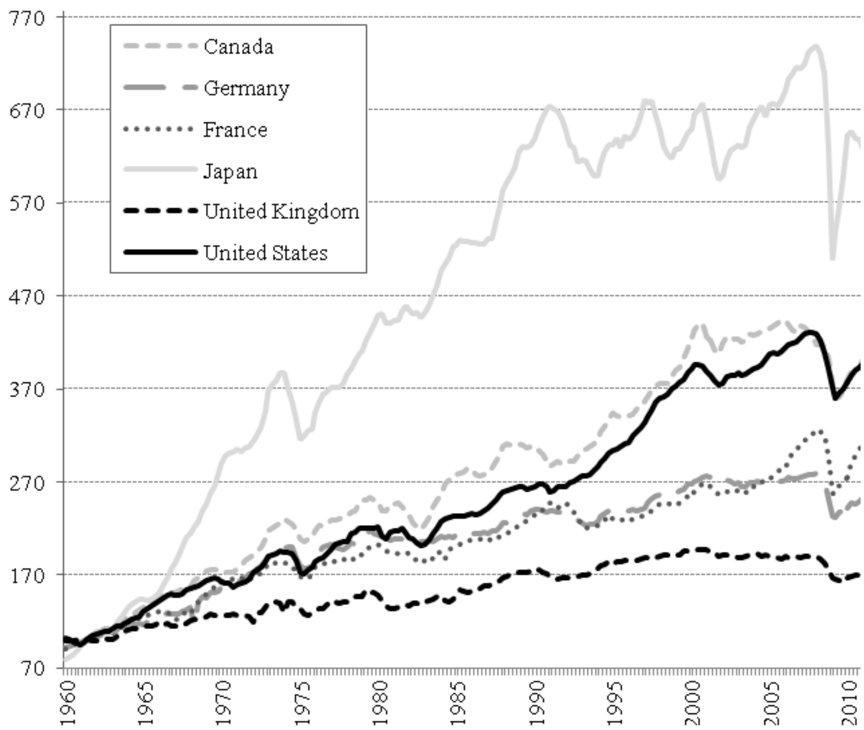

Figure 1.1: Industrial production index. Six major countries 1960-2011. 1961 = 100. Source: OECD. Note: Industrial production includes manufacturing, mining and quarrying, electricity, gas, and water, and construction.

els with a time horizon long enough such that changes in the capital stock, population, and technology matter. Apart from a few illustrative examples, we leave the *sources* of changes in technology out of consideration, which is tantamount to regarding these changes as exogenous.

It may be argued that this separation of different "runs" and themes involves a danger of overlooking the possibility that endogenous technical change plays a role in − and interacts with − business cycles. This is in fact a key element in "New Schumpeterian" theory. Ideas like this are not yet, however, established as part of core macroeconomic modelling. Hence we shall only briefly refer to such viewpoints and models when occasion arises.

In addition to the time scale dimension the national-international dimension is important for macroeconomics. Most industrialized economies participate in international trade of goods and financial assets. This results in considerable mutual dependency and co-movement of these economies. Downturns as well as upturns occur at about the same time, as indicated by Fig. 1.1. In particular the economic recessions triggered by the oil price shocks in 1973 and 1980 and by the disruption of credit markets in the outbreak of the financial crisis 2007-2008 are visible across the countries. Many of the models and mechanisms treated in this text will therefore be considered not only in a closed economy setup, but also from the point of view of open economies.

1.2 Components of macroeconomic models

1.2.1 Basics

(Incomplete)

Basic categories

Agents: We use simple descriptions of the economic agents: A *household* is generally either a one-parent family or an adult individual single. A *firm* is described as an abstract entity that carries out production, represented by a production function where the administrative staff and sales personnel are treated along with the production workers as an undifferentiated labor input.

Goods, labor, and assets markets

Stocks vs. flows (cf. a bathtub).

Types of model relations

Although model relations can take different forms, in macroeconomics they often have the form of equations. A taxonomy for such equations is the following:

1. *Technology equations,* describe relations between inputs and output (production functions and similar).

2. *Preference equations,* express preferences, e.g. $U = \sum_{t=0}^{T} \frac{u(c_t)}{(1+\rho)^t}$, $\rho > 0, u' > 0, u'' < 0$.

3. *Institutional equations,* refer to relationships required by law (e.g., how the tax levied depends on income) and similar.

4. *Behavioral equations,* describe the behavioral response to the determinants of behavior. This includes an agent's optimizing behavior written

as a function of its determinants. A consumption function is an example. Whether first-order conditions in optimization problems should be considered behavioral equations or just separate first-order conditions is a matter of taste.

5. *Identity equations,* are true by definition of the variables involved. National income accounting equations are an example.

6. *Equilibrium equations,* define the condition for equilibrium of some kind, for instance equality of Walrasian demand and Walrasian supply. No-arbitrage conditions for the asset markets also belong under the heading equilibrium condition.

A model equation may also represent a combination of two or more of these types of equations.

Analysis

Endogenous vs. exogenous variables.

Statics vs. dynamics. Comparative dynamics vs. study of dynamic effects of a parameter shift in historical time.

Modern macroeconomics studies processes in real time. Hence, the emphasis in this book is on *dynamic* models, that is, models that link a state of the economic system to a change in the system. So a dynamic model generates an evolution over time of the endogenous variables. A *static model* is a model where time does not enter or where all variables refer to the same point in time. Occasionally we do consider static models, but only as a prelude to a more elaborate dynamic model concerned with the same issues as they come into being in real time.

Partial equilibrium vs. general equilibrium:

We say that a given single market is in *partial equilibrium* at a given point in time if for arbitrarily given prices and quantities in the other markets, the agents' chosen actions in this market are mutually compatible. In contrast the concept of general equilibrium take the mutual dependencies between markets into account. We say that a given economy is in *general equilibrium* at a given point in time if in all markets the actions chosen by all the agents are mutually compatible.

An analyst trying to clarify a partial equilibrium problem is doing *partial equilibrium analysis.* Thus partial equilibrium analysis does not take into account the feedbacks from these actions to the rest of the economy and the feedbacks from these feedbacks − and so on. In contrast, an analyst trying to clarify a general equilibrium problem is doing *general equilibrium analysis.*

This requires considering the mutual dependencies in the system of markets as a whole.

Sometimes even the analysis of the constrained maximization problem of a single decision maker is called partial equilibrium analysis. Consider for instance the consumption-saving decision of a household. Then the analytical derivation of the saving function of the household is by some authors included under the heading partial equilibrium analysis, which may seem natural since the real wage and real interest rate appearing as arguments in the derived saving function are arbitrary. Indeed, what the actual saving of the young will be in the end, depends on the real wage and real interest rate formed in the general equilibrium.

In this book we call the analysis of a single decision maker's problem *partial analysis*, not partial equilibrium analysis. The motivation for this is that transparency is improved if one preserves the notion of equilibrium for a state of a *market* or a state of a *system of markets*.

1.2.2 The time dimension of inputs and outputs

In macroeconomic theory the production of a firm, a sector, or the economy as a whole is often represented by a two-factor production function given by

$$Y = F(K, L), \tag{1.1}$$

where Y is output (value added), K is capital input, and L is labor input ($K \geq 0$, $L \geq 0$). The idea is that for many issues it is useful in a first approach to think of output as a homogeneous good which is produced by two inputs, one of which is a *producible* durable means of production, the other being a *non-producible* human input. Simple as it looks, an equation like (1.1) is not always interpreted in the right way.

First, how are the variables entering (1.1) *denominated,* that is, in what units are the variables measured? It is most satisfactory, both from a theoretical and empirical point of view, to think of both outputs and inputs as *flows:* quantities per unit of time. This is generally recognized as far as Y is concerned. Unfortunately, it is less recognized concerning K and L, a circumstance which is probably related to a tradition in macroeconomic notation, as we will now explain.

Let the time unit be one year. Then the K appearing in the production function should be seen as the number of machine hours per year. Similarly, L should be seen as the number of labor hours per year. Unless otherwise specified, it should be understood that the rate of utilization of the production factors is constant over time; for convenience, one can then *normalize* *the rate of utilization of each factor to equal one.* That is, K = stock of capital $\times$ 1 and L = number of laborers $\times$ 1. Thus, with one year as our time unit, we imagine that normally a machine is in operation in h hours during a year. Then, it is natural to define one *machine-year* as the service of a machine in operation h hours a year. If K machines are in operation and on average deliver one machine year per year, then the total capital input is K machine-years per year:

$$K \text{ (machine-yrs/yr)} = K \text{ (machines)} \times 1 \text{ ((machine-yrs/yr)/machine)}, \tag{1.2}$$

where the denomination of the variables is indicated in brackets. Similarly, if the stock of laborers is L men and on average they deliver one *man-year* (say h hours) per year, then the total labor input is L man-years per year:

$$L\text{(man-yrs/yr)} = L\text{(men)} \times 1\text{((man-yrs/yr)/man)}. \tag{1.3}$$

One of the reasons that confusion of stocks and flows may arise is the tradition in macroeconomics to use the same symbol, K, for the capital *input* (the number of machine hours per year), in (1.1) as for the capital *stock* in an accumulation equation like

$$K_{t+1} = K_t + I_t - \delta K_t. \tag{1.4}$$

Here the interpretation of K_t is as a capital *stock* (number of machines) at the beginning of period t, I_t is gross investment, and δ is the rate of physical capital depreciation due to wear and tear ($0 \leq \delta \leq 1$). In (1.4) there is no role for the rate of *utilization* of the capital stock, which is, however, of key importance in (1.1). Similarly, there is a tradition in macroeconomics to denote the number of heads in the labor force by L and write, for example, $L_t = L_0(1+n)^t$, where n is a constant growth rate of the labor force. Here the interpretation of L_t is as a stock (number of persons). There is no role for the average rate of utilization in actual employment of this stock over the year.

This text will not attempt a break with this tradition of using the same symbol for two in principle different variables. But we insist on interpretations such that the notation is *consistent*. This requires normalization of the utilization rates for capital and labor in the production function to equal one, as indicated in (1.2) and (1.3) above. We are then allowed to use the same symbol for a stock and the corresponding flow because the *values* of the two variables will coincide.

An illustration of the importance of being aware of the distinction between stock and flows appears when we consider the following measure of per capita income in a given year:

$$\frac{GDP}{N} = \frac{GDP}{\#\text{hours}} \cdot \frac{\#\text{hours}}{\#\text{workers}} \cdot \frac{\#\text{workers}}{N}, \tag{1.5}$$

where N and #workers indicate, say, the average size of the population and the workforce, respectively, during the year. That is, aggregate per capita income equals average labor productivity times average labor intensity times the workforce participation rate. An increase from one year to the next in the ratio on the left-hand side of the equation reflects the net effect of changes in the three ratios on the right-hand side. Similarly, a fall in per capita income (a ratio between a flow and a stock) need not reflect a fall in productivity ($GDP/\#$hours, a ratio of two flows), but may reflect, say, a fall in the number of hours per member of the workforce due to a rise in unemployment or a rise in the preference for leisure.

When talking about the "labor force", we shall generally think of aggregate labor supply, i.e., #workers·(#hours supplied per worker per year), whether or not these supplied hours are also demanded (and thereby enter "employment") or not.

A *second* conceptual issue concerning the production function in (1.1) relates to the question: what about land and other natural resources? As farming requires land and factories and office buildings require building sites, a third argument, a natural resource input, should in principle appear in (1.1). In theoretical macroeconomics for industrialized economies this third factor is often left out because it does not vary much as an input to production and tends to be of relatively minor importance in value terms.

A *third* conceptual issue concerning the production function in (1.1) relates to the question: what about raw materials and other intermediate goods? Indeed, it may seem strange to regard output as produced by only capital and labor. Certainly, raw materials (non-durable means of production) are generally necessary inputs at the micro level. In macroeconomics, however, we normally abstract from the engineering input-output relations, involving raw materials and intermediate products, in the different branches of the economy. We imagine that at a lower stage of production materials are continuously produced by capital and labor, but are then immediately used up at a higher stage of production, again using capital and labor. The value of these materials are not part of value added in the sector or in the economy as a whole. Since value added is what macroeconomics usually focuses at and what the Y in (1.1) represents, materials therefore are often not explicit in the model.[3]

[3] On the other hand, if of interest for the problems studied, the analysis *should,* of course,

To further clarify this point as well as more general aspects of how macroeconomic models are related to national income and product accounts, the next section gives a review of national income accounting.

1.3 Macroeconomic models and national income accounting

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Stylized national income and product accounts

We give here a stylized picture of national income and product accounts with emphasis on the conceptual structure. The basic thing to be aware of is that national income accounting looks at output from *three sides*:

- the production side (value added),
- the use side,
- the income side.

Consider a closed economy with three production sectors (careful, see Raa p. 80). Sector 1 produces raw materials (or energy) in the amount Q_1 per time unit, Sector 2 produces durable capital goods in the amount Q_2 per time unit, and the third sector produces consumption goods in the amount Q_3 per time unit. The production factors are "land" (or non-producible natural resources more generally), labor, raw materials, and fixed capital.

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1.4 Some terminological points

On the vocabulary used in this book:

(Incomplete)

Economic terms

Physical capital refers to stocks of *reproducible durable* means of production such as machines and structures. Reproducible *non-durable* means of production include raw materials and energy and are sometimes called intermediate goods. Non-reproducible means of production, such as land and

take into account that at the aggregate level in real world situations, there will generally be a minor difference between produced and used-up materials which then constitutes net investment in inventories of materials.

other natural resources, are in this book not included under the heading "capital" but just called *natural resources*.

We follow the convention in macroeconomics and, unless otherwise specified, use "capital" for physical capital, that is, a production factor. In other branches of economics and in everyday language "capital" may mean the funds (sometimes called "financial capital") that finance purchases of physical capital.

By a household's *wealth, W*, we mean the value of the total stock of resources possessed by the household at a given point in time. This wealth generally has two main components, the *human wealth*, which is the present value of the stream of future labor income, and the *non-human wealth.* The latter is the sum of the value of the household's *physical assets* (also called *real* assets) and its *net financial assets*. Typically, housing wealth is the dominating component in households' physical assets. By *net financial assets* is meant the difference between the value of financial assets and the value of financial liabilities. *Financial assets* include cash as well as paper claims that entitles the owner to future transfers from the issuer of the claim, perhaps conditional on certain events. Bonds and shares are examples. And a *financial liability* of a household (or other type of agent) is an obligation to transfer resources to others in the future. A mortgage loan is an example.

In spite of this distinction between what is called physical assets and what is called financial assets, often in macroeconomics (and in this book unless otherwise indicated) the household's "financial wealth" is used synonymous with its non-human wealth, that is, including purely physical assets like land, buildings, machines, and other equipment. Somewhat at odds with this convention macroeconomics (including this book) generally uses "investment" as synonymous with "physical capital investment", that is, procurement of new machines and plants by firms and new houses or apartments by households. Then, when having purchases of *financial* assets in mind, macroeconomists talk of *financial investment*.

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Modeling terms

Types of variables:

A *predetermined* variable is a variable whose value is determined historically at any point in time. For example, the stock (quantity) of water as of time t in a bath tub is historically determined as the accumulated quantity of water stemming from the previous inflow and outflow. But if y_t is a variable which is not tied down by its own past but, on the contrary, can immediately adjust if new conditions or new information emerge, then y_t is a non-predetermined variable, also called a *jump variable,* and y_0 can then in principle immediately take on any value within some range. Returning to our bath tub example: in the moment we pull out the waste plug, the outflow of water per time unit will jump from zero to a positive value − it is a jump variable.

Math terms: locally-globally; stable, asymptotically stable.

1.5 Brief history of macroeconomics

Text not yet available.

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Akerlof and Shiller (2009),
Gali (2008)

1.6 Literature notes

The modern theory of economic growth ("new growth theory", "endogenous growth theory") is extensively covered in dedicated textbooks like Aghion and Howitt (1998), Jones (2002), Barro and Sala-i Martin (2004), Acemoglu (2009), and Aghion and Howitt (2009).

Snowdon and Vane (1997), Blanchard (2000), and Woodford (2000) present useful overviews of the history of macroeconomics. For surveys on recent developments on the research agenda within theory as well as practical policy analysis, see Mankiw (2006), Blanchard (2008), and Woodford (2009). Somewhat different perspectives, from opposite poles, are offered by Chari et al. (2009) and Colander et al. (2008).

Chapter 2

Review of technology and firms

The aim of this chapter is threefold. First, we shall introduce this book's vocabulary concerning firms' technology and technological change. Second, we shall refresh our memory of key notions from microeconomics relating to firms' behavior and factor market equilibrium under perfect competition. Finally, to prepare for the many cases where perfect competition is not a good approximation to reality, we give a simple account of firms' behavior and factor market equilibrium under monopolistic competition.

The vocabulary pertaining to other aspects of the economy, for instance households' preferences and behavior, is better dealt with in close connection with the specific models to be discussed in the subsequent chapters. Regarding the distinction between discrete and continuous time analysis, the definitions contained in this chapter are applicable to both.

2.1 The production technology

Consider a two-factor production function given by

$$Y = F(K, L), \tag{2.1}$$

where Y is output (value added) per time unit, K is capital input per time unit, and L is labor input per time unit ($K \geq 0$, $L \geq 0$). We may think of (2.1) as describing the output of a firm, a sector, or the economy as a whole. It is in any case a very simplified description, ignoring the heterogeneity of output, capital, and labor. Yet, for many macroeconomic questions it may be a useful first approach. Note that in (2.1) not only Y but also K and L represent *flows,* that is, quantities per unit of time. If the time unit is one year, we think of K as measured in machine hours per year. Similarly, we think of L as measured in labor hours per year. Unless otherwise specified, it is understood that the rate of utilization of the production factors is constant over time and normalized to one for each production factor. As explained in Chapter 1, we can then use the same symbol, K, for the *flow* of capital services as for the *stock* of capital. Similarly with L.

2.1.1 A neoclassical production function

By definition, K and L are non-negative. It is generally understood that a production function, $Y = F(K, L)$, is *continuous* and that $F(0, 0) = 0$ (no input, no output). Sometimes, when specific functional forms are used to represent a production function, that function may not be defined at points where $K = 0$ or $L = 0$ or both. In such a case we adopt the convention that the domain of the function is understood extended to include such boundary points whenever it is possible to assign function values to them such that continuity is maintained. For instance the function $F(K, L) = \alpha L + \beta KL/(K + L)$, where $\alpha > 0$ and $\beta > 0$, is not defined at $(K, L) = (0, 0)$. But by assigning the function value 0 to the point $(0, 0)$, we maintain both continuity and the "no input, no output" property, cf. Exercise 2.4.

We call the production function *neoclassical* if for all (K, L), with $K > 0$ and $L > 0$, the following additional conditions are satisfied:

(a) $F(K, L)$ has continuous first- and second-order partial derivatives satisfying:

$$F_K > 0, \quad F_L > 0, \tag{2.2}$$

$$F_{KK} < 0, \quad F_{LL} < 0. \tag{2.3}$$

(b) $F(K, L)$ is strictly quasiconcave (i.e., the level curves, also called isoquants, are strictly convex to the origin).

In words: (a) says that a neoclassical production function has continuous substitution possibilities between K and L and the *marginal productivities* are positive, but diminishing in own factor. Thus, for a given number of machines, adding one more unit of labor, adds to output, but less so, the higher is already the labor input. And (b) says that every isoquant, $F(K, L) = \bar{Y}$, has a form qualitatively similar to that shown in Fig. 2.1.[1] When we speak of for example F_L as the marginal *productivity* of labor, it is because the "pure" partial derivative, $\partial Y/\partial L = F_L$, has the denomination of a productivity (output units/yr)/(man-yrs/yr). It is quite common, however, to refer

[1] A refresher on mathematical terms such as *boundary point, convex function,* etc. is contained in Math Tools at the end of this book.

to F_L as the marginal *product* of labor. Then a unit marginal increase in the labor input is understood: $\Delta Y \approx (\partial Y/\partial L)\Delta L = \partial Y/\partial L$ when $\Delta L = 1$. Similarly, F_K can be interpreted as the marginal *productivity* of capital or as the marginal *product* of capital. In the latter case it is understood that $\Delta K = 1$, so that $\Delta Y \approx (\partial Y/\partial K)\Delta K = \partial Y/\partial K$.

The definition of a neoclassical production function can be extended to the case of n inputs. Let the input quantities be $X_1, X_2, \ldots, X_n$ and consider a production function $Y = F(X_1, X_2, \ldots, X_n)$. Then F is called neoclassical if all the marginal productivities are positive, but diminishing, and F is strictly quasiconcave (i.e., the upper contour sets are strictly convex, cf. Appendix A).

Returning to the two-factor case, since $F(K, L)$ presumably depends on the level of technical knowledge and this level depends on time, t, we might want to replace (2.1) by

$$Y_t = F^t(K_t, L_t), \tag{2.4}$$

where the superscript on F indicates that the production function may shift over time, due to changes in technology. We then say that $F^t(\cdot)$ is a neoclassical production function if it satisfies the conditions (a) and (b) for all pairs (K_t, L_t). *Technological progress* can then be said to occur when, for K_t and L_t held constant, output increases with t.

For convenience, to begin with we skip the explicit reference to time and level of technology.

The marginal rate of substitution Given a neoclassical production function F, we consider the isoquant defined by $F(K, L) = \bar{Y}$, where $\bar{Y}$ is a positive constant. The *marginal rate of substitution*, MRS_{KL}, of K for L at the point (K, L) is defined as the absolute slope of the isoquant at that point, cf. Fig. 2.1. The equation $F(K, L) = \bar{Y}$ defines K as an implicit function of L. By implicit differentiation we find $F_K(K, L)dK/dL + F_L(K, L) = 0$, from which follows

$$MRS_{KL} = -\frac{dK}{dL}_{|Y=\bar{Y}} = \frac{F_L(K, L)}{F_K(K, L)} > 0. \tag{2.5}$$

That is, MRS_{KL} measures the amount of K that can be saved (approximately) by applying an extra unit of labor. In turn, this equals the ratio of the marginal productivities of labor and capital, respectively.[2] Since F is neoclassical, by definition F is strictly quasi-concave and so the marginal rate of substitution is diminishing as substitution proceeds, i.e., as the labor

[2]The subscript $|Y = \bar{Y}$ in (2.5) indicates that we are moving along a given isoquant, $F(K, L) = \bar{Y}$.

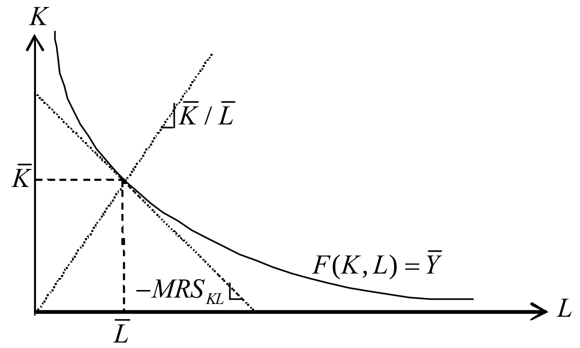

Figure 2.1: MRS_{KL} as the absolute slope of the isoquant.

input is further increased along a given isoquant. Notice that this feature characterizes the marginal rate of substitution for any neoclassical production function, whatever the returns to scale (see below).

When we want to draw attention to the dependency of the marginal rate of substitution on the factor combination considered, we write $MRS_{KL}(K, L)$. Sometimes in the literature, the marginal rate of substitution between two production factors, K and L, is called the *technical* rate of substitution (or the technical rate of transformation) in order to distinguish from a consumer's marginal rate of substitution between two consumption goods.

As is well-known from microeconomics, a firm that minimizes production costs for a given output level and given factor prices, will choose a factor combination such that MRS_{KL} equals the ratio of the factor prices. If $F(K, L)$ is homogeneous of degree q, then the marginal rate of substitution depends only on the factor proportion and is thus the same at any point on the ray $K = (\bar{K}/\bar{L})L$. That is, in this case the expansion path is a straight line.

The Inada conditions A continuously differentiable production function is said to satisfy the *Inada conditions*[3] if

$$\lim_{K\to 0} F_K(K, L) = \infty, \lim_{K\to\infty} F_K(K, L) = 0, \tag{2.6}$$
$$\lim_{L\to 0} F_L(K, L) = \infty, \lim_{L\to\infty} F_L(K, L) = 0. \tag{2.7}$$

In this case, the marginal productivity of either production factor has no upper bound when the input of the factor becomes infinitely small. And the marginal productivity is vanishing when the input of the factor increases without bound. Actually, (2.6) and (2.7) express *four* conditions, which it is

[3]After the Japanese economist Ken-Ichi Inada, 1925-2002.

preferable to consider separately and label one by one. In (2.6) we have two *Inada conditions for MPK* (the marginal productivity of capital), the first being a *lower*, the second an *upper* Inada condition for MPK. And in (2.7) we have two *Inada conditions for MPL* (the marginal productivity of labor), the first being a *lower*, the second an *upper* Inada condition for MPL. In the literature, when a sentence like "the Inada conditions are assumed" appears, it is sometimes not made clear which, and how many, of the four are meant. Unless it is evident from the context, it is better to be explicit about what is meant.

The definition of a neoclassical production function we gave above is quite common in macroeconomic journal articles and convenient because of its flexibility. There are economic growth textbooks that define a neoclassical production function more narrowly by including the Inada conditions as a requirement for calling the production function neoclassical. In contrast, when in a given context we need one or another Inada condition, we state it explicitly as an additional assumption.

2.1.2 Returns to scale

If all the inputs are multiplied by some factor, is output then multiplied by the same factor? There may be different answers to this question, depending on circumstances. We consider a production function $F(K, L)$ where $K > 0$ and $L > 0$. Then F is said to have *constant returns to scale* (CRS for short) if it is homogeneous of degree one, i.e., if for all (K, L) and all $\lambda > 0$,

$$F(\lambda K, \lambda L) = \lambda F(K, L).$$

As all inputs are scaled up or down by some factor, output is scaled up or down by the same factor.[4] The assumption of CRS is often defended by the *replication argument.* Before discussing this argument, lets us define the two alternative "pure" cases.

The production function $F(K, L)$ is said to have *increasing returns to scale* (IRS for short) if, for all (K, L) and all $\lambda > 1$,

$$F(\lambda K, \lambda L) > \lambda F(K, L).$$

That is, IRS is present if, when all inputs are scaled up by some factor, output is scaled up by *more* than this factor. The existence of gains by

[4] In their definition of a neoclassical production function some textbooks add constant returns to scale as a requirement besides (a) and (b). This book follows the alternative terminology where, if in a given context an assumption of constant returns to scale is needed, this is stated as an additional assumption.

specialization and division of labor, synergy effects, etc. sometimes speak in support of this assumption, at least up to a certain level of production. The assumption is also called the *economies of scale* assumption.

Another possibility is *decreasing returns to scale* (DRS). This is said to occur when for all (K, L) and all $\lambda > 1$,

$$F(\lambda K, \lambda L) < \lambda F(K, L).$$

That is, DRS is present if, when all inputs are scaled up by some factor, output is scaled up by *less* than this factor. This assumption is also called the *diseconomies of scale* assumption. The underlying hypothesis may be that control and coordination problems confine the expansion of size. Or, considering the "replication argument" below, DRS may simply reflect that behind the scene there is an additional production factor, for example land or a irreplaceable quality of management, which is tacitly held fixed, when the factors of production are varied.

EXAMPLE 1 The production function

$$Y = AK^{\alpha}L^{\beta}, \qquad A > 0, 0 < \alpha < 1, 0 < \beta < 1, \tag{2.8}$$

where A, α, and β are given parameters, is called a *Cobb-Douglas production function*. The parameter A depends on the choice of measurement units; for a given such choice it reflects the "total factor productivity". Exercise 2.2 asks the reader to verify that (2.8) satisfies (a) and (b) above and is therefore a neoclassical production function. The function is homogeneous of degree $\alpha + \beta$. If $\alpha + \beta = 1$, there are CRS. If $\alpha + \beta < 1$, there are DRS, and if $\alpha + \beta > 1$, there are IRS. Note that α and β must be less than 1 in order not to violate the diminishing marginal productivity condition. □

EXAMPLE 2 The production function

$$Y = \min(AK, BL), \qquad A > 0, B > 0, \tag{2.9}$$

where A and B are given parameters, is called a *Leontief production function* or a *fixed-coefficients production function;* A and B are called the *technical coefficients*. The function is not neoclassical, since the conditions (a) and (b) are not satisfied. Indeed, with this production function the production factors are not substitutable at all. This case is also known as the case of *perfect complementarity* between the production factors. The interpretation is that already installed production equipment requires a fixed number of workers to operate it. The inverse of the parameters A and B indicate the required capital input per unit of output and the required labor input per unit of output, respectively. Extended to many inputs, this type of production function is often used in multi-sector input-output models (also called Leontief models). In aggregate analysis neoclassical production functions, allowing substitution between capital and labor, are more popular than Leontief functions. But sometimes the latter are preferred, in particular in short-run analysis with focus on the use of already installed equipment where the substitution possibilities are limited.[5] As (2.9) reads, the function has CRS. A generalized form of the Leontief function is $Y = \min(AK^\gamma, BL^\gamma)$, where $\gamma > 0$. When $\gamma < 1$, there are DRS, and when $\gamma > 1$, there are IRS. □

The replication argument The assumption of CRS is widely used in macroeconomics. The model builder may appeal to the *replication argument.* To explain the content of this argument we have to first clarify the distinction between rival and nonrival inputs or more generally the distinction between rival and nonrival goods. A good is *rival* if its character is such that one agent's use of it inhibits other agents' use of it at the same time. A pencil is thus rival. Many production inputs like raw materials, machines, labor etc. have this property. In contrast, however, technical knowledge like a farmaceutical formula or an engineering principle are *nonrival.* An unbounded number of factories can simultaneously use the same farmaceutical formula.

The replication argument now says that by, conceptually, doubling all the rival inputs, we should always be able to double the output, since we just "replicate" what we are already doing. One should be aware that the CRS assumption is about *technology* in the sense of functions linking inputs to outputs − limits to the availability of resources is another matter. The fact that for example managerial talent may be in limited supply does not preclude the thought experiment that *if* a firm could double all its inputs, including managerial talent, then the output level could also be doubled.

The CRS assumption and the replication argument presuppose first that *all* the relevant inputs are explicit as arguments in the production function, second that these are changed equiproportionately. This, however, exhibits the weakness of the replication argument as a defence for assuming CRS of our present production function, $F(\cdot)$. One could easily make the case that besides capital and labor, also land is a necessary input and should appear as a separate argument.[6] If an industrial firm decides to duplicate what it

[5] See Section 2.5.2.

[6] Recall from Chapter 1 that we think of "capital" as producible means of production, whereas "land" refers to non-producible natural resources, including for example building sites.

has been doing, it needs a piece of land to build another plant like the first. Then, on the basis of the replication argument we should in fact expect DRS w.r.t. capital and labor alone. In manufacturing and services, empirically, this and other possible sources for departure from CRS may be minor and so many macroeconomists feel comfortable enough with assuming CRS w.r.t. K and L alone, at least as a first approximation. This approximation is, however, less applicable to poor countries, where natural resources may be a quantitatively important production factor.

There is a further problem with the replication argument. Strictly speaking, the CRS claim is that by changing all the inputs equiproportionately by *any* positive factor, λ, which does not have to be an integer, the firm should be able to get output changed by the same factor. Hence, the replication argument requires that indivisibilities are negligible, which is certainly not always the case. In fact, the replication argument is more an argument *against* DRS than *for* CRS in particular. The argument does not rule out IRS due to synergy effects as size is increased.

Sometimes the replication line of reasoning is given a more subtle form. This builds on a useful *local* measure of returns to scale.

The elasticity of scale To allow for indivisibilities and mixed cases (for example IRS at low levels of production and CRS or DRS at higher levels), we need a local measure of returns to scale. One defines the *elasticity of scale*, $\eta(K, L)$, of F at the point (K, L), where $F(K, L) > 0$, as

$$\eta(K,L) = \frac{\lambda}{F(K,L)}\frac{dF(\lambda K, \lambda L)}{d\lambda} \approx \frac{\Delta F(\lambda K, \lambda L)/F(K,L)}{\Delta\lambda/\lambda}, \text{ evaluated at } \lambda = 1. \tag{2.10}$$

So the elasticity of scale at a point (K, L) indicates the (approximate) percentage increase in output when both inputs are increased by 1 percent. We say that

$$\text{if } \eta(K,L) \begin{cases} > 1, & \text{then there are locally } \mathit{IRS}, \\ = 1, & \text{then there are locally } \mathit{CRS}, \\ < 1, & \text{then there are locally } \mathit{DRS}. \end{cases} \tag{2.11}$$

The production function *may* have the same elasticity of scale everywhere. This is the case if and only if the production function is homogeneous. If F is homogeneous of degree h, then $\eta(K, L) = h$ and h is called the *elasticity of scale parameter.*

Note that the elasticity of scale at a point (K, L) will always equal the sum of the partial output elasticities at that point:

$$\eta(K,L) = \frac{F_K(K,L)K}{F(K,L)} + \frac{F_L(K,L)L}{F(K,L)}. \tag{2.12}$$

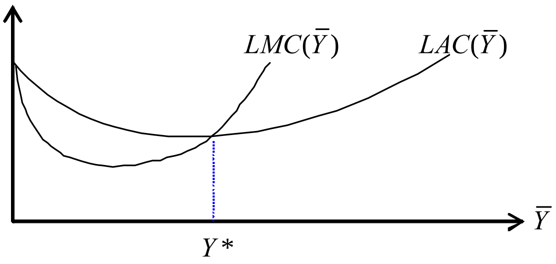

Figure 2.2: Locally CRS at optimal plant size.

This follows from the definition in (2.10) by taking into account that

$$\begin{aligned}\frac{dF(\lambda K, \lambda L)}{d\lambda} &= F_K(\lambda K, \lambda L)K + F_L(\lambda K, \lambda L)L \\ &= F_K(K, L)K + F_L(K, L)L, \text{ when evaluated at } \lambda = 1.\end{aligned}$$

Fig. 2.2 illustrates a popular case from microeconomics, a U-shaped average cost curve from the perspective of the individual firm (or plant): at low levels of output there are falling average costs (thus IRS), at higher levels rising average costs (thus DRS). Given the input prices, w_K and w_L, and a specified output level, $\bar{Y}$, we know that the cost minimizing factor combination $(\bar{K}, \bar{L})$ is such that $F_L(\bar{K}, \bar{L})/F_K(\bar{K}, \bar{L}) = w_L/w_K$. It is shown in Appendix A that the elasticity of scale at $(\bar{K}, \bar{L})$ will satisfy:

$$\eta(\bar{K}, \bar{L}) = \frac{LAC(\bar{Y})}{LMC(\bar{Y})}, \tag{2.13}$$

where $LAC(\bar{Y})$ is average costs (the minimum unit cost associated with producing $\bar{Y}$) and $LMC(\bar{Y})$ is marginal costs at the output level $\bar{Y}$. The L in LAC and LMC stands for "long-run", indicating that both capital and labor are considered variable production factors within the period considered. At the optimal plant size, Y^*, there is equality between LAC and LMC, implying a unit elasticity of scale, that is, locally we have CRS.

This provides a more subtle replication argument for CRS at the aggregate level. Even though technologies may differ across firms, the surviving firms in a competitive market will have the same average costs at the optimal plant size. In the medium and long run, changes in aggregate output will take place primarily by entry and exit of optimal-size plants. Then, with a large number of relatively small plants, each producing at approximately constant unit costs for small output variations, we can without substantial error assume constant returns to scale at the aggregate level. So the argument goes. Notice, however, that even in this form the replication argument is not entirely convincing since the question of indivisibility remains. The optimal plant size may be large relative to the market − and is in fact so in many industries. Besides, in this case also the perfect competition premise breaks down.

The empirical evidence concerning returns to scale is mixed (see the literature notes at the end of the chapter). Notwithstanding the theoretical and empirical ambiguities, the assumption of CRS w.r.t. capital and labor has a prominent role in macroeconomics. In many contexts it is regarded as an acceptable approximation and a convenient simple background for studying the question at hand.

2.1.3 Properties of the production function under CRS

Expedient inferences of the CRS assumption include:

(i) marginal costs are constant and equal to average costs (so the right-hand side of (2.13) equals unity);

(ii) if production factors are paid according to their marginal productivities, factor payments exactly exhaust total output so that pure profits are neither positive nor negative (so the right-hand side of (2.12) equals unity);

(iii) a production function known to exhibit CRS and satisfy property (a) from the definition of a neoclassical production function above, will automatically satisfy also property (b) and consequently *be* neoclassical;

(iv) a neoclassical two-factor production function with CRS has always $F_{KL} > 0$, i.e., it exhibits "direct complementarity" between K and L;

(v) a two-factor production function known to have CRS and to be twice continuously differentiable with positive marginal productivity of each factor everywhere in such a way that all isoquants are strictly convex to the origin, *must* have *diminishing* marginal productivities everywhere.[7]

A principal implication of the CRS assumption is that it allows a reduction of dimensionality. Considering a neoclassical production function, $Y = F(K, L)$ with $L > 0$, we can under CRS write $F(K, L) = LF(K/L, 1)$

[7]Proof of claim (iii) is in Appendix A and proofs of claim (iv) and (v) are in Appendix B.

$\equiv Lf(k)$, where $k \equiv K/L$ is called the *capital-labor ratio* (sometimes the *capital intensity*) and $f(k)$ is the *production function in intensive form* (sometimes named the per capita production function). Thus output per unit of labor depends only on the capital intensity:

$$y \equiv \frac{Y}{L} = f(k).$$

When the original production function F is neoclassical, under CRS the expression for the marginal productivity of capital simplifies:

$$F_K(K,L) = \frac{\partial Y}{\partial K} = \frac{\partial\left[Lf(k)\right]}{\partial K} = Lf'(k)\frac{\partial k}{\partial K} = f'(k). \tag{2.14}$$

And the marginal productivity of labor can be written

$$\begin{aligned} F_L(K,L) &= \frac{\partial Y}{\partial L} = \frac{\partial\left[Lf(k)\right]}{\partial L} = f(k) + Lf'(k)\frac{\partial k}{\partial L} \\ &= f(k) + Lf'(k)K(-L^{-2}) = f(k) - f'(k)k. \end{aligned} \tag{2.15}$$

A neoclassical CRS production function in intensive form always has a positive first derivative and a negative second derivative, i.e., $f' > 0$ and $f'' < 0$. The property $f' > 0$ follows from (2.14) and (2.2). And the property $f'' < 0$ follows from (2.3) combined with

$$F_{KK}(K,L) = \frac{\partial f'(k)}{\partial K} = f''(k)\frac{\partial k}{\partial K} = f''(k)\frac{1}{L}.$$

For a neoclassical production function with CRS, we also have

$$f(k) - f'(k)k > 0 \text{ for all } k > 0, \tag{2.16}$$

in view of $f(0) \geq 0$ and $f'' < 0$. Moreover,

$$\lim_{k\to 0}\left[f(k) - f'(k)k\right] = f(0). \tag{2.17}$$

Indeed, from the mean value theorem[8] we know there exists a number $a \in (0,1)$ such that for any given $k > 0$ we have $f(k) - f(0) = f'(ak)k$. From this follows $f(k) - f'(ak)k = f(0) < f(k) - f'(k)k$, since $f'(ak) > f'(k)$ by $f'' < 0$. In view of $f(0) \geq 0$, this establishes (2.16). And from $f(k) > f(k) - f'(k)k > f(0)$ and continuity of f follows (2.17).

[8]This theorem says that if f is continuous in $[\alpha,\beta]$ and differentiable in (α,β), then there exists at least one point γ in (α,β) such that $f'(\gamma) = (f(\beta) - f(\alpha))/(\beta - \alpha)$.

Under CRS the Inada conditions for MPK can be written

$$\lim_{k\to 0} f'(k) = \infty, \qquad \lim_{k\to\infty} f'(k) = 0. \tag{2.18}$$

An input which must be positive for positive output to arise is called an *essential input*. The second part of (2.18), representing the upper Inada condition for MPK under CRS, has the implication that *labor* is an essential input; but capital need not be, as the production function $f(k) = a + bk/(1+k)$, $a > 0, b > 0$, illustrates. Similarly, under CRS the upper Inada condition for MPL implies that *capital* is an essential input. These claims are proved in Appendix C. Combining these results, when *both* the upper Inada conditions hold and CRS obtain, then both capital and labor are essential inputs.[9]

Fig. 2.3 is drawn to provide an intuitive understanding of a neoclassical CRS production function and at the same time illustrate that the lower Inada conditions are more questionable than the upper Inada conditions. The left panel of Fig. 2.3 shows output per unit of labor for a CRS neoclassical production function satisfying the Inada conditions for MPK. The $f(k)$ in the diagram could for instance represent the Cobb-Douglas function in Example 1 with $\beta = 1-\alpha$, i.e., $f(k) = Ak^{\alpha}$. The right panel of Fig. 2.3 shows a non-neoclassical case where only two alternative Leontief techniques are available, technique 1: $y = \min(A_1 k, B_1)$, and technique 2: $y = \min(A_2 k, B_2)$. In the exposed case it is assumed that $B_2 > B_1$ and $A_2 < A_1$ (if $A_2 \geq A_1$ at the same time as $B_2 > B_1$, technique 1 would not be efficient, because the same output could be obtained with less input of at least one of the factors by shifting to technique 2). If the available K and L are such that $k < B_1/A_1$ or $k > B_2/A_2$, some of either L or K, respectively, is idle. If, however, the available K and L are such that $B_1/A_1 < k < B_2/A_2$, it is efficient to *combine* the two techniques and use the fraction μ of K and L in technique 1 and the remainder in technique 2, where $\mu = (B_2/A_2 - k)/(B_2/A_2 - B_1/A_1)$. In this way we get the "labor productivity curve" OPQR (the envelope of the two techniques) in Fig. 2.3. Note that for $k \to 0$, MPK stays equal to $A_1 < \infty$, whereas for all $k > B_2/A_2$, $MPK = 0$. A similar feature remains true, when we consider *many,* say n, alternative efficient Leontief techniques available. Assuming these techniques cover a considerable range w.r.t. the B/A ratios, we get a labor productivity curve looking more like that of a neoclassical CRS production function. On the one hand, this gives some intuition of what lies behind the assumption of a neoclassical CRS production function. On the other hand, it remains true that for all $k > B_n/A_n$, $MPK = 0$,[10] whereas

[9] Given a Cobb-Douglas production function, both production factors are essential whether we have DRS, CRS, or IRS.

[10] Here we assume the techniques are numbered according to ranking with respect to the size of B.

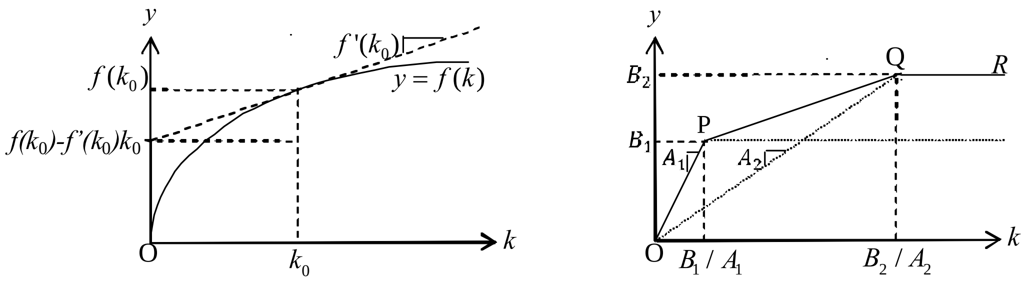

Figure 2.3: Two labor productivity curves based on CRS technologies. Left: neoclassical technology with Inada conditions for MPK satisfied. Right: a combination of two efficient Leontief techniques.

for $k \to 0$, MPK stays equal to $A_1 < \infty$, thus questioning the lower Inada condition.

The implausibility of the lower Inada conditions is also underlined if we look at their implication in combination with the more reasonable upper Inada conditions. Indeed, the four Inada conditions taken *together* imply, under CRS, that output has no upper bound when either input goes to infinity for fixed amount of the other input (see Appendix C).

2.2 Technological change

When considering the movement over time of the economy, we shall often take into account the existence of *technological change*. When technological change occurs, the production function becomes time-dependent. Over time the production factors tend to become more productive: more output for given inputs. To put it differently: the isoquants move inward. When this is the case, we say that the technological change displays *technological progress*.

Concepts of neutral technological change

A first step in taking this into account is to replace (2.1) by (2.4). In macroeconomics, however, technological change is often (and not too unrealistically) assumed to take the specific form of *Harrod-neutral technological change*.[11] This amounts to assuming we can write (2.4) in the form

$$Y_t = F(K_t, T_t L_t), \tag{2.19}$$

[11] The name refers to the English economist Roy F. Harrod, 1900−1978.

where F is a (time-independent) neoclassical production function, Y_t, K_t, and L_t are output, capital, and labor input, respectively, at time t, while T_t is the efficiency of labor and indicates the "technology level". Although one can imagine natural disasters implying a fall in T_t, generally T_t tends to rise over time and then we say that (2.19) represents *Harrod-neutral technological progress.* An alternative name for this is *labor-augmenting* technological progress (it acts *as if* the labor input were augmented).

If the function F in (2.19) is homogeneous of degree one (so that the technology for all t exhibits CRS w.r.t. capital and labor), we may write

$$\tilde{y}_t \equiv \frac{Y_t}{T_t L_t} = F(\frac{K_t}{T_t L_t}, 1) = F(\tilde{k}_t, 1) \equiv f(\tilde{k}_t), \qquad f' > 0, f'' < 0.$$

where $\tilde{k}_t \equiv K_t/(T_t L_t) \equiv k_t/T_t$ (habitually called the "effective" capital intensity or, if there is no risk of confusion, just the capital intensity). In rough accordance with a general trend in aggregate productivity data for industrialized countries we often assume that T grows at a constant rate, g, so that, in discrete time $T_t = T_0(1+g)^t$ and in continuous time $T_t = T_0 e^{gt}$, where $g > 0$. The popularity in macroeconomics of the hypothesis of labor-augmenting technological progress derives from its consistency with Kaldor's "stylized facts", cf. Chapter 4.

There exists two alternative concepts of neutral technological progress. *Hicks-neutral* technological progress is said to occur if the technological development is such that (2.4) can be written in the form

$$Y_t = T_t F(K_t, L_t), \tag{2.20}$$

where, again, F is a (time-independent) neoclassical production function, while T_t is the growing technology level.[12] The assumption of Hicks-neutrality has been used more in microeconomics and partial equilibrium analysis than in macroeconomics. Finally, *Solow-neutral* technological progress[13] is said to occur if technological development is such that (2.4) can be written in the form

$$Y_t = F(T_t K_t, L_t). \tag{2.21}$$

Another name for the same is *capital-augmenting* technological progress (because the assumption is essentially that technological change acts as if the

[12] The name refers to the English economist and Nobel Prize winner John R. Hicks, 1904−1989.

[13] The name refers to the American economist and Nobel Prize winner Robert Solow (1924−).

capital input were augmented).[14]

It is easily shown (Exercise 2.5) that the Cobb-Douglas production function satisfies all three neutrality criteria at the same time, if it satisfies one of them (which it does if technological change does not affect α and β, cf. (2.8) above). It can also be shown that within the class of neoclassical CRS production functions the Cobb-Douglas function is the only one with this property (see Exercise 4.? in Chapter 4).

Note that the neutrality concepts do not say anything about the *source* of technological progress, only about the quantitative form in which it materializes. For instance, the occurrence of Harrod-neutrality should not be interpreted as indicating that the technological change emanates specifically from the labor input in some sense. Harrod-neutrality only means that technological innovations predominantly are such that not only do labor and capital in combination become more productive, but this happens to *manifest itself* in the form (2.19).

Before proceeding, a brief remark about how the capital stock, K_t, is typically measured may be useful. While data on gross investment, I, is available in national income and product accounts, data on K_t usually is not. One approach to the measurement of K_t is the *perpetual inventory method* which builds on the accounting relationship

$$K_t = I_{t-1} + (1-\delta)K_{t-1}. \tag{2.22}$$

Assuming a constant capital depreciation rate δ, backward substitution gives

$$K_t = I_{t-1} + (1-\delta)\left[I_{t-2} + (1-\delta)K_{t-2}\right] = \ldots = \sum_{i=1}^{T}(1-\delta)^{i-1}I_{t-i} + (1-\delta)^T K_{t-T}. \tag{2.23}$$

Based on a long time series for I and an estimate of δ, one can insert these observed values in the formula and calculate K_t, starting from a rough conjecture about the initial value K_{t-T}. The result will not be very sensitive to this conjecture since, for large T the last term in (2.23) becomes very small.

[14] Macroeconomists' use of the value-laden term "technological progress" in connection with technological change may seem suspect. But the term should be interpreted as merely a label for certain types of shifts of isoquants in an abstract universe. At a more concrete and disaggregate level analysts of course make use of more refined notions about technological change, recognizing for example not only benefits of new technologies, but also the risks, including risk of fundamental mistakes (think of the introduction and later abandonment of asbestos in the construction industry).

Embodied vs. disembodied technological progress

There is an important taxonomy of technological change not yet considered. We say that technological change is *embodied*, if taking advantage of new technical knowledge requires construction of new investment goods. The new technology is incorporated in the design of newly produced equipment, but this equipment will not participate in subsequent technological progress. An example: only the most recent vintage of a computer series incorporates the most recent advance in information technology. Then investment goods produced later (investment goods of a later "vintage") have higher productivity than investment goods produced earlier at the same resource cost. Thus investment becomes an important driving force in productivity increases.

We way formalize embodied technological progress by writing capital accumulation in the following way:

$$K_{t+1} - K_t = Q_t I_t - \delta K_t, \tag{2.24}$$

where I_t is gross investment in period t, i.e., $I_t = Y_t - C_t$, and Q_t measures the "quality" (productivity) of newly produced investment goods. The rising level of technology implies rising Q so that a given level of investment gives rise to a greater and greater addition to the capital stock, K, measured in efficiency units. Even if technological change does not directly appear in the production function, that is, even if for instance (2.19) is replaced by $Y_t = F(K_t, L_t)$, the economy may thus experience a rising standard of living.

In contrast, *disembodied technological change* occurs when new technical and organizational knowledge increases the combined productivity of the production factors independently of when they were constructed or educated. If the K_t appearing in (2.19), (2.20), and (2.21) above refers to the total, historically accumulated capital stock, then these expressions represent disembodied technological change. All vintages of the capital equipment benefit from a rise in the technology level T_t. No new investment is needed to benefit. The perpetual inventory method, as described by (2.23), for calculating K_t makes sense in this case.

Based on data for the U.S. 1950-1990, and taking quality improvements into account, Greenwood et al. (1997) estimate that embodied technological progress explains about 60% of the growth in output per man hour. So, empirically, *embodied* technological progress seems to play the dominant role. Then the perpetual inventory method no longer fits the situation. Or at least, the resulting K will not indicate the capital stock measured in efficiency units.

For many issues in macroeconomics the distinction between embodied and disembodied technological progress is not so important and therefore the analyst is inclined to assume the simplest form, which definitely is the disembodied form. In this book we will follow this practice and unless specified otherwise, technological change is understood to be disembodied.

2.3 The concepts of a representative firm and an aggregate production function

Many macroeconomic models make use of the simplifying notion of a *representative firm.* By this is meant a hypothetical firm whose production "represents" aggregate production (value added) in a sector or in society as a whole. Let n be the actual number of firms in the sector or in society and let Y_i, K_i, and L_i be output, capital input and labor input (per time unit) for firm i, $i = 1, 2, \ldots, n$. Further, let $Y = \Sigma_{i=1}^n Y_i$, $K = \Sigma_{i=1}^n K_i$, and $L = \Sigma_{i=1}^n L_i$. Ignoring technological change, suppose these aggregate variables in a given society turn out to be related through some production function, $F^*(\cdot)$, in the following way:

$$Y = F^*(K, L).$$

Then $F^*(K, L)$ is called the *aggregate production function* or the production function of the *representative* firm. It is *as if* aggregate production is the result of the behavior of such a single firm.

A simple example where the aggregate production function is well-defined is the following. Suppose that all firms have the same production function $F(\cdot)$ with CRS so that $Y_i = F(K_i, L_i)$, $i = 1, 2, \ldots, n$. In view of CRS,

$$Y_i = F(K_i, L_i) = L_i F(k_i, 1) \equiv L_i f(k_i),$$

where $k_i \equiv K_i/L_i$. Hence, facing given factor prices, all cost minimizing firms will choose the same capital intensity: $k_i = k$, for all i. From $K_i = kL_i$ then follows $\sum_i K_i = k \sum_i L_i$ so that $k = K/L$. Thence,

$$Y \equiv \sum Y_i = \sum L_i f(k_i) = f(k) \sum L_i = f(k)L = F(k, 1)L = F(K, L).$$

In this (trivial) case it is thus easy to construct an aggregate production function and this function turns out to be exactly the same as the (identical) CRS production functions of the individual firms.

Allowing for the existence of *different* output goods, capital goods, and technologies makes the issue more intricate, of course. Yet, if firms are price taking profit maximizers and there are nonincreasing returns to scale, then the aggregate outcome is *as if* the firms jointly maximized aggregate profit on the basis of their combined production technology. But the problem is that the conditions needed for an aggregate production function to be *well-behaved* (in the sense of inheriting simple properties from its constituent parts) are quite restrictive.[15]

One aspect of the difficulties concerns the aggregation of the different kinds of equipment into one variable, the capital stock "K". In the 1960s there was a heated debate (the "Cambridge controversy") about these aggregation issues between a group of economists from Cambridge University, UK, and a group from Massachusetts Institute of Technology (MIT), which is located in Cambridge, USA. The former group questioned the theoretical robustness of several of the neoclassical tenets, including the proposition that rising aggregate capital intensity tends to be associated with a falling rate of interest. Starting at the disaggregate level, an association of this sort is not a logical necessity because the relative prices tend to change, when the interest rate changes. While acknowledging the possibility of "paradoxical" relationships, the latter group maintained that in a macroeconomic context they are likely to cause devastating problems only under exceptional circumstances. In the end this is a matter of empirical assessment.[16]

Since there is today no well-tried alternative, this book is about models that use aggregate constructs like "Y", "K", and "L" as simplifying devices, assuming they are, for a broad class of cases, acceptable in a first approximation. Of course there are cases where some disaggregation is pertinent. When for example the role of imperfect competition is in focus, we shall be ready to disaggregate the production side of the economy into several product lines producing different goods (see Section 2.5.3).

The notion of a representative firm (within a sector or at the economy-wide level) is certainly not always applicable. In some contexts the center of attention is a collection of firms each of which produces its own differentiated product. The firms have market power, but this is constrained by the imperfect substitutes supplied by the other firms and by the entrance of new firms. In some models of economic growth a key ingredient is the phenomenon of "creative destruction" where an incumbent technological leader is competed out by an entrant with a new technology.

Like the representative firm, the *representative household* is a simplifying notion that should be applied only when it does not get in the way of the

[15] Naturally, there are similar problems with the concept of an aggregate consumption function (in fact even more involved problems, in view of the role of individual budget constraints).

[16] In his review of the Cambridge controversy Mas-Colell (1989) concluded that: "What the 'paradoxical' comparative statics [of disaggregate capital theory] has taught us is simply that modelling the world as having a single capital good is not *a priori* justified. So be it."

issue to be studied. It is of course not an appropriate notion if we aim at understanding, say, the *interaction,* perhaps via financial intermediaries, between lending and borrowing households. Similarly, if the theme is conflicts of interests between firm owners and employees, the existence of *different* types of households should be taken into account. Or if we want to assess the welfare costs of business cycle fluctuations, a representative household is not a good starting point in view of the fact that exposure to unemployment risk is very unevenly distributed.

2.4 The neoclassical competitive one-sector setup

Many *long-run* macromodels, including those in the first chapters to follow, share the same abstract setup regarding the firms and the market environment in which they are placed. We give an account here which will serve as a reference point for these later chapters.

The setup is characterized by the following simplifications:

(a) there is only one produced good, an all-purpose good that can be used for consumption as well as investment; physical capital is just the accumulated amount of what is left of the produced good after consumption; models using this simplification are called one-sector models; one may think of "corn", a good that can be used for consumption as well as investment in the form of seed to yield corn next period;

(b) a representative firm maximizes profit subject to a neoclassical production function under non-increasing returns to scale;

(c) capital goods become productive immediately upon purchase or renting (so installation costs are ignorable);

(d) in all markets *perfect competition* rules and so the economic actors are price takers, perceiving no constraint on how much they can sell or buy at the going price;

(e) factor supplies are inelastic;

(f) there is no uncertainty; when a choice of action is made, the consequences are known.

We call such a setup a *neoclassical competitive one-sector setup.*

2.4.1 Profit maximization

We consider a single period. Let the representative firm have the neoclassical production function

$$Y = F(K, L). \tag{2.25}$$

We ignore technological change. Although generally CRS will be assumed, we may throw the outcome under CRS in relief by starting with a broader view. From microeconomics we know that equilibrium with perfect competition is compatible with producers operating under the condition of locally *nonincreasing returns* to scale (cf. Fig. 2.2). It greatly simplifies the analysis to assume that F is a *concave* function.[17] Concavity of F ensures non-increasing returns to scale *everywhere*, thus ruling out the possibility of locally IRS (see Appendix D).

Since F is neoclassical, we have $F_{KK} < 0$ and $F_{LL} < 0$ everywhere. To guarantee concavity it is then necessary and sufficient to add the assumption that

$$D \equiv F_{KK}(K, L)F_{LL}(K, L) - F_{KL}(K, L)^2 \geq 0, \tag{2.26}$$

holds for all (K, L). This is a simple application of a general theorem on concave functions (see Math Tools).

Let the factor prices be denoted w_K and w_L, respectively. For the time being we assume the firm rents the machines it uses; then the price, w_K, of capital services is called the *rental rate*. As *numeraire* (unit of account) we apply the output good. So all prices are measured in terms of the output good which itself has the price 1. Then *profit*, defined as revenue minus costs, is

$$\Pi = F(K, L) - w_K K - w_L L. \tag{2.27}$$

We assume both production inputs are *variable* inputs. Taking the factor prices as given from the factor markets, the firm's problem is to choose (K, L), where $K \geq 0$ and $L \geq 0$, so as to maximize Π. An interior solution will satisfy the first-order conditions

$$\frac{\partial \Pi}{\partial K} = F_K(K, L) - w_K = 0 \quad \text{or} \quad F_K(K, L) = w_K, \tag{2.28}$$

$$\frac{\partial \Pi}{\partial L} = F_L(K, L) - w_L = 0 \quad \text{or} \quad F_L(K, L) = w_L. \tag{2.29}$$

Since F is concave, so is the profit function. The first-order conditions are then *sufficient* for (K, L) to be a solution.

[17] By definition *concavity* means that by applying a weighted average of two factor combinations, (K_1, L_1) and (K_2, L_2), the obtained output is at least as large as the weighted average of the original outputs, Y_1 and Y_2. So, if $0 < \lambda < 1$ and $(K, L) = \lambda(K_1, L_1)$ $+(1 - \lambda)(K_2, L_2)$, then $F(K, L) \geq \lambda F(K_1, L_1)$ $+(1 - \lambda)F(K_2, L_2)$.

It is now convenient to proceed by considering the two cases, DRS and CRS, separately.

The DRS case

Suppose the production function satisfies (2.26) with strict inequality everywhere, i.e., $D > 0$. In combination with the neoclassical property of diminishing marginal productivities this implies that F is *strictly concave,* which in turns implies DRS everywhere. The factor demands are now unique. Indeed, the equations (2.28) and (2.29) define the factor demands K^d and L^d ("d" for demand) as implicit functions of the factor prices:

$$K^d = K(w_K, w_L), \qquad L^d = L(w_K, w_L).$$

An easy way to find the partial derivatives of these functions is to first take the differential of both sides of (2.28) as well as (2.29):[18]

$$\begin{aligned} F_{KK}dK^d + F_{KL}dL^d &= dw_K, \\ F_{LK}dK^d + F_{LL}dL^d &= dw_L. \end{aligned}$$

Then we interpret these conditions as a system of two linear equations with two unknowns, the variables dK^d and dL^d. The determinant of the coefficient matrix equals D in (2.26) and is in this case positive everywhere. Using Cramer's rule (see Math Tools), we find

$$\begin{aligned} dK^d &= \frac{F_{LL}dw_K - F_{KL}dw_L}{D}, \\ dL^d &= \frac{F_{KK}dw_L - F_{LK}dw_K}{D}, \end{aligned}$$

so that

$$\frac{\partial K^d}{\partial w_K} = \frac{F_{LL}}{D} < 0, \qquad \frac{\partial K^d}{\partial w_L} = -\frac{F_{KL}}{D} < 0 \text{ if } F_{KL} > 0, \tag{2.30}$$

$$\frac{\partial L^d}{\partial w_K} = -\frac{F_{KL}}{D} < 0 \text{ if } F_{KL} > 0, \qquad \frac{\partial L^d}{\partial w_L} = \frac{F_{KK}}{D} < 0, \tag{2.31}$$

[18] Applying the full content of the *implicit function theorem* (see Math Tools), one can directly write down the results (2.30) and (2.31) below and will not need the procedure outlined here, based on differentials. On the other hand this procedure is probably easier to remember and use in practice.

The *differential* of a differentiable function $y = f(x)$ is denoted dy (or df) and is defined as $f'(x)dx$, where dx is some arbitrary real number (interpreted as the change in x). For a differentiable function of two variables, $z = f(x, y)$, the *differential* of the function is denoted dz (or df) and is defined as $dz = f_x(x, y)dx + f_y(x, y)dy$, where dx and dy are arbitrary real numbers.

in view of $F_{LK} = F_{KL}$. In contrast to the cases of CRS and IRS, here we cannot be sure that direct complementarity ($F_{KL} > 0$) holds everywhere. In any event, the rule for both factors is that when a factor price increases, the demand for the factor in question decreases and under direct complementarity also the demand for the other factor will decrease. Although there is a substitution effect towards higher demand for the factor whose price has not been increased, this is more than offset by the negative output effect, which is due to the higher marginal costs. This is an implication of perfect competition. In a different market structure output may be determined from the demand side (think of a Keynesian short-run model) and then only the substitution effect will be operative; an increase in one factor price will then *increase* the demand for the other factor.

The CRS case

Under CRS D in (2.26) equals zero everywhere, as shown in Appendix B. Then the factor prices no longer determine the factor demands uniquely. But the *relative* factor demand, $k^d \equiv K^d/L^d$, is determined uniquely by the *relative* factor price, w_L/w_K. Indeed, by (2.28) and (2.29),

$$MRS = \frac{F_L(K,L)}{F_K(K,L)} = \frac{f(k) - f'(k)k}{f'(k)} \equiv mrs(k) = \frac{w_L}{w_K}. \tag{2.32}$$

By straightforward calculation,

$$mrs'(k) = -\frac{f(k)f''(k)}{f'(k)^2} = -\frac{kf''(k)/f'(k)}{\alpha(k)} > 0,$$

where $\alpha(k) \equiv kf'(k)/f(k)$ is the elasticity of f w.r.t. k and the numerator is the elasticity of f' w.r.t. k. For instance, in the Cobb-Douglas case $f(k) = Ak^\alpha$, we get $mrs'(k) = (1-\alpha)/\alpha$. Given w_L/w_K, the last equation in (2.32) gives k^d as an implicit function $k^d = k(w_L/w_K)$, where $k'(w_L/w_K) = 1/mrs'(k) > 0$. The solution is illustrated in Fig. 2.4. Under CRS (indeed, for any homogeneous neoclassical production function) the desired capital-labor ratio is an increasing function of the inverse factor price ratio and independent of the output level.

To determine K^d and L^d separately we need to know the level of output. And here we run into the general problem of indeterminacy under perfect competition combined with CRS. Saying that the output level is so as to maximize profit is pointless. Of course, if at the going factor prices attainable profit is negative, exit from the market is profit maximizing (loss minimizing), which amounts to $K^d = L^d = 0$. But if the profit is positive, there will be

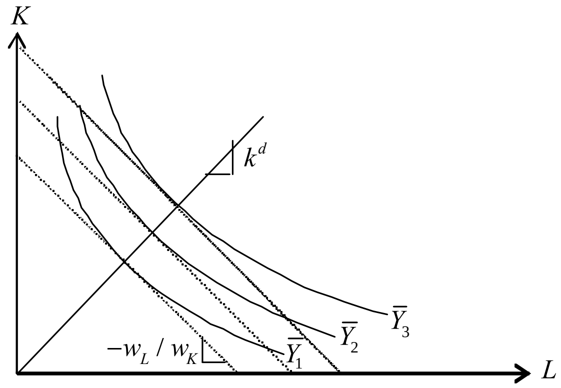

Figure 2.4: Constancy of MRS along rays when the production function is homogeneous of degree h (the cost-minimizing capital intensity is the same at all output levels).

no upper bound to the factor demands. Owing to CRS, doubling the factor inputs will double the profits of a price taking firm. An equilibrium with positive production is only possible if profit is zero. And then the firm is indifferent w.r.t. the level of output. Solving the indeterminacy problem requires a look at the factor markets.

2.4.2 Clearing in factor markets

Considering a closed economy, we denote the available supplies of physical capital and labor K^s and L^s, respectively, and assume these supplies are inelastic. W.r.t. capital this is a "natural" assumption since in a closed economy in the short term the available amount of capital will be *predetermined,* that is, historically determined by the accumulated previous investment in the economy. W.r.t. labor supply it is just a simplifying assumption introduced because the question about possible responses of labor supply to changes in factor prices is a secondary issue in the present context.

The factor markets clear when

$$K^d = K^s, \tag{2.33}$$
$$L^d = L^s. \tag{2.34}$$

Achieving this equilibrium requires that the factor prices adjust to their equilibrium levels, which are

$$w_K = F_K(K^s, L^s), \tag{2.35}$$
$$w_L = F_L(K^s, L^s), \tag{2.36}$$

by (2.28) and (2.29). This says that in equilibrium the real factor prices are determined by the *marginal productivities of the respective factors at full utilization of the given supplies.* This holds under DRS as well as CRS. So, under non-increasing returns to scale there is, at the macroeconomic level, a unique equilibrium (w_K, w_L, K^d, L^d) given by the above four equilibrium conditions.[19] It is an *equilibrium* in the sense that no agent has an incentive to "deviate".

As to *comparative statics*, since $F_{KK} < 0$, a larger capital supply implies a lower w_K, and since $F_{LL} < 0$, a larger labor supply implies a lower w_L.

The intuitive mechanism behind the *attainment* of equilibrium is that if, for example, for a short moment $w_K < F_K(K^s, L^s)$, then $K^d > K^s$ and so competition between the firms will generate an upward pressure on w_K until equality is obtained. And if for a short moment $w_K > F_K(K^s, L^s)$, then $K^d < K^s$ and so competition between the *suppliers* of capital will generate a downward pressure on w_K until equality is obtained.

Looking more carefully at the matter, however, we see that this intuitive reasoning fits at most the DRS case. In the CRS case we have $F_K(K^s, L^s) = f(k^s)$, where $k^s \equiv K^s/L^s$. Here we can only argue that for instance $w_K < F_K(K^s, L^s)$ implies $k^d > k^s$. And even if this leads to upward pressure on w_K until $k^d = k^s$ is achieved, and even if both factor prices have obtained their equilibrium levels given by (2.35) and (2.36), there is nothing to induce the representative firm (or the many firms in the actual economy taken together) to choose the "right" input *levels* so as to satisfy the clearing conditions (2.33) and (2.34). In this way the indeterminacy under CRS pops up again, this time as a problem endangering stability of the equilibrium.

Stability not guaranteed*

To substantiate this point, let us consider a standard Walrasian *tâtonnement* adjustment process.[20] We imagine that our period is sub-divided into many short time intervals $(t,\, t + \Delta t)$. In the initial short time interval the factor markets may not be in equilibrium. It is assumed that no capital or labor

[19] At the microeconomic level, under CRS, industry structure remains indeterminate in that firms are indifferent as to their size.

[20] *Tâtonnement* is a French word meaning "groping". A section headline marked by * indicates that the section can be skipped in a first reading.

is hired out of equilibrium. To allow an analysis in continuous time, we let $\Delta t \to 0$. A dot over a variable will denote the time derivative, i.e., $\dot{x}(t) = dx(t)/dt$. The adjustment process assumed is the following:

$$\begin{aligned}
\dot{K}^d(t) &= \lambda_1 \left[F_K(K^d(t), L^d(t)) - w_K(t)\right], \qquad \lambda_1 > 0,\\
\dot{L}^d(t) &= \lambda_2 \left[F_L(K^d(t), L^d(t)) - w_L(t)\right], \qquad \lambda_2 > 0,\\
\dot{w}_K(t) &= K^d(t) - K^s,\\
\dot{w}_L(t) &= L^d(t) - L^s,
\end{aligned}$$

where the initial values, $K^d(0)$, $L^d(0)$, $w_K(0)$, and $w_L(0)$, are given. The parameters λ_1 and λ_2 are constant adjustment speeds. The corresponding adjustment speeds for the factor prices are set equal to one by choice of measurement units of the inputs. Of course, the four endogenous variables should be constrained to be nonnegative, but that is not important for the discussion here. The system has a unique stationary state: $K^d(t) = K^s, L^d(t) = L^s, w_K(t) = K_K(K^s, L^s)$, $w_L(t) = K_L(K^s, L^s)$.

A widespread belief, even in otherwise well-informed circles, seems to be that with such adjustment dynamics around the stationary state, this state is at least *locally asymptotically stable.* By this is meant that there exists a small neighborhood, $\mathcal{N}$, of the stationary state with the property that if the initial state, ($K^d(0)$, $L^d(0)$, $w_K(0)$, $w_L(0)$), belongs to $\mathcal{N}$, then the solution ($K^d(t)$, $L^d(t)$, $w_K(t)$, $w_L(t)$) will converge to the stationary state for $t \to \infty$?

Unfortunately, however, this property is *not* guaranteed. To bear this out, it is enough to present a counterexample. Let $F(K, L) = K^{\frac{1}{2}}L^{\frac{1}{2}}$, $\lambda_1 = \lambda_2 = K^s = L^s = 1$, and suppose $K^d(0) = L^d(0) > 0$ and $w_K(0) = w_L(0) > 0$. All this symmetry implies that $K^d(t) = L^d(t) = x(t) > 0$ and $w_K(t) = w_L(t) = w(t)$ for all $t \geq 0$. So $F_K(K^d(t), L^d(t)) = 0.5x(t)^{-0.5}x(t)^{0.5} = 0.5$, and similarly $F_L(K^d(t), L^d(t)) = 0.5$ for all $t \geq 0$. Now the system is equivalent to the two-dimensional system,

$$\dot{x}(t) = 0.5 - w(t), \tag{2.37}$$
$$\dot{w}(t) = x(t) - 1. \tag{2.38}$$

Using the theory of coupled linear differential equations, the solution is[21]

$$x(t) = 1 + (x(0) - 1)\cos t - (w(0) - 0.5)\sin t, \tag{2.39}$$
$$w(t) = 0.5 + (w(0) - 0.5)\cos t + (x(0) - 1)\sin t. \tag{2.40}$$

The solution exhibits undamped oscillations and never settles down at the stationary state, $(1, 0.5)$, if not being there from the beginning. In fact, the

[21] For details, see Exercise 2.6 or Sydsaeter et al. (2008), pp. 245-246, where the solution of essentially the same system is derived.

solution curves in the (x, w) plane will be circles around the stationary state. This is so whatever the size of the initial distance, $\sqrt{(x(0)-1)^2 + (w(0)-0.5)^2}$, to the stationary point. In Exercise 2.6 the reader is asked to show this and to draw a phase diagram that is useful for visualizing what is going on.

The mechanism is as follows. Suppose for instance that $x(0) < 1$ and $w(0) < 0.5$. Then to begin with there is excess supply and so w will be falling while, with w below marginal products, x will be increasing. When x reaches its potential equilibrium value, 1, w is at its trough and so induces further increases in the factor demands, thus bringing about a phase where $x > 1$. This excess demand causes w to begin an upturn. When w reaches its potential equilibrium value, 0.5, however, excess demand, $x - 1$, is at its peak and this induces further increases in factor prices, w. This brings about a phase where $w > 0.5$ so that factor prices exceed marginal products, which leads to declining factor demands. But as x comes back to its potential equilibrium value, w is at its peak and drives x further down. Thus excess supply arises which in turn triggers a downturn of w. This continues in never ending oscillations where the overreaction of one variable carries the seed to an overreaction of the other variable soon after and so on.

This possible outcome underlines that the theoretical *existence* of equilibrium is one thing and *stability* of the equilibrium is another. Especially under CRS, where demand *functions* for inputs are absent, the issue of stability can be more intricate than one might at first glance expect.

The link between capital costs and the interest rate*

Returning to the description of equilibrium, we shall comment on the relationship between the factor price w_K and the more everyday concept of an interest rate. The factor price w_K is the cost per unit of capital service. It has different names in the literature such as the *rental rate,* the *unit capital cost,* or the *user cost.* It is related to the capital depreciation and the interest costs that the owner of the capital good in question defrays. In the simple neoclassical setup considered here, it does not matter whether the firm rents the capital it uses or owns it; in the latter case, w_K, is the *imputed capital cost,* i.e., the forgone interest plus depreciation.

As to depreciation it is common in simple macroeconomics to apply the approximation that, due to wear and tear, a constant fraction δ (where $0 \leq \delta \leq 1$) of a given capital stock evaporates per period. If for instance the period length is one year and $\delta = 0.1$, this means that a given machine in the next year has only the fraction 0.9 of its productive capacity in the current year. Otherwise the productive characteristics of a capital good are assumed to be the same whatever its time of birth. Sometimes δ is referred to as the rate of *physical* capital depreciation or the *deterioration rate*. When changes in relative prices can occur, this must be distinguished from the *economic depreciation* of capital which refers to the loss in economic value of a machine after one year.

Let p_t be the price of a certain type of machine in terms of the same numeraire as that in which the interest rate, r, is measured. And let p_{t+1} be the price of the same type of machine one period later. Then the *economic depreciation* is

$$p_t - (1 - \delta)p_{t+1} = \delta p_{t+1} - (p_{t+1} - p_t).$$

The economic depreciation thus equals the value of the physical wear and tear minus the capital gain (positive or negative) on the machine.

By holding the machine the owner faces an opportunity cost, namely the forgone interest on the value p_t placed in the machine during period t. If r_t is the interest rate on a loan from the beginning of period t to the beginning of period $t + 1$, this interest cost is $r_t p_t$. The benefit of holding the (new) machine is that it can be rented out to the representative firm and provide the return w_{Kt} at the end of the period. Since there is no uncertainty, in equilibrium we must then have $w_{Kt} = r_t p_t + \delta p_{t+1} - (p_{t+1} - p_t)$, or

$$\frac{w_{Kt} - \delta p_{t+1} + p_{t+1} - p_t}{p_t} = r_t. \tag{2.41}$$

This is a *no-arbitrage* condition saying that the rate of return on holding the machine equals the rate of return obtainable in the loan market (no profitable arbitrage opportunities are available).[22]

In the simple setup considered so far, the capital good and the produced good are physically identical and thus have the same price. As the produced good is our numeraire, we have $p_t = p_{t+1} = 1$. This has two implications. *First*, the interest rate, r_t, is a real interest rate so that $1 + r_t$ measures the rate at which future units of output can be traded for current units of output. *Second,* (2.41) simplifies to

$$w_{Kt} - \delta = r_t.$$

Combining this with equation (2.35), we see that in the simple neoclassical setup the equilibrium real interest rate is determined as

$$r_t = F_K(K_t^s, L_t^s) - \delta, \tag{2.42}$$

[22] In continuous time analysis the rental rate, the interest rate, and the price of the machine are considered as differentiable functions of time, $w_K(t)$, $r(t)$, and $p(t)$, respectively. In analogy with (2.41) we then get $w_K(t) = (r(t) + \delta)p(t) - \dot{p}(t)$, where $\dot{p}(t)$ denotes the time derivative of $p(t)$.

where K_t^S and L_t^s are predetermined. Under CRS this takes the form $r_t = f'(k_t^s) - \delta$, where $k_t^s \equiv K_t^s/L_t^s$.

We have assumed that the firms rent capital goods from their owners, presumably the households. But as long as there is no uncertainty, no capital adjustment costs, and no taxation, it will have no consequences for the results if instead we assume that the firms own the physical capital they use and finance capital investment by issuing bonds or shares. Then such bonds and shares would constitute financial assets, owned by the households and offering a rate of return r_t as given by (2.42).

2.5 More complex model structures

The neoclassical setup described above may be useful as a first way of organizing one's thoughts about the production side of the economy. To come closer to a model of how modern economies function, however, many modifications and extensions are needed.

2.5.1 Convex capital installation costs

In the real world the capital goods used by a production firm are usually owned by the firm itself rather than rented for single periods on rental markets. This is because inside the specific plant in which these capital goods are an integrated part, they are generally worth much more than outside. So in practice firms acquire and install fixed capital equipment with a view on maximizing discounted expected profits in the future. The cost associated with this fixed capital investment not only includes the purchase price of new equipment, but also the *installation costs* (the costs of setting up the new fixed equipment in the firm and the associated costs of reorganizing work processes).

In this context the firm has to solve an *intertemporal* optimization problem. Forward-looking expectations thus become important. And to the extent that the capital installation costs are strictly convex (as often assumed), the model structure changes radically. This has implications for how equilibrium in the output market is established and how the equilibrium interest rate is determined. Indeed, in the simple neoclassical setup above, the interest rate equilibrates the market for capital services. The value of the interest rate is simply tied down by the equilibrium condition (2.39) in this market. What happens in the output market is a trivial consequence of this. How the presence of convex capital installation costs changes these features is taken up in chapters 14 and 15.

2.5.2 Long-run vs. short-run production functions

The production functions considered up to now are based on the questionable assumption that the substitutability between capital and labor are the same "ex ante" and "ex post". By ex ante is meant "when plant and machinery are to be decided upon" and by ex post is meant "after the equipment is designed and constructed". In the standard neoclassical competitive setup like in (2.32) there is a presumption that also after the construction and installation of the equipment in the firm, the ratio of the factor inputs can be fully adjusted to a change in the relative factor price. In practice, however, when some machinery has been constructed and installed, its functioning will often require a more or less fixed number of machine operators. What can be varied is just the *degree of utilization* of the machinery per week, say. That is, after construction and installation of the machinery, the choice opportunities are no longer described by the neoclassical production function, but by a Leontief production function,

$$Y = \min(Au\bar{K}, BL), \qquad A > 0, B > 0, \tag{2.43}$$

where $\bar{K}$ is the size of the installed machinery (a fixed factor in the short run) measured in efficiency units, u is its utilization rate ($0 \leq u \leq 1$), and A and B are given technical coefficients (cf. Section 2.1.2).

So in the short run the choice variables are u and L. Under "full capacity utilization" we have $u = 1$ (the machine is used 24 hours per day seven days per week, say); "capacity" is given as $A\bar{K}$ per week and producing at capacity requires $L = A\bar{K}/B$. But if demand, Y^d, is *less* than capacity, satisfying this demand efficiently requires $u = Y^d/(A\bar{K}) < 1$ and $L = Y^d/B$. As long as $u < 1$, the marginal productivity of labor is a *constant*, B.

The various efficient input proportions that are possible *ex ante* may be approximately described by a neoclassical CRS production function. Let this function on intensive form be denoted $y = f(k)$. When investment is decided upon and undertaken, there is thus a choice between alternative efficient pairs of the technical coefficients A and B in (2.43). These pairs satisfy

$$f(k) = Ak = B. \tag{2.44}$$

So, for an increasing sequence of k's, $k_1, k_2, \ldots, k_i, \ldots$, the corresponding pairs are $(A_i, B_i) = (f(k_i)/k_i, f(k_i))$, $i = 1, 2, \ldots$.[23] We say that ex ante, depending on the relative factor prices as they are "now" and are expected

[23] The points P and Q in the right-hand panel of Fig. 2.3 can be interpreted as constructed this way from the neoclassical production function in the left-hand panel of the figure.

to evolve in the future, a suitable technique, (A_i, B_i), is chosen from an opportunity set described by the given neoclassical production function. But ex post, i.e., when the equipment corresponding to this technique is installed, the production opportunities are described by a Leontief production function with $(A, B) = (A_i, B_i)$.

In the picturesque language of Phelps (1963), technology is in this case *putty-clay.* Ex ante the technology involves capital which is "putty" in the sense of being in a malleable state which can be transformed into a range of various machinery requiring capital-labor ratios of different magnitude. But once the machinery is constructed, it enters a "hardened" state and becomes "clay". Then factor substitution is no longer possible; the capital-labor ratio at full capacity utilization is fixed at the level $k = B_i/A_i$, as in (2.43). Following the terminology of Johansen (1972), we say that a putty-clay technology involves a "long-run production function" which is neoclassical and a "short-run production function" which is Leontief.

In the standard neoclassical setup, however, the same range of substitutability between capital and labor reigns ex ante and ex post. Then the technology is called *putty-putty.* This term may also be used if ex post there is at least *some* substitutability although less than ex ante. At the opposite pole of putty-putty we may consider a technology which is *clay-clay.* Here neither ex ante nor ex post is factor substitution possible. Table 1 gives an overview of the alternative cases.

Table 1. Technologies classified according to factor substitutability ex ante and ex post

	Ex post substitution	
Ex ante substitution	possible	impossible
possible	putty-putty	putty-clay
impossible		clay-clay

To take technological change as time proceeds into account, we may replace (2.44) and (2.43) by $f(k,t) = A_t k = B_t$ and $Y = \min(A_t u\bar{K}_t, B_t L)$, respectively. If a new pair of Leontief coefficients, (A_{t_2}, B_{t_2}), efficiency-dominates its predecessor (by satisfying $A_{t_2} \geq A_{t_1}$ and $B_{t_2} \geq B_{t_1}$ with at least one strict equality), it may pay the firm to invest in the new technology at the same time as some old plant and machinery are dismantled.

The ex post clay aspect of many technologies seems clearly relevant for short-run analysis. In long-run analysis it is less important since within a long time horizon it is the accumulated new investments that matter.

2.5.3 A simple portrayal of price-making firms*

Another modification which is important in short- and medium-run analysis, relates to the assumed market forms. Perfect competition is not a good approximation to market conditions in manufacturing and service industries. To bring perfect competition on the output market in perspective, we give here a brief review of firms' behavior under a form of monopolistic competition that is applied in many short-run models.

Suppose there is a large number of differentiated goods, $i = 1, 2, \ldots, n$, each produced by a separate firm. In the short run n is given. Each firm has monopoly on its own good (supported, say, by a trade mark, patent protection, or simply secrecy regarding the production recipe). The goods are imperfect substitutes to each other and so indirect competition prevails. Each firm is small in relation to the "sum" of competing firms and perceives that these other firms do not respond to its actions.

In the given period let firm i face a given downward-sloping demand curve for its product,

$$Y_i \leq \left(\frac{P_i}{P}\right)^{-\varepsilon} \frac{Y}{n} \equiv \mathcal{D}(P_i), \qquad \varepsilon > 1. \tag{2.45}$$

Here Y_i is the produced quantity and the expression on the right-hand side of the inequality is the demand as a function of the price P_i chosen by the firm.[24] The "general price level" P (a kind of average across the different goods) and the "general demand level", given by the index Y, matter for the position of the demand curve in the (Y_i, P_i) plan, cf. Fig. 2.5. The price elasticity of demand, ε, is assumed constant and higher than one (otherwise there is no solution to the monopolist's decision problem). Variables that the monopolist perceives as exogenous are implicit in the demand function symbol $\mathcal{D}$. We imagine prices are expressed in terms of money (so they are "nominal" prices, hence denoted by capital letters whereas we use small letters for "real" prices).

For simplicity, factor markets are still assumed competitive. Given the nominal factor prices, W_K and W_L, firm i wants to maximize its profit

$$\Pi_i = P_i Y_i - W_K K_i - W_L L_i,$$

subject to (2.45) and the neoclassical production function $Y_i = F(K_i, L_i)$ (presumably, the technology would differ across the different product lines and F thus be replaced by F^i, but for notational convenience we ignore this). It is no serious restriction on the problem to assume the monopolist will want to produce the amount demanded so that $Y_i = \mathcal{D}(P_i)$. It is convenient to solve the problem in two steps.

[24] We ignore production for inventory holding.

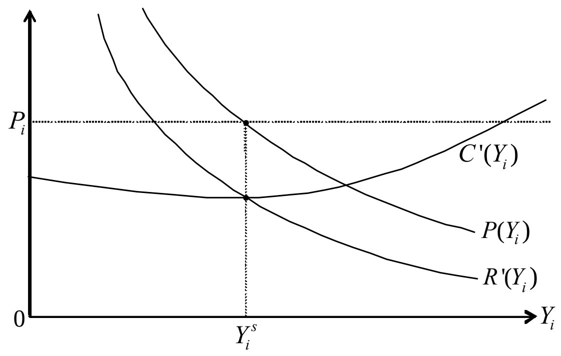

Figure 2.5: Determination of the monopolist price and output.

Step 1. Imagine the monopolist has already chosen the output level Y_i. Then the problem is to minimize cost:

$$\min_{K_i, L_i} W_K K_i + W_L L_i \text{ s.t. } F(K_i, L_i) = Y_i.$$

An interior solution (K_i, L_i) will satisfy the first-order conditions

$$\lambda F_K(K_i, L_i) = W_K, \qquad \lambda F_L(K_i, L_i) = W_L, \tag{2.46}$$

where λ is the Lagrange multiplier. Since F is neoclassical and thereby strictly quasiconcave, the first-order conditions are not only necessary but also sufficient for (K_i, L_i) to be a solution, and (K_i, L_i) will be unique so that we can write these conditional factor demands as functions, $K_i^d = K(W_K, W_L, Y_i)$ and $L_i^d = L(W_K, W_L, Y_i)$. This gives rise to the cost function $\mathcal{C}(Y_i) = W_K K(W_K, W_L, Y_i) + W_L L(W_K, W_L, Y_i)$.

Step 2. Solve

$$\max_{Y_i} \mathcal{P}(Y_i) Y_i - \mathcal{C}(Y_i),$$

where we have introduced the inverse demand function $P_i = \mathcal{D}^{-1}(Y_i) = \left[P(Y/n)^{1/\varepsilon}\right] Y_i^{-1/\varepsilon} \equiv \mathcal{P}(Y_i)$. With "total revenue", $\mathcal{P}(Y_i)Y_i$, denoted $R(Y_i)$, the first-order condition is

$$R'(Y_i) = \mathcal{P}(Y_i) + \mathcal{P}'(Y_i) Y_i = \mathcal{C}'(Y_i), \tag{2.47}$$

where the left-hand side is *marginal revenue* and the right-hand side is *marginal cost*. Marginal revenue can also be written

$$R'(Y_i) = P_i \left(1 + \frac{Y_i}{P_i} \mathcal{P}'(Y_i)\right) = P_i \left(1 - \frac{1}{\varepsilon}\right) = P_i \frac{\varepsilon - 1}{\varepsilon}.$$

So the pricing rule is $P_i = (1+\mu)\mathcal{C}'(Y_i)$, where $\mu \equiv \varepsilon/(\varepsilon-1) - 1 > 0$ is the markup on marginal cost. An analytical very convenient feature is that the markup is thus a *constant*.

A sufficient second-order condition is that $R''(Y_i) - \mathcal{C}''(Y_i) < 0$, i.e., the marginal revenue curve crosses the marginal cost curve from above. In the present case this is surely satisfied if we assume $\mathcal{C}''(Y_i) \geq 0$, which also ensures existence and uniqueness of a solution to (2.47). Substituting this solution, which we denote Y_i^s, cf. Fig. 2.5, into the conditional factor demand functions from Step 1, we find the factor demands, K_i^d and L_i^d. Owing to the downward-sloping demand curves the factor demands are unique whether the technology exhibits DRS, CRS, or IRS. Thus, contrary to the perfect competition case, neither CRS nor IRS pose particular problems.

In parallel with (2.28) and (2.29) the solution is characterized by the marginal productivity conditions

$$R'(Y_i^s)F_K(K_i^d, L_i^d) = W_K, \tag{2.48}$$

$$R'(Y_i^s)F_L(K_i^d, L_i^d) = W_L, \tag{2.49}$$

where $Y_i^s = F(K_i^d, L_i^d)$. These conditions follow from (2.46), since the Lagrange multiplier equals marginal cost (see Appendix A), which equals marginal revenue. That is, at profit maximum the marginal revenue products of capital and labor, respectively, equal the corresponding factor prices. Since $P_i > R'(Y_i^s)$, the values of the marginal products are above the factor prices. This reflects the market power of the firms.

In simple macro models a lot of symmetry is often assumed. If there is complete symmetry across product lines and if factor markets clear as in (2.33) and (2.34) with inelastic factor supplies, K^s and L^s, then $K_i^d = K^s/n$ and $L_i^d = L^s/n$. Furthermore, all firms will choose the same price so that $P_i = P$, $i = 1, 2, \ldots, n$. Then the given factor supplies, together with (2.48) and (2.49), determine the equilibrium *real* factor prices:

$$\begin{aligned} w_K &\equiv \frac{W_K}{P} = \frac{\varepsilon - 1}{\varepsilon} F_K\left(\frac{K^s}{n}, \frac{L^s}{n}\right), \\ w_L &\equiv \frac{W_L}{P} = \frac{\varepsilon - 1}{\varepsilon} F_L\left(\frac{K^s}{n}, \frac{L^s}{n}\right), \end{aligned}$$

where we have used that $R'(Y_i^s) = P(\varepsilon - 1)/\varepsilon$ under these circumstances. As under perfect competition, the real factor prices are proportional to the corresponding marginal productivities, although with a factor of proportionality less than one, namely equal to the inverse of the markup. This observation is sometimes used as a defence for applying the simpler perfect-competition framework for studying certain long-run aspects of the economy. For these aspects, the size of the proportionality factor may be immaterial, at least as long as it is relatively constant over time. Indeed, the constant markups open up for a simple transformation of the many of the perfect competition results to monopolistic competition results by inserting the markup factor $1 + \mu$ the relevant places in the formulas.

If in the short term only labor is a variable production factor, then (2.48) need not hold. As claimed by Keynesian and New Keynesian thinking, also the prices chosen by the firms may be more or less fixed in the short term because the firms face price adjustment costs ("menu costs") and are reluctant to change prices too often. Then in the short term only the produced quantity will adjust to changes in demand. As long as the output level is within the range where marginal cost is below the price, such adjustments are still beneficial to the firm. As a result, even (2.49) need not hold any more. These matters are dealt with in Part V of this book.

In practice, market power and other market imperfections also play a role in the factor markets, implying that further complicating elements enter the picture. One of the tasks of theoretical and empirical macroeconomics is to clarify the aggregate implications of market imperfections and sort out which market imperfections are quantitatively important in different contexts.

2.5.4 The financing of firms' operations

We have so far talked about aspects related to production and pricing. What about the *financing* of a firm's operations? To acquire not only its fixed capital (structures and machines) but also its raw material and other intermediate inputs, a firm needs *funds* (there are expenses before the proceeds of production arrive). These funds ultimately come from the accumulated saving of households. In long-run macromodels to be considered in the next chapters, uncertainty as well as non-neutrality of corporate taxation are ignored. In that context the capital structure (the debt-equity ratio) of firms is indeterminate and irrelevant for production outcomes.[25] In those chapters we shall therefore concentrate on the latter. Later chapters, dealing with short- and medium-run issues, take up cases where capital structure and bankruptcy risk matter and financial intermediaries enter the scene.

[25] In chapter 14 we return to this irrelevance theorem, called the Modigliani-Miller theorem.

2.6 Literature notes

As to the question of the empirical validity of the constant returns to scale assumption, Malinvaud (1998) offers an accessible account of the econometric difficulties associated with estimating production functions. Studies by Basu (1996) and Basu and Fernald (1997) suggest returns to scale are about constant or decreasing. Studies by Hall (1990), Caballero and Lyons (1992), Harris and Lau (1992), Antweiler and Treffler (2002), and Harrison (2003) suggest there are quantitatively significant increasing returns, either internal or external. On this background it is not surprising that the case of IRS (at least at industry level), together with market forms different from perfect competition, has in recent years received more attention in macroeconomics and in the theory of economic growth.

For general discussions of the limitations of representative agent approaches, see Kirman (1992) and Gallegati and Kirman (1999). A critical review of concepts such as aggregate capital, aggregate production function etc. is contained in Mas-Colell (1989) and Felipe and Fisher (2003). These authors find the conditions required for the well-behavedness of these constructs so stringent that it is difficult to believe that actual economies are in any sense close to satisfy them.[26] For a less distrustful view, see for instance Ferguson (1969), Johansen (1972), Malinvaud (1998), Jorgenson et al. (2005), and Jones (2005).

The counterexample to guaranteed stability of the neoclassical factor market equilibrium presented towards the end of Section 2.4 is taken from Bliss (1975), where further perspectives are discussed. It may be argued that this kind of stability questions should be studied on the basis of adjustment processes of a less mechanical nature than a Walrasian tâtonnement process. The view would be that trade out of equilibrium should be incorporated in the analysis and agents' behavior out of equilibrium should be founded on some kind of optimization or "satisficing", incorporating adjustment costs and imperfect information. The field is complicated and the theory far from settled. Yet it seems fair to say that the studies of adjustment processes out of equilibrium indicate that the equilibrating force of Adam Smith's invisible hand is not without its limits. See Fisher (1983), Osborne and Rubinstein (1990), and Negishi (2008) for reviews and elaborate discussion of these issues.

We introduced the assumption that physical capital depreciation can be

[26] Some authors draw the conclusion that macroeconomics should abandon the aggregate constructs altogether: " 'It's crooked, but it's the only wheel in town' is not a scientific argument. The profession needs to find a different 'wheel' " (Felipe and Fisher, 2008, p. 10).

described as geometric (in continuous time exponential) evaporation of the capital stock. This formula is popular in macroeconomics, more so because of its simplicity than its realism. An introduction to more general approaches to depreciation is contained in, e.g., Nickell (1978).

2.7 Appendix

A. Strict quasiconcavity

Consider a function, $f(x)$, defined on the convex set $\mathcal{A} \subseteq \mathbb{R}^n$.[27] Given a real number a, if $f(x) = a$, the *upper contour set* is defined as $\{x \in \mathcal{A} | \, f(x) \geq a\}$ (the set of input bundles that can produce at least the amount a of output). The function $f(x)$ is called *quasiconcave* if its upper contour sets, for any constant a, are convex sets. If all these sets are strictly convex, $f(x)$ is called *strictly quasiconcave*.

Average and marginal costs To show that (2.13) holds with n production inputs, $n = 1, 2, \ldots$, we derive the cost function of a firm with a neoclassical production function, $Y = F(X_1, X_2, \ldots, X_n)$. Given a vector of strictly positive input prices $\mathbf{w} = (w_1, \ldots, w_n) >> 0$, the firm faces the problem of finding a cost-minimizing way to produce a given positive output level $\bar{Y}$ within the range of F. The problem is

$$\min \sum_{i=1}^{n} w_i X_i \text{ s.t. } F(X_1, \ldots, X_n) = \bar{Y} \text{ and } X_i \geq 0, \; i = 1, 2, \ldots, n.$$

An interior solution, $\mathbf{X}^* = (X_1^*, \ldots, X_n^*)$, to this problem satisfies the first-order conditions $\lambda F_i'(\mathbf{X}^*) = w_i$, where λ is the Lagrange multiplier, $i = 1, \ldots, n$.[28] Since F is neoclassical and thereby strictly quasiconcave in the interior of $\mathbb{R}_+^n$, the first-order conditions are not only necessary but also sufficient for the vector $\mathbf{X}^*$ to be a solution, and $\mathbf{X}^*$ will be unique[29] so that we can write it as a function, $\mathbf{X}^*(\bar{Y}) = (X_1^*(\bar{Y}), \ldots, X_n^*(\bar{Y}))$. This gives rise to the *cost function* $\mathcal{C}(\bar{Y}) = \sum_{i=1}^{n} w_i X_i^*(\bar{Y})$. So *average cost* is $\mathcal{C}(\bar{Y})/\bar{Y}$. We

[27] Recall that a set S is said to be *convex* if $x, y \in S$ and $\lambda \in [0, 1]$ implies $\lambda x + (1 - \lambda) y \in S$.

[28] Since in this section we use a bit of vector notation, we exceptionally mark first-order partial derivatives by a prime in order to clearly distinguish from the elements of a vector (that is, here we write F_i' instead of our usual F_i).

[29] See Sydsaeter et al. (2008), pp. 74, 75, and 125.

find *marginal cost* to be

$$\mathcal{C}'(\bar{Y}) = \sum_{i=1}^{n} w_i X_i^{*\prime}(\bar{Y}) = \lambda \sum_{i=1}^{n} F_i'(\mathbf{X}^*) X_i^{*\prime}(\bar{Y}) = \lambda,$$

where the third equality comes from the first-order conditions, and the last equality is due to the constraint $F(\mathbf{X}^*(\bar{Y})) = \bar{Y}$, which, by taking the total derivative on both sides, gives $\sum_{i=1}^{n} F_i'(\mathbf{X}^*) X_i^{*\prime}(\bar{Y}) = 1$. Consequently, the ratio of average to marginal costs is

$$\frac{\mathcal{C}(\bar{Y})/\bar{Y}}{\mathcal{C}'(\bar{Y})} = \frac{\sum_{i=1}^{n} w_i X_i^*(\bar{Y})}{\lambda \bar{Y}} = \frac{\sum_{i=1}^{n} F_i'(\mathbf{X}^*) X_i^*(\bar{Y})}{F(\mathbf{X}^*)},$$

which in analogy with (2.12) is the elasticity of scale at the point $\mathbf{X}^*$. This proves (2.13).

Sufficient conditions for strict quasiconcavity The claim (iii) in Section 2.1.3 was that a continuously differentiable two-factor production function $F(K, L)$ with CRS, satisfying $F_K > 0, F_L > 0$, and $F_{KK} < 0, F_{LL} < 0$, will automatically also be strictly quasi-concave in the interior of $\mathbb{R}^2$ and thus neoclassical.

To prove this, consider a function of two variables, $z = f(x, y)$, that is twice continuously differentiable with $f_1 \equiv \partial z/\partial x > 0$ and $f_2 \equiv \partial z/\partial y > 0$, everywhere. Then the equation $f(x, y) = a$, where a is a constant, defines an isoquant, $y = g(x)$, with slope $g'(x) = -f_1(x, y)/f_2(x, y)$. Substitute $g(x)$ for y in this equation and take the derivative w.r.t. x. By straightforward calculation we find

$$g''(x) = -\frac{f_1^2 f_{22} - 2 f_1 f_2 f_{21} + f_2^2 f_{11}}{f_2^3} \tag{2.50}$$

If the numerator is negative, then $g''(x) > 0$; that is, the isoquant is strictly convex to the origin. And if this holds for all (x, y), then f is strictly quasi-concave in the interior of $\mathbb{R}^2$. A sufficient condition for a negative numerator is that $f_{11} < 0$, $f_{22} < 0$ and $f_{21} \geq 0$. All these conditions, including the last three are satisfied by the given function F. Indeed, F_K, F_L, F_{KK}, and F_{LL} have the required signs. And when F has CRS, F is homogeneous of degree 1 and thereby $F_{KL} > 0$, see Appendix B. Hereby claim (iii) in Section 2.1.3 is proved.

B. Homogeneous production functions

The claim (iv) in Section 2.1.3 was that a two-factor production function with CRS, satisfying $F_K > 0, F_L > 0$, and $F_{KK} < 0, F_{LL} < 0$, has always $F_{KL} >$

0, i.e., there is *direct complementarity* between K and L. This assertion is implied by the following observations on homogeneous functions.

Let $Y = F(K,\ L)$ be a twice continuously differentiable production function with $F_K > 0$ and $F_L > 0$ everywhere. Assume F is homogeneous of degree $h > 0$, that is, for all possible (K, L) and all $\lambda > 0$, $F(\lambda K, \lambda L) = \lambda^h F(K, L)$. According to Euler's theorem (see Math Tools) we then have:

CLAIM 1 For all $(K,\ L)$, where $K > 0$ and $L > 0$,

$$KF_K(K, L) + LF_L(K, L) = hF(K, L). \tag{2.51}$$

Euler's theorem also implies the inverse:

CLAIM 2 If (2.51) is satisfied for all $(K,\ L)$, where $K > 0$ and $L > 0$, then $F(K, L)$ is homogeneous of degree h.

Partial differentiation w.r.t. K and L, respectively, gives, after ordering,

$$KF_{KK} + LF_{LK} = (h-1)F_K \tag{2.52}$$
$$KF_{KL} + LF_{LL} = (h-1)F_L. \tag{2.53}$$

In (2.52) we can substitute $F_{LK} = F_{KL}$ (by Young's theorem). In view of Claim 2 this shows:

CLAIM 3 The marginal products, F_K and F_L, considered as functions of K and L, are homogeneous of degree $h - 1$.

We see also that when $h \geq 1$ and K and L are positive, then

$$F_{KK} < 0 \text{ implies } F_{KL} > 0, \tag{2.54}$$
$$F_{LL} < 0 \text{ implies } F_{KL} > 0. \tag{2.55}$$

For $h = 1$ this establishes the direct complementarity result, (iv) in Section 2.1.3, to be proved. A by-product of the derivation is that also when a neoclassical production function is homogeneous of degree $h > 1$ (which implies IRS), does direct complementarity between K and L hold.

Remark. The terminology around complementarity and substitutability may easily lead to confusion. In spite of K and L exhibiting *direct complementarity* when $F_{KL} > 0$, K and L are still *substitutes* in the sense that cost minimization for a given output level implies that a rise in the price of one factor results in higher demand for the other factor.

The claim (v) in Section 2.1.3 was the following. Suppose we face a CRS production function, $Y = F(K, L)$, that has positive marginal products,

F_K and F_L, everywhere and isoquants, $K = g(L)$, satisfying the condition $g''(L) > 0$ everywhere (i.e., F is strictly quasi-concave). Then the partial second derivatives must satisfy the neoclassical conditions:

$$F_{KK} < 0, F_{LL} < 0. \tag{2.56}$$

The proof is as follows. The first inequality in (2.56) follows from (2.50) combined with (2.52). Indeed, for $h = 1$, (2.52) and (2.53) imply $F_{KK} = -F_{LK}L/K = -F_{KL}L/K$ and $F_{KL} = -F_{LL}L/K$, i.e., $F_{KK} = F_{LL}(L/K)^2$ (or, in the notation of Appendix A, $f_{22} = f_{11}(x/y)^2$), which combined with (2.50) gives the conclusion $F_{KK} < 0$, when $g'' > 0$. The second inequality in (2.56) can be verified in a similar way.

Note also that for $h = 1$ the equations (2.52) and (2.53) entail

$$KF_{KK} = -LF_{LK} \text{ and } KF_{KL} = -LF_{LL}, \tag{2.57}$$

respectively. By dividing the left- and right-hand sides of the first of these equations with those of the second we conclude that $F_{KK}F_{LL} = F_{KL}^2$ in the CRS case. We see also from (2.57) that, under CRS, the implications in (2.54) and (2.55) can be turned round.

Finally, we asserted in § 2.1.1 that when the neoclassical production function $Y = F(K, L)$ is homogeneous of degree h, then the marginal rate of substitution between the production factors depends only on the factor proportion $k \equiv K/L$. Indeed,

$$MRS_{KL}(K, L) = \frac{F_L(K, L)}{F_K(K, L)} = \frac{L^{h-1}F_L(k, 1)}{L^{h-1}F_K(k, 1)} = \frac{F_L(k, 1)}{F_K(k, 1)} \equiv mrs(k), \tag{2.58}$$

where $k \equiv K/L$. The result (2.58) follows even if we only assume $F(K, L)$ is *homothetic*. When $F(K, L)$ is homothetic, by definition we can write $F(K, L) \equiv \varphi(G(K, L))$, where G is homogeneous of degree 1 and φ is an increasing function. In view of this, we get

$$MRS_{KL}(K, L) = \frac{\varphi' G_L(K, L)}{\varphi' G_K(K, L)} = \frac{G_L(k, 1)}{G_K(k, 1)},$$

where the last equality is implied by Claim 3 for $h = 1$.

C. The Inada conditions combined with CRS

We consider a neoclassical production function, $Y = F(K, L)$, exhibiting CRS. Defining $k \equiv K/L$, we can then write $Y = LF(k, 1) \equiv Lf(k)$, where $f(0) \geq 0, f' > 0$, and $f'' < 0$.

Essential inputs In Section 2.1.2 we claimed that the upper Inada condition for MPL together with CRS implies that without capital there will be no output:

$$F(0, L) = 0 \qquad \text{for any } L > 0.$$

In other words: in this case capital is an essential input. To prove this claim, let $K > 0$ be fixed and let $L \to \infty$. Then $k \to 0$, implying, by (2.15) and (2.17), that $F_L(K, L) = f(k) - f'(k)k \to f(0)$. But from the upper Inada condition for MPL we also have that $L \to \infty$ implies $F_L(K, L) \to 0$. It follows that

$$\text{the upper Inada condition for } MPL \text{ implies } f(0) = 0. \tag{2.59}$$

Since under CRS, for any $L > 0$, $F(0, L) = LF(0, 1) \equiv Lf(0)$, we have hereby shown our claim.

Similarly, we can show that the upper Inada condition for MPK together with CRS implies that labor is an essential input. Consider the output-capital ratio $x \equiv Y/K$. When F has CRS, we get $x = F(1, \ell) \equiv g(\ell)$, where $\ell \equiv L/K$, $g' > 0$, and $g'' < 0$. Thus, by symmetry with the previous argument, we find that under CRS, the upper Inada condition for MPK implies $g(0) = 0$. Since under CRS $F(K, 0) = KF(1, 0) \equiv Kg(0)$, we conclude that the upper Inada condition for MPK together with CRS implies

$$F(K, 0) = 0 \qquad \text{for any } K > 0,$$

that is, without labor, no output.

Sufficient conditions for output going to infinity when either input goes to infinity Here our first claim is that when F exhibits CRS and satisfies the upper Inada condition for MPL and the lower Inada condition for MPK, then

$$\lim_{L\to\infty} F(K, L) = \infty \qquad \text{for any } K > 0.$$

To prove this, note that Y can be written $Y = Kf(k)/k$, since $K/k = L$. Here,

$$\lim_{k\to 0} f(k) = f(0) = 0,$$

by continuity and (2.59), presupposing the upper Inada condition for MPL. Thus, for any given $K > 0$,

$$\lim_{L\to\infty} F(K, L) = K \lim_{L\to\infty} \frac{f(k)}{k} = K \lim_{k\to 0} \frac{f(k) - f(0)}{k} = K \lim_{k\to 0} f'(k) = \infty,$$

by the lower Inada condition for MPK. This verifies the claim.

Our second claim is symmetric with this and says: when F exhibits CRS and satisfies the upper Inada condition for MPK and the lower Inada condition for MPL, then

$$\lim_{K\to\infty} F(K,L) = \infty \qquad \text{for any } L > 0.$$

The proof is analogue. So, in combination, the four Inada conditions imply, under CRS, that output has no upper bound when either input goes to infinity.

D. Concave neoclassical production functions

Two claims made in Section 2.4 are proved here.

CLAIM 1 When a neoclassical production function $F(K,L)$ is concave, it has non-increasing returns to scale everywhere.

Proof. We consider a concave neoclassical production function, F. Let $\mathbf{x} = (x_1, x_2) = (K, L)$. Then we can write $F(K,L)$ as $F(\mathbf{x})$. By concavity, for all pairs $\mathbf{x}^0, \mathbf{x} \in \mathbb{R}^2_+$, we have $F(\mathbf{x}^0) - F(\mathbf{x}) \leq \sum_{i=1}^2 F_i'(\mathbf{x})(x_i^0 - x_i)$. In particular, for $\mathbf{x}^0 = (0,0)$, since $F(\mathbf{x}^0) = F(0,0) = 0$, we have

$$-F(\mathbf{x}) \leq -\sum_{i=1}^{2} F_i'(\mathbf{x}) x_i. \tag{2.60}$$

Suppose $\mathbf{x} \in \mathbb{R}^2_{++}$. Then $F(\mathbf{x}) > \mathbf{0}$ in view of F being neoclassical so that $F_K > 0$ and $F_L > 0$. From (2.60) we now find the elasticity of scale to be

$$\sum_{i=1}^{2} F_i'(\mathbf{x}) x_i / F(\mathbf{x}) \leq 1. \tag{2.61}$$

In view of (2.12) and (2.11), this implies non-increasing returns to scale everywhere. □

CLAIM 2 When a neoclassical production function $F(K,L)$ is strictly concave, it has decreasing returns to scale everywhere.

Proof. The argument is analogue to that above, but in view of strict concavity the inequalities in (2.60) and (2.61) become strict. This implies that F has DRS everywhere. □

2.8 Exercises

2.1 *(stocks versus flows)* Consider an economy with output Y_t, capital input K_t, and labor input L_t, all in period t, $t = 0, 1, 2, \ldots$. The technology is given by the aggregate production function (*) $Y_t = F(K_t, L_t)$, and accumulation of capital in the economy is given by (**) $K_{t+1} = Y_t - C_t + (1-\delta)K_t$, $0 < \delta < 1$.

a) Might there be a consistency problem in the notation? Explain.

b) Suggest an interpretation such that there is no consistency problem.

c) Consider two countries in period t. They have the same technology, the same capital stock and the same number of man-hours per worker per year. Country A does not use shift work, but country B uses shift work in the form of two work teams per day. Adapt formula (*) so that it can be applied to both countries.

d) Compare output per man-hour of A and B. Comment.

2.2 Consider a firm with the production function $Y = AK^\alpha L^\beta$, where $A > 0$, $0 < \alpha < 1$, $0 < \beta < 1$.

a) Is the production function neoclassical?

b) Find the marginal rate of substitution at a given (K, L).

c) Draw in the same diagram three isoquants and draw the expansion path for the firm, assuming it is cost-minimizing and faces a given factor price ratio.

d) Check whether the four Inada conditions hold for this function?

e) Suppose that instead of $0 < \alpha < 1$ we have $\alpha \geq 1$. Check whether the function is still neoclassical?

2.3 Consider the production function $Y = \min(AK, BL)$, $A > 0, B > 0$. In a diagram draw two different isoquants and a ray with slope A/B. Comment.

2.4 Consider the production function $Y = \alpha L + \beta KL/(K+L)$, where $\alpha > 0$ and $\beta > 0$.

a) Does the function imply constant returns to scale?

b) Is the production function neoclassical? *Hint:* after checking criterion (a) of the definition of a neoclassical production function, you may use claim (iii) of Section 2.1.3 together with your answer to a).

c) Given this production function, is capital an essential production factor? Is labor?

d) If we want to extend the domain of definition of the production function to include $(K, L) = (0, 0)$, how can this be done while maintaining continuity of the function?

2.5 Show that the Cobb-Douglas production function has the property that, under technological progress, it satisfies all three neutrality criteria if it satisfies one of them.

Next, write down a Cobb-Douglas production function that allows non-neutral technological change.

2.6 (For a reader not familiar with continuous time phase diagrams, this exercise may be postponed to Chapter 10).

a) In (x, w)-plane draw a phase diagram for the tâtonnement process (2.37) - (2.38).

b) Define $\tilde{x}(t) \equiv x(t) - 1$ and $\tilde{w}(t) \equiv w(t) - 0.5$ (the deviations from the stationary state). Using the solution in (2.39) - (2.40), show that $\tilde{x}(t)^2 + \tilde{w}(t)^2 = \tilde{x}(0)^2 + \tilde{w}(0)^2$ for all $t \geq 0$. *Hint:* $\cos^2 t + \sin^2 t = 1$.

c) Use this result to draw a solution curve in the phase diagram and indicate by an arrow the direction of movement. Comment.

d) Show that for $(\tilde{x}(0), \tilde{w}(0)) \neq (0, 0)$ the solution is periodic with period 2π. *Hint:* The solution $(\tilde{x}(t), \tilde{w}(t))$ is called *periodic* if, for $(\tilde{x}(0), \tilde{w}(0)) \neq (0, 0)$, there exists a positive number T such that $(\tilde{x}(t+T), \tilde{w}(t+T)) = (\tilde{x}(t), \tilde{w}(t))$ for all t.

Chapter 3

The Diamond OLG model

In today's macroeconomics there are two main analytical frameworks for analyzing the basic intertemporal choice, consumption versus saving, and its general equilibrium implications in the longer run: *overlapping generations* models and *representative agent* models. In the first class of models the focus is on (a) the interaction between different generations alive at the same time, and (b) the never-ending entrance of new generations. In the second class of models the household sector is modelled as consisting of a finite number of infinitely-lived agents. One interpretation is that these agents are dynasties where parents take the utility of their descendants fully into account by leaving bequests. This approach, which is also called the Ramsey approach (after the British mathematician and economist Frank Ramsey, 1903-1930), will be discussed in Chapter 8 (discrete time) and Chapter 10 (continuous time).

In the present chapter we introduce the overlapping generations approach which has shown its usefulness for analysis of questions associated with public debt problems, taxation of capital income, financing of social security (pensions), design of educational systems, non-neutrality of money, and the possibility of speculative bubbles. We present an overlapping generations model (OLG model for short) called Diamond's OLG model[1] after the American economist Peter A. Diamond (1940-). The model extends the original contributions of Allais (1947) and Samuelson (1957) by including physical capital. Among the strengths of the model are:

- The *life-cycle* aspect of human behavior is taken into account. Although the economy is infinitely-lived, the individual agents have finite time horizons. During lifetime one's educational level, working capacity, income, and needs change and this is reflected in the individual

[1] Diamond (1965).

labor supply and saving behavior. The aggregate implications of the life-cycle behavior of individual agents at different stages in their life is at the centre of the OLG approach.

- The model takes elementary forms of *heterogeneity* into account − young versus old, currently alive versus unborn whose preferences are not reflected in current market transactions. Questions relating to the distribution of income and wealth across generations can be studied. For example, how does the investment in capital and environmental protection by current generations affect the conditions for succeeding generations?

3.1 Motives for saving

Before going to the specifics of Diamond's model, let us briefly consider what may motivate people to save:

(a) The *life-cycle motive for saving.* Typically, individual income has a hump-shaped lifetime pattern; by saving and dissaving the individual attempts to obtain the desired smoothing of consumption across lifetime. This is the essence of the *life-cycle saving hypothesis* put forward by Nobel laureate Franco Modigliani (1918-2003) and associates in the 1950s.[2] This hypothesis states that consumers plan their saving and dissaving in accordance with anticipated variations in income and needs over lifetime. Because needs vary less over lifetime than income, the time profile of saving tends to be hump-shaped with some dissaving early in life (while studying etc.), positive saving during the years of peak earnings and then dissaving after retirement where people live off the accumulated wealth.

(b) The *precautionary motive for saving.* Income as well as needs may vary due to *uncertainty* (sudden unemployment, illness, or other kinds of bad luck). By saving, the individual can obtain a buffer against such unwelcome events.

(c) Saving enables the purchase of *durable consumption goods* and owner-occupied housing as well as repayment of debt.

(d) Saving may be motivated by the *desire to leave bequests* to heirs.

[2] See for example Modigliani and Brumberg (1954).

(e) Saving may simply be motivated by the fact that financial wealth may lead to *social prestige* or economic and political *power*.

Diamond's OLG model aims at simplicity and concentrates on motive (a). In fact, only a part of motive (a) is considered, namely the saving for retirement. People live for two periods only, as "young", working full-time, and as "old", having retired and living by their savings, and so there is little room for considering education and dissaving in the early years of life. The young work full-time while the old have retired and live by their savings. The Diamond model also abstracts from a possible bequest motive as well as non-economic motives for saving. Uncertainty is absent in the model. Numerous extensions of the framework, however, have taken up issues relating to the motives (b) - (e).[3]

Now to the details of the model.

3.2 The model framework

The flow of time is divided into successive periods of equal length, taken as the time unit. Given the two-period lifetime of (adult) individuals, the period length is understood to be around, say, 30 years. Further assumptions are:

1. The number of young people in period t, denoted L_t, changes over time according to $L_t = L_0(1+n)^t$, $t = 0, 1, 2, ...$, where n is a constant, $n > -1$. As is common in economic models, indivisibility problems are ignored and so L_t is just considered a positive real number.

2. Only young people work. Each young supplies one unit of labor inelastically. So the division of available time between work and leisure is considered as exogenous.

3. Output is homogeneous and can be used for consumption as well as investment in physical capital. Physical capital is the only asset in the economy; it is owned by the households and rented out to the firms. Money (means of payment) is ignored.

4. The economy is closed.

5. Firms' technology has constant returns to scale.

[3] Horioka and Watanabe (1997) find that empirically, the saving motives (a) and (b) are of dominant importance (Japanese data).

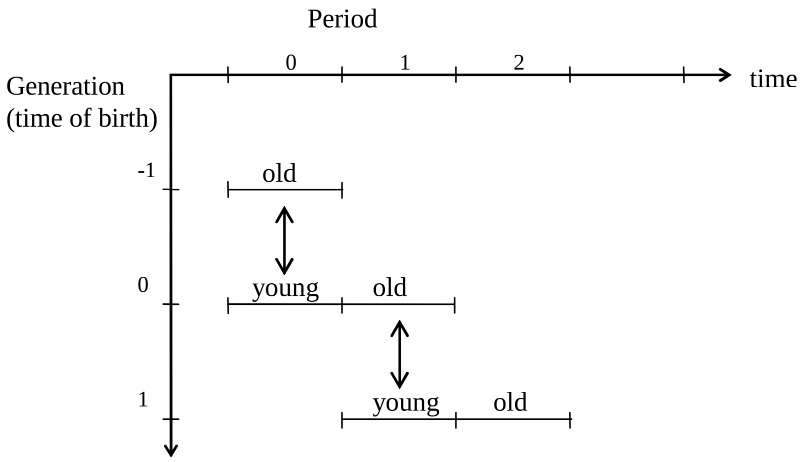

Figure 3.1: The two-period model's time structure.

6. There are three markets, a market for output, a market for labor services, and a market for capital services. Perfect competition rules in all markets.

The model ignores recurrent stochastic influences on the system in which the agents act. When a decision is made, its consequences are known. The agents are assumed to have "rational expectations" or, with a better name, "model-consistent expectations". That is, forecasts made by the agents coincide with the forecasts that can be calculated from the model. As there are no stochastic elements in the model, these forecasts are point estimates rather than probabilistic forecasts. In this situation model-consistent expectations are equivalent with *perfect foresight.* We say that agents have *perfect foresight* if they agree in their expectations about the future evolution of the economy and these expectations are point estimates that coincide with the subsequent actual evolution of the economy. So the seventh assumption is:

7. Agents have perfect foresight.

The time structure of the model is illustrated in Fig. 3.1. In every period two generations are alive and interact with each other as indicated by the arrows. The young supply labor to the firms and these use capital goods owned and rented out by the old. The young save for retirement, thereby offsetting the dissaving of the old and possibly bringing about positive net investment in the economy.

The households place their savings directly in new capital goods which constitute the non-consumed part of aggregate output. The households rent their stock of capital goods out to the firms. Let the homogeneous output good be the numeraire and let $\hat{r}_t$ denote the rental rate for capital in period t; that is, $\hat{r}_t$ is the real price a firm has to pay at the end of period t for the right to use one unit of someone else's physical capital through period t. So the owner of K_t units of physical capital receives a

$$\text{real (net) rate of return on capital } = \frac{\hat{r}_t K_t - \delta K_t}{K_t} = \hat{r}_t - \delta, \tag{3.1}$$

where δ is the rate of physical capital depreciation which is assumed constant, $0 \leq \delta \leq 1$.

Let us imagine there is an active market for loans, the "credit market". Assume you have lent out one unit of output from the end of period $t-1$ to the end of period t. If the *real interest rate* in the loan market is r_t, then, at the end of period t you will get back $1 + r_t$ units of output. In the absence of uncertainty, equilibrium requires that capital and loans give the same rate of return,

$$\hat{r}_t - \delta = r_t. \tag{3.2}$$

This no-arbitrage condition indicates how, in equilibrium, the rental rate for capital and the more everyday concept, the interest rate, are related. And although no loan market is operative in the present model, we follow the tradition and call the right-hand side of (3.2) the *interest rate*.

Table 3.1 provides an overview of the notation. As to our timing convention, notice that any stock variable dated t indicates the amount held at the beginning of period t. That is, the capital stock accumulated by the end of period $t-1$ and available for production in period t is denoted K_t. Thus we write $K_t = (1-\delta)K_{t-1} + I_{t-1}$ and $Y_t = F(K_t, L_t)$, where $F(\cdot)$ is an aggregate production function. In this context it is useful to think of "period t" as running from date t to date $t+1$, i.e., the time interval $[t, t+1)$ on a continuous time axis. Still, all decisions are taken at discrete points in time, $t = 0, 1, 2, ...$ ("dates"). We imagine that both receipts for work and lending and payment for consumption in period t occur at the end of the period. These timing conventions are common in discrete-time growth and business cycle theory.[4] They are convenient because they make switching between discrete and continuous time analysis fairly easy.

Table 3.1. List of main variable symbols

[4]In contrast, in the standard *finance* literature, K_t would denote the *end-of-period* t stock that begins to yield its services *next* period.

Variable	*Meaning*
L_t	the number of young people in period t
n	generation growth rate
K_t	aggregate capital available in period t
c_{1t}	consumption as young in period t
c_{2t}	consumption as old in period t
w_t	real wage in period t
r_t	real interest rate (from end of per. $t-1$ to end of per. t)
ρ	rate of time preference
θ	elasticity of marginal utility
s_t	saving of each young in period t
Y_t	aggregate output in period t
$C_t = c_{1t}L_t + c_{2t}L_{t-1}$	aggregate consumption in period t
$S_t = Y_t - C_t$	aggregate gross saving in period t
$\delta \in [0,1]$	capital depreciation rate
$K_{t+1} - K_t = I_t - \delta K_t$	aggregate net investment in period t

3.3 The saving of the young

The decision problem of the young in period t, given w_t and r_{t+1}, is:

$$\max_{c_{1t},c_{2t+1}} U(c_{1t}, c_{2t+1}) = u(c_{1t}) + (1+\rho)^{-1}u(c_{2t+1}) \quad \text{s.t.} \tag{3.3}$$

$$c_{1t} + s_t = w_t \qquad (w_t > 0), \tag{3.4}$$

$$c_{2t+1} = (1 + r_{t+1})s_t \qquad (r_{t+1} > -1), \tag{3.5}$$

$$c_{1t} \geq 0, c_{2t+1} \geq 0. \tag{3.6}$$

The interpretation of the variables is given in Table 3.1 above. We may think of the "young" as a household of one adult and $1 + n$ children whose consumption is included in c_{1t}. Note that "utility" appears at two levels. There is a *lifetime utility function*, $U(c_{1t}, c_{2t+1})$, and a *period utility function*, $u(c)$.[5] The latter is assumed to be the same in both periods of life (this simplifies notation and has no effects on the qualitative results). The period utility function is assumed continuous and twice continuously differentiable with $u' > 0$ and $u'' < 0$ (positive, but diminishing marginal utility of consumption). Many popular specifications of u, e.g., $u(c) = \ln c$, have the property that $\lim_{c\to 0} u(c) = -\infty$; then we *define* $u(0) = -\infty$.

[5] Other names for these two functions are the *intertemporal utility function* and the *subutility function*, respectively.

The parameter ρ is called the *rate of time preference.* It acts as a utility discount *rate,* whereas $(1+\rho)^{-1}$ is a utility discount *factor.* Thus ρ indicates the degree of *impatience* w.r.t. utility. By definition, $\rho > -1$, but we often assume $\rho > 0$. When preferences can be represented in this additive way, they are called *time-separable.* In principle, as seen from period t, the interest rate appearing in (3.5) should be interpreted as an *expected* real interest rate. But as long as we assume perfect foresight, there is no need to distinguish between actual and expected magnitudes.

In (3.5) the interest rate acts as a rate of return on saving.[6] We may also interpret the interest rate as a discount rate relating to consumption over time. For example, by isolating s_t in (3.5) and substituting into (3.4), we consolidate the two period budget constraints of the individual into one *intertemporal budget constraint,*

$$c_{1t} + \frac{1}{1+r_{t+1}} c_{2t+1} = w_t. \tag{3.7}$$

To avoid the possibility of corner solutions, cf. Lemma 1 below, we impose the "No Fast" assumption

$$\lim_{c \to 0} u'(c) = \infty. \qquad \text{(No Fast).} \tag{A1}$$

In view of the sizeable period length this is definitely plausible!

Box 3.1. A discount rate depends on what is to be discounted

By a *discount rate* is meant an interest rate applied in the construction of a discount factor. A *discount factor* is a factor by which future benefits or costs, measured in some unit of account, are converted into present equivalents. The higher the discount rate the lower the discount factor.

One should be aware that a discount rate depends on what is to be discounted. In (3.3) the unit of account is utility and ρ acts as a *utility discount rate.* In (3.7) the unit of account is the consumption good and r_{t+1} acts as a *consumption discount rate.* If people also work as old, the right-hand side of (3.7) would read $w_t + (1 + r_{t+1})^{-1} w_{t+1}$ and thus r_{t+1} would act as an *earnings discount rate.* If income were defined in monetary units, there would be a *nominal earnings discount rate*, namely the nominal interest rate. Unfortunately, confusion of different discount rates is not rare.

[6] While s_t in (3.4) appears as a *flow* (non-consumed income), in (3.5) s_t appears as a *stock* (the accumulated financial wealth at the begining of period $t + 1$). This notation is legitimate because the magnitude of the two is the same.

Solving the saving problem

Inserting the two budget constraints into the objective function in (3.3), we get $U(c_{1t}, c_{2t+1}) = u(w_t - s_t) + (1+\rho)^{-1}u((1+r_{t+1})s_t) \equiv \tilde{U}_t(s_t)$, a function of only one decision variable, s_t, where $0 \leq s_t \leq w_t$ according to the non-negativity constraint on consumption in both periods, (3.6).[7] We find the first-order condition

$$\frac{d\tilde{U}_t}{ds_t} = -u'(w_t - s_t) + (1+\rho)^{-1}u'((1+r_{t+1})s_t)(1+r_{t+1}) = 0. \qquad \text{(FOC)}$$

The second derivative of $\tilde{U}_t$ is

$$\frac{d^2\tilde{U}_t}{ds_t^2} = u''(w_t - s_t) + (1+\rho)^{-1}u''((1+r_{t+1})s_t)(1+r_{t+1})^2 < 0. \qquad \text{(SOC)}$$

Hence there can at most be one s_t satisfying (FOC). Moreover, for a positive wage income there always exists such an s_t. Indeed:

LEMMA 1 Let $w_t > 0$ and suppose the No Fast assumption (A1) applies. Then the saving problem of the young has a unique solution s_t. The solution is interior, i.e., $0 < s_t < w_t$, and s_t satisfies (FOC).

Proof. Assume (A1). For any $s \in (0, w_t)$, $d\tilde{U}_t(s)/ds > -\infty$. Now consider the endpoints $s = 0$ and $s = w_t$. By (FOC) and (A1),

$$\begin{aligned}
\lim_{s\to 0}\frac{d\tilde{U}_t}{ds} &= -u'(w_t) + (1+\rho)^{-1}(1+r_{t+1})\lim_{s\to 0}u'((1+r_{t+1})s) = \infty, \\
\lim_{s\to w}\frac{d\tilde{U}_t}{ds} &= -\lim_{s\to w_t}u'(w_t - s) + (1+\rho)^{-1}(1+r_{t+1})u'((1+r_{t+1})w_t) = -\infty.
\end{aligned}$$

By continuity of $\tilde{U}_t$, it follows that there exists an $s_t \in (0, w_t)$ such that for $s = s_t$, $d\tilde{U}_t/ds = 0$, hence (FOC) holds for this s_t. By (SOC), s_t is unique. □

The consumption Euler equation

The first-order condition (FOC) can conveniently be written

$$u'(c_{1t}) = (1+\rho)^{-1}u'(c_{2t+1})(1+r_{t+1}). \qquad (3.8)$$

[7] The simple optimization method used here is called the *substitution method*: by substitution of the constraints into the objective function an unconstrained maximization problem is obtained. Alternatively, one can use the *Lagrange method*.

This is an example of what is called an *Euler equation,* after the Swiss mathematician L. Euler (1707-1783) who was the first to study dynamic optimization problems. In the present context the condition is called a *consumption Euler equation.*

The equation says that in an optimal plan the marginal utility cost of saving equals the marginal utility benefit obtained by doing that. More specifically: the opportunity cost (in terms of current utility) of saving one more unit of account in the current period must be equal to the benefit of having $1 + r_{t+1}$ more units of account in the next period. This benefit is the discounted extra utility that can be obtained next period through the increase in consumption by $1 + r_{t+1}$ units.

It may seem odd to express the intuitive interpretation of an optimality condition the way we just did, that is, in terms of "utility units". The utility concept is just a convenient mathematical device used to represent the assumed *preferences*. Our interpretation is only meant as an as-if interpretation: as if utility were something concrete. A full interpretation in terms of measurable quantities goes like this. We rewrite (3.8) as

$$\frac{u'(c_{1t})}{(1+\rho)^{-1}u'(c_{2t+1})} = 1 + r_{t+1}. \tag{3.9}$$

The left-hand side measures the *marginal rate of substitution* (MRS) of consumption as old for consumption as young, defined as the increase in period $t+1$ consumption needed to compensate for a one-unit marginal decrease in period t consumption. That is,

$$MRS_{c_2 c_1} = -\frac{dc_{2t+1}}{dc_{1t}}\,|_{U=\bar{U}} = \frac{u'(c_{1t})}{(1+\rho)^{-1}u'(c_{2t+1})}. \tag{3.10}$$

And the right-hand side of (3.9) indicates the marginal rate of transformation, MRT, which is the rate at which saving allows an agent to shift consumption from period t to period $t+1$. In an optimal plan MRS must equal MRT. Note that when consumption is constant, (3.9) gives $1+\rho = 1+r_{t+1}$. In this (and only this) case will the utility discount rate coincide with the income discount rate, the real interest rate.

Even though interpretations in terms of "MRS equal to MRT" are more satisfactory, we will often use "as if" interpretations like the one before. They are a convenient short-hand for the more elaborate interpretation.

The Euler equation (3.8) is a discrete time analogue to what is called the Keynes-Ramsey rule in continuous time models (cf. Chapter 9). A simple implication of the rule is that

$$\rho \lesseqgtr r_{t+1} \text{ causes } u'(c_{1t}) \gtreqless u'(c_{2t+1}), \text{ i.e., } c_{1t} \lesseqgtr c_{2t+1},$$

in view of $u'' < 0$. That is, absent uncertainty the optimal plan entails either increasing, constant or decreasing consumption over time according to whether the rate of time preference is below, equal to, or above the market interest rate, respectively. For example, when $\rho < r_{t+1,}$ the plan is to start with relatively low consumption in order to take advantage of the relatively high rate of return on saving.

The saving function

There are infinitely many pairs (c_{1t}, c_{2t+1}) satisfying the Euler equation (3.8). Only when requiring the two period budget constraints, (3.4) and (3.5), satisfied, do we get a unique solution for c_{1t} and c_{2t+1} or, equivalently, for s_t as in Lemma 1. We now consider the properties of saving as a function of the market prices faced by the decision maker.

With the budget constraints inserted as in (FOC), this equation determines the saving of the young as an implicit function of w_t and r_{t+1}, i.e., $s_t = s(w_t, r_{t+1})$. The partial derivatives of this function can be found by using the *implicit function theorem* on (FOC). A practical procedure is the following. We first write $d\tilde{U}_t/ds_t$ as a function of the variables involved, s_t, w_t, and r_{t+1}, i.e.,

$$\frac{d\tilde{U}_t}{ds_t} = -u'(w_t - s_t) + (1+\rho)^{-1}u'((1+r_{t+1})s_t)(1+r_{t+1}) \equiv f(s_t, w_t, r_{t+1}).$$

By (FOC), $f(s_t, w_t, r_{t+1}) = 0$ and so the implicit function theorem implies

$$\frac{\partial s_t}{\partial w_t} = -\frac{\partial f(\cdot)/\partial w_t}{D} \quad \text{and} \quad \frac{\partial s_t}{\partial r_{t+1}} = -\frac{\partial f(\cdot)/\partial r_{t+1}}{D},$$

where $D \equiv \partial f(\cdot)/\partial s_t \equiv d^2\tilde{U}_t/ds_t^2 < 0$ by (SOC). We find

$$\begin{aligned} \frac{\partial f(\cdot)}{\partial w_t} &= -u''(c_{1t}) > 0, \\ \frac{\partial f(\cdot)}{\partial r_{t+1}} &= (1+\rho)^{-1}\left[u'(c_{2t+1}) + u''(c_{2t+1})s_t(1+r_{t+1})\right]. \end{aligned}$$

Consequently, the partial derivatives of the saving function $s_t = s(w_t, r_{t+1})$ are

$$s_w \equiv \frac{\partial s_t}{\partial w_t} = \frac{u''(c_{1t})}{D} > 0 \quad (\text{but } < 1), \tag{3.11}$$

$$s_r \equiv \frac{\partial s_t}{\partial r_{t+1}} = -\frac{(1+\rho)^{-1}[u'(c_{2t+1}) + u''(c_{2t+1})c_{2t+1}]}{D}, \tag{3.12}$$

where in the last expression we have used (3.5).[8]

We see that $0 < s_w < 1$, which implies that $0 < \partial c_{1t}/\partial w_t < 1$ and $0 < \partial c_{2t}/\partial w_t < 1 + r_{t+1}$. The positive sign of these two derivatives indicate that consumption in each of the periods is a *normal* good (which certainly is plausible since we are talking about the total consumption of the individual in each period).[9] The sign of s_r is seen to be ambiguous. This ambiguity reflects that the Slutsky substitution and income effects on consumption as young of a rise in the interest rate are of opposite signs. The *substitution effect* on c_{1t} is negative because the higher interest rate makes future consumption cheaper in terms of current consumption. And the *income effect* on c_{1t} is positive because a given budget, cf. (3.7), can buy more consumption in both periods. Generally there would be a third Slutsky effect, a wealth effect of a rise in the interest rate, but such an effect is ruled out in this model. This is because there is no labor income in the second period of life. Indeed, as indicated by (3.4), the human wealth of a member of generation t is simply w_t, which is independent of r_{t+1}.

Rewriting (3.12) gives

$$s_r = \frac{(1+\rho)^{-1}u'(c_{2t+1})[\theta(c_{2t+1}) - 1]}{D} \gtreqqless 0 \text{ for } \theta(c_{2t+1}) \lesseqqgtr 1, \tag{3.13}$$

respectively, where $\theta(c_{2t+1})$ is the absolute *elasticity of marginal utility* of consumption in the second period, that is,

$$\theta(c_{2t+1}) \equiv -\frac{c_{2t+1}}{u'(c_{2t+1})}u''(c_{2t+1}) \approx -\frac{\Delta u'(c_{2t+1})/u'(c_{2t+1})}{\Delta c_{2t+1}/c_{2t+1}} > 0,$$

where the approximation is valid for a "small" change, Δc_{2t+1}, in c_{2t+1}. The inequalities in (3.13) show that when the absolute elasticity of marginal utility is below one, then the substitution effect on consumption as young of an

[8] This derivation could alternatively be based on "total differentiation". First, keeping r_{t+1} fixed, one calculates the total derivative w.r.t. w_t on both sides of (FOC). Next, with w_t fixed, one calculates the total derivative w.r.t. r_{t+1} on both sides of (FOC). Yet another approach could be based on differentiation in terms of *differentials*. Taking the differential w.r.t. s_t, w_t, and r_{t+1} on both sides of (FOC) gives $-u''(c_{1t})(dw_t - ds_t) + (1+\rho)^{-1} \cdot \{u''(c_{2t+1})[(1+r_{t+1})ds_t + s_t dr_{t+1}](1+r_{t+1}) + u'(c_{2t+1})dr_{t+1}\} = 0$. By ordering we find the ratios ds_t/dw_t and ds_t/dr_{t+1}, which will indicate the value of the partial derivatives (3.11) and (3.12).

[9] Recall, a consumption good is called *normal* for a given consumer if the demand for it is an increasing function of the consumer's wealth. Since in this model the consumer is born without any financial wealth, the consumer's wealth is simply the present value of labor earnings through life, which in turn consists only of w_t as there is no labor income in the second period of life, cf. (3.7).

increase in the interest rate dominates the income effect and saving increases. The opposite is true if the elasticity of marginal utility is above one.

The reason that $\theta(c_{2t+1})$ has this role is that it reflects how sensitive marginal utility of c_{2t+1} is to changes in c_{2t+1}. To see the intuition, consider the case where consumption as young, and thus saving, happens to be unaffected by an increase in the interest rate. Even in this case, consumption as old, c_{2t+1}, is automatically increased (in view of the higher income as old through the higher rate of return on the unchanged saving); and the marginal utility of c_{2t+1} is thus decreased in response to a higher interest rate. The point is that this outcome can only be optimal if the elasticity of marginal utility of c_{2t+1} is of medium size. A very high elasticity of marginal utility of c_{2t+1} would result in a sharp decline in marginal utility − so sharp that not much would be lost by dampening the automatic rise in c_{2t+1} and instead increase c_{1t}, thus reducing saving. On the other hand, a very low elasticity of marginal utility of c_{2t+1} would result in only a small decline in marginal utility − so small that it is beneficial to take advantage of the higher rate of return and save *more*, thus accepting the first-period utility loss brought about by a lower c_{1t}.

We see from (3.12) that an absolute elasticity of marginal utility equal to exactly one is the case leading to the interest rate being *neutral* vis-a-vis the saving of the young. What is the intuition behind this? Neutrality vis-a-vis the saving of the young of a rise in the interest rate requires that c_{1t} remains unchanged since $c_{1t} = w_t - s_t$. In turn this requires that the marginal utility, $u'(c_{2t+1})$, on the right-hand side of (3.8) falls by the same percentage as $1+r_{t+1}$ rises. The budget (3.5) as old, however, tells us that c_{2t+1} must rise by the same percentage as $1+r_{t+1}$ is s_t should remain unchanged. Altogether we thus need that $u'(c_{2t+1})$ falls by the same percentage as c_{2t+1} rises. But this requires that the absolute elasticity of $u'(c_{2t+1})$ w.r.t. c_{2t+1} is exactly one.

The elasticity of marginal utility, also called the marginal utility flexibility, will generally depend on the level of consumption, as implicit in the notation $\theta(c_{2t+1})$. A popular special case is however the following.

EXAMPLE 1 *The CRRA utility function.* If we impose the requirement that $u(c)$ should have an absolute elasticity of marginal utility of consumption equal to a constant $\theta > 0$, then one can show (see Appendix A) that the utility function must be of the so-called CRRA form

$$u(c) = \begin{cases} \frac{c^{1-\theta}-1}{1-\theta}, & \text{when } \theta \neq 1, \\ \ln c, & \text{when } \theta = 1. \end{cases}, \tag{3.14}$$

It may seem odd that for $\theta \neq 1$ we subtract the constant $1/(1-\theta)$ from

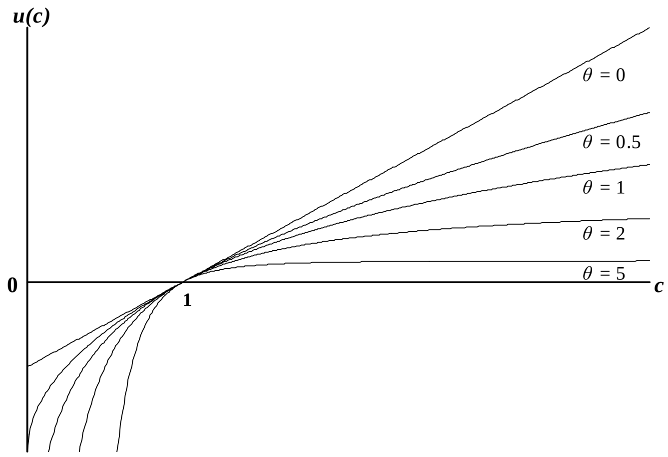

Figure 3.2: The CRRA family of utility functions.

$c^{1-\theta}/(1-\theta)$. Adding or subtracting a constant from a utility function does not affect the marginal rate of substitution and consequently not behavior. But notwithstanding that we could do without this constant, its presence in (3.14) has two advantages. One is that in contrast to $c^{1-\theta}/(1-\theta)$ the expression $(c^{1-\theta}-1)/(1-\theta)$ can be interpreted as valid even for $\theta = 1$, namely as identical to $\ln c$. This is because $(c^{1-\theta}-1)/(1-\theta) \to \ln c$ for $\theta \to 1$ (by L'Hôpital's rule for "0/0"). Another advantage is that the kinship between the different members of the CRRA family, indexed by θ, becomes more transparent. Indeed, by defining $u(c)$ as in (3.14), all graphs of $u(c)$ will go through the same point as the log function, namely $(1, 0)$, cf. Fig. 3.2.

The higher is θ, the more "curvature" does the corresponding curve in Fig. 3.2 have which in turn reflects a higher incentive to smooth consumption across time. The reason is that a large curvature means that the marginal utility will drop sharply if consumption rises and will increase sharply if consumption falls. Consequently, there is not so much utility to be lost by lowering consumption when it is relatively high but there is a lot to be gained by raising it when it is relatively low. So the curvature θ indicates the degree of *aversion towards variation in consumption*. Or we may say that θ indicates the *desire for consumption smoothing*.[10] Given (3.14), from (FOC) we get

[10] The name CRRA is a short form of *Constant Relative Risk Aversion* and comes from the theory of behavior under uncertainty. Also in that theory does the CRRA function constitute an important benchmark case. And θ is in this context called the *degree of relative risk aversion*.

an explicit solution for the saving of the young:

$$s_t = \frac{1}{1 + (1+\rho)^{\frac{1}{\theta}}(1 + r_{t+1})^{\frac{\theta-1}{\theta}}} w_t. \tag{3.15}$$

We see that the signs of $\partial s_t / \partial w_t$ and $\partial s_t / \partial r_{t+1}$, shown in (3.11) and (3.13), respectively, are confirmed. Moreover, in this special case the saving of the young is proportional to income with a factor of proportionality that depends on the interest rate (as long as $\theta \neq 1$). But in the general case the saving-income ratio depends also on the income level.

A major part of the attempts at empirically estimating θ suggests that $\theta > 1$. Based on U.S. data, Hall (1988) provides estimates above 5, while Attanasio and Weber (1993) suggest $1.25 \leq \theta \leq 3.33$. For Japanese data Okubo (2011) suggests $2.5 \leq \theta \leq 5.0$. According this evidence we should expect the income effect on current consumption of an increase in the interest rate to dominate the substitution effect, thus implying $s_r < 0$ *as long as there is no wealth effect*. (On the other hand, it is not obvious that these econometric estimates are applicable to a period length as long as that in the Diamond model, about 30 years.) □

When the elasticity of marginal utility of consumption is a constant, θ, its inverse, $1/\theta$, equals the *elasticity of intertemporal substitution* in consumption. This concept refers to the willingness to substitute consumption over time when the interest rate changes. Formally, the elasticity of intertemporal substitution is defined as the elasticity of the ratio c_{2t+1}/c_{1t} w.r.t. $1 + r_{t+1}$ when we move along a given indifference curve. The next subsection, which can be omitted in a first reading, goes more into detail with this concept.

Digression: The elasticity of intertemporal substitution*

Consider a two-period consumption problem like the one above. Fig. 3.3 depicts a particular indifference curve, $u(c_1) + (1+\rho)^{-1}u(c_2) = \bar{U}$. At a given point, (c_1, c_2), on the curve, the marginal rate of substitution of period-2 consumption for period-1 consumption, MRS, is given by

$$MRS = -\frac{dc_2}{dc_1}\,|_{U=\bar{U}}\ ,$$

that is, MRS at the point (c_1, c_2) is the absolute value of the slope of the tangent to the indifference curve at that point.[11] Under the "normal" assumption of strictly convex preferences, MRS is rising along the curve when

[11] When the meaning is clear from the context, to save notation we just write MRS instead of the more precise $MRS_{c_2 c_1}$.

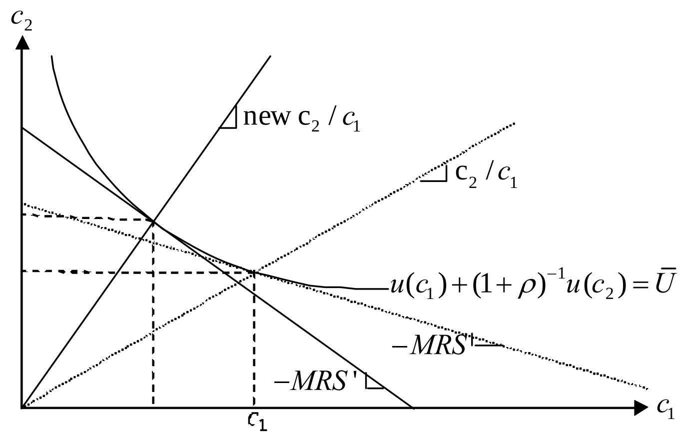

Figure 3.3: Substitution of period 2-consumption for period 1-consumption as MRS increases to MRS'.

c_1 decreases (and thereby c_2 increases). Conversely, we can let MRS be the independent variable and consider the corresponding point on the indifference curve, and thereby the ratio c_2/c_1, as a function of MRS. If we raise MRS along the indifference curve, the corresponding value of the ratio c_2/c_1 will also rise.

The *elasticity of intertemporal substitution in consumption* at a given point is defined as the elasticity of the ratio c_2/c_1 w.r.t. the marginal rate of substitution of c_2 for c_1, when we move along the indifference curve through the point (c_1, c_2). Let the elasticity of a differentiable function $f(x)$ w.r.t. x be denoted $\mathrm{E}\ell_x f(x)$. Then the elasticity of intertemporal substitution in consumption is

$$\mathrm{E}\ell_{MRS}\left(c_2/c_1\right) = \frac{MRS}{c_2/c_1}\frac{d\left(c_2/c_1\right)}{dMRS}\,|_{U=\bar{U}} \approx \frac{\frac{\Delta(c_2/c_1)}{c_2/c_1}}{\frac{\Delta MRS}{MRS}},$$

where the approximation is valid for a "small" increase, ΔMRS, in MRS.

A more concrete understanding is obtained when we take into account that in the consumer's optimal plan, MRS equals the ratio of the discounted prices of good 1 and good 2, that is, the ratio $1/(1/(1+r))$ given in (3.7). Indeed, from (3.10) and (3.9), omitting the time indices we have

$$MRS = -\frac{dc_2}{dc_1}\,|_{U=\bar{U}} = \frac{u'(c_1)}{(1+\rho)^{-1}u'(c_2)} = 1 + r \equiv R. \tag{3.16}$$

Letting $\sigma(c_1, c_2)$ denote the elasticity of intertemporal substitution, evaluated at the point (c_1, c_2), we then have

$$\sigma(c_1, c_2) = \frac{R}{c_2/c_1}\frac{d(c_2/c_1)}{dR}\,|_{U=\bar{U}} \approx \frac{\frac{\Delta(c_2/c_1)}{c_2/c_1}}{\frac{\Delta R}{R}}. \tag{3.17}$$

Thus, the elasticity of intertemporal substitution can be interpreted as the approximate percentage increase in the consumption ratio, c_2/c_1, triggered by a one percentage increase in the inverse price ratio, holding the utility level unchanged.[12]

Given $u(c)$, we let $\theta(c)$ be the absolute elasticity of marginal utility of consumption, i.e., $\theta(c) \equiv -cu''(c)/u'(c)$. As shown in Appendix B, we then find the elasticity of intertemporal substitution to be

$$\sigma(c_1, c_2) = \frac{c_2 + Rc_1}{c_2\theta(c_1) + Rc_1\theta(c_2)}. \tag{3.18}$$

We see that if $u(c)$ belongs to the CRRA class, i.e., $\theta(c_1) = \theta(c_2) = \theta$, then $\sigma(c_1, c_2) = 1/\theta$. In this case (as well as whenever $c_1 = c_2$) the elasticity of marginal utility and the elasticity of intertemporal substitution are inversely related to each other.

3.4 Production

The specification of technology and production conditions follows the simple competitive one-sector setup discussed in Chapter 2. Although the Diamond model is a long-run model, we shall in this chapter for simplicity ignore technological change.

The representative firm

There is a representative firm with a neoclassical production function and constant returns to scale (CRS). Omitting the time argument t when not needed for clarity, we have

$$Y = F(K, L) = LF(k, 1) \equiv Lf(k), \qquad f' > 0, f'' < 0, \tag{3.19}$$

where Y is output (GDP) per period, K is capital input, L is labor input, and $k \equiv K/L$ is the capital-labor ratio. Finally, the derived function, f,

[12] This characterization is equivalent to saying that an elasticity of substitution measures the percentage *decrease* in the ratio of the chosen quantities of goods (when moving along a given indifference curve) induced by a one-percentage *increase* in the *corresponding* price ratio.

is called the production function on intensive form. Capital installation and other adjustment costs are ignored. Hence the profit function is $\Pi \equiv F(K, L)$ $-\hat{r}K$ $-wL$. The firm maximizes Π under perfect competition. This gives, first, $\partial\Pi/\partial K = F_K(K, L) - \hat{r} = 0$, that is,

$$F_K(K, L) = \frac{\partial [Lf(k)]}{\partial K} = f'(k) = \hat{r}. \tag{3.20}$$

Second, $\partial\Pi/\partial L = F_L(K, L) - w = 0$, that is,

$$F_L(K, L) = \frac{\partial [Lf(k)]}{\partial L} = f(k) - kf'(k) = w. \tag{3.21}$$

The interpretation is that the firm will in every period use capital up to the point where the marginal product of capital equals the rental rate given from the market. Similarly, the firm will employ labor up to the point where the marginal product of labor equals the wage rate given from the market.

In view of $f'' < 0$, a k satisfying (3.20) is unique. We will call it the *desired capital-labor ratio.* Owing to CRS, however, at this stage the separate factor inputs, K and L, are indeterminate; only their ratio, k, is determinate.[13] We will now see how the equilibrium conditions for the factor markets select the factor prices and the level of factor inputs consistent with equilibrium.

Clearing in the factor markets

Let the aggregate demand for capital services and labor services be denoted K^d and L^d, respectively. Clearing in factor markets in period t implies

$$K_t{}^d = K_t, \tag{3.22}$$

$$L_t{}^d = L_t = L_0(1+n)^t, \tag{3.23}$$

where K_t is the aggregate supply of capital services and L_t the aggregate supply of labor services. As was called attention to in Chapter 1, unless otherwise specified it is understood that the rate of utilization of each production factor is constant over time and normalized to one. So the quantity K_t will at one and the same time measure both the capital input, a flow, and the available capital stock. Similarly, the quantity L_t will at one and the

[13] It might seem that k is overdetermined because we have two equations, (3.20) and (3.21), but only one unknown. This reminds us that for *arbitrary* factor prices, $\hat{r}$ and w, there will *not* exist a k satisfying both (3.20) and (3.21). But in equilibrium the factor prices faced by the firm are not arbitrary. They are equilibrium prices, i.e., they are adjusted so that (3.20) and (3.21) become consistent.

same time measure both the labor input, a flow, and the size of the labor force as a stock (= the number of young people).

The aggregate input demands, K^d and L^d, are linked through the desired capital-labor ratio, k^d, and in equilibrium we have $K_t^d/L_t^d = k_t{}^d = K_t/L_t$, by (3.22) and (3.23). Therefore, k in (3.20) and (3.21) can be identified with the ratio of the stock supplies, $k_t \equiv K_t/L_t$, which is a predetermined variable. Interpreted this way, (3.20) and (3.21) *determine* the equilibrium factor prices $\hat{r}_t$ and w_t in each period. In view of (3.2), $r_t = \hat{r}_t - \delta$, and so we end up, for $k_t > 0$, with

$$r_t = f'(k_t) - \delta \equiv r(k_t), \qquad \text{where} \qquad r'(k_t) = f''(k_t) < 0, \tag{3.24}$$

$$w_t = f(k_t) - f'(k_t)k_t \equiv w(k_t), \quad \text{where} \quad w'(k_t) = -k_t f''(k_t) > 0. \tag{3.25}$$

Technical Remark. In these formulas it is understood that $L > 0$, but we may allow $K = 0$, i.e., $k = 0$. In case $f'(0)$ is not immediately well-defined, we interpret $f'(0)$ as $\lim_{k\to 0^+} f'(k)$ if this limit exists. If it does not, it must be because we are in a situation where $\lim_{k\to 0^+} f'(k) = \infty$, since $f''(k) < 0$ (an example is the Cobb-Douglas function, $f(k) = Ak^\alpha$, $0 < \alpha < 1$, where $\lim_{k\to 0^+} f'(k) = A\alpha k^{\alpha-1} = +\infty$). In this situation we simply include $+\infty$ in the range of $r(k)$ and define $r(0) \cdot 0 \equiv \lim_{k\to 0^+}(f'(k) - \delta)k = 0$, where the last equality comes from the general property that $\lim_{k\to 0^+} f'(k)k = 0$, cf. (2.17) of Chapter 2. Letting $r(0) \cdot 0 = 0$ also fits well with intuition since, when $k = 0$, nobody receives capital income anyway. Note that since $\delta \in [0, 1]$, $r(k) > -1$ for all $k \geq 0$. What about $w(0)$? We interpret $w(0)$ as $\lim_{k\to 0} w(k)$. From (2.17) of Chapter 2 we have that $\lim_{k\to 0} w(k) = f(0) \equiv F(0, 1) \geq 0$. If capital is essential, $F(0, 1) = 0$. Otherwise, $F(0, 1) > 0$. Finally, since $w' > 0$, we have for $k > 0$, $w(k) > 0$ as also noted in Chapter 2. $\square$

To fix ideas we have assumed that the households own the physical capital and rent it out to the firms. But as long as the model ignores uncertainty and capital installation costs, the results will be unaffected if instead we let the firms themselves own the physical capital and finance capital investment by issuing bonds and shares. These bonds and shares would then be accumulated by the households and constitute their financial wealth instead of the capital goods. The equilibrium rate of return, r_t, would be the same.

3.5 The dynamic path of the economy

As in other fields of economics, it is important to distinguish between the set of technically feasible allocations and an allocation brought about, within this set, by a specific economic institution (the set of rules of the game). The economic institution assumed by the Diamond model is the private-ownership perfect-competition market institution. We shall now introduce three different concepts concerning allocations over time in the economy, that is, sequences $\{(k_t, c_{1t}, c_{2t})\}_{t=0}^{\infty}$. The three concepts are: technically feasible path, temporary equilibrium, and equilibrium path. These concepts are mutually related in the sense that there is a whole *set* of technically feasible paths or sequences, *within* which there may exist a unique equilibrium path, which in turn is a sequence of *states* with a certain property (temporary equilibria).

3.5.1 Technically feasible paths

When we speak of technically feasible paths, we disregard aspects not relating to the available technology and exogenous resources: the agents' preferences, the optimizing behavior given the constraints, the market forces etc. The focus is merely upon what is feasible, given the technology and exogenous resources. The technology is represented by (3.19) and there are two exogenous resources, the labor force, $L_t = L_0(1+n)^t$, and the initial capital stock, K_0.

Aggregate consumption can be written $C_t \equiv C_{1t} + C_{2t} = c_{1t}L_t + c_{2t}L_{t-1}$. With S_t denoting aggregate gross saving, from national accounting we have $S_t \equiv Y_t - C_t = F(K_t, L_t) - C_t$ by (3.19). In a closed economy aggregate gross saving equals (ex post) aggregate gross investment, $K_{t+1} - K_t + \delta K_t$. So

$$C_t = F(K_t, L_t) - (K_{t+1} - K_t + \delta K_t). \tag{3.26}$$

Let c_t denote aggregate consumption per unit of labor in period t, i.e.,

$$c_t \equiv \frac{C_t}{L_t} = \frac{c_{1t}L_t + c_{2t}L_{t-1}}{L_t} = c_{1t} + \frac{c_{2t}}{1+n}.$$

Combining this with (3.26) and using the definitions of k and $f(k)$, we obtain the dynamic resource constraint of the economy:

$$c_{1t} + \frac{c_{2t}}{1+n} = f(k_t) + (1-\delta)k_t - (1+n)k_{t+1}. \tag{3.27}$$

DEFINITION 1 Let $\bar{k}_0 \geq 0$ be the historically given initial capital-labor ratio. The path $\{(k_t, c_{1t}, c_{2t})\}_{t=0}^{\infty}$ is called *technically feasible* if $k_0 = \bar{k}_0$ and for $t = 0, 1, 2, \ldots$, (3.27) holds with $k_t \geq 0$, $c_{1t} \geq 0$, and $c_{2t} \geq 0$.

Next we consider how, for given household preferences, the private-ownership market institution with profit-maximizing firms under perfect competition generates a *selection* within the class of technically feasible paths. A member (sometimes the unique member) of this selection is called an *equilibrium path* and constitutes a sequence of states, temporary equilibria, with a certain property.

3.5.2 A temporary equilibrium

Standing in a given period, we may think of next period's interest rate as an *expected* interest rate that provisionally can deviate from the ex post realized one. We let r_{t+1}^e denote the expected real interest rate of period $t+1$ as seen from period t.

Essentially, by a temporary equilibrium in period t is meant a state where for a *given* r_{t+1}^e, all markets clear. There are three markets, namely two factor markets and a market for produced goods. We have already described the two factor markets. In the market for produced goods the representative firm supplies the amount $F(K_t^d, L_t^d)$ in period t. The demand side in this market has two components, consumption, C_t, and gross investment, I_t. There are two types of agents on the demand side, the young and the old. We have $C_t = c_{1t}L_t + c_{2t}L_{t-1}$. Saving is in this model an act of acquiring capital goods and is therefore directly an act of capital investment. So the net investment by the young equals their net saving, s_tL_t. The old in period t disinvest by consuming not only their interest income but also the financial wealth with which they entered period t. We claim that this financial wealth, $s_{t-1}L_{t-1}$, must equal the aggregate capital stock at the beginning of period t:

$$s_{t-1}L_{t-1} = K_t. \tag{3.28}$$

Indeed, there is no bequest motive and so the old in any period consume all they have and leave nothing as bequests. It follows that the young in any period enter the period with no financial wealth. So any financial wealth existing at the beginning of a period must belong to the old in that period and be the result of their saving as young in the previous period. As K_t constitutes the aggregate financial wealth in our closed economy at the beginning of period t, (3.28) follows.

Recalling that net saving is by definition the same as the increase in financial wealth, the net saving of the old in period t is thus $-K_t$. At the same time this is the (negative) net investment of the old. Aggregate net investment is thus $s_tL_t + (-K_t)$. By definition, aggregate *gross* investment equals aggregate net investment plus capital depreciation, i.e.,

$$I_t = s_tL_t - K_t + \delta K_t. \tag{3.29}$$

Equilibrium in the goods market, $C_t + I_t = F(K_t^d, L_t^d)$, therefore obtains when

$$c_{1t}L_t + c_{2t}L_{t-1} + s_tL_t - K_t + \delta K_t = F(K_t^d, L_t^d). \tag{3.30}$$

DEFINITION 2 For a given period t with capital stock $K_t \geq 0$, let the expected real interest rate be given as $r_{t+1}^e > -1$. A *temporary equilibrium* in period t is a state, $(k_t, c_{1t}, c_{2t}, w_t, r_t)$, of the economy such that (3.30), (3.22), and (3.23) hold (i.e., all markets clear) for $c_{1t} = w_t - s(w_t, r_{t+1}^e)$ and $c_{2t} = (k_t + r_tk_t)(1+n)$, where $w_t = w(k_t) > 0$ and $r_t = r(k_t)$, as defined in (3.25) and (3.24), respectively, and with $k_t = K_t/L_t$.

Speaking about "equilibrium" in this context is appropriate because (a) the agents optimize, given their expectations and the constraints they face, and (b) the aggregate demands and supplies are mutually consistent, i.e., markets clear. The reason for the requirement $w_t > 0$ in the definition is that if $w_t = 0$, people would have nothing to live on as young and save from for retirement. The system would not be economically viable in this case. With regard to the equation for c_{2t} in the definition, note that (3.28) gives $s_{t-1} = K_t/L_{t-1} = (K_t/L_t)(L_t/L_{t-1}) = k_t(1+n)$, the wealth of each old at the beginning of period t. Substituting into $c_{2t} = (1+r_t)s_{t-1}$, we get $c_{2t} = (1+r_t)k_t(1+n)$, which can also be written $c_2 = (k_t + r_tk_t)(1+n)$. This last way of writing c_2 has the advantage of being applicable even if $k_t = 0$, cf. Technical Remark in Section 3.4. The remaining conditions for a temporary equilibrium are self-explanatory.

PROPOSITION 1 Suppose the No Fast assumption (A1) applies. Consider a given period t. Then for any $r_{t+1}^e > -1$,
(i) if $k_t > 0$, there exists a temporary equilibrium $(k_t, c_{1t}, c_{2t}, w_t, r_t)$, and c_{1t} and c_{2t} are positive;
(ii) if and only if $f(0) > 0$ (i.e., capital not essential), does a temporary equilibrium exist even for $k_t = 0$; in that case $w_t = w(k_t) = w(0) = f(0) > 0$ and c_{1t} and s_t are positive, while $c_{2t} = 0$;
(iii) whenever a temporary equilibrium exists, it is unique.

Proof. We begin with (iii). That there is at most one temporary equilibrium is immediately obvious since w_t and r_t are functions of k_t, $w_t = w(k_t)$ and $r_t = r(k_t)$. And given w_t and r_t, c_{1t} and c_{2t} are uniquely determined.

(i) Let $k_t > 0$ be given. Then, by (3.25), $w(k_t) > 0$. We claim that the state $(k_t, c_{1t}, c_{2t}, w_t, r_t)$, with $w_t = w(k_t)$, $r_t = r(k_t)$, $c_{1t} = w(k_t) - s(w(k_t), r_{t+1}^e)$, and $c_{2t} = (1 + r(k_t))k_t(1+n)$, is a temporary equilibrium. Indeed, Section 3.4 showed that the factor prices $w_t = w(k_t)$ and $r_t = r(k_t)$ are consistent with clearing in the factor markets in period t. Given that these markets clear, it follows by Walras' law (see Appendix C) that also the third market, the goods market, clears in period t. So all criteria in Definition 2 are satisfied. That $c_{1t} > 0$, follows from $w(k_t) > 0$ and the No Fast assumption (A1), in view of Lemma 1. That $c_{2t} > 0$, follows immediately from $c_{2t} = (1 + r(k_t))k_t(1 + n)$ when $k_t > 0$, since $r(k_t) > -1$ always.

(ii) Let $k_t = 0$ be given. Suppose $f(0) > 0$. Then, by Technical Remark in Section 3.4, $w_t = w(0) = f(0) > 0$ and $c_{1t} = w_t - s(w_t, r^e_{t+1})$ is well-defined, positive, and less than w_t, in view of Lemma 1; so $s_t = s(w_t, r^e_{t+1}) > 0$. The old in period 0 will starve since $c_{2t} = (0 + 0)(1 + n)$, in view of $r(0) \cdot 0 = 0$, cf. Technical Remark in Section 3.4. Even though this is a bad situation for the old, it is consistent with the criteria in Definition 2. On the other hand, if $f(0) = 0$, we get $w_t = f(0) = 0$, which violates one of the criteria in Definition 2. □

Point (ii) of the proposition says that a temporary equilibrium *may* exist even in a period where $k = 0$. The old in this period will starve and not survive very long. But if capital is not essential, the young get positive labor income out of which they will save a part for their old age and be able to maintain life also next period which will be endowed with positive capital, and so on in every future period.

3.5.3 An equilibrium path

An equilibrium path, also called an intertemporal equilibrium, requires more conditions satisfied. The concept of an equilibrium path refers to a sequence of temporary equilibria such that *expectations* of the agents are *fulfilled* in all periods:

DEFINITION 3 An *equilibrium path* or, equivalently, an *intertemporal equilibrium* is a technically feasible path $\{(k_t, c_{1t}, c_{2t})\}_{t=0}^{\infty}$ such that for $t = 0, 1, 2, \ldots$, the state $(k_t, c_{1t}, c_{2t}, w_t, r_t)$ is a temporary equilibrium with $r^e_{t+1} = r(k_{t+1})$.

To characterize such a path, we forward (3.28) one period and rearrange so as to get

$$K_{t+1} = s_t L_t. \tag{3.31}$$

Since $K_{t+1} \equiv k_{t+1}L_{t+1} = k_{t+1}L_t(1 + n)$, this can be written

$$k_{t+1} = \frac{s(w(k_t), r(k_{t+1}))}{1 + n}, \tag{3.32}$$

using that $s_t = s(w_t, r^e_{t+1})$, $w_t = w(k_t)$, and $r^e_{t+1} = r_{t+1} = r(k_{t+1})$ in a sequence of temporary equilibria with fulfilled expectations. Equation (3.32) is a first-order difference equation, known as the *fundamental difference equation* or the *law of motion* of the Diamond model.

PROPOSITION 2 Suppose the No Fast assumption (A1) applies. Then,
(i) for any $k_0 > 0$ there exists at least one equilibrium path;
(ii) if and only if $f(0) > 0$ (i.e., capital not essential), does an equilibrium path exist even for $k_0 = 0$;
(iii) in any case, an equilibrium path has positive real wage in all periods and positive capital in all periods except possibly the first;
(iv) an equilibrium path satisfies the first-order difference equation (3.32).

Proof. As to (i) and (ii), see Appendix D. (iii) For a given t, let $k_t \geq 0$. Then, since an equilibrium path is a sequence of temporary equilibria, we have $w_t = w(k_t) > 0$ and $s_t = s(w(k_t), r^e_{t+1})$, where $r^e_{t+1} = r(k_{t+1})$. Hence, by Lemma 1, $s(w(k_t), r^e_{t+1}) > 0$, which implies $k_{t+1} > 0$, in view of (3.32). This shows that only for $t = 0$ is $k_t = 0$ possible along an equilibrium path. Finally, (iv) was shown in the text above (3.32). □

The formal proofs of point (i) and (ii) of the proposition are placed in appendix because they are rather technical. But the graphs in the ensuing figures 3.4-3.7 provide an intuitive verification. The "only if" part of point (ii) reflects the not very surprising fact that *if* capital were an essential production factor, no capital "now" would imply no income "now", hence no investment and thus no capital in the next period and so on. On the other hand, the "if" part of point (ii) says that when capital is not essential, an equilibrium path can set off even from an initial period with no capital. Then point (iii) adds that an equilibrium path will have positive capital in all subsequent periods. Finally, as to point (iv), note that the fundamental difference equation, (3.32), rests on equation (3.31). The economic logic behind this key equation is as follows: Since capital is the only financial asset in the economy and the young are born without any inheritance, the aggregate capital stock at the beginning of period $t + 1$ must be owned by the old generation in that period and thus be equal to the aggregate saving these people had in the previous period when they were young.

The transition diagram

To be able to further characterize equilibrium paths, we construct a transition diagram in the (k_t, k_{t+1}) plane. The *transition curve* is defined as the set of points, (k_t, k_{t+1}), satisfying (3.32). Fig. 3.4 shows a *possible*, but not necessary configuration of this curve. A complicating circumstance is that the equation (3.32) has k_{t+1} on both sides. Sometimes we are able to solve the equation explicitly for k_{t+1} as a function of k_t, but sometimes we can do so only implicitly. What is even worse is that there are cases where k_{t+1} is not unique for given k_t. We will proceed step by step.

First, what can we say about the *slope* of the transition curve? In general a point on the transition curve has the property that at least in a neighborhood of this point the equation (3.32) will define k_{t+1} as an implicit function of k_t.[14] Taking the total derivative w.r.t. k_t on both sides of (3.32), we get

$$\frac{dk_{t+1}}{dk_t} = \frac{1}{1+n}\left[s_w\left(\cdot\right)w'\left(k_t\right) + s_r(\cdot)r'\left(k_{t+1}\right)\frac{dk_{t+1}}{dk_t}\right]. \tag{3.33}$$

By ordering and using (3.24) and (3.25), the slope of the transition curve can be written

$$\frac{dk_{t+1}}{dk_t} = \frac{-s_w\left(w\left(k_t\right), r\left(k_{t+1}\right)\right)k_t f''\left(k_t\right)}{1+n-s_r\left(w\left(k_t\right), r\left(k_{t+1}\right)\right)f''\left(k_{t+1}\right)}, \tag{3.34}$$

when $s_r[w(k_t), r(k_{t+1})]f''(k_{t+1}) \neq 1+n$. Since $s_w > 0$, the numerator in (3.34) is always positive and we have

$$\frac{dk_{t+1}}{dk_t} \gtrless 0 \text{ for } s_r[w(k_t), r(k_{t+1})] \gtrless \frac{1+n}{r'(k_{t+1})},$$

respectively, where $(1+n)/r'(k_{t+1}) = (1+n)/f''(k_{t+1}) < 0$.

It follows that the transition curve is universally upward-sloping if and only if $s_r[w(k_t), r(k_{t+1})] > (1+n)/r'(k_{t+1})$ everywhere along the transition curve. The intuition behind this becomes visible by rewriting (3.33) in terms of differentials w.r.t. k_{t+1} and k_t:

$$(1+n-s_r\left(\cdot\right)\ r'(k_{t+1}))dk_{t+1} = s_w\left(\cdot\right)\ w'(k_t)dk_t,$$

where $s_w\left(\cdot\right) > 0, w'(k_t) = -k_t\ f''(k_t) > 0$, and $r'(k_{t+1}) = f''(k_{t+1}) < 0$. Now, a rise in k_t will always raise wage income and, via the resulting rise in s_t, raise k_{t+1}, everything else equal. Everything else is *not* equal, however, since a rise in k_{t+1} implies a fall in the rate of interest. Yet, if $s_r\left(\cdot\right) = 0$, there is no feedback effect from this and so the tendency to a rise in k_{t+1} is neither offset nor fortified. If $s_r\left(\cdot\right) > 0$, the tendency to a rise in k_{t+1} will be partly offset through the *dampening* effect on saving resulting in *this* case from the fall in the interest rate. This negative feedback effect can not fully or more than fully offset the tendency to a rise in k_{t+1}. This is because the negative feedback on the saving of the young will only be there *if* the interest falls in the first place. We cannot in the same period have both a *fall* in the interest rate triggering lower saving *and* a *rise* in the interest rate (via a lower k_{t+1}) *because* of the lower saving.

[14] An exception occurs if the denominator in (3.34) below vanishes.

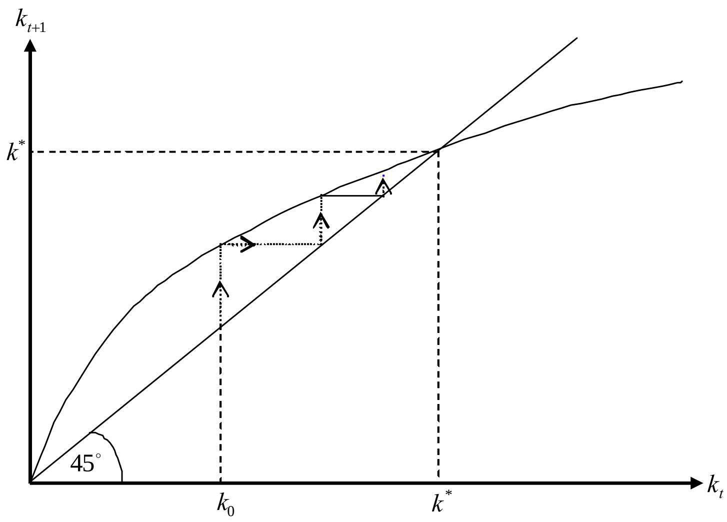

Figure 3.4: Transition curve and the resulting dynamics in the log utility Cobb-Douglas case.

On the other hand, if $s_r(\cdot)$ is sufficiently *negative*, then the initial tendency to a rise in k_{t+1} via the higher wage income in response to a rise in k_t *can* be more than fully offset by a *rise* in the interest rate leading in this case to *lower* saving by the young, hence *lower* k_{t+1}.

So a *sufficient* condition for a universally upward-sloping transition curve is that the saving of the young is a non-decreasing function of the interest rate.

PROPOSITION 3 *(the transition curve is nowhere flat)* For all $k_t > 0$, $dk_{t+1}/dk_t \neq 0$.

Proof. Since $s_w > 0$ always, the numerator in (3.34) is always positive. □

The implication is that no part of the transition curve can be horizontal.[15]

When the transition curve crosses the 45° degree line for some $k_t > 0$, as in the example in Fig. 3.4, we have a steady state at this k_t. Formally:

DEFINITION 4 An equilibrium path $\{(k_t, c_{1t}, c_{2t})\}_{t=0}^{\infty}$ is in a *steady state* with capital-labor ratio $k^* > 0$ if the fundamental difference equation, (3.32), is satisfied with k_t as well as k_{t+1} replaced by k^*.

This is simply an application of the notion of a steady state as a stationary point in a dynamic process. Some economists use the term "dynamic

[15] This would not necessarily hold if the utility function were not separable in time.

equilibrium" instead of "steady state". In this book the term "equilibrium" is used in the more general sense of situations where the constraints and decided actions of the market participants are compatible with each other. In this terminology an economy can be in equilibrium without being in a steady state. A steady state is seen as a *special* sequence of equilibria, namely one with the property that the variable(s), here k, entering the fundamental difference equation(s) does not change over time.

EXAMPLE 2 (the log utility Cobb-Douglas case) Let $u(c) = \ln c$ and $Y = AK^{\alpha}L^{1-\alpha}$, where $0 < \alpha < 1$ and $A > 0$. Since $u(c) = \ln c$ is the case $\theta = 1$ in Example 1, we have $s_r = 0$ by (3.15). Indeed, with logarithmic utility the substitution and income effects on s_t offset each other; and, as discussed above, in the Diamond model there can be no wealth effect of a rise in r_{t+1}. Further, (3.32) reduces to a simple transition function

$$k_{t+1} = \frac{(1-\alpha)Ak_t^{\alpha}}{(1+n)(2+\rho)}. \tag{3.35}$$

The transition curve is shown in Fig. 3.4 and there is for $k_0 > 0$ both a unique equilibrium path and a unique steady state with capital-labor ratio $k^* > 0$. At $k_t = k^*$ the slope of the transition curve is necessarily less than one and therefore the steady state is globally asymptotically stable. In the steady state the interest rate is $r^* = f'(k^*) - \delta = \alpha(1+n)(2+\rho)/(1-\alpha) - \delta$. □

Because the Cobb-Douglas production function implies that capital is essential, (3.35) implies $k_{t+1} = 0$ if $k_t = 0$. The state $k_{t+1} = k_t = 0$ is thus a stationary point of the difference equation (3.35) considered in isolation. This state is not, however, an equilibrium path as defined above (not a steady state of an *economic* system). Such a steady state is often called a *trivial* steady state in contrast to the economically viable steady state $k_{t+1} = k_t = k^*$ which is then called a *non-trivial* steady state.

Theoretically, there may be more than one (non-trivial) steady state. Non-existence of a steady state is also possible. But before considering these steady state questions we will face an even more defiant feature which is that for a given k_0 there may exist more than one equilibrium path.

The possibility of multiple equilibrium paths

It turns out that a transition curve like that in Fig. 3.5 is possible within the model. Not only are there two steady states but for $k_t \in (\underline{k}, \overline{k})$ there are *three temporary equilibria* with self-fulfilling expectations. That is, there are three different values of k_{t+1} that for the given k_t are consistent with self-fulfilling

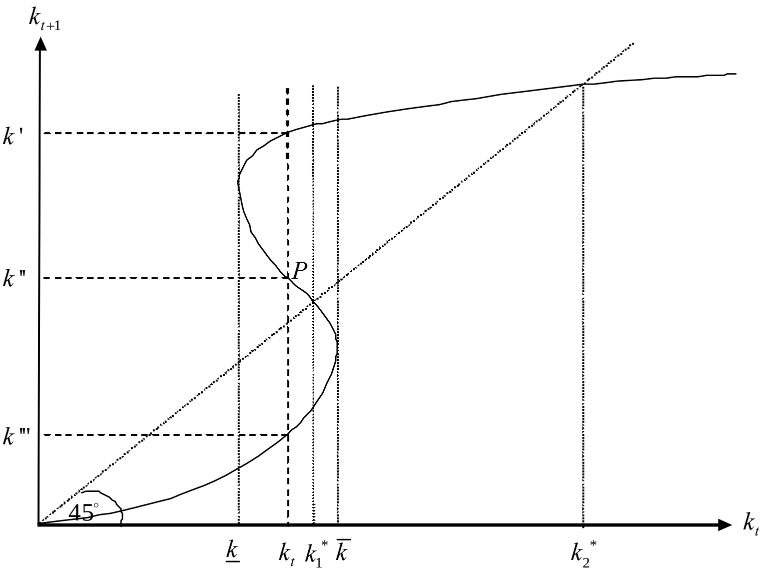

Figure 3.5: Multiple temporary equilibria with selfulfilling expectations.

expectations. Some of the exercises at the end of the chapter document this possibility by way of numerical examples.

The theoretical possibility of multiple equilibria with self-fulfilling expectations requires that there is at least one interval on the horizontal axis where a section of the transition curve has negative slope. Let us see if we can get an intuitive understanding of why in this situation multiple equilibria can arise. Consider the specific configuration in Fig. 3.5 where k', k'', and k''' are the possible values for the capital-labor ratio next period when $\underline{k} < k_t < \bar{k}$. In a neighborhood of the point P associated with the intermediate value, k'', the slope of the transition curve is negative. As we saw above, this requires not only that in this neighborhood $s_r(w_t, r(k_{t+1})) < 0$, but that the stricter condition $s_r(w_t, r(k_{t+1})) < (1+n)/f''(k'')$ holds (we take w_t as given since k_t is given and $w_t = w(k_t)$). That the point P with coordinates (k_t, k'') is on the transition curve indicates that given $w_t = w(k_t)$ and an expected interest rate $r_{t+1}^e = r(k'')$, the induced saving by the young, $s(w_t, r(k'')$, will be such that $k_{t+1} = k''$, that is, the expectation is fulfilled. The fact that also the point (k_t, k'), where $k' > k''$, is on transition curve indicates that also a lower interest rate, $r(k')$, can be self-fulfilling. By this is meant that *if* an interest rate at the level $r(k')$ is expected, then this expectation induces *more* saving by the young – just enough more to make $k_{t+1} = k' > k''$, thus confirming the expectation of the lower interest rate level $r(k')$. What makes this possible is exactly the negative dependency of s on r^e_{t+1}. The fact that also the point (k_t, k'''), where $k''' < k''$, is on transition curve can be similarly interpreted. It is again $s_r < 0$ that makes it possible that a *lower* saving than at P can be induced by an expected *higher* interest rate, $r(k''')$, than at P.

These ambiguities point to a serious problem with the assumption of perfect foresight. The model presupposes that all the young *agree* in their expectations. Only then will one of the three mentioned temporary equilibria appear. But the model is silent about how the needed coordination of expectations is brought about, and if it is, why this coordination ends up in one rather than another of the three possible equilibria with self-fulfilling expectations. Each single young is isolated in the market and will not know what the others will expect. The market mechanism as such provides no coordination of expectations. As it stands, the model cannot determine how the economy will evolve in this situation.

This ambiguity in the simple model version presented here is of course a weakness. Yet the encountered phenomenon itself − that multiple self-fulfilling equilibrium paths are theoretically possible − is certainly of interest and plays an important role in a variety of business cycle theories of booms and busts. We shall have occasion to touch upon these both exiting and complicated issues of expectation formation and self-fulfilling optimism or pessimism later in this book.

For now we plainly want to circumvent non-uniqueness. There are at least two alternative ways to rule out the possibility of multiple equilibrium paths. One simple approach is to discard the assumption of perfect foresight. Instead, some kind of adaptive expectations may be assumed, for example in the form of *myopic foresight* (sometimes called *static expectations*). This means that the expectation formed by the agents this period about the value of a variable next period is that it will stay the same as in this period.[16] So here the assumption would be that the young have the expectation $r^e_{t+1} = r_t$. Then, given $k_0 > 0$, a unique sequence of temporary equilibria $\{(k_t, c_{1t}, c_{2t}, w_t, r_t)\}_{t=0}^{\infty}$ is generated by the model. *Oscillations* in the sense of repetitive movements up and down of k_t are possible. Even *chaotic* trajectories are possible (see Exercise 3.6).

Outside steady state the agents will experience that their expectations are systematically wrong. And the assumption of myopic foresight rules out that learning occurs. Whether this is an acceptable approximation depends

[16] This expectation *will* in certain contexts be rational (model consistent). This will for instance be the case if the variable about which the expectation is held follows a random walk. In the present context the myopic expectation is not rational, however, unless the economy is already from the beginning in steady state.

on the circumstances. In the context of the Diamond model we might say that although the old may be disappointed when they realize that their expectations turned out wrong, it is too late to learn because next period they will be dead. On the other hand, it is natural to imagine that social interaction occurs in a society and so the young might learn from the mistakes by the old. But such aspects are not part of the model as it stands.

Another approach to the indeterminacy problem is motivated by the presumption that the possibility of multiple equilibria in the Diamond model is basically due to the rough time structure of the model. Each period in the model corresponds to half of an adult person's lifetime. Moreover, in the first period of life there is no capital income, in the second there is no labor income. This coarse notion of time may artificially generate multiplicity of equilibria or, with myopic foresight, oscillations. An expanded model where people live many periods might "smooth" the responses of the system to the different events impinging on it. The analyst may nevertheless in a first approach want to stay with the rough time structure of the model because of its analytical convenience and then make the most of it by imposing conditions that rule out multiple equilibria.

Following this approach we stay with the assumption of perfect foresight, but assume that circumstances are such that multiple temporary equilibria with self-fulfilling expectations do not arise.

Conditions for uniqueness of the equilibrium path

Sufficient for the equilibrium path to be unique is that preferences and technology in combination are such that the slope of the transition curve is everywhere positive. As we saw in connection with (3.34), this requires that the dependency of the saving of the young on r is not *too* negative, that is,

$$s_r[w(k_t), r(k_{t+1})] > \frac{1+n}{f''(k_{t+1})} \tag{A2}$$

everywhere along an equilibrium path. This condition is of course always satisfied when $s_r \geq 0$ (reflecting an elasticity of marginal utility of consumption not above one) and *can* be satisfied even if $s_r < 0$ (as long as s_r is small in absolute value). Essentially, it is an assumption that the income effect on consumption as young of a rise in the interest rate does not dominate the substitution effect "too much".

Unfortunately, a condition like (A2) is not in itself very informative. This is because it is expressed in terms of an *endogenous* variable, k_{t+1}, for given k_t. A model assumption should preferably be stated in terms of what is *given*, also called the "primitives" of the model, that is, the exogenous elements which in this model comprise the assumed preferences, demography, technology, and the market form. But we can at least state sufficient conditions, in terms of the "primitives", such that (A2) is ensured. Here we state two such conditions, both involving a CRRA period utility function with parameter θ as defined in (3.14):

(a) If $0 < \theta \leq 1$, then (A2) holds for all $k_t > 0$ along an equilibrium path.

(b) If the production function is of CES-type,[17] i.e., $f(k) = A(\alpha k^{\gamma}+1-\alpha)^{1/\gamma}$, $A > 0$, $0 < \alpha < 1$, $-\infty < \gamma < 1$, then (A2) holds along an equilibrium path even for $\theta > 1$, if the elasticity of substitution between capital and labor, $1/(1-\gamma)$, is not too small, i.e., if

$$\frac{1}{1-\gamma} > \frac{1-1/\theta}{1+(1+\rho)^{-1/\theta}(1+f'(k)-\delta)^{(1-\theta)/\theta}} \tag{3.36}$$

for all $k > 0$. In turn, sufficient for this is that $(1-\gamma)^{-1} > 1-1/\theta$.

That (a) is sufficient for (A2) is immediately visible in (3.15). The sufficiency of (b) is proved in Appendix D. The elasticity of substitution between capital and labor is a concept analogue to the elasticity of intertemporal substitution in consumption. It indicates the sensitivity of the chosen $k = K/L$ with respect to the relative factor price. Section 4.4 of the next chapter goes more into detail with the concept and shows, among other things, that the Cobb-Douglas production function corresponds to $\gamma = 0$. So the Cobb-Douglas production function will satisfy (3.36) since $\theta > 0$.

With these or other sufficient conditions in the back of our mind we shall now proceed assuming (A2), which we will call the Positive Slope assumption. To summarize:

PROPOSITION 4 *(uniqueness)* Suppose the No Fast and Positive Slope assumptions (A1) and (A2) apply. Then, if either $k_0 > 0$ or $k_0 = 0$ with $f(0) > 0$, there exists a unique equilibrium path.

When the conditions of Proposition 4 hold, the fundamental difference equation, (3.32), of the model defines k_{t+1} as an implicit function of k_t,

$$k_{t+1} = \varphi(k_t),$$

for all $k_t > 0$, where $\varphi(k_t)$ is called a *transition function.* The derivative of this implicit function is given by (3.34) with k_{t+1} on the right-hand side

[17] CES stands for Constant Elasticity of Substitution. CES production functions are considered in detail in Chapter 4.

replaced by $\varphi(k_t)$, i.e.,

$$\varphi'(k_t) = \frac{-s_w\left(w\left(k_t\right), r\left(\varphi(k_t)\right)\right) k_t f''\left(k_t\right)}{1+n-s_r\left(w\left(k_t\right), r\left(\varphi(k_t)\right)\right) f''\left(\varphi(k_t)\right)}. \tag{3.37}$$

From now, our transition curve will represent this transition function and thus have positive slope everywhere.

Existence and stability of a steady state?

To address the question of existence of steady states, we examine the possible configurations of the transition curve in more detail. A useful observation is that the transition curve will always, for $k_t > 0$, be situated strictly below the solid curve, $k_{t+1} = w(k_t)/(1+n)$, in Fig. 3.6. In turn, the latter curve is always, for $k_t > 0$, strictly below the stippled curve, $k_{t+1} = f(k_t)/(1+n)$, in the figure. To be precise:

PROPOSITION 5 *(ceiling and roof)* Suppose the No Fast assumption (A1) applies. Along an equilibrium path, whenever $k_t > 0$,

$$0 < k_{t+1} < \frac{w(k_t)}{1+n} < \frac{f(k_t)}{1+n}, \qquad t = 0, 1, \ldots.$$

Proof. From (iii) of Proposition 2, an equilibrium path has $w_t = w(k_t) > 0$ and $k_{t+1} > 0$ for $t = 0, 1, 2, \ldots$. Thus,

$$0 < k_{t+1} = \frac{s_t}{1+n} < \frac{w_t}{1+n} = \frac{w(k_t)}{1+n} = \frac{f(k_t) - f'(k_t)k_t}{1+n} < \frac{f(k_t)}{1+n},$$

where the first equality comes from (3.32), the second inequality from Lemma 1 in Section 3.3, and the last inequality from the fact that $f'(k_t)k_t > 0$ when $k_t > 0$. $\square$

We will call the graph $(k_t, w(k_t)/(1+n))$ in Fig. 3.6 a *ceiling*. It acts as a ceiling on k_{t+1} simply because the saving of the young cannot exceed the income of the young, $w(k_t)$. And we will call the graph $(k_t, f(k_t)/(1+n))$ a *roof,* because "everything of interest" occurs below it. The roof is easy to draw directly on the basis of the production function $f(k_t)$.

To characterize the position of the roof relative to the 45° line, we consider the lower Inada condition, $\lim_{k\to 0} f'(k) = \infty$.

LEMMA 2 The roof, $\mathcal{R}(k) \equiv f(k)/(1+n)$, has positive slope everywhere, crosses the 45° line for at most one $k > 0$ and can only do that from above. A necessary and sufficient condition for the *roof* to be above the 45° line for

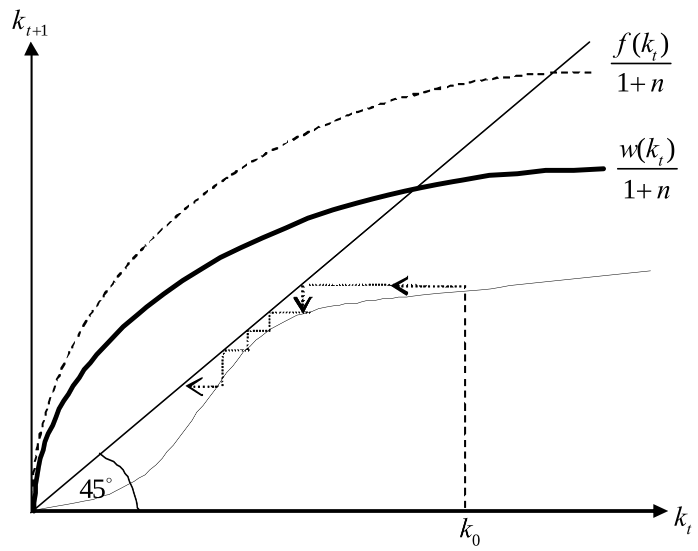

Figure 3.6: A case where both the roof and the ceiling cross the 45° line, but the transition curve does not (no steady state exists).

small k is that either $\lim_{k\to 0} f'(k) > 1+n$ or $f(0) > 0$ (capital not essential).

Proof. Since $f' > 0$, the roof has positive slope. Since $f'' < 0$, it can only cross the 45° line once and only from above. If and only if $\lim_{k\to 0} f'(k) > 1 + n$, then for small k_t, the roof is steeper than the 45° line. Obviously, if $f(0) > 0$, then close to the origin, the roof will be above the 45° line. $\square$

LEMMA 3 Given $w(k) = f(k) - f'(k)k$ for all $k \geq 0$, where $f(k)$ satisfies $f(0) \geq 0$, $f' > 0$, $f'' < 0$, the following holds:
(i) $\lim_{k\to\infty} w(k)/k = 0$;
(ii) the ceiling, $\mathcal{C}(k) \equiv w(k)/(1+n)$, is positive and has positive slope for all $k > 0$; moreover, there exists $\bar{k} > 0$ such that $\mathcal{C}(k) < k$ for all $k > \bar{k}$.

Proof. (i) In view of $f(0) \geq 0$ combined with $f'' < 0$, we have $w(k) > 0$ for all $k > 0$ (as noted several times before). Hence, $\lim_{k\to\infty} w(k)/k \geq 0$ if this limit exists. Consider an arbitrary $k_1 > 0$. We have $f'(k_1) > 0$. For all $k > k_1$, it holds that $0 < f'(k) < f'(k_1)$, in view of $f' > 0$ and $f'' < 0$. Hence, $\lim_{k\to\infty} f'(k)$ exists and

$$0 \leq \lim_{k\to\infty} f'(k) < f'(k_1). \tag{3.38}$$

We have

$$\lim_{k\to\infty} \frac{w(k)}{k} = \lim_{k\to\infty} \frac{f(k)}{k} - \lim_{k\to\infty} f'(k). \tag{3.39}$$

There are two cases to consider. Case 1: $f(k)$ has an upper bound. Then, $\lim_{k\to\infty} f(k)/k = 0$ so that $\lim_{k\to\infty} w(k)/k = -\lim_{k\to\infty} f'(k) = 0$, by (3.39) and (3.38), as $w(k)/k > 0$ for all $k > 0$. Case 2: $\lim_{k\to\infty} f(k) = \infty$. Then, by L'Hôpital's rule for "∞/∞", $\lim_{k\to\infty}(f(k)/k) = \lim_{k\to\infty} f'(k)$ so that (3.39) implies $\lim_{k\to\infty} w(k)/k = 0$.

(ii) As $n > -1$ and $w(k) > 0$ for all $k > 0$, $\mathcal{C}(k) > 0$ for all $k > 0$. From $w'(k) = -kf''(k) > 0$ follows $\mathcal{C}'(k) = -kf''(k)/(1+n) > 0$ for all $k > 0$; that is, the ceiling has positive slope everywhere. For $k > 0$, the inequality $\mathcal{C}(k) < k$ is equivalent with $w(k)/k < 1+n$. By (i) follows that for all $\varepsilon > 0$, there exists $k_\varepsilon > 0$ such that $w(k)/k < \varepsilon$ for all $k > k_\varepsilon$. Now, letting $\varepsilon = 1+n$ and $\bar{k} = k_\varepsilon$ proves that there exists $\bar{k} > 0$ such that $w(k)/k < 1+n$ for all $k > \bar{k}$. $\square$

In view of the ceiling being always an upper bound on k_{t+1}, what is the point of also introducing the roof? The point is that the roof is a more convenient construct since it is directly given from the production function and is always strictly concave. The ceiling is generally a more complex construct and can have convex sections.

The roof being above the 45^0 line for small k_t is not sufficient for also the transition curve to be above the 45^0 line for small k_t. Fig. 3.6 illustrates this. Here the transition curve is in fact everywhere below the 45^0 line. In this case no steady state exists and the dynamics imply convergence towards the "catastrophic" point $(0,0)$. Given the rate of population growth, the saving of the young is not sufficient to avoid famine in the long run. This will for example happen if the technology implies so low productivity that even if all income of the young were saved, we would have $k_{t+1} < k_t$ for all $k_t > 0$, cf. Exercise 3.2. The Malthusian mechanism will be at work and bring down n (outside the model). This exemplifies that even a trivial steady state (the point (0,0)) may be of interest in so far as it may be the point the economy is heading to without ever reaching it.

To help existence of a steady state we will impose the condition that either capital is not essential or preferences and technology fit together in such a way that the slope of the transition curve is larger than one for small k_t. That is, we assume that either

$$\begin{array}{llcll} \text{(i)} & f(0) & > & 0 \quad \text{or} & \text{(A3)} \\ \text{(ii)} & \lim_{k_t\to 0} \varphi'(k_t) & > & 1, & \end{array}$$

where $\varphi'(k_t)$ is implicitly given in (3.37). There are cases where we can find an explicit transition function and try out whether (i) or (ii) of (A3) holds (like in Example 2 above). But generally we can not. Then we may state

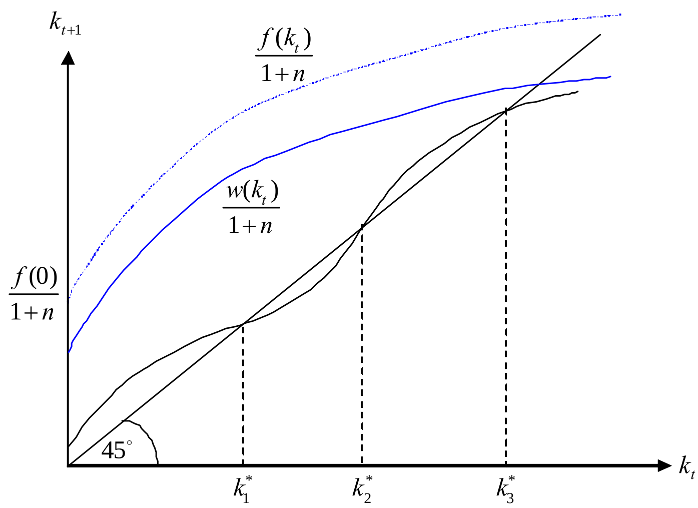

Figure 3.7: A case of multiple steady states (one of which is a poverty trap).

sufficient conditions for (A3), expressed in terms of the "primitives". For example, if the period utility function belongs to the CRRA class and the production function is Cobb-Douglas at least for small k_t, then (ii) of (A3) holds (see Appendix E).

PROPOSITION 6 *(existence and stability of a steady state)* Assume that the No Fast assumption (A1) and the Positive Slope assumption (A2) apply as well as assumption (A3). Then there exists at least one locally asymptotically stable steady state $k^* > 0$. Oscillations do not occur.

Proof. By (A1) Proposition 5 holds. From Proposition 2 we know that if (i) of (A3) holds, then $k_{t+1} = s_t/(1+n) > 0$ even for $k_t = 0$. Alternatively, (ii) of (A3) is enough to ensure that the transition curve lies above the 45° line for small k_t. By Lemma 2 the roof then does the same. According to (ii) of Lemma 3, for large k_t the ceiling is below the 45° line. Being below the ceiling, cf. Proposition 5, the transition curve must therefore cross the 45° line at least once. Let k^* denote the smallest k_t at which it crosses. Then $k^* > 0$ is a steady state with the property $0 < \varphi'(k^*) < 1$. By graphical inspection we see that this steady state is asymptotically stable. For oscillations to come about there must exist a steady state, k^{**}, with $\varphi'(k^{**}) < 0$, but this is impossible in view of (A2). □

We conclude that, given k_0, the assumptions (A1) - (A3) ensure existence, uniqueness, and convergence of the equilibrium path towards *some* steady state. Thus with these assumptions, for any $k_0 > 0$, sooner or later the system settles down at some steady state $k^* > 0$. For the factor prices we therefore have

$$\begin{aligned} r_t &= f'(k_t) - \delta \to f'(k^*) - \delta \equiv r^*, \quad \text{and} \\ w_t &= f(k_t) - k_t f'(k_t) \to f(k^*) - k^* f'(k^*) \equiv w^*, \end{aligned}$$

for $t \to \infty$. But there may be more than one steady state and therefore only *local* stability is guaranteed. This can be shown by examples, where the utility function, the production function, and parameters are specified in accordance with the assumptions (A1) - (A3) (see Exercise 3.5).

Fig. 3.7 illustrates such a case (with $f(0) > 0$ so that capital is not essential). Moving West-East in the figure, the first steady state, k_1^*, is stable, the second, k_2^*, unstable, and the third, k_3^*, stable. In which of the two stable steady states the economy ends up depends on the initial capital-labor ratio, k_0. The lower steady state, k_1^*, is known as a *poverty trap*. If $0 < k_0 < k_2^*$, the economy is caught in the trap and converges to the low steady state. But with high enough k_0 ($k_0 > k_2^*$), perhaps obtained by foreign aid, the economy avoids the trap and converges to the high steady state. Looking back at Fig. 3.6, we can interpret that figure's scenario as exhibiting an *inescapable* poverty trap.

It turns out that CRRA utility combined with a Cobb-Douglas production function ensures both that (A1) - (A3) hold and that a unique (non-trivial) steady state exists. So in this case *global* asymptotic stability of the steady state is ensured.[18] Example 2 and Fig. 3.4 above display a special case of this, the case $\theta = 1$.

This is of course a convenient case for the analyst. A Diamond model satisfying assumptions (A1) - (A3) *and* featuring a unique steady state is called a *well-behaved Diamond model.*

Before proceeding to the next section we need check the "technical" question whether along an equilibrium path aggregate consumption might for some periods exceed aggregate income. We shall see that this is indeed possible in the model if K_0 (financial wealth of the old in the initial period) is large enough.

Indeed, from national accounting we have:

$$C_{10}+C_{20} = F(K_0, L_0)-I_0 > F(K_0, L_0) \Leftrightarrow I_0 < 0 \Leftrightarrow K_1 < (1-\delta)K_0 \Leftrightarrow K_0-K_1 > \delta K_0.$$

So aggregate consumption in period 0 being greater than aggregate income is equivalent with a fall in the capital stock from period 0 to period 1 greater

[18] See last section of Appendix E.

than the capital depreciation in period 0. Consider the log utility Cobb-Douglas case in Fig. 3.4 and suppose $n = 0$ and $\delta < 1$. Then $k_t = K_t$ for all t and by (3.35) $K_{t+1} = \frac{(1-\alpha)A}{2+\rho} K_t^{\alpha}$. Thus $K_1 < (1-\delta)K_0$ for

$$K_0 > \left(\frac{(1-\alpha)A}{(2+\rho)(1-\delta)} \right)^{1/(1-\alpha)}.$$

As initial K is arbitrary, this situation is possible. When it occurs, it reflects that the financial wealth of the old is so large that their consumption (recall they consume all their financial wealth as well as the interest on this wealth) exceeds what is left of current aggregate production after subtracting the amount consumed by the young. So aggregate gross investment in the economy will be negative. Of course this is only feasible if capital goods can be converted back into consumption goods. We assume this to be the case. In some sense this is in line with the general setup since the output good is homogeneous and can either be consumed or piled up as capital.

We now turn to efficiency problems.

3.6 The golden rule and dynamic inefficiency

An economy described by the Diamond model has the property that even though there is perfect competition and no externalities, the outcome brought about by the market mechanism may not be Pareto optimal.[19] Indeed, the economy may *overaccumulate* forever and thus suffer from a distinctive form of production inefficiency.

The key element in understanding the concept of overaccumulation is the concept of a *golden rule capital-labor ratio*. Overaccumulation occurs when aggregate saving maintains a capital-labor ratio above the golden rule value forever. Let us consider these concepts in detail.

The golden-rule capital-labor ratio

The golden rule is a principle that relates to technically feasible paths. The principle does not depend on the market form.

Consider the economy-wide resource constraint $C_t = Y_t - S_t = F(K_t, L_t) - (K_{t+1} - K_t + \delta K_t)$. Accordingly, aggregate consumption per unit of labor can

[19] Recall that a *Pareto optimal* state or path is a technically feasible path with the property that no other technically feasible path will make at least one individual better off without making someone else worse off. A technically feasible path which is not Pareto optimal is called *Pareto inferior*.

be written

$$c_t \equiv \frac{C_t}{L_t} = f(k_t) + (1-\delta)k_t - (1+n)k_{t+1}. \tag{3.40}$$

Note that C_t will generally be greater than the workers' consumption. One should simply think of C_t as the flow of produced consumption goods in the economy and c_t as this flow divided by aggregate employment, including the labor that produces the necessary capital goods. How the consumption goods are distributed to different members of society is not our concern here.

DEFINITION 5 By the *golden rule capital-labor ratio*, k_{GR}, is meant that capital-labor ratio $k \equiv K/L$ which results in the highest sustainable consumption level per unit of labor.

Sustainability requires replicability forever. We therefore consider a steady state. In a steady state $k_{t+1} = k_t = k$ so that (3.40) simplifies to

$$c = f(k) - (\delta + n)k \equiv c(k). \tag{3.41}$$

Maximizing gives the first-order condition

$$c'(k) = f'(k) - (\delta + n) = 0. \tag{3.42}$$

In view of $c''(k) = f''(k) < 0$, the condition (3.42) is both necessary and sufficient for an interior maximum. Assuming $\delta + n > 0$ and that f satisfies the Inada conditions, (3.42) has a solution in k (and it is unique because $c''(k) < 0$). The solution is called k_{GR} so that

$$f'(k_{GR}) - \delta = n.$$

That is:

PROPOSITION 7 *(the golden rule)* The highest sustainable consumption level per unit of labor in society is obtained when in steady state the net marginal productivity of capital equals the growth rate of the economy.

It follows that if a society aims at the highest sustainable level of consumption, it should increase its capital-labor ratio up to the point where the extra output obtainable by a further small increase is exactly offset by the extra gross investment needed to maintain the capital-labor ratio at that level. The intuition is visible from (3.41). The golden rule capital-labor ratio, k_{GR}, strikes the right balance in the trade-off between high output per unit of labor and a not too high investment requirement. Although a steady state with $k > k_{GR}$ would imply higher output per unit of labor, it would also imply that a large part of that output is set aside for investment (namely the amount $(\delta + n)k$ per unit of labor) to counterweigh capital depreciation and

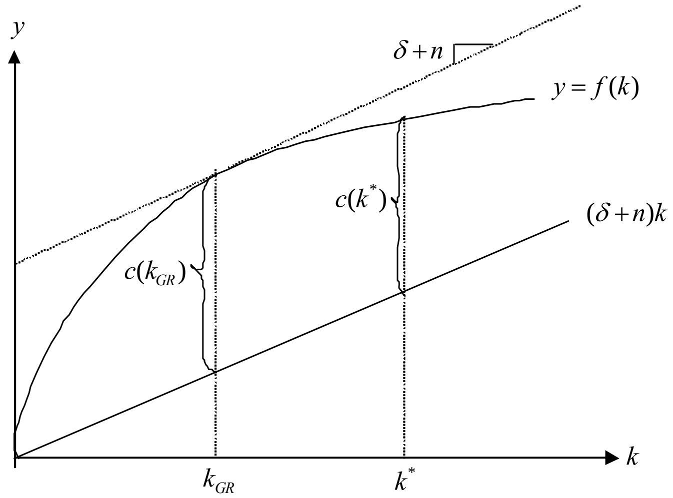

Figure 3.8: A steady state with overaccumulation.

growth in the labor force. Without this investment the high capital-labor ratio k^* would not be maintained. With $k > k_{GR}$ this feature would dominate the first effect so that consumption per unit of labor ends up low. Fig. 3.8 illustrates.

The name golden rule hints at the golden rule from the Bible: "Do unto others as you would have them to do unto you." We imagine that God asks the newly born generation: "What capital-labor ratio would you prefer to be presented with, given that you must hand over the same capital-labor ratio to the next generation?" The appropriate answer is: the golden rule capital-labor ratio!

The possibility of overaccumulation in a competitive market economy

The equilibrium path in the Diamond model with perfect competition implies an interest rate $r^* = f'(k^*) - \delta$ in a steady state. As an implication,

$$r^* \gtreqqless n \Leftrightarrow f'(k^*) - \delta \gtreqqless n \Leftrightarrow k^* \lesseqqgtr k_{GR}, \text{ respectively,}$$

in view of $f'' < 0$. Hence, a long-run interest rate below the growth rate of the economy indicates that $k^* > k_{GR}$. This amounts to a Pareto-inferior state of affairs. Indeed, everyone can be made better off if by a coordinated reduction of saving and investment, k is reduced. A formal proof of this is given in connection with Proposition 8 in the next subsection. Here we give an account in more intuitive terms.

Consider Fig. 3.8. Let k be gradually reduced to the level k_{GR} by refraining from investment in period t_0 and forward until this level is reached. When this happens, let k be maintained at the level k_{GR} forever by providing for the needed investment per young, $(\delta + n)k_{GR}$. Then there would be higher aggregate consumption in period t_0 and every future period. Both the immediate reduction of saving and a resulting lower capital-labor ratio to be maintained contribute to this result. There is thus scope for both young and old to consume more in every future period.

A simple policy implementing such a Pareto improvement in the case where $k^* > k_{GR}$ (i.e., $r^* < n$) is to incur a lump-sum tax on the young, the revenue of which is immediately transferred lump sum to the old, hence, fully consumed.[20] Suppose this amounts to a transfer of one good from each young to the old. Since there are (on average) $1 + n$ young people for each old person, every old receives in this way $1 + n$ goods in the same period. Let this transfer be repeated every future period. By decreasing their saving by one unit, the young can maintain unchanged consumption in their youth, and when becoming old, they receive $1 + n$ goods from the next period's young and so on. In effect, the "return" on the initial payment by the young is $1 + n$, which is more than the $1 + r^*$ that could be obtained through own saving.

A proof that $k^* > k_{GR}$ is indeed theoretically possible can be based on the log utility-Cobb-Douglas case from Example 2 in Section 3.5.3. As indicated by the formula for r^* in that example, the outcome $r^* < n$, which is equivalent to $k^* > k_{GR}$, can always be obtained by making the parameter $\alpha \in (0, 1)$ in the Cobb-Douglas function small enough. The intuition is that a small α implies a high $1 - \alpha$, that is, a high wage income $wL = (1 - \alpha)K^{\alpha}L^{-\alpha} \cdot L = (1 - \alpha)Y$; this leads to high saving by the young, since $s_w > 0$. The result is a high k_{t+1} which generates a high real wage also next period and may in this manner be sustained forever.

An intuitive understanding of the fact that the perfectly competitive market mechanism can thus lead to overaccumulation, can also be based on the following argument. Assume, first, that $s_r < 0$. In this case, if the young in period t expects the rate of return on their saving to end up small (less than n), the decided saving will be large in order to provide for consumption after retirement. But the aggregate result of this behavior is a high k_{t+1} and therefore a low $f'(k_{t+1})$. In this way the expectation of a low r_{t+1} is confirmed by the actual events. The young persons each do the best they can as atom-

[20] In this model with no utility of leisure, a tax on wage income, or a mandatory pay-as-you-go pension contribution (see Chapter 5) would act like a lump-sum tax on the young.

istic individuals, taking the market conditions as given. Yet the aggregate outcome is an equilibrium with overaccumulation, hence a Pareto-inferior outcome.

Looking at the issue more closely, we see that $s_r < 0$ is not crucial for this outcome. Suppose $s_r = 0$ (the log utility case) and that in the current period, k_t is, for some historical reason, considerably above k_{GR}. Thus, current wages are high, hence, s_t is relatively high (there is in this case no offsetting effect on s_t from the relatively low expected r_{t+1}). Again, the aggregate result is a high k_{t+1} and thus the expectation is confirmed. Consequently, the situation in the next period is the same and so on. By continuity, even if $s_r > 0$, the argument goes through as long as s_r is not too large.

Dynamic inefficiency and the double infinity

Another name for the overaccumulation phenomenon is *dynamic inefficiency.*

DEFINITION 6 Any technically feasible path $\{(c_t, k_t)\}_{t=0}^{\infty}$ such that there does not exist another technically feasible path with higher c_t in some periods without smaller c_t in other periods is called *dynamically efficient.* A technically feasible path $\{(c_t, k_t)\}_{t=0}^{\infty}$ which is not dynamically efficient is called *dynamically inefficient.*

PROPOSITION 8 Any technically feasible path $\{(c_t, k_t)\}_{t=0}^{\infty}$ such that for $t \to \infty$, $k_t \to k^* > k_{GR}$, is dynamically inefficient.

Proof. Let $k^* > k_{GR}$. Then there exists an $\varepsilon > 0$ such that $k \in (k^* - 2\varepsilon, k^* + 2\varepsilon)$ implies $f'(k) - \delta < n$ since $f'' < 0$. By concavity of f,

$$f(k) - f(k - \varepsilon) \leq f'(k - \varepsilon)\varepsilon. \tag{3.43}$$

Consider a technically feasible path $\{(c_t, k_t)\}_{t=0}^{\infty}$ with $k_t \to k^*$ for $t \to \infty$ (the reference path). Then there exists a t_0 such that for $t \geq t_0$, $k_t \in (k^* - \varepsilon, k^* + \varepsilon)$, $f'(k_t) - \delta < n$ and $f'(k_t - \varepsilon) - \delta < n$. Consider an alternative feasible path $\left\{(\hat{c}_t, \hat{k}_t)\right\}_{t=0}^{\infty}$, where (a) for $t = t_0$ consumption is increased relative to the reference path such that $\hat{k}_{t_0+1} = k_{t_0} - \varepsilon$; and (b) for all $t > t_0$, consumption is such that $\hat{k}_{t+1} = k_t - \varepsilon$. We now show that after period t_0, $\hat{c}_t > c_t$. Indeed, for all $t > t_0$, by (3.40),

$$\begin{aligned}
\hat{c}_t &= f(\hat{k}_t) + (1-\delta)\hat{k}_t - (1+n)\hat{k}_{t+1} \\
&= f(k_t - \varepsilon) + (1-\delta)(k_t - \varepsilon) - (1+n)(k_{t+1} - \varepsilon) \\
&\geq f(k_t) - f'(k_t - \varepsilon)\varepsilon + (1-\delta)(k_t - \varepsilon) - (1+n)(k_{t+1} - \varepsilon) \quad \text{(by (3.43))} \\
&> f(k_t) - (\delta + n)\varepsilon + (1-\delta)k_t - (1+n)k_{t+1} + (\delta + n)\varepsilon \\
&= f(k_t) + (1-\delta)k_t - (1+n)k_{t+1} = c_t,
\end{aligned}$$

by (3.40)). □

Moreover, it can be shown that:

PROPOSITION 9 Any technically feasible path $\{(c_t, k_t)\}_{t=0}^{\infty}$ such that for $t \to \infty$, $k_t \to k^* \leq k_{GR}$, is dynamically efficient.[21]

Accordingly, a steady state with $k^* < k_{GR}$ is never dynamically inefficient. This is because increasing k from this level always has its price in terms of a decrease in *current* consumption; and at the same time decreasing k from this level always has its price in terms of lost *future* consumption. But a steady state with $k^* > k_{GR}$ is always dynamically inefficient.

The fact that $k^* > k_{GR}$, and therefore dynamic inefficiency, cannot be ruled out might seem to contradict the First Welfare Theorem. This is the theorem saying that when increasing returns to scale are absent, markets are competitive and complete, no goods are of public good character, and there are no other kinds of externalities, then market equilibria are Pareto optimal. In fact, however, the First Welfare Theorem also presupposes a finite number of periods or, if the number of periods is infinite, then a finite number of agents.[22] In contrast, in the OLG model there is a *double infinity*: an infinite number of periods *and* agents. Hence, the First Welfare Theorem breaks down. Indeed, the case $r^* < n$, i.e., $k^* > k_{GR}$, can arise under *laissez-faire.* Then, as we have seen, everyone can be made better off by a coordinated intervention by some social arrangement (a government for instance) such that k is reduced.

The essence of the matter is that the double infinity opens up for technically feasible reallocations which are definitely beneficial when $r^* < n$ and which a central authority can accomplish but the market can not. That *nobody* need loose by the described kind of redistribution is due to the double infinity: the economy goes on forever and there is no last generation. Nonetheless, some kind of centralized *coordination* is required to accomplish a solution.

As noted by Shell (1971) there is an analogy in "Gamow's bed problem": There are an infinite number of inns along the road, each with one bed. On a certain rainy night all innkeepers have committed their beds. A late guest comes to the first inn and asks for a bed. "Sorry, full up!" But the minister

[21] Both this proposition and Proposition 8 were shown in a stronger form by the American economist David Cass (1937-2008). Cass established the *general* necessary and sufficient condition for a feasible path $\{(c_t, k_t)\}_{t=0}^{\infty}$ to be dynamically efficient (Cass 1972). Our two propositions are more restrictive in that they are limited to paths that converge.

[22] As we shall see later in this book, in the Ramsey model (which has a finite number of infinitely lived households or "dynasties") a situation with $r^* < n$ (dynamic inefficiency) cannot be a market equilibrium.

of welfare hears about it and suggests that each guest move down the road one inn.[23]

Whether the theoretical possibility of overaccumulation should be a matter of practical concern is an empirical question about the relative size of rates of return and economic growth. To answer the question meaningfully we need an extension of the criterion for overaccumulation such that the presence of technological progress and rising per capita consumption in the long run is taken into account. This is one of the topics of the next chapter. We can already here, however, reveal that there is no indication that overaccumulation has ever been an actual problem in industrialized market economies.

A final remark before concluding seems in place. It can be shown that Proposition 7 about the golden rule can be generalized to the case where instead of one there are n different capital goods in the economy. Essentially the generalization says that assuming concave neoclassical production functions with n different capital goods as inputs, one consumption good, and perfectly competitive markets, a steady state in which per capita consumption is maximized has interest rate equal to the growth of the labor force (Mas-Colell, 1989).

3.7 Concluding remarks

(Unfinished)

In several respects the conclusions we get from OLG models are different than those from representative agent models (RA models for short). The underlying reason is that in an RA model, many aggregate quantities are just a multiple of the actions of the representative household. In OLG models this is not so; the aggregate quantities are the outcome of the interplay of agents at different stages in their life; further, the turnover in the population plays a crucial role. The OLG approach lays bare the possibility of coordination failure on a grand scale.

Regarding analytical tractability, the complexity implied by having in every period two different coexisting generations is in some respects more than compensated by the fact that the finite time horizon of the households make the *dynamics* of the model *one-dimensional*: we end up with a first-order difference equation in the capital-labor ratio, k_t, in the economy. In contrast, the dynamics of the basic representative agent model (Chapter 10)

[23] George Gamow (1904-1968) was a Russian physicist. The problem is also known as *Hilbert's hotel problem*, after the German mathematician David Hilbert (1862-1943).

is two-dimensional (owing to the assumed infinite horizon of the households considered as dynasties).

3.8 Literature notes

1. The Nobel Laureate Paul A. Samuelson is one of the pioneers of OLG models. In his famous article, Samuelson (1958), he was concerned with a kind of financial market imperfection. Imagine a two-period OLG economy where, as in the Diamond model, only the young have an income (which in turn is, by Samuelson, assumed exogenous). Contrary to the Diamond model, however, there is neither capital nor other stores of value. Then, in the laissez-faire market economy the old have to starve. This is clearly a Pareto-inferior allocation; if each member of the young generation hands over to the old generation one unit of account, and this is repeated every period, everyone will be better off. Since for every old there are $1 + n$ young, the implied rate of return would be n, the population growth rate. Such transfers do not arise under laissez-faire. A kind of social contract is required. As Samuelson pointed out, a government could in period 0 issue paper notes, "money", and transfer them to the members of the old generation who would then use them to buy goods from the young. Provided the young believed the notes to be valuable in the next period, they would accept them in exchange for some of their goods in order to use them in the next period for buying from the new young generation and so on. We have here an example of how a social institution can solve a coordination problem.

This gives a flavour of Samuelson's contribution although in his original article he assumed three periods of life.

2. Diamond (1965) extended Samuelson's contribution by adding capital accumulation to the model. In our exposition of Diamond's OLG model above we draw upon clarifications by Galor and Ryder (1989) and de la Croix and Michel (2002). The last mentioned contribution is an extensive exploration of discrete-time OLG models and their applications.

3. Diamond has also contributed to other fields in economics, including search theory for labor markets. In 2010 Diamond, together with Dale Mortensen and Christopher Pissarides, was awarded the Nobel price in economics.

4. A review of the empirics of life-cycle behavior and attempts at refining the life-cycle model are given in Browning and Crossley (2001).

From here very incomplete:

Multiple equilibria, self-fulfilling expectations, optimism and pessimism..

Dynamic inefficiency, see also Burmeister (1980).

Two-sector OLG: Galor (1992). Galor's book??
On the golden rule, see Mas-Colell (1989).

3.9 Appendix

A. On the CRRA utility function

Derivation of the CRRA function Consider a utility function $u(c)$, defined for all $c > 0$ and satisfying $u'(c) > 0$, $u''(c) < 0$. Let the absolute value of the elasticity of marginal utility be denoted $\theta(c)$, that is, $\theta(c) \equiv -cu''(c)/u'(c) > 0$. We claim that if $\theta(c)$ is a positive constant, θ, then up to a positive linear transformation $u(c)$ must be of the form

$$u(c) = \begin{cases} \frac{c^{1-\theta}}{1-\theta}, & \text{when } \theta \neq 1, \\ \ln c, & \text{when } \theta = 1, \end{cases} \tag{*}$$

i.e., of CRRA form.

Proof. Suppose $\theta(c) = \theta > 0$. Then, $u''(c)/u'(c) = -\theta/c$. By integration, $\ln u'(c) = -\theta \ln c + A$, where A is an arbitrary constant. Take the antilogarithm function on both sides to get $u'(c) = e^A e^{-\theta \ln c} = e^A c^{-\theta}$. By integration we get

$$u(c) = \begin{cases} e^A \frac{c^{1-\theta}}{1-\theta} + B, & \text{when } \theta \neq 1, \\ e^A \ln c + B, & \text{when } \theta = 1, \end{cases}$$

where B is an arbitrary constant. This proves the claim. Letting $A = B = 0$, we get (*). □

When we want to make the kinship between the members of the CRRA family transparent, we maintain $A = 0$ and for $\theta = 1$ also $B = 0$, whereas for $\theta \neq 1$ we set $B = -1/(1-\theta)$. In this way we achieve that all members of the CRRA family will be represented by curves going through the same point as the log function, namely the point $(1, 0)$, cf. Fig. 3.2. And adding or subtracting a constant does not affect marginal rates of substitution and consequently not behavior.

The domain of the CRRA function We want to extend the domain to include $c = 0$. If $\theta \geq 1$, the CRRA function, whether in the form $u(c) = (c^{1-\theta} - 1)/(1-\theta)$ or in the form (*), is defined only for $c > 0$, not for $c = 0$. This is because for $c \to 0$ we get $u(c) \to -\infty$. In this case we simply define $u(0) = -\infty$. This will create no problems since the CRRA function anyway has the property that $u'(c) \to \infty$, when $c \to 0$ (whether θ is larger or smaller than one). The marginal utility thus becomes very large as c becomes very small, that is, the No Fast assumption is satisfied. This will ensure that the chosen c is strictly positive whenever there is a positive budget. So we define the domain of the CRRA function to be $[0, \infty)$.

The range of the CRRA function Considering the CRRA function $u(c) \equiv \left(c^{1-\theta} - 1\right)(1-\theta)^{-1}$ for $c \in [0, \infty)$, we have:

$$
\begin{aligned}
\text{for } 0 &< \theta < 1, \text{ the range of } u(c) \text{ is } \left[-(1-\theta)^{-1}, \infty\right), \\
\text{for } \theta &= 1, \text{ the range of } u(c) \text{ is } [-\infty, \infty), \\
\text{for } \theta &> 1, \text{ the range of } u(c) \text{ is } [-\infty, -(1-\theta)^{-1}).
\end{aligned}
$$

Thus, in the latter case $u(c)$ is bounded above and so allows asymptotic "saturation" to occur.

B. Deriving the elasticity of intertemporal substitution in consumption

Referring to Section 3.3, we here show that the definition of $\sigma(c_1, c_2)$ in (3.17) gives the result (3.18). Let $x \equiv c_2/c_1$ and $\beta \equiv (1+\rho)^{-1}$. Then the first-order condition (3.16) and the equation describing the considered indifference curve constitute a system of two equations

$$
\begin{aligned}
u'(c_1) &= \beta u'(xc_1)R, \\
u(c_1) + \beta u(xc_1) &= \bar{U}.
\end{aligned}
$$

For a fixed utility level $U = \bar{U}$ these equations define c_1 and x as implicit functions of R, $c_1 = c(R)$ and $x = x(R)$. We calculate the total derivative w.r.t. R in both equations and get, after ordering,

$$
\begin{aligned}
& [u''(c_1) - \beta R u''(xc_1)x]\, c'(R) - \beta R u''(xc_1) c_1 x'(R) \\
&= \beta u'(xc_1), \qquad (3.44) \\
[u'(c_1) + \beta u'(xc_1)x]\, c'(R) &= -\beta u'(xc_1) c_1 x'(R). \qquad (3.45)
\end{aligned}
$$

Substituting $c'(R)$ from (3.45) into (3.44) and ordering now yields

$$
-\left[x\frac{c_1 u''(c_1)}{u'(c_1)} + R\frac{xc_1 u''(xc_1)}{u'(xc_1)}\right]\frac{R}{x}x'(R) = x + R.
$$

Since $-cu''(c)/u'(c) \equiv \theta(c)$, this can be written

$$
\frac{R}{x}x'(R) = \frac{x + R}{x\theta(c_1) + R\theta(xc_1)}.
$$

Finally, in view of $xc_1 = c_2$ and the definition of $\sigma(c_1, c_2)$, this gives (3.18).

C. Walras' law

In the proof of Proposition 1 we referred to Walras' law. Here is how Walras' law works in each period in a model like this. We consider period t, but for simplicity we skip the time index t on the variables. Suppose a "Walrasian auctioneer" calls out the price vector $(\hat{r}, w, 1)$, where $\hat{r} > 0$ and $w > 0$, and asks all agents, i.e., the young, the old, and the representative firm, to declare their supplies and demands.

The supplies of capital and labor are by assumption inelastic and equal to K units of capital services and L units of labor services. But the demand for capital and labor depends on the announced $\hat{r}$ and w. Let the potential pure profit of the representative firm be denoted Π. If $\hat{r}$ and w are so that $\Pi < 0$, the firm declares $K^d = 0$ and $L^d = 0$. If on the other hand at the announced $\hat{r}$ and w, $\Pi = 0$ (as when $\hat{r} = r(k) + \delta$ and $w = w(k)$), the desired capital-labor ratio is given as $k^d = f'^{-1}(\hat{r})$ from (3.20), but the firm is indifferent w.r.t. the absolute level of the factor inputs. In this situation the auctioneer tells the firm to declare $L^d = L$ (recall L is the given labor supply) and $K^d = k^d L^d$ which is certainly acceptable for the firm. Finally, if $\Pi > 0$, the firm is tempted to declare infinite factor demands, but to avoid that, the auctioneer imposes the rule that the maximum allowed demands for capital and labor are $2K$ and $2L$, respectively. Within these constraints the factor demands will be uniquely determined by $\hat{r}$ and w and we have

$$\Pi = \Pi(\hat{r}, w, 1) = F(K^d, L^d) - \hat{r}K^d - wL^d. \tag{3.46}$$

The owners of both the capital stock K and the representative firm must be those who saved in the previous period, namely the currently old. These elderly will together declare the consumption $c_2 L_{-1} = (1 + \hat{r} - \delta)K + \Pi$ and the net investment $-K$ (which amounts to disinvestment). The young will declare the consumption $c_1 L = wL - s(w, r^e_{+1})L$ and the net investment $sL = s(w, r^e_{+1})L$. So aggregate declared consumption will be $C = (1 + \hat{r} - \delta)K + \Pi + wL - s(w, r^e_{+1})L$ and aggregate net investment $I - \delta K = s(w, r^e_{+1})L - K$. It follows that $C + I = wL + \hat{r}K + \Pi$. The aggregate declared supply of output is $Y^s = F(K^d, L^d)$. The values of excess demands in the three markets now add to

$$\begin{aligned}
Z(\hat{r}, w, 1) &\equiv w(L^d - L) + \hat{r}(K^d - K) + C + I - Y^s \\
&= wL^d - wL + \hat{r}K^d - \hat{r}K + wL + \hat{r}K + \Pi - F(K^d, L^d) \\
&= wL^d + \hat{r}K^d + \Pi - F(K^d, L^d) = 0,
\end{aligned}$$

by (3.46).

This is a manifestation of Walras' law for each period: *whatever the announced price vector for the period is, the aggregate value of excess demands* *in the period is zero.* The reason is the following. When each household satisfies its budget constraint and each firm pays out its pure profit, then the economy as a whole has to satisfy an aggregate budget constraint for the period considered.

The demands and budget constraints operating in this thought experiment (and in Walras' law in general) are the *Walrasian* demands and *Walrasian* budget constraints. Outside equilibrium these are somewhat artificial constructs. A Walrasian budget constraint, for instance, is based on the assumption that the desired actions can be realized. This assumption will be wrong unless $\hat{r}$ and w are already at their equilibrium levels. But the assumption that desired actions can be realized is never falsified because the thought experiment does not allow trades to take place outside equilibrium. Similarly, the Walrasian consumption demand by the worker is rather hypothetical outside equilibrium. This demand is based on the income the worker *would* get if fully employed at the announced real wage, not on the actual employment (or unemployment) at that real wage.

These ambiguities notwithstanding, the important message of Walras' law goes through, namely that when two of the three markets clear, so does the third.

D. Proof of (i) and (ii) of Proposition 2

For convenience we repeat the fundamental difference equation characterizing an equilibrium path:

$$k_{t+1} = \frac{s\left(w\left(k_t\right), r\left(k_{t+1}\right)\right)}{1+n},$$

where $w(k) \equiv f(k) - f'(k)k > 0$ for all $k > 0$ and $r(k) \equiv f'(k) - \delta > -1$ for all $k \geq 0$. The key to the proof of Proposition 2 about existence of an equilibrium path is the following lemma.

LEMMA D1 Suppose the No Fast assumption (A1) applies and let $w > 0$ and $n > -1$ be given. Then the equation

$$\frac{s\left(w, r\left(k\right)\right)}{k} = 1 + n. \tag{3.47}$$

has at least one solution $k > 0$.

Proof. Note that $1 + n > 0$. From Lemma 1 in Section 3.3 follows that for all possible values of $r(k)$, $0 < s(w, r(k)) < w$. Hence, for any $k > 0$,

$$0 < \frac{s\left(w, r\left(k\right)\right)}{k} < \frac{w}{k}.$$

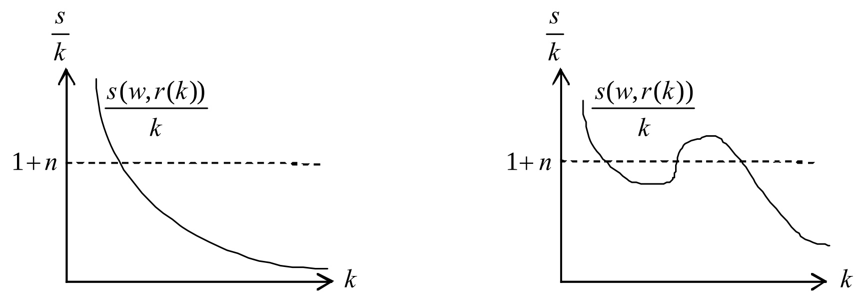

Figure 3.9: Existence of a solution to equation (3.47).

Letting $k \to \infty$ we then have $s(w, r(k))/k \to 0$ since $s(w, r(k))/k$ is squeezed between 0 and 0 (as indicated in the two graphs in Fig. 3.9).

Next we consider $k \to 0$. There are two cases.

Case 1: $\lim_{k\to 0} s(w, r(k)) > 0$.[24] Then obviously $\lim_{k\to 0} s(w, r(k))/k = \infty$.

Case 2: $\lim_{k\to 0} s(w, r(k)) = 0$.[25] In this case we have

$$\lim_{k\to 0} r(k) = \infty. \tag{3.48}$$

Indeed, since $f'(k)$ rises monotonically as $k \to 0$, the only alternative would be that $\lim_{k\to 0} r(k)$ exists and is $< \infty$; then, by Lemma 1 in Section 3.3, we would be in case 1 rather than case 2. By the second-period budget constraint, with $r = r(k)$, consumption as old is $c_2 = s(w, r(k))(1 + r(k)) \equiv c(w, k) > 0$ so that

$$\frac{s(w, r(k))}{k} = \frac{c(w, k)}{[1 + r(k)]\, k}.$$

The right-hand side of this equation goes to ∞ for $k \to 0$ since $\lim_{k\to 0} [1 + r(k)]\, k = 0$ by Technical Remark in Section 3.4 and $\lim_{k\to 0} c(w, k) = \infty$; this latter fact follows from the first-order condition (3.8), which can be written

$$0 \leq u'(c(w, k)) = (1 + \rho)\frac{u'(w - s(w, r(k))}{1 + r(k)} \leq (1 + \rho)\frac{u'(w)}{1 + r(k)}.$$

[24] If the limit does not exist, the proof applies to the *limit inferior* of $s(w, r(k))$ for $k \to 0$. The limit inferior for $i \to \infty$ of a sequence $\{x_i\}_{i=0}^{\infty}$ is defined as $\lim_{i\to\infty} \inf \{x_j | j = i, i + 1, \dots\}$, where inf of a set $S_i = \{x_j | j = i, i + 1, \dots\}$ is defined as the greatest lower bound for S_i.

[25] If the limit does not exist, the proof applies to the *limit inferior* of $s(w, r(k))$ for $k \to 0$.

Taking limits on both sides gives

$$\lim_{k\to 0} u'(c(w,k)) = (1+\rho)\lim_{k\to 0}\frac{u'(w-s\,(w,r\,(k)))}{1+r(k)} = (1+\rho)\lim_{k\to 0}\frac{u'(w)}{1+r(k)} = 0,$$

where the second equality comes from the fact that we are in case 2 and the third comes from (3.48). But since $u'(c) > 0$ and $u''(c) < 0$ for all $c > 0$, $\lim_{k\to 0} u'(c(w,k)) = 0$ requires $\lim_{k\to 0} c(w,k) = \infty$, as was to be shown.

In both Case 1 and Case 2 we thus have that $k \to 0$ implies $s\,(w, r\,(k))\,/k \to \infty$. Since $s\,(w, r\,(k))\,/k$ is a continuous function of k, there must be at least one $k > 0$ such that (3.47) holds (as illustrated by the two graphs in Fig. 3.14). □

Now, to prove (i) of Proposition 2, consider an arbitrary $k_t > 0$. We have $w(k_t) > 0$. In (3.47), let $w = w(k_t)$. By Lemma C1, (3.47) has a solution $k > 0$. Set $k_{t+1} = k$. Starting with $t = 0$, from a given $k_0 > 0$ we thus find a $k_1 > 0$ and letting $t = 1$, from the now given k_1 we find a k_2 and so on. The resulting infinite sequence $\{k_t\}_{t=0}^{\infty}$ is an equilibrium path. In this way we have proved existence of an equilibrium path if $k_0 > 0$. Thereby (i) of Proposition 2 is proved.

But what if $k_0 = 0$? Then, if $f(0) = 0$, no temporary equilibrium is possible in period 0, in view of (ii) of Proposition 1; hence there can be no equilibrium path. Suppose $f(0) > 0$. Then $w(k_0) = w(0) = f(0) > 0$, as explained in Technical Remark in Section 3.4. Let w in equation (3.47) be equal to $f(0)$. By Lemma C1 this equation has a solution $k > 0$. Set $k_1 = k$. Letting period 1 be the new initial period, we are back in the case with initial capital positive. This proves (ii) of Proposition 2.

E. Sufficient conditions for the transition curve to obtain certain properties

Positive slope everywhere For convenience we repeat here the condition (3.36):

$$\frac{1}{1-\gamma} > \frac{1-\sigma}{1+(1+\rho)^{-\sigma}(1+f'(k)-\delta)^{\sigma-1}}, \tag{*}$$

where we have substituted $\sigma \equiv 1/\theta$. In Section 3.5.3 we claimed that in the CRRA-CES case this condition is sufficient for the transition curve to be positively sloped everywhere. We here prove the claim.

Consider an arbitrary $k_t > 0$ and let $w \equiv w(k_t) > 0$. Knowing that $w'(k_t) > 0$ for all $k_t > 0$, we can regard k_{t+1} as directly linked to w. With k representing k_{t+1}, k must satisfy the equation $k = s(w, r(k))/(1+n)$. A sufficient condition for this equation to implicitly define k as an increasing function of w is also a sufficient condition for the transition curve to be positively sloped for all $k_t > 0$.

When $u(c)$ belongs to the CRRA class, by (3.15) with $\sigma \equiv 1/\theta$, we have $s(w, r(k)) = \left[1 + (1+\rho)^{\sigma}(1+r(k))^{1-\sigma}\right]^{-1} w$. The equation $k = s(w, r(k))/(1+n)$ therefore implies

$$\frac{w}{1+n} = k\left[1 + (1+\rho)^{\sigma} R(k)^{1-\sigma}\right] \equiv h(k), \tag{3.49}$$

where $R(k) \equiv 1 + r(k) \equiv 1 + f'(k) - \delta > 0$ for all $k > 0$. It remains to provide a sufficient condition for obtaining $h'(k) > 0$ for all $k > 0$. We have

$$h'(k) = 1 + (1+\rho)^{\sigma} R(k)^{1-\sigma}\left[1 - (1-\sigma)\eta(k)\right], \tag{3.50}$$

since $\eta(k) \equiv -kR'(k)/R(k) > 0$, the sign being due to $R'(k) = f''(k) < 0$. So $h'(k) > 0$ if and only if $1 - (1-\sigma)\eta(k) > -(1+\rho)^{-\sigma} R(k)^{\sigma-1}$, a condition equivalent with

$$\frac{1}{\eta(k)} > \frac{1-\sigma}{1 + (1+\rho)^{-\sigma} R(k)^{\sigma-1}}. \tag{3.51}$$

To make this condition more concrete, consider the CES production function

$$f(k) = A(\alpha k^{\gamma} + 1 - \alpha), \qquad A > 0, 0 < \alpha < 1, \gamma < 1. \tag{3.52}$$

Then $f'(k) = \alpha A^{\gamma}(f(k)/k)^{1-\gamma}$ and defining $\pi(k) \equiv f'(k)k/f(k)$ we find

$$\eta(k) = (1-\gamma)\frac{(1-\pi(k))f'(k)}{1-\delta+f'(k)} \leq (1-\gamma)(1-\pi(k)) < 1-\gamma, \tag{3.53}$$

where the first inequality is due to $0 \leq \delta \leq 1$ and the second to $0 < \pi(k) < 1$, which is an implication of strict concavity of f combined with $f(0) \geq 0$. Thus, $\eta(k)^{-1} > (1-\gamma)^{-1}$ so that if (*) holds for all $k > 0$, then so does (3.51), i.e., $h'(k) > 0$ for all $k > 0$. We have hereby shown that (*) is sufficient for the transition curve to be positively sloped everywhere.

Transition curve steep for small k Here we specialize further and consider the CRRA-Cobb-Douglas case: $u(c) = (c^{1-\theta} - 1)/(1-\theta)$, $\theta > 0$, and $f(k) = Ak^{\alpha}$, $A > 0$, $0 < \alpha < 1$. In the prelude to Proposition 6 in Section 3.5 it was claimed that if this combined utility and technology condition holds at least for small k, then (ii) of (A3) is satisfied. We now show this.

Letting $\gamma \to 0$ in (3.52) gives the Cobb-Douglas function $f(k) = Ak^{\alpha}$ (this is proved in the appendix to Chapter 4). With $\gamma = 0$, clearly $(1-\gamma)^{-1} = 1 > 1 - \sigma$, where $\sigma \equiv \theta^{-1} > 0$. This inequality implies that (*) above holds and so the transition curve is positively sloped everywhere. As an implication there is a transition function, φ, such that $k_{t+1} = \varphi(k_t)$, $\varphi'(k_t) > 0$. Moreover, since $f(0) = 0$, we have, by Proposition 5, $\lim_{k_t \to 0} \varphi(k_t) = 0$.

Given the imposed CRRA utility, the fundamental difference equation of the model is

$$k_{t+1} = \frac{w(k_t)}{(1+n)\left[1 + (1+\rho)^{\sigma} R(k_{t+1})^{1-\sigma}\right]} \tag{3.54}$$

or, equivalently,

$$h(k_{t+1}) = \frac{w(k_t)}{1+n},$$

where $h(k_{t+t})$ is defined as in (3.49). By implicit differentiation we find $h'(k_{t+1})\varphi'(k_t) = w'(k_t)/(1+n)$, i.e.,

$$\varphi'(k_t) = \frac{w'(k_t)}{(1+n)h'(k_{t+1})} > 0.$$

If $k^* > 0$ is a steady-state value of k_t, (3.54) implies

$$1 + (1+\rho)^{\sigma} R(k^*)^{1-\sigma} = \frac{w(k^*)}{(1+n)k^*}, \tag{3.55}$$

and the slope of the transition curve at the steady state will be

$$\varphi'(k^*) = \frac{w'(k^*)}{(1+n)h'(k^*)} > 0. \tag{3.56}$$

If we can show that such a $k^* > 0$ exists, is unique, and implies $\varphi'(k^*) < 1$, then the transition curve crosses the 45° line from above, and so (ii) of (A3) follows in view of $\lim_{k_t \to 0} = 0$.

Defining $x(k) \equiv f(k)/k = Ak^{\alpha-1}$, where $x'(k) = (\alpha-1)Ak^{\alpha-2} < 0$, and using that $f(k) = Ak^{\alpha}$, we have $R(k) = 1 + \alpha x(k) - \delta$ and $w(k)/k = (1-\alpha)x(k)$. Hence, (3.55) can be written

$$1 + (1+\rho)^{\sigma}(1 + \alpha x^* - \delta)^{1-\sigma} = \frac{1-\alpha}{1+n}x^*, \tag{3.57}$$

where $x^* = x(k^*)$. It is easy to show graphically that this equation has a unique solution $x^* > 0$ whether $\sigma < 1$, $\sigma = 1$, or $\sigma > 1$. Then $k^* = (x^*/A)^{1/(\alpha-1)} > 0$ is also unique.

By (3.50) and (3.57),

$$\begin{aligned} h'(k^*) &= 1 + (\frac{1-\alpha}{1+n}x^* - 1)\left[1 - (1-\sigma)\eta(k^*)\right] > 1 + (\frac{1-\alpha}{1+n}x^* - 1)(1 - \eta(k^*)) \\ &\geq 1 + (\frac{1-\alpha}{1+n}x^* - 1)\alpha, \end{aligned}$$

where the first inequality is due to $\sigma > 0$ and the second to the fact that $\eta(k) \leq 1 - \alpha$ in view of (3.53) with $\gamma = 0$ and $\pi(k) = \alpha$. Substituting this together with $w'(k^*) = (1-\alpha)\alpha x^*$ into (3.56) gives

$$0 < \varphi'(k^*) < \frac{\alpha x^*}{1 + n + \alpha x^*} < 1, \tag{3.58}$$

as was to be shown.

The CRRA-Cobb-Douglas case is well-behaved For the case of CRRA utility and Cobb-Douglas technology with CRS, existence and uniqueness of a steady state has just been proved. Asymptotic stability follows from (3.58). So the CRRA-Cobb-Douglas case is well-behaved.

3.10 Exercises

3.1 The dynamic accounting relation for a closed economy is

$$K_{t+1} = K_t + S^N \tag{*}$$

where K_t is the aggregate capital stock and S_t^N is aggregate net saving. In the Diamond model, let S_{1t} be aggregate net saving of the young in period t and S_{2t} aggregate net saving of the old in the same period. On the basis of (*) give a direct proof that the link between two successive periods takes the form $k_{t+1} = s_t/(1+n)$, where s_t is the saving of each young, n is the population growth rate, and k_{t+1} is the capital/labor ratio at the beginning of period $t+1$. *Hint:* by definition, net saving is the same as the increase in financial wealth.

3.2 Suppose the production function in Diamond's OLG model is $Y = A(\alpha K^\gamma + (1-\alpha)L^\gamma)^{1/\gamma}$, $A > 0, 0 < \alpha < 1, \gamma < 0$, and $A\alpha^{1/\gamma} < 1+n$. a) Given $k \equiv K/L$, find the equilibrium real wage, $w(k)$. b) Show that $w(k) < (1+n)k$ for all $k > 0$. *Hint:* consider the roof. c) Comment on the implication for the long-run evolution of the economy. *Hint:* consider the ceiling.

3.3 *(multiple temporary equilibria with self-fulfilling expectations)* Fig. 3.10 shows the transition curve for a Diamond OLG model with $u(c) = c^{1-\theta}/(1-\theta)$, $\theta = 8$, $\rho = 0.4$, $n = 0.2$, $\delta = 0.6$, $f(k) = A(bk^p + 1 - b)^{1/p}$, $A = 7$, $b = 0.33$, $p = -0.4$.

a) Let $t = 0$. For a given k_0 slightly below 1, how many temporary equilibria with self-fulfilling expectations are there?

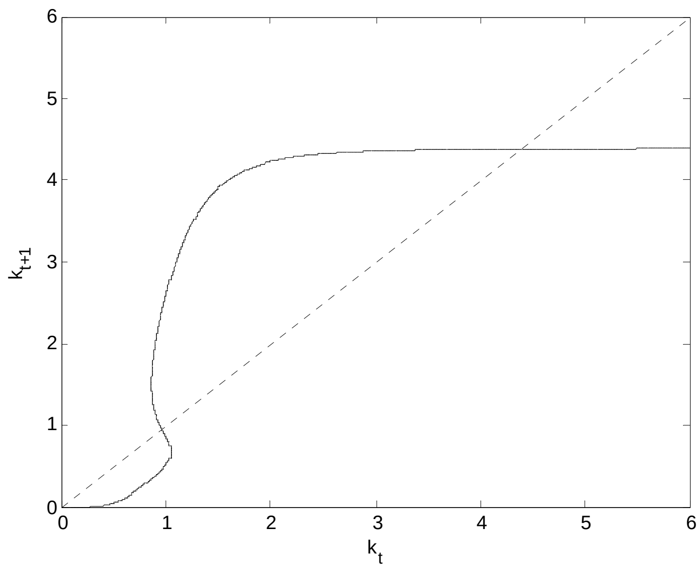

Figure 3.10: Transition curve for Diamond's OLG model in the case described in Exercise 3.3.

b) Suppose the young in period 0 expect the real interest rate on their saving to be relatively low. Describe by words the resulting equilibrium path in this case. Comment (what is the economic intuition behind the path?).

c) In the first sentence under b), replace "low" by "high". How is the answer to b) affected? What kind of difficulty arises?

3.4 *(plotting the transition curve by MATLAB)* This exercise requires computation by a computer. You may use *MATLAB OLG program.*[26]

a) Enter the model specification from Exercise 3.3 and plot the transition curve.

b) Plot examples for two other values of the substitution parameter: $p = -1.0$ and $p = 0.5$. Comment.

c) Find the approximate largest lower bound for p such that higher values of p eliminates multiple equilibria.

[26] Made by Marc P. B. Klemp and available at the address http://www.econ.ku.dk/okocg/Computation/main.htm.

d) In continuation of c), what is the corresponding elasticity of factor substitution, ψ? *Hint:* as shown in §4.4, the formula is $\psi = 1/(1-p)$.

e) The empirical evidence for industrialized countries suggests that $0.4 < \psi < 1.0$. Is your ψ from d) empirically realistic? Comment.

3.5 *(one stable and one unstable steady state)* Consider the following Diamond model: $u(c) = \ln c$, $\rho = 2.3$, $n = 2.097$, $\delta = 1.0$, $f(k) = A(bk^p+1-b)^{1/p}$, $A = 20$, $b = 0.5$, $p = -1.0$.

a) Plot the transition curve of the model. *Hint:* you may use either a program like *MATLAB OLG Program* (available on the course website) or first a little algebra and then Excel (or similar simple software).

b) Comment on the result you get. Will there exist a poverty trap? Why or why not?

c) At the stable steady state calculate numerically the output-capital ratio, the aggregate saving-income ratio, the real interest rate, and the capital income share of gross national income.

d) Briefly discuss how your results in c) comply with your knowledge of corresponding empirical magnitudes in industrialized Western countries?

e) There is one feature which this model, as a long-run model, ought to incorporate, but does not. Extend the model, taking this feature into account, and write down the fundamental difference equation for the extended model in algebraic form.

f) Plot the new transition curve. *Hint:* given the model specification, this should be straightforward if you use Excel (or similar); and if you use MATLAB OLG Program, note that by a simple "trick" you can transform your new model into the "old" form.

g) The current version of the MATLAB OLG Program is not adapted to this question. So at least here you need another approach, for instance based on a little algebra and then Excel (or similar simple software). Given $k_0 = 10$, calculate numerically the time path of k_t and plot the *time profile* of k_t, i.e., the graph (t, k_t) in the tk-plane. Next, do the same for $k_0 = 1$. Comment.

3.6 *(dynamics under myopic foresight)*
(incomplete) Show the possibility of a chaotic trajectory.

3.7 Given the period utility function is CRRA, derive the saving function of the young in Diamond's OLG model. *Hint:* substitute the period budget constraints into the Euler equation.

3.8 *Short questions* a) A steady-state capital-labor ratio can be in the "dynamically efficient" region or in the "dynamically inefficient" region. How are the two mentioned regions defined? b) Give a simple characterization of the two regions. c) The First Welfare Theorem states that, given certain conditions, any competitive equilibrium ($\equiv$ Walrasian equilibrium) is Pareto optimal. Give a list of circumstances that each tend to obstruct Pareto optimality of a competitive equilibrium.

3.9

3.10

Chapter 4

A growing economy

In the previous chapter we ignored technological progress. An incontestable fact of real life in industrialized countries is, however, the presence of a persistent rise in GDP per capita – on average between 1.5 and 2.5 percent per year since 1870 in many developed economies. In regard to U.K., U.S., and Japan, see Fig. 4.1. And in regard to Denmark, see Fig. 4.2. This fact should be taken into account in a model which, like the Diamond model, aims at dealing with long-run issues. For example, in relation to the question of dynamic inefficiency, cf. Section 3.6 of the previous chapter, the cut-off value of the steady-state interest rate is the steady-state GDP growth rate of the economy and this growth rate increases one-to-one with the rate of technological progress. We shall therefore now introduce technological progress.

On the basis of a summary of "stylized facts" about growth, Section 4.1 motivates the assumption that technological progress at the aggregate level take the Harrod-neutral form. Section 4.2 then extends the Diamond OLG model by incorporating this form of technological progress in the model. In Section 4.3 general aspects of the neoclassical theory of the functional income distribution are addressed and an expedient analytical tool, the elasticity of factor substitution, is presented. Section 4.4 goes into detail with the CES production function. The name CES stands for Constant Elasticity of factor Substitution. Finally, Section 4.5 concludes.

4.1 Harrod-neutrality and Kaldor's stylized facts

We assume the technology changes over time in such a way that we can write the aggregate production function as

$$Y_t = F(K_t, T_t L_t), \tag{4.1}$$

where the level of technology is represented by the factor T_t which is growing over time, and where Y_t, K_t, and L_t stand for output, capital input, and labor input, respectively. When technological change takes this purely labor-augmenting form, it is known as *Harrod-neutral technological progress*.

The reason that macroeconomists often assume that technological change at the aggregate level takes the Harrod-neutral form and not for example the form $Y_t = F(X_t K_t, T_t L_t)$ (where both X and T are changing over time), is the following. One wants the long-run properties of the model to comply with Kaldor's "stylized facts" (Kaldor 1961) concerning the long-run evolution of industrialized economies. Abstracting from short-run fluctuations,

1. the growth rates in K/L and Y/L are roughly constant;
2. the output-capital ratio, Y/K, the income share of labor, wL/Y, and the average rate of return, $(Y - wL - \delta K)/K$,[1] are roughly constant;
3. the growth rate of Y/L can vary substantially across countries for quite long time.

Ignoring the conceptual difference between the path of Y/L and that of Y *per capita* (a difference not important in this context), the figures 4.1 and 4.2 illustrate Kaldor's "fact 1" about the long-run property of the Y/L path for the more developed countries. Japan had an extraordinarily high growth rate for a couple of decades after World War II, usually explained by fast technology transfer from the most developed countries (the catching-up process which can only last until the technology gap is eliminated). Fig. 4.3 gives rough support for Kaldor's "fact 2" about factor income shares.

To comply with these features the model should be capable of generating a *balanced growth path*. As we shall see, this in turn requires that technological change takes the Harrod-neutral form.

[1] In this formula, land (and/or similar natural resources) is ignored. For countries where land is a quantitatively important production factor, the denominator should be replaced by $K + p_J J$, where p_J is the real price of land, J.

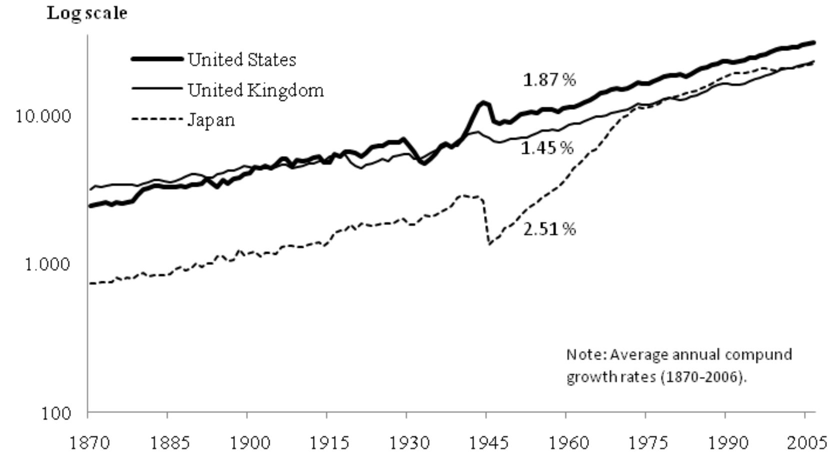

Figure 4.1: GDP per capita. U.S., U.K., and Japan 1870-2006. Source: Maddison (2009), Statistics on World Population, GDP and Per Capita GDP, 1-2006 AD, www.ggdc.net/maddison.

DEFINITION 1 A *balanced growth path* is a path $\{(K_t, Y_t, C_t)\}_{t=0}^{\infty}$ along which the variables K_t, Y_t, and C_t are positive and grow at constant rates (not necessarily positive).

Before describing the exact association between Harrod-neutral technological change and balanced growth, we will draw attention to a general equivalence relationship in a closed economy with disembodied technological progress, namely the equivalence between balanced growth and constancy of certain key ratios like Y/K and S/Y. This relationship is a simple implication of accounting based on the aggregate dynamic resource constraint:

$$K_{t+1} - K_t = S_t - \delta K_t \equiv Y_t - C_t - \delta K_t, \qquad K_0 > 0 \text{ given}, \tag{4.2}$$

where δ is a constant capital depreciation rate, $0 \leq \delta \leq 1$. This dynamic resource constraint is valid for a closed economy with disembodied technological change.[2]

[2]In case of *embodied* technical progress, cf. Section 2.2 of Chapter 2, we would have $K_{t+1} - K_t = Q_t I_t - \delta K_t$, where Q_t measures the "quality" (i.e., productivity) of newly produced investment goods and I_t is gross investment (which in a closed economy equals gross saving, S_t). Taking a rising Q_t into account, the K in the "Kaldor facts" should be replaced by $p_K K$, where p_K ($\approx 1/Q$) is the (real) price index for capital goods measured in efficiency units.

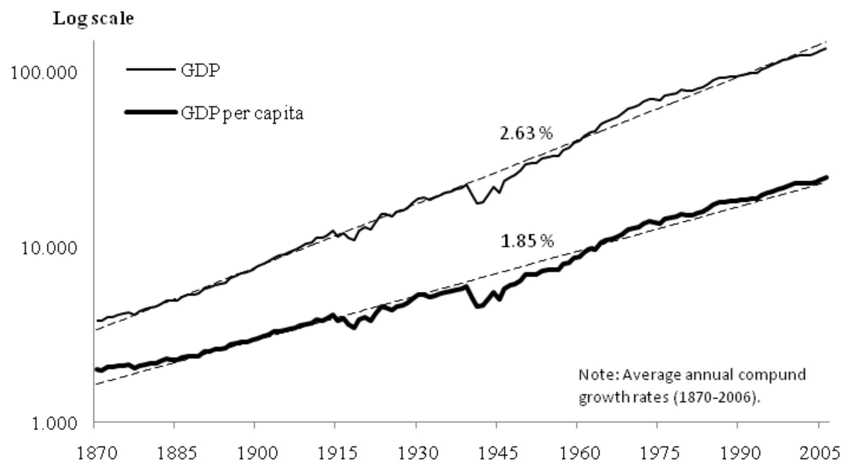

Figure 4.2: GDP and GDP per capita. Denmark 1870-2006. Source: Maddison (2009), Statistics on World Population, GDP and Per Capita GDP, 1-2006 AD, www.ggdc.net/maddison.

We will denote the growth rate of a positive variable, x, between t and $t+1$, $g_x(t+1)$, i.e., $g_x(t+1) \equiv (x_{t+1} - x_t)/x_t \equiv \Delta x_t/x_t$. When there is no risk of confusion, we suppress the explicit dating and write $g_x \equiv \Delta x/x$.

PROPOSITION 1 *(the balanced growth equivalence theorem).* Let $\{(K_t, Y_t,\ C_t)\}_{t=0}^{\infty}$ be a path along which Y_t, K_t, C_t, and $S_t \equiv Y_t - C_t$ are positive for all $t = 0, 1, 2, \ldots$. Then, given the dynamic resource constraint (4.2), the following holds:
(i) if there is balanced growth, then $g_Y = g_K = g_C = g_S$ and the ratios Y/K, C/K, C/Y, and S/Y are constant;
(ii) if Y/K and C/Y are constant, then Y, K, C, and S all grow at the same constant rate, i.e., not only are g_Y, g_K, g_C, and g_S constant (so that there is balanced growth), but $g_Y = g_K = g_C = g_S$.

Proof Consider a path $\{(K_t, Y_t, C_t)\}_{t=0}^{\infty}$ along which K, Y, C, and $S_t \equiv Y - C_t$ are positive for all $t = 0, 1, 2, \ldots$. (i) Suppose the path is a balanced growth path. Then, by definition, g_Y, g_K, and g_C are constant. Hence, by (4.2), $S/K = g_K + \delta$ must be constant, implying[3]

$$g_S = g_K. \tag{4.3}$$

[3]The ratio between two positive variables is constant if and only if the variables have the same growth rate (not necessarily constant or positive). For this and similar simple growth arithmetic rules, see Appendix A.

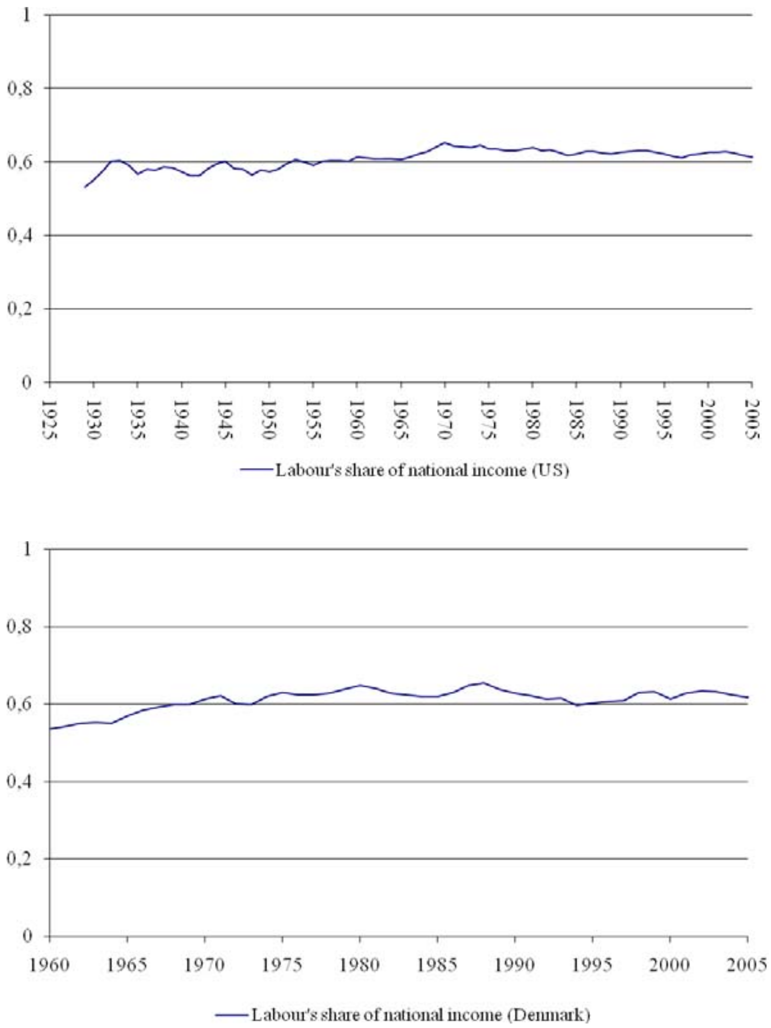

Figure 4.3: Labor income's share of gross national factor income in USA (upper) and Denmark (lower). Source: US Department of Commerce and AMECO (the database of the European Commission).

By (4.2), $Y \equiv C + S$, and so

$$\begin{aligned} g_Y &= \frac{\Delta Y}{Y} = \frac{\Delta C}{Y} + \frac{\Delta S}{Y} = \frac{C}{Y}g_C + \frac{S}{Y}g_S = \frac{C}{Y}g_C + \frac{S}{Y}g_K \qquad \text{(by (4.3))} \\ &= \frac{C}{Y}g_C + \frac{Y-C}{Y}g_K = \frac{C}{Y}(g_C - g_K) + g_K. \end{aligned} \tag{4.4}$$

Now, let us provisionally assume that $g_C \neq g_K$. Then (4.4) gives

$$\frac{C}{Y} = \frac{g_Y - g_K}{g_C - g_K}, \tag{4.5}$$

a constant since g_Y, g_K, and g_C are constant. Now, constancy of C/Y implies $g_C = g_Y$, hence, (4.5), $C/Y = 1$. In view of $Y \equiv C + S$, however, this result contradicts the given condition that $S > 0$. Hence, our provisional assumption is falsified, and we have $g_C = g_K$, so that (4.5) is not valid and $C/Y = 1$ not implied. Now (4.4) gives $g_Y = g_K = g_C$. It follows that Y/K and C/K are constant. Then, also $C/Y = (Y/K)/(C/K)$ is constant and so is S/Y since by (4.2), $S/Y \equiv (Y - C)/Y = 1 - C/Y$.

(ii) Suppose Y/K and C/Y are positive constants. Using that the ratio between two variables is constant if and only if the variables have the same (not necessarily constant or positive) growth rate, we can conclude that $g_Y = g_K = g_C$. By constancy of C/Y follows that $S/Y \equiv 1 - C/Y$ is constant. So $g_S = g_Y = g_K$, which in turn implies that S/K is constant. By (4.2),

$$\frac{S}{K} = \frac{\Delta K + \delta K}{K} = g_K + \delta,$$

so that also g_K is constant. By the above result that $g_Y = g_K = g_C = g_S$ follows that g_Y, g_C, and g_S are constant as well. □

For many long-run closed-economy models, including the Diamond OLG model, it holds that if and only if the dynamic system implied by the model is in steady state (as defined in Chapter 2), will the economy feature balanced growth. The reason is that when the dynamic resource constraint (4.2) applies, the dynamic system generally can be in steady state if and only if Y/K and C/Y are constant.

Note that Proposition 1 pertains to *any* model for which (4.2) is valid. No assumption about market form is involved. And except for the assumed constancy of the capital depreciation rate δ, no assumption about the technology is involved, neither constant returns to scale nor any neoclassical property. The proposition presupposes, however, that gross saving, S_t, is positive. In case of zero gross saving permanently (a very extreme case), we would get $g_K = -\delta$, which is automatically constant. Then we would have $C = Y$ and these two variables might have a constant growth rate but generally different from that of K.[4] In spite of g_K, g_Y, and g_C thus being constant, a constant output-capital ratio is no longer ensured. Hence (i) of the proposition would no longer apply.

Proposition 1 suggests that if one accepts Kaldor's stylized facts as a description of the past century's growth experience and therefore wants a model consistent with them, one should construct the model such that it can generate balanced growth. Our next proposition states that for a model to be capable of generating balanced growth, technological progress must take the Harrod-neutral form (i.e., be labor-augmenting). Also this proposition holds in a fairly general environment, but not as general as that of Proposition 1. Constant returns to scale and constant growth in the labor force, about which Proposition 1 is silent, will now have a role to play.[5]

Consider an aggregate production function

$$Y_t = \tilde{F}(K_t, L_t; t), \qquad \frac{\partial \tilde{F}}{\partial t} \geq 0, \tag{4.6}$$

where $\tilde{F}$ is homogeneous of degree one w.r.t. the first two arguments (CRS). The third argument, t, represents technological progress: as time proceeds, unchanged inputs of capital and labor result in more and more output (when $\partial \tilde{F}/\partial t > 0$). Let the labor force grow at a constant rate n,

$$L_t = L_0(1+n)^t, \quad n > -1, \tag{4.7}$$

where $L_0 > 0$. The Japanese economist Hirofumi Uzawa (1928–) is famous for several contributions, not least his balanced growth theorem (Uzawa 1961), which we here state in a modernized form.

PROPOSITION 2 *(Uzawa's balanced growth theorem).* Let $\{(K_t, Y_t, C_t)\}_{t=0}^{\infty}$ be a path along which Y_t, K_t, C_t, and $S_t \equiv Y_t - C_t$ are positive for all $t = 0, 1, 2, \ldots$, and satisfy the dynamic resource constraint (4.2), given the production function (4.6) and the labor force (4.7). Then:
(i) a *necessary* condition for this path to be a balanced growth path is that along the path it holds that

$$Y_t = \tilde{F}(K_t, T_t L_t; 0), \tag{4.8}$$

where $T_t = (1+g)^t$ with $g \equiv g_T = [(1+g_Y)/(1+n)] - 1$ and g_Y being the constant growth rate of output along the balanced growth path;

[4] If for instance $Y = K^{\alpha}L^{1-\alpha}, 0 < \alpha < 1$, we would get $g_C = g_Y = -\alpha\delta + (1-\alpha)n > -\delta$ for $n \geq 0$. That is, Y/K would necessarily be rising.

[5] On the other hand we do not imply that CRS is *always* necessary for a balanced growth path (see Exercise 4.??).

(ii) for any $\gamma \geq 0$ such that there is a $q > (1+\gamma)(1+n)+\delta$ with the property that $\tilde{F}(1, k^{-1}; 0) = q$ for some $k > 0$ (i.e., at any t, hence also at $t = 0$, the production function $\tilde{F}$ in (4.6) allows an output-capital ratio equal to q), a *sufficient* condition for the existence of a balanced growth path with output-capital ratio q is that the technology can be written as in (4.8) with $T_t = (1+g)^t$.

Proof See Appendix B.

The form (4.8) indicates that along a balanced growth path (BGP from now), technological progress must be purely labor augmenting, that is, Harrod-neutral. Moreover, by defining a new CRS production function F by $F(K_t, T_tL_t) \equiv \tilde{F}(K_t, T_tL_t; 0)$, we see that (i) of the proposition implies that at least in a neighborhood of a BGP with technology growth, we can rewrite the original production function this way:

$$Y_t = \tilde{F}(K_t, L_t; t) = F(K_t, T_tL_t), \tag{4.9}$$

where T_t represents the growing level of technology.

As noted also in Chapter 2, presence of Harrod-neutrality says nothing about what the *source* of technological progress is. Harrod-neutrality does not mean that the technological change emanates specifically from the labor input. It only means that technical innovations predominantly are such that not only do labor and capital in combination become more productive, but this *manifests itself* such that we can rewrite the production function as in (4.9).

What is the intuition behind the Uzawa result that for balanced growth to be possible, technological progress must have the purely labor-augmenting form? First, notice that there is an asymmetry between capital and labor. Capital is an accumulated amount of non-consumed output. In contrast, labor is a non-produced production factor which (at least in this context) grows in an exogenous way. Second, because of CRS, the original production function $\tilde{F}$ implies that

$$1 = \tilde{F}(\frac{K_t}{Y_t}, \frac{L_t}{Y_t}; t). \tag{4.10}$$

Now, since capital is accumulated non-consumed output, it inherits the trend in output such that K_t/Y_t must be constant along a BGP (this is what Proposition 1 is about). Labor does not inherit the trend in output; indeed, the ratio L_t/Y_t is free to adjust as t proceeds. If there is technological progress ($\partial\tilde{F}/\partial t > 0$) along a BGP, this progress must manifest itself in the form of a falling L_t/Y_t (rising Y_t/L_t) in (4.10) as t proceeds, precisely because K_t/Y_t *must* be constant along the path. This is what (4.10) shows. Indeed, the fall in L_t/Y_t must exactly offset the effect on $\tilde{F}$ of the rising t, for a fixed capital-output ratio. It follows that along the BGP, Y_t/L_t is an increasing implicit function of t. If we denote this function T_t, we end up with (4.9).

The generality of Uzawa's theorem is noteworthy. The theorem assumes CRS, but does not presuppose that the technology is neoclassical, not to speak of satisfying the Inada conditions. And the theorem holds for exogenous as well as endogenous technological progress.

A simple implication of the theorem is the following. Let labor productivity, in the sense of output per unit of labor, be denoted $y_t \equiv Y_t/L_t$. We have:

COROLLARY Along a BGP with positive gross saving and the technology level growing at a constant rate, g, output grows at the rate $(1+g)(1+n)-1$ ($\approx g+n$ for g and n small) and both labor productivity, y, and the capital-labor ratio, k, grow at the rate g.

Proof That $g_Y = (1+g)(1+n)-1$ follows from (i) of the proposition. As to g_y we have

$$y_t = \frac{Y_0(1+g_Y)^t}{L_0(1+n)^t} = y_0(1+g)^t,$$

showing that labor productivity, y, grows at the rate g. By Proposition 1, Y/K is constant along the BGP. As $y/k = Y/K$, k then grows at the same rate as y. $\square$

There is one facet of Kaldor's stylized facts which we have not yet related to Harrod-neutral technological progress, namely the long-run "approximate" constancy of the income share of labor and the rate of return on capital. At least, assuming neoclassical technology, profit maximizing firms, and perfect competition in the output and factor markets, these properties are inherent in the combination of constant returns to scale and balanced growth.

To see this, let the aggregate production function be $Y_t = F(K_t, T_tL_t)$ where F is neoclassical and has CRS. With w_t denoting the real wage at time t, in equilibrium under perfect competition the labor income share will be

$$\frac{w_tL_t}{Y_t} = \frac{\frac{\partial Y_t}{\partial L_t}L_t}{Y_t} = \frac{F_2(K_t, T_tL_t)T_tL_t}{Y_t}. \tag{4.11}$$

When the capital good is nothing but non-consumed output, the rate of return on capital at time t can be written

$$r_t = \frac{Y_t - w_tL_t - \delta K_t}{K_t}. \tag{4.12}$$

Since land as a production factor is ignored, gross capital income equals non-labor income, $Y_t - w_t L_t$. Denoting the gross capital income share by α_t, we thus have

$$\begin{aligned}\alpha_t &= \frac{Y_t - w_t L_t}{Y_t} = \frac{(r_t + \delta)K_t}{Y_t} = \frac{F(K_t, T_t L_t) - F_2(K_t, T_t L_t)T_t L_t}{Y_t} \\ &= \frac{F_1(K_t, T_t L_t)K_t}{Y_t} = \frac{\frac{\partial Y_t}{\partial K_t}K_t}{Y_t}, \qquad (4.13)\end{aligned}$$

where the second equality comes from (4.12), the third from (4.11), and the fourth from Euler's theorem.[6]

PROPOSITION 3 *(factor income shares)* Suppose the path $\{(K_t, Y_t, C_t)\}_{t=0}^{\infty}$ is a BGP with positive saving in this economy. Then $\alpha_t = \alpha$, a constant $\in (0, 1)$. The labor income share will be $1 - \alpha$ and the rate of return on capital $\alpha q - \delta$, where q is the constant output-capital ratio along the BGP.

Proof We have $Y_t = F(K_t, T_t L_t) = T_t L_t F(\tilde{k}_t, 1) \equiv T_t L_t f(\tilde{k}_t)$. Along the BGP, from Proposition 1 we know that Y_t/K_t is some constant, q. Since $Y_t/K_t = f(\tilde{k}_t)/\tilde{k}_t$ and $f'' < 0$, this implies $\tilde{k}_t$ constant, say equal to $\tilde{k}^*$. Along the BGP $\partial Y_t/\partial K_t = f'(\tilde{k}_t)$ thus equals the constant $f'(\tilde{k}^*)$. From (4.13) then follows that $\alpha_t = f'(\tilde{k}^*)/q \equiv \alpha$. Moreover, $0 < \alpha < 1$, since $0 < \alpha$ is implied by $f' > 0$ and $\alpha < 1$ by the fact that $q = Y/K = f(\tilde{k}^*)/\tilde{k}^* > f'(\tilde{k}^*)$, in view of $f'' < 0$ and $f(0) \geq 0$. So, by the first equality in (4.13), the labor income share will be $w_t L_t/Y_t = 1 - \alpha_t = 1 - \alpha$ and the rate of return on capital $r_t = (1 - w_t L_t/Y_t)Y_t/K_t - \delta = \alpha q - \delta$. $\square$

Although this proposition implies constancy of the factor income shares, it does not *determine* them. The proposition expresses the factor income shares in terms of the unknown constants α and q. These constants will generally depend on the effective capital-labor ratio in steady state, $\tilde{k}^*$, which will generally be an unknown as long as we have not specified a theory of saving. This takes us back to Diamond's OLG model which provides such a theory.

4.2 The Diamond model with Harrod-neutral technological progress

We recall from the previous chapter that in the Diamond model people live in two periods, as young and as old. Only the young work and each young supplies one unit of labor inelastically. The period utility function, $u(c)$, satisfies

[6] Indeed, from Euler's theorem follows that $F_1 K + F_2 TL = F(K, TL)$, when F is homogeneous of degree one.

the No Fast assumption. The saving function of the young is $s_t = s(w_t, r_{t+1})$. To allow for technological progress, we write aggregate production as

$$Y_t = F(K_t, T_t L_t), \tag{4.14}$$

where F is neoclassical with CRS. The employment equals L_t which is the number of young and grows at the constant rate $n > -1$. There is disembodied Harrod-neutral technological progress at a constant rate $g > 0$ so that $T_t = T_0(1+g)^t$. The initial level of technology, T_0, is historically given.[7]

Suppressing for a while the explicit dating of the variables, in view of CRS w.r.t. K and TL, we have

$$\tilde{y} \equiv \frac{Y}{TL} = F(\frac{K}{TL}, 1) = F(\tilde{k}, 1) \equiv f(\tilde{k}), \qquad f' > 0, f'' < 0,$$

where TL is the *labor input in efficiency units* and $\tilde{k} \equiv K/(TL)$ is known as the "effective" or "technology corrected" capital-labor ratio – sometimes called the effective capital-labor ratio or the capital intensity. There is perfect competition in all markets and in each period the firm maximizes profit, $\Pi = F(K, TL) - \hat{r}K - wL$. This leads to the first-order conditions

$$\frac{\partial Y}{\partial K} = \frac{\partial \left[TLf(\tilde{k})\right]}{\partial K} = f'(\tilde{k}) = r + \delta, \tag{4.15}$$

and

$$\frac{\partial Y}{\partial L} = \frac{\partial \left[TLf(\tilde{k})\right]}{\partial L} = \left[f(\tilde{k}) - f'(\tilde{k})\tilde{k}\right] T = w. \tag{4.16}$$

In view of $f'' < 0$, a $\tilde{k}$ satisfying (4.15) is unique. Let us name its value in period t, $\tilde{k}_t^d$. Assuming equilibrium in the factor markets, this desired effective capital-labor ratio equals the effective capital-labor ratio from the supply side, $\tilde{k}_t \equiv K_t/(T_t L_t) \equiv k_t/T_t$, which is predetermined in every period. The equilibrium interest rate and real wage in period t are thus given by

$$r_t = f'(\tilde{k}_t) - \delta \equiv r(\tilde{k}_t), \quad \text{where } r'(\tilde{k}_t) = f''(\tilde{k}_t) < 0, \tag{4.17}$$

$$w_t = \left[f(\tilde{k}_t) - f'(\tilde{k}_t)\tilde{k}\right] T_t \equiv \tilde{w}(\tilde{k}_t)T_t, \text{ where } \tilde{w}'(\tilde{k}_t) = -\tilde{k}_t f''(\tilde{k}_t) > 0. \tag{4.18}$$

Here, $w_t/T_t = \tilde{w}(\tilde{k}_t)$ is the "technology-corrected" real wage.

[7] The value of T_0 then depends only on the choice of measurement units. In connection with (4.8) measurement units were chosen such that $T_0 = 1$. But to improve transparency, here we let measurement units be arbitrary.

The equilibrium path

The aggregate capital stock at the beginning of period $t+1$ must still be owned by the old generation in that period and thus be equal to the aggregate saving these people had as young in the previous period, $s_t L_t$. Hence, as before, $K_{t+1} = s(w_t, r_{t+1})L_t$. In view of $K_{t+1} \equiv \tilde{k}_{t+1}T_{t+1}L_{t+1} = \tilde{k}_{t+1}T_t(1+g)L_t(1+n)$, together with (4.17) and (4.18), we get

$$\tilde{k}_{t+1} = \frac{s(\tilde{w}(\tilde{k}_t)T_t, r(\tilde{k}_{t+1}))}{T_t(1+g)(1+n)}. \tag{4.19}$$

This is the general version of the law of motion of the Diamond OLG model with Harrod-neutral technological progress.

For the model to comply with Kaldor's "stylized facts", the model should be capable of generating balanced growth. Essentially, this capability is equivalent with being able to generate a steady state. In the presence of technological progress this latter capability requires a restriction on the lifetime utility function, U. Indeed, we see from (4.19) that the model is consistent with existence of a steady state only if the time-dependent technology level, T_t, in the numerator and denominator cancels out. This requires that the saving function is homogeneous of degree one in its first argument such that $s(\tilde{w}(\tilde{k}_t)T_t, r(\tilde{k}_{t+1})) = s(\tilde{w}(\tilde{k}_t), r(\tilde{k}_{t+1}))T_t$. In turn this is so if and only if the lifetime utility function of the young is *homothetic;* this property entails that if wealth (here w_t) is multiplied by a positive factor, then the chosen c_{1t} and c_{2t+1} are also multiplied by this factor (see Appendix C); it then follows that s_t is multiplied by this factor as well.

In addition to the No Fast Assumption from Chapter 3 we will thus assume:

$$\text{the lifetime utility function } U \text{ is homothetic.} \tag{A4}$$

Then we can write

$$s_t = s(1, r(\tilde{k}_{t+1}))\tilde{w}(\tilde{k}_t)T_t \equiv \hat{s}(r(\tilde{k}_{t+1}))\tilde{w}(\tilde{k}_t)T_t, \tag{4.20}$$

where $\hat{s}(r(\tilde{k}_{t+1}))$ is the saving *rate* (saving-income ratio) of the young. The distinctive feature is that this saving rate is independent of income (but in general it depends on the interest rate). By (4.19), the law of motion of the economy reduces to

$$\tilde{k}_{t+1} = \frac{\hat{s}(r(\tilde{k}_{t+1}))}{(1+g)(1+n)}\tilde{w}(\tilde{k}_t). \tag{4.21}$$

The equilibrium path of the economy can be analyzed in a similar way as in the model without technological progress. In the assumptions (A2) and

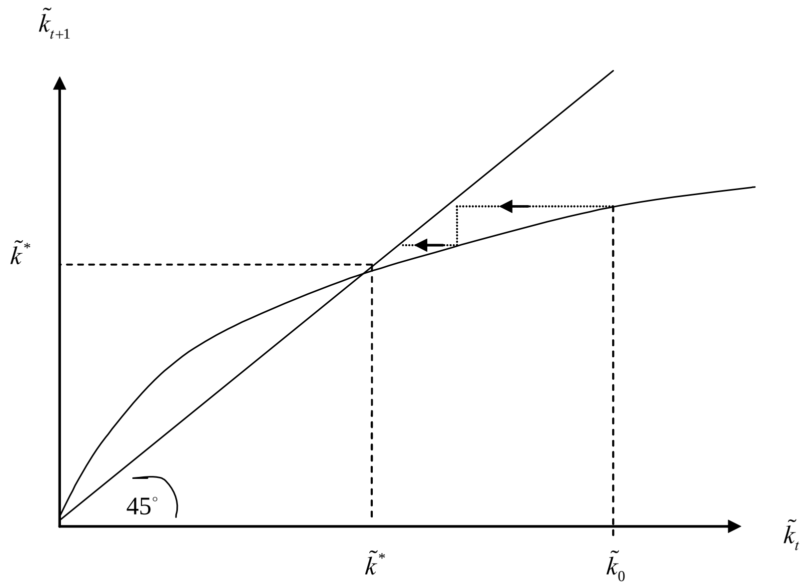

Figure 4.4: Transition curve for a well-behaved Diamond OLG model with Harrod-neutral technical progress.

(A3) from Chapter 3 we replace k by $\tilde{k}$ and $1+n$ by $(1+g)(1+n)$. Then for instance Proposition 6 of Chapter 3 applies to the present economy when (A4) holds. If the No Fast Assumption and the extended versions of (A2) and (A3) hold, then there exists at least one locally asymptotically stable steady state $\tilde{k}^* > 0$.

With these assumptions satisfied, the prediction of the model is that the economy will in the long run behave in accordance with Kaldor's stylized facts. Indeed, this follows by Proposition 3 and Corollary to Proposition 2 from Chapter 3 together with equivalence in the model between converging to a steady state and converging to a BGP. This equivalence follows from:

LEMMA 1 If the Diamond economy is in a steady state $\tilde{k}^* > 0$, then it features balanced growth.

Proof Suppose the Diamond economy is in a (non-trivial) steady state, i.e., given (4.21), $\tilde{k}_t = \tilde{k}_{t+1} = \tilde{k}^*$ for some $\tilde{k}^* > 0$. The constancy of $\tilde{k} \equiv K/(TL)$ and $\tilde{y} \equiv Y/(TL) = f(\tilde{k})$ implies that both g_K and g_Y must equal $(1+g)(1+n)-1 > 0$. As K and Y thus grow at the same rate, Y/K is constant. As $S/K = (\Delta K + \delta K)/K = g_K + \delta$, S/K is constant so that S also grows at the rate g_K and thereby at the same rate as output. Hence S/Y is constant. Because $C/Y \equiv 1 - S/Y$, also C grows at the constant rate g_Y. All criteria for a balanced growth path are thus satisfied. □

Let us consider a transition diagram in the $(\tilde{k}_t, \tilde{k}_{t+1})$ plane. Fig. 4.4 shows a well-behaved case. In this specific example the initial effective capital-labor ratio, $\tilde{k}_0$, is assumed to be relatively large. This need not be interpreted as if the economy is highly developed and has a high initial capital-labor ratio, K_0/L_0. Indeed, the reason that $\tilde{k}_0 \equiv K_0/(T_0L_0)$ is large relative to its steady-steady value may be that the economy is "backward" in the sense of having a relatively low initial level of technology. Growing at a given rate g, the technology will in this situation grow faster than the capital-labor ratio, K/L, so that the effective capital-labor ratio declines over time. This continues until the steady state is reached with a real interest rate r^* $= f'(\tilde{k}^*) - \delta$. This is to remind the reader that from an empirical point of view, the adjustment towards a steady state can be from above as well as from below.

The output growth rate in steady state, $(1+g)(1+n)-1$, is sometimes called the "natural rate of growth". Since $(1+g)(1+n)-1 = g+n+gn \approx g+n$ for g and n small, the natural rate of growth is "approximately" equal to the sum of rate of technological progress and the growth rate of the labor force. *Warning*: When measured on an *annual* basis, the growth rates of technology and labor force, $\bar{g}$ and $\bar{n}$, *are* typically small, say $\bar{g} = 0.02$ and $\bar{n} = 0.005$, so that $\bar{g} + \bar{n} + \bar{g}\bar{n} = 0.0251 \approx 0.0250 = \bar{g} + \bar{n}$. But in the context of models like Diamond's, the period length is, say, 30 years. Then the corresponding g and n will satisfy the equations $1+g = (1+\bar{g})^{30} = 1.02^{30}$ $= 1.8114$ and $1+n = (1+\bar{n})^{30} = 1.005^{30} = 1.1614$, respectively. We get $g+n = 0.973$, which is about 10 per cent smaller than the true output growth rate over 30 years, which is $g+n+gn = 1.104$.

We end our account of Diamond's OLG model with some remarks on a popular special case of a homothetic utility function.

An example: CRRA utility

An example of a homothetic lifetime utility function is obtained by letting the period utility function take the CRRA form introduced in the previous chapter. Then

$$U(c_1, c_2) = \frac{c_1^{1-\theta}-1}{1-\theta} + (1+\rho)^{-1}\frac{c_2^{1-\theta}-1}{1-\theta}, \qquad \theta > 0. \tag{4.22}$$

Recall that the CRRA utility function with parameter θ has the property that the (absolute) elasticity of marginal utility of consumption equals the constant $\theta > 0$ for all $c > 0$. Up to a positive linear transformation it is, in fact, the only period utility function with this property.[8] A proof that the utility function (4.22) is indeed homothetic is given in Appendix C.

One of the reasons that the CRRA function is popular in macroeconomics is that in *representative* agent models, the period utility function *must* have this form to obtain consistency with balanced growth and Kaldor's stylized facts (we show this in Chapter 7). In contrast, a heterogeneous agent model, like the Diamond model, does not need CRRA utility to comply with the Kaldor facts. CRRA utility is just a convenient special case leading to homothetic lifetime utility. It is this more general property which is needed.

Given the CRRA assumption in (4.22), the saving-income ratio of the young becomes

$$\hat{s}(r) = \frac{1}{1 + (1+\rho)^{1/\theta}(1+r)^{(\theta-1)/\theta}}. \tag{4.23}$$

It follows that $\hat{s}'(r) \gtreqless 0$ for $\theta \lesseqgtr 1$.

When $\theta = 1$ (the log utility case), $\hat{s}(r) = 1/(2+\rho) \equiv \hat{s}$, a constant, and the law of motion simplifies to

$$\tilde{k}_{t+1} = \frac{1}{(1+g)(1+n)(2+\rho)}\tilde{w}(\tilde{k}_t).$$

A convenient feature of the log utility case is thus that independently of the production function, $\tilde{k}_{t+1}$ enters only at the left-hand side of the fundamental difference equation, which reduces to a simple transition function. Since $\tilde{w}'(\tilde{k}) > 0$, the transition curve is positively sloped everywhere. If the production function is Cobb-Douglas, $Y_t = K_t^\alpha(T_tL_t)^{1-\alpha}$, then $\tilde{w}(\tilde{k}_t) = (1-\alpha)\tilde{k}_t^\alpha$. Combining this with the case $\theta = 1$ yields a well-behaved Diamond model with Harrod-neutral technological progress, cf. Fig. 4.4 above. In fact, as noted in Chapter 3, given the technology is Cobb-Douglas, well-behavedness results whatever the value of $\theta > 0$. (Although absent in Chapter 3, Harrod-neutral technological progress at constant rate g can easily be incorporated simply by replacing n with $n' = (1+g)(1+n) - 1$ in the formulas in Section 3.5 of Chapter 3.)

4.3 The golden rule under Harrod-neutral technological progress

Given that there is technological progress, consumption per unit of labor is likely to grow over time. Therefore the definition of the golden rule capital-

[8] As noted in Chapter 3, the expression $u(c) = (c^{1-\theta} - 1)/(1-\theta)$ can be interpreted as valid also for $\theta = 1$, in which case the expression should be interpreted as $\ln c$ (this is seen by using L'Hôpital's rule for "0/0").

labor ratio from Chapter 3 has to be extended to cover the case of technological progress. To allow existence of steady states and balanced growth paths, we maintain the assumption that technological progress is Harrod-neutral, that is, we maintain (4.14) where the technology level, T, grows at a constant rate, g.

DEFINITION 2 The golden rule capital intensity, $\tilde{k}_{GR}$, is that level of $\tilde{k} \equiv K/(TL)$ which gives the highest sustainable path for consumption per unit of labor in the economy.

As before, we consider aggregate consumption per unit of labor

$$\begin{aligned} c_t &\equiv \frac{C_t}{L_t} = \frac{F(K_t, T_t L_t) - S_t}{L_t} = \frac{f(\tilde{k}_t) T_t L_t - (K_{t+1} - K_t + \delta K_t)}{L_t} \\ &= f(\tilde{k}_t) T_t - (1+g) T_t (1+n) \tilde{k}_{t+1} + (1-\delta) T_t \tilde{k}_t \\ &= \left[f(\tilde{k}_t) + (1-\delta)\tilde{k}_t - (1+g)(1+n)\tilde{k}_{t+1} \right] T_t. \end{aligned}$$

In a steady state we have $\tilde{k}_{t+1} = \tilde{k}_t = \tilde{k}^*$ and therefore

$$c_t = \left[f(\tilde{k}^*) + (1-\delta)\tilde{k}^* - (1+g)(1+n)\tilde{k}^* \right] T_t \equiv \tilde{c}(\tilde{k}^*) T_t,$$

where $\tilde{c}(\tilde{k}^*)$ is the "technology-corrected" level of aggregate consumption per unit of labor in steady state. We see that in steady state, consumption per unit of labor will grow at the same rate as the technology. Thus, $\ln c_t = \ln \tilde{c}(\tilde{k}^*) + \ln T_0 + t \ln(1+g)$. Fig. 4.5 illustrates.

Since the path of technology, represented by T_0 and g, is exogenous, the highest possible path of c_t is found by maximizing $\tilde{c}(\tilde{k}^*)$. This gives the first-order condition

$$\tilde{c}'(\tilde{k}^*) = f'(\tilde{k}^*) + (1-\delta) - (1+g)(1+n) = 0. \tag{4.24}$$

Assuming, for example, $n \geq 0$, we have $(1+g)(1+n) - (1-\delta) > 0$ since $g > 0$. Then, by continuity the equation (4.24) necessarily has a unique solution in $\tilde{k}^* > 0$, if the production function satisfies the condition

$$\lim_{\tilde{k} \to 0} f'(\tilde{k}) > (1+g)(1+n) - (1-\delta) > \lim_{\tilde{k} \to \infty} f'(\tilde{k}),$$

a considerable milder condition than the Inada conditions. The solution is the golden rule capital intensity, $\tilde{k}_{GR}$, and we have

$$f'(\tilde{k}_{GR}) - \delta = (1+g)(1+n) - 1 \approx g + n, \tag{4.25}$$

where the right-hand side is the "natural rate of growth". Thus, the golden rule capital intensity is that level of the capital intensity at which the net marginal productivity of capital equals the output growth rate in steady state.

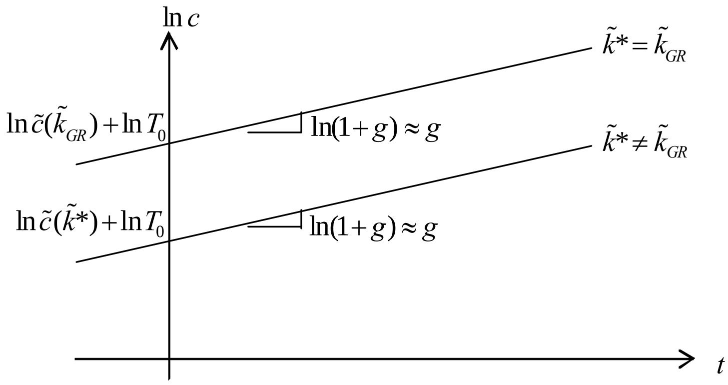

Figure 4.5: The higest sustainable path of consumption is where $\tilde{k}^* = \tilde{k}_{GR}$.

Is dynamic inefficiency a problem in practice? As in the Diamond model without technological progress, it is theoretically possible that the economy ends up in a steady state with $\tilde{k}^* > \tilde{k}_{GR}$.[9] If this happens, the economy is dynamically inefficient and $r^* < (1 + g)(1 + n) - 1 \approx g + n$. To check whether dynamic inefficiency is a realistic outcome in an industrialized economy or not, we should compare the observed average GDP growth rate over a long stretch of time to the average real interest rate. For the period after the Second World War the average GDP growth rate ($\approx g + n$) in Western countries typically is about 3 per cent per year. But what interest rate should one choose? In simple macro models, like the Diamond model, there is no uncertainty and no need for money to carry out trades. In such models all assets earn the same rate of return, r. In the real world there is a spectrum of interest rates, reflecting the different risk and liquidity properties of the different assets. The expected real rate of return on a short-term government bond is 1-3 per cent per year (a relatively safe and liquid asset). This is much lower than the expected real rate of return on corporate stock, 7-9 per cent per year. Our model cannot tell which rate of return we should choose, but the conclusion hinges on that choice.

Abel et al. (1989) study the problem on the basis of a model with uncertainty. They show that a sufficient condition for dynamic efficiency is that gross investment, I, does not exceed the gross capital income in the long run, that is $I \leq Y - wL$. They find that for the U.S. and six other major OECD nations this seems to hold. Indeed, for the period 1929-85 the U.S. has, on average, $I/Y = 0.15$ and $(Y - wL)/Y = 0.29$. A similar difference is found for other industrialized countries, suggesting that they are dynamically effi-

[9] The proof is analogue to that in Chapter 3 for the case $g = 0$.

cient. Thus, at least in the U.S., the potential coordination failure laid bare by OLG models does not seem to have been operative in practice.

4.4 The functional distribution of income

.....Text to be inserted

The neoclassical theory

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How the labor income share depends on the capital-labor ratio

To begin with we ignore technological progress and write aggregate output as $Y = F(K, L)$, where F is neoclassical and CRS obtains. From Euler's theorem follows that $F(K, L) = F_1 K + F_2 L = f'(k)K + (f(k) - kf'(k))L$, where $k \equiv K/L$. In equilibrium under perfect competition we have

$$Y = \hat{r}K + wL,$$

where $\hat{r} = r + \delta$ is the cost per unit of capital input and w is the real wage, i.e., the cost per unit of labor input. The labor income share is

$$\frac{wL}{Y} = \frac{f(k) - kf'(k)}{f(k)} \equiv \frac{w(k)}{f(k)} \equiv SL(k) = \frac{wL}{\hat{r}K + wL} = \frac{\frac{w/\hat{r}}{k}}{1 + \frac{w/\hat{r}}{k}},$$

where the function $SL(\cdot)$ is the share of labor function and $w/\hat{r}$ is the factor price ratio.

Suppose that capital tends to grow faster than labor so that k rise over time. Unless the production function is Cobb-Douglas, this will under perfect competition affect the labor income share. But apriori it is not obvious in what direction. If the proportionate rise in the factor price ratio $w/\hat{r}$ is greater (smaller) than that in k, then SL goes up (down). Indeed, if we let $\mathrm{E}\ell_x g(x)$ denote the elasticity of a function $g(x)$ w.r.t. x, then

$$SL'(k) \gtreqless 0 \text{ for } \mathrm{E}\ell_k(\frac{w}{\hat{r}}) \gtreqless 1,$$

respectively. Usually, however, the inverse elasticity is considered, namely $\mathrm{E}\ell_{w/\hat{r}}k$. This elasticity, which indicates how sensitive the cost minimizing capital-labor ratio, k, is to a given factor price ratio $w/\hat{r}$, coincides with the *elasticity of factor substitution* (for a general definition, see below). The latter is often denoted σ. Since in the CRS case, σ will be a function of only k, we write $\mathrm{E}\ell_{w/\hat{r}}k = \sigma(k)$. We therefore have

$$SL'(k) \gtreqless 0 \text{ for } \sigma(k) \lesseqgtr 1,$$

respectively.

We may put forward another example that illustrates the importance of the size of $\sigma(k)$. Consider a closed economy with perfect competition and a given aggregate capital stock K. Suppose that for some reason aggregate labor supply, L, goes down. In what direction will aggregate labor income wL then change? The effect of the smaller L is to some extent offset by a higher w brought about by the higher capital-labor ratio. Indeed, $dw/dk = -kf''(k) > 0$. So we cannot apriori sign the change in wL. Since $w = w(k)$ and $k \equiv KL^{-1}$, we can write wL as a function $wL = W(K, L)$. In Exercise 4.?? the reader is asked to show that

$$\frac{\partial W}{\partial L} = (1 - \frac{\alpha(k)}{\sigma(k)})w \gtreqless 0 \text{ for } \alpha(k) \lesseqgtr \sigma(k), \tag{4.26}$$

respectively, where $a(k) \equiv kf'(k)/f(k)$ is the output elasticity w.r.t. capital which under perfect competition equals the gross capital income share. It follows that the lower L will not be fully offset by the higher w as long as the elasticity of factor substitution, $\sigma(k)$, exceeds the gross capital income share, $\alpha(k)$. This condition seems confirmed by most of the empirical evidence (see, e.g., Antras, 2004).

The elasticity of factor substitution

We shall here discuss the concept of elasticity of factor substitution at a more general level. Fig. 4.6 depicts an isoquant, $F(K, L) = \bar{Y}$, for a given neoclassical production function, $F(K, L)$, which need not have CRS. Let MRS denote the marginal rate of substitution of K for L, i.e., $MRS = F_L(K, L)/F_K(K, L)$.[10] At a given point (K, L) on the isoquant curve, MRS is given by the absolute value of the slope of the tangent to the isoquant at that point. This tangent coincides with that isocost line which, given the factor prices, has minimal intercept with the vertical axis while at the same time touching the isoquant. In view of $F(\cdot)$ being neoclassical, the isoquants are by definition strictly convex to the origin. Consequently, MRS is rising along the curve when L decreases and thereby K increases. Conversely, we can let MRS be the independent variable and consider the corresponding

[10] When there is no risk of confusion as to what is up and what is down, we use MRS as a shorthand for the more correct MRS_{KL}.

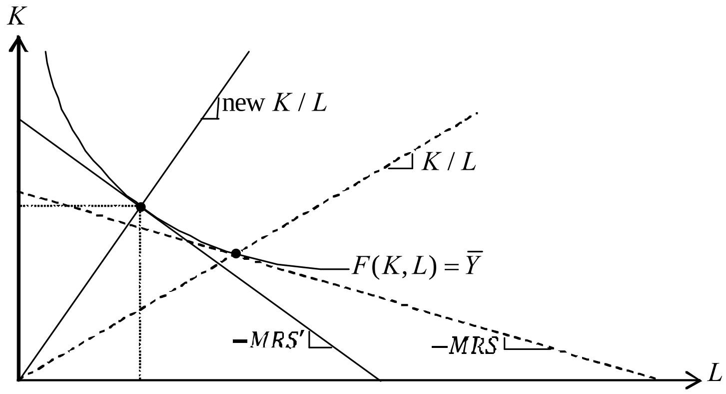


Figure 4.6: Substitution of capital for labor as the marginal rate of substitution increases from MRS to MRS'.

point on the indifference curve, and thereby the ratio K/L, as a function of MRS. If we let MRS rise along the given isoquant, the corresponding value of the ratio K/L will also rise.

The *elasticity of substitution* between capital and labor is defined as the elasticity of the ratio K/L with respect to MRS when we move along a given isoquant, evaluated at the point (K, L). Let this elasticity be denoted $\tilde{\sigma}(K, L)$. Thus,

$$\tilde{\sigma}(K,L) = \frac{MRS}{K/L}\frac{d(K/L)}{dMRS}_{|Y=\bar{Y}} = \frac{\frac{d(K/L)}{K/L}}{\frac{dMRS}{MRS}}_{|Y=\bar{Y}}. \tag{4.27}$$

Although the elasticity of factor substitution is a characteristic of the technology as such and is here defined without reference to markets and factor prices, it helps the intuition to refer to factor prices. At a cost-minimizing point, MRS equals the factor price ratio $w/\hat{r}$. Thus, the *elasticity of factor substitution* will under cost minimization coincide with *the percentage increase in the ratio of the cost-minimizing factor ratio induced by a one percentage increase in the inverse factor price ratio, holding the output level unchanged.*[11] The elasticity of factor substitution is thus a positive number and reflects how sensitive the capital-labor ratio K/L is under cost minimization to an increase in the factor price ratio $w/\hat{r}$ for a given output level. The less curvature the isoquant has, the greater is the elasticity of factor substitu-

[11]This characterization is equivalent to interpreting the elasticity of substitution as the percentage *decrease* in the factor ratio (when moving along a given isoquant) induced by a one-percentage *increase* in the *corresponding* factor price ratio.

tion. In an analogue way, in consumer theory one considers the elasticity of substitution between two consumption goods or between consumption today and consumption tomorrow, cf. Chapter 3. In that context the role of the given isoquant is taken over of an indifference curve. That is also the case when we consider the intertemporal elasticity of substitution in labor supply, cf. Chapter 5.

Calculating the elasticity of substitution between K and L at the point (K, L), we get

$$\tilde{\sigma}(K, L) = -\frac{F_K F_L (F_K K + F_L L)}{KL\left[(F_L)^2 F_{KK} - 2F_K F_L F_{KL} + (F_K)^2 F_{LL}\right]}, \tag{4.28}$$

where all the derivatives are evaluated at the point (K, L). When $F(K, L)$ has CRS, the formula (4.28) simplifies to

$$\tilde{\sigma}(K, L) = \frac{F_K(K, L)F_L(K, L)}{F_{KL}(K, L)F(K, L)} = -\frac{f'(k)\left(f(k) - f'(k)k\right)}{f''(k)kf(k)} \equiv \sigma(k), \tag{4.29}$$

where $k \equiv K/L$.[12] We see that under CRS, the elasticity of substitution depends only on the capital-labor ratio k, not on the output level. We will now consider the case where the elasticity of substitution is independent also of the capital-labor ratio.

4.5 The CES production function

It can be shown[13] that if a neoclassical production function with CRS has a constant elasticity of factor substitution different from one, it must be of the form

$$Y = A\left[\alpha K^{\beta} + (1 - \alpha)L^{\beta}\right]^{\frac{1}{\beta}}, \tag{4.30}$$

where A, α, and β are parameters satisfying $A > 0$, $0 < \alpha < 1$, and $\beta < 1$, $\beta \neq 0$. This function has been used intensively in empirical studies and is called a *CES production function* (CES for Constant Elasticity of Substitution). The parameter A depends on the choice of measurement units; given this choice, it reflects the "total factor productivity". The parameters α and β are called the *distribution parameter* and the *substitution parameter,* respectively. The restriction $\beta < 1$ ensures that the isoquants are strictly convex to the origin. Note that if $\beta < 0$, the right-hand side of (4.30) is not defined when either K or L (or both) equal 0. We can circumvent this

[12] The formulas (4.28) and (4.29) are derived in Appendix D.

[13] See, e.g., Allen (1973), pp. 52-53.

problem by extending the domain of the CES function and assign the function value 0 to these points when $\beta < 0$. Continuity is maintained in the extended domain (see Appendix E).

By taking partial derivatives in (4.30) and substituting back we get

$$\frac{\partial Y}{\partial K} = \alpha A^{\beta} \left(\frac{Y}{K}\right)^{1-\beta} \quad \text{and} \quad \frac{\partial Y}{\partial L} = (1-\alpha)A^{\beta} \left(\frac{Y}{L}\right)^{1-\beta}, \tag{4.31}$$

where $Y/K = A\left[\alpha + (1-\alpha)k^{-\beta}\right]^{\frac{1}{\beta}}$ and $Y/L = A\left[\alpha k^{\beta} + 1 - \alpha\right]^{\frac{1}{\beta}}$. The marginal rate of substitution of K for L therefore is

$$MRS = \frac{\partial Y/\partial L}{\partial Y/\partial K} = \frac{1-\alpha}{\alpha} k^{1-\beta} > 0.$$

Consequently,

$$\frac{dMRS}{dk} = \frac{1-\alpha}{\alpha}(1-\beta)k^{-\beta},$$

where the inverse of the right-hand side is the value of $dk/dMRS$. Substituting these expressions into (4.27) gives

$$\sigma(K, L) = \frac{1}{1-\beta} \equiv \sigma, \tag{4.32}$$

confirming the constancy of the elasticity of substitution. Since $\beta < 1$, $\sigma > 0$ always. A higher substitution parameter, β, results in a higher elasticity of factor substitution, σ. And $\sigma \lesseqgtr 1$ for $\beta \lesseqgtr 0$, respectively.

Since $\beta = 0$ is not allowed in (4.30), at first sight we cannot get $\sigma = 1$ from this formula. Yet, $\sigma = 1$ can be introduced as the *limiting* case of (4.30) when $\beta \to 0$, which turns out to be the Cobb-Douglas function. Indeed, one can show[14] that, for fixed K and L,

$$A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} \to AK^{\alpha}L^{1-\alpha}, \text{ for } \beta \to 0.$$

By a similar procedure as above we find that a Cobb-Douglas function always has elasticity of substitution equal to 1; this is exactly the value taken by σ in (4.32) when $\beta = 0$. In addition, the Cobb-Douglas function is the *only* production function that has unit elasticity of substitution everywhere.

Another interesting limiting case of the CES function appears when, for fixed K and L, we let $\beta \to -\infty$ so that $\sigma \to 0$. We get

$$A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} \to A \min(K, L), \text{ for } \beta \to -\infty. \tag{4.33}$$

[14] Proofs of this and the further claims below are in Appendix E.

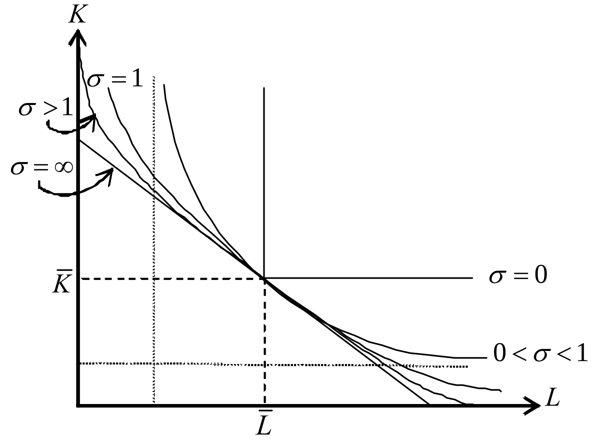


Figure 4.7: Isoquants for the CES production function for alternative values of $\sigma = 1/(1-\beta)$.

So in this case the CES function approaches a Leontief production function, the isoquants of which form a right angle, cf. Fig. 4.7. In the limit there is *no* possibility of substitution between capital and labor. In accordance with this the elasticity of substitution calculated from (4.32) approaches zero when β goes to $-\infty$.

Finally, let us consider the "opposite" transition. For fixed K and L we let the substitution parameter rise towards 1 and get

$$A\left[\alpha K^{\beta}+(1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} \rightarrow A\left[\alpha K+(1-\alpha)L\right], \text{ for } \beta \rightarrow 1.$$

Here the elasticity of substitution calculated from (4.32) tends to ∞ and the isoquants tend to straight lines with slope $-(1-\alpha)/\alpha$. In the limit, the production function thus becomes linear and capital and labor become *perfect substitutes.*

Fig. 4.7 depicts isoquants for alternative CES production functions and their limiting cases. In the Cobb-Douglas case, $\sigma = 1$, the horizontal and vertical asymptotes of the isoquant coincide with the coordinate axes. When $\sigma < 1$, the horizontal and vertical asymptotes of the isoquant belong to the interior of the positive quadrant. This implies that both capital and labor are essential inputs. When $\sigma > 1$, the isoquant terminates in points *on* the coordinate axes. Then neither capital, nor labor are essential inputs. Empirically there is not complete agreement about the "normal" size of the elasticity of factor substitution for industrialized economies. The elasticity also differs across the production sectors. A recent thorough econometric study (Antràs,

2004) of U.S. data indicate the aggregate elasticity of substitution to be in the interval $(0.5, 1.0)$.

The CES production function on intensive form

Dividing through by L on both sides of (4.30), we obtain the CES production function in intensive form,

$$y \equiv \frac{Y}{L} = A(\alpha k^{\beta} + 1 - \alpha)^{\frac{1}{\beta}}, \tag{4.34}$$

where $k \equiv K/L$. The marginal productivity of capital can be written

$$MPK = \frac{dy}{dk} = \alpha A \left[\alpha + (1-\alpha)k^{-\beta}\right]^{\frac{1-\beta}{\beta}} = \alpha A^{\beta} \left(\frac{y}{k}\right)^{1-\beta},$$

which of course equals $\partial Y/\partial K$ in (4.31). We see that the CES function violates either the lower or the upper Inada condition for MPK, depending on the sign of β. Indeed, when $\beta < 0$ (i.e., $\sigma < 1$), then for $k \rightarrow 0$ both y/k and dy/dk approach an upper bound equal to $A\alpha^{1/\beta} < \infty$, thus violating the lower Inada condition for MPK (see left panel of Fig. 2.3 of Chapter 2). It is also noteworthy that in this case, for $k \rightarrow \infty$, y approaches an upper bound equal to $A(1-\alpha)^{1/\beta} < \infty$. These features reflect the low degree of substitutability when $\beta < 0$.

When instead $\beta > 0$, there is a high degree of substitutability ($\sigma > 1$). Then, for $k \rightarrow \infty$ both y/k and $dy/dk \rightarrow A\alpha^{1/\beta} > 0$, thus violating the upper Inada condition for MPK (see right panel of Fig. 4.8). It is also noteworthy that for $k \rightarrow 0$, y approaches a lower bound equal to $A(1-\alpha)^{1/\beta} > 0$. Thus, in this case capital is not essential. At the same time $dy/dk \rightarrow \infty$ for $k \rightarrow \infty$ (so the lower Inada condition for the marginal productivity of capital holds). Details are in Appendix E.

Since (4.30) is symmetric in K and L, we get a series of symmetric results by considering output per unit of capital as $x \equiv Y/K = A\left[\alpha + (1-\alpha)(L/K)^{\beta}\right]^{1/\beta}$. In total, therefore, when there is low substitutability ($\beta < 0$), for fixed input of either of the production factors, there is an upper bound for how much an unlimited input of the other production factor can increase output. And when there is high substitutability ($\beta > 0$), there is no such bound and an unlimited input of either production factor take output to infinity.

The Cobb-Douglas case, i.e., the limiting case for $\beta \rightarrow 0$, constitutes in several respects an intermediate case in that *all* four Inada conditions are satisfied and we have $y \rightarrow 0$ for $k \rightarrow 0$, and $y \rightarrow \infty$ for $k \rightarrow \infty$.

Under perfect competition, with (4.34), the equilibrium real wage will be

$$w = (1-\alpha)A(\alpha k^{\beta} + 1 - \alpha)^{(1-\beta)/\beta} = w(k),$$

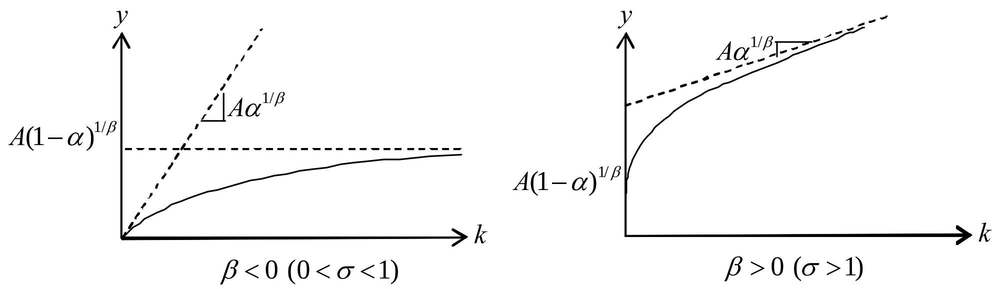


Figure 4.8: The CES production function for $\sigma < 1$ (left panel) and $\sigma > 1$ (right panel).

from (4.31).

Generalizations

The CES production function considered above has CRS. By adding an elasticity of scale parameter, γ, we get the generalized form

$$Y = A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{\gamma}{\beta}}, \qquad \gamma > 0. \tag{4.35}$$

In this form the CES function is homogeneous of degree γ. For $0 < \gamma < 1$, there are DRS, for $\gamma = 1$ CRS, and for $\gamma > 1$ IRS. If $\gamma \neq 1$, it may be convenient to consider $Q \equiv Y^{1/\gamma} = A^{1/\gamma}\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{1/\beta}$ and $q \equiv Q/L$ $= A^{1/\gamma}(\alpha k^{\beta} + 1 - \alpha)^{1/\beta}$.

The elasticity of substitution between K and L is $\sigma = 1/(1-\beta)$ whatever the value of γ. So including the limiting cases as well as non-constant returns to scale in the "family" of production functions with constant elasticity of substitution, we have the simple classification displayed in Table 4.1.

Table 4.1 The family of production functions with constant elasticity of substitution.

$\sigma = 0$	$0 < \sigma < 1$	$\sigma = 1$	$\sigma > 1$
Leontief	CES	Cobb-Douglas	CES

Note that only for $\gamma \leq 1$ is (4.35) a *neoclassical* production function. This is because, when $\gamma > 1$, the conditions $F_{KK} < 0$ and $F_{NN} < 0$ do not hold everywhere.

We may generalize further by assuming there are n inputs, in the amounts $X_1, X_2, ..., X_n$. Then the CES production function takes the form

$$Y = A\left[\alpha_1 X_1{}^{\beta} + \alpha_2 X_2{}^{\beta} + ...\alpha_n X_n{}^{\beta}\right]^{\frac{\gamma}{\beta}}, \ \alpha_i > 0 \text{ for all } i, \sum_i \alpha_i = 1, \gamma > 0. \tag{4.36}$$

In analogy with (4.27), for an n-factor production function the *partial elasticity of substitution* between factor i and factor j is defined as

$$\sigma_{ij} = \frac{MRS_{ij}}{X_i/X_j}\frac{d(X_i/X_j)}{dMRS_{ij}}_{|Y=\bar{Y}},$$

where it is understood that not only the output level but also all X_k, $k \neq i, j$, are kept constant. Note that $\sigma_{ji} = \sigma_{ij}$. In the CES case considered in (4.36), all the partial elasticities of substitution take the same value, $1/(1-\beta)$. For more general and flexible production functions applied in econometric work, see, e.g., Nadiri (1982).

4.6 Concluding remarks

(Incomplete)

OLG gives theoretical insights concerning macroeconomic implications of life cycle behavior, allows heterogeneity, provides training in seeing the economy as consisting of a heterogeneous population where the *distribution* of agent characteristics matters for the aggregate outcome.

Farmer (1993), p. 125, notes that OLG models are difficult to apply and for this reason much empirical work in applied general equilibrium theory has regrettably instead taken the representative agent approach.

Outlook: Rational speculative bubbles in general equilibrium, cf. Chapter 27.

4.7 Literature notes

1. We introduced the assumption that at the macroeconomic level the "direction" of technological progress tends to be Harrod-neutral. Otherwise the model will not be consistent with Kaldor's stylized facts. The Harrod-neutrality of the "direction" of technological progress is in the present model just an exogenous feature. This raises the question whether there are *mechanisms* tending to generate Harrod-neutrality. Fortunately new growth theory provides clues as to the sources of the speed as well as the direction of technological change. A facet of this theory is that the direction of technological

change is linked to the same economic forces as the speed, namely profit incentives. See Acemoglu (2003) and Jones (2006).

2. In Section 4.2 we claimed that from an empirical point of view, the adjustment towards a steady state can be from above as well as from below. Indeed, Cho and Graham (1996) find that "on average, countries with a lower income per adult are above their steady-state positions, while countries with a higher income are below their steady-state positions".

As to the assessment of the role of uncertainty for the condition that dynamic efficiency is satisfied, in addition to Abel et al. (1989) other useful sources include Ball et al. (1998) and Blanchard and Weil (2001).

3. In the Diamond OLG model as well as in many other models for a closed economy, a steady state and a balanced growth path imply each other. Indeed, they are two sides of the same process. There *exist* cases, however, where this equivalence does not hold (some open economy models and some models with *embodied* technological change, see Groth et al., 2010). Therefore, it is recommendable always to maintain a terminological distinction between the two concepts.

4. The definition of balanced growth refers to *aggregate* variables. At the same time as there is balanced growth at the aggregate level, *structural change* may occur. That is, a changing sectorial composition of the economy is under certain conditions compatible with balanced growth (in a generalized sense) at the aggregate level, cf. the "Kuznets facts" (see Kongsamut et al., 2001, and Acemoglu, 2009).

From here incomplete:

Demange and Laroque (1999, 2000) extend Diamond's OLG model to uncertain environments.

Keeping-up-with-the-Jones externalities. Do we work too much?

Blanchard, O., (2004) The Economic Future of Europe, J. Economic Perspectives, vol. 18 (4), 3-26.

Prescott, E. (2003), Why do Americans work so much more than Europeans? Federal Reserve Bank of Minneapolis Research Department Staff Report No. 321. I Ch. 5?

Chari, V. V., and P. J. Kehoe (2006), Modern macroeconomics in practice: How theory is shaping policy, J. of Economic Perspectives, vol. 20 (4), 3-28.

For expositions in depth of OLG modeling and dynamics in discrete time, see Azariadis (1993), de la Croix and Michel (2002), and Bewley (2007).

Dynamic inefficiency, see also Burmeister (1980).

Two-sector OLG: Galor (1992). Galor's book??

Bewley (2007).

Uzawa's theorem: Uzawa (1961), Schlicht (2006).

The way the intuition behind the Uzawa theorem was presented in Section 4.1 draws upon Jones and Scrimgeour (2008).

La Grandville's normalization of the CES function.

Other aspects of life cycle behavior: education. OLG where people live three periods.

4.8 Appendix

A. Growth arithmetic in discrete time

Let $t = 0, \pm 1, \pm 2, \ldots,$ and consider the variables $z_t, x_t,$ and y_t, assumed positive for all t. Define $\Delta z_t = z_t - z_{t-1}$ and Δx_t and Δy_t similarly. These Δ's need not be positive. The *growth rate* of x_t from period $t-1$ to period t is defined as $\Delta x_t / x_{t-1}$. The *growth factor* for x_t from period $t-1$ to period t is defined as $x_t/x_{t-1} = 1 + \Delta x_t/x_{t-1}$.

PRODUCT RULE If $z_t = x_t y_t$, then $1 + \Delta z_t/z_{t-1} = (1 + \Delta x_t/x_{t-1})(1 + \Delta y_t/y_{t-1})$ and $\Delta z_t/z_{t-1} \approx \Delta x_t/x_{t-1} + \Delta y_t/y_{t-1}$, when $\Delta x_t/x_{t-1}$ and $\Delta y_t/y_{t-1}$ are small.

Proof. By definition, $z_t = x_t y_t$, which implies $z_{t-1} + \Delta z_t = (x_{t-1} + \Delta x_t)(y_{t-1} + \Delta y_t)$. Dividing by $z_{t-1} = x_{t-1} y_{t-1}$ gives $1 + \Delta z_t/z_{t-1} = (1 + \Delta x_t/x_{t-1})(1 + \Delta y_t/y_{t-1})$ as claimed. By carrying out the multiplication on the right-hand side of this equation, we get $1 + \Delta z_t/z_{t-1} = 1 + \Delta x_t/x_{t-1} + \Delta y_t/y_{t-1} + (\Delta x_t/x_{t-1})(\Delta y_t/y_{t-1}) \approx 1 + \Delta x_t/x_{t-1} + \Delta y_t/y_{t-1}$ when $\Delta x_t/x_{t-1}$ and $\Delta y_t/y_{t-1}$ are small. Subtracting 1 on both sides gives the stated approximation. □

So the product of two positive variables will grow at a rate approximately equal to the sum of the growth rates of the two variables.

FRACTION RULE If $z_t = \frac{x_t}{y_t}$, then $1 + \Delta z_t/z_{t-1} = \frac{1+\Delta x_t/x_{t-1}}{1+\Delta y_t/y_{t-1}}$ and $\Delta z_t/z_{t-1} \approx \Delta x_t/x_{t-1} - \Delta y_t/y_{t-1}$, when $\Delta y_t/y_{t-1}$ is small.

Proof. By interchanging z and x in Product Rule and rearranging, we get $1 + \Delta z_t/z_{t-1} = \frac{1+\Delta x_t/x_{t-1}}{1+\Delta y_t/y_{t-1}}$, as stated in the first part of the claim. Subtracting 1 on both sides gives $\Delta z_t/z_{t-1} = \frac{\Delta x_t/x_{t-1} - \Delta y_t/y_{t-1}}{1+\Delta y_t/y_{t-1}} \approx \Delta x_t/x_{t-1} - \Delta y_t/y_{t-1}$, when $\Delta x_t/x_{t-1}$ and $\Delta y_t/y_{t-1}$ are small. This proves the stated approximation. □

So the ratio between two positive variables will grow at a rate approximately equal to the excess of the growth rate of the numerator over that of the denominator. An implication of the first part of Claim 2 is that the ratio between two positive variables is constant if and only if the variables have the same growth rate (not necessarily constant or positive).

Given a time series $x_0, x_1, ..., x_n$ with a more or less monotonous trend, by the *average growth rate* per period (with discrete compounding) is normally meant a g which satisfies $x_n = x_0(1+g)^n$. The solution for g is $g = (x_n/x_0)^{1/n} - 1$.

Finally, the following approximation may be useful if used with caution:

THE GROWTH FACTOR With n denoting a positive integer above 1 but "not too large", the growth factor $(1+g)^n$ can be approximated by $1+ng$ when g is small. For $g \neq 0$, the approximation error is larger the larger is n.

Proof. (i) We prove the claim by induction. Suppose the claim holds for a fixed $n \geq 2$, i.e., $(1+g)^n \approx 1+ng$ for g small. Then $(1+g)^{n+1} = (1+g)^n(1+g)$ $\approx (1+ng)(1+g) = 1+ng+g+ng^2 \approx 1+(n+1)g$ for g small. So the claim holds also for $n+1$. Since $(1+g)^2 = 1+2g+g^2 \approx 1+2g$, for g small, the claim does indeed hold for $n = 2$. (ii) □

B. Proof of Uzawa's theorem

For convenience we restate the theorem here:

PROPOSITION 2 *(Uzawa's balanced growth theorem).* Let $\{(K_t, Y_t, C_t)\}_{t=0}^{\infty}$ be a path along which Y_t, K_t, C_t, and $S_t \equiv Y_t - C_t$ are positive for all $t = 0, 1, 2, \ldots$, and satisfy the dynamic resource constraint (4.2), given the production function (4.6) and the labor force (4.7). Then:
(i) a *necessary* condition for this path to be a balanced growth path is that along the path it holds that

$$Y_t = \tilde{F}(K_t, T_t L_t; 0), \tag{*}$$

where $T_t = (1+\gamma)^t$ with $\gamma \equiv [(1+g_Y)/(1+n)] - 1$ and g_Y being the constant growth rate of output along the balanced growth path;
(ii) for any $\gamma \geq 0$ such that there is a $q > (1+\gamma)(1+n)+\delta$ with the property that $\tilde{F}(1, k^{-1}; 0) = q$ for some $k > 0$ (i.e., at $t = 0$ the production function $\tilde{F}$ in (4.6) allows an output-capital ratio equal to q), a *sufficient* condition for the existence of a balanced growth path with output-capital ratio q is that the technology can be written as in (*) with $T_t = (1+\gamma)^t$.

Proof (i) Suppose the given path $\{(K_t, Y_t, C_t)\}_{t=0}^{\infty}$ is a balanced growth path. By definition, g_K and g_Y are then constant, so that $K_t = K_0(1+g_K)^t$ and $Y_t = Y_0(1+g_Y)^t$. With $t = 0$ in (4.6) we then have

$$Y_t(1+g_Y)^{-t} = Y_0 = \tilde{F}(K_0, L_0; 0) = \tilde{F}(K_t(1+g_K)^{-t}, L_t(1+n)^{-t}; 0). \tag{4.37}$$

In view of the assumption that $S_t \equiv Y_t - C_t > 0$, we know from (i) of Proposition 1, that Y/K is constant so that $g_Y = g_K$. By CRS, (4.37) then implies

$$Y_t = \tilde{F}(K_t, (1+g_Y)^t(1+n)^{-t}L_t; 0).$$

We see that (*) holds for $T_t = (1+\gamma)^t$ with $\gamma \equiv [(1+g_Y)/(1+n)] - 1$.

(ii) Suppose (*) holds with $T_t = (1+\gamma)^t$. Let $\gamma \geq 0$ be given such that there is a $q > (1+\gamma)(1+n)+\delta$ with the property that $\tilde{F}(1, k^{-1}; 0) = q$ for some $k > 0$. We claim that with $K_0 = kL_0$, $s \equiv [(1+\gamma)(1+n) - (1-\delta)]/q$, and $S_t = sY_t$, (4.2), (4.7), and (*) imply $Y_t/K_t = q$ for all $t = 0, 1, 2, \ldots$. We use induction to show this.[15] First, by (*)

$$\frac{Y_0}{K_0} = \frac{\tilde{F}(K_0, L_0; 0)}{K_0} = \tilde{F}(1, k^{-1}; 0) = q,$$

where the second equality comes from CRS. Next, suppose that for some t, $Y_t/K_t = q$. From (4.2) and (*) we then get

$$\begin{aligned} K_{t+1} &= sY_t + (1-\delta)K_t = \left(s\frac{Y_t}{K_t} + 1 - \delta\right)K_t = (sq+1-\delta)K_t \\ &= (1+\gamma)(1+n)K_t. \end{aligned} \tag{4.38}$$

With this K_{t+1}, (*) gives

$$\begin{aligned} Y_{t+1} &= \tilde{F}((1+\gamma)(1+n)K_t, (1+\gamma)^{t+1}L_t(1+n); 0) \\ &= (1+\gamma)(1+n)\tilde{F}(K_t, (1+\gamma)^t L_t; 0) = (1+\gamma)(1+n)Y_t, \end{aligned} \tag{4.39}$$

where the second equality comes from CRS and the last from (*). Thus, $Y_{t+1}/K_{t+1} = Y_t/K_t$ and so $Y_{t+1}/K_{t+1} = q$ when $Y_t/K_t = q$. By induction, $Y_t/K_t = q$ for all $t = 0, 1, 2, \ldots$, as was to be shown. Finally, (4.38) and (4.39) show that g_K and g_Y are constant along the constructed path; since $C_t \equiv Y_t - S_t = (1-s)Y_t$, also g_C is constant. Hence, the constructed path is a balanced growth path. $\square$

It is noteworthy that the proof of the sufficiency part of the theorem is *constructive*. It provides a method for constructing a hypothetical balanced growth path.

[15] Induction is the following principle: if we can show that (a) a certain property holds at $t+1$, if it holds at t, and (b) the property holds at $t = 0$, then it must hold for all $t = 0, 1, 2, \ldots$.

C. Homothetic utility functions

Generalities A set C in $\mathbb{R}^n$ is called a *cone* if $x \in C$ and $\lambda > 0$ implies $\lambda x \in C$. A function $f(\mathbf{x}) = f(x_1,\ldots,x_n)$ is *homothetic* in the cone C if for all $\mathbf{x}, \mathbf{y} \in C$ and all $\lambda > 0$, $f(\mathbf{x}) = f(\mathbf{y})$ implies $f(\lambda\mathbf{x}) = f(\lambda\mathbf{y})$.

Consider the continuous utility function $U(x_1, x_2)$, defined in $\mathbb{R}^2_+$. Suppose U is *homothetic* and that the consumption bundles (x_1, x_2) and (y_1, y_2) are on the same indifference curve, i.e., $U(x_1, x_2) = U(y_1, y_2)$. Then for any $\lambda > 0$ we have $U(\lambda x_1, \lambda x_2) = U(\lambda y_1, \lambda y_2)$ so that the bundles $(\lambda x_1, \lambda x_2)$ and $(\lambda y_1, \lambda y_2)$ are also on the same indifference curve.

For a continuous utility function $U(x_1, x_2)$, defined in $\mathbb{R}^2_+$ and increasing in each of its arguments (as is our life time utility function in the Diamond model), one can show that U is homothetic if and only if U can be written $U(x_1, x_2) \equiv F(f(x_1, x_2))$ where the function f is homogeneous of degree one and F is an increasing function. The "if" part is easily shown. Indeed, if $U(x_1, x_2) = U(y_1, y_2)$, then $F(f(x_1, x_2)) = F(f(y_1, y_2))$. Since F is increasing, this implies $f(x_1, x_2) = f(y_1, y_2)$. Because f is homogeneous of degree one, if $\lambda > 0$, then $f(\lambda x_1, \lambda x_2) = \lambda f(x_1, x_2)$ and $f(\lambda y_1, \lambda y_2) = \lambda f(y_1, y_2)$ so that $U(\lambda x_1, \lambda x_2) = F(f(\lambda x_1, \lambda x_2)) = F(f(\lambda y_1, \lambda y_2)) = U(\lambda y_1, \lambda y_2)$, which shows that U is homothetic. As to the "only if" part, see Sydsaeter et al. (2002).

Using differentiability of our homothetic time utility function $U(x_1, x_2) \equiv F(f(x_1, x_2))$, we find the marginal rate of substitution of good 2 for good 1 to be

$$MRS = \frac{dx_2}{dx_1}_{|U=\bar{U}} = \frac{\partial U/\partial x_1}{\partial U/\partial x_2} = \frac{F' f_1(x_1, x_2)}{F' f_2(x_1, x_2)} = \frac{f_1(1, \frac{x_2}{x_1})}{f_2(1, \frac{x_2}{x_1})}. \tag{4.40}$$

The last equality is due to Euler's theorem saying that when f is homogeneous of degree 1, then the first-order partial derivatives of f are homogeneous of degree 0. Now, (4.40) implies that for a given MRS, in optimum reflecting a given relative price of the two goods, the same consumption ratio, x_2/x_1, will be chosen whatever the budget. For a given relative price, a rising budget (rising wealth) will change the position of the budget line, but not its slope. So MRS will not change, which implies that the chosen pair (x_1, x_2) will move outward along a given ray in $\mathbb{R}^2_+$. Indeed, as the intercepts with the axes rise proportionately with the budget (the wealth), so will x_1 and x_2.

Proof that the utility function in (4.22) is homothetic In Section 4.2 we claimed that (4.22) is a homothetic utility function. This can be proved in the following way. There are two cases to consider. *Case 1:* $\theta > 0$, $\theta \neq 1$.

We rewrite (4.22) as

$$U(c_1, c_2) = \frac{1}{1-\theta}\left[(c_1^{1-\theta} + \beta c_2^{1-\theta})^{1/(1-\theta)}\right]^{1-\theta} - \frac{1+\beta}{1-\theta},$$

where $\beta \equiv (1+\rho)^{-1}$. The function $x = g(c_1, c_2) \equiv (c_1^{1-\theta} + \beta c_2^{1-\theta})^{1/(1-\theta)}$ is homogeneous of degree one and the function $G(x) \equiv (1/(1-\theta))x^{1-\theta} - (1+\beta)/(1-\theta)$ is an increasing function, given $\theta > 0$, $\theta \neq 1$. *Case 2:* $\theta = 1$. Here we start from $U(c_1, c_2) = \ln c_1 + \beta \ln c_2$. This can be written

$$U(c_1, c_2) = (1+\beta)\ln\left[(c_1 c_2^{\beta})^{1/(1+\beta)}\right],$$

where $x = g(c_1, c_2) = (c_1 c_2^{\beta})^{1/(1+\beta)}$ is homogeneous of degree one and $G(x) \equiv (1+\beta)\ln x$ is an increasing function. $\square$

D. General formulas for the elasticity of factor substitution

Given the neoclassical production function $F(K, L)$, the slope of the isoquant $F(K, L) = \bar{Y}$ at the point $(\bar{K}, \bar{L})$ is

$$\frac{dK}{dL}_{|Y=\bar{Y}} = -MRS = -\frac{F_L(\bar{K}, \bar{L})}{F_K(\bar{K}, \bar{L})}. \tag{4.41}$$

We consider this slope as a function of the value of $k \equiv K/L$ as we move along the isoquant. The derivative of this function is

$$\begin{aligned} -\frac{dMRS}{dk}_{|Y=\bar{Y}} &= -\frac{dMRS}{dL}_{|Y=\bar{Y}}\frac{dL}{dk}_{|Y=\bar{Y}} \\ &= -\frac{(F_L)^2 F_{KK} - 2F_K F_L F_{KL} + (F_K)^2 F_{LL}}{F_K^3}\frac{dL}{dk}_{|Y=\bar{Y}} \end{aligned} \tag{4.42}$$

by (2.50). In view of $L \equiv K/k$ we have

$$\frac{dL}{dk}_{|Y=\bar{Y}} = \frac{k\frac{dK}{dk}_{|Y=\bar{Y}} - K}{k^2} = \frac{k\frac{dK}{dL}_{|Y=\bar{Y}}\frac{dL}{dk}_{|Y=\bar{Y}} - K}{k^2} = \frac{-kMRS\frac{dL}{dk}_{|Y=\bar{Y}} - K}{k^2}.$$

From this we find

$$\frac{dL}{dk}_{|Y=\bar{Y}} = -\frac{K}{(k+MRS)k},$$

to be substituted into (4.42). Finally, we substitute the inverse of (4.42) together with (4.41) into the definition of the elasticity of factor substitution:

$$\begin{aligned}
\sigma(K,L) &\equiv \frac{MRS}{k}\frac{dk}{dMRS}_{|Y=\bar{Y}} \\
&= -\frac{F_L/F_K}{k}\frac{(k+F_L/F_K)k}{K}\frac{F_K^3}{[(F_L)^2F_{KK}-2F_KF_LF_{KL}+(F_K)^2F_{LL}]} \\
&= -\frac{F_KF_L(F_KK+F_LL)}{KL\,[(F_L)^2F_{KK}-2F_KF_LF_{KL}+(F_K)^2F_{LL}]},
\end{aligned}$$

which is the same as (4.28).

Under CRS, this reduces to

$$\begin{aligned}
\sigma(K,L) &= -\frac{F_KF_LF(K,L)}{KL\,[(F_L)^2F_{KK}-2F_KF_LF_{KL}+(F_K)^2F_{LL}]} \text{ (from (2.51) with } h=1) \\
&= -\frac{F_KF_LF(K,L)}{KLF_{KL}\,[-(F_L)^2L/K-2F_KF_L-(F_K)^2K/L]} \text{ (from (2.57))} \\
&= \frac{F_KF_LF(K,L)}{F_{KL}(F_LL+F_KK)^2} = \frac{F_KF_L}{F_{KL}F(K,L)}, \text{ (using (2.51) with } h=1)
\end{aligned}$$

which proves the first part of (4.29). The second part is a simple consequence of the results (2.14) and (2.15) in the text.

E. Properties of the CES production function

The generalized CES production function is

$$Y = A\left[\alpha K^{\beta}+(1-\alpha)L^{\beta}\right]^{\frac{\gamma}{\beta}}, \tag{4.43}$$

where A, α, and β are parameters satisfying $A>0$, $0<\alpha<1$, and $\beta<1$, $\beta\neq 0, \gamma>0$. If $\gamma<1$, there is DRS, if $\gamma=1$, CRS, and if $\gamma>1$, IRS. The elasticity of substitution is always $\sigma = 1/(1-\beta)$. Throughout below, k means K/L.

The limiting functional forms We claimed in the text that, for fixed $K>0$ and $L>0$, (4.43) implies:

$$\begin{aligned}
\lim_{\beta\to 0} Y &= A(K^{\alpha}L^{1-\alpha})^{\gamma} = AL^{\gamma}k^{\alpha\gamma}, && (*) \\
\lim_{\beta\to -\infty} Y &= A\min(K^{\gamma},L^{\gamma}) = AL^{\gamma}\min(k^{\gamma},1). && (**)
\end{aligned}$$

Proof. Let $q \equiv Y/(AL^\gamma)$. Then $q = (\alpha k^\beta + 1 - \alpha)^{\gamma/\beta}$ so that

$$\ln q = \frac{\gamma \ln(\alpha k^\beta + 1 - \alpha)}{\beta} \equiv \frac{m(\beta)}{\beta}, \tag{4.44}$$

where

$$m'(\beta) = \frac{\gamma \alpha k^\beta \ln k}{\alpha k^\beta + 1 - \alpha} = \frac{\gamma \alpha \ln k}{\alpha + (1-\alpha)k^{-\beta}}. \tag{4.45}$$

Hence, by L'Hôpital's rule for "0/0",

$$\lim_{\beta \to 0} \ln q = \lim_{\beta \to 0} \frac{m'(\beta)}{1} = \gamma \alpha \ln k = \ln k^{\gamma\alpha},$$

so that $\lim_{\beta \to 0} q = k^{\gamma\alpha}$, which proves (*). As to (**), note that

$$\lim_{\beta \to -\infty} k^\beta = \lim_{\beta \to -\infty} \frac{1}{k^{-\beta}} \to \begin{cases} 0 \text{ for } k > 1, \\ 1 \text{ for } k = 1, \\ \infty \text{ for } k < 1. \end{cases}$$

Hence, by (4.44),

$$\lim_{\beta \to -\infty} \ln q = \begin{cases} 0 \text{ for } k \geq 1, \\ \lim_{\beta \to -\infty} \frac{m'(\beta)}{1} = \gamma \ln k = \ln k^\gamma \text{ for } k < 1, \end{cases}$$

where the result for $k < 1$ is based on L'Hôpital's rule for "∞/-∞". Consequently,

$$\lim_{\beta \to -\infty} q = \begin{cases} 1 \text{ for } k \geq 1, \\ k^\gamma \text{ for } k < 1, \end{cases}$$

which proves (**). $\square$

Properties of the isoquants of the CES function The absolute value of the slope of an isoquant for (4.43) in (L, K) space is

$$MRS_{KL} = \frac{\partial Y/\partial L}{\partial Y/\partial K} = \frac{1-\alpha}{\alpha} k^{1-\beta} \to \begin{cases} 0 \text{ for } k \to 0, \\ \infty \text{ for } k \to \infty. \end{cases}$$

This holds whether $\beta < 0$ or $\beta > 0$.

Concerning the asymptotes and terminal points, if any, of the isoquant $Y = \bar{Y}$ we have from (4.43) $\bar{Y}^{\beta/\gamma} = A\left[\alpha K^\beta + (1-\alpha)L^\beta\right]$. Hence,

$$\begin{aligned} K &= \left(\frac{\bar{Y}^{\frac{\beta}{\gamma}}}{A\alpha} - \frac{1-\alpha}{\alpha} L^\beta\right)^{\frac{1}{\beta}}, \\ L &= \left(\frac{\bar{Y}^{\frac{\beta}{\gamma}}}{A(1-\alpha)} - \frac{\alpha}{1-\alpha} K^\beta\right)^{\frac{1}{\beta}}. \end{aligned}$$

From these two equations follows, when $\beta < 0$ (i.e., $0 < \sigma < 1$), that

$$\begin{aligned} K &\rightarrow (A\alpha)^{-\frac{1}{\beta}}\bar{Y}^{\frac{1}{\gamma}} \text{ for } L \rightarrow \infty, \\ L &\rightarrow [A(1-\alpha)]^{-\frac{1}{\beta}}\bar{Y}^{\frac{1}{\gamma}} \text{ for } K \rightarrow \infty. \end{aligned}$$

When instead $\beta > 0$ (i.e., $\sigma > 1$), the same limiting formulas obtain for $L \rightarrow 0$ and $K \rightarrow 0$, respectively.

Properties of the CES function on intensive form Given $\gamma = 1$, i.e., CRS, we have $y \equiv Y/L = A(\alpha k^{\beta} + 1 - \alpha)^{1/\beta}$ from (4.43). Then

$$\frac{dy}{dk} = A\frac{1}{\beta}(\alpha k^{\beta} + 1 - \alpha)^{\frac{1}{\beta}-1}\alpha\beta k^{\beta-1} = A\alpha\left[\alpha + (1-\alpha)k^{-\beta}\right]^{\frac{1-\beta}{\beta}}.$$

Hence, when $\beta < 0$ (i.e., $0 < \sigma < 1$),

$$\begin{aligned} y &= \frac{A}{(ak^{\beta} + 1 - \alpha)^{-1/\beta}} \rightarrow \begin{cases} 0 \text{ for } k \rightarrow 0, \\ A(1-\alpha)^{1/\beta} \text{ for } k \rightarrow \infty. \end{cases} \\ \frac{dy}{dk} &= \frac{A\alpha}{[\alpha + (1-\alpha)k^{-\beta}]^{(\beta-1)/\beta}} \rightarrow \begin{cases} A\alpha^{1/\beta} \text{ for } k \rightarrow 0, \\ 0 \text{ for } k \rightarrow \infty. \end{cases} \end{aligned}$$

If instead $\beta > 0$ (i.e., $\sigma > 1$),

$$\begin{aligned} y &\rightarrow \begin{cases} A(1-\alpha)^{1/\beta} \text{ for } k \rightarrow 0, \\ \infty \text{ for } k \rightarrow \infty. \end{cases} \\ \frac{dy}{dk} &\rightarrow \begin{cases} \infty \text{ for } k \rightarrow 0, \\ A\alpha^{1/\beta} \text{ for } k \rightarrow \infty. \end{cases} \end{aligned}$$

The output-capital ratio is $y/k = A\left[\alpha + (1-\alpha)k^{-\beta}\right]^{\frac{1}{\beta}}$ and has the same limiting values as dy/dk, when $\beta > 0$.

Continuity at the boundary of $\mathbb{R}^2_+$ When $\beta > 0$, the right-hand side of (4.43) is defined and continuous also on the boundary of $\mathbb{R}^2_+$. When $\beta < 0$, however, the right-hand side is not defined on the boundary. We circumvent this problem by redefining the CES function in the following way when $\beta < 0$:

$$Y = F(K, L) = \begin{cases} A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{\gamma}{\beta}} \text{ when } K > 0 \text{ and } L > 0, \\ 0 \text{ when either } K \text{ or } L \text{ equals } 0. \end{cases} \tag{4.46}$$

We now show that continuity holds in the extended domain. When $K > 0$ and $L > 0$, we have

$$Y^{\frac{\beta}{\gamma}} = A^{\frac{\beta}{\gamma}}\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right] \equiv A^{\frac{\beta}{\gamma}}G(K, L). \tag{4.47}$$

Let $\beta < 0$ and $(K, L) \to (0, 0)$. Then, $G(K, L) \to \infty$, and so $Y^{\beta/\gamma} \to \infty$. Since $\beta/\gamma < 0$, this implies $Y \to 0 = F(0, 0)$, where the equality follows from the definition in (4.46). Next, consider a fixed $L > 0$ and rewrite (4.47) as

$$\begin{aligned} Y^{\frac{1}{\gamma}} &= A^{\frac{1}{\gamma}} \left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} = A^{\frac{1}{\gamma}} L(\alpha k^{\beta} + 1 - \alpha)^{\frac{1}{\beta}} \\ &= \frac{A^{\frac{1}{\gamma}} L}{(ak^{\beta} + 1 - \alpha)^{-1/\beta}} \to 0 \text{ for } k \to 0, \end{aligned}$$

when $\beta < 0$. Since $1/\gamma > 0$, this implies $Y \to 0 = F(0, L)$, from (4.46). Finally, consider a fixed $K > 0$ and let $L/K \to 0$. Then, by an analogue argument we get $Y \to 0 = F(K, 0)$, (4.46). So continuity is maintained in the extended domain.

4.9 Exercises

4.1 *(the aggregate saving rate in steady state)*

a) In a well-behaved Diamond OLG model let n be the rate of population growth and k^* the steady state capital-labor ratio (until further notice, we ignore technological progress). Derive a formula for the long-run aggregate net saving rate, S^N/Y, in terms of n and k^*. *Hint:* use that for a closed economy $S^N = K_{t+1} - K_t$.

b) In the Solow growth model without technological change a similar relation holds, but with a different interpretation of the causality. Explain.

c) Compare your result in a) with the formula for S^N/Y in steady state one gets in *any* model with the same CRS-production function and no technological change. Comment.

d) Assume that $n = 0$. What does the formula from a) tell you about the level of net aggregate savings in this case? Give the intuition behind the result in terms of the aggregate saving by any generation in two consecutive periods. One might think that people's rate of impatience (in Diamond's model the rate of time preference ρ) affect S^N/Y in steady state. Does it in this case? Why or why not?

e) Suppose there is Harrod-neutral technological progress at the constant rate $g > 0$. Derive a formula for the aggregate net saving rate in the long run in a well-behaved Diamond model in this case.

f) Answer d) with "from a)" replaced by "from e)". Comment.

g) Consider the statement: "In Diamond's OLG model any generation saves as much when young as it dissaves when old." True or false? Why?

4.2 (*increasing returns to scale and balanced growth*)

Chapter 6

Long-run aspects of fiscal policy and public debt

6.1 Introduction

We consider an economy with a government that provides public goods and services and finances its spending by taxation and borrowing. The term *fiscal policy* refers to policy that involves decisions about the government's spending and the financing of this spending, be it by taxes or debt issue. The government's choice concerning the level and composition of its spending and how to finance it, may aim at:

1 affecting resource allocation (deliver public goods that would otherwise not be supplied in a sufficient amount, correct externalities and other markets failures, prevent monopoly inefficiencies, provide social insurance);

2 affecting income distribution, be it (a) within generations and/or (b) between generations;

3 providing macroeconomic stabilization (dampen business cycle fluctuations through aggregate demand policies).

The design of fiscal policy with regard to the aims 1 and 2 at a disaggregate level is a major theme within public economics. Macroeconomics deals with aim 3 and the big-picture aspects of 1 and 2, like policy to enhance economic growth.

In this chapter we address issues of fiscal sustainability and long-run implications of debt finance. Section 6.2 introduces the basics of government budgeting and Section 6.3 defines the concepts of *government solvency* and *fiscal sustainability.* In Section 6.4 the analytics of debt dynamics is presented. As an example the Stability and Growth Pact of the EMU (the Economic and Monetary Union of the European Union) is discussed. Section 6.5 looks more closely at the link between government solvency and the government's *No-Ponzi-Game condition* and *intertemporal budget constraint.* This is applied in Section 6.6 to a study of the *Ricardian equivalence* issue; applying the Diamond OLG framework we address questions like: Is Ricardian equivalence likely to be a good approximation to reality? If not, why?

Before entering the specialized sections, here are some factors that constrain public financing instruments:

(i) financing by debt issue is constrained by the need to remain solvent and avoid catastrophic debt dynamics;

(ii) financing by taxes is limited by problems arising from:

  (a) distortionary supply-side effects of many kinds of taxes;

  (b) tax evasion (cf. the rise of the shadow economy, tax havens used by multinationals, etc.).

(iii) time lags (such as recognition lag, decision lag, implementation lag, and effect lag);

(iv) credibility problems due to time-inconsistency;

(v) limitations imposed by political processes, bureaucratic self-interest, and rent seeking.

Point (i) is discussed in detail in the sections below. The remaining points, (ii) $-$ (v) are not addressed specifically in this chapter. They should always be kept in mind, however, when discussing fiscal policy. Hence a few remarks here. First, the points (ii.a) and (ii.b) give rise to the so-called *Laffer curve* (after the American economist Arthur Laffer, 1940-). The Laffer curve refers to a hump-shaped relationship between the income tax rate and the tax revenue. For simplicity, suppose the tax revenue equals income times a given average tax rate. A 0% tax rate and probably also a 100% tax rate generate no tax revenue. As the tax rate increases from a low initial level, a rising tax revenue is obtained. But after a certain point some people may begin to work less (in the legal economy) or stop reporting all their income. While Laffer was seemingly wrong about where USA was on the curve (see Fullerton 1982), surely there is likely to be a point above which tax revenue depends negatively on the tax rate. On the other hand, in practice there is no such thing as *the* Laffer curve. This is because a lot of contingencies are involved. Income taxes are typically progressive, marginal tax rates differ from average tax rates, it matters how the tax revenue is spent, etc.

That time lags, point (iii), may be a constraining factor is especially important for macroeconomic stabilization policy aiming at dampening business cycle fluctuations. If these potential lags are ignored, there is a risk that the government intervention comes too late and ends up amplifying the fluctuations instead of dampening them. In particular the *monetarists*, lead by the American economist Milton Friedman (1912-2006), have warned against this risk.

Point (iv) hints at the fact that when outcomes depend on expectations in the private sector, governments sometimes face a so-called *time-inconsistency* problem. *Time-inconsistency* refers to the possible temptation of the government to deviate from its previously announced course of action once the private sector has acted. An example: With the purpose of stimulating private saving, the government announces that it will not tax financial wealth. Nevertheless, when financial wealth has reached a certain level, it constitutes a tempting base for taxation and so a tax on wealth might be levied. To the extent the private sector anticipates this, the attempt to stimulate private saving in the first place fails. We return to this kind of problems in Chapter 13.

Finally, political processes, bureaucratic self-interest, and rent seeking may interfere with fiscal policy. This is a theme in the branch of economics called *political economy* and is outside the scope of this chapter.

6.2 The government budget

The basics of budget accounting cannot be described without including money, nominal prices, and inflation. Indeed, money and inflation sometimes play a key role in budget policy and fiscal rules − for example in connection with the Stability and Growth Pact of the EMU. Elementary aspects of money and inflation will therefore be included in this section. We shall not, however, consider money and inflation in any systematic way until later chapters.

Table 6.1 lists key variables of government budgeting.

Table 6.1. List of main variable symbols

Variable	*Meaning*
Y_t	real GDP
C_t^g	public consumption
I_t^g	public fixed capital investment
G_t	$\equiv C_t^g + I_t^g$ real public spending on goods and services
X_t	real transfer payments
$\tilde{T}_t$	real gross tax revenue
$T_t \equiv \tilde{T}_t - X_t$	real net tax revenue
M_t	the monetary base
P_t	price level (in money) for goods and services (the GDP deflator)
D_t	nominal net public debt
$B_t \equiv \frac{D_t}{P_{t-1}}$	real net public debt
$b_t \equiv \frac{B_t}{Y_t}$	debt-income ratio
i_t	nominal short-term interest rate
Δx_t	$= x_t - x_{t-1}$ (where x is some arbitrary variable)
$\pi_t \equiv \frac{\Delta P_t}{P_{t-1}}$	$\equiv \frac{P_t - P_{t-1}}{P_{t-1}}$ inflation rate
$1 + r_t \equiv \frac{P_{t-1}(1+i_t)}{P_t}$	$\equiv \frac{1+i_t}{1+\pi_t}$ real short-term interest rate

Note that Y_t, G_t, and T_t are quantities defined *per period*, or more generally, *per time unit,* and are thus flow variables. On the other hand, M_t, D_t, and B_t are stock variables, that is, quantities defined at a given point in time, here at the *beginning* of period t. We measure D and B *net* of financial claims held by the government. The terms public debt and government debt are used synonymously. The monetary base, M_t, is currency plus cash reserves held in the central bank, available at the beginning of period t. While most countries have positive government debt, there are exceptions (...). In contrast, M_t is by definition nonnegative.

Ignoring uncertainty and risk of default, the nominal interest rate on government bonds must be the same as that on other interest-bearing assets in the economy. For ease of exposition we imagine that all debt is short-term, i.e., one-period bonds. Given the debt D_t at the start of period t, the nominal expenditure to be made at the end of the period to redeem the outstanding debt thus is $D_t + i_t D_t$.[1] The total nominal government expenditure in period t will be

$$P_t(G_t + X_t) + D_t(1 + i_t).$$

[1] One may define the quantity of government bonds at time t such that *one* government bond has face value (value at maturity) equal to $1 + i_t$. Then D_t can be interpreted as the *quantity* of outstanding government bonds in period t as well as the nominal market value of this quantity at the start of period t.

It is common to refer to this as expenditure "in period t". Yet, in a discrete time framework (with a period length of a year or a quarter corresponding to typical macroeconomic data) one has to imagine that the *payment* for goods and services delivered in the period occurs either at the beginning or the end of the period. We follow the latter interpretation and so the nominal price level P_t for period-t goods and services refers to payment occurring at the *end* of period t. As an implication, the real value, B_t, of government debt at the beginning of period t (= end of period $t-1$) is D_t/P_{t-1}. This may look a little awkward but is nevertheless entirely meaningful. Indeed, D_t is a *stock* of liabilities at the beginning of period t while P_{t-1} is a price referring to a *flow* paid for at the end of period $t-1$ which is essentially the same point in time as the beginning of period t. Anyway, whatever timing convention is chosen, some kind of awkwardness will always arise in discrete time analysis. This is because the discrete time approach artificially treats the continuous flow of time as a sequence of discrete points in time.[2]

The government expenditure is financed by a combination of taxes, bonds issue, and change in the monetary base:

$$P_t\tilde{T}_t + \Delta M_{t+1} + D_{t+1} = P_t(G_t + X_t) + D_t(1 + i_t). \tag{6.1}$$

By rearranging we have

$$\Delta D_{t+1} + \Delta M_{t+1} = P_t(G_t + X_t - \tilde{T}_t) + i_t D_t. \tag{6.2}$$

In standard government budget accounting the nominal *government budget deficit, GBD*, is defined as the excess of total government spending over government revenue, $P\tilde{T}$. That is, according to this definition the right-hand side of (6.2) is the nominal budget deficit in period t, GBD_t. The first term on the right-hand side, $P_t(G_t + X_t - \tilde{T}_t)$, is named the *primary budget deficit* (non-interest spending less taxes). The second term, $i_t D_t$, is called the *debt service*.[3] Similarly, $P_t(\tilde{T}_t - X_t - G_t)$ is called the *primary budget surplus.* Note that negative values of "deficits" and "surpluses" represent positive values of "surpluses" and "deficits", respectively.

We immediately see that this budget accounting is different from "normal" budgeting principles. Private companies, for example, typically have separate capital and operating budgets. In contrast, the budget deficit defined above treats that part of G which represents government *net investment* parallel to government consumption. Government net investment is

[2] This kind of problems is avoided when government budgeting is formulated in continuous time, cf. Chapter 13.

[3] If $D_t < 0$, the government has positive net financial claims on the private sector and earns interest on these claims − which is then an additional source of government revenue besides taxation.

attributed as a *cost* in a single year's account; according to "normal" budgeting principles it is only the *depreciation* on the public capital that should figure as a cost. Similarly, it is not taken into account that a part of D (or perhaps more than D) is backed by the value of public physical capital. That is, the cost and asset aspects of government net investment are not properly dealt with in the standard public accounting. Yet, in our treatment below we will stick to the traditional vocabulary. Where this might create logical problems (as it almost inevitably will do), it helps to imagine that:

(a) all of G is public consumption, i.e., $G_t = C_t^g$ for all t.

(b) there is no public physical capital.

An additional anomaly relates to what the accounting normally includes under "public consumption". A sizeable part of this is in fact investment in an economic sense: expenses on education, research, and health. To avoid confusion we shall treat C_t^g as being consumption in a narrow sense.

Now, from (6.16.2) and the definition $T_t \equiv \tilde{T}_t - X_t$ (net tax revenue) follows that real government debt at the beginning of period $t+1$ is:

$$\begin{aligned} B_{t+1} &\equiv \frac{D_{t+1}}{P_t} = G_t + X_t - \tilde{T}_t + (1+i_t)\frac{D_t}{P_t} - \frac{\Delta M_{t+1}}{P_t} \\ &= G_t - T_t + \frac{1+i_t}{1+\pi_t}B_t - \frac{\Delta M_{t+1}}{P_t} \equiv (1+r_t)B_t + G_t - T_t - \frac{\Delta M_{t+1}}{P_t}. \end{aligned}$$

The last term, $\Delta M_{t+1}/P_t$, in this expression is *seigniorage,* i.e., public sector revenue obtained by issuing base money.

Suppose real output grows at the constant rate g_Y so that $Y_{t+1} = (1 + g_Y)Y_t$. Then the debt-income ratio can be written

$$b_{t+1} \equiv \frac{B_{t+1}}{Y_{t+1}} = \frac{1+r_t}{1+g_Y}b_t + \frac{G_t - T_t}{(1+g_Y)Y_t} - \frac{\Delta M_{t+1}}{P_t(1+g_Y)Y_t}. \qquad (6.3)$$

The last term here is the (growth-corrected) seigniorage-income ratio,

$$\frac{\Delta M_{t+1}}{P_t(1+g_Y)Y_t} = (1+g_Y)^{-1}\frac{\Delta M_{t+1}}{M_t}\frac{M_t}{P_tY_t}.$$

If in the long run the base money growth rate, $\Delta M_{t+1}/M_t$, and the nominal interest rate (i.e., the opportunity cost of holding money, cf. Chapter 16) are constant, then the velocity of money and its inverse, the money-income ratio, $M_t/(P_tY_t)$, are also likely to be more or less constant. So is, therefore, the seigniorage-income ratio. For more developed countries this ratio is generally a small number.[4] For some of the emerging economies that have poor institutions for collection of taxes it is considerably higher; and in situations of budget crisis and hyperinflation it may be sizeable (see Montiel, 2003).

Generally budget deficits are financed primarily by debt creation but also to some extent by money creation, as envisioned in the above equations. However, from now on we will in the main ignore the seigniorage term in (6.3). Equivalently, we shall focus on a separate government sector, not a consolidated government and central bank. In fact, the legislation for central banking nowadays typically allows central banks a considerable degree of operational independence from their governments.[5]

Thus we proceed with the simple government accounting equation:

$$B_{t+1} - B_t = r_t B_t + G_t - T_t, \qquad \text{(DGBC)}$$

where the right-hand side is the *real budget deficit.* This equation is in macroeconomics often called the *dynamic government budget constraint* (or DGBC for short). It is in fact just a book-keeping relation, saying that if the real budget deficit is positive and there is no financing by money creation, then the public debt grows. It comes closer to being a constraint when combined with the requirement that the government stays *solvent.*

6.3 Government solvency and fiscal sustainability

To be *solvent* means being able to meet the financial commitments as they fall due. In practice this concept is closely related to the government's No-Ponzi-Game condition and intertemporal budget constraint (to which we return below), but at the theoretical level it is more general.

We can view the public sector as an infinitely-lived agent in the sense that there is no last date where all public debt has to be repaid. Nevertheless, as we shall see, there tends to be stringent constraints on government debt creation in the long run.

[4]In Denmark seigniorage was around 0.2 per cent of GDP during the 1990s (*Kvartalsoversigt 4. kvartal 2000*, Danmarks Nationalbank).

[5]The existing fiscal-financial framework for the EMU member countries completely excludes the central bank. Yet, for the EMU area as a whole seigniorage *is*, of course, income, albeit small, for the aggregate consolidated public sector and is afterwards divided by the central banks of the member countries.

6.3.1 The role of the growth-corrected interest rate

Very much depends on whether the real interest rate in the long-run is higher than the growth rate of GDP or not. To see this, suppose the country considered has positive government debt at time 0 and that the government levies taxes equal to its non-interest spending:

$$\tilde{T}_t = G_t + X_t \quad \text{or} \quad T_t \equiv \tilde{T}_t - X_t = G_t \text{ for all } t \geq 0. \tag{6.4}$$

This means that taxes cover only the primary expenses while interest payments (and debt repayments when necessary) are financed by issuing new debt. That is, the government attempts a permanent *roll-over* of the debt including the interest due for payment. This implies that the debt grows at the rate r_t according to (DGBC).

Assuming, for simplicity, that $r_t = r$ (a constant), the law of motion for the debt-income ratio is

$$b_{t+1} \equiv \frac{B_{t+1}}{Y_{t+1}} = \frac{1+r}{1+g_Y}\frac{B_t}{Y_t} \equiv \frac{1+r}{1+g_Y}b_t, \qquad b_0 > 0,$$

where b_0 is positive and assumed historically given and we maintained the assumption of a constant output growth rate, g_Y. The solution to this linear difference equation is $b_t = b_0(\frac{1+r}{1+g_Y})^t$, where we consider both r and g_Y as exogenous. We see that the growth-corrected interest rate, $\frac{1+r}{1+g_Y} - 1 \approx r - g_Y$ (for g_Y small) plays a key role. There are two contrasting cases to discuss:

Case I: $r > g_Y$. In this case, $b_t \to \infty$ for $t \to \infty$. But this is not a feasible path because, due to compound interest, the debt grows so large in the long run that the government will be unable to find buyers for all the debt. Imagine for example an economy described by the Diamond OLG model. Then the buyers of the debt are the young who place part of their saving in government bonds. But if the stock of these bonds grows at a higher rate than income, the saving of the young cannot in the long run keep track with the fast growing government debt. The private sector sooner or later understands that bankruptcy is threatening and nobody will buy any new government bonds except at a low price, which means a high interest rate. The high interest rate only aggravates the problem. That is, the fiscal policy (6.4) breaks down. Either the government defaults on the debt or T must be increased or G decreased (or both) until the growth rate of the debt is no longer higher than g_Y.

If the debt is denominated in the domestic currency, as is usual, and the government directly controls the central bank, then an alternative way out is of course a shift to money financing of the budget deficit, that is, seigniorage.

In a situation with full capacity utilization this leads to higher and higher inflation and thus the real value of the debt is eroded. Although the interest payments on the debt are hereby facilitated for a while, this policy has its own costs namely the economic and social costs generated by high inflation. And this route comes to a dead end if seigniorage reaches the backward-bending part of the "seigniorage Laffer curve".[6]

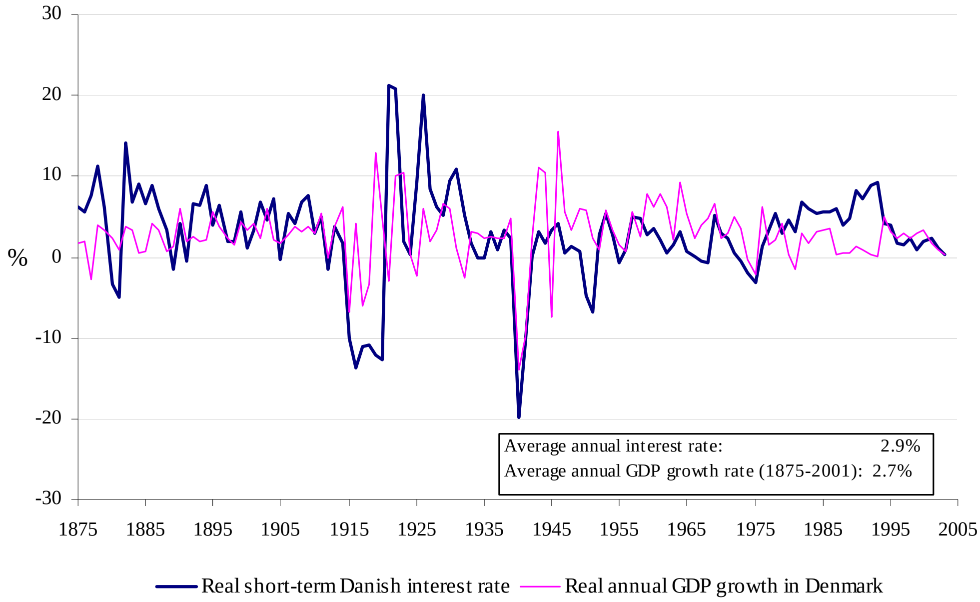
Case II: $r \leq g_Y$. If $r = g_Y$, we get $b_t = b_0$ for all $t \geq 0$. Since the debt, rising at the rate r, does not in the long run increase faster than national income, the government has no problem finding buyers of its newly issued debt. Thereby the government is able to finance its interest payments − it stays solvent. The growing debt is passed on to ever new generations with higher income and saving and the debt roll-over implied by (6.4) can continue forever. If $r < g_Y$, we get $b_t \rightarrow 0$ for $t \rightarrow \infty$, and the same conclusion holds *a fortiori*.

The government can thus pursue a permanent debt roll-over policy as implied by (6.4) and still remain solvent if in the long run the interest rate is not higher than the growth rate of the economy. But in the opposite case, permanent debt roll-over is not possible and sooner or later at least part of the interest payments must be tax financed.

Which of the two cases is relevant for the real world? The answer is not obvious if by the "interest rate" is meant the risk-free interest rate. Fig. 6.1 shows for Denmark (upper panel) and the US (lower panel) the time paths of the real short-term interest rate and the GDP growth rate, both on an annual basis. Overall the levels of the two are more or less the same, although on average the interest rate is in Denmark slightly higher but in the US somewhat lower than the growth rate.

Nevertheless, it seems generally believed that in "normal" situations there is good reason for paying attention to the case $r > g_Y$, also for a country like the US. This is because we live in a world of *uncertainty,* an aspect the above line of reasoning does not take into account. Indeed, the prudent debt policy needed in the case $r > g_Y$ can be shown to apply for a larger range of circumstances when uncertainty is present (see Abel et al. 1989, Ball et al. 1998, Blanchard and Weil 2001). On the other hand there is a different feature which draws the matter somewhat in the opposite direction. This is the possibility that an effective tax, $\tau \in (0, 1)$, on interest income is in force so that the net interest rate on the government debt is $(1 - \tau)r$ rather than r.

[6] See Chapter 18.



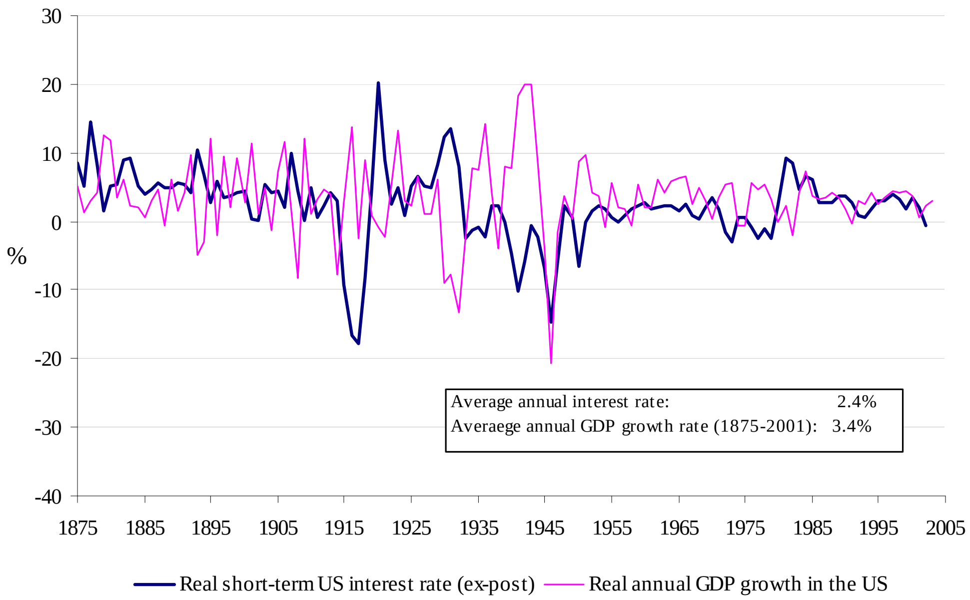


Figure 6.1: Real short-term interest rate and annual growth rate of real GDP in Denmark and the US since 1875. The real short-term interest rate is calculated as the money market rate minus the contemporaneous rate of consumer price inflation. Source: Abildgren (2005) and Maddison (2003).

6.3.2 Sustainable fiscal policy

The concept of sustainable fiscal policy is closely related to the concept of government solvency. A given fiscal policy is called *sustainable* if by applying its tax and spending rules forever, the government stays solvent.

To be specific, suppose G_t and T_t are determined by fiscal policy rules represented by the functions

$$G_t = \mathcal{G}(x_{1t}, ..., x_{nt}, t), \quad \text{and} \quad T_t = \mathcal{T}(x_{1t}, ..., x_{nt}, t),$$

where $t = 0, 1, 2, \ldots$, and x_1,..., x_n are key macroeconomic and demographic variables (like net national income, old-age dependency ratio, rate of unemployment, extraction of natural resources, say oil in the ground, etc.). In this way a given fiscal policy is characterized by the rules $\mathcal{G}(\cdot)$ and $\mathcal{T}(\cdot)$. Suppose further that we have a model, $\mathcal{M}$, of how the economy functions.

DEFINITION 1 Let initial public debt B_0 be given. Then, given a forecast of the evolution of the demographic and foreign economic environment in the future and given the model $\mathcal{M}$, the fiscal policy $(\mathcal{G}(\cdot), \mathcal{T}(\cdot))$ is *sustainable* if the forecast calculated on the basis of $\mathcal{M}$ is that the government stays solvent under this policy. The fiscal policy $(\mathcal{G}(\cdot), \mathcal{T}(\cdot))$ is called *unsustainable*, if it is not sustainable.

Thus the fiscal policy is sustainable only if the fiscal policy rules imply a stable debt-income ratio. The terms "sustainable"/"unsustainable" conveys the intuitive meaning. It is all about the question: can the current tax and spending rules continue forever?

Our definition of fiscal sustainability is silent about the presence of uncertainty. Without going into detail about the difficult uncertainty issue, we may elaborate a little on the definition by letting "stays solvent" be reformulated as "stays solvent with 100-ε percent probability", where ε is a small number.

Owing to the increasing pressure on public finances caused by reduced birth rates, increased life expectancy, and a fast-growing demand for medical care, many industrialized countries have for a long time been assessed to be in a situation where their fiscal policy is not sustainable (Elmendorf and Mankiw 1999). The implication is that sooner or later one or more expenditure rules and/or tax rules (in a broad sense) will have to be changed.

Two major kinds of strategies have been suggested. One kind of strategy is the *pre-funding strategy* (a Danish example was the so-called "Denmark 2010" plan). By ensuring a fiscal consolidation prior to the expected future demographic changes this strategy may prevent sharp future tax increases. Another strategy (alternative or complementary to the former) is to attempt a gradual increase in the labor force by letting the age limits for retirement and pension increase along with expected lifetime − this is the *indexed retirement strategy*. The first strategy implies that current generations bear a large part of the adjustment cost. In the second strategy the costs are shared by current and future generations in a way more similar to the way the benefits in the form of increasing life expectancy are shared. We shall not here go into detail about these matters, but refer the reader to the growing literature about securing fiscal sustainability in the ageing society, see Literature notes.

6.4 Debt arithmetic

A key tool for evaluating fiscal sustainability is debt arithmetic, i.e., the analytics of debt dynamics. The previous section described the important role of the growth-corrected interest rate. Here we will consider the minimum primary budget surplus required for fiscal sustainability in different situations.

6.4.1 The required primary budget surplus

Let γ denote the spending-income ratio, G/Y, and τ the net tax-income ratio, T/Y. Then from (6.3) with $\Delta M = 0$ follows that the debt-income ratio $b_t \equiv B_t/Y_t$ changes over time according to

$$b_{t+1} \equiv \frac{B_{t+1}}{Y_{t+1}} = \frac{1+r}{1+g_Y} b_t + \frac{\gamma - \tau}{1+g_Y}. \tag{6.5}$$

Suppose, until further notice, that g_Y, r, γ, and τ are constant. There are three cases to consider. *Case 1:* $r > g_Y$. As emphasized above this case is generally considered the one of most practical relevance. And it is in this case that *latent debt instability* is present and the government has to pay attention to the danger of runaway debt dynamics. To see this, note that the solution of the linear difference equation (6.5) is

$$b_t = (b_0 - b^*) \left(\frac{1+r}{1+g_Y} \right)^t + b^*, \quad \text{where} \quad b^* = \frac{\tau - \gamma}{r - g_Y}. \tag{6.6}$$

Here b_0 is historically given. But the steady-state debt-income ratio, b^*, depends on fiscal policy. The important feature is that the growth-corrected interest factor is in this case higher than 1 and has the exponent t. Therefore, if fiscal policy is such that $b^* < b_0$, the debt-income ratio explodes.[7] The solid curve in the uppermost panel in Fig. 6.2 shows such a case, that is, a case where fiscal policy implies $\tau - \gamma < (r - g_Y)b_0$.[8]

The only way to avoid the snowball effects of compound interest when the growth-corrected interest rate is positive is to ensure a primary budget surplus as a share of GDP, $\tau - \gamma$, high enough such that $b^* \geq b_0$. That is, the minimum required primary surplus as a share of GDP, $\hat{s}$, is

$$\hat{s} = \frac{T - G}{Y} = \tau - \gamma = (r - g_Y)b_0. \tag{6.7}$$

Note that $\hat{s}$ will be larger:

- the higher is the initial level of debt; and,
- when $b_0 > 0$, the higher is the growth-corrected interest rate, $r - g_Y$.

Given the spending-income ratio γ, the minimum needed net tax rate is $\hat{\tau} = \gamma + (r - g_Y)b_0$. The difference, $\hat{\tau} - \tau$, indicates the size of the needed adjustment, were it to take place at time 0. A more involved indicator of the *sustainability gap*, taking the room for manoeuvre into account, is $(\hat{\tau} - \tau)/(1 - \tau)$.[9] Delaying the adjustment increases the size of the needed policy action, since the debt-income ratio has become higher in the meantime.

Suppose that the debt build-up is prevented already from time 0 by ensuring that the primary surplus as a share of income, $\tau - \gamma$, at least equals $\hat{s}$ so that $b^* \geq b_0$. The solid curve in the second from the top panel in Fig. 6.2 illustrates the resulting evolution of the debt-income ratio if b^* is at the level corresponding to the hatched horizontal line. Presumably, the government would in such a state of affairs relax its fiscal policy after a while in order not to accumulate large government net wealth. Yet, the pre-funding strategy vis-a-vis the fiscal challenge of population ageing (referred to above) is in fact based on accumulating some positive public financial net wealth as a buffer before the substantial effects of population ageing set in. In this context, the higher the growth-corrected interest rate, the shorter the time needed to reach a given positive net wealth position.

[7] With reference to the net asset position and fiscal stance of the US government, economist and Nobel laureate George Akerlof remarked: "It takes some time after running off the cliff before you begin to fall. But the law of gravity works, and that fall is a certainty" (Akerlof 2004, p. 6).

[8] Somewhat surprisingly, perhaps, there can be debt explosion in the long run even if $\tau > \gamma$, namely if $0 < \tau - \gamma < (r - g_Y)b_0$. Debt explosion can even arise if $b_0 < 0$, namely if $\tau - \gamma < (r - g_Y)b_0 < 0$.

[9] With a Laffer curve in mind, in this formula one should in principle replace the number 1 by a tax rate estimated to maximize the tax revenue in the country considered. This is easier said than done, however, because as noted in the introduction to this chapter there are many uncertainties and contingencies involved in the construction of a Laffer curve.

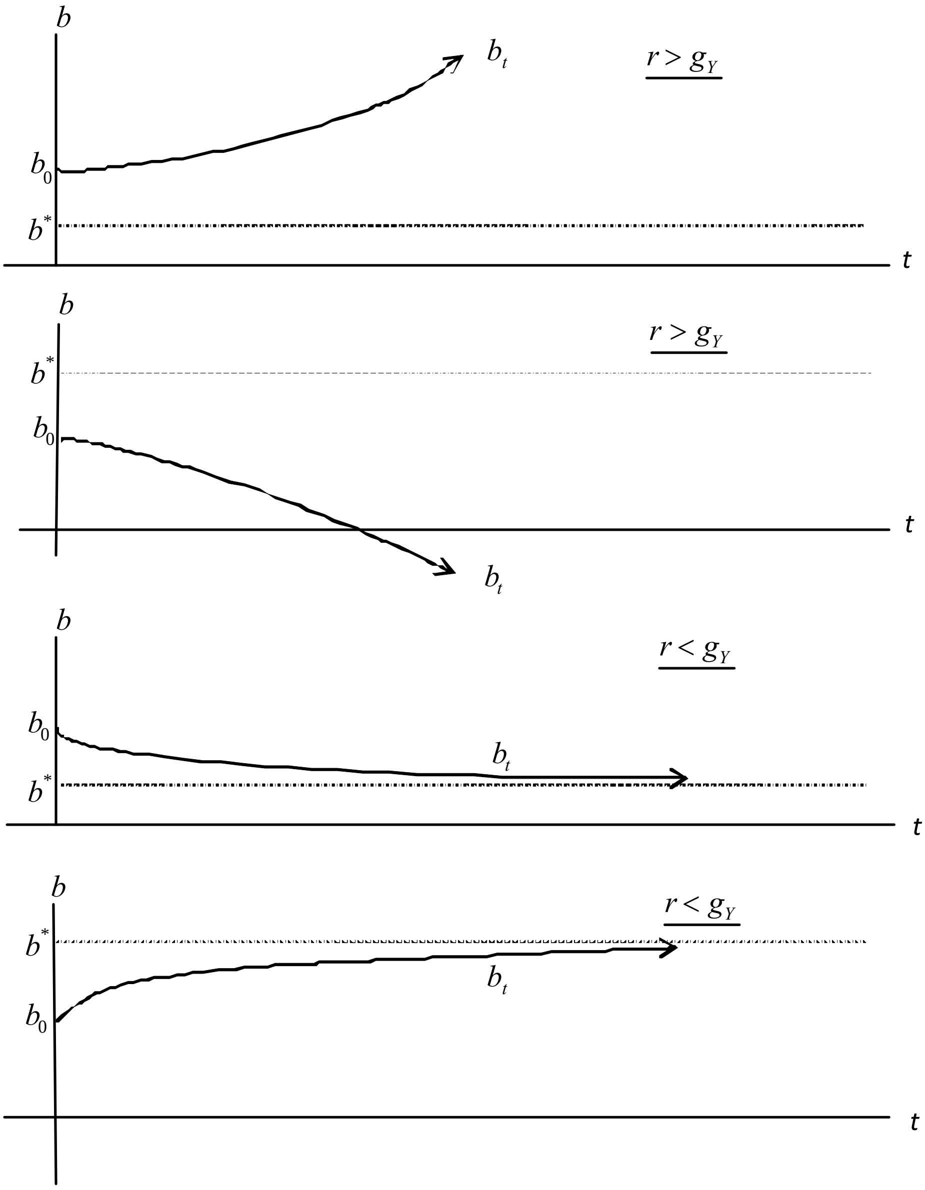


Figure 6.2: Debt dynamics in the cases $r > g_Y$ (the two upper panels) and $r < g_Y$ (the two lower panels), respectively.

Case 2: $r = g_Y$. In this knife-edge there is still a danger of instability, but less explosive. The formula (6.6) is no longer valid. Instead the solution of (6.5) is $b_t = b_0 + (\gamma - \tau)t/(1 + g_Y)$. Here, a non-negative primary surplus is sufficient to avoid $b_t \to \infty$ for $t \to \infty$.

Case 3: $r < g_Y$. This is the case of stable debt dynamics. The formula (6.6) is again valid, but now implying a non-exploding debt-income ratio even if there is a *negative* primary budget surplus and initial debt is large. The curve in the second from bottom panel in Fig. 6.2 illustrates such a situation. In spite of $b^* < b_0$, with unchanged fiscal policy, b will converge to b^*. This is so whether b^* is positive (as in the figure) or negative. In fact, if the growth-corrected interest rate remains negative, permanent debt roll-over can handle the financing and taxes need never be levied.[10] Finally, the bottom panel in Fig. 6.2 shows the case where with a *large* primary deficit ($\gamma - \tau > 0$), the excess of output growth over the interest rate still implies convergence towards a constant debt-income ratio, albeit a high one.

In the above analysis we have simplified by assuming that several variables, including γ, τ, and r, are given constants. The ongoing rise in the dependency ratio, due to a decreased birth rate and rising life expectancy, together with a rising request for medical care is likely to generate upward pressure on γ and downward pressure on τ. Thereby a high initial debt-income ratio becomes *more* challenging. A sustained government budget deficit may also raise the interest rate in the economy. Incorporating these aspects requires *general equilibrium* analysis of budget deficits and public debt, a topic to which we return in Chapter 13.

6.4.2 The debate over the Stability and Growth Pact of the EMU

The Maastrict criteria, after the Treaty of Maastrict 1992, for joining the Economic and Monetary Union (EMU) of the EU specified both a government deficit rule and a government debt rule. The first is the rule saying that the nominal budget deficit must not be above 3 percent of nominal GDP. The debt rule says that the debt should not be above 60 percent of GDP. The deficit rule and the debt rule were upheld in the Stability and Growth Pact (SGP) which was implemented in 1997 as one of the "great pillars" of the EMU institutional framework.[11]

[10] When the GDP growth rate exceeds the interest rate on government debt, a large debt-income ratio can be brought down quite fast, as witnessed by the evolution of both UK and US government debt in the first three decades after the second world war.

[11] The other "great pillar" is the European Central Bank (ECB). Notice that some of the EMU member states (Greece, Italy, and Belgium) have had debt-income ratios above

The two rules (with associated detailed arrangements and contingencies) are meant as discipline devices aiming at "sound budgetary policy". The declared goal is protection of the European Central Bank (ECB) against political demands on the ECB to loosen monetary policy and "bail out" a member country "too big to fail". An extreme fiscal crisis in one or more of the euro-zone countries could set in and entail a state of affairs approaching default on government debt. This is in fact what we have seen in southern Europe in the wake of the Great Recession triggered by the financial crisis in 2008. Such a situation is likely to generate open or concealed political pressure on the ECB to curb soaring interest rates and spreading of financial problems through buying government bonds from the country in trouble. The lid on deficit spending was meant as a means to avoid such a state of affairs which might interfere with the ECB's one and (up to "now", i.e., winter 2011-12) only concern with price stability.

The pact has an exemption clause referring to "exceptional" circumstances. These circumstances are defined as "severe economic recession" the interpretation of which was, by the reform of the SGP in March 2005, changed from an annual fall of real GDP of at least 1-2% to simply "negative growth". Thus, owing to the international economic crisis that broke out in 2008, the deficit rule was suspended temporarily for many of the EMU countries.

Whatever the virtues or vices of the SGP and whatever its future status, let us ask the plain question: what is the logical relationship, if any, between the 3 percent and 60 percent tenets?

To answer this, consider a deficit rule saying that the (total) nominal budget deficit must never be above $\alpha \cdot 100$ percent of nominal GDP. By (6.2) with $\Delta M_{t+1} = 0$ this is equivalent to the requirement

$$D_{t+1} - D_t = GBD_t \leq \alpha P_t Y_t. \tag{6.8}$$

In the Maastrict Treaty as well as the SGP, $\alpha = 0.03$. Here we consider the general case: $\alpha > 0$. To see the implication for the debt-income ratio in the long run, let us first imagine a situation where the deficit ceiling, α, is always

100 percent since the early 1990s. Yet they became full members of the EMU. The 60 percent debt rule in the SGP is to be understood as a long-run ceiling and, by the stock nature of debt, cannot be a here-and-now requirement.

Moreover, the measure of government debt, called the EMU debt, used in the SGP criterion is based on the book value of the financial liabilities rather than the market value. In addition, the EMU debt is more of a gross nature than the standard government net debt measure, corresponding to our D. The EMU debt measure allows fewer of the government financial assets to be subtracted from the government financial liabilities in order to reach a net debt measure. In our discussion we ignore these complications.

binding for the economy we look at. Then $D_{t+1} = D_t + \alpha P_t Y_t$ and so

$$b_{t+1} \equiv \frac{B_{t+1}}{Y_{t+1}} \equiv \frac{D_{t+1}}{P_t Y_{t+1}} = \frac{D_t}{(1+\pi)P_{t-1}(1+g_Y)Y_t} + \frac{\alpha}{1+g_Y},$$

assuming constant output growth rate, g_Y, and inflation rate π. This reduces to

$$b_{t+1} = \frac{1}{(1+\pi)(1+g_Y)} b_t + \frac{\alpha}{1+g_Y}. \tag{6.9}$$

Assuming (realistically) that $(1+\pi)(1+g_Y) > 1$, this linear difference equation has the stable solution

$$b_t = (b_0 - b^*)\left(\frac{1}{(1+\pi)(1+g_Y)}\right)^t + b^* \rightarrow b^* \text{ for } t \rightarrow \infty, \tag{6.10}$$

where

$$b^* = \frac{(1+\pi)\alpha}{(1+\pi)(1+g_Y) - 1}.$$

Consequently, if the deficit rule (6.8) is always binding, the debt-income ratio tends in the long run to be proportional to the deficit bound α; the factor of proportionality is a decreasing function of the long-run growth rate of real GDP and the inflation rate. This result confirms the general tenet that if there is economic growth, perpetual budget deficits need not lead to fiscal problems.

If on the other hand the deficit rule is *not* always binding, then the budget deficit is on average smaller than above so that the debt-income ratio will in the long run be *smaller* than b^*.

The conclusion is the following. With one year as the time unit, suppose the deficit rule is $\alpha = 0.03$ and that $g_Y = 0.03$ and $\pi = 0.02$ (the upper end of the inflation interval aimed at by the ECB). Further, suppose the deficit rule is never violated. Then in the long run the debt-income ratio will be *at most* $b^* = 1.02 \times 0.03/(1.02 \times 1.03 - 1) \approx 0.60$. This is in agreement with the debt rule of the Maastrict Treaty and the SGP according to which the maximum value allowed for the debt-income ratio is 60%.

By (6.10) we see that if α and g_Y are given, then the long-run debt-income ratio depends negatively on the rate of inflation. In this way a deficit rule like (6.8) is more restrictive the higher is inflation. This reflects that from (6.8) together with (6.2) with $\Delta M_{t+1} = 0$, we have

$$\Delta D_{t+1} \equiv D_{t+1} - D_t = iD_t + P_t(G_t - T_t) = \left[i(1+\pi)^{-1}b_t + \gamma - \tau\right] P_t Y_t \leq \alpha P_t Y_t. \tag{6.11}$$

where we have used that $D_t/(P_t Y_t) \equiv D_t P_{t-1}/((P_{t-1}Y_t)P_t) \equiv (1+\pi)^{-1}b_t$. Since, for given r, $i(1+\pi)^{-1} = [(1+r)(1+\pi) - 1](1+\pi)^{-1} = 1+r-(1+\pi)^{-1}$

rises when π rises, in (6.11) either a lower b_t or a lower $\gamma - \tau$ is needed to satisfy the bound. Observe also, that the deficit rule is, for given π, more restrictive the higher is r.

Although there is nothing sacred about either of the numbers 0.60 or 0.03, they are at least mutually consistent as long as 0.60 is considered an upper bound for the debt-income ratio. On the other hand, if a debt-income ratio at 60 percent is considered *acceptable* for a given country, then it is the *average* deficit over the business cycle that should be 3 percent, not the deficit ceiling.

After the founding of the SGP, critics have maintained that when considering the need for fiscal stimuli even in a mild recession, a ceiling at 0.03 may be too low, giving too little scope for counter-cyclical fiscal policy (including the free working of the automatic fiscal stabilizers). As an economy moves towards recession, the deficit rule may, bizarrely, force the government to tighten fiscal policy although the situation calls for a stimulation of aggregate demand. The pact has therefore sometimes been called the Instability and Depression Pact.

Besides it has been objected that there is no reason to have a flow rule (a deficit rule) on top of the stock rule (the debt rule), since it is the long-run stock that matters for whether the fiscal accounts are worrying. In this perspective, a criterion for a deficit rule should relate the deficit to the *trend* nominal GDP, which we may denote $(PY)^*$. Such a criterion would imply

$$\Delta D \leq \alpha (PY)^*. \tag{6.12}$$

Then

$$\frac{\Delta D}{PY} \leq \alpha \frac{(PY)^*}{PY}.$$

In recessions the ratio $(PY)^*/(PY)$ is high, in booms it is low. This has the advantage of allowing more room for budget deficits when they are needed − without necessarily interfering with the long-run aim of stabilizing government debt below some specified ceiling.

A further step in this direction is a rule directly in terms of the *structural* or *cyclically adjusted* budget deficit rather than the actual year-by-year deficit. The cyclically adjusted budget deficit in a given year is defined as the value the deficit would take in case actual output were equal to trend output in that year. Denoting the cyclically adjusted budget deficit ΔD^*, the rule would be

$$\Delta D^* \leq \alpha (PY)^*.$$

In fact, in its original version the SGP also contained a rule like that, but in the very strict form of $\alpha \approx 0$. This requirement was implicit in the directive that the cyclically adjusted budget "should be close to balance or in surplus". By this bound it is imposed that the debt-income ratio should be close to zero in the long run. Many EMU countries certainly had − and have − larger cyclically adjusted deficits. Taking steps to comply with such a low structural deficit bound could be costly.[12] The minor reform of the SGP endorsed in March 2005 allowed more contingencies, also concerning this structural bound.

At a more general level critics have contended that bureaucratic budget rules imposed on sovereign nations will never be able to do their job unless they encompass incentive compatible elements. Regarding the *composition* of government expenditure, critics have argued that the SGP pact entails a problematic disincentive to public investment. In this context Blanchard and Giavazzi (2004) maintain that a fiscal rule should be based on a proper accounting of public investment instead of simply ignoring the composition of government expenditure. This issue is taken up in Chapter 13.

A general counterargument raised against the criticisms of the SGP has been that the potential benefits of the proposed alternative rules are more than offset by the costs in terms of reduced simplicity, measurability, and transparency (Schuknecht, 2005). Space does not allow us to go into more detail with the issues; the reader is referred to the discussions in for example Buiter (2003), Buiter and Grafe (2004), Fogel and Saxena (2004), and Wyplosz (2005).

We close this section by a remark on the recent 2010-11 European sovereign debt crisis". In the first months of 2010 anxiety broke out about the Greek government debt crisis to spill over to Spain, Portugal, Italy, and Ireland, thus widening bond yield spreads in these countries in the midst of a serious economic recession. Moreover, the solvency of big German banks that were among the prime creditors of Greece was endangered. The major Eurozone governments and the International Monetary Fund (IMF) reached an agreement to help Greece (and indirectly its creditors) with loans and guarantees for loans, conditional on the government of Greece imposing yet another round of fiscal austerity measures. In the wake of these events the European Commission took steps to intensify the monitoring of the fiscal policies of the EMU members.

The whole crisis has pointed to the difficulties faced by the Euro zone: in spite of the member countries being economically very different, they are subordinate to the same one-size-fits-all monetary policy. Bureaucratic policy rules and surveillance procedures for the fiscal policy of sovereign nations can

[12] Moreover, to insist on exactly a zero debt-income ratio in the long run has no foundation in public economics or general welfare theory.

hardly replace self-regulation, and enforcement mechanism are bound to be weak. Moreover, abiding by the fiscal rules of the SGP was certainly no assurance of not ending up in a fiscal crisis in the wake of the crisis in the financial sector, as witnessed by Ireland and Spain.

6.5 Solvency and the intertemporal government budget constraint

Up to now we have considered the issue of government solvency in the perspective of the dynamics of the debt-income ratio. It is sometimes useful to view government solvency from another angle, that of the intertemporal budget constraint (GIBC). Under a certain condition given below, the intertemporal budget constraint − and the associated No-Ponzi-Game (NPG) condition − are just as relevant for a government as they are for private agents. We first consider the relationship between solvency and the NPG condition. We concentrate on the government debt measured in real terms.

6.5.1 When is the NPG condition necessary for solvency?

Consider a situation with a constant interest rate $r > 0$. The relevant NPG condition is that the present discounted value of the public debt in the far future is not positive, i.e.,

$$\lim_{t\to\infty} B_t(1+r)^{-t} \leq 0. \tag{NPG}$$

This condition says that government debt is not allowed to grow in the long run at a rate above (or just equal to) the interest rate.[13] That is, a fiscal policy satisfying the NPG condition rules out a permanent debt roll-over. The designation No-Ponzi-Game condition refers to a guy from Boston, Charles Ponzi, who in the 1920s made a fortune out of an investment scam based on the chain letter principle. The principle is to pay off old investors with money from the new investors. Ponzi was sentenced many years in prison for his transactions; he died poor − and without friends.

Note, however, that since there is no final date, the NPG condition does *not* say that all debt ultimately has to be repaid or even that the debt has ultimately to be non-increasing. It "only" says that the debtor, here the

[13] If there is taxation of interest income at the rate $\tau_r \in (0, 1)$, then the after-tax interest rate, $(1 - \tau_r)r$, is the relevant discount rate.

government, is not allowed to let the debt grow forever at a rate as high as (or higher than) the interest rate. And for instance the U.K. as well as the U.S. governments have had positive debt for centuries.

Let $\lim_{t\to\infty} Y_{t+1}/Y_t = 1 + g_Y$. We have:

PROPOSITION 1 Assume an initial debt $B_0 > 0$. Then:

(i) if $r > g_Y$, solvency requires (NPG) to be satisfied;

(ii) if $r \leq g_Y$, the government can remain solvent without satisfying (NPG).

Proof. For the debt-income ratio we have $b_{t+1} \equiv B_{t+1}/Y_{t+1}$ so that

$$\lim_{t\to\infty} \frac{b_{t+1}}{b_t} = \lim_{t\to\infty} \frac{B_{t+1}/Y_{t+1}}{B_t/Y_t} = \lim_{t\to\infty} \frac{B_{t+1}/B_t}{Y_{t+1}/Y_t} = \lim_{t\to\infty} \frac{B_{t+1}/B_t}{1+g_Y} \text{ for } t \to \infty. \tag{6.13}$$

Case (i): $r > g_Y$. If $\lim_{t\to\infty} B_t \leq 0$, then (NPG) is trivially satisfied. Assume $\lim_{t\to\infty} B_t > 0$. For this situation we prove the statement by contradiction. Suppose (NPG) is not satisfied. Then, $\lim_{t\to\infty} B_t(1+r)^{-t} > 0$, implying that $\lim_{t\to\infty} B_{t+1}/B_t \geq 1+r$. In view of (6.13) this implies that $\lim_{t\to\infty} b_{t+1}/b_t \geq (1+r)/(1+g_Y) > 1$. Thus, $b_t \to \infty$, which violates solvency. By contradiction, this proves that solvency implies (NPG) in this case.

Case (ii): $r \leq g_Y$. Consider the permanent debt roll-over policy $T_t = G_t$ for all $t \geq 0$. By (DGBC) of Section 6.2 this policy yields $B_{t+1}/B_t = 1 + r$; hence, in view of (6.13), $\lim_{t\to\infty} b_{t+1}/b_t = (1+r)/(1+g_Y) \leq 1$. The policy consequently implies solvency. On the other hand the solution of the difference equation $B_{t+1} = (1+r)B_t$ is $B_t = B_0(1+r)^t$. Thus $B_t(1+r)^{-t} = B_0 > 0$ for all t, thus violating (NPG). □

Hence imposition of the NPG condition on the government relies on the interest rate being in the long run higher than the growth rate of GDP. If instead $r \leq g_Y$, the government can cut taxes, run a budget deficit, and postpone the tax burden indefinitely. So in that case the government can run a Ponzi Game and still stay solvent. But as alluded to earlier, if uncertainty is added to the picture, matters become more complicated and qualifications to Proposition 1 are needed (Blanchard and Weil, 2001). The prevalent view among macroeconomists is that imposition of the NPG condition on the government is generally warranted.

While in the case $r > g_Y$, the NPG condition is *necessary* for solvency, it is *not sufficient*. Indeed, we could have

$$1 + g_Y < \lim_{t\to\infty} B_{t+1}/B_t < 1 + r. \tag{6.14}$$

Here (NPG) is satisfied, yet, by (6.13), $\lim_{t\to\infty} b_{t+1}/b_t > 1$ so that the debt-income ratio explodes. (Note that we are here dealing with a partial equilibrium analysis, which takes the interest rate as given.)

A way to *obtain* solvency is for example to impose that the primary budget surplus as a share of GDP should be constant during the debt stabilization. Thus, ignoring short-run differences between Y_{t+1}/Y_t and $1+g_Y$ and between r_t and its long-run value, r, the minimum primary surplus as a share of GDP, $\hat{s}$, required to obtain $b_{t+1}/b_t \leq 1$ for all $t \geq 0$ is $\hat{s} = (r - g_Y)b_0$ as in (6.7). This $\hat{s}$ is a measure of the burden that the government debt imposes on the economy. If the fiscal policy steps needed to realize $\hat{s}$ are not taken, the debt-income ratio will grow further, thus worsening the fiscal stance in the future by increasing $\hat{s}$.

6.5.2 Equivalence of NPG and IBC

The condition under which the NPG condition is relevant for solvency is also the condition under which the government's intertemporal budget constraint is relevant. To show this we let t denote the current period and $t+i$ denote a period in the future. From Section 6.2, debt accumulation is described by

$$B_{t+1} = (1+r)B_t + G_t + X_t - \tilde{T}_t, \qquad \text{where } B_t \text{ is given.} \tag{6.15}$$

The *government intertemporal budget constraint* (GIBC), as seen from the beginning of period t, is the requirement

$$\sum_{i=0}^{\infty}(G_{t+i} + X_{t+i})(1+r)^{-(i+1)} \leq \sum_{i=0}^{\infty}\tilde{T}_{t+i}(1+r)^{-(i+1)} - B_t. \tag{GIBC}$$

This says that the present value of current and expected future government spending cannot exceed the government's net wealth (the present value of current and expected future tax revenue minus existing government debt).By the symbol $\sum_{i=0}^{\infty} x_i$ we mean $\lim_{I\to\infty}\sum_{i=0}^{I} x_i$, assuming this limit exists in the improper interval $[-\infty, +\infty]$.

To see the connection between the dynamic accounting relationship (6.15) and the intertemporal budget constraint, note that by rearranging (6.15) and using forward substitution we get

$$\begin{aligned} B_t &= (1+r)^{-1}(\tilde{T}_t - G_t - X_t) + (1+r)^{-1}B_{t+1} \\ &= \sum_{i=0}^{j}(1+r)^{-(i+1)}(\tilde{T}_{t+i} - G_{t+i} - X_{t+i}) + (1+r)^{-(j+1)}B_{t+j+1} \\ &\leq \sum_{i=0}^{\infty}(1+r)^{-(i+1)}(\tilde{T}_{t+i} - G_{t+i} - X_{t+i}), \end{aligned} \tag{6.16}$$

if and only if government debt ultimately grows at a rate which is less than r, so that

$$\lim_{j\to\infty}(1+r)^{-(j+1)}B_{t+j+1} \leq 0. \tag{6.17}$$

This latter condition is exactly the NPG condition above. Reordering (6.16) gives (GIBC). We have thus shown:

PROPOSITION 2 Given the book-keeping relation (6.15), then:

(i) (NPG) is satisfied if and only if (GIBC) is satisfied;

(ii) there is strict equality in (NPG) if and only if there is strict equality in (GIBC).

We know from Proposition 1 that in the "normal case" where $r > g_Y$, (NPG) is needed for government solvency. The message of (i) of Proposition 2 is that then also (GIBC) need be satisfied. So, when $r > g_Y$, to remain solvent a government has to realistically plan taxation and spending profiles such that the present value of current and future primary budget surpluses matches the current debt. Otherwise debt default threatens.[14] In view of the remarks around the inequalities in (6.14), however, satisfying the GIBC is only a necessary condition (if $r > g_Y$), not in itself a sufficient condition for solvency. A simple condition under which the GIBC *is* sufficient for solvency is that both G_t and T_t are proportional to Y_t, cf. Example 1 below.

On the other hand, if $r \leq g_Y$, it follows from propositions 1 and 2 together that the government can remain solvent *without* satisfying its intertemporal budget constraint.

Returning to the "normal case" where $r > g_Y$, it is certainly *not* required that the budget is balanced *all* the time. The point is "only" that for a given planned expenditure path, a government should plan realistically a stream of future tax revenues the PV of which equals the PV of planned expenditure *plus* the current debt. If during an economic recession for example, a budget deficit is run so that public debt accumulates, then higher taxes than otherwise must be raised in the future.

EXAMPLE 1 Consider a small open economy facing an exogenous constant real interest rate r. Suppose that at time t government debt is $B_t > 0$. GDP, Y_t, grows at the constant rate g_Y, and $r > g_Y$. Assume $G_t = \gamma Y_t$ and $T_t \equiv \tilde{T}_t - X_t = \tau Y_t$, where γ and τ are positive constants. What is the minimum size

[14] Government debt defaults have their own economic as well as political costs. Yet, they occur now and then. Recent examples include Russia in 1998 and Argentina in 2001-2002. Since 2010, Greece has been on the brink of debt default.

of the primary budget surplus as a share of GDP required for satisfying the government's intertemporal budget constraint as seen from time t? Inserting into the formula (6.16), with strict equality, yields $\sum_{i=0}^{\infty}(1+r)^{-(i+1)}(\tau-\gamma)Y_{t+i}$ $= B_t$. This gives $\frac{\tau-\gamma}{1+g_Y}Y_t\sum_{i=0}^{\infty}\left(\frac{1+g_Y}{1+r}\right)^{(i+1)} = \frac{\tau-\gamma}{r-g_Y}Y_t = B_t$, so that

$$\tau-\gamma = (r-g_Y)\frac{B_t}{Y_t}.$$

This is the same result as in (6.7) above if we substitute $\hat{s} = \tau - \gamma$ and $t = 0$. Thus, satisfying the government's intertemporal budget constraint ensures in this example a constant debt-income ratio and thereby government solvency. □

To the extent that the tax revenue $\tilde{T}$ and the spending $G + X$ do not in the long run grow at a higher rate than GDP does, the integrals in (GIBC) are bounded if the interest rate is in the long run higher than the output growth rate. On the other hand, we have seen that it is not difficult for the government to remain solvent if the growth-corrected interest rate is non-positive. Perhaps, the background for this fact becomes more apparent, when we recognize how the condition $r \leq g_Y$ affects the constraint (GIBC). Indeed, to the extent that tax revenue tends to grow at the same rate as national income, we have $\tilde{T}_{t+i} = \tilde{T}_t(1+g_Y)^i$ and therefore

$$\sum_{i=0}^{\infty}\tilde{T}_{t+i}(1+r)^{-(i+1)} = \frac{\tilde{T}_t}{1+g_Y}\sum_{i=0}^{\infty}\left(\frac{1+g_Y}{1+r}\right)^{(i+1)} = \infty,$$

for a constant $r \leq g_Y$. That is, the PV of future tax revenues is infinite and so is the PV of future spending. In this case, as (6.16) shows, any primary budget surplus, ever so small, can repay any level of initial debt in finite time. Yet, as emphasized time and again, this should probably not be seen as more than a theoretical possibility.

If the real interest rate varies over time, the above formulas remain valid if $(1+r)^{-(i+1)}$ is replaced with $\Pi_{j=0}^{i}(1+r_{t+j})^{-1}$.

6.6 Ricardian non-equivalence

We now turn to the question how budget policy affects resource allocation and intergenerational distribution. The role of budget policy for economic activity within a time horizon corresponding to the business cycle is not our topic here. Our question is about the longer run: does it matter for aggregate consumption and aggregate saving, and thereby resource allocation, in an economy without cyclical unemployment whether the government finances its current spending by taxes or borrowing?

6.6.1 Two differing views

In overlapping generations models the answer turns out to be affirmative. The essential reason is that those persons who benefit from lower taxes today will at most be a fraction of those who bear the higher tax burden in the future.

To spell this out, we consider the interesting situation where $r > g_Y$, regarding this as the "normal circumstances". Then, to remain solvent, the government has to satisfy its no-Ponzi-game condition or, equivalently, its intertemporal budget constraint. This follows from (i) of Proposition 1 and Proposition 2. We will concentrate on the "ordinary" case where the government does not tax so heavily that public net debt turns to public net financial wealth ($B < 0$) and as such accumulates in the long run at a speed equal to the interest rate or faster. Excluding this kind of "capitalist government", strict equality instead of weak inequality will rule in both the no-Ponzi-game condition, (6.17), and the intertemporal budget constraint of the government, (GIBC). Then there is equality also in (6.16), which may thus, with period 0 as the initial period and $T_t \equiv \tilde{T}_t - X_t$, be written

$$\sum_{t=0}^{\infty} T_t(1+r)^{-(t+1)} = \sum_{t=0}^{\infty} G_t(1+r)^{-(t+1)} + B_0. \tag{6.18}$$

The initial debt, B_0, is historically given. For a given planned time path of G_t, equation (6.18) shows that a tax cut in any period has to be met by an increase in future taxes of the same present discounted value as the tax cut. Nevertheless, since taxes levied at different times are levied at different sets of agents, the timing of taxes generally matters. The current tax cut makes current tax payers feel wealthier and this tends to increase their consumption and lead to a decrease in aggregate saving. The present generations benefit and future tax payers (partly future generations) bear the cost in the form of access to less national wealth than otherwise.

This conclusion is called *absence of debt neutrality* or *absence of Ricardian equivalence*. The latter name comes from a − seemingly false − association of the opposite view, that the timing of taxes does not matter, with the early nineteenth-century British economist David Ricardo. It is true that Ricardo articulated the possible logic behind debt neutrality. But he suggested several reasons that debt neutrality would not hold in practice and in fact he warned against high public debt levels (Ricardo, 1969, pp. 161-164). Therefore, to cite Elmendorf and Mankiw (1999, p. 1643), it is doubtful whether Ricardo was a Ricardian.

Debt neutrality was rejuvenated, however, by the American economist Robert Barro in a paper entitled "Are government bonds net wealth [of the private sector]?", a question which Barro answered in the negative (Barro 1974). Barro's debt neutrality hypothesis rests on a representative agent model, that is, a model with a fixed number of infinitely-lived forward-looking "dynasties". A change in the timing of (lump-sum) taxes does not change the present value of the infinite stream of taxes imposed on the individual dynasty. A cut in current taxes is offset by the expected higher future taxes. Though current government saving $(T - G - rB)$ goes down, private saving and bequests left to the members of the next generation go up. And private saving goes up just as much as taxation is reduced because the altruistic parents make sure that the next generation is fully compensated for the higher taxes in the future. Consumption is thus not affected and aggregate saving in society stays the same.[15]

But there are several reasons to expect that the debt neutrality hypothesis breaks down in practice and one of them comes from the demography of OLG models as outlined above. The next subsection provides a simple example illustrating how a change in the timing of taxes affect aggregate private consumption and saving in an OLG framework. The economy considered is a "small open economy".

6.6.2 A small open OLG economy with budget deficits

We consider a Diamond-style OLG model of a small open economy (henceforth SOE) with a government sector. As earlier we let L_t denote the size of the young generation and $L_t = L_0(1+n)^t$, $n \geq 0$.

Some national accounting for the open economy

We start with a little national accounting for our SOE. In the notation from Chapter 3 gross national saving is

$$S_t = Y_t - C_t - G_t = Y_t - (c_{1t}L_t + c_{2t}L_{t-1}) - G_t, \tag{6.19}$$

where G_t is the amount of the public good in period t. It is assumed that the production of G_t uses the same technology, and therefore involves the same unit production costs, as the other components of GDP. Since our focus here is not on the distortionary effects of taxation, taxes are assumed to be lump sum, i.e., levied on individuals irrespective of their economic behavior.

With A_t denoting private financial (net) wealth and B_t government (net) debt, *national wealth,* V_t, of SOE at the beginning of period t is by definition the sum of private financial wealth and government financial wealth. The

[15] A formal account of Barro's argument is given in Chapter 7.

latter equals minus the government debt if the government has no physical assets which we assume. Thus,

$$V_t = A_t + (-B_t). \tag{6.20}$$

With NFD_t denoting (net) foreign debt (also called external debt) at the beginning of period t, we also have

$$V_t = K_t - NFD_t,$$

saying, if $NFD_t > 0$, that some of the capital stock is directly or indirectly owned by foreigners. On the other hand, if $NFD_t < 0$, SOE has positive net claims on resources in the rest of the world.

When the young save, they accumulate private financial wealth. The private financial wealth at the start of period $t+1$ must in our Diamond framework equal the saving by the young in the previous period, i.e.,

$$A_{t+1} = s_t L_t \equiv S_{1t}. \tag{6.21}$$

So, by (6.20) forwarded one period, we have

$$V_{t+1} = A_{t+1} + (-B_{t+1}) = s_t L_t - B_{t+1}. \tag{6.22}$$

We may alternatively view this conclusion from the perspective of *national saving*. By definition the increase in *national* wealth equals net *national* saving, $S_t - \delta K_t$, which in turn equals net private saving, $s_t L_t + (-A_t)$, minus net public dissaving. The latter equals the size of the government budget deficit, GBD_t. That is,

$$\begin{aligned} V_{t+1} - V_t &= S_t - \delta K_t = S_{1t} + S_{2t} - GBD_t = S_{1t} + (-A_t) - GBD_t \\ &= A_{t+1} - A_t - (B_{t+1} - B_t), \end{aligned}$$

where the last equality comes from (6.21) and the maintained assumption that budget deficits are fully financed by debt issue.

Regarding the relations of SOE to international markets we re-introduce the three standard assumptions (from Chapter 5):

(a) There is perfect mobility of goods and financial capital across borders.

(b) There is no uncertainty and domestic and foreign financial claims are perfect substitutes.

(c) There is no labor mobility across borders.

The assumptions (a) and (b) imply *real interest rate equality.* That is, the real interest rate in our SOE must equal the real interest rate in the world financial market, r. And by saying that the SOE is "small" we mean it is small enough to not affect the world market interest rate as well as other world market factors. We imagine that all countries trade one and the same homogeneous good. As argued in Section 5.3 of Chapter 5, international trade will then only be *intertemporal* trade, i.e., international borrowing and lending of this good. Suppose r is constant over time and that $r > n$.

Technology and preferences

GDP is produced by an aggregate neoclassical production function with CRS:

$$Y_t = F(K_t, L_t),$$

where K_t and L_t are input of capital and labor, respectively. Technological change is ignored. Imposing perfect competition in all markets, markets clear so that L_t can be interpreted as both employment and labor supply. Profit maximization leads to $f'(k_t) = r + \delta$, where $k_t \equiv K_t/L_t$, $f(k_t) \equiv F(k_t, 1)$, and δ is a constant capital depreciation rate, $0 \leq \delta \leq 1$. When f satisfies the condition $\lim_{k\to 0} f'(k) > r + \delta > \lim_{k\to\infty} f'(k)$, there is always a solution in k to this equation and it is unique (since $f'' < 0$) and constant over time (as long as r and δ are constant). Thus,

$$k_t = f'^{-1}(r + \delta) \equiv k, \text{ for all } t. \tag{6.23}$$

The stock of capital, K_t, is the determined as $K_t = kL_t$.

In view of firms' profit maximization, the equilibrium real wage before tax is

$$w_t = \frac{\partial Y_t}{\partial L_t} = f(k) - f'(k)k \equiv w, \tag{6.24}$$

a constant. GDP will evolve according to

$$Y_t = f(k)L_t = f(k)L_0(1 + n)^t = Y_0(1 + n)^t = Y_0(1 + n)^t$$

and thus grow at the same rate as the labor force, i.e., $g_Y = n$.

Suppose the role of the government sector is to deliver a service, a "public good". Let G_t be the size of this service. Think of a non-rival good like "rule of law", TV-transmitted theatre, or another public service free of charge etc. Suppose

$$G_t = G_0(1 + n)^t,$$

where $0 < G_0 < F(K_0, L_0)$. Without consequences for the qualitative results, we assume a CRRA period utility function $u(c) = (c^{1-\theta} - 1)/(1-\theta)$, where $\theta >$

0. To keep things simple, the utility of the public good enters additively in lifetime utility so that it does not affect marginal utilities of private consumption. In addition we assume that the public good does not affect productivity in the private sector. There is a lump-sum tax on the young as well as the old in period t, τ_{1t} and τ_{2t}, respectively (a negative value of either τ_{1t} or τ_{2t} represents a transfer).

The consumption-saving decision of the young will be the solution to the following problem:

$$\begin{aligned}
\max U(c_{1t}, c_{2t+1}) &= \frac{c_{1t}^{1-\theta} - 1}{1-\theta} + v(G_t) + (1+\rho)^{-1}\left[\frac{c_{2t+1}^{1-\theta} - 1}{1-\theta} + v(G_{t+1})\right] \text{ s.t.} \\
c_{1t} + s_t &= w - \tau_{1t}, \\
c_{2t+1} &= (1+r)s_t - \tau_{2t+1}, \\
c_{1t} &\geq 0, c_{1t+1} \geq 0,
\end{aligned}$$

where the function $v(\cdot)$ represents the utility contribution of the public good. The implied Euler equation can be written

$$\frac{c_{2t+1}}{c_{1t}} = \left(\frac{1+r}{1+\rho}\right)^{1/\theta}.$$

Inserting the two budget constraints and ordering gives

$$s_t = \frac{w - \tau_{1t} + \tau_{2t+1}\left(\frac{1+\rho}{1+r}\right)^{1/\theta}}{1 + (1+\rho)^{1/\theta}(1+r)^{(\theta-1)/\theta}}.$$

Consumption in the first and second period then is

$$c_{1t} = w - \tau_1 - s_t = z_1 h_t \tag{6.25}$$

and

$$c_{2t+1} = z_2 h_t, \tag{6.26}$$

respectively, where

$$\begin{aligned}
z_1 &\equiv \frac{(1+\rho)^{1/\theta}(1+r)^{(\theta-1)/\theta}}{1 + (1+\rho)^{1/\theta}(1+r)^{(\theta-1)/\theta}} \in (0,1), \\
z_2 &= \left(\frac{1+r}{1+\rho}\right)^{1/\theta} z_1 = \frac{1+r}{1 + (1+\rho)^{1/\theta}(1+r)^{(\theta-1)/\theta}},
\end{aligned}$$

and

$$h_t \equiv w - \tau_{1t} - \frac{\tau_{2t+1}}{1+r}. \tag{6.27}$$

This last equation says that h_t is the after-tax human wealth, that is, the present value of disposable lifetime income; we assume τ_{1t} and τ_{2t} in every period are such that $h_t > 0$. The consumption in both the first and the second period of life is thus proportional to individual human wealth.[16]

The tax revenue in period t is $T_t = \tau_{1t}L_t + \tau_{2t}L_{t-1}$. Suppose $B_0 = 0$ and that in the reference path the budget is and remains balanced for all t, i.e., $T_t = G_t$. For simplicity, let the lump-sum taxes be constants, τ_1 and τ_2. Then the tax scheme is some pair (τ_1, τ_2) satisfying

$$\left(\tau_1 + \frac{\tau_2}{1+n}\right) L_0 = G_0.$$

In view of the tax scheme (τ_1, τ_2) being time independent, so are individual human wealth and the individual consumption levels, which we may hence denote c_1 and c_2. In the reference path aggregate private consumption now grows at the same constant rate as GDP and public consumption, the rate n. Indeed,

$$C_t = c_1 L_1 + \frac{c_2}{1+n} L_t = (c_1 + \frac{c_2}{1+n}) L_0 (1+n)^t = C_0 (1+n)^t.$$

A one-time tax cut

What are the consequences of a one-time cut in taxation by $x > 0$ for every individual, whether young or old? Letting the tax cut occur in period 0, it amounts to creating a budget deficit in period 0 equal to $(L_0 + L_{-1})x$. At the start of period 1 there is thus a debt $B_1 = (L_0 + L_{-1})x$. Since we assumed $r > n = g_Y$, government solvency now requires a rise in the present value of future taxes equal to $(L_0 + L_{-1})x(1+r)^{-1}$. This may be accomplished by for instance raising the tax on all individuals from period 1 onward by m. The required value of m will satisfy

$$\sum_{t=1}^{\infty} (L_0 + L_{-1})(1+n)^t m (1+r)^{-(t+1)} = (L_0 + L_{-1})x(1+r)^{-1}.$$

This gives

$$m \sum_{t=1}^{\infty} \left(\frac{1+n}{1+r}\right)^t = x, \qquad (r > n)$$

so that

$$m = \frac{r-n}{1+n} x \equiv \bar{m}. \tag{6.28}$$

[16] This was to be expected in view of the homothetic life time utility function and the constant interest rate.

The higher the growth-corrected interest rate, $r-n$, the higher the needed rise in future taxes.

Let the value of the variables in this alternative fiscal regime be marked with a prime. In period 0 the tax cut unambiguously benefit the old whose increase in consumption equals the saved taxes:

$$c_{20}' - c_{20} = x > 0.$$

The young in period 0 enjoy an increase in after-tax human wealth equal to

$$\begin{aligned} h_0' - h_0 &= w - \tau_1 + x - \frac{\tau_2 + \bar{m}}{1+r} - \left(w - \tau_1 - \frac{\tau_2}{1+r} \right) \\ &= \left(1 - \frac{r-n}{(1+r)(1+n)} \right) x \qquad \text{(by (6.28))} \\ &= \frac{1+(2+r)n}{(1+r)(1+n)} x > 0. \end{aligned}$$

But all future generations are worse off, since they do not benefit from the tax relief in period 0, but instead have to pay for this by a *reduction* in individual after-tax human wealth. Indeed, for $t \geq 1$,

$$\begin{aligned} h_t' - h_t &= w - \tau_1 - \bar{m} - \frac{\tau_2 + \bar{m}}{1+r} - \left(w - \tau_1 - \frac{\tau_2}{1+r} \right) \\ &= -\left(\bar{m} + \frac{\bar{m}}{1+r} \right) < 0. \end{aligned}$$

All things considered, since both the young and the old in period 0 increase their consumption, aggregate consumption rises and aggregate saving ($\equiv$ national saving) therefore falls, implying less national wealth and less aggregate consumption in the future. Thus Ricardian equivalence fails. Indeed, according to this analysis, budget deficits crowd out private saving and reduce future national wealth.

In our model example the lower aggregate saving in period 0 results in higher net foreign debt at the beginning of period 1 than otherwise. Essentially, the reason that the generations 1, 2,..., are worse off than otherwise is this debt, which requires interest payments. In a closed economy these future generations would also be worse off, but that would be because the national capital stock at the beginning of period 1 would be smaller than otherwise, in view of the smaller aggregate saving in period 0.

The fundamental point underlined by OLG models is that there is a difference between the public sector's future tax base, including the resources of individuals yet to be born, and the future tax base emanating from individuals alive today. This difference implies a tendency to non-neutrality of variations over time in the pattern of lump-sum taxation. There is little theoretical as well as empirical foundation for the debt neutrality view that this tendency is fully offset by parental altruism or other factors.

In an economic depression with high unemployment, temporary budget deficits also have real effects contrary to Barro's debt neutrality hypothesis. As accounted for in Keynesian short-run theory, this is due to a different mechanism, namely the positive effect on aggregate demand of allowing a deficit to arise (possibly solely through by the operation of the automatic stabilizers). Additional factors of relevance for the debate about Ricardian equivalence or non-equivalence are discussed in Chapter 13.

Further perspectives

One might add that in the real world taxes are *not* lump sum, but usually distortionary. This fact should, of course, always be kept in mind when discussing *practical issues* of fiscal policy. Yet, it is not an argument against the possible *theoretical* validity of the Ricardian equivalence view. This is so because Ricardian equivalence (in the proper sense of the word) claims absence of allocational effects of deficit financing *when* taxes are lump sum. There is hardly disagreement that *distortionary* taxes and their *timing* may matter.

In the end, whether Ricardian equivalence is a good or bad approximation to reality is an empirical question. Sometimes an observed a negative correlation between public and private *financial* saving is taken as an argument for Ricardian equivalence. That is a mistake, however. A sector's *financial saving* is defined as the sector's total saving minus its capital investment. For a closed economy a negative relationship between public and private financial saving is a trivial implication of accounting. Indeed, the closed-economy identity $I = S$ implies $I_p + I_g = S_p + S_g$, where the subindices p and g stand for "private" and "government", respectively. Thus,

$$(S_p - I_p) + (S_g - I_g) = 0,$$

saying that private financial saving plus government financial saving must necessarily sum to zero. Moreover, if net capital investment by the government is close to negligible, the financial saving of the government approximately equals the budget surplus, i.e., $S_g - I_g \approx GBS$. Then we have, essentially from mere accounting,

$$S_p - I_p \approx -GBS.$$

So there will almost automatically be a negative correlation between the government budget surplus and private financial saving whether or not Ricardian equivalence holds.

For an open economy we have that the *current account surplus*, CAS, equals $S - I = S_p + S_g - (I_p + I_g)$, so that

$$(S_p - I_p) + (S_g - I_g) = CAS.$$

This says that unless the current account surplus moves one-to-one (or more) with government financial saving, there will by mere accounting (i.e., independently of whether Ricardian equivalence holds or not) be a negative relationship between government and private financial saving.

The Ricardian equivalence hypothesis is however the statement that neither aggregate saving nor aggregate investment depends on government saving. In view of the accounting relation

$$CAS = S - I,$$

there should thus, according to the Ricardian equivalence hypothesis, be no relationship between budget deficits and current account deficits. Yet, the simple cross-country regression analysis for 19 OECD countries over the 1981-86 period reported in Obstfeld and Rogoff (1996, p. 144) indicates a positive relationship between budget and current account surpluses. Warning about the omitted variable problem, the authors also consider two historical episodes of drastic shifts in tax policy, namely the U.S. tax cuts in the early 1980s and the large debt-financed transfer program from the western part to the eastern part of unified Germany starting in 1990. These episodes, which mirror a Ricardian experiment of a ceteris paribus tax cut, support the debt-*non*neutrality view that private saving does not rise one-to-one in response to government dissaving.[17] On the other hand, there certainly exists evidence that consumers sometimes worry about the future fiscal stance of the country; and government spending multipliers tend to be *smaller* in highly indebted countries (see Ilzetzki et al., 2009).

6.7 Literature notes

[17] Obstfeld and Rogoff (1996, p. 144-45).

Chapter 9

Dynamic optimization in discrete and continuous time

In the next two chapters we shall discuss the continuous-time version of the basic representative agent model, the Ramsey model, and some of its applications. As a preparation for this, the present chapter gives an account of the transition from discrete time to continuous time analysis and of the application of optimal control theory for solving the household's consumption/saving problem in continuous time.

There are many fields of study where a setup in continuous time is preferable to one in discrete time. One reason is that continuous time opens up for application of the mathematical apparatus of differential equations; this apparatus is more powerful than the corresponding apparatus of difference equations. Another reason is that optimal control theory is more developed and potent in its continuous time version than in its discrete time version, considered in Chapter 8. In addition, many formulas in continuous time are simpler than the corresponding ones in discrete time (cf. the growth formulas in Appendix A).

As a vehicle for comparing continuous time analysis with discrete time analysis we consider a standard household consumption/saving problem. How does the household assess the choice between consumption today and consumption in the future? In contrast to the preceding chapters we allow for an arbitrary number of periods within the time horizon of the household. The period length may thus be much shorter than in the previous models. This opens up for capturing additional aspects of economic behavior and for carrying out the transition to continuous time in a smooth way.

First, we specify the market environment in which the optimizing household operates.

9.1 Market conditions

In the Diamond OLG model no loan market was active and wealth effects on consumption through changes in the interest rate were absent. It is different in a setup where agents live for many periods and realistically have a hump-shaped income profile through life. This motivates a look at the financial market and more refined notions related to intertemporal choice.

We maintain the assumption of perfect competition in all markets, i.e., households take all prices as given from the markets. Ignoring uncertainty, the various assets (real capital, stocks, loans etc.) in which households invest give the same rate of return in equilibrium.

A perfect loan market Consider a given household. Suppose it can at any date take a loan or provide loans to others at the going interest rate, i_t, measured in money terms. That is, monitoring, administration, and other transaction costs are absent so that one faces the same interest rate whether borrowing or lending and one can not influence this rate. There are no borrowing restrictions other than the requirement on the part of borrowers to comply with their intertemporal budget constraint. The borrowers can somehow be forced to repay their debt and so there is no default risk. A loan market satisfying these idealized conditions is called a *perfect loan market.* The implications of such a market are:

1. various payment streams can be subject to comparison; if they have the same present value (PV for short), they are equivalent;

2. any payment stream can be converted into another one with the same present value;

3. payment streams can be compared with the value of stocks.

Consider a payment stream $\{x_t\}_{t=0}^{T-1}$ over T periods, where x_t is the payment at the *end* of period t. As in the previous chapters, period t runs from time t to time $t+1$ for $t = 0, 1, ..., T-1$; and i_t is defined as the interest rate on a loan from time t to time $t+1$. Then the present value, PV_0, as seen from the beginning of period 0, of the payment stream is defined as[1]

$$PV_0 = \frac{x_0}{1+i_0} + \frac{x_1}{(1+i_0)(1+i_1)} + \cdots + \frac{x_{T-1}}{(1+i_0)(1+i_1)\cdots(1+i_{T-1})}. \quad (9.1)$$

[1] We use "present value" as synonymous with "present discounted value". As usual our timing convention is such that PV_0 denotes the time-0 value of the payment stream, including the discounted value of the payment (or dividend) indexed by 0.

If Ms. Jones is entitled to the income stream $\{x_t\}_{t=0}^{T-1}$, but wishes to buy today a durable consumption good of value PV_0, she can borrow this amount and use the income stream $\{x_t\}_{t=0}^{T-1}$ to repay the debt over the periods $t = 0, 1, 2, ..., T - 1$. In general, when Jones wishes to have a time profile on the payment stream different from the income stream, she can attain this through appropriate transactions in the loan market, leaving her with any stream of payments of the same present value as the income stream.

The good which is traded in the loan market will here be referred to as a *bond*. The borrower issues bonds and the lender buys them. In this chapter all bonds are assumed to be short-term, i.e., one-period bonds. For every unit of account borrowed in the form of a one-period loan at the end of period $t-1$, the borrower pays back with certainty (1 + short-term interest rate) units of account at the end of period t. If a borrower wishes to maintain debt through several periods, new bonds are issued and the obtained loans are spent rolling over the older loans at the going market interest rate. For the lender, who lends in several periods, this is equivalent to a variable-rate demand deposit in a bank.[2]

Real versus nominal rate of return As in the preceding chapters our analysis will be in "real" terms, also called "inflation-corrected" terms. In principle the unit of account is a fixed bundle of consumption goods. In the simple macroeconomic models to be studied in this and subsequent chapters, such a bundle is reduced to *one* consumption good because the models assume there *is* only one consumption good in the economy. Moreover, there will only be *one produced good*, "the" output good, which can be used for both consumption and capital investment. Then, whether we say our unit of account is the consumption good or the output good does not matter. To fix our language, we will say the latter.

The *real* (net) rate of return on an investment is the rate of return in units of the output good. More precisely, the *real rate of return* in period t, r_t, is the (proportionate) rate at which the *real* value of an investment, made at the end of period $t-1$, has grown after one period.

The link between this rate of return and the more common place nominal rate of return is the following. Imagine that at the end of period $t-1$ you make a deposit of value V_t euro on an account in a bank. The *real value* of the deposit when you invest is then V_t/P_{t-1}, where P_{t-1} is the price in euro of the output good at the end of period $t-1$. If the nominal short-term interest rate is i_t, the deposit is worth $V_{t+1} = V_t(1+i_t)$ euro at the end of period t. By

[2]Unless otherwise specified, we use terms like "loan market", "credit market, and "bond market" interchangeably.

definition of r_t, the factor by which the deposit in real terms has expanded is

$$1 + r_t = \frac{V_{t+1}/P_t}{V_t/P_{t-1}} = \frac{V_{t+1}/V_t}{P_t/P_{t-1}} = \frac{1+i_t}{1+\pi_t}, \tag{9.2}$$

where $\pi_t \equiv (P_t - P_{t-1})/P_{t-1}$ is the inflation rate in period t. So the real (net) *rate* of return on the investment is $r_t = (i_t - \pi_t)/(1+\pi_t) \approx i_t - \pi_t$ for i_t and π_t "small". The number $1 + r_t$ is called the real interest *factor* and measures the rate at which current units of output can be traded for units of output one period later.

In the remainder of this chapter we will think in terms of *real* values and completely ignore monetary aspects of the economy.

9.2 Maximizing discounted utility in discrete time

We assume that the consumption/saving problem faced by the household involves only one consumption good. So the composition of consumption in the single periods is not part of the problem. What remains is the question how to distribute consumption over time.

The intertemporal utility function

A plan for consumption in the periods $0, 1, ..., T-1$ is denoted $\{c_t\}_{t=0}^{T-1}$, where c_t is the consumption in period t. We say the plan has *time horizon* T. We assume the preferences of the household can be represented by a time-separable intertemporal utility function with a constant utility discount rate and no utility from leisure. The latter assumption implies that the labor supply of the household in each period is exogenous (and thereby inelastic). The time-separability itself just means that the intertemporal utility function is additive, i.e., $U(c_0, c_1, \ldots, c_{T-1}) = u^{(0)}(c_0) + u^{(1)}(c_1) + \ldots + u^{(T-1)}(c_{T-1})$, where $u^{(t)}(c_t)$ is the utility contribution from period-t consumption, $t = 0, 1, \ldots, T-1$. But in addition we assume there is a constant utility discount rate, $\rho > -1$, implying that $u^{(t)}(c_t) = u(c_t)(1+\rho)^{-t}$, where $u(c)$ is a time-independent period utility function. Together these two assumptions amount to

$$U(c_0, c_1, \cdots, c_{T-1}) = u(c_0) + \frac{u(c_1)}{1+\rho} + \ \ldots + \frac{u(c_{T-1})}{(1+\rho)^{T-1}} = \sum_{t=0}^{T-1} \frac{u(c_t)}{(1+\rho)^t}. \tag{9.3}$$

The period utility function, also called a *subutility* function, is assumed to satisfy $u'(c) > 0$ and $u''(c) < 0$.

To avoid corner solutions we impose the No Fast assumption $\lim_{c \to 0} u'(c) = \infty$. As (9.3) indicates, the number $1 + \rho$ tells how many extra units of utility in the next period the household insists on to compensate for a decrease of one unit of utility in the current period. So, a $\rho > 0$ will reflect that if the level of consumption is the same in two periods, then the individual always appreciates a marginal unit of consumption higher if it arrives in the earlier period. This explains why ρ is named "the rate of time preference" or even more to the point "the rate of impatience". The *utility discount factor,* $1/(1 + \rho)^t$, indicates how many units of utility the household is at most willing to give up in period 0 to get one additional unit of utility in period t (see Box 9.1).[3]

It is generally believed that human beings are impatient and that ρ should therefore be assumed positive; indeed, it seems intuitively reasonable that the distant future does not matter much for current private decisions.[4] There is, however, a growing body of evidence suggesting that the discount rate is not constant, but declining with the time distance from now to the period in question (see, e.g., Loewenstein and Thaler, 1989). Since this last point complicates the models considerably, macroeconomics often, as a first approach, ignores it and assumes a constant ρ to keep things simple. Here we follow this practice. Except where needed, we shall not, however, impose any other constraint on ρ than the definitional requirement in discrete time that $\rho > -1$.

[3] Multiplying through in (9.3) by $(1+\rho)^{-1}$ would make the objective function appear in a way similar to (9.1) in the sense that also the first term in the sum becomes discounted. At the same time the ranking of all possible alternative consumption paths would remain unaffected. For ease of notation, however, we use the form (9.3) which is more standard.

[4] If uncertainty were included in the model, $(1 + \rho)^{-1}$ might be seen as reflecting the probability of surviving to the next period and in this perspective $\rho > 0$ seems a plausible assumption.

Box 9.1. Admissible transformations of the period utility function
When preferences, as assumed here, can be represented by *discounted utility*, the concept of utility appears at two levels. The function $U(\cdot)$ in (9.3) is defined on the set of alternative feasible consumption paths and corresponds to an ordinary utility function in general microeconomic theory. That is, $U(\cdot)$ will express the same ranking between alternative consumption paths as any increasing transformation of $U(\cdot)$. The period utility function, $u(\cdot)$, defined on the consumption in a single period, is a less general concept, requiring that reference to "utility units" is legitimate. That is, the *size* of the difference in terms of period utility between two outcomes has significance for choices. Indeed, the essence of the discounted utility hypothesis is that we have, for example,

$$u(c_0) - u(c_0') > 0.95\left[u(c_1') - u(c_1)\right] \Leftrightarrow (c_0, c_1) \succ (c_0', c_1'),$$

meaning that the household, having a utility discount factor $1/(1+\rho) = 0.95$, strictly prefers consuming (c_0, c_1) to (c_0', c_1') in the first two periods, if and only if the utility differences satisfy the indicated inequality. (The notation $x \succ y$ means that x is strictly preferred to y.)

Only a *linear* positive transformation of the utility function $u(\cdot)$, that is, $v(c) = au(c) + b$, where $a > 0$, leaves the ranking of all possible alternative consumption paths, $\{c_t\}_{t=0}^{T-1}$, unchanged.[5]

The saving problem in discrete time

Suppose the household considered has income from two sources: work and financial wealth. Let a_t denote the real value of financial wealth held by the household at the beginning of period t (a for "assets"). We treat a_t as predetermined at time t and in this respect similar to a variable-interest deposit in a bank. The initial financial wealth, a_0, is thus *given*, independently of whatever might happen to expected future interest rates. And a_0 can be positive as well as negative (in the latter case the household is initially in debt).

The labor income of the household in period t is denoted $w_t \geq 0$ and may follow a typical life-cycle pattern with labor income first rising, then more or less stationary, and finally vanishing due to retirement. Thus, in contrast to previous chapters where w_t denoted the real wage per unit of labor, here a broader interpretation of w_t is allowed. Whatever the time profile of the

[5] This is because a linear positive transformation does not affect the *ratios* of marginal period utilities (the marginal rates of substitution across time).

amount of labor delivered by the household through life, in this chapter, where the focus is on individual saving, we regard this time profile, as well as the hourly wage as exogenous. The present interpretation of w_t will coincide with the one in the other chapters if we imagine that the household in each period delivers one unit of labor.

Since uncertainty is by assumption ruled out, the problem is to choose a plan $(c_0, c_1, \ldots, c_{T-1})$ so as to maximize

$$\begin{aligned} U &= \sum_{t=0}^{T-1} u(c_t)(1+\rho)^{-t} \qquad \text{s.t.} & (9.4)\\ c_t &\geq 0, & (9.5)\\ a_{t+1} &= (1+r_t)a_t + w_t - c_t, \qquad a_0 \text{ given}, & (9.6)\\ a_T &\geq 0, & (9.7) \end{aligned}$$

where r_t is the interest rate. The control region (9.5) reflects the definitional nonnegativity of consumption. The dynamic equation (9.6) is an accounting relation telling how financial wealth moves over time. Indeed, income in period t is $r_t a_t + w_t$ and saving is then $r_t a_t + w_t - c_t$. Since saving is by definition the same as the increase in financial wealth, $a_{t+1} - a_t$, we obtain (9.6). Finally, the terminal condition (9.7) is a solvency requirement that no financial debt be left over at the terminal date. We shall refer to this decision problem as the *standard discounted utility maximization problem without uncertainty.*

Solving the problem

To solve the problem, let us use the substitution method. From (9.6) we have $c_t = (1 + r_t)a_t + w_t - a_{t+1}$, for $t = 0, 1, \ldots, T - 1$. Substituting this into (9.4), we obtain a function of $a_1, a_2, \ldots, a_T$. Since $u' > 0$, saturation is impossible and so an optimal solution cannot have $a_T > 0$. Hence we can put $a_T = 0$ and the problem is reduced to an essentially unconstrained problem of maximizing a function $\tilde{U}$ w.r.t. $a_1, a_2, \ldots, a_{T-1}$. Thereby we indirectly choose $c_0, c_1, \ldots, c_{T-2}$. Given a_{T-1}, consumption in the last period is trivially given as

$$c_{T-1} = (1 + r_{T-1})a_{T-1} + w_{T-1},$$

ensuring $a_T = 0$.

To obtain first-order conditions we put the partial derivatives of $\tilde{U}$ w.r.t. a_{t+1}, $t = 0, 1, \ldots, T - 2$, equal to 0:

$$\frac{\partial \tilde{U}}{\partial a_{t+1}} = (1+\rho)^{-t}\left[u'(c_t)\cdot(-1) + (1+\rho)^{-1}u'(c_{t+1})(1+r_{t+1})\right] = 0.$$

Reordering gives the Euler equations describing the trade-off between consumption in two succeeding periods,

$$u'(c_t) = (1+\rho)^{-1}u'(c_{t+1})(1+r_{t+1}), \qquad t = 0, 1, 2, ..., T-2. \tag{9.8}$$

Interpretation The interpretation of (30.16) is as follows. Let the consumption path $(c_0, c_1, \ldots, c_{T-1})$ be our "reference path". Imagine an alternative path which coincides with the reference path except for the periods t and $t+1$. If it is possible to obtain a higher total discounted utility than in the reference path by varying c_t and c_{t+1} within the constraints (9.5), (9.6), and (9.7), at the same time as consumption in the other periods is kept unchanged, then the reference path cannot be optimal. That is, "local optimality" is a necessary condition for "global optimality". So the optimal plan must be such that the current utility loss by decreasing consumption c_t by one unit equals the discounted expected utility gain next period by having $1 + r_{t+1}$ extra units available for consumption, namely the gross return on saving one more unit in the current period.

A more concrete interpretation, avoiding the notion of "utility units", is obtained by rewriting (30.16) as

$$\frac{u'(c_t)}{(1+\rho)^{-1}u'(c_{t+1})} = 1 + r_{t+1}. \tag{9.9}$$

The left-hand side indicates the marginal rate of substitution, MRS, of period-$(t+1)$ consumption for period-t consumption, namely the increase in period-$(t+1)$ consumption needed to compensate for a one-unit marginal decrease in period-t consumption:

$$MRS_{t+1,t} = -\frac{dc_{t+1}}{dc_t}\,|_{U=\bar{U}} = \frac{u'(c_t)}{(1+\rho)^{-1}u'(c_{t+1})}. \tag{9.10}$$

And the right-hand side of (9.9) indicates the marginal rate of transformation, MRT, which is the rate at which the loan market allows the household to shift consumption from period t to period $t+1$. In an optimal plan MRS must equal MRT.

The formula (9.10) for MRS indicates why the assumption of a constant utility discount rate is convenient (but also restrictive). The marginal rate of substitution between consumption this period and consumption next period is independent of the level of consumption as long as this level is the same in the two periods. The formula for MRS between consumption this period and consumption two periods ahead is

$$MRS_{t+2,t} = -\frac{dc_{t+2}}{dc_t}\,|_{U=\bar{U}} = \frac{u'(c_t)}{(1+\rho)^{-2}u'(c_{t+2})}.$$

This displays one of the reasons that the time-separability of the intertemporal utility function is a strong assumption. It implies that the trade-off between consumption this period and consumption two periods ahead is independent of consumption in the interim.

Deriving a consumption function The first-order conditions (30.16) tell us about the relative consumption levels over time, not the absolute level. The latter is determined by the condition that initial consumption, c_0, must be highest possible, given that the first-order conditions *and* the constraints (9.6) and (9.7) must be satisfied.

To find an explicit solution we have to specify the period utility function. As an example we choose the CRRA function $u(c) = c^{1-\theta}/(1-\theta)$, where $\theta > 0$.[6] Moreover we simplify by assuming $r_t = r$, a constant > -1. Then the Euler equations take the form $(c_{t+1}/c_t)^\theta = (1+r)(1+\rho)^{-1}$ so that

$$\frac{c_{t+1}}{c_t} = \left(\frac{1+r}{1+\rho}\right)^{1/\theta} \equiv \gamma, \tag{9.11}$$

and thereby $c_t = \gamma^t c_0$, $t = 0, 1, \ldots, T-1$. Substituting into the accounting equation (9.6), we thus have $a_{t+1} = (1+r)a_t + w_t - \gamma^t c_0$. By backward substitution we find the solution of this difference equation to be

$$a_t = (1+r)^t \left[a_0 + \sum_{i=0}^{t-1}(1+r)^{-(i+1)}(w_i - \gamma^i c_0)\right].$$

Optimality requires that the left-hand side of this equation vanishes for $t = T$. So we can solve for c_0 :

$$c_0 = \frac{1+r}{\sum_{i=0}^{T-1}\left(\frac{\gamma}{1+r}\right)^i}\left[a_0 + \sum_{i=0}^{T-1}(1+r)^{-(i+1)}w_i\right] = \frac{1+r}{\sum_{i=0}^{T-1}\left(\frac{\gamma}{1+r}\right)^i}(a_0 + h_0), \tag{9.12}$$

where we have inserted the human wealth of the household (present value of expected lifetime labor income) as seen from time zero:

$$h_0 = \sum_{i=0}^{T-1}(1+r)^{-(i+1)}w_i. \tag{9.13}$$

[6] In later sections of this chapter we will let the time horizon of the decision maker go to infinity. To ease convergence of an infinite sum of discounted utilities, it is an advantage not to have to bother with additive constants in the period utilities and therefore we write the CRRA function as $c^{1-\theta}/(1-\theta)$ instead of the form, $(c^{1-\theta}-1)/(1-\theta)$, introduced in Chapter 3. As implied by Box 9.1, the two forms represent the same preferences.

Thus (9.12) says that initial consumption is proportional to initial total wealth, the sum of financial wealth and human wealth at time 0. To allow positive consumption we need $a_0 + h_0 > 0$, which we assume satisfied.

As (9.12) indicates, the propensity to consume out of total wealth depends on:

$$\sum_{i=0}^{T-1}\left(\frac{\gamma}{1+r}\right)^i = \begin{cases} \frac{1-\left(\frac{\gamma}{1+r}\right)^T}{1-\frac{\gamma}{1+r}} & \text{when } \gamma \neq 1+r, \\ T & \text{when } \gamma = 1+r. \end{cases} \tag{9.14}$$

where the result for $\gamma \neq 1+r$ follows from the formula for the sum of a finite geometric series. Inserting this together with (9.11) into (9.12), we end up with a *candidate* consumption function,

$$c_0 = \begin{cases} \frac{(1+r)\left[1-(1+\rho)^{-1/\theta}(1+r)^{(1-\theta)/\theta}\right]}{1-(1+\rho)^{-T/\theta}(1+r)^{(1-\theta)T/\theta}}(a_0+h_0) & \text{when } \left(\frac{1+r}{1+\rho}\right)^{1/\theta} \neq 1+r, \\ \frac{1+r}{T}(a_0+h_0) & \text{when } \left(\frac{1+r}{1+\rho}\right)^{1/\theta} = 1+r. \end{cases} \tag{9.15}$$

For the subsequent periods, $c_t = ((1+r)/(1+\rho))^{t/\theta}\, c_0$, $t = 1,\ldots, T-1$.

EXAMPLE 1 Let $\rho > 0$ and $\theta = 1$ (interpreted as $u(c) = \ln c$). Then the upper case in (9.15) is the relevant one, so that period-0 consumption will be

$$c_0 = \frac{(1+r)(1-(1+\rho)^{-1})}{1-(1+\rho)^{-T}}(a_0+h_0) \qquad \text{for } \theta = 1.$$

We see that $c_0 \to (1+r)\rho(1+\rho)^{-1}(a_0+h_0)$ for $T \to \infty$. We have assumed that payment for consumption occurs at the end of the period at the price 1 per consumption unit. To compare with the corresponding result in continuous time with continuous compounding (see Section 9.4), we might want to have initial consumption in the same present value terms as a_0 and h_0. That is, we consider $\tilde{c}_0 \equiv c_0(1+r)^{-1} = \rho(1+\rho)^{-1}(a_0+h_0)$. □

That our candidate consumption function is indeed an optimal solution when $a_0 + h_0 > 0$ follows by concavity of the objective function (or by concavity of the Hamiltonian if one applies optimal control theory, cf. the previous chapter). The conclusion is that under the idealized conditions assumed, including a perfect loan market and perfect foresight, it is only initial wealth and the interest rate that affect the time profile of consumption. The time profile of income does not matter because consumption can be smoothed over time by drawing on the bond market.

Alternative approach based on the intertemporal budget constraint

There is another approach to the household's saving problem. With its choice of consumption plan the household must act in conformity with its intertemporal budget constraint (IBC for short). The present value of the consumption plan $(c_1, ..., c_{T-1})$, as seen from time zero, is

$$PV(c_0, c_1, ..., c_{T-1}) \equiv \sum_{t=0}^{T-1} \frac{c_t}{\Pi_{\tau=0}^{t}(1+r_\tau)}. \tag{9.16}$$

This value cannot exceed the household's total initial wealth, $a_0 + h_0$. So the household's *intertemporal budget constraint* is

$$\sum_{t=0}^{T-1} \frac{c_t}{\Pi_{\tau=0}^{t}(1+r_\tau)} \leq a_0 + h_0. \tag{9.17}$$

In this setting the household's problem is to choose its consumption plan so as to maximize U in (9.4) subject to this budget constraint.

This way of stating the problem is equivalent to the approach above based on the dynamic budget condition (9.6) and the solvency condition (9.7). Indeed, given the accounting equation (9.6), the consumption plan of the household will satisfy the intertemporal budget constraint (9.17) if and only if it satisfies the solvency condition (9.7). And there will be strict equality in the intertemporal budget constraint if and only if there is strict equality in the solvency condition (the proof is left as Exercise 9.?).

Moreover, since in our specific saving problem saturation is impossible, an optimal solution must imply strict equality in (9.17). So it is straightforward to apply the substitution method also within the IBC approach. Alternatively one can introduce the Lagrange function associated with the problem of maximizing $U = \sum_{t=0}^{T-1}(1+\rho)^{-t}u(c_t)$ s.t. (9.17) with strict equality.

9.3 Transition to continuous time analysis

In the discrete time framework the run of time is divided into successive periods of equal length, taken as the time-unit. Let us here index the periods by $i = 0, 1, 2, ...$ Thus financial wealth accumulates according to

$$a_{i+1} - a_i = s_i, \qquad a_0 \text{ given},$$

where s_i is (net) saving in period i.

Multiple compounding per year

With time flowing continuously, we let $a(t)$ refer to financial wealth at time t. Similarly, $a(t+\Delta t)$ refers to financial wealth at time $t+\Delta t$. To begin with, let Δt equal one time unit. Then $a(i\Delta t)$ equals $a(i)$ and is of the same value as a_i. Consider the *forward* first difference in a, $\Delta a(t) \equiv a(t+\Delta t) - a(t)$. It makes sense to consider this change in a in relation to the length of the time interval involved, that is, to consider the *ratio* $\Delta a(t)/\Delta t$. As long as $\Delta t = 1$, with $t = i\Delta t$ we have $\Delta a(t)/\Delta t = (a_{i+1} - a_i)/1 = a_{i+1} - a_i$. Now, keep the time unit unchanged, but let the length of the time interval $[t, t+\Delta t)$ approach zero, i.e., let $\Delta t \to 0$. When $a(\cdot)$ is a differentiable function, we have

$$\lim_{\Delta t \to 0} \frac{\Delta a(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{a(t+\Delta t) - a(t)}{\Delta t} = \frac{da(t)}{dt},$$

where $da(t)/dt$, often written $\dot{a}(t)$, is known as the *derivative of* $a(\cdot)$ at the point t. Wealth accumulation in continuous time can then be written

$$\dot{a}(t) = s(t), \qquad a(0) = a_0 \text{ given}, \tag{9.18}$$

where $s(t)$ is the saving at time t. For Δt "small" we have the approximation $\Delta a(t) \approx \dot{a}(t)\Delta t = s(t)\Delta t$. In particular, for $\Delta t = 1$ we have $\Delta a(t) = a(t+1) - a(t) \approx s(t)$.

As time unit choose one year. Going back to discrete time we have that if wealth grows at a constant (proportionate) rate $g > 0$ per year, then after i periods of length one year, with annual compounding,

$$a_i = a_0(1+g)^i, \quad i = 0, 1, 2, \ldots . \tag{9.19}$$

If instead compounding (adding saving to the principal) occurs n times a year, then after i periods of length of $1/n$ year,

$$a_i = a_0(1+\frac{g}{n})^i. \tag{9.20}$$

With t still denoting time (measured in years) that has passed since the initial date (here date 0), we have $i = nt$ periods. Substituting into (9.20) gives

$$a(t) = a_{nt} = a_0(1+\frac{g}{n})^{nt} = a_0\left[(1+\frac{1}{m})^m\right]^{gt}, \qquad \text{where } m \equiv \frac{n}{g}.$$

We keep g and t fixed, but let $n \to \infty$ (and thus $m \to \infty$). Then, in the limit there is continuous compounding and

$$a(t) = a_0 e^{gt}, \tag{9.21}$$

where e is the base of the natural logarithm defined as $e \equiv \lim_{m\to\infty}(1+1/m)^m \simeq 2.7182818285....$

The formula (9.21) is the analogue in continuous time (with continuous compounding) to the discrete time formula (9.19) with annual compounding. Thus, a geometric growth factor is replaced by an exponential growth factor.

We can also view the formulas (9.19) and (9.21) as the solutions to a difference equation and a differential equation, respectively. Thus, (9.19) is the solution to the linear difference equation $a_{i+1} = (1+g)a_i$, given the initial value a_0. And (9.21) is the solution to the linear differential equation $\dot{a}(t) = ga(t)$, given the initial condition $a(0) = a_0$. With a time dependent growth rate, $g(t)$, the corresponding differential equation is $\dot{a}(t) = g(t)a(t)$ with solution

$$a(t) = a(0)e^{\int_0^t g(\tau)d\tau}, \tag{9.22}$$

where the exponent, $\int_0^t g(\tau)d\tau$, is the definite integral of the function $g(\tau)$ from 0 to t. The result (9.22) is called the *basic accumulation formula* in continuous time and the factor $e^{\int_0^t g(\tau)d\tau}$ is called the *growth factor* or the *accumulation factor*.[7]

Compound interest and discounting in continuous time

Let $r(t)$ denote the *short-term real interest rate in continuous time* at time t. To clarify what is meant by this, consider a deposit of $V(t)$ euro on a drawing account in a bank at time t. If the general price level in the economy at time t is $P(t)$ euro, the *real* value of the deposit is $a(t) = V(t)/P(t)$ at time t. By definition the *real rate of return* on the deposit in continuous time (with continuous compounding) at time t is the (proportionate) instantaneous rate at which the real value of the deposit expands per time unit when there is no withdrawal from the account. Thus, if the instantaneous nominal interest rate is $i(t)$, we have $\dot{V}(t)/V(t) = i(t)$ and so, by the fraction rule in continuous time (cf. Appendix A),

$$r(t) = \frac{\dot{a}(t)}{a(t)} = \frac{\dot{V}(t)}{V(t)} - \frac{\dot{P}(t)}{P(t)} = i(t) - \pi(t), \tag{9.23}$$

where $\pi(t) \equiv \dot{P}(t)/P(t)$ is the instantaneous inflation rate. In contrast to the corresponding formula in discrete time, this formula is exact. Sometimes $i(t)$ and $r(t)$ are referred to as the nominal and real *interest intensity*, respectively, or the nominal and real *force of interest.*

[7]Sometimes the accumulation factor with time-dependent growth rate is written in a different way, see Appendix B.

How to treat cases where a jump in an exogenous variable occurs so that a discontinuity in a responding endogenous variable occurs is dealt with in Chapter 11.

Calculating the terminal value of the deposit at time $t_1 > t_0$, given its value at time t_0 and assuming no withdrawal in the time interval $[t_0, t_1]$, the accumulation formula (9.22) immediately yields

$$a(t_1) = a(t_0)e^{\int_{t_0}^{t_1} r(t)dt}.$$

When calculating *present values* in continuous time analysis, we use compound discounting. So we simply reverse the accumulation formula and go from the compounded or terminal value to the present value $a(t_0)$. Similarly, given a consumption plan, $(c(t))_{t=t_0}^{t_1}$, the present value of this plan as seen from time t_0 is

$$PV = \int_{t_0}^{t_1} c(t)\ e^{-rt}dt, \tag{9.24}$$

presupposing a constant interest rate. Instead of the geometric discount factor, $1/(1+r)^t$, from discrete time analysis, we have now an exponential discount factor, $1/(e^{rt}) = e^{-rt}$, and instead of a sum, an integral. When the interest rate varies over time, (9.24) is replaced by

$$PV = \int_{t_0}^{t_1} c(t)\ e^{-\int_{t_0}^{t} r(\tau)d\tau}dt.$$

In (9.24) $c(t)$ is discounted by $e^{-rt} \approx (1+r)^{-t}$. This might not seem analogue to the discrete-time discounting in (9.16) where c_t is discounted by $(1+r)^{-(t+1)}$, assuming a constant interest rate. When taking into account the timing convention that payment for c_t in period analysis occurs at the end of the period (= beginning of period $t+1$), there is no discrepancy, however. The relevant discount factor for c_t as seen from time 0 *is* $(1+r)^{-(t+1)}$ and thereby analogue to the continuous-time form (9.24) where $c(t+1)$ is discounted by $e^{-r(t+1)} \approx (1+r)^{-(t+1)}$.

The allowed range for parameter values

The allowed range for parameters may change when we go from discrete time to continuous time with continuous compounding. For example, the usual equation for aggregate capital accumulation in continuous time is

$$\dot{K}(t) = I(t) - \delta K(t), \qquad K(0) = K_0 \text{ given}, \tag{9.25}$$

where $K(t)$ is the capital stock, $I(t)$ is the gross investment at time t and $\delta \geq 0$ is the (physical) capital depreciation rate. Unlike in discrete time, here $\delta > 1$ is conceptually allowed. Indeed, suppose for simplicity that $I(t) = 0$ for all $t \geq 0$; then (9.25) gives $K(t) = K_0 e^{-\delta t}$ (exponential decay). This formula is meaningful for any $\delta \geq 0$. Usually, the time unit used in continuous time macro models is one year (or, in business cycle theory, perhaps a quarter of a year) and then a realistic value of δ is of course < 1 (say, between 0.05 and 0.10). However, if the time unit applied to the model is large (think of a Diamond-style OLG model), say 30 years, then $\delta > 1$ may fit better, empirically, if the model is converted into continuous time with the same time unit. Suppose, for example, that physical capital has a half-life of 10 years. Then with 30 years as our time unit, inserting into the formula $1/2 = e^{-\delta/3}$ gives $\delta = (\ln 2) \cdot 3 \simeq 2$.

In many simple macromodels, where the level of aggregation is high, the relative price of a unit of physical capital in terms of the consumption good is 1 and thus constant. But if, more generally, we let the relative price of the capital good at time t be $p(t)$ and allow $\dot{p}(t) \neq 0$, then we have to distinguish between the physical depreciation of capital, δ, and the *economic depreciation*, that is, the loss in economic value of a machine per time unit. The economic depreciation will be $d(t) = p(t)\delta - \dot{p}(t)$, namely the economic value of the physical wear and tear minus the capital gain (positive or negative) on the machine.

Other variables and parameters that are constrained by definition in discrete time analysis, but not so in continuous time analysis, include rates of return and discount rates in general.

Stocks and flows

An advantage of continuous time analysis is that it forces the analyst to make a clear distinction between *stocks* (say wealth) and *flows* (say consumption or saving). Recall, a *stock* variable is a variable measured as just a quantity at a given point in time. The variables $a(t)$ and $K(t)$ considered above are stock variables. A *flow* variable is a variable measured as quantity *per time unit* at a given point in time. The variables $s(t)$, $\dot{K}(t)$ and $I(t)$ above are flow variables.

One can not add a stock and a flow, because they have *different denominations.* What exactly is meant by this? The elementary measurement units in economics are *quantity units* (so and so many machines of a certain kind or so and so many liters of oil or so and so many units of payment) and *time units* (months, quarters, years). On the basis of these we can form *composite measurement units.* Thus, the capital stock, K, has the denomination "quantity of machines", whereas investment, I, has the denomination "quantity of machines per time unit" or, shorter, "quantity/time". A growth rate or interest rate has the denomination "(quantity/time)/quantity" = "time^{-1}". If we change our time unit, say from quarters to years, the value of a flow variable

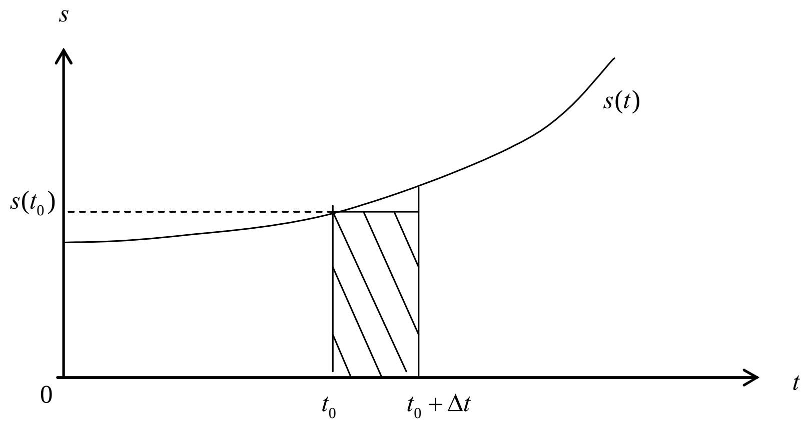


Figure 9.1: With Δt "small" the integral of $s(t)$ from t_0 to $t_0 + \Delta t$ is $\approx$ the hatched area.

as well as a growth rate is changed, in this case quadrupled (presupposing annual compounding).

In continuous time analysis expressions like $K(t)+I(t)$ or $K(t)+\dot{K}(t)$ are thus illegitimate. But one can write $K(t+\Delta t) \approx K(t)+(I(t)-\delta K(t))\Delta t$, or $\dot{K}(t)\Delta t \approx (I(t) - \delta K(t))\Delta t$. In the same way, suppose a bath tub at time t contains 50 liters of water and that the tap pours $\frac{1}{2}$ liter per second into the tub for some time. Then a sum like $50\ \ell + \frac{1}{2}$ (ℓ/sec) does not make sense. But the *amount* of water in the tub after one minute is meaningful. This amount would be $50\ \ell + \frac{1}{2} \cdot 60$ ((ℓ/sec)×sec) $= 90\ \ell$. In analogy, economic flow variables in continuous time should be seen as *intensities* defined for every t in the time interval considered, say the time interval $[0, T)$ or perhaps $[0, \infty)$. For example, when we say that $I(t)$ is "investment" at time t, this is really a short-hand for "investment intensity" at time t. The actual investment in a time interval $[t_0, t_0 + \Delta t)$, i.e., the invested amount *during* this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} I(t)dt \approx I(t_0)\Delta t$. Similarly, $s(t)$, that is, the flow of individual saving, should be interpreted as the saving *intensity* at time t. The actual saving in a time interval $[t_0, t_0 + \Delta t)$, i.e., the saved (or accumulated) amount during this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} s(t)dt$. If Δt is "small", this integral is approximately equal to the product $s(t_0) \cdot \Delta t$, cf. the hatched area in Fig. 9.1.

The notation commonly used in discrete time analysis blurs the distinction between stocks and flows. Expressions like $a_{i+1} = a_i + s_i$, without further comment, are usual. Seemingly, here a stock, wealth, and a flow, saving, are added. In fact, however, it is wealth at the beginning of period i and the saved *amount during* period i that are added: $a_{i+1} = a_i + s_i \cdot \Delta t$. The tacit condition is that the period length, Δt, is the time unit, so that $\Delta t = 1$. But suppose that, for example in a business cycle model, the period length is one quarter, but the time unit is one year. Then saving in quarter i is $s_i = (a_{i+1} - a_i) \cdot 4$ per year.

The choice between discrete and continuous time analysis

In empirical economics, data typically come in discrete time form and data for flow variables typically refer to periods of constant length. One could argue that this discrete form of the data speaks for discrete time rather than continuous time modelling. And the fact that economic actors often think and plan in period terms, may seem a good reason for putting at least microeconomic analysis in period terms. Nonetheless real time is continuous. And it can hardly be said that the *mass* of economic actors think and plan with one and the same period. In macroeconomics we consider the *sum* of the actions. In this perspective the continuous time approach has the advantage of allowing variation *within* the usually artificial periods in which the data are chopped up. And for example centralized asset markets equilibrate almost instantaneously and respond immediately to new information. For such markets a formulation in continuous time seems preferable.

There is also a risk that a discrete time model may generate *artificial* oscillations over time. Suppose the "true" model of some mechanism is given by the differential equation

$$\dot{x} = \alpha x, \qquad \alpha < -1. \tag{9.26}$$

However, the analyst takes a discrete time approach and sets up the seemingly "corresponding" discrete time model

$$x_{t+1} - x_t = \alpha x_t.$$

This yields the difference equation $x_{t+1} = (1 + \alpha)x_t$, where $1 + \alpha < 0$. Thus oscillations arise in spite of the "true" model generating monotonous convergence towards the steady state $x^* = 0$.

It should be added, however, that this potential problem *can* always be avoided within discrete time models by choosing a sufficiently *short* period length. Indeed, the solution to a differential equation can always be obtained as the limit of the solution to a corresponding difference equation for the period length approaching zero. In the case of (9.26) the approximating difference equation is $x_{i+1} = (1 + \alpha\Delta t)x_i$, where Δt is the period length, $i = t/\Delta t$, and $x_i = x(i\Delta t)$. By choosing Δt small enough, the solution comes arbitrarily close to the solution of (9.26). It is generally more difficult to go in the opposite direction and find a differential equation that approximates a given difference equation. But the problem is solved as soon as a differential equation has been found that has the initial difference equation as an approximating difference equation.

In his discussion of these modelling issues Allen (1967) concluded that from the point of view of the economic contents, the choice between discrete time or continuous time analysis may be a matter of taste. From the point of view of mathematical convenience, however, the continuous time formulation, which has worked so well in the natural sciences, seems preferable. At least this is so in the absence of uncertainty. For problems where uncertainty is important, discrete time formulations are easier unless one is familiar with stochastic calculus.

9.4 Maximizing discounted utility in continuous time

9.4.1 The saving problem in continuous time

The analogue in continuous time to the intertemporal budget constraint, (IBC), in Section 9.2 is

$$\int_0^T c(t)e^{-\int_0^t r(\tau)d\tau}dt \leq a_0 + h_0. \tag{9.27}$$

As above, a_0 is the historically given initial financial wealth (the value of the stock of short-term bonds held at time 0), while h_0 is the given human wealth,

$$h_0 = \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt. \tag{9.28}$$

The analogue in continuous time to the intertemporal utility function, (9.3), is

$$U_0 = \int_0^T u(c(t))e^{-\rho t}dt. \tag{9.29}$$

In continuous time it is common to name the utility flow, $u(\cdot)$, the *instantaneous utility function*.

The household's problem is then to choose a consumption plan $(c(t))_{t=0}^T$ so as to maximize discounted utility, U_0, subject to the budget constraint (9.27).

Infinite time horizon In for example the Ramsey model of the next chapter the idea is used that households may have an *infinite* time horizon. One interpretation of this is that parents care about their children's future welfare and leave bequests accordingly. This gives rise to a series of intergenerational links. Indeed, the household may be seen as a family dynasty with a time horizon far beyond the life time of the current members of the family. Barro's bequest model in Chapter 7 is a discrete time application of this idea. As a mathematical approximation one can then use an infinite planning horizon. Introducing a positive constant utility discount rate, less weight is attached to circumstances further away in the future and it may be ensured that the integral of achievable discounted utility is bounded. Yet, infinity is a long time. The sun will eventually (in some billion years) burn out and life on earth become extinct.

Nonetheless an infinite time horizon may provide a useful mathematical approximation. This is because the solution to an optimization problem for T "large" will in many cases most of the time be close to the solution for $T \rightarrow \infty$.[8] And an infinite time horizon may make aggregation easier. An infinite time horizon can also be a convenient notion when in any given period there is a positive probability that there will also be a next period to be concerned about. The lower this probability is, the higher will the utility discount rate be. This idea is applied in chapters 12 and 13.

We perform the transition to infinite horizon by letting $T \rightarrow \infty$ in *(9.29)*, (9.27), and (9.28). In the limit the household's (or dynasty's) problem becomes one of choosing a plan $(c(t))_{t=0}^{\infty}$, which maximizes

$$U_0 = \int_0^\infty u(c(t))e^{-\rho t}dt \qquad \text{s.t.} \tag{9.30}$$

$$\int_0^\infty c(t)e^{-\int_0^t r(\tau)d\tau}dt \leq a_0 + h_0, \tag{IBC}$$

where h_0 emerges by letting T in (9.28) approach ∞. Working with infinite horizons, there may exist technically feasible paths along which the integrals in (9.28), (9.30), and (IBC) go to ∞ for $T \rightarrow \infty$. In that case maximization is not well-defined. However, the assumptions that we are going to make when working with the Ramsey model will guarantee that the integrals converge as $T \rightarrow \infty$ (or at least that *some* feasible paths have $-\infty < U_0 < \infty$, while the remainder have $U_0 = -\infty$ and are thus clearly inferior). The essence of the matter is that the rate of time preference, ρ, must be assumed sufficiently high relative to the potential growth in instantaneous utility so as to ensure

[8] The turnpike proposition in Chapter 8 exemplified this.

that the long-run interest rate becomes higher than the long-run growth rate of income.

Generally, we define a person *solvent* if she is able to meet her financial obligations as they fall due. Each person is considered "small" relative to the economy as a whole. As long as all agents in the economy remain "small", they will in general equilibrium remain solvent if and only if their *gross* debt does not exceed their gross assets. Here "assets" should be understood in the broadest possible sense, that is, including the present value of the expected future labor income. Considering *net* debt, d_0, the solvency requirement becomes

$$d_0 \leq \int_0^\infty (w(t) - c(t))e^{-\int_0^t r(\tau)d\tau}dt,$$

where the right-hand side of the inequality is the present value of the expected future primary saving.[9] By use of the definition in (9.28), it can be seen that this requirement is identical to the stock budget constraint (IBC) which consequently expresses solvency.

The budget constraint in flow terms

The mathematical method which is particularly apt for solving intertemporal decision problems in continuous time is *optimal control theory.* To apply the method, we have to convert the household's budget constraint from the present value formulation considered above into flow terms.

By mere accounting, in every short time interval $(t, t+\Delta t)$ the household's consumption plus saving equals the household's total income, that is,

$$(c(t) + \dot{a}(t))\Delta t = (r(t)a(t) + w(t))\Delta t.$$

Here, $\dot{a}(t) \equiv da(t)/dt$ is saving and thus the same as the increase per time unit in financial wealth. If we divide through by Δt and isolate saving on the left-hand side of the equation, we get for all $t \geq 0$

$$\dot{a}(t) = r(t)a(t) + w(t) - c(t), \qquad a(0) = a_0 \text{ given.} \tag{9.31}$$

This equation in itself is just a dynamic budget identity. It tells us by how much and in which direction the financial wealth is changing due to the difference between current income and current consumption. The equation *per se* does not impose any restriction on consumption over time. If this equation were the only "restriction", one could increase consumption indefinitely by incurring an increasing debt without limits. It is not until the

[9] By *primary* saving is meant the difference between current *earned* income and current consumption, where earned income means income before interest transfers.

requirement of solvency is added to (9.31) that we get a *constraint.* When $T < \infty$, the relevant solvency requirement is $a(T) \geq 0$ (that is, no debt left over at the terminal date). This is equivalent to satisfying the intertemporal budget constraint (9.27). When $T = \infty$, the relevant solvency requirement is a No-Ponzi-Game condition:

$$\lim_{t\to\infty} a(t)e^{-\int_0^t r(\tau)d\tau} \geq 0, \tag{NPG}$$

i.e., the present value of *debts,* measured as $-a(t)$, infinitely far out in the future, is not permitted to be positive. Indeed, we have the following equivalency:

PROPOSITION 1 *(equivalence of NPG condition and intertemporal budget constraint)* Let the time horizon be infinite and assume that the integral (9.28) remains finite for $T \to \infty$. Then, given the accounting relation (9.31), we have:

(i) the requirement (NPG) is satisfied if and only if the intertemporal budget constraint, (IBC), is satisfied; and
(ii) there is strict equality in (NPG) if and only if there is strict equality in (IBC).

Proof. See Appendix C.

The condition (NPG) does not preclude that the household (or rather the family dynasty) can remain in debt. This would also be an unnatural requirement as the dynasty is infinitely-lived. The condition does imply, however, that there is an upper bound for the speed whereby debts can increase in the long term. In the long term, debts cannot grow at a rate greater than (or just equal to) the interest rate.

To understand the implication, let us look at the case where the interest rate is a constant, $r > 0$. Assume that the household at time t has net debt $d(t) > 0$, i.e., $a(t) \equiv -d(t) < 0$. If $d(t)$ were persistently growing at a rate equal to or greater than the interest rate, (NPG) would be violated.[10] Equivalently, one can interpret (NPG) as an assertion that lenders will only issue loans if the borrowers in the long run are able to cover at least part of their interest payments by other means than by taking up new loans. In this way, it is avoided that $\dot{d}(t) \geq rd(t)$ in the long run, that is, the debt does not explode.

As mentioned in Chapter 6 the name "No-Ponzi-Game condition" refers to a guy, Charles Ponzi, who in Boston in the 1920s temporarily became

[10]Starting from a given initial positive debt, d_0, when $\dot{d}(t)/d(t) \geq r > 0$, we have $d(t) \geq d_0 e^{rt}$ so that $d(t)e^{-rt} \geq d_0 > 0$ for all $t \geq 0$. Consequently, $a(t)e^{-rt} = -d(t)e^{-rt} \leq -d_0 < 0$ for all $t \geq 0$, that is, $\lim_{t\to\infty} a(t)e^{-rt} < 0$, which violates (NPG).

very rich by a loan arrangement based on the chain letter principle. The fact that debts grow without bounds is irrelevant for the lender *if* new lenders can always be found and their outlay be used to pay off old lenders. In the real world, endeavours to establish this sort of financial eternity machine tend sooner or later to break down because the flow of new lenders dries up. It is exactly such a financial eternity machine the constraint (NPG) precludes. The so-called pyramid companies are nowadays-examples of attempts to evade the NPG condition are.[11]

9.4.2 Solving the saving problem

The household's consumption/saving problem is one of choosing a path for the *control variable* $c(t)$ so as to maximize a *criterion function,* in the form an integral, subject to constraints that include a first-order differential equation. This equation determines the evolution of a *state variable*, $a(t)$. Optimal control theory, which in Chapter 8 was applied to a related discrete time problem, is a well-suited apparatus for solving this kind of optimization problems. We will make use of a special case of the continuous time *Maximum Principle* (the basic tool of optimal control theory). We shall consider the case with a finite time horizon as well as the case with an infinite time horizon.

For $T < \infty$ the problem is: choose a plan $(c(t))_{t=0}^{T}$ so as to maximize

$$
\begin{aligned}
U_0 &= \int_0^T u(c(t))e^{-\rho t}dt \qquad \text{s.t.} && (9.32)\\
c(t) &\geq 0, \qquad \text{(control region)} && (9.33)\\
\dot{a}(t) &= r(t)a(t) + w(t) - c(t), \quad a(0) = a_0 \text{ given,} && (9.34)\\
a(T) &\geq 0. && (9.35)
\end{aligned}
$$

With infinite time horizon, T in (9.32) is replaced by ∞ and the solvency condition (9.35) is replaced by

$$\lim_{t\to\infty} a(t)e^{-\int_0^t r(\tau)d\tau} \geq 0. \qquad \text{(NPG)}$$

Let I denote the time interval $[0, T]$ if $T < \infty$ and the time interval $[0, \infty)$ if $T = \infty$. If $c(t)$ and the corresponding evolution of $a(t)$ fulfil (9.33) and (9.34) for all $t \in I$ as well as the relevant solvency condition, we call $(a(t), c(t))_{t=0}^{T}$ an *admissible path.* If a given admissible path $(a(t), c(t))_{t=0}^{T}$

[11] A Danish example, though on a smaller scale, could be read in the Danish newpaper *Politiken* on the 21st of August 1992. "A twenty-year-old female student from Tylstrup in Northern Jutland is charged with fraud. In an ad, she offered 200 DKK to tell you how to make easy money. Some hundred people responded and received the reply: Do like me."

solves the problem, it is referred to as an *optimal path*.[12] We assume that $w(t)$ and $r(t)$ are piecewise continuous functions of t and that $w(t)$ is positive for all t. No condition on the impatience parameter, ρ, is imposed (in this chapter).

First-order conditions

The solution procedure is as follows:[13]

1. Set up the so-called *current-value Hamiltonian* function:

$$H(a, c, \lambda, t) \equiv u(c) + \lambda(ra + w - c),$$

   where λ is an *adjoint variable* (also called a *co-state variable*) associated with the dynamic constraint (9.34), that is, λ is an auxiliary variable which is a function of t (it is analogous to the Lagrange multiplier in static optimization).

2. At every point in time, maximize the Hamiltonian w.r.t. the *control variable*, in the present case c. Focusing on an *interior* optimal path,[14] put the partial derivative of H with respect to c equal to zero:

$$\frac{\partial H}{\partial c} = u'(c) - \lambda = 0,$$

   that is, at every $t \in I$,

$$u'(c) = \lambda. \tag{9.36}$$

3. Put the partial derivative of H with respect to the *state variable*, in the present case a, equal to the difference between the discount rate (as it appears in the integrand of the criterion function) multiplied by λ and the time derivative of the adjoint variable, λ :

$$\frac{\partial H}{\partial a} = \lambda r = \rho\lambda - \dot{\lambda}.$$

   That is, at any point in time, the adjoint variable λ should fulfil the differential equation

$$\dot{\lambda} = (\rho - r)\lambda. \tag{9.37}$$

[12] The term "path", sometimes "trajectory", is common in the natural sciences for a solution to a differential equation because one may think of this solution as the path of a particle moving in two- or three-dimensional space.

[13] The explicit dating of the time-dependent variables a, c, and λ is omitted where not needed for clarity.

[14] A path, $(a_t, c_t)_{t=0}^{T}$, is an *interior* path if $c_t > 0$ for all $t \geq 0$.

4. Apply the *Maximum Principle* which (in this case) says: an interior optimal path $(a(t), c(t))_{t=0}^{T}$ will satisfy that there exits a continuous function $\lambda = \lambda(t)$ such that for all $t \in I$, (9.36) and (9.37) hold along the path, and the *transversality condition,*

$$\begin{aligned} e^{-\rho T}\lambda(T)a_T &= 0, \text{ if } T < \infty, \text{ or} \\ \lim_{t\to\infty} e^{-\rho t}\lambda(t)a(t) &= 0, \text{ if } T = \infty, \end{aligned} \tag{TVC}$$

is satisfied.

An optimal path is thus characterized as a path that for every t maximizes the Hamiltonian associated with the problem. The intuition is that the Hamiltonian weighs the direct contribution of the marginal unit of the control variable to the criterion function in the "right" way relative to the indirect contribution, which comes from the generated change in the state variable (here financial wealth); "right" means in accordance with the opportunities offered by the rate of return vis-a-vis the time preference rate, ρ. The optimality condition (9.36) can be seen as a $MC = MB$ condition: on the margin one unit of account (here the consumption good) must be equally valuable in its two uses: consumption and wealth accumulation. Together with the optimality condition (9.37) this signifies that the adjoint variable λ can be interpreted as the *shadow price* (measured in units of current utility) of financial wealth along the optimal path.[15]

Remark. The current-value Hamiltonian function is often just called the *current-value Hamiltonian.* More importantly the prefix "current-value" is used to distinguish it from the so-called *present-value Hamiltonian.* The latter can be defined as $\tilde{H} \equiv He^{-\rho t}$ with $\lambda e^{-\rho t}$ substituted by μ, which is the associated (discounted) adjoint variable. The solution procedure is similar except that step 3 is replaced by $\partial\tilde{H}/\partial a = -\dot{\mu}$. The two methods are formally equivalent, but for continuous time problems we generally prefer the *current-value* Hamiltonian. The reason is that it makes both the calculations and the interpretation slightly simpler. The adjoint variable, $\lambda(t)$, which as mentioned acts as a shadow price of the state variable, becomes a *current* price along with the other prices in the problem, $w(t)$ and $r(t)$, instead of a *discounted* price. □

Reordering (9.37) gives

$$\frac{r\lambda + \dot{\lambda}}{\lambda} = \rho. \tag{9.38}$$

[15] Recall, a *shadow price* (measured in some unit of account) of a good is the number of units of account that the optimizing agent is just willing to offer for one extra unit of the good.

This can be interpreted as a no-arbitrage condition. The left-hand side gives the *actual* rate of return, measured in utility units, on the marginal unit of saving. Indeed, $r\lambda$ can be seen as a dividend and $\dot{\lambda}$ as a capital gain. The-right hand side is the *required* rate of return measured in utility terms, ρ. Along an optimal path the two must coincide. The household is willing to save the marginal unit only if the actual rate of return on saving equals the required rate.

We may also write the optimality condition (9.37) as

$$r = \rho - \frac{\dot{\lambda}}{\lambda}. \tag{9.39}$$

Now we have on the left-hand-side the *actual real* rate of return on saving. The right-hand-side can be interpreted as the *required real* rate of return. To see this, suppose Ms. Jones makes a deposit of $v(t)$ utility units in a "bank" that offers a utility interest rate, that is, a proportionate rate of expansion per time unit of the utility value of the deposit equal to $\hat{\rho}$ (assuming no withdrawal occurs):

$$\frac{\dot{v}}{v} = \hat{\rho}.$$

To calculate the *real* value of the deposit at time t, we divide the number of utility units, v, by λ, since λ is the (shadow) price, measured in utility units, of one unit of real financial wealth (we may interpret the financial wealth as an accumulated amount of consumption goods). Thus, a financial wealth consisting of $v/\lambda \equiv m$ real goods is worth v utility units. Hence, the real rate of return on the deposit is

$$r = \frac{\dot{m}}{m} = \frac{\dot{v}}{v} - \frac{\dot{\lambda}}{\lambda} = \hat{\rho} - \frac{\dot{\lambda}}{\lambda}.$$

Ms. Jones is then just willing to save the marginal unit of income if this real rate of return on saving equals the required rate, that is, the right-hand side of (9.39) (in turn this requires $\hat{\rho} = \rho$ of course).

Substituting (9.36) into the transversality condition for the case $T < \infty$, gives

$$a(T)e^{-\rho T}u'(c(T)) = 0. \tag{9.40}$$

This can be read as a standard complementary slackness condition, because we can replace the factor $a(T)$ by $(a(T)-0)$. Our solvency condition, $a(T) \geq 0$, can be seen as an example of a general inequality constraint, $a(T) \geq a_T$, where here a_T happens to equal 0. Since $u'(c(T))$ is always positive, an optimal plan must satisfy $a(T) = 0$. The alternative, $a(T) > 0$, would imply that consumption, and thereby U_0, could be increased by a decrease in $a(T)$ without violating the solvency requirement.

Now let $T \to \infty$. Then in the limit the solvency requirement is (NPG), and (9.40) is replaced by

$$\lim_{T\to\infty} a(T)e^{-\rho T}u'(c(T)) = 0. \tag{9.41}$$

This is the same as (TVC) (replace T by t). Intuitively, a plan that violates this condition by having ">" instead "=" indicates scope for improvement and thus cannot be optimal. Generally, however, care must be taken when extending a necessary transversality condition from a finite to an infinite horizon. But for the present problem, the extension *is* valid. Indeed, (TVC) is just a requirement that the NPG condition is not "over-satisfied":

PROPOSITION 2 *(the transversality condition with infinite time horizon)* Let $T = \infty$ and assume the integral (9.28) remains finite for $T \to \infty$. Provided the adjoint variable, $\lambda(t)$, satisfies the optimality conditions (9.36) and (9.37), (TVC) holds if and only if (NPG) holds with strict equality.

Proof. See Appendix D.

In view of this proposition, we can write the transversality condition for $T = \infty$ as the NPG condition with strict equality:

$$\lim_{t\to\infty} a(t)e^{-\int_0^t r(\tau)d\tau} = 0. \tag{TVC'}$$

9.4.3 The Keynes-Ramsey rule

The first-order conditions have interesting implications. Differentiate both sides of (9.36) w.r.t. t to get $u''(c)\dot{c} = \dot{\lambda}$ which can be written as $u''(c)\dot{c}/u'(c) = \dot{\lambda}/\lambda$ by drawing on (9.36) again. Applying (9.37) now gives

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta(c(t))}(r(t) - \rho), \tag{9.42}$$

where $\theta(c)$ is the (absolute) *elasticity of marginal utility* w.r.t. consumption,

$$\theta(c) \equiv -\frac{c}{u'(c)}u''(c) > 0. \tag{9.43}$$

As in discrete time, $\theta(c)$ indicates the strength of the consumer's desire to smooth consumption. The inverse of $\theta(c)$ measures the *instantaneous intertemporal elasticity of substitution* in consumption, which in turn indicates the willingness to accept variation in consumption over time when the interest rate changes, see Appendix F.

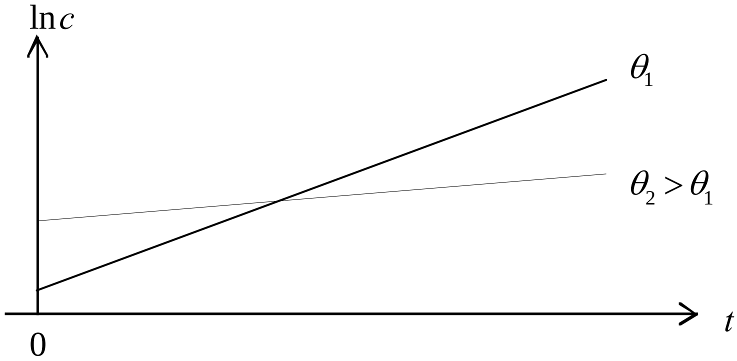


Figure 9.2: Optimal consumption paths for a low and a high constant θ, given a constant $r > \rho$.

The result (9.42) says that an optimal consumption plan is characterized in the following way. The household will completely smooth consumption over time if the rate of time preference equals the real interest rate. The household will choose an upward-sloping time path for consumption if and only if the rate of time preference is less than the real interest rate. Indeed, in this case the household would accept a relatively low level of current consumption with the purpose of enjoying more consumption in the future. The lower the rate of time preference relative to the real interest rate, the more favorable it becomes to defer consumption. Moreover, by (9.42) we see that the greater the elasticity of marginal utility (that is, the greater the curvature of the utility function), the greater the incentive to smooth consumption for a given value of $r(t) - \rho$. The reason for this is that a large curvature means that the marginal utility will drop sharply if consumption increases, and will rise sharply if consumption falls. Fig. 9.2 illustrates this in the CRRA case where $\theta(c) = \theta$, a positive constant. For a given constant $r > \rho$, the consumption path chosen when θ is high has lower slope, but starts from a higher level, than when θ is low.

The condition (9.42), which holds for a finite as well as an infinite time horizon, is referred to as the *Keynes-Ramsey rule*. The name springs from the English mathematician Frank Ramsey who derived the rule in as early as 1928, while John Maynard Keynes suggested a simple and intuitive way of presenting it.

The rule reflects the general microeconomic principle that the consumer equates the marginal rate of substitution between any two goods with the corresponding price ratio. In the present context the principle is applied to a situation where the "two goods" refer to the same consumption good delivered at two different dates. In Section 9.2 we used the principle to interpret the optimal saving behavior in discrete time. How can the principle be translated into a continuous time setting?

Local optimality in continuous time Let $(t, t+\Delta t)$ and $(t+\Delta t, t+2\Delta t)$ be two short successive time intervals. The marginal rate of substitution, $MRS_{t+\Delta t,t}$, of consumption in the second time interval for consumption in the first is[16]

$$MRS_{t+\Delta t,t} \equiv -\frac{dc(t+\Delta t)}{dc(t)}|_{U=\bar{U}} = \frac{u'(c(t))}{e^{-\rho\Delta t}u'(c(t+\Delta t))}, \tag{9.44}$$

approximately. On the other hand, by saving $-\Delta c(t)$ more per time unit (where $\Delta c(t) < 0$) in the short time interval $(t,\ t+\Delta t)$, one can via the market transform $-\Delta c(t) \cdot \Delta t$ units of consumption in this time interval into

$$\Delta c(t+\Delta t) \cdot \Delta t \approx -\Delta c(t)\Delta t \ e^{\int_t^{t+\Delta t} r(\tau)d\tau} \tag{9.45}$$

units of consumption in the time interval $(t+\Delta t,\ t+2\Delta t)$. The marginal rate of transformation is therefore

$$\begin{aligned} MRT_{t+\Delta t,t} &\equiv -\frac{dc(t+\Delta t)}{dc(t)}|_{U=\bar{U}} \approx \\ &= e^{\int_t^{t+\Delta t} r(\tau)d\tau}. \end{aligned}$$

In the optimal plan we must have $MRS_{t+\Delta t,t} = MRT_{t+\Delta t,t}$ which gives

$$\frac{u'(c(t))}{e^{-\rho\Delta t}u'(c(t+\Delta t))} = e^{\int_t^{t+\Delta t} r(\tau)d\tau}, \tag{9.46}$$

approximately. When $\Delta t = 1$ and ρ and $r(t)$ are small, this relation can be approximated by (9.9) from discrete time (generally, by a first-order Taylor approximation, $e^x \approx 1 + x$, when x is close to 0).

Taking logs on both sides of (9.46), dividing through by Δt, inserting (9.45), and letting $\Delta t \to 0$, we get (see Appendix E)

$$\rho - \frac{u''(c(t))}{u'(c(t))}\dot{c}(t) = r(t). \tag{9.47}$$

With the definition of $\theta(c)$ in (9.43), this is exactly the same as the Keynes-Ramsey rule (9.42) which, therefore, is merely an expression of the general

[16] The underlying analytical steps can be found in Appendix E.

optimality condition $MRS = MRT$. The household is willing to sacrifice some consumption today for more consumption tomorrow, that is, to go for $\dot{c} > 0$, only if it is compensated by an interest rate sufficiently above ρ. Naturally, the required compensation is higher, the faster marginal utility declines ($-u''/u'$ large) when consumption increases. Indeed, a higher c_t in the future than today implies a lower marginal utility of consumption in the future than of consumption today. So saving of the marginal unit of income today is only warranted if the rate of return is sufficiently above ρ, and this is what (9.47) indicates.

9.4.4 The consumption function

We have not yet fully solved the saving problem. The Keynes-Ramsey rule gives only the optimal rate of *change* of consumption over time. It says nothing about the *level* of consumption. In order to determine the level, that is, $c(0)$, we implicate the solvency condition which limits the amount the household can borrow in the long term. Among the infinitely many consumption paths satisfying the Keynes-Ramsey rule, the household will choose the "highest" one that fulfils the solvency requirement (NPG). Thus, the household acts so that strict equality in (NPG) obtains. As we saw, this is equivalent to satisfying the transversality condition.

Before considering examples, a remark on *sufficient* conditions for optimality is appropriate. The first-order and transversality conditions, (9.36), (9.37), and (TVC), are only *necessary* conditions for an interior path to be optimal. Hence, up to this point, we have only claimed that if the consumption-saving problem has an interior solution, then it satisfies the Keynes-Ramsey rule and the transversality condition (TVC'). Are these conditions also *sufficient*? The answer is yes in the present case. This follows from Mangasarian's sufficiency theorem which applied to the present problem tells us that if the Hamiltonian is *concave* in (a, c) for every t, then the listed necessary conditions, including the transversality condition, are also sufficient. Because the instantaneous utility function, the first term in the Hamiltonian, is strictly concave and the second term is linear in (a, c), the Hamiltonian *is* concave in (a, c). Thus if we have found a path satisfying the Keynes-Ramsey rule and the (TVC'), we have a *candidate solution*. And by the Mangasarian theorem this candidate *is* an optimal solution. In fact the strict concavity of the Hamiltonian with respect to the control variable ensures that the optimal solution is unique (Exercise 9.?).

To avoid any misunderstanding, the examples below should not be interpreted such that for *any* evolution of wages and interest rates there exists a solution to the household's maximization problem with infinite horizon. There is generally no guarantee that integrals have an upper bound for $T \to \infty$. The evolution of wages and interest rates which prevails in *general equilibrium* is not arbitrary, however. It is determined by the requirement of equilibrium. In turn, of course *existence* of an equilibrium imposes restrictions on the utility discount rate relative to the potential growth in instantaneous utility. We shall return to these issues in the next chapter.

EXAMPLE 2 *(constant elasticity of marginal utility; infinite time horizon).* We impose that the elasticity of marginal utility $\theta(c)$, as defined in (9.43), is a constant $\theta > 0$. From Appendix A of Chapter 3 we know that this requirement implies that up to a positive linear transformation the utility function must be of the form:

$$u(c) = \begin{cases} \frac{c^{1-\theta}}{1-\theta}, & \text{when } \theta > 0, \theta \neq 1, \\ \ln c, & \text{when } \theta = 1. \end{cases} \tag{9.48}$$

The Keynes-Ramsey rule now implies $\dot{c}(t) = \theta^{-1}(r(t) - \rho)c(t)$. Solving this linear differential equation yields

$$c(t) = c(0)e^{\frac{1}{\theta}\int_0^t (r(\tau)-\rho)d\tau}, \tag{9.49}$$

cf. the general accumulation formula (9.22).

We know from Proposition 2 that the transversality condition is equivalent to the intertemporal budget constraint being satisfied with strict equality, i.e.,

$$\int_0^\infty c(t)e^{-\int_0^t r(\tau)d\tau} dt = a_0 + h_0. \tag{IBC'}$$

This result can be used to determine $c(0)$. Multiply both sides of (9.49) by the discount factor, $e^{-\int_0^t r(\tau)d\tau}$, integrate both sides, and finally substitute into (IBC').[17] We get

$$c(0)\int_0^\infty e^{\frac{1}{\theta}\int_0^t [(1-\theta)r(\tau)-\rho]d\tau} dt = a_0 + h_0,$$

where

$$h_0 = \int_0^\infty w(t)e^{-\int_0^t r(\tau)d\tau} dt. \tag{9.50}$$

The consumption function is thus

$$\begin{aligned} c(0) &= \beta_0(a_0 + h_0), \qquad \text{where} \\ \beta_0 &\equiv \frac{1}{\int_0^\infty e^{\int_0^t [\frac{1}{\theta}(r(\tau)-\rho)-r(\tau)]d\tau} dt} = \frac{1}{\int_0^\infty e^{\frac{1}{\theta}\int_0^t [(1-\theta)r(\tau)-\rho]d\tau} dt} \end{aligned} \tag{9.51}$$

[17] This method also applies if instead of $T = \infty$, we have $T < \infty$.

is the marginal propensity to consume out of wealth. We have here assumed that these improper integrals over an infinite horizon are upward bounded for all admissible paths.

Generally, an increase in the interest rate level, for given total wealth, $a_0 + h_0$, can effect $c(0)$ both positively and negatively.[18] On the one hand, such an increase makes future consumption cheaper in present value terms. This change in the trade-off between current and future consumption entails a negative *substitution effect* on $c(0)$. On the other hand, the increase in the interest rates decreases the present value of a given consumption plan, allowing for higher consumption both today and in the future, for given total wealth, cf. (IBC'). This entails a positive *pure income effect* on consumption today as consumption is a normal good. If $\theta < 1$ (small curvature of the utility function), the substitution effect will dominate the pure income effect, and if $\theta > 1$ (large curvature), the reverse will hold. This is because the larger is θ, the stronger is the propensity to smooth consumption over time.

In the intermediate case $\theta = 1$ (the logarithmic case) we get from (9.51) that $\beta_0 = \rho$, hence

$$c(0) = \rho(a_0 + h_0). \tag{9.52}$$

In this special case the marginal propensity to consume is time independent and equal to the rate of time preference. For a given *total* wealth, $a_0 + h_0$, current consumption is thus independent of the expected path of the interest rate. That is, in the logarithmic case the *substitution* and *pure income effects* on current consumption exactly offset each other. Yet, on top of this comes the negative *wealth effect* on current consumption of an increase in the interest rate level. The present value of future wage incomes becomes lower (similarly with expected future dividends on shares and future rents in the housing market in a more general setup). Because of this, h_0 (and so $a_0 + h_0$) becomes lower, which adds to the negative substitution effect.[19] Thus, even in the logarithmic case, and *a fortiori* when $\theta < 1$, the *total effect* of an increase in the interest rate level is unambiguously negative on $c(0)$.

If, for example, $r(t) = r$ and $w(t) = w$ (positive constants), we get $\beta_0 = [(\theta - 1)r + \rho]/\theta$ and $a_0 + h_0 = a_0 + w/r$. When $\theta = 1$, the negative effect of a higher r on h_0 is decisive. When $\theta < 1$, a higher r reduces both β_0 and

[18] By an increase in the interest rate *level* we mean an upward shift in the time-profile of the interest rate. That is, there is at least one time interval within $[0, \infty)$ where the interest rate is higher than in the original situation and no time interval within $[0, \infty)$ where the interest rate is lower.

[19] If $a_0 < 0$ and this debt is not a variable-rate loan (as hitherto assumed), but like a fixed-rate mortgage loan, then a rise in the interest rate lowers the present value of the debt and thereby raises wealth (ceteris paribus) and *counteracts* the negative substitution effect on current consumption.

h_0, hence the total effect on $c(0)$ is even "more negative". When $\theta > 1$, a higher r gives a higher β_0 and this more or less offsets the lower h_0, so that the total effect on $c(0)$ becomes ambiguous. As referred to in Chapter 3, available empirical studies generally suggest a value of θ somewhat above 1. □

EXAMPLE 3 (*constant absolute semi-elasticity of marginal utility; infinite time horizon*). The requirement is that the sensitivity of marginal utility, $-u''(c)/u'(c) \approx -(\Delta u'/u')/\Delta c$, is a positive constant, α.[20] The utility function must then, up to a positive linear transformation, be of the form,

$$u(c) = -\alpha^{-1}e^{-\alpha c}, \alpha > 0. \tag{9.53}$$

The Keynes-Ramsey rule becomes $\dot{c}(t) = \alpha^{-1}(r(t) - \rho)$. When the interest rate is a constant $r > 0$, we find, through (IBC') and partial integration, $c(0) = r(a_0 + h_0) - (r - \rho)/(\alpha r)$, presupposing $r \geq \rho$ and $a_0 + h_0 > (r - \rho)/(ar^2)$.

This hypothesis of a "constant absolute variability aversion" implies that the degree of *relative* variability aversion is $\theta(c) = \alpha c$ and thus greater, the larger is c. In the theory of behavior under uncertainty, (9.53) is referred to as the CARA function ("Constant Absolute Risk Aversion"). One of the theorems of expected utility theory is that the degree of absolute risk aversion, $-u''(c)/u'(c)$, is proportional to the risk premium which the economic agent will require to be willing to exchange a specified amount of consumption received with certainty for an uncertain amount having the same mean value. Empirically this risk premium seems to be a decreasing function of the level of consumption. Therefore the CARA function is generally considered less realistic than the CRRA function of the previous example. □

EXAMPLE 4 *(logarithmic utility; finite time horizon; retirement).* We consider a life-cycle saving problem. A worker enters the labor market at time 0 with a financial wealth of 0, has finite lifetime T (assumed known) and does not wish to pass on bequests. For simplicity, we assume that $r_t = r > 0$ for all $t \in [0, T]$ and $w(t) = w > 0$ for $t \leq t_1 \leq T$, while $w(t) = 0$ for $t > t_1$ (no wage income after retirement, which takes place at time t_1). The decision problem is

$$\begin{aligned}
\max_{(c(t))_{t=0}^T} U_0 &= \int_0^T (\ln c(t))e^{-\rho t}dt \quad \text{s.t.} \\
c(t) &> 0, \\
\dot{a}(t) &= ra(t) + w(t) - c(t), \qquad\qquad a(0) = 0, \\
a(T) &\geq 0.
\end{aligned}$$

[20] By the *semi-elasticity* of a differentiable function $f(x)$ we mean the absolute value of $f'(x)/f(x)$.

The Keynes-Ramsey rule becomes $\dot{c}_t/c_t = r-\rho$. A solution to the problem will thus fulfil

$$c(t) = c(0)e^{(r-\rho)t}. \tag{9.54}$$

Inserting this into the differential equation for a, we get a first-order linear differential equation the solution of which (for $a(0) = 0$) can be reduced to

$$a(t) = e^{rt}\left[\frac{w}{r}(1-e^{-rz}) - \frac{c_0}{\rho}(1-e^{-\rho t})\right], \tag{9.55}$$

where $z = t$ if $t \leq t_1$, and $z = t_1$ if $t > t_1$. We need to determine $c(0)$. The transversality condition implies $a(T) = 0$. Having $t = T$, $z = t_1$ and $a_T = 0$ in (9.55), we get

$$c(0) = (\rho w/r)(1-e^{-rt_1})/(1-e^{-\rho T}). \tag{9.56}$$

Substituting this into (9.54) gives the optimal consumption plan.[21]

If $r = \rho$, consumption is constant over time at the level given by (9.56). If, in addition, $t_1 < T$, this consumption level is less than the wage income per year up to t_1 (in order to save for retirement); in the last years the level of consumption is maintained although there is no wage income; the retired person uses up both the return on financial wealth and this wealth itself (dissaving). □

The examples illustrate the importance of forward-looking *expectations*, here the expected evolution of interest rates and wages. The expectations affect $c(0)$ both through their impact on the marginal propensity to consume (cf. β_0 in Example 2) and through their impact on the present value, h_0, of expected future labor income (or of expected future dividends on shares in a more general setup). Yet the examples – and the consumption theory in this chapter in general – should only be seen as a first, crude approximation to consumption/saving behavior. Real world factors such as uncertainty and credit constraints (absence of perfect loan markets) also affect the behavior. Including these factors in the analysis tend to make current income an additional and independent determinant of the consumption by a large fraction of the population, as is recognized in many short- and medium-run macro models.

9.5 Literature notes

The (strong) assumptions regarding the underlying intertemporal preferences which allow them to be represented by the present value of period utilities

[21] For $t_1 = T$ and $T \to \infty$ we get in the limit $c(0) = \rho w/r \equiv \rho h_0$, which is also what (9.51) gives when $a(0) = 0$ and $\theta = 1$.

discounted at a constant rate are dealt with by Koopmans (1960), Fishburn and Rubinstein (1982), and – in summary form – by Heal (1998).

Rigorous and more general treatments of the Maximum Principle in continuous time are available in, e.g., Seierstad and Sydsaeter (1987) and Sydsaeter et al. (2005).

Allen (1967), see Section 9.3.

Goldberg (1958).

9.6 Appendix

A. Growth formulas in continuous time

Let the variables z, x, and y be differentiable functions of time t. Suppose $z(t)$, $x(t)$, and $y(t)$ are positive for all t. Then:

PRODUCT RULE $z(t) = x(t)y(t) \Rightarrow \frac{\dot{z}(t)}{z(t)} = \frac{\dot{x}(t)}{x(t)} + \frac{\dot{y}(t)}{y(t)}$.

Proof. Taking logs on both sides of the equation $z(t) = x(t)y(t)$ gives $\ln z(t) = \ln x(t) + \ln y(t)$. Differentiation w.r.t. t, using the chain rule, gives the conclusion. □

The procedure applied in this proof is called *logarithmic differentiation* w.r.t. t.

FRACTION RULE $z(t) = \frac{x(t)}{y(t)} \Rightarrow \frac{\dot{z}(t)}{z(t)} = \frac{\dot{x}(t)}{x(t)} - \frac{\dot{y}(t)}{y(t)}$.

The proof is similar.

POWER FUNCTION RULE $z(t) = x(t)^{\alpha} \Rightarrow \frac{\dot{z}(t)}{z(t)} = \alpha\frac{\dot{x}(t)}{x(t)}$.

The proof is similar.

In continuous time these simple formulas are exactly true. In discrete time the analogue formulas are only approximately true and the approximation can be quite bad unless the growth rates of x and y are small, cf. Appendix A to Chapter 4.

B. The cumulative mean of growth and interest rates

Sometimes in the literature the basic accumulation formula, (9.22), is expressed in terms of the arithmetic average (also called the cumulative mean) of the growth rates in the time interval $[0, t]$. This average is $\bar{g}_{0,t} = (1/t)\int_0^t g(\tau)d\tau$. So we can write

$$a(t) = a(0)e^{\bar{g}_{0,t}t},$$

which has form more similar to (9.21). Similarly, let $\bar{r}_{0,t}$ denote the arithmetic average of the (short-term) interest rates from time 0 to time t, i.e., $\bar{r}_{0,t} = (1/t)\int_0^t r(\tau)d\tau$. Then we can write the present value of the consumption stream $(c(t))_{t=0}^T$ as $PV = \int_0^T c(t)e^{-\bar{r}_{0,t}t}dt$.

In discrete time the arithmetic average of growth rates can at best be used as an approximation. The relevant average annual growth rate from year 0 to year t is $\hat{g}_{0,t}$, which is defined by the requirement $(1+\hat{g}_{0,t})^t = (1+g_0)(1+g_1)\cdots(1+g_{t-1})$, which gives $\hat{g}_{0,t} = \sqrt[t]{(1+g_0)(1+g_1)\cdots(1+g_{t-1})} - 1$. Hence, $1+\hat{g}_{0,t}$ is a geometric average of the growth factors for the periods involved. If the period length is short, however, say a quarter of a year, the growth rates $g_1, g_2, \ldots,$ will generally be not far from zero so that the approximation $\ln(1+g_t) \approx g_t$ is acceptable. Then $\hat{g}_{0,t} \approx \frac{1}{t}(g_0 + g_1 + \ldots + g_{t-1})$, a simple *arithmetic* average. This is only an approximation, as compounding is left out; the approximation is not good if there are many periods unless the growth rates are very small numbers.

Similarly with interest rates in discrete time.

C. Notes on Proposition 1 (equivalence between the No-Ponzi-Game condition and the intertemporal budget constraint)

We consider the book-keeping relation

$$\dot{a}(t) = r(t)a(t) + w(t) - c(t), \tag{9.57}$$

where $a(0) = a_0$ (given), and the solvency requirement

$$\lim_{t\to\infty} a(t)e^{-\int_0^t r(\tau)d\tau} \geq 0. \tag{NPG}$$

First a clarifying remark. The expression in (NPG) is understood to include the possibility that $a(t)e^{-\int_0^t r(\tau)d\tau} \to \infty$ for $t \to \infty$. Moreover, if full generality were aimed at, we should allow for infinitely fluctuating paths in both the (NPG) and (TVC) and therefore replace "$\lim_{t\to\infty}$" by "$\liminf_{t\to\infty}$", i.e., the *limit inferior*. The limit inferior for $t \to \infty$ of a function $f(t)$ on $[0,\infty)$ is defined as $\lim_{t\to\infty} \inf\{f(s)|\ s \geq t\}$.[22] As noted in Appendix ?? of the previous chapter, however, infinitely fluctuating paths never turn up in the optimization problems considered in this book, whether in discrete or continuous time. Hence, we apply the simpler concept "lim" rather than "lim inf".

[22] By "inf" is meant *infinum* of the set, that is, the largest number less than or equal to all numbers in the set.

On the background of (9.57) Proposition 1 claimed that (NPG) is equivalent with the intertemporal budget constraint,

$$\int_0^\infty c(t)e^{-\int_0^t r(\tau)d\tau}dt \leq h_0 + a_0, \tag{IBC}$$

being satisfied, where h_0 is defined as in (9.50) and is assumed to be a finite number. In addition, Proposition 1 in Section 9.4 claimed that there is strict equality in (IBC) if and only there is strict equality in (NPG). We now prove these claims.

Proof. Isolate $c(t)$ in (9.57) and multiply through by $e^{-\int_0^t r(\tau)d\tau}$ to obtain

$$c(t)e^{-\int_0^t r(\tau)d\tau} = w(t)e^{-\int_0^t r(\tau)d\tau} - (\dot{a}(t) - r(t)a(t))e^{-\int_0^t r(\tau)d\tau}.$$

Integrate from 0 to $T > 0$ to get $\int_0^T c(t)e^{-\int_0^t r(\tau)d\tau}dt$

$$\begin{aligned}
&= \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt - \int_0^T \dot{a}(t)e^{-\int_0^t r(\tau)d\tau}dt + \int_0^T r(t)a(t)e^{-\int_0^t r(\tau)d\tau}dt \\
&= \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt - \left(\left[a(t)e^{-\int_0^t r(\tau)d\tau}\right]_0^T - \int_0^T a(t)e^{-\int_0^t r(\tau)d\tau}(-r(t))dt\right) \\
&+ \int_0^T r(t)a(t)e^{-\int_0^t r(\tau)d\tau}dt \\
&= \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt - (a(T)e^{-\int_0^T r(\tau)d\tau} - a(0)),
\end{aligned}$$

where the second last equality follows from integration by parts. If we let $T \to \infty$ and use the definition of h_0 and the initial condition $a(0) = a_0$, we get (IBC) if and only if (NPG) holds. It follows that when (NPG) is satisfied with strict equality, so is (IBC), and vice versa. □

An alternative proof is obtained by using the general solution to a linear inhomogeneous first-order differential equation and then let $T \to \infty$. Since this is a more generally applicable approach, we will show how it works and use it for Claim 1 below (an extended version of Proposition 1) and for the proof of Proposition 2. Claim 1 will for example prove useful in Exercise 9.1 and in the next chapter.

CLAIM 1 Let $f(t)$ and $g(t)$ be given continuous functions of time, t. Consider the differential equation

$$\dot{x}(t) = g(t)x(t) + f(t), \tag{9.58}$$

with $x(t_0) = x_{t_0}$, a given initial value. Then the inequality

$$\lim_{t\to\infty} x(t)e^{-\int_{t_0}^{t} g(s)ds} \geq 0 \tag{9.59}$$

is equivalent to

$$-\int_{t_0}^{\infty} f(\tau)e^{-\int_{t_0}^{\tau} g(s)ds} d\tau \leq x_{t_0}. \tag{9.60}$$

Moreover, if and only if (9.59) is satisfied with strict equality, then (9.60) is also satisfied with strict equality.

Proof. The linear differential equation, (9.58), has the solution

$$x(t) = x(t_0)e^{\int_{t_0}^{t} g(s)ds} + \int_{t_0}^{t} f(\tau)e^{\int_{\tau}^{t} g(s)ds} d\tau. \tag{9.61}$$

Multiplying through by $e^{-\int_{t_0}^{t} g(s)ds}$ yields

$$x(t)e^{-\int_{t_0}^{t} g(s)ds} = x(t_0) + \int_{t_0}^{t} f(\tau)e^{-\int_{t_0}^{\tau} g(s)ds} d\tau.$$

By letting $t \to \infty$, it can be seen that if and only if (9.59) is true, we have

$$x(t_0) + \int_{t_0}^{\infty} f(\tau)e^{-\int_{t_0}^{\tau} g(s)ds} d\tau \geq 0.$$

Since $x(t_0) = x_{t_0}$, this is the same as (9.60). We also see that if and only if (9.59) holds with strict equality, then (9.60) also holds with strict equality. □

COROLLARY Let n be a given constant and let

$$h_{t_0} \equiv \int_{t_0}^{\infty} w(\tau)e^{-\int_{t_0}^{\tau}(r(s)-n)ds} d\tau, \tag{9.62}$$

which we assume is a finite number. Then, given

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t), \text{ where } a(t_0) = a_{t_0}, \tag{9.63}$$

it holds that

$$\lim_{t\to\infty} a(t)e^{-\int_{t_0}^{t}(r(s)-n)ds} \geq 0 \Leftrightarrow \int_{t_0}^{\infty} c(\tau)e^{-\int_{t_0}^{\tau}(r(s)-n)ds} d\tau \leq a_{t_0} + h_{t_0}, \tag{9.64}$$

where a strict equality on the left of "⇔" implies a strict equality on the right, and vice versa.

Proof. Let $x(t) = a(t)$, $g(t) = r(t) - n$ and $f(t) = w(t) - c(t)$ in (9.58), (9.59) and (9.60). Then the conclusion follows from Claim 1. □

By setting $t_0 = 0$ in the corollary and replacing τ by t and n by 0, we have hereby provided an alternative proof of Proposition 1.

D. Proof of Proposition 2 (the transversality condition with an infinite time horizon)

In the differential equation (9.58) we let $x(t) = \lambda(t)$, $g(t) = -(r(t) - \rho)$, and $f(t) = 0$. This gives the linear differential equation $\dot{\lambda}(t) = (\rho - r(t))\lambda(t)$, which is identical to the first-order condition (9.37) in Section 9.3. The solution is

$$\lambda(t) = \lambda(t_0)e^{-\int_{t_0}^{t}(r(s)-\rho)ds}.$$

Substituting this into (TVC) in Section 9.3 yields

$$\lambda(t_0)\lim_{t\to\infty} a(t)e^{-\int_{t_0}^{t}(r(s)-n)ds} = 0. \tag{9.65}$$

From the first-order condition (9.36) in Section 9.3 we have $\lambda(t_0) = u'(c(t_0)) > 0$ so that $\lambda(t_0)$ in (9.65) can be ignored. Thus (TVC) in Section 9.3 is equivalent to the condition (NPG) in that section being satisfied with strict equality (let $t_0 = 0 = n$). $\square$

E. Intertemporal consumption smoothing

We claimed in Section 9.4 that equation (9.44) gives approximately the marginal rate of substitution of consumption in the time interval $(t+\Delta t, t+2\Delta t)$ for consumption in $(t, t+\Delta t)$. This can be seen in the following way. To save notation we shall write our time-dependent variables as c_t, r_t, etc., even though they are continuous functions of time. The contribution from the two time intervals to the criterion function is

$$\begin{aligned}
\int_t^{t+2\Delta t} u(c_\tau)e^{-\rho\tau}d\tau &\approx e^{-\rho t}\left(\int_t^{t+\Delta t} u(c_t)e^{-\rho(\tau-t)}d\tau + \int_{t+\Delta t}^{t+2\Delta t} u(c_{t+\Delta t})e^{-\rho(\tau-t)}d\tau\right) \\
&= e^{-\rho t}\left(u(c_t)\left[\frac{e^{-\rho(\tau-t)}}{-\rho}\right]_t^{t+\Delta t} + u(c_{t+\Delta t})\left[\frac{e^{-\rho(\tau-t)}}{-\rho}\right]_{t+\Delta t}^{t+2\Delta t}\right) \\
&= \frac{e^{-\rho t}(1-e^{-\rho\Delta t})}{\rho}\left[u(c_t) + u(c_{t+\Delta t})e^{-\rho\Delta t}\right].
\end{aligned}$$

Requiring unchanged utility integral $U_0 = \bar{U}_0$ is thus approximately the same as requiring $\Delta[u(c_t) + u(c_{t+\Delta t})e^{-\rho\Delta t}] = 0$, which by carrying through the differentiation and rearranging gives (9.44).

The instantaneous local optimality condition, equation (9.47), can be interpreted on the basis of (9.46). Take logs on both sides of (9.46) to get

$$\ln u'(c_t) + \rho\Delta t - \ln u'(c_{t+\Delta t}) = \int_t^{t+\Delta t} r_\tau d\tau.$$

Dividing by Δt, substituting (9.45), and letting $\Delta t \to 0$ we get

$$\rho - \lim_{\Delta t \to 0} \frac{\ln u'(c_{t+\Delta t}) - \ln u'(c_t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{R_{t+\Delta t} - R_t}{\Delta t}, \tag{9.66}$$

where R_t is the antiderivative of r_t. By the definition of a time derivative, (9.66) can be written

$$\rho - \frac{d \ln u'(c_t)}{dt} = \frac{dR_t}{dt}.$$

Carrying out the differentiation, we get

$$\rho - \frac{1}{u'(c_t)} u''(c_t)\dot{c}_t = r_t,$$

which was to be shown.

F. Elasticity of intertemporal substitution in continuous time

The relationship between the elasticity of marginal utility and the concept of *instantaneous elasticity of intertemporal substitution* in consumption can be exposed in the following way: consider an indifference curve for consumption in the non-overlapping time intervals $(t,\, t+\Delta t)$ and $(s,\, s+\Delta t)$. The indifference curve is depicted in Fig. 9.3. The consumption path outside the two time intervals is kept unchanged. At a given point $(c_t\Delta t, c_s\Delta t)$ on the indifference curve, the marginal rate of substitution of s-consumption for t-consumption, MRS_{st}, is given by the absolute slope of the tangent to the indifference curve at that point. In view of $u''(c) < 0$, MRS_{st} is rising along the curve when c_t decreases (and thereby c_s increases).

Conversely, we can consider the ratio c_s/c_t as a function of MRS_{st} along the given indifference curve. The elasticity of this consumption ratio w.r.t. MRS_{st} as we move along the given indifference curve then indicates the *elasticity of substitution* between consumption in the time interval $(t,\, t+\Delta t)$ and consumption in the time interval $(s,\, s+\Delta t)$. Denoting this elasticity by $\sigma(c_t, c_s)$, we thus have:

$$\sigma(c_t, c_s) = \frac{MRS_{st}}{c_s/c_t} \frac{d(c_s/c_t)}{dMRS_{st}} \approx \frac{\frac{\Delta(c_s/c_t)}{c_s/c_t}}{\frac{\Delta MRS_{st}}{MRS_{st}}}.$$

At an optimum point, MRS_{st} equals the ratio of the discounted prices of good t and good s. Thus, the elasticity of substitution can be interpreted as approximately equal to the percentage increase in the ratio of the chosen goods, c_s/c_t, generated by a one percentage increase in the inverse price

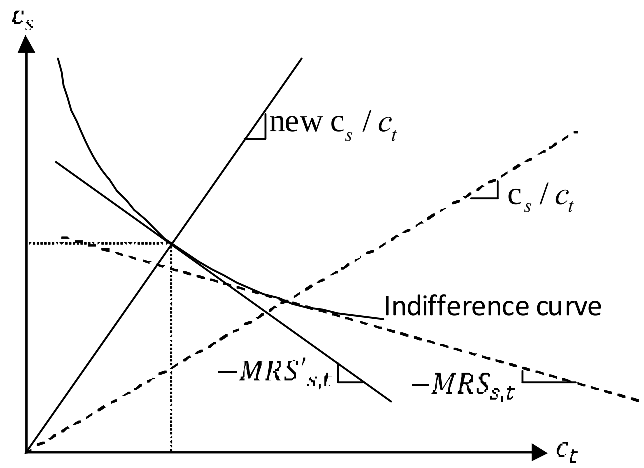


Figure 9.3: Substitution of s-consumption for t-consumption as MRS_{st} increases til MRS'_{st}.

ratio, holding the utility level and the amount of other goods unchanged. If $s = t + \Delta t$ and the interest rate from date t to date s is r, then (with continuous compounding) this price ratio is $e^{r\Delta t}$, cf. (9.46). Inserting MRS_{st} from (9.44) with $t + \Delta t$ replaced by s, we get

$$\begin{aligned}
\sigma(c_t, c_s) &= \frac{u'(c_t)/[e^{-\rho(s-t)}u'(c_s)]}{c_s/c_t} \frac{d(c_s/c_t)}{d\{u'(c_t)/[e^{-\rho(s-t)}u'(c_s)]\}} \\
&= \frac{u'(c_t)/u'(c_s)}{c_s/c_t} \frac{d(c_s/c_t)}{d(u'(c_t)/u'(c_s))}, \qquad (9.67)
\end{aligned}$$

since the factor $e^{-\rho(t-s)}$ cancels out.

We now interpret the d's in (9.67) as differentials (recall, the differential of a differentiable function $y = f(x)$ is denoted dy and defined as $dy = f'(x)dx$ where dx is some arbitrary real number). Calculating the differentials we get

$$\sigma(c_t, c_s) \approx \frac{u'(c_t)/u'(c_s)}{c_s/c_t} \frac{(c_t dc_s - c_s dc_t)/c_t^2}{[u'(c_s)u''(c_t)dc_t - u'(c_t)u''(c_s)dc_s]/u'(c_s)^2}.$$

Hence, for $s \to t$ we get $c_s \to c_t$ and

$$\sigma(c_t, c_s) \to \frac{c_t(dc_s - dc_t)/c_t^2}{u'(c_t)u''(c_t)(dc_t - dc_s)/u'(c_t)^2} = -\frac{u'(c_t)}{c_t u''(c_t)} \equiv \tilde{\sigma}(c_t).$$

This limiting value is known as the *instantaneous elasticity of intertemporal substitution* of consumption. It reflects the opposite of the desire for consumption smoothing. Indeed, we see that $\tilde{\sigma}(c_t) = 1/\theta(c_t)$, where $\theta(c_t)$ is the elasticity of marginal utility at the consumption level $c(t)$.

9.7 Exercises

9.1 We look at a household (or dynasty) with infinite time horizon. The household's labor supply is inelastic and grows at the constant rate $n > 0$. The household has a constant rate of time preference $\rho > n$ and the individual instantaneous utility function is $u(c) = c^{1-\theta}/(1-\theta)$, where θ is a positive constant. There is no uncertainty. The household maximizes the integral of per capita utility discounted at the rate $\rho - n$. . Set up the household's optimization problem. Show that the optimal consumption plan satisfies

$$\begin{aligned}
c(0) &= \beta_0(a_0 + h_0), \qquad \text{where} \\
\beta_0 &= \frac{1}{\int_0^\infty e^{\int_0^t (\frac{(1-\theta)r(\tau)-\rho}{\theta}+n)d\tau} dt}, \qquad \text{and} \\
h_0 &= \int_0^\infty w(t) e^{-\int_0^t (r(\tau)-n)d\tau} dt,
\end{aligned}$$

where $w(t)$ is the real wage per unit of labor and otherwise the same notation as in this chapter is used. *Hint:* use the corollary to Claim 1 in Appendix C and the method of Example 2.

Macroeconomics 2 Short note 2.

12.10.2012 Christian Groth

Solution formulas for linear differential equations of first order

1. $\dot{x}(t) + ax(t) = b$, with $a \neq 0$ and initial condition $x(0) = x_0$. Solution:

$$x(t) = (x_0 - x^*)e^{-at} + x^*, \text{ where } x^* = \frac{b}{a}.$$

2. $\dot{x}(t) + ax(t) = b(t)$, with initial condition $x(0) = x_0$. Solution:

$$x(t) = x_0 e^{-at} + e^{-at} \int_0^t b(s)e^{as} ds.$$

Special case: $b(t) = ce^{ht}$, with $h \neq -a$ and initial condition $x(0) = x_0$. Solution:

$$x(t) = x_0 e^{-at} + e^{-at} c \int_0^t e^{(a+h)s} ds = (x_0 - \frac{c}{a+h})e^{-at} + \frac{c}{a+h} e^{ht}.$$

3. $\dot{x}(t) + a(t)x(t) = b(t)$, with initial condition $x(0) = x_0$. Solution:

$$x(t) = x_0 e^{-\int_0^t a(\tau)d\tau} + e^{-\int_0^t a(\tau)d\tau} \int_0^t b(s) e^{\int_0^s a(\tau)d\tau} ds.$$

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Chapter 10

The Ramsey model

As early as 1928 a sophisticated model of a society's optimal saving was published by the British mathematician and economist Frank Ramsey (1903-1930). Ramsey's contribution was mathematically demanding and did not experience much response at the time. Three decades had to pass until his contribution was taken up seriously (Samuelson and Solow, 1956). The model was fusioned with Solow's simpler growth model (Solow 1956) and became a cornerstone in neoclassical growth theory from the mid 1960s. The version of the model which we present below was completed by the work of Cass (1965) and Koopmans (1965). Hence the model is also known as the *Ramsey-Cass-Koopmans model*.

The model is one of the basic workhorse models in macroeconomics. It can be seen as placed at one end of a line segment. At the other end appears another basic workhorse model, namely Diamond's overlapping generations model. In the Diamond model there is an *infinite* number of agents (since in every new period a new generation enters the economy) and these have a *finite* time horizon. In the Ramsey model there is a *finite* number of agents with an *infinite* time horizon; further, these agents are completely alike. The Ramsey model is thus a *representative agent* model, whereas the Diamond model has heterogeneous agents, young and old, interacting in every period. There are important economic questions where these differences in the model setup lead to salient differences in the answers.

Along the line segment where these two frameworks are polar cases, less abstract models are scattered, some being closer to the one pole and others closer to the other. A given model may open up for different regimes, one close to Ramsey's pole, another close to Diamond's. An example is Robert Barro's model with parental altruism presented in Chapter 7. When the bequest motive in the Barro model is operative, the model coincides with a Ramsey model (in discrete time) as was shown in Chapter 8. But when the bequest motive is not operative, the Barro model coincides with a Diamond OLG model. The Blanchard (1985) OLG model in continuous time (the chapters 12, 13, and 15) also belongs to the interior of this line segment, but closer to the Diamond pole than the Ramsey pole.

The present chapter presents the continuous-time version of the Ramsey model. We first study the Ramsey framework under the conditions of a perfectly competitive market economy. In this context we will see, for example, that the Solow growth model comes out as a special case of the Ramsey model. Next we consider the Ramsey framework in a setting with an "all-knowing and all-powerful" social planner.

10.1 Preliminaries

We consider a closed economy. Time is continuous. We assume that the households own the capital goods and hire them out to firms at a market rental rate $\hat{r}$. This is just to have something concrete in mind. If instead the capital goods were owned by the firms using them in production and the capital investment by these firms were financed by issuing shares and bonds, the conclusions would remain unaltered as long as we ignore uncertainty.

The variables in the model are considered as (piecewise) continuous and differentiable functions of time, t. Yet, to save notation, we shall write them as w_t, $\hat{r}_t$, etc. instead of $w(t)$, $\hat{r}(t)$, etc. In every short time interval $(t, t+\Delta t)$, the individual firm employs labor at the market wage w_t and rents capital goods at the rental rate $\hat{r}_t$. The combination of labor and capital produces the homogeneous output good. This good can be used for consumption as well as investment. So in every short time interval there are at least three active markets, one for the "all-purpose" output good, one for labor, and one for capital services (the rental market for capital goods). For the sake of intuition it may be useful to imagine that there is also a market for loans. There is a time-dependent short-term interest rate, r_t, on these loans. As all households are alike, however, the loan market will not be active in general equilibrium. There is perfect competition in all markets, that is, prices are exogenous to the individual households and firms. Any need for means of payment − money − is abstracted away. Prices are measured in units of the homogeneous current output.

There are no stochastic elements in the model. We assume households understand precisely how the economy works and can predict the future path of wages and interest rates. That is, we assume "rational expectations" which in our non-stochastic setting amounts to perfect foresight.

As uncertainty is absent by assumption, rates of return on alternative assets must in equilibrium be the same. So if otherwise the loan market were active, the interest rate would equal the rate of return on holding capital goods, i.e.,

$$r_t = \hat{r}_t - \delta, \tag{10.1}$$

where δ (≥ 0) is a constant rate of capital depreciation. This no-arbitrage condition indicates how the rental rate of capital is related to the short-term interest rate.

Below we present, first, the households' behavior and next the firms' behavior. After this, the interaction between households and firms in general equilibrium and the resulting dynamics will be analyzed.

10.2 The agents

10.2.1 Households

There is a fixed number of identical households with an infinite time horizon. This feature makes aggregation very simple: we just have to multiply the behavior of a single household with the number of households. Every household has L_t (adult) members and L_t changes over time at a constant rate, n :

$$L_t = L_0 e^{nt}, \qquad L_0 > 0. \tag{10.2}$$

Indivisibility problems are ignored.

Each household member supplies inelastically one unit of labor per time unit. Equation (10.2) therefore describes the growth of the population as well as the labor force. Since there is only one consumption good, the only decision problem is how to distribute current income between consumption and saving.

Intertemporal utility function

From now we consider a single household. Its preferences can be represented by an additive intertemporal utility function with a constant rate of time preference, ρ. Seen from time 0, the intertemporal utility function is

$$U_0 = \int_0^\infty u(c_t) L_t e^{-\rho t} dt,$$

where $c_t \equiv C_t/L_t$ is consumption per family member. The instantaneous utility function, $u(c)$, has $u'(c) > 0$ and $u''(c) < 0$, i.e., positive but diminishing marginal utility of consumption. The utility contribution from consumption per family member is weighted by the number of family members, L_t. So it is the sum of the family members' utility that counts. Such a utility function is called a *classical-utilitarian* utility function (with discounting).

The infinite horizon of the "household" may be seen as reflecting an altruistic bequest motive. That is, the household is seen as an infinitely-lived family, a family dynasty. The current members of the dynasty act in unity and are concerned about the utility from own consumption as well as the utility of the future generations within the dynasty.[1] Note that in this setup, births (into adult life) do not reflect the emergence of *new* economic agents with independent interests. Births and population growth are seen as just an expansion of the size of already existing infinitely-lived households. In contrast, in the Diamond OLG model births imply entrance of new economic decision makers whose preferences no-one has cared about in advance.

Because of (10.2), U_0 can be written as

$$U_0 = \int_0^\infty u(c_t)e^{-(\rho-n)t}dt, \tag{10.3}$$

where the unimportant positive factor L_0 has been eliminated. Here $\bar{\rho} \equiv \rho - n$ is known as the *effective* rate of time preference while ρ is the *pure* rate of time preference. We later introduce a restriction on $\rho - n$ to ensure upward boundedness of the utility integral in general equilibrium.

The household chooses a consumption-saving plan which maximizes U_0 subject to its budget constraint. Let $A_t \equiv a_t L_t$ be the household's (net) financial wealth in real terms at time t. It is of no consequence whether we imagine the components of this wealth are capital goods or loans to other agents in the economy. We have

$$\dot{A}_t = r_t A_t + w_t L_t - c_t L_t, \qquad A_0 \text{ given.} \tag{10.4}$$

This equation is a book-keeping relation telling how financial wealth or debt $(-A)$ is evolving depending on how consumption relates to current income. The equation merely says that the increase in financial wealth per time unit equals saving which equals income minus consumption. Income is the sum of the net return on financial wealth, $r_t A_t$, and labor income, $w_t L_t$, where w_t is the real wage.[2] Saving can be negative. In that case the household "dissaves" and does so simply by selling a part of its stock of capital goods or by taking loans in the loan market.[3]

[1] The Barro model of Chapter 7 exemplifies such a structure in discrete time. In that chapter we also discussed the shortcomings of the dynasty setup.

[2] Since the technology exhibits constant returns to scale, in competitive equilibrium the firms make no (pure) profit to pay out to their owners (presumably the households).

[3] The market prices, w_t and r_t, faced by the household are assumed to be piecewise continuous functions of time.

When the dynamic budget identity (10.4) is combined with a requirement of solvency, we have a budget *constraint*. The relevant solvency requirement is the No-Ponzi-Game condition (NPG for short):

$$\lim_{t\to\infty} A_t e^{-\int_0^t r_s ds} \geq 0. \tag{10.5}$$

This condition says that financial wealth far out in the future cannot have a negative present value. That is, in the long run, debt is at most allowed to rise at a rate *less* than the real interest rate r. The NPG condition thus precludes permanent financing of the interest payments by new loans.[4]

The decision problem is: choose a plan $(c_t)_{t=0}^{\infty}$ so as to achieve a maximum of U_0 subject to non-negativity of the control variable, c, and the constraints (10.4) and (10.5). The problem is a slight generalization of the problem studied in Section 9.4 of the previous chapter.

To solve the problem we shall apply the Maximum Principle. This method can be applied directly to the problem as stated above or to an equivalent problem with constraints expressed in per capita terms. Let us follow the latter approach. From the definition $a_t \equiv A_t/L_t$ we get by differentiation w.r.t. t

$$\dot{a}_t = \frac{L_t \dot{A}_t - A_t \dot{L}_t}{L_t^2} = \frac{\dot{A}_t}{L_t} - a_t n.$$

Substitution of (10.4) gives the dynamic budget identity in per capita terms:

$$\dot{a}_t = (r_t - n)a_t + w_t - c_t, \qquad a_0 \text{ given.} \tag{10.6}$$

By inserting $A_t \equiv a_t L_t = a_t L_0 e^{nt}$, the No-Ponzi-Game condition (10.5) can be rewritten as

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n) ds} \geq 0, \tag{10.7}$$

where the unimportant factor L_0 has been eliminated.

We see that in both (10.6) and (10.7) a kind of corrected interest rate appears, namely the interest rate, r, minus the family size growth rate, n. Although deferring consumption gives a real interest rate of r, this return is diluted on a per head basis because it will have to be shared with more members of the family when $n > 0$. In the form (10.7) the NPG condition requires that debt, if any, in the long run rises at most at a rate *less* than $r - n$.

[4] In the previous chapter we saw that the NPG condition, in combination with (10.4), is equivalent to an ordinary *intertemporal* budget constraint which says that the present value of the planned consumption path cannot exceed initial total wealth, i.e., the sum of the initial financial wealth and the present value of expected future labor income.

Solving the consumption/saving problem

The decision problem is now: choose $(c_t)_{t=\infty}^{0}$ so as to a maximize U_0 subject to the constraints: $c_t \geq 0$ for all $t \geq 0$, (10.6), and (10.7). The solution procedure is similar to that in the simpler problem of the previous chapter:

1) We set up the current-value Hamiltonian

$$H(a, c, \lambda, t) = u(c) + \lambda \left[(r - n)\, a + w - c\right],$$

where λ is the *adjoint variable* associated with the dynamic constraint (10.6).

2) We differentiate H partially w.r.t. the control variable and put the result equal to zero:

$$\frac{\partial H}{\partial c} = u'(c) - \lambda = 0. \tag{10.8}$$

3) We differentiate H partially w.r.t. the state variable, a, and put the result equal to the effective discount rate (appearing in the integrand of the criterion function) multiplied by λ minus the time derivative of the adjoint variable λ:

$$\frac{\partial H}{\partial a} = \lambda(r - n) = (\rho - n)\lambda - \dot{\lambda}. \tag{10.9}$$

4) Finally, we apply the Maximum Principle: an interior optimal path $(a_t, c_t)_{t=0}^{\infty}$ will satisfy that there exists a continuous function $\lambda(t)$ such that for all $t \geq 0$, (10.8) and (10.9) hold along the path and the transversality condition,

$$\lim_{t\to\infty} a_t \lambda_t e^{-(\rho-n)t} = 0, \tag{10.10}$$

is satisfied.

The interpretation of these optimality conditions is as follows. The condition (10.8) can be considered a $MC = MB$ condition (in utility terms). It illustrates together with (10.9) that the adjoint variable λ can be seen as the shadow price, measured in current utility, of per head financial wealth along the optimal path. Rearranging (10.9) gives, $r_t = \rho - \dot{\lambda}_t/\lambda_t$; the left-hand-side of this equation is the market rate of return on saving while the right-hand-side is the *required* rate of return (as in the previous chapter, by subtracting the shadow price "inflation rate" from the required utility rate of return, ρ, we get the required real rate of return). The household is willing to save the marginal unit only if the actual return equals the required return.

The transversality condition (10.10) says that for $t \to \infty$ the present shadow value of per head financial wealth should go to zero. Combined with (10.8), the condition is that

$$\lim_{t\to\infty} a_t u'(c_t) e^{-(\rho-n)t} = 0 \tag{10.11}$$

must hold along the optimal path. This requirement is not surprising if we compare with the case where instead $\lim_{t\to\infty} a_t u'(c_t)e^{-(\rho-n)t} > 0$. In this case there would be over-saving; U_0 could be increased by reducing the "ultimate" a_t and thereby consume more and save less. The opposite case, $\lim_{t\to\infty} a_t u'(c_t)e^{-(\rho-n)t} < 0$, will not even satisfy the NPG condition in view of Proposition 2 of the previous chapter. In fact, from that proposition we know that the transversality condition (10.11) is equivalent with the NPG condition (10.7) being satisfied with strict equality, i.e.,

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} = 0. \tag{10.12}$$

Recall that the Maximum Principle gives only *necessary* conditions for an optimal plan. But since the Hamiltonian is jointly concave in (a, c) for every t, the necessary conditions are also *sufficient*, by Mangasarian's sufficiency theorem.

The first-order conditions (10.8) and (10.9) give the Keynes-Ramsey rule:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta(c_t)}(r_t - \rho), \tag{10.13}$$

where $\theta(c_t)$ is the (absolute) elasticity of marginal utility,

$$\theta(c_t) \equiv -\frac{c_t}{u'(c_t)} u''(c_t) > 0. \tag{10.14}$$

As we know from previous chapters, this elasticity indicates the consumer's wish to smooth consumption over time. The inverse of $\theta(c_t)$ is the elasticity of intertemporal substitution in consumption. It indicates the willingness to incur variation in consumption over time in response to a change in the interest rate.

Interestingly, the population growth rate, n, does not appear in the Keynes-Ramsey rule. Going from $n = 0$ to $n > 0$ implies that r_t is replaced by $r_t - n$ in the dynamic budget identity and ρ is replaced by $\rho - n$ in the criterion function. This implies that n cancels out in the Keynes-Ramsey rule. Yet n appears in the transversality condition and thereby also in the *level* of consumption for given wealth, cf. (10.18) below.

CRRA utility

In order that the model can accommodate Kaldor's stylized facts, it should be able to generate a balanced growth path. When the population grows at the same constant rate as the labor force, here n, by definition balanced growth requires that per capita output, per capita capital, and per capita consumption grow at constant rates. This will generally require that the real interest rate is constant in the process. But (10.13) shows that having a constant per capita consumption growth rate at the same time as r is constant, is only possible if the elasticity of marginal utility does *not* vary with c. Hence, it makes sense to assume that the right-hand-side of (10.14) is a positive constant, θ. So we will assume that the instantaneous utility function is of CRRA form:

$$u(c) = \frac{c^{1-\theta}}{1-\theta}, \qquad \theta > 0; \tag{10.15}$$

here, for $\theta = 1$, the right-hand side should be interpreted as $\ln c$. So our Keynes-Ramsey rule simplifies to

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho). \tag{10.16}$$

The Keynes-Ramsey rule characterizes the optimal *rate of change* of consumption. The optimal initial *level* of consumption, c_0, will be the highest feasible c_0 which is compatible with both the Keynes-Ramsey rule and the NPG condition. And for this reason the choice will exactly comply with the transversality condition (10.12). Although an explicit determination of c_0 is not actually necessary to pin down the equilibrium path of the economy, we note in passing that c_0 can be found by the method described at the end of Chapter 9. Indeed, given the book-keeping relation (10.6), we have by Proposition 1 of Chapter 9 that the transversality condition (10.12) is equivalent with satisfying the following intertemporal budget constraint (with strict equality):

$$\int_0^\infty c_t e^{-\int_0^t (r_\tau - n)d\tau} dt = a_0 + h_0. \tag{10.17}$$

Solving the differential equation (11.40) we get $c_t = c_0 e^{\frac{1}{\theta}\int_0^t (r_\tau - \rho)d\tau}$, which we substitute for c_t in (10.17). Isolating c_0 now gives[5]

$$\begin{aligned} c_0 &= \beta_0(a_0 + h_0), \qquad \text{where} \\ \beta_0 &= \frac{1}{\int_0^\infty e^{\int_0^t (\frac{(1-\theta)r_\tau - \rho}{\theta} + n)d\tau} dt}, \qquad \text{and} \\ h_0 &= \int_0^\infty w_t e^{-\int_0^t (r_\tau - n)d\tau} dt. \end{aligned} \tag{10.18}$$

[5] These formulas can also be derived directly from Example 1 of Section 9.4.4 of Chapter 9 by replacing $r(\tau)$ and ρ by $r(\tau) - n$ and $\rho - n$, respectively.

Thus, the entire expected future evolution of wages and interest rates determines c_0. The marginal propensity to consume out of wealth, β_0, is less, the greater is the population growth rate, n.[6] The explanation is that the effective utility discount rate, $\rho - n$, is less, the greater is n. The propensity to save is greater the more mouths to feed in the future. The initial saving level will be $r_0 a_0 + w_0 - c_0 = r_0 a_0 + w_0 - \beta_0(a_0 + h_0)$.

In the Solow growth model the saving-income ratio is parametrically given and constant over time. The Ramsey model endogenizes the saving-income ratio. Solow's parametric saving rate is replaced by two "deeper" parameters, the rate of impatience, ρ, and the desire for consumption smoothing, θ. As we shall see the resulting saving-income ratio will not generally be constant outside the steady state of the dynamic system implied by the Ramsey model.

10.2.2 Firms

There is a large number of firms which maximize profits under perfect competition. All firms have the same neoclassical production function with CRS,

$$Y_t = F(K_t^d, T_t L_t^d) \tag{10.19}$$

where Y_t is supply of output, K_t^d is capital input, and L_t^d is labor input, all measured per time unit, at time t. The superscript d on the two inputs indicates that these inputs are seen from the demand side. The factor T_t represents the economy-wide level of technology as of time t and is exogenous. We assume there is technological progress at a constant rate g (≥ 0) :

$$T_t = T_0 e^{gt}, \qquad T_0 > 0. \tag{10.20}$$

Thus the economy features Harrod-neutral technological progress, as is needed for compliance with Kaldor's stylized facts.

Necessary and sufficient conditions for the factor combination (K_t^d, L_t^d), where $K_t^d > 0$ and $L_t^d > 0$, to maximize profits are that

$$\begin{aligned} F_1(K_t^d, T_t L_t^d) &= \hat{r}_t, && (10.21) \\ F_2(K_t^d, T_t L_t^d) T_t &= w_t. && (10.22) \end{aligned}$$

10.3 General equilibrium

We now consider the economy as a whole and thereby the interaction between households and firms in the various markets. For simplicity, we assume that

[6] This holds also if $\theta = 1$, since in that case $\beta_0 = \rho - n$.

the number of households is the same as the number of firms. We normalize this number to one so that $F(\cdot, \cdot)$ can from now on be interpreted as the aggregate production function and C_t as aggregate consumption.

Factor markets

In the short term, that is, for fixed t, the available quantities of labor, $L_t = L_0 e^{nt}$, and capital, K_t, are predetermined. The factor markets clear at all points in time, that is,

$$K_t^d = K_t, \quad \text{and} \quad L_t^d = L_t, \tag{10.23}$$

for all $t \geq 0$. It is the rental rate, $\hat{r}_t$, and the wage rate, w_t, which adjust (immediately) so that this is achieved. Aggregate output can be written

$$Y_t = F(K_t, T_t L_t) = T_t L_t F(\tilde{k}_t, 1) \equiv T_t L_t f(\tilde{k}_t), \qquad f' > 0, f'' < 0, \tag{10.24}$$

where $\tilde{k}_t \equiv K_t/(T_t L_t)$. Substituting (10.23) into (10.21) and (10.22), we find the equilibrium interest rate and wage rate:

$$r_t = \hat{r}_t - \delta = \frac{\partial(T_t L_t f(\tilde{k}_t))}{\partial K_t} - \delta = f'(\tilde{k}_t) - \delta, \tag{10.25}$$

$$w_t = \frac{\partial(T_t L_t f(\tilde{k}_t))}{\partial L_t} T_t = \left[f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t) \right] T_t \equiv \tilde{w}(\tilde{k}_t) T_t, \tag{10.26}$$

where $\tilde{k}_t$ is at any point in time predetermined and where in (10.25) we have used the no-arbitrage condition (10.1).

Capital accumulation

From now we leave out the explicit dating of the variables when not needed for clarity. By national product accounting we have

$$\dot{K} = Y - C - \delta K. \tag{10.27}$$

Let us check whether we get the same result from the wealth accumulation equation of the household. Because physical capital is the only asset in the economy, aggregate financial wealth, A, at time t equals the total quantity of capital, K, at time t.[7] From (10.4) we thus have

$$\begin{aligned}
\dot{K} &= rK + wL - cL \\
&= (f'(\tilde{k}) - \delta)K + (f(\tilde{k}) - \tilde{k} f'(\tilde{k}))TL - cL \qquad \text{(from (10.25) and (10.26))} \\
&= f(\tilde{k})TL - \delta K - cL \qquad \text{(by rearranging and use of } K \equiv \tilde{k}TL) \\
&= F(K, TL) - \delta K - C = Y - C - \delta K \qquad \text{(by } C \equiv cL).
\end{aligned}$$

[7] Whatever financial claims on each other the households might have, they net out for the household sector as a whole.

Hence, the book-keeping is in order (the national income account is consistent with the national product account).

We now face a fundamental difference as compared with models where households have a finite horizon, such as the Diamond OLG model. Current consumption cannot be determined independently of the expected *long-term* evolution of the economy. This is because consumption and saving, as we saw in Section 10.2, depend on the expectations of the entire future evolution of wages and interest rates. And given the presumption of perfect foresight, the households' expectations are identical to the prediction that can be calculated from the model. In this way there is interdependence between expectations and the level and evolution of consumption. We can determine the level of consumption only in the context of the overall dynamic analysis. In fact, the economic agents are in some sense in the same situation as the outside analyst. They, too, have to think through the entire dynamics of the economy in order to form their rational expectations.

The dynamic system

We get a concise picture of the dynamics by reducing the model to the minimum number of coupled differential equations. This minimum number is two. The key endogenous variables are $\tilde{k} \equiv K/(TL)$ and $\tilde{c} \equiv C/(TL) \equiv c/T$. Using the rule for the growth rate of a fraction, we get

$$\begin{aligned}
\frac{\dot{\tilde{k}}}{\tilde{k}} &= \frac{\dot{K}}{K} - \frac{\dot{T}}{T} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - (g+n) \qquad \text{(from (10.2) and (10.20))} \\
&= \frac{F(K,TL) - C - \delta K}{K} - (g+n) \qquad \text{(from (10.27))} \\
&= \frac{f(\tilde{k}) - \tilde{c}}{\tilde{k}} - (\delta + g + n) \qquad \text{(from (10.24))}.
\end{aligned}$$

The associated differential equation for $\tilde{c}$ is obtained in a similar way:

$$\begin{aligned}
\frac{\dot{\tilde{c}}}{\tilde{c}} &= \frac{\dot{c}}{c} - \frac{\dot{T}}{T} = \frac{1}{\theta}(r_t - \rho) - g \quad \text{(from the Keynes-Ramsey rule)} \\
&= \frac{1}{\theta}\left[f'(\tilde{k}) - \delta - \rho - \theta g\right] \qquad \text{(from (10.25))}.
\end{aligned}$$

We thus end up with the dynamic system

$$\dot{\tilde{k}} = f(\tilde{k}) - \tilde{c} - (\delta + g + n)\tilde{k}, \qquad \tilde{k}_0 > 0 \quad \text{given}, \tag{10.28}$$

$$\dot{\tilde{c}} = \frac{1}{\theta}\left[f'(\tilde{k}) - \delta - \rho - \theta g\right]\tilde{c}. \tag{10.29}$$

The lower panel of Fig. 10.1 shows the *phase diagram* of the system. The curve OEB represents the points where $\dot{\tilde{k}} = 0$; from (10.28) we see that

$$\dot{\tilde{k}} = 0 \text{ for } \tilde{c} = f(\tilde{k}) - (\delta + g + n)\tilde{k} \equiv \tilde{c}(\tilde{k}). \tag{10.30}$$

The upper panel of Fig. 10.1 displays the value of $\tilde{c}(\tilde{k})$ as the vertical distance between the curve $\tilde{y} = f(\tilde{k})$ and the line $\tilde{y} = (\delta + g + n)\tilde{k}$ (to save space the proportions are distorted).[8] The maximum value of $\tilde{c}(\tilde{k})$, if it exists, is reached at the point where the tangent to the OEB curve in the lower panel is horizontal, i.e., where $\tilde{c}'(\tilde{k}) = f'(\tilde{k}) - (\delta + g + n) = 0$ or $f'(\tilde{k}) - \delta = g + n$. The value of $\tilde{k}$ which satisfies this is the golden rule capital intensity, $\tilde{k}_{GR}$:

$$f'(\tilde{k}_{GR}) - \delta = g + n. \tag{10.31}$$

From (10.28) we see that for points above the $\dot{\tilde{k}} = 0$ locus we have $\dot{\tilde{k}} < 0$, whereas for points below the $\dot{\tilde{k}} = 0$ locus, $\dot{\tilde{k}} > 0$. The horizontal arrows in the figure indicate these directions of movement.

From (10.29) we see that

$$\dot{\tilde{c}} = 0 \text{ for } f'(\tilde{k}) = \delta + \rho + \theta g \qquad \text{or} \qquad \tilde{c} = 0. \tag{10.32}$$

Let $\tilde{k}^* > 0$ satisfy the equation $f'(\tilde{k}^*) - \delta = \rho + \theta g$. Then the vertical line $\tilde{k} = \tilde{k}^*$ represents points where $\dot{\tilde{c}} = 0$ (and so does of course the horizontal half-line $\tilde{c} = 0, \tilde{k} \geq 0$). For points to the left of the $\tilde{k} = \tilde{k}^*$ line we have, according to (10.29), $\dot{\tilde{c}} > 0$ and for points to the right of the $\tilde{k} = \tilde{k}^*$ line we have $\dot{\tilde{c}} < 0$. The vertical arrows in Fig. 10.1 indicate these directions of movement. Four illustrative examples of solution curves (*I*, *II*, *III*, and *IV*) are drawn in the figure.

[8] As the graph is drawn, $f(0) = 0$, i.e., capital is assumed essential. But none of the conclusions we are going to consider depends on this.

Steady state

The point E has coordinates $(\tilde{k}^*, \tilde{c}^*)$ and represents the unique steady state.[9] From (10.32) and (10.30) follows that

$$\begin{aligned} f'(\tilde{k}^*) &= \delta + \rho + \theta g, \qquad \text{and} & (10.33)\\ \tilde{c}^* &= f(\tilde{k}^*) - (\delta + g + n)\tilde{k}^*. & (10.34) \end{aligned}$$

From (10.33) it can be seen that the real interest rate in steady state is

$$r^* = f'(\tilde{k}^*) - \delta = \rho + \theta g. \tag{10.35}$$

The capital intensity satisfying this equation is known as the *modified golden rule* capital intensity, $\tilde{k}_{MGR}$. The modified golden rule is the rule saying that for a representative agent economy to be in steady state, the capital intensity must be such that the net marginal product of capital equals the required rate of return, taking into account the pure rate of time preference, ρ, and the desire for consumption smoothing, measured by θ.[10]

We show below that the steady state is, in a specific sense, asymptotically stable. First we have to make sure, however, that the steady state exists and is consistent with general equilibrium. This consistency requires that the household's transversality condition (10.12) holds in the point E. Using $a_t = K_t/L_t \equiv \tilde{k}_t T_t = \tilde{k}_t T_0 e^{gt}$ and $r_t = f'(\tilde{k}_t) - \delta$, we see that (10.11) is equivalent with

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t \left(f'(\tilde{k}_s) - \delta - g - n\right)ds} = 0. \tag{10.36}$$

In the point E $\tilde{k}_t = \tilde{k}^*$ and $f'(\tilde{k}_t) - \delta = \rho + \theta g$ for all t. So the condition (10.36) becomes

$$\lim_{t\to\infty} \tilde{k}^* e^{-(\rho+\theta g - g - n)t} = 0. \tag{10.37}$$

This is fulfilled if and only if $\rho + \theta g > g + n$, that is,

$$\rho - n > (1 - \theta)g. \tag{A1}$$

[9] As (10.32) shows, if $\tilde{c}_t = 0$, then $\dot{\tilde{c}} = 0$. Therefore, mathematically, point B (if it exists) in Fig. 10.1 is also a stationary point of the dynamic system. And if $f(0) = 0$, then, according to (10.29) and (10.30), also the point $(0, 0)$ in the figure is a stationary point. But these stationary points have zero consumption forever and are therefore not steady states of any *economic* system. From an economic point of view they are "trivial" steady states.

[10] Note that the ρ of the Ramsey model corresponds to the intergenerational discount rate R of the Barro dynasty model in Chapter 7. Indeed, in the discrete time Barro model we have $1 + r^* = (1 + R)(1 + g)^\theta$, which, by taking logs on both sides and using first-order Taylor approximations around 1 gives $r^* \approx \ln(1 + r^*) = \ln(1 + R) + \theta \ln(1 + g) \approx R + \theta g$. Recall, however, that in view of the considerable period length (about 25-30 years) of the Barro model, this approximation will not be very good.

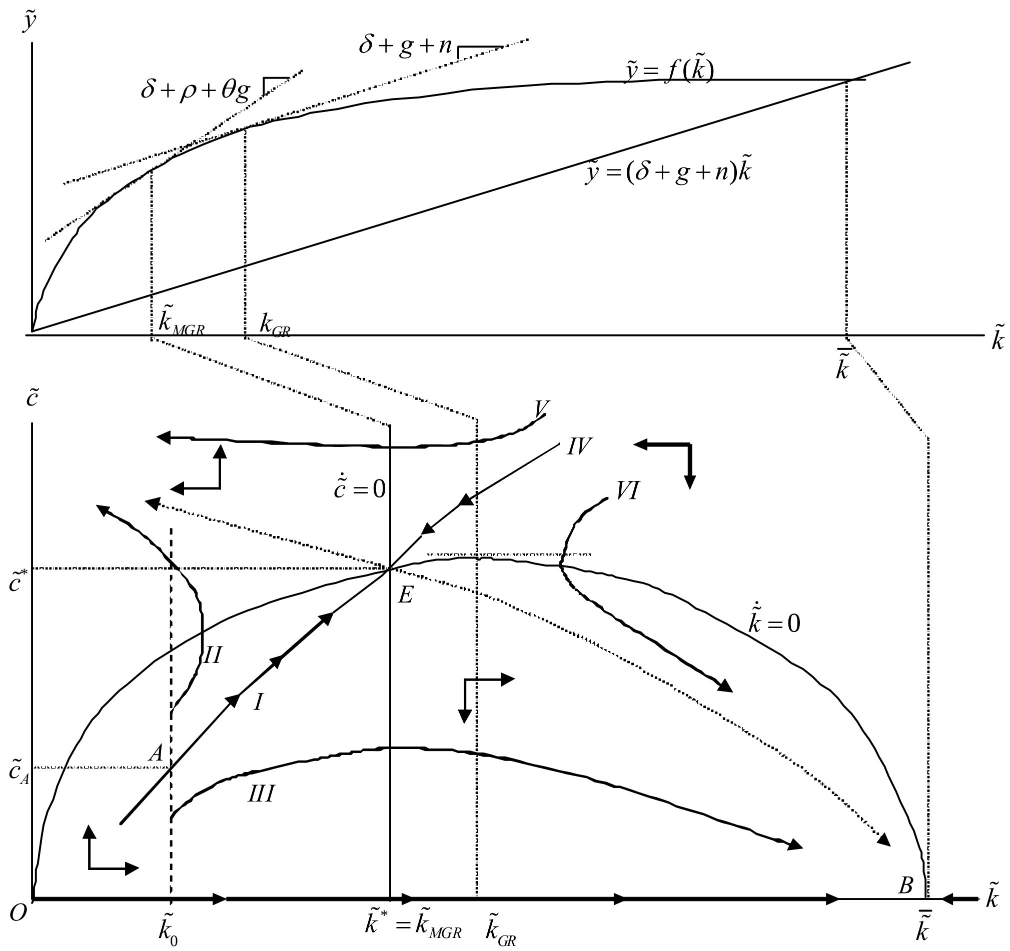

Figure 10.1: Phase portrait of the Ramsey model.

This condition ensures also that the improper integral U_0 is bounded, at least when the system is in steady state (see Appendix B). If $\theta \geq 1$, (A1) is fulfilled as soon as the effective utility discount rate, $\rho - n$, is positive; (A1) may even hold for a negative $\rho - n$ if not "too" negative). If $\theta < 1$, (A1) requires $\rho - n$ to be "sufficiently positive".

Since the parameter restriction (A1) can be written $\rho + \theta g > g + n$, it implies that the steady-state interest rate, r^*, given in (10.35), is higher than the "natural" growth rate, $g + n$. If this did not hold, the transversality condition (10.12) would fail in the steady state. Indeed, along the steady state path we have

$$a_t e^{-r^* t} = a_0 e^{(g+n)t} e^{-r^* t} = k_0 e^{(g+n-r^*)t},$$

which would take the value $k_0 > 0$ for all $t \geq 0$ if $r^* = g+n$ and would go to ∞ for $t \to \infty$ if $r^* < g+n$. The individual households would be over-saving. Each household would in this situation alter its behavior and the steady state could thus not be an equilibrium path.

Another way of seeing that $r^* \leq g+n$ can never be an equilibrium in a Ramsey model is to recognize that this condition would make the household's human wealth infinite because wage income, wL, would grow at a rate, $g+n$, at least as high as the real interest rate, r^*. This would motivate an immediate increase in consumption and so the considered steady-state path would again not be an equilibrium.

To have a model of interest, from now on we assume that the parameters satisfy the inequality (A1). As an implication, the capital intensity in steady state, $\tilde{k}^*$, is less than the golden rule value $\tilde{k}_{GR}$. Indeed, $f'(\tilde{k}^*)-\delta = \rho+\theta g > g+n = f'(\tilde{k}_{GR}) - \delta$, so that $\tilde{k}^* < \tilde{k}_{GR}$, in view of $f'' < 0$.

So far we have only ensured that if the steady state, E, exists, it is consistent with general equilibrium. Existence of both a steady state and a golden rule capital intensity requires that the marginal productivity of capital is sufficiently sensitive to variation in the capital intensity. We therefore assume that f has the properties

$$\lim_{\tilde{k}\to 0} f'(\tilde{k}) - \delta > \rho + \theta g \quad \text{and} \quad \lim_{\tilde{k}\to\infty} f'(\tilde{k}) - \delta < g + n. \tag{A2}$$

Together with (A1) this implies $\lim_{\tilde{k}\to 0} f'(\tilde{k}) - \delta > \rho + \theta g > g + n > \lim_{\tilde{k}\to\infty} f'(\tilde{k}) - \delta$. By continuity of f', these inequalities ensure the existence of both $\tilde{k}^* > 0$ and $\tilde{k}_{GR} > 0$.[11] Moreover, the inequalities ensure the existence of a $\bar{\tilde{k}} > 0$ with the property that $f(\bar{\tilde{k}}) - (\delta + g + n)\tilde{k} = 0$, as in Fig. 10.1.[12] Because $f'(\tilde{k}) > 0$ for all $\tilde{k} > 0$, it is implicit in the technology assumption (A2) that $\delta+g+n > 0$. Even without deciding on the sign of n (a decreasing workforce should not be excluded in our days), this seems like a plausible presumption.

Trajectories in the phase diagram

A first condition for a path $(\tilde{k}_t,\ \tilde{c}_t)$, with $\tilde{k}_t > 0$ and $\tilde{c}_t > 0$ for all $t \geq 0$, to be a solution to the model is that it satisfies the system of differential equations (10.28)-(10.29). Indeed, it must satisfy (10.28) to be technically

[11] The often presumed Inada conditions, $\lim_{\tilde{k}\to 0} f'(\tilde{k}) = \infty$ and $\lim_{\tilde{k}\to\infty} f'(\tilde{k}) = 0$, are stricter than (A2) and not necessary.

[12] We claim that $\bar{\tilde{k}} > \tilde{k}_{GR}$ *must* hold. Indeed, this inequality follows from $f'(\tilde{k}_{GR}) = \delta + n + g \equiv f(\bar{\tilde{k}})/\bar{\tilde{k}} > f'(\bar{\tilde{k}})$, the latter inequality being due to $f'' < 0$ and $f(0) \geq 0$.

feasible and it must satisfy (10.29) to comply with the Keynes-Ramsey rule. Technical feasibility of the path also requires that its initial value for $\tilde{k}$ equals the historically given (pre-determined) value $\tilde{k}_0 \equiv K_0/(T_0 L_0)$. In contrast, for $\tilde{c}$ we have no exogenously given initial value. This is because $\tilde{c}_0$ is a so-called *jump variable* or *forward-looking variable.* By this is meant an endogenous variable which can immediately shift to another value if new information arrives so as to alter expectations about the future. We shall see that the terminal condition (10.36), reflecting the transversality condition of the households, makes up for this lack of an initial condition for c.

In Fig. 10.1 we have drawn some possible paths that could be solutions as t increases. We are especially interested in the paths which are consistent with the historically given $\tilde{k}_0$, that is, paths starting at some point on the stippled vertical line in the figure. If the economy starts out with a high value of $\tilde{c}$, it will follow a curve like II in the figure. The low level of saving implies that the capital stock goes to zero in finite time (see Appendix C). If the economy starts out with a low level of $\tilde{c}$, it will follow a curve like III in the figure. The high level of saving implies that the capital intensity converges towards $\bar{\tilde{k}}$ in the figure.

All in all this suggests the existence of an initial level of consumption somewhere in between, which gives a path like I. Indeed, since the curve II emerged with a high $\tilde{c}_0$, then by lowering this $\tilde{c}_0$ slightly, a path will emerge in which the maximal value of $\tilde{k}$ on the $\dot{\tilde{k}} = 0$ locus is greater than curve II's maximal $\tilde{k}$ value.[13] We continue lowering $\tilde{c}_0$ until the path's maximal $\tilde{k}$ value is exactly equal to $\tilde{k}^*$. The path which emerges from this, namely the path I, starting at the point A, is special in that it converges towards the steady-state point E. No other path starting at the stippled line, $\tilde{k} = \tilde{k}_0$, has this property. Paths starting above A do not, as we just saw. It is similar for a path starting below A, like path III. Either this path never reaches the consumption level $\tilde{c}_A$ and then it can not converge to E, of course. Or, after a while its consumption level reaches $\tilde{c}_A$, but at the same time it has $\tilde{k} > \tilde{k}_0$. From then on, as long as $\tilde{k} \leq \tilde{k}^*$, for every $\tilde{c}$-value that path III has in common with path I, path III has a higher $\dot{\tilde{k}}$ and a lower $\dot{\tilde{c}}$ than path I (use (10.28) and (10.29)). Hence, path III diverges from point E.

Equivalently, had we considered a value of $\tilde{k}_0 > \tilde{k}^*$, there would also be a unique value of $\tilde{c}_0$ such that the path starting from $(\tilde{k}_0, \tilde{c}_0)$ would converge to E (see path IV in Fig. 10.1).

The point E is a *saddle point.* By this is meant a steady-state point with

[13] As an implication of the uniqueness theorem for differential equations, two solution paths in the phase plan cannot intersect.

the following property: there exists exactly two paths, one from each side of $\tilde{k}^*$, that converge towards the steady-state point; all other paths (at least starting in a neighborhood of the steady state) move away from the steady state and asymptotically approach one of the two diverging paths, the dotted North-West and South-East curves in Fig. 10.1. The two converging paths together make up what is known as the *stable branch* (or *stable arm)*; on their own they are referred to as *saddle paths* (sometimes referred to in the singular as the saddle path).[14] The dotted diverging paths in Fig. 10.1 together make up the *unstable branch (or unstable arm).*

The equilibrium path

A solution to the model is a path which is technically feasible and in addition satisfies a set of equilibrium conditions. In analogy with the definition in discrete time (see Chapter 3) a path $(\tilde{k}_t,\ \tilde{c}_t)_{t=0}^{\infty}$ is called a *technically feasible path* if (i) the path has $\tilde{k}_t \geq 0$ and $\tilde{c}_t \geq 0$ for all $t \geq 0$; (ii) it satisfies the accounting equation (10.28); and (iii) it starts out, at $t = 0$, with the historically given initial capital intensity. An *equilibrium path* with perfect foresight is then a technically feasible path $(\tilde{k}_t,\ \tilde{c}_t)_{t=0}^{\infty}$ with the properties that the path (a) is consistent with the households' optimization given their expectations; (b) is consistent with market clearing for all $t \geq 0$; and (c) has the property that the evolution over time of the pair $(w_t,\ r_t)$, where $w_t = \tilde{w}(\tilde{k}_t)T_t$ and $r_t = f'(\tilde{k}_t) - \delta$, is as expected by the households. The condition (a) in this definition requires the transformed Keynes-Ramsey rule (10.29) and the transversality condition (10.36) to hold for all $t \geq 0$.

Consider the case where $0 < \tilde{k}_0 < \tilde{k}^*$, as illustrated in Fig. 10.1. Then, the path starting at point A and following the saddle path towards the steady state is an equilibrium path because, by construction, it is technically feasible and in addition has the required properties, (a), (b), and (c). More intuitively: if the households expect an evolution of w_t and r_t corresponding to this path (that is, expect a corresponding underlying movement of $\tilde{k}_t$, which we know unambiguously determines r_t and w_t), then these expectations will induce a behavior the aggregate result of which is an actual path for $(\tilde{k}_t, \tilde{c}_t)$ that confirms the expectations. And along this path the households find no reason to correct their behavior because the path allows both the Keynes-Ramsey rule and the transversality condition to be satisfied.

No other path than the saddle path can be an equilibrium. This is because no other technically feasible path is compatible with the households' in-

[14] An algebraic definition of a saddle point, in terms of eigenvalues, is given in Appendix A. There it is also shown that if $\lim_{\tilde{k}\to 0} f(\tilde{k}) = 0$, then the saddle path on the left side of the steady state in Fig. 10.1 will start out infinitely close to the origin.

dividual utility maximization under perfect foresight. An initial point above point A can be excluded in that the implied path, II, does not satisfy the household's NPG condition (and, consequently, not at all the transversality condition).[15] So, if the individual household expected an evolution of r_t and w_t corresponding to path II, then the household would immediately choose a *lower* level of consumption, that is, the household would *deviate* in order not to suffer the same fate as Charles Ponzi. In fact *all* the households would react in this way. Thus path II would not be realized and the expectation that it would, can not be a rational expectation.

Likewise, an initial point below point A can be ruled out because the implied path, III, does not satisfy the household's transversality condition but implies over-saving. Indeed, at some point in the future, say at time t_1, the economy's capital intensity would pass the golden rule value so that for all $t > t_1$, $r_t < g + n$. But with a rate of interest permanently below the growth rate of wage income of the household, the present value of human wealth is *infinite*. This motivates a *higher* consumption level than that along the path. Thus, if the household expects an evolution of r_t and w_t corresponding to path III, then the household will immediately *deviate* and choose a higher initial level of consumption. But so will *all* the households react and the expectation that the economy will follow path III can not be rational.

We have presumed $0 < \tilde{k}_0 < \tilde{k}^*$. If instead $\tilde{k}_0 > \tilde{k}^*$, the economy would move along the saddle path *from above*. Paths like V and VI in Fig. 10.1 can be ruled out because they violate the NPG condition and the transversality condition, respectively. With this we have shown:

PROPOSITION 1 Assume (A1) and (A2). Let there be a given $\tilde{k}_0 > 0$. Then the Ramsey model exhibits a unique equilibrium path, characterized by $(\tilde{k}_t,\ \tilde{c}_t)$ converging, for $t \to \infty$, towards a unique steady state with a capital intensity $\tilde{k}^*$ satisfying $f'(\tilde{k}^*) - \delta = \rho + \theta g$. In the steady state the real interest rate is given by the modified golden rule formula, $r^* = \rho + \theta g$, the per capita consumption path is $c_t^* = \tilde{c}^* T_0 e^{gt}$, where $\tilde{c}^* = f(\tilde{k}^*) - (\delta + g + n)\tilde{k}^*$, and the real wage path is $w_t^* = \tilde{w}(\tilde{k}^*) T_0 e^{gt}$.

A numerical example based on one year as the time unit: $\theta = 2$, $g = 0.02$, $n = 0.01$ and $\rho = 0.01$. Then, $r^* = 0.05 > 0.03 = g + n$.

So output per capita, $y_t \equiv Y_t/L_t \equiv \tilde{y}_t T_t$, tends to grow at the rate of technological progress, g :

$$\frac{\dot{y}_t}{y_t} \equiv \frac{\dot{\tilde{y}}_t}{\tilde{y}_t} + \frac{\dot{T}_t}{T_t} = \frac{f'(\tilde{k}_t)\dot{\tilde{k}}_t}{f(\tilde{k}_t)} + g \to g \quad \text{for} \quad t \to \infty,$$

[15] This is shown in Appendix C.

in view of $\dot{\tilde{k}}_t \to 0$. This is also true for the growth rate of consumption per capita and the real wage, since $c_t \equiv \tilde{c}_t T_t$ and $w_t = \tilde{w}(\tilde{k}_t)T_t$.

The intuition behind the convergence lies in the neoclassical principle that starting from a low capital intensity and therefore high marginal and average product of capital, the resulting high aggregate saving[16] will be more than enough to maintain the capital intensity which therefore increases. But when this happens, the marginal and average product of capital decreases and the resulting saving, as a proportion of the capital stock, declines until eventually it is only sufficient to replace worn-out machines and equip new "effective" workers with enough machines to just maintain the capital intensity. If instead we start from a high capital intensity a similar story can be told in reverse. Thus in the Ramsey model the long-run state is attained when the marginal saving and investment yields a return as great as the representative household's willingness to postpone the marginal unit of consumption.

The equilibrium path generated by the Ramsey model is necessarily dynamically efficient and satisfies the modified golden rule in the long run. Why this contrast to Diamonds OLG model where equilibrium paths *may* be dynamically inefficient? The reason lies in the fact that only a "single infinity", not a "double infinity", is involved in the Ramsey model. The time horizon of the economy is infinite but the number of decision makers is finite. Births (into adult life) do not reflect the emergence of new economic agents with separate interests. It is otherwise in the Diamond OLG model where births imply the entrance of new economic decision makers whose preferences no-one has cared about in advance. In that model neither is there any final date, nor any final decision maker. It is this difference that lies behind that the two models in some respects give different results. A type of equilibria, namely dynamically inefficient ones, can be realized in the Diamond model but not so in the Ramsey model. A rate of time preference low enough to generate a *tendency* to a long-run interest rate below the income growth rate is inconsistent with the conditions needed for general equilibrium in the Ramsey model. And such a low rate of time preference is in fact ruled out in the Ramsey model by the parameter restriction (A1).

The concept of saddle-point stability

The steady state of the model is globally asymptotically stable for arbitrary initial values of the capital intensity (the phase diagram only verifies local asymptotic stability, but the extension to global asymptotic stability is verified

[16] Saving will be high because the substitution and wealth effects on current consumption of the high interest rate dominate the income effect.

in Appendix A). If $\tilde{k}$ is hit by a shock at time 0 (say by a discrete jump in the technology level T_0), the economy will converge toward the same unique steady state as before. At first glance this might seem peculiar considering that the steady state is a saddle point. Such a steady state is unstable for arbitrary initial values of *both* variables, $\tilde{k}$ and $\tilde{c}$. But the crux of the matter is that it is only the initial $\tilde{k}$ that *is* arbitrary. The model assumes that the decision variable c_0, and therefore the value of $\tilde{c}_0 \equiv c_0/T_0$, immediately adjusts to the given circumstances and information about the future. That is, the model assumes that $\tilde{c}_0$ always takes the value needed for the household's transversality condition under perfect foresight to be satisfied. This ensures that the economy is initially on the saddle path, cf. the point A in Fig. 10.1. In the language of differential equations *conditional* asymptotic stability is present. The condition that ensures the stability in our case is the transversality condition.

We shall follow the common terminology in macroeconomics and call a steady state of a two-dimensional dynamic system (locally) *saddle-point stable* if:

1. the steady state is a saddle point;
2. there is one predetermined variable and one jump variable; and
3. the saddle path is not parallel to the jump variable axis.

Thus, to establish saddle-point stability all three properties must be verified. If for instance point 1 and 2 hold but, contrary to point 3, the saddle path is parallel to the jump variable axis, then saddle-point stability does not obtain. Indeed, given that the predetermined variable initially deviated from its steady-state value, it would not be possible to find any initial value of the jump variable such that the solution of the system would converge to the steady state for $t \to \infty$

In the present case, we have already verified point 1 and 2. And as the phase diagram indicates, the saddle path is not vertical. So also point 3 holds. Thus, the Ramsey model is saddle-point stable. In Appendix A it is shown that the positively-sloped saddle path in Fig. 10.1 ranges over *all* $\tilde{k} > 0$ (there is nowhere a vertical asymptote to the saddle path). Hence, the steady state is *globally* saddle point stable. All in all, these characteristics of the Ramsey model are analogue to those of Barro's dynasty model in discrete time when the bequest motive is operative.

10.4 Comparative analysis

10.4.1 The role of key parameters

The conclusion that in the long run the real interest rate is given by the modified golden rule formula, $r^* = \rho + \theta g$, tells us that only three parameters matter: the rate of time preference, the elasticity of marginal utility, and the rate of technological progress. A higher ρ, i.e., more impatience and thereby less willingness to defer consumption, implies less capital accumulation and thus smaller capital intensity and in the long run a higher interest rate and lower consumption than otherwise. The long-run growth rate is unaffected.

A higher desire for consumption smoothing, θ, will have a similar effect in that it implies that a larger part of the greater consumption opportunities in the future (reflecting a positive g) will be consumed immediately. Similarly, the long-run interest rate will depend positively on g (the growth rate of labor productivity) because the higher this is, the greater is the expected future wage income and the associated consumption possibilities even without any current saving. This discourages current saving and we end up with lower capital accumulation and lower effective capital intensity in the long run, hence higher interest rate. It is also true that the higher is g, the higher is the rate of return needed to induce the saving required for maintaining a steady state and resist the desire for more consumption smoothing.

The long-run interest rate is independent of the particular form of the aggregate production function, f. This function matters for *what* effective capital intensity and *what* consumption level per unit of effective labor are compatible with the long-run interest rate. This kind of results are specific to representative agent models. This is because only in these models will the Keynes-Ramsey rule hold not only for the individual household, but also at the aggregate level.

Unlike the Solow growth model, the Ramsey model provides a *theory* of the evolution and long-run level of the saving rate. The endogenous gross saving rate of the economy is

$$\begin{aligned} s_t &\equiv \frac{Y_t - C_t}{Y_t} = \frac{\dot{K}_t + \delta K_t}{Y_t} = \frac{\dot{K}_t/K_t + \delta}{Y_t/K_t} = \frac{\dot{\tilde{k}}_t/\tilde{k}_t + g + n + \delta}{f(\tilde{k}_t)/\tilde{k}_t} \\ &\rightarrow \frac{g + n + \delta}{f(\tilde{k}^*)/\tilde{k}^*} \equiv s^* \quad \text{for} \quad t \rightarrow \infty. \end{aligned} \tag{10.38}$$

By determining the path of $\tilde{k}_t$, the Ramsey model determines how s_t moves over time and adjusts to its constant long-run level. Indeed, for any given $\tilde{k} > 0$, the equilibrium value of $\tilde{c}_t$ is uniquely determined by the requirement that the economy must be on the saddle path. Since this defines $\tilde{c}_t$ as a function, $\tilde{c}(\tilde{k}_t)$, of $\tilde{k}_t$, there is a corresponding function for the saving rate in that $s_t = 1 - \tilde{c}(\tilde{k}_t)/f(\tilde{k}_t) \equiv s(\tilde{k}_t)$; so $s(\tilde{k}^*) = s^*$.

We note that the long-run saving rate is a decreasing function of the rate of impatience, ρ, and the desire of consumption smoothing, θ; it is an increasing function of the capital depreciation rate, δ, and the rate of population growth, n.

For an example with an explicit formula for the long-run saving rate, consider:

EXAMPLE 1 Suppose the production function is Cobb-Douglas:

$$\tilde{y} = f(\tilde{k}) = A\tilde{k}^{\alpha}, \qquad A > 0, 0 < \alpha < 1. \tag{10.39}$$

Then $f'(\tilde{k}) = A\alpha\tilde{k}^{\alpha-1} = \alpha f(\tilde{k})/\tilde{k}$. In steady state we get, by use of the steady-state result (10.33),

$$\frac{f(\tilde{k}^*)}{\tilde{k}^*} = \frac{1}{\alpha}f'(\tilde{k}^*) = \frac{\delta + \rho + \theta g}{\alpha}.$$

Substitution in (10.38) gives

$$s^* = \alpha\frac{\delta + g + n}{\delta + \rho + \theta g} < \alpha, \tag{10.40}$$

where the inequality follows from our parameter restriction (A1). Indeed, (A1) implies $\rho + \theta g > g + n$. The long-run saving rate depends positively on the following parameters: the elasticity of production w.r.t. to capital, α, the capital depreciation rate, δ, and the population growth rate, n. The long-run saving rate depends negatively on the rate of impatience, ρ, and the desire for consumption smoothing, θ. The role of the rate of technological progress is ambiguous.[17]

It can be shown (see Appendix D) that if, by coincidence, $\theta = 1/s^*$, then $s'(\tilde{k}) = 0$, that is, the saving rate s_t is also outside of steady state equal to s^*. In view of (10.40), the condition $\theta = 1/s^*$ is equivalent to the "knife-edge" condition $\theta = (\delta+\rho)/\left[\alpha(\delta + g + n) - g\right] \equiv \bar{\theta}$. More generally, assuming $\alpha(\delta + g + n) > g$ (which seems likely empirically), we have that if $\theta \lesseqqgtr 1/s^*$ (i.e., $\theta \lesseqqgtr \bar{\theta}$), then $s'(\tilde{k}) \lesseqqgtr 0$, respectively (and if instead $\alpha(\delta + g + n) \leq g$, then $s'(\tilde{k}) < 0$, unconditionally).[18] Data presented in Barro and Sala-i-Martin

[17] Partial differentiation w.r.t. g yields $\partial s^*/\partial g = \alpha\left[\rho - \theta n - (\theta - 1)\delta\right]/(\delta + \rho + \theta g)^2$, the sign of which cannot be determined in general.

[18] See Appendix D.

(2004, p. 15) indicate no trend for the US saving rate, but a positive trend for several other developed countries since 1870. One interpretation is that whereas the US has for a long time been close to its steady state, the other countries are still in the adjustment process toward the steady state. As an example, consider the parameter values $\delta = 0.05$, $\rho = 0.02$, $g = 0.02$ and $n = 0.01$. In this case we get $\bar{\theta} = 10$ if $\alpha = 0.33$; given $\theta < 10$, these other countries should then have $s'(\tilde{k}) < 0$ which, according to the model, is compatible with a rising saving rate over time only if these countries are approaching their steady state from *above* (i.e., they should have $\tilde{k}_0 > \tilde{k}^*$). It may be argued that α should also reflect the role of education and R&D in production and thus be higher; with $\alpha = 0.75$ we get $\bar{\theta} = 1.75$. Then, if $\theta > 1.75$, these countries would have $s'(\tilde{k}) > 0$ and thus approach their steady state from *below* (i.e., $\tilde{k}_0 < \tilde{k}^*$). $\square$

10.4.2 Solow's growth model as a special case

The above results give a hint that Solow's growth model, with a given constant saving rate $s \in (0, 1)$ and given δ, g, and n (with $\delta + g + n > 0$), can, under certain circumstances, be interpreted as a special case of the Ramsey model. The Solow model is given by

$$\dot{\tilde{k}}_t = sf(\tilde{k}_t) - (\delta + g + n)\tilde{k}_t.$$

The constant saving rate implies proportionality between consumption and income which, in growth-corrected terms, per capita consumption is

$$\tilde{c}_t = (1 - s)f(\tilde{k}_t).$$

For the Ramsey model to yield this, the production function must be like in (10.39) (i.e., Cobb-Douglas) with $\alpha > s$. And the elasticity of marginal utility, θ, must satisfy $\theta = 1/s$. Finally, the rate of time preference, ρ, must be such that (10.40) holds with s^* replaced by s, which implies $\rho = \alpha(\delta+g+n)/s - \delta - \theta g$. It remains to show that this ρ satisfies the inequality, $\rho - n > (1 - \theta)g$, which is necessary for existence of an equilibrium in the Ramsey model. Since $\alpha/s > 1$, the chosen ρ satisfies $\rho > \delta+g+n - \delta - \theta g = n+(1-\theta)g$, which was to be proved. Thus, in this case the Ramsey model generates an equilibrium path which implies an evolution identical to that generated by the Solow model with $s = 1/\theta$.[19]

[19] A more elaborate account of the Solow model as a special case of the Ramsey model is given in Appendix D.

With this foundation of the Solow model, it will always hold that $s = s^* < s_{GR}$, where s_{GR} is the golden rule saving rate. Indeed, from (10.38) and (10.31), respectively,

$$s_{GR} = \frac{(\delta + g + n)\tilde{k}_{GR}}{f(\tilde{k}_{GR})} = \frac{f'(\tilde{k}_{GR})\tilde{k}_{GR}}{f(\tilde{k}_{GR})} = \alpha > s^*,$$

from the Cobb-Douglas specification and (10.40), respectively.

A point of the Ramsey model vis-a-vis the Solow model is to replace a mechanical saving rule by maximization of discounted utility and thereby, on the one hand, open up for a wider range of possible evolutions and on the other hand at the same time narrow the range in certain respects.

, the model opens up for studying welfare consequences of alternative economic policies.

10.5 A social planner's problem

Another implication of the Ramsey setup is that the decentralized market equilibrium (within the idealized presumptions of the model) brings about the same allocation of resources as would a social planner with the same criterion function as the representative household. As in Chapter 8, by a social planner we mean a hypothetical central authority who is "all-knowing and all-powerful". The social planner is constrained only by the limitations arising from technology and initial resources. Within these confines the social planner can fully decide on the resource allocation. Since we consider a closed economy, the social planner has no access to an international loan market.

Let the economy be closed and let the social welfare function be time separable with constant elasticity, $\hat{\theta}$, of marginal utility and a pure rate of time preference $\hat{\rho}$.[20] Then the social planner's optimization problem is

$$\max_{(c_t)_{t=0}^{\infty}} W_0 = \int_0^{\infty} \frac{c_t^{1-\hat{\theta}}}{1-\hat{\theta}} e^{-(\hat{\rho}-n)t} dt \quad \text{s.t.} \tag{10.41}$$

$$c_t \geq 0, \tag{10.42}$$

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \frac{c_t}{T_t} - (\delta + g + n)\tilde{k}_t, \tag{10.43}$$

$$\tilde{k}_t \geq 0 \quad \text{for all } t \geq 0. \tag{10.44}$$

We assume $\hat{\theta} > 0$ and $\hat{\rho} - n > (1 - \hat{\theta})g$ in line with the assumption (A1) for the market economy above. In case $\hat{\theta} = 1$, the expression $c_t^{1-\hat{\theta}}/(1-\hat{\theta})$ should

[20] Possible reasons for allowing these two preference parameters to deviate from the corresponding parameters in the private sector are given Chapter 8.

be interpreted as $\ln c_t$. The dynamic constraint (10.43) reflects the national product account. Because the economy is closed, the social planner does not have the opportunity of borrowing or lending from abroad and hence there is no solvency requirement. Instead we just impose the definitional constraint (10.44) of non-negativity of the state variable $\tilde{k}$. The problem is the continuous time analogue of the social planner's problem in discrete time in Chapter 8. Note, however, a minor conceptual difference, namely that in continuous time there is in the short run no *upper* bound on the *flow* variable c_t, that is, no bound like $c_t \leq T_t \left[f(\tilde{k}_t) - (\delta + g + n)\tilde{k}_t\right]$. A consumption intensity c_t which is higher than the right-hand side of this inequality will just be reflected in a negative value of the flow variable $\dot{\tilde{k}}_t$.[21]

To solve the problem we apply the Maximum Principle. The current-value Hamiltonian is

$$H(\tilde{k}, c, \lambda, t) = \frac{c^{1-\hat{\theta}}}{1-\hat{\theta}} + \lambda \left[f(\tilde{k}) - \frac{c}{T} - (\delta + g + n)\tilde{k}\right],$$

where λ is the adjoint variable associated with the dynamic constraint (10.43). An interior optimal path $(\tilde{k}_t, c_t)_{t=0}^{\infty}$ will satisfy that there exists a continuous function $\lambda = \lambda(t)$ such that, for all $t \geq 0$,

$$\frac{\partial H}{\partial c} = c^{-\hat{\theta}} - \frac{\lambda}{T} = 0, \text{ i.e., } c^{-\hat{\theta}} = \frac{\lambda}{T}, \quad \text{and} \tag{10.45}$$

$$\frac{\partial H}{\partial \tilde{k}} = \lambda(f'(\tilde{k}) - \delta - g - n) = (\hat{\rho} - n)\lambda - \dot{\lambda} \tag{10.46}$$

hold along the path and the transversality condition,

$$\lim_{t\to\infty} \tilde{k}_t \lambda_t e^{-(\hat{\rho}-n)t} = 0, \tag{10.47}$$

is satisfied.[22]

The condition (10.45) can be seen as a $MC = MB$ condition and illustrates that λ_t is the social planner's shadow price, measured in terms

[21] As usual we presume that capital can be "eaten". That is, we consider the capital good to be instantaneously convertible to a consumption good. Otherwise there *would* be at any time an upper bound on c, namely $c \leq Tf(\tilde{k})$, saying that the per capita consumption flow cannot exceed the per capita output flow. The role of such constraints is discussed in Feichtinger and Hartl (1986).

[22] The infinite-horizon Maximum Principle itself does not guarantee validity of such a straightforward extension of a necessary transversality condition from a finite horizon to an infinite horizon. Yet, this extension *is* valid for the present problem when $\hat{\rho} - n > (1-\hat{\theta})g$, cf. Appendix E.

of current utility, of $\tilde{k}_t$ along the optimal path.[23] The differential equation (10.46) tells us how this shadow price evolves over time. The transversality condition, (11.50), together with (10.45), entails the condition

$$\lim_{t\to\infty} \tilde{k}_t c_t^{-\hat{\theta}} e^{gt} e^{-(\hat{\rho}-n)t} = 0,$$

where the unimportant factor T_0 has been eliminated. Imagine the opposite were true, namely that $\lim_{t\to\infty} \tilde{k}_t c_t^{-\hat{\theta}} e^{[g-(\hat{\rho}-n)]t} > 0$. Then, intuitively U_0 could be increased by reducing the long-run value of $\tilde{k}_t$, i.e., consume more and save less.

By taking logs in (10.45) and differentiating w.r.t. t, we get $-\hat{\theta}\dot{c}/c = \dot{\lambda}/\lambda - g$. Inserting (10.46) and rearranging gives the condition

$$\frac{\dot{c}}{c} = \frac{1}{\hat{\theta}}(g - \frac{\dot{\lambda}}{\lambda}) = \frac{1}{\hat{\theta}}(f'(\tilde{k}) - \delta - \hat{\rho}). \tag{10.48}$$

This is the social planner's Keynes-Ramsey rule. If the rate of time preference, $\hat{\rho}$, is lower than the net marginal product of capital, $f'(\tilde{k}) - \delta$, the social planner will let per capita consumption be relatively low in the beginning in order to attain greater per capita consumption later. The lower the impatience relative to the return on capital, the more favorable it becomes to defer consumption.

Because $\tilde{c} \equiv c/T$, we get from (11.51) qualitatively the same differential equation for $\tilde{c}$ as we obtained in the decentralized market economy. And the dynamic resource constraint (10.43) is of course identical to that of the decentralized market economy. Thus, the dynamics are in principle unaltered and the phase diagram in Fig. 10.1 is still valid. The solution of the social planner implies that the economy will move along the saddle path towards the steady state. This trajectory, path I in the diagram, satisfies both the first-order conditions and the transversality condition. However, paths such as III in the figure do not satisfy the transversality condition of the social planner but imply permanent over-saving. And paths such as II in the figure will experience a sudden end when all the capital has been used up. Intuitively, they cannot be optimal. A rigorous argument is given in Appendix E, based on the fact that the Hamiltonian is *strictly concave* in $(\tilde{k}, \tilde{c})$. Thence, not only is the saddle path an optimal solution, it is the *unique* optimal solution.

Comparing with the market solution of the previous section, we have established:

[23] Decreasing c_t by one unit, increases $\tilde{k}_t$ by $1/T_t$ units, each of which are worth λ_t utility units to the social planner.

PROPOSITION 2 *(equivalence theorem)* Assume (A1) and (A2) with θ and ρ replaced by $\hat{\theta}$ and $\hat{\rho}$, respectively. Let there be a given $\tilde{k}_0 > 0$. Then the perfectly competitive market economy with a representative infinitely-lived household, without externalities, brings about the same resource allocation as that brought about by a social planner with the same criterion function as the representative household, i.e., with $\hat{\theta} = \theta$ and $\hat{\rho} = \rho$.

This is a continuous time analogue to the discrete time equivalence theorem of Chapter 8.

The capital intensity $\tilde{k}$ in the social planner's solution will not converge towards the golden rule level, $\tilde{k}_{GR}$, but towards a level whose distance to the golden rule level depends on how much $\hat{\rho} + \hat{\theta}g$ exceeds the natural growth rate, $g + n$. Even if society would be able to consume more in the long term if it aimed for the golden rule level, this would not compensate for the reduction in current consumption which would be necessary to achieve it. This consumption is relatively more valuable, the greater is the social planner's effective rate of time preference, $\hat{\rho} - n$. In line with the market economy, the social planner's solution ends up in a *modified golden rule.* In the long term, net marginal productivity of capital is determined by preference parameters and productivity growth and equals $\hat{\rho} + \hat{\theta}g > g + n$. Hereafter, given the net marginal productivity of capital, the capital intensity and the level of the consumption path is determined by the production function.

Classical versus average utilitarianism

In the above analysis the social planner maximizes the sum of discounted per capita utilities *weighed* by generation size. This is known as discounted *classical utilitarianism.* As an implication, the *effective* utility discount rate, $\rho - n$, varies negatively (one to one) with the population growth rate. Since this corresponds to how the per capita rate of return on saving, $r - n$, is "diluted" by population growth, the net marginal product of capital in steady state becomes independent of n, namely equal to $\hat{\rho} + \hat{\theta}g$.

An alternative to discounted classical utilitarianism is to maximize discounted *per capita* utility. This accords with the principle of discounted *average utilitarianism.* Here the social planner maximizes the sum of discounted per capita utilities without weighing by generation size. Then the effective utility discount rate is independent of the population growth rate, n. With $\hat{\rho}$ still denoting the pure rate of time preference, the criterion function becomes

$$W_0 = \int_0^{\infty} \frac{c_t{}^{1-\hat{\theta}}}{1-\hat{\theta}} e^{-\hat{\rho}t} dt.$$

The social planner's solution then converges towards a steady state with the net marginal product of capital

$$f'(\tilde{k}^*) - \delta = \hat{\rho} + n + \hat{\theta}g. \tag{10.49}$$

Here, an increase in n will imply higher long-run net marginal product of capital and lower capital intensity, everything else equal.

The representative household in the Ramsey model may of course also have a criterion function in line with discounted average utilitarianism, that is, $U_0 = \int_0^\infty u(c_t)e^{-\rho t}dt$. Then, the interest rate in the economy will in the long run be $r^* = \rho + n + \theta g$ and so an increase in n will increase r^* and decrease $\tilde{k}^*$.

Ramsey's original zero discount rate and the overtaking criterion

It was mostly the perspective of a social planner, rather than the market mechanism, which was at the center of Ramsey's original analysis. The case considered by Ramsey has $g = n = 0$. Ramsey maintained that the social planner should "not discount later enjoyments in comparison with earlier ones, a practice which is ethically indefensible and arises merely from the weakness of the imagination" (Ramsey 1928). So Ramsey has $\rho - n = \rho = 0$. Given the instantaneous utility function, u, where $u' > 0, u'' < 0$, and given $\rho = 0$, Ramsey's original problem was: choose $(c_t)_{t=0}^\infty$ so as to optimize (in some sense, see below)

$$\begin{aligned} W_0 &= \int_0^\infty u(c_t)dt \qquad \text{s.t.} \\ c_t &\geq 0, \\ \dot{k}_t &= f(k_t) - c_t - \delta k_t, \\ k_t &\geq 0 \qquad \text{for all } t \geq 0. \end{aligned}$$

A condition corresponding to our assumption (A1) above does not apply. So the improper integral W_0 will generally not be bounded and Ramsey can not use maximization of W_0 as an optimality criterion. Instead he considers a criterion akin to the overtaking criterion we considered in a discrete time context in the previous chapter. We only have to reformulate this criterion for a continuous time setting.

Let $(c_t)_{t=0}^\infty$ be the consumption path associated with an arbitrary technically feasible path and let $(\hat{c}_t)$ be the consumption path associated with our candidate as an optimal path, that is, the path we wish to test for optimality. Define

$$D_T \equiv \int_0^T u(\hat{c}_t)dt - \int_0^T u(c_t)dt. \tag{10.50}$$

Then the feasible path $(\hat{c}_t)_{t=0}^{\infty}$ is *overtaking optimal*, if for any feasible path, $(c_t)_{t=0}^{\infty}$, there exists a number $T' \geq 0$ such that $D_T \geq 0$ for all $T \geq T'$. That is, if for every alternative feasible path, the candidate path has from some date on, cumulative utility up to *all* later dates at least as great as that of the alternative feasible path, then the candidate path is overtaking optimal.

We say that the candidate path is *weakly preferred* in case we just know that $D_T \geq 0$ for all $T \geq T'$. If $D_T \geq 0$ can be replaced by $D_T > 0$, we say it is *strictly preferred*.[24]

Optimal control theory is also applicable with this criterion. The Hamiltonian is

$$H(k, c, \lambda, t) = u(c) + \lambda \left[f(k) - c - \delta k \right].$$

The Maximum Principle states that an interior overtaking-optimal path will satisfy that there exists an adjoint variable λ such that for all $t \geq 0$ it holds along this path that

$$\frac{\partial H}{\partial c} = u'(c) - \lambda = 0, \text{ and} \tag{10.51}$$

$$\frac{\partial H}{\partial k} = \lambda(f'(k) - \delta) = -\dot{\lambda}. \tag{10.52}$$

Since $\rho = 0$, the Keynes-Ramsey rule reduces to

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta(c_t)}(f'(k_t) - \delta), \qquad \text{where } \theta(c) \equiv -\frac{c}{u'(c)}u''(c).$$

One might conjecture that also the transversality condition,

$$\lim_{t \to \infty} k_t \lambda_t = 0, \tag{10.53}$$

is necessary for optimality but, as we will see below, this turns out to be wrong in this case with no discounting.

Our assumption (A2) here reduces to $\lim_{k \to 0} f'(k) > \delta > \lim_{k \to \infty} f'(k)$ (which requires $\delta > 0$). Apart from this, the phase diagram is fully analogue to that in Fig. 10.1, except that the steady state, E, is now at the top of the $\dot{k} = 0$ curve. This is because in steady state, $f'(k^*) - \delta = 0$, and this equation also defines k_{GR} in this case. It can be shown that the saddle path is again the unique solution to the optimization problem (by essentially the same method as in the discrete time case of Chapter 8).

A noteworthy feature is that in this case the Ramsey model constitutes a counterexample to the widespread presumption that an optimal plan with

[24] A more generally applicable optimality criterion is the *catching-up* criterion. The meaning of this criterion in continuous time is analogue to its meaning in discrete time, cf. Chapter 8.

infinite horizon *must* satisfy a transversality condition like (10.53). Indeed, by (10.51), $\lambda_t = u'(c_t) \to u'(c^*)$ for $t \to \infty$ along the overtaking-optimal path (the saddle path). Thus, instead of (10.53), we get

$$\lim_{t\to\infty} k_t \lambda_t = u'(c^*)k^* > 0.$$

With CRRA utility it is straightforward to generalize these results to the case $g \geq 0, n \geq 0$ and $\hat{\rho} - n = (1-\hat{\theta})g$. The social planner's overtaking-optimal solution is still the saddle path approaching the golden rule steady state; and this solution violates the seemingly "natural" transversality condition.

Note also that with zero effective utility discounting, there can not be equilibrium in the *market* economy version of this story. The real interest rate would in the long run be zero and thus the human wealth of the infinitely-lived household would be infinite. But then the demand for consumption goods would be unbounded and equilibrium thus be impossible.

10.6 Concluding remarks

The Ramsey model has played an important role as a way of structuring economists' thoughts about many macrodynamic phenomena including economic growth. The model should not be considered directly descriptive but rather as an examination of a benchmark case. As just noted this case is in some sense the opposite of the Diamond OLG model. Both models build on very idealized assumptions. Whereas the Diamond model ignores any bequest motive and emphasizes life-cycle behavior and at least some heterogeneity in the population, the Ramsey model implicitly assumes an altruistic bequest motive which is always operative and which turns households into homogeneous, infinitely-lived agents. In this way the Ramsey model ends up as an easy-to-apply framework, implying, among other things, a clear-cut theory of the level of the real interest rate in the long run. The model's usefulness lies in allowing general equilibrium analysis of an array of problems in a "vacuum". The fact that the assumption of perfect foresight in the Ramsey model always generates a *unique* equilibrium path makes the strong assumption of perfect foresight slightly less problematic than in the Diamond OLG model. In the latter model, as we saw in Chapter 3, multiple equilibria paths could arise under certain circumstances (and with this multiplicity the mystery about how coordination of expectations was brought about).

The next chapter discusses different applications of the Ramsey model. Because of the model's simplicity, one should always be aware of the risk of non-robust conclusions. The assumption of a representative household is a main limitation of the Ramsey model. It is not easy to endow the dynasty portrait of households with plausibility. One of the problems is, as argued by Bernheim and Bagwell (1988), that this portrait does not comply with the fact that families are interconnected in a complex way via marriage of partners coming from different parent families. And the lack of heterogeneity in the model's population of households implies a danger that important interdependencies between different classes of agents are unduly neglected. For some problems these interdependencies may be of only secondary importance, but for others (for instance, issues concerning public debt or interaction between private debtors and creditors) they are crucial.

Another critical limitation of the model comes from its reliance on saddle-point stability with the associated presumption of perfect foresight infinitely far out in the future. There can be good reasons for bearing in mind the following warning (by Solow, 1990, p. 221) against overly reliance on the Ramsey framework in the analysis of a market economy:

> "The problem is not just that perfect foresight into the indefinite future is so implausible away from steady states. The deeper problem is that in practice − if there is any practice − miscalculations about the equilibrium path may not reveal themselves for a long time. The mistaken path gives no signal that it will be "ultimately" infeasible. It is natural to comfort oneself: whenever the error is perceived there will be a jump to a better approximation to the converging arm. But a large jump may be required. In a decentralized economy it will not be clear who knows what, or where the true converging arm is, or, for that matter, exactly where we are now, given that some agents (speculators) will already have perceived the need for a mid-course correction while others have not. This thought makes it hard even to imagine what a long-run path would look like. It strikes me as more or less devastating for the interpretation of quarterly data as the solution of an infinite time optimization problem."

10.7 Literature notes

1. Frank Ramsey died at the age of 26 but he published several important articles. Ramsey discussed economic issues with, among others, John Maynard Keynes. In an obituary published in the Economic Journal (March 1932) some months after Ramsey's death, Keynes described Ramsey's article about the optimal savings as "one of the most remarkable contributions to mathematical economics ever made, both in respect of the intrinsic importance and difficulty of its subject, the power and elegance of the technical methods

employed, and the clear purity of illumination with which the writer's mind is felt by the reader to play about its subject".

2. The version of the Ramsey model we have considered is in accordance with the general tenet of neoclassical preference theory: saving is motivated only by higher consumption in the future. Other versions assume accumulation of wealth is also motivated by a desire for social prestige and economic and political power rather than consumption. In Kurz (1968b) an extended Ramsey model is studied where wealth is an independent argument in the instantaneous utility function.

3. The equivalence in the Ramsey model between the decentralized market equilibrium and the social planner's solution can be seen as an extension of the first welfare theorem as it is known from elementary textbooks, to the case where the market structure stretches infinitely far out in time, and the finite number of economic agents (families) face an infinite time horizon: in the absence of externalities etc., the allocation of resources under perfect competition will lead to a Pareto optimal allocation. The Ramsey model is indeed a special case in that all households are identical. But the result can be shown in a far more general setup, cf. Debreu (1954). The result, however, does not hold in overlapping generations models where an infinite number of new generations enter and the "interests" of the new households have not been accounted for in advance.

4. Cho and Graham (1996) consider the empirical question whether countries tend to be above or below their steady state. Based on the Penn World Table they find that *on average*, countries with a relatively low income per adult are above their steady state and that countries with a higher income are below.

10.8 Appendix

A. Algebraic analysis of the dynamics around the steady state

To supplement the graphical approach of Section 10.3 with an exact analysis of the adjustment dynamics of the model, we compute the Jacobian matrix for the system of differential equations (10.28) - (10.29):

$$J(\tilde{k},\tilde{c}) = \begin{bmatrix} \partial\dot{\tilde{k}}/\partial\tilde{k} & \partial\dot{\tilde{k}}/\partial\tilde{c} \\ \partial\dot{\tilde{c}}/\partial\tilde{k} & \partial\dot{\tilde{c}}/\partial\tilde{c} \end{bmatrix} = \begin{bmatrix} f'(\tilde{k}) - (\delta+g+n) & -1 \\ \frac{1}{\theta}f''(\tilde{k})\tilde{c} & \frac{1}{\theta}(f'(\tilde{k}) - \delta - \rho + \theta g) \end{bmatrix}.$$

Evaluated in the steady state this reduces to

$$J(\tilde{k}^*,\tilde{c}^*) = \begin{bmatrix} \rho - n - (1-\theta)g & -1 \\ \frac{1}{\theta}f''(\tilde{k}^*)\tilde{c}^* & 0 \end{bmatrix}$$

This matrix has the determinant

$$\frac{1}{\theta}f''(\tilde{k}^*)\tilde{c}^* < 0.$$

Since the product of the eigenvalues of the matrix equals the determinant, the eigenvalues are real and opposite in sign.

In standard math terminology a steady-state point in a two dimensional continuous-time dynamic system is called a *saddle point* if the associated eigenvalues are opposite in sign.[25] For the present case we conclude that the steady state is a saddle point. This mathematical definition of a saddle point is equivalent to that given in the text of Section 10.3. Indeed, with two eigenvalues of opposite sign, there exists, in a small neighborhood of the steady state, a stable arm consisting of two saddle paths which point in opposite directions. From the phase diagram in Fig. 10.1 we know that the stable arm has a positive slope. At least for $\tilde{k}_0$ sufficiently close to $\tilde{k}^*$ it is thus possible to start out on a saddle path. Consequently, there is a (unique) value of $\tilde{c}_0$ such that $(\tilde{k}_t, \tilde{c}_t) \to (\tilde{k}^*, \tilde{c}^*)$ for $t \to \infty$. Finally, the dynamic system has exactly one jump variable, $\tilde{c}$, and one predetermined variable, $\tilde{k}$. It follows that the steady state is (locally) *saddle-point stable.*

We claim that for the present model this can be strengthened to *global* saddle-point stability. Indeed, for *any* $\tilde{k}_0 > 0$, it is possible to start out on the saddle path. For $0 < \tilde{k}_0 \leq \tilde{k}^*$, this is obvious in that the extension of the saddle path towards the left reaches the y-axis at a non-negative value of $\tilde{c}^*$. That is to say that the extension of the saddle path cannot, according to the uniqueness theorem for differential equations, intersect the $\tilde{k}$-axis for $\tilde{k} > 0$ in that the positive part of the $\tilde{k}$-axis is a solution of (10.28) - (10.29).[26]

For $\tilde{k}_0 > \tilde{k}^*$, our claim can be verified in the following way: suppose, contrary to our claim, that there exists a $\tilde{k}_1 > \tilde{k}^*$ such that the saddle path does not intersect that region of the positive quadrant where $\tilde{k} \geq \tilde{k}_1$. Let $\tilde{k}_1$ be chosen as the smallest possible value with this property. The slope, $d\tilde{c}/d\tilde{k}$, of the saddle path will then have no upper bound when $\tilde{k}$ approaches $\tilde{k}_1$ from the left. Instead $\tilde{c}$ will approach ∞ along the saddle path. But then $\ln \tilde{c}$ will also approach ∞ along the saddle path for $\tilde{k} \to \tilde{k}_1$ $(\tilde{k} < \tilde{k}_1)$. It follows that $d \ln \tilde{c}/d\tilde{k} = (d\tilde{c}/d\tilde{k})/\tilde{c}$, computed along the saddle path, will have no upper

[25] Note the difference compared to the discrete time system in Chapter 8. In the discrete time system we have next period's $\tilde{k}$ and $\tilde{c}$ on the left-hand side of the dynamic equations, not the increase in $\tilde{k}$ and $\tilde{c}$, respectively. Therefore, the criterion for a saddle point is different in discrete time.

[26] Because the extension of the saddle path towards the left in Fig. 10.1 can not intersect the $\tilde{c}$-axis at a value of $\tilde{c} > f(0)$, it follows that if $f(0) = 0$, the extension of the saddle path ends up in the origin.

bound. Nevertheless, we have

$$\frac{d\ln\tilde{c}}{d\tilde{k}} = \frac{d\ln\tilde{c}/dt}{d\tilde{k}/dt} = \frac{\dot{\tilde{c}}/\tilde{c}}{\dot{\tilde{k}}} = \frac{\frac{1}{\theta}(f'(\tilde{k}) - \delta - \rho - \theta g)}{f(\tilde{k}) - \tilde{c} - (\delta + g + n)\tilde{k}}.$$

When $\tilde{k} \to \tilde{k}_1$ and $\tilde{c} \to \infty$, the numerator in this expression is bounded, while the denominator will approach $-\infty$. Consequently, $d\ln\tilde{c}/d\tilde{k}$ will approach zero from above, as $\tilde{k} \to \tilde{k}_1$. But this contradicts that $d\ln\tilde{c}/d\tilde{k}$ has no upper bound, when $\tilde{k} \to \tilde{k}_1$. Thus, the assumption that such a $\tilde{k}_1$ exists is false and our original hypothesis holds true.

B. Boundedness of the utility integral

We claimed in Section 10.3 that if the parameter restriction

$$\rho - n > (1 - \theta)g \tag{A1}$$

holds, then the utility integral, $U_0 = \int_0^\infty \frac{c^{1-\theta}}{1-\theta} e^{-(\rho-n)t} dt$, is bounded along the steady-state path, $c_t = \tilde{c}^* T_t$. The proof is as follows. For $\theta \neq 1$,

$$\begin{aligned}(1-\theta)U_0 &= \int_0^\infty c_t^{1-\theta} e^{-(\rho-n)t} dt = \int_0^\infty (c_0 e^{gt})^{1-\theta} e^{-(\rho-n)t} dt \\ &= c_0 \int_0^\infty e^{[(1-\theta)g-(\rho-n)]t} dt \\ &= \frac{c_0}{\rho - n - (1-\theta)g},\end{aligned} \tag{10.54}$$

by (A1). If $\theta = 1$, we get

$$U_0 = \int_0^\infty (\ln c_0 + gt) e^{-(\rho-n)t} dt,$$

which is also finite, in view of (A1) implying $\rho - n > 0$ in *this* case. It follows that also any path converging to the steady state will entail bounded utility, when (A1) holds.

On the other hand, suppose that (A1) does *not* hold, i.e., $\rho - n \leq (1-\theta)g$. Then by the third equality in (10.54) and $c_0 > 0$ follows that $U_0 = \infty$.

C. The diverging paths

In Section 10.3 we stated that paths of types *II* and *III* in the phase diagram in Fig. 10.1 can not be equilibria with perfect foresight. Given the expectation corresponding to any of these paths, every single household will choose to *deviate* from the expected path (i.e., deviate from the expected "average behavior" in the economy). We will now show this formally.

We first consider a path of type *III.* A path of this type will not be able to *reach* the horizontal axis in Fig. 10.1. It will only *converge* towards the point $(\bar{\bar{\tilde{k}}}, 0)$ for $t \rightarrow \infty$. This claim follows from the uniqueness theorem for differential equations with continuously differentiable right-hand sides. The uniqueness implies that two solution curves cannot intersect. And we see from (10.29) that the positive part of the x-axis is from a mathematical point of view a solution curve (and the point $(\bar{\bar{\tilde{k}}}, 0)$ is a trivial steady state). This rules out another solution curve hitting the x-axis.

The convergence of $\tilde{k}$ towards $\bar{\bar{\tilde{k}}}$ implies $\lim_{t\rightarrow\infty} r_t = f'(\bar{\bar{\tilde{k}}}) - \delta < g + n$, where the inequality follows from $\bar{\bar{\tilde{k}}} > \tilde{k}_{GR}$. So,

$$\lim_{t\rightarrow\infty} a_t e^{-\int_0^t (r_s - n)ds} = \lim_{t\rightarrow\infty} \tilde{k}_t e^{-\int_0^t (r_s - g - n)ds} = \lim_{t\rightarrow\infty} \tilde{k}_t e^{-\int_0^t (f'(\tilde{k}_s) - \delta - g - n)ds} = \bar{\bar{\tilde{k}}} e^{\infty} > 0. \tag{10.55}$$

Hence the transversality condition of the households is violated. Consequently, the household will choose higher consumption than along this path and can do so without violating the NPG condition.

Consider now instead a path of type *II.* We shall first show that if the economy follows such a path, then depletion of all capital occurs in finite time. Indeed, in the text it was shown that any path of type *II* will pass the $\dot{\tilde{k}} = 0$ locus in Fig. 10.1. Let t_0 be the point in time where this occurs. If path *II* lies above the $\dot{\tilde{k}} = 0$ locus for all $t \geq 0$, then we set $t_0 = 0$. For $t > t_0$, we have

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \tilde{c}_t - (\delta + g + n)\tilde{k}_t < 0.$$

By differentiation w.r.t. t we get

$$\ddot{\tilde{k}}_t = f'(\tilde{k}_t)\dot{\tilde{k}}_t - \dot{c}_t - (\delta + g + n)\dot{\tilde{k}}_t = [f'(\tilde{k}_t) - \delta - g - n]\dot{\tilde{k}}_t - \dot{c}_t < 0,$$

where the inequality comes from $\dot{\tilde{k}}_t < 0$ combined with $\tilde{k}_t < \tilde{k}_{GR} \Rightarrow f'(\tilde{k}_t) - \delta > f'(\tilde{k}_{GR}) - \delta = g + n$. Therefore, there exists a $t_1 > t_0 \geq 0$ such that

$$\tilde{k}_{t_1} = \tilde{k}_{t_0} + \int_{t_0}^{t_1} \dot{\tilde{k}}_t dt = 0,$$

as was to be shown. At time t_1, $\tilde{k}$ cannot fall any further and $\tilde{c}_t$ immediately drops to $f(0)$ and stay there hereafter.

Yet, this result does not in itself explain why the individual household will deviate from such a path. The individual household has a negligible impact on the movement of $\tilde{k}_t$ in society and correctly perceives r_t and w_t as essentially independent of its own consumption behavior. Indeed, the economy-wide $\tilde{k}$ is not the household's concern. What the household cares about is its own financial wealth and budget constraint. Nothing prevents the household from planning a negative financial wealth, a, and possibly a continuously declining financial wealth, if only the NPG condition,

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s-n)ds} \geq 0,$$

is satisfied.

But we can show that paths of type *II* will *violate* the NPG condition. The reasoning is as follows. The household plans to follow the Keynes-Ramsey rule. Given an expected evolution of r_t and w_t corresponding to path *II*, this will imply a planned gradual transition from positive financial wealth to debt. The transition to positive net debt, $\tilde{d}_t \equiv -\tilde{a}_t \equiv -a_t/T_t > 0$, takes place at time t_1 defined above.

The continued growth in the debt will meanwhile be so fast that the NPG condition is violated. To see this, note that the NPG condition implies the requirement

$$\lim_{t\to\infty} \tilde{d}_t e^{-\int_0^t (r_s-g-n)ds} \leq 0, \tag{NPG}$$

that is, the productivity-corrected debt, $\tilde{d}_t$, is allowed to grow in the long run only at a rate *less* than the growth-corrected real interest rate. For $t > t_1$ we get from the accounting equation $\dot{a}_t = (r_t - n)a_t + w_t - c_t$ that

$$\dot{\tilde{d}}_t = (r_t - g - n)\tilde{d}_t + \tilde{c}_t - \tilde{w}_t > 0,$$

where $\tilde{d}_t > 0$, $r_t > \rho+\theta g > g+n$, and where $\tilde{c}_t$ grows exponentially according to the Keynes-Ramsey rule, while $\tilde{w}_t$ is non-increasing in that $\tilde{k}_t$ does not grow. This implies

$$\lim_{t\to\infty} \frac{\dot{\tilde{d}}_t}{\tilde{d}_t} \geq \lim_{t\to\infty}(r_t - g - n),$$

which is in conflict with (NPG).

Consequently, the household will choose a lower consumption path and thus *deviate* from the reference path considered. Every household will do this and the evolution of r_t and w_t corresponding to path *II* is thus *not* an equilibrium with perfect foresight.

The conclusion is that all individual households understand that the only evolution which can be expected rationally is the one corresponding to the saddle path.

D. A constant saving rate as a special case

As we noted in Section 10.4 Solow's growth model can be seen as a special case of the Ramsey model. Indeed, a constant saving rate may, under certain conditions, emerge as an endogenous result in the Ramsey model.

Let the rate of saving, $(Y_t - C_t)/Y_t$, be s_t. We have generally

$$\tilde{c}_t = (1 - s_t)f(\tilde{k}_t), \qquad \text{and so} \tag{10.56}$$

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \tilde{c}_t - (\delta + g + n)\tilde{k}_t = s_t f(\tilde{k}_t) - (\delta + g + n)\tilde{k}_t. \tag{10.57}$$

In the Solow model the rate of saving is a constant, s, and we then get, by differentiating with respect to t in (10.56) and using (10.57),

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = f'(\tilde{k}_t)[s - \frac{(\delta + g + n)\tilde{k}_t}{f(\tilde{k}_t)}]. \tag{10.58}$$

By maximization of discounted utility in the Ramsey model, given a rate of time preference ρ and an elasticity of marginal utility θ, we get in equilibrium

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \frac{1}{\theta}(f'(\tilde{k}_t) - \delta - \rho - \theta g). \tag{10.59}$$

There will not generally exist a constant, s, such that the right-hand sides of (10.58) and (10.59), respectively, are the same for varying $\tilde{k}$ (that is, outside steady state). But Kurz (1968a) showed the following:

CLAIM Let δ, g, n, α, and θ be given. If the elasticity of marginal utility θ is greater than 1 and the production function is $\tilde{y} = A\tilde{k}^{\alpha}$ with $\alpha \in (1/\theta, 1)$, then a Ramsey model with $\rho = \theta\alpha(\delta + g + n) - \delta - \theta g$ will generate a constant saving rate $s = 1/\theta$. Thereby the same resource allocation and transitional dynamics arise as in the corresponding Solow model with $s = 1/\theta$.

Proof. Let $1/\theta < \alpha < 1$ and $f(\tilde{k}) = A\tilde{k}^{\alpha}$. Then $f'(\tilde{k}) = A\alpha\tilde{k}^{\alpha-1}$. The right-hand-side of the Solow equation, (10.58), becomes

$$A\alpha\tilde{k}^{\alpha-1}[s - \frac{(\delta + g + n)\tilde{k}_t}{A\tilde{k}^{\alpha}}] = sA\alpha\tilde{k}^{\alpha-1} - \alpha(\delta + g + n). \tag{10.60}$$

The right-hand-side of the Ramsey equation, (10.59), becomes

$$\frac{1}{\theta}A\alpha\tilde{k}^{\alpha-1} - \frac{\delta + \rho + \theta g}{\theta}.$$

By inserting $\rho = \theta\alpha(\delta + g + n) - \delta - \theta g$, this becomes

$$\begin{aligned} & \frac{1}{\theta} A\alpha \tilde{k}^{\alpha-1} - \frac{\delta + \theta\alpha(\delta + g + n) - \delta - \theta g + \theta g}{\theta} \\ = & \frac{1}{\theta} A\alpha \tilde{k}^{\alpha-1} - \alpha(\delta + g + n). \end{aligned} \tag{10.61}$$

For the chosen ρ we have $\rho = \theta\alpha(\delta + g + n) - \delta - \theta g > n + (1-\theta)g$, because $\theta\alpha > 1$ and $\delta + g + n > 0$. Thus, $\rho - n > (1-\theta)g$ and existence of equilibrium in the Ramsey model with this ρ is ensured. We can now make (10.60) and (10.61) the same by inserting $s = 1/\theta$. This also ensures that the two models require the same $\tilde{k}^*$ to obtain a *constant* $\tilde{c} > 0$. With this $\tilde{k}^*$, the requirement $\dot{\tilde{k}}_t = 0$ gives the same steady-state value of $\tilde{c}$ in both models, in view of (10.57). It follows that $(\tilde{k}_t, \tilde{c}_t)$ is the same in the two models for all $t \geq 0$. $\square$

On the other hand, maintaining $\tilde{y} = A\tilde{k}^\alpha$, but allowing $\rho \neq \theta\alpha(\delta + g + n) - \delta - \theta g$, so that $\theta \neq 1/s^*$, then $s'(\tilde{k}) \neq 0$, i.e., the Ramsey model does not generate a constant saving rate except in steady state. Defining s^* as in (10.40) and $\bar{\theta} \equiv (\delta + \rho)/\left[\alpha(\delta + g + n) - g\right]$, we have: When $\alpha(\delta + g + n) > g$ (which seems likely empirically), it holds that if $\theta \lesseqgtr 1/s^*$ (i.e., if $\theta \lesseqgtr \bar{\theta}$), then $s'(\tilde{k}) \lesseqgtr 0$, respectively; if instead $\alpha(\delta + g + n) \leq g$, then $\theta < 1/s^*$ and $s'(\tilde{k}) < 0$, unconditionally. These results follow by considering the slope of the saddle path in a phase diagram in the $(\tilde{k}, \tilde{c}/f(\tilde{k}))$ plane and using that $s(\tilde{k}) = 1 - \tilde{c}/f(\tilde{k})$, cf. Exercise 10.?? The intuition is that when $\tilde{k}$ is rising over time (i.e., society is becoming wealthier), then, when the desire for consumption smoothing is "high" (θ "high"), the prospect of high consumption in the future is partly taken out as high consumption already today, implying that saving is initially low, but rising over time until it eventually settles down in the steady state. But if the desire for consumption smoothing is "low" (θ "low"), saving will initially be high and then gradually fall in the process towards the steady state. The case where $\tilde{k}$ is falling over time gives symmetric results.

E. The social planner's solution

In the text of Section 10.5 we postponed some of the more technical details. First, by (A2), the existence of the steady state, E, and the saddle path in Fig. 10.1 is ensured. Solving the linear differential equation (10.46) gives $\lambda_t = \lambda_0 e^{-\int_0^t (f'(\tilde{k}_s) - \delta - \hat{\rho} - g)ds}$. Substituting this into the transversality condition (11.50) gives

$$\lim_{t\to\infty} e^{-\int_0^t (f'(\tilde{k}_s) - \delta - g - n)ds} \tilde{k}_t = 0, \tag{10.62}$$

where we have eliminated the unimportant positive factor $\lambda_0 = c_0^{-\hat{\theta}} T_0$.

This condition is essentially the same as the transversality condition (10.36) for the market economy and holds in the steady state, given the parameter restriction $\hat{\rho} - n > (1 - \hat{\theta})g$, which is satisfied in view of (A1). Thus, (10.62) also holds along the saddle path. Since we must have $\tilde{k} \geq 0$ for all $t \geq 0$, (10.62) has the form required by Mangasarian's sufficiency theorem. Thus, if we can show that the Hamiltonian is concave in $(\tilde{k}, c)$ for all $t \geq 0$, then the saddle path *is* a solution to the social planner's problem. And if we can show strict concavity, the saddle path is the *unique* solution. We have:

$$\begin{aligned}
\frac{\partial H}{\partial \tilde{k}} &= \lambda(f'(\tilde{k}) - (\delta + g + n)), \quad \frac{\partial H}{\partial c} = c^{-\hat{\theta}} - \frac{\lambda}{T}, \\
\frac{\partial^2 H}{\partial \tilde{k}^2} &= \lambda f''(\tilde{k}) < 0 \quad (\text{by } \lambda = c^{-\hat{\theta}} T > 0), \quad \frac{\partial^2 H}{\partial c^2} = -\hat{\theta} c^{-\hat{\theta}-1} < 0, \\
\frac{\partial^2 H}{\partial \tilde{k} \partial c} &= 0.
\end{aligned}$$

Thus, the leading principal minors of the Hessian matrix of H are

$$D_1 = -\frac{\partial^2 H}{\partial \tilde{k}^2} > 0, \quad D_2 = \frac{\partial^2 H}{\partial \tilde{k}^2} \frac{\partial^2 H}{\partial c^2} - \left(\frac{\partial^2 H}{\partial \tilde{k} \partial c} \right)^2 > 0.$$

Hence, H is strictly concave in $(\tilde{k}, c)$ and the saddle path is the unique optimal solution.

It also follows that the transversality condition (11.50) *is* a *necessary* optimality condition. Note that we have had to derive this conclusion in a different way than when solving the household's consumption/saving problem in Section 10.2. There we could appeal to a link between the No-Ponzi-Game condition (with strict equality) and the transversality condition to verify necessity of the transversality condition. But that proposition does not cover the social planner's problem where there is no NPG condition.

As to the diverging paths in Fig. 10.1, note that paths of type II (those paths which, as shown in Appendix C, in finite time deplete all capital) can not be optimal, in spite of the temporarily high consumption level. This follows from the fact that the saddle path is the unique solution. Finally, paths of type III in Fig. 10.1 behave as in (10.55) and thus violate the transversality condition (11.50), as claimed in the text.

10.9 Exercises

Macroeconomics 2 Short Note 1

30.09.2012. Christian Groth

Theory of the rate of return

This short note gives a summary of different circumstances that give rise to differences in the rate of return on different assets. We also provide a brief sketch of what macroeconomics can say about the general level around which these rates of return fluctuate.

In non-monetary models without uncertainty there is in equilibrium only one rate of return, r. If in addition there is perfect competition in all markets and no capital adjustment costs, as in simple neoclassical models (like the Diamond OLG model and the Ramsey model), then the equilibrium real interest rate is at any time equal to the net marginal product of capital ($r = \partial Y/\partial K - \delta$, standard notation). Moreover, under conditions ensuring "well-behavedness" of these models, they predict that the capital intensity, and thereby the marginal product of capital, adjusts over time to some long-run level (on which more below).

Different rates of return It goes without saying that the link between the real interest rate and the marginal product of capital is loosened by the need for means of payments. Or rather, this link is loosened by many factors among which money is just one. As we will see in Chapter 14, existence of convex *capital adjustment costs* loosens the link between r and $\partial Y/\partial K$. The adjustment costs create a wedge between the price of investment goods and the market value of the marginal unit of installed capital. Besides the marginal product of capital, the possible capital gain in the market value of installed capital as well as the effect of the marginal unit of installed capital on future installation costs enter as co-determinants of the current rate of return on capital.

When *imperfect competition* on the output markets rules, prices are typically set as a mark-up on marginal cost. This implies a wedge between the net marginal product of capital and capital costs. And when *uncertainty* and limited opportunities for risk spreading are added to the model, a wide spectrum of expected rates of return on different financial assets and expected marginal products of capital in different production

	Arithmetic average	Standard deviation	Geometric average
	---------------- Percent	----------------	----------------
Nominal values			
Small Company Stocks	17,3	33,2	12,5
Large Company Stocks	12,7	20,2	10,7
Long-Term Corporate Bonds	6,1	8,6	5,8
Long-Term Government Bonds	5,7	9,4	5,3
Intermediate-Term Government Bonds	5,5	5,7	5,3
U.S. Treasury Bills	3,9	3,2	3,8
Cash	0,0	0,0	0,0
Inflation rate	3,1	4,4	3,1
Real values			
Small Company Stocks	13,8	32,6	9,2
Large Company Stocks	9,4	20,4	7,4
Long-Term Corporate Bonds	3,1	9,9	2,6
Long-Term Government Bonds	2,7	10,6	2,2
Intermediate-Term Government Bonds	2,5	7,0	2,2
U.S. Treasury Bills	0,8	4,1	0,7
Cash	-2,9	4,2	-3,0

Table 1: Average annual rates of return on a range of U.S. asset portfolios, 1926-2001. Source: Stocks, Bonds, Bills, and Inflation: Yearbook 2002, Valuation Edition. Ibbotson Associates, Inc.

sectors arise, depending on the risk profiles of the different assets and production sectors. Moreover, the presence of taxation, and sometimes differential taxation on different asset returns, complicates the picture.

Table 1 reports the nominal and real average annual rates of return on a range of U.S. asset portfolios for the period 1926–2001. The portfolio of small company stocks had an average annual real rate of return of 13.8 per cent (the arithmetic average) or 9.2 per cent (the geometric average) throughout the period 1926-2001. This is more than the annual rate of return of any of the other considered portfolios. Small company stocks are also seen to be the most volatile. The standard deviation of the annual real rate of return of the portfolio of small company stocks is almost eight times higher than that of the portfolio of U.S. Treasury bills (government zero coupon bonds with 30 days to maturity), with an average annual real return of only 0.8 per cent (arithmetic average) or 0.7 per cent (geometric average) throughout the period. Explanation in terms of risk aversion is in line with the displayed positive relation between high returns and high volatility. Yet, interpreting volatility as a rough measure of risk, the pattern is not without exceptions. The portfolio of long-term corporate bonds has performed better than the portfolio of long-term government bonds, although they have been slightly less volatile as here measured. But the data is historical, expectations are not always met, and risk depends significantly on the covariance of asset returns for assets in broader defined portfolios as well as correlation of asset returns with the business cycle, a feature about which Table 1 has nothing to say; share prices are in fact very sensitive to business cycle fluctuations.

The need for means of payment – money – further complicates the picture. That is, besides differences in risk and expected return across different assets, also dissimilarities in their degree of liquidity are important, not least in times of financial crisis. The expected real rate of return on cash holding is minus the expected rate of inflation and is therefore negative in an economy with inflation, cf. the last row in Table 1. When agents nevertheless hold money in their portfolios, it is because the low rate of return is compensated by the *liquidity* services of money. In the Sidrauski model of Chapter 17 this is modeled in a simple (albeit ad hoc) way by including real money holdings directly as an argument in the utility function. Another dimension along which the presence of money interferes with returns is through inflation. Real assets, like physical capital, land, houses, etc. are better protected against fluctuating inflation than are nominally denominated bonds (and money of course).

Without claiming too much we can say that investors facing these multiple rates of return choose a portfolio composition so as to balance the need for liquidity, the wish for a high expected return, and the wish for low risk. Finance theory teaches us that adjusted for differences in risk and liquidity, asset returns tend to be the same. This raises the question: at what level? This is where macroeconomics – as a theory about the economy as a whole – comes to the fore.

Macroeconomic theory of the trend level of rates of return The point of departure is that market forces by and large tend to anchor the rate of return of an average portfolio to the net marginal product of capital in an aggregate production function, assuming a closed economy. Some popular phrases are:

- the net marginal product of capital acts as a centre of gravitation for asset returns; and
- movements of the rates of return are in the long run held in check by the net marginal product of capital.

Though such phrases seem to convey the right flavour, in themselves they are not very informative. The net marginal product of capital is not a given, but an endogenous variable which, via changes in the capital intensity, adjusts through time to more fundamental factors in the economy.

The different macroeconomic models we have studied in previous chapters bring to mind different presumptions about what these fundamental factors are.

1. Solow's growth model The Solow growth model (Solow 1956) leads to the fundamental differential equation (standard notation)

$$\dot{\tilde{k}}_t = sf(\tilde{k}_t) - (\delta + g + n)\tilde{k}_t, \qquad \tilde{k}_0 > 0 \text{ given},$$

where s is an exogenous and constant aggregate saving rate, $0 < s < 1$. In steady state

$$r^* = f'(\tilde{k}^*) - \delta, \tag{1}$$

where $\tilde{k}^*$ is the unique steady state value of the (effective) capital intensity, $\tilde{k}$, satisfying

$$sf(\tilde{k}^*) = (\delta + g + n)\tilde{k}^*. \tag{2}$$

In society there is a debate and a concern that changed demography and less growth in the source of new technical ideas, i.e., the stock of educated human beings, will in the future result in lower n and lower g, making financing old-age security harder. On the basis of the Solow model we find by implicit differentiation in (2) $\partial\tilde{k}^*/\partial n = \partial\tilde{k}^*/\partial g = -\tilde{k}^*\left[\delta + g + n - sf'(\tilde{k}^*)\right]^{-1}$, which is negative since $sf'(\tilde{k}^*) < sf(\tilde{k}^*)/\tilde{k}^* = \delta + g + n$. Hence, by (1),

$$\frac{\partial r^*}{\partial n} = \frac{\partial r^*}{\partial g} = \frac{\partial r^*}{\partial \tilde{k}^*}\frac{\partial \tilde{k}^*}{\partial n} = f''(\tilde{k}^*)\frac{-\tilde{k}^*}{\delta + g + n - sf'(\tilde{k}^*)} > 0,$$

since $f''(\tilde{k}^*) < 0$. It follows that

$$n \downarrow \text{ or } g \downarrow \Rightarrow r^* \downarrow. \tag{3}$$

2. The Diamond OLG model The Diamond OLG model also just concludes that $r^* = f'(\tilde{k}^*) - \delta$. Like in the Solow model, the long-run rate of return thus depends on the aggregate production function and on $\tilde{k}^*$, which in turn may depend in a complicated way on the lifetime utility function and the production function. The steady state of a well-behaved Diamond model will nevertheless have the same qualitative property as indicated in (3).

3. The Ramsey model In contrast to the Solow and Diamond models, the Ramsey model implies not only that $r^* = f'(\tilde{k}^*) - \delta$, but also that the net marginal product of capital converges in the long run to a specific value given by the *modified golden rule* formula. In a continuous time framework this formula says:

$$r^* = \rho + \theta g, \tag{4}$$

where the new parameter, θ, is the (absolute) elasticity of marginal utility of consumption. Because the Ramsey model is a representative agent model, the Keynes-Ramsey rule holds not only at the individual level, but also at the aggregate level. This is what gives rise to this simple formula for r^*.

Here there is no role for n, only for g. On the other hand, there is an alternative specification of the Ramsey model, namely the "discounted average utilitarianism" specification. In this version of the Ramsey model, we get $r^* = f'(\tilde{k}^*) - \delta = \rho + n + \theta g$, so that not only a lower g, but also a lower n implies lower r^*.

Also the Sidrauski model, i.e., the monetary Ramsey model of Chapter 17, results in the *modified golden rule* formula.

4. Blanchard's OLG model A continuous time model with OLG structure and emphasis on life-cycle aspects is Blanchard's OLG model (Blanchard 1985). In that model the net marginal product of capital adjusts to a value within an interval:

$$\rho + g - \lambda < r^* < \rho + g + b,$$

where two additional parameters appear, the retirement rate λ (reflecting how early in life the "average" person retire from the labor market) and the crude birth rate b ($\theta = 1$ for simplicity). The population growth rate is the difference between the crude birth rate, b, and the crude mortality rate, m, so that $n = b - m$. The qualitative property indicated in (3) becomes conditional. It still holds if the fall in n reflects a lower b, but not necessarily if it reflects a higher m.

5. What if technological change is embodied? The models in the list above assume a neoclassical aggregate production function with CRS and *disembodied* Harrod-neutral technological progress, that is,

$$Y = F(K, TL) \equiv TLf(\tilde{k}), \qquad f' > 0, f'' < 0. \tag{5}$$

This amounts to assuming that new technical knowledge advances the combined productivity of capital and labor *independently* of whether the workers operate old or new machines.

In contrast, we say that technological change is *embodied* if taking advantage of new technical knowledge requires construction of new investment goods. The newest technology is incorporated in the design of newly produced equipment; and this equipment will not participate in subsequent technological progress. Both intuition and empirics suggest that most technological progress is of this form. Indeed, Greenwood et al. (1997) estimate for the U.S. 1950-1990 that embodied technological change explains 60% of the growth in output per man hour.

So a theory of the rate of return should take this into account. Fortunately, this can be done with only minor modifications. We assume that the link between investment and capital accumulation takes the form

$$\dot{K}_t = Q_t I_t - \delta K_t, \tag{6}$$

where I_t is gross investment ($I = Y - C$) and Q_t measures the "quality" (productivity) of newly produced investment goods. Suppose for instance that $Q_t = Q_0 e^{\gamma t}$, $\gamma > 0$. Then, even if no technological change directly appears in the production function, that is, even if (5) is replaced by

$$Y = F(K, L) = K^{\alpha} L^{1-\alpha}, \quad 0 < \alpha < 1,$$

the economy will still experience a rising standard of living.[1] A given level of gross investment will give rise to a greater and greater additions to the capital stock K, measured in efficiency units. Since at time t, Q_t capital goods can be produced at the same cost as one consumption good, the price, p_t, of capital goods in terms of the consumption good must in competitive equilibrium equal the inverse of Q_t, that is, $p_t = 1/Q_t$. In this way embodied technological progress results in a steady decline in the relative price of capital equipment.

This prediction is confirmed by the data. Greenwood et al. (1997) find for the U.S. that the relative price of capital equipment has been declining at an average rate of 0.03 per year in the period 1950-1990, a trend that has seemingly been fortified in the wake of the computer revolution.

[1] We specify F to be Cobb-Douglas, because otherwise a model with embodied technical progress in the form (6) will not be able to generate balanced growth and comply with Kaldor's stylized facts.

Along a balanced growth path the constant growth rate of K will now exceed that of Y, and Y/K thus be falling. The output-capital ratio in value terms, $Y/(pK)$, will be constant, however. Embedding these features in a Ramsey-style framework, we find the long-run rate of return to be[2]

$$r^* = \rho + \theta\frac{\alpha\gamma}{1-\alpha}.$$

This is of exactly the same form as (4), if we define $g = \alpha\gamma/(1-\alpha)$.

0.0.1 Adding uncertainty and risk of bankruptcy

Although absent from many simple macroeconomic models, uncertainty and risk of bankruptcy are significant features of reality. Bankruptcy risk may lead to a conflict of interest between share owners and managers. Managers may want less debt and more equity than the share owners because bankruptcy can be very costly to managers who loose a well-paid job and a promising carrier. So managers are unwilling to finance all new capital investment by new debt in spite of the associated lower capital cost (there is generally a lower rate of return on debt than on equity). In this way the excess of the rate of return on equity over that on debt, the equity premium, is sustained.

A rough, behavioral theory of the equity premium goes as follows.[3] Firm managers prefer a payout structure with a fraction, s_f, going to equity and the remaining fraction, $1 - s_f$, to debt (corporate bonds). That is, out of each unit of expected operating profit, managers are unwilling to commit more than $1 - s_f$ to bond owners. This is to reduce the risk of a failing payment ability in case of a bad market outcome. And those who finance firms by loans definitely also want debtor firms to have some equity at stake.

We let households' preferred portfolio consist of a fraction s_h in equities and the remainder, $1 - s_h$, in bonds. In view of households' risk aversion and memory of historical stock market crashes, it is plausible to assume that $s_h < s_f$.

As a crude adaptation of for instance the Blanchard OLG model to these features, we interpret the model's r^* as an average rate of return across firms. Let time be discrete and let aggregate financial wealth be $A = pK$, where p is the price of capital equipment in terms of consumption goods. In standard versions of the above models we have $p \equiv 1$, but under item 5 above relative price. Anyway, given A at time t, the aggregate gross return or payout is $(1 + r^*)A$. Out of this, $(1 + r^*)As_f$ constitutes the gross return to the

[2]See Appendix.

[3]This builds on Baker et al. (2005).

equity owners and $(1+r^*)A(1-s_f)$ the gross return to the bond owners. Let r_e denote the rate of return on equity and r_b the rate of return on bonds.

To find r_e and r_b we have

$$\begin{aligned}(1+r_e)As_h &= (1+r^*)As_f, \\ (1+r_b)A(1-s_h) &= (1+r^*)A(1-s_f).\end{aligned}$$

Thus,

$$\begin{aligned}1+r_e &= (1+r^*)\frac{s_f}{s_h} > 1+r^*, \\ 1+r_b &= (1+r^*)\frac{1-s_f}{1-s_h} < 1+r^*.\end{aligned}$$

We may define the *equity premium,* π, by $1+\pi \equiv (1+r_e)/(1+r_b)$. Then

$$\pi = \frac{s_f(1-sh)}{s_h(1-s_f)} - 1 > 0.$$

Of course these formulas have their limitations. The key variables s_f and s_h will depend on a lot of economic circumstances and should be endogenous in an elaborate model. Yet, the formulas may be helpful as a way of organizing one's thoughts about rates of return in a world with asymmetric information and risk of bankruptcy.

There is evidence that in the last decades of the twentieth century the equity premium had become lower than in the long aftermath of the Great Depression in the 1930s.[4] A likely explanation is that s_h had gone up along with rising confidence; the computer and the World Wide Web have made it much easier for individuals to invests in stocks of shares. On the other hand, the recent global financial and economic crisis, the Great Recession 2008- , and the associated rise in mistrust may have halted and possibly reversed this tendency for some time.

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[4] Blanchard (2003, p. 333).

Chapter 11

Applications of the Ramsey model

General introduction not yet available, except this:

There are at present two main sections:

11.1 Market economy with a public sector.

11.2 Learning by investing, welfare comparisons, and first-best policy.

11.1 Market economy with a public sector

In this section we extend the Ramsey model of a competitive market economy by adding a government that spends on goods and services, makes transfers to the private sector, and levies taxes.

Section 11.1.1 considers the effect of government spending on goods and services, assuming a balanced budget where all taxes are lump sum. The section also has a subsection about how to model effects of once-for-all shocks in a perfect foresight model. In the next two sections the focus shifts to income taxation. Finally, Section 11.1.4 introduces financing by temporary budget deficits. In view of the Ramsey model being a representative agent model, it is not surprising that Ricardian equivalence will hold in the model.

11.1.1 The effect of public spending

The representative household has $L_t = L_0 e^{nt}$ members each of which supplies one unit of labor inelastically per time unit. The household's preferences can be represented by a time separable utility function

$$\int_0^\infty \tilde{u}(c_t, G_t) L_t e^{-\rho t} dt,$$

where $c_t \equiv C_t/L_t$ is consumption per family member and G_t is the flow of a public service delivered by the government, while ρ is the pure rate of time preference. We assume that the instantaneous utility function is additive: $\tilde{u}(c, G) = u(c) + v(G)$, where $u' > 0, u'' < 0$, $v' > 0$, i.e., there is positive but diminishing marginal utility of private consumption and positive marginal utility of public consumption (whether $v'' < 0$ or not is immaterial for the questions we are going to study). The public service consists in making a certain public good (a non-rival good, say TV-transmitted theatre) available for everybody free of charge.

To allow balanced growth under technological progress we further assume that u is a CRRA function. Thus, the criterion function of the representative household can be written

$$U_0 = \int_0^\infty \left(\frac{c_t^{1-\theta}}{1-\theta} + v(G_t) \right) e^{-(\rho-n)t} dt, \tag{11.1}$$

where $\theta > 0$ is the constant (absolute) elasticity of marginal utility of private consumption.

In this section the government budget is always balanced and the spending, G_t, is financed by a per capita lump-sum tax, τ_t, so that

$$\tau_t L_t = G_t.$$

As usual, let the real interest rate and the real wage be denoted r_t and w_t, respectively. The household's dynamic book-keeping equation reads

$$\dot{a}_t = (r_t - n)a_t + w_t - \tau_t - c_t, \qquad a_0 \text{ given}, \tag{11.2}$$

where a_t is per capita financial wealth in the household and τ_t is the per capita lump-sum tax rate at time t. The financial wealth is assumed held in financial claims of a form similar to a variable-rate deposit in a bank. Hence, at any point in time a_t is historically determined and independent of the current and future interest rates.

GDP is produced by an aggregate neoclassical production function with CRS:

$$Y_t = F(K_t, \mathcal{T}_t L_t),$$

where K_t and L_t are input of capital and labor, respectively, and $\mathcal{T}_t$ is the technology level, assumed to grow at the constant rate $g \geq 0$. For simplicity, we assume that F satisfies the Inada conditions. It is further assumed that the production of G_t applies the same technology, and therefore involves the same unit production costs, as the other components of GDP. For simplicity, a possible role of G_t for productivity is ignored. The economy is closed and there is perfect competition in all markets.

General equilibrium

The increase in the capital stock, K, per time unit equals aggregate gross saving:

$$\dot{K}_t = Y_t - C_t - G_t - \delta K_t = F(K_t, \mathcal{T}_t L_t) - c_t L_t - G_t - \delta K_t, \qquad K_0 > 0 \text{ given.} \tag{11.3}$$

We assume G_t is proportional to the work force measured in efficiency units, that is $G_t = \tilde{\gamma}\mathcal{T}_t L_t$, where $\tilde{\gamma} \geq 0$ is decided by the government. In growth-corrected form the dynamic aggregate resource constraint (11.3) is

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \tilde{c}_t - \tilde{\gamma} - (\delta + g + n)\tilde{k}_t, \qquad \tilde{k}_0 > 0 \text{ given,} \tag{11.4}$$

where $\tilde{k}_t \equiv K_t/(\mathcal{T}_t L_t) \equiv k_t/\mathcal{T}_t$, $\tilde{c}_t \equiv c_t/\mathcal{T}_t$, and f is the production function on intensive form, $f' > 0, f'' < 0$. As F satisfies the Inada conditions, we have

$$\lim_{\tilde{k}\to 0} f'(\tilde{k}) = \infty, \quad \lim_{\tilde{k}\to\infty} f'(\tilde{k}) = 0.$$

In view of the additive instantaneous utility function in (11.1), marginal utility of private consumption is not affected by G_t. Therefore the Keynes-Ramsey rule of the household will be as if there were no government sector:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho).$$

In equilibrium the real interest rate, r_t, equals $f'(\tilde{k}_t) - \delta$. Expressed in terms of technology-corrected per capita consumption, $\tilde{c}$, the Keynes-Ramsey rule thus becomes

$$\dot{\tilde{c}} = \frac{1}{\theta}\left[f'(\tilde{k}) - \delta - \rho - \theta g\right]\tilde{c}. \tag{11.5}$$

Expressed in terms of $\tilde{k}$, the transversality condition of the household can be written

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t \left(f'(\tilde{k}_s) - \delta - n - g\right)ds} = 0. \tag{11.6}$$

The phase diagram of the dynamic system (11.4) - (11.5) is shown in Fig. 11.1. Apart from a vertical downward shift of the $\dot{\tilde{k}} = 0$ locus, due to $\tilde{\gamma} > 0$, the phase diagram is similar to that of the Ramsey model without government. The lump-sum tax has no immediately visible effect on resource allocation. At least the lump-sum tax is not visible in the reduced form of the model consisting of (11.4), (11.5), and (11.6). Yet, the lump-sum tax is indirectly present because it ensures that for all $t \geq 0$, the $\tilde{c}_t$ and $\dot{\tilde{k}}_t$ appearing in (11.4) represent exactly the consumption demand and net saving coming from the households' intertemporal budget constraint (which depends on these taxes). Otherwise, equilibrium would not be maintained.

We assume that $\tilde{\gamma}$ is of "moderate size" compared to the productive capacity of the economy so as to not rule out the existence of a steady state. Moreover, to guarantee bounded discounted utility and existence of general equilibrium, we impose the parameter restriction

$$\rho - n > (1-\theta)g. \tag{A1}$$

How to model effects of once-for-all policy shifts

In a perfect foresight model, as the present one, agents' expectations and actions never incorporate that unanticipated events, "shocks", may arrive. That is, if a shock occurs in historical time, it must be treated as a complete surprise, a "once-for-all" shock not expected to be replicated in any sense.

Suppose that up until time $t_0 > 0$ government spending maintains the constant given ratio $G_t/(\mathcal{T}_t L_t) = \tilde{\gamma}$. Suppose further that before time t_0, the households expected this state of affairs to continue forever. But, unexpectedly, at time t_0 there is a shift to a higher constant spending ratio, $\tilde{\gamma}'$, which is maintained for a long time.

We assume that the upward shift in public spending goes hand in hand with higher lump-sum taxes so that the after-tax human wealth of the household is at time t_0 immediately reduced. As the household sector is thereby less wealthy, private consumption drops.

Mathematically, the time path of c_t will have a discontinuity at $t = t_0$. To fix ideas we will generally consider *control* variables, e.g., consumption, to be *right-continuous* functions of time in such cases. That is, here $c_{t_0} = \lim_{t \to t_0^+} c_t$. Likewise, at such switch points the time derivative of the *state* variable a in (11.2) should be interpreted as the *right-hand time derivative*, i.e., $\dot{a}_{t_0} = \lim_{t \to t_0^+}(a_t - a_{t_0})/(t - t_0)$.[1] We say that the control variable has a *jump* at time t_0 and the state variable, which remains a continuous function of t, has a *kink* at time t_0.

Correspondingly, control variables are in economics often called *jump variables* or *forward-looking variables.* The latter name comes from the notion that a decision variable can immediately shift to another value if new information arrives so as to alter the expected circumstances conditioning

[1] While these conventions help intuition, they are mathematically inconsequential. Indeed, the value of the consumption intensity at each isolated point of discontinuity will affect neither the utility integral of the household nor the value of the state variable, a.

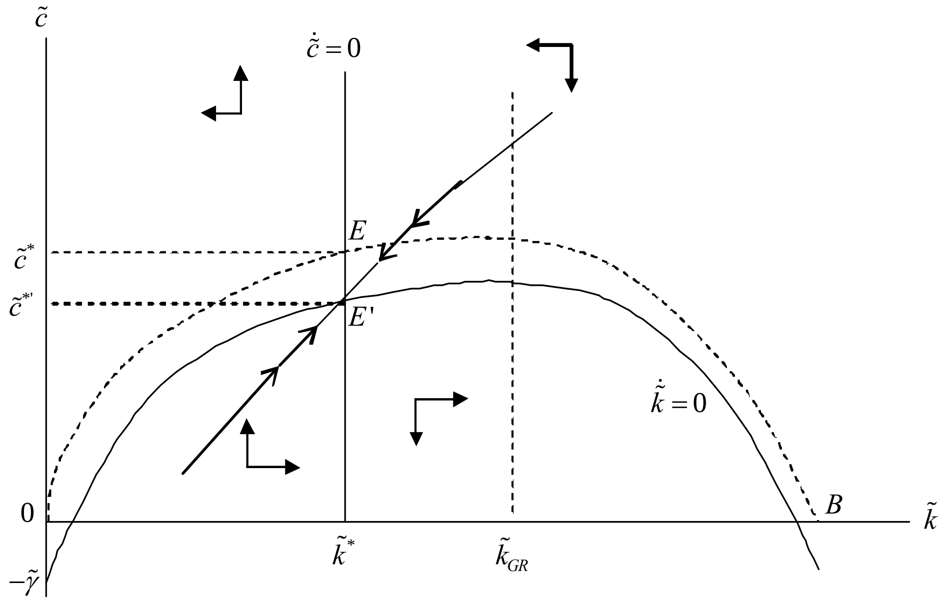

Figure 11.1: Phase portrait of an unanticipated permanent increase in government spending from 0 to $\tilde{\gamma}$.

the decision. In contrast, a state variable is often said to be *pre-determined* because its value is an outcome of the past and it cannot jump.

An unanticipated rise in government spending Returning to our specific example, suppose that the economy has been in steady state for $t < t_0$. Then, unexpectedly, the new spending policy $\tilde{\gamma}' > \tilde{\gamma}$ is introduced. Let the households rightly expect this policy to be continued forever. As a consequence, the $\dot{\tilde{c}} = 0$ locus in Fig. 11.1 remains where it is, while the $\dot{\tilde{k}} = 0$ locus is shifted downwards. It follows that $\tilde{k}$ remains unchanged at its old steady state level, $\tilde{k}^*$, while $\tilde{c}$ jumps down to the new steady state value, $\tilde{c}^{*\prime}$. There is crowding out of private consumption to the exact extent of the rise in public consumption.[2] The mechanism is that the upward shift in public spending is accompanied by higher lump-sum taxes forever, implying that the after-tax human wealth of the household is reduced, which in turn reduces consumption.

Often a disturbance of a steady state will result in a gradual adjustment

[2]The conclusion is modified, of course, if G_t encompasses public investments and if these have an impact on the productivity of the private sector.

process, either to a new steady state or back to the original steady state. But in this example there is an *immediate jump* to a new steady state.

11.1.2 The effect of income taxation

We now replace the assumed lump-sum taxation by income taxation of different kinds. The assumption of a balanced budget is maintained.

A labor income tax

Consider a tax on wage income at the constant rate τ_w, $0 < \tau_w < 1$. In that the labor supply is presumed to be inelastic, it is unaffected by the wage income tax. For simplicity we temporarily ignore government spending on goods and services and assume that the tax revenue is used to instead finance lump-sum transfers, x_t, of income. So the household's dynamic book-keeping equation reads

$$\dot{a}_t = (r_t - n)a_t + (1 - \tau_w)w_t + x_t - c_t, \qquad a_0 \text{ given.}$$

In view of $\tilde{\gamma} = 0$, (11.4) reduces to

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \tilde{c}_t - (\delta + g + n)\tilde{k}_t. \tag{11.7}$$

Human wealth at time t per member of the representative household is

$$h_t = \int_t^\infty \left[(1 - \tau_w)w_s + x_s\right] e^{-\int_t^s (r_z - n)dz} ds = \int_t^\infty w_s e^{-\int_t^s (r_z - n)dz} ds, \tag{11.8}$$

where x_s is per capita income transfers at time s and where growth in the size of the household (family dynasty) at the rate n implies a growth-corrected income discount rate equal to $r_s - n$. The second equality in (11.8) is due to the symmetry implied by the representative agent assumption and the balanced government budget. The latter entails $x_s L_s = \tau_w w_s L_s$, that is, $x_s = \tau_w w_s$, for all $s \geq t$. Hence, h_t will not be affected by a change in τ_w.

Per capita consumption of the household is

$$c_t = \beta_t(a_t + h_t), \tag{11.9}$$

where β_t is the propensity to consume out of wealth,

$$\beta_t = \frac{1}{\int_t^\infty e^{\int_t^s \left(\frac{(1-\theta)r_z - \rho}{\theta} + n\right)dz} ds}, \tag{11.10}$$

as derived in the previous chapter, and h_t is given in (11.8). We see that none of the determinants of per capita consumption are affected by the wage income tax τ_w, which thus leaves the saving behavior of the household unaffected. Indeed, τ_w does not enter the model in its reduced form, consisting of (11.7), (11.5), and (11.6), and so the evolution of the economy unaffected by the size of τ_w. The intuitive explanation is that since labor supply is inelastic, a labor income tax used to finance transfers to the homogeneous household sector neither affects the marginal trade-offs nor the intertemporal budget restriction. In this setting, whether income takes the form of disposable income or transfers does not matter and so there are no real effects on the economy. If the model were extended with endogenous labor supply, the result would be different.

A capital income tax

It is different when it comes to a tax on capital income because saving in the Ramsey model responds to incentives. Consider a capital income tax at the rate τ_r, $0 < \tau_r < 1$. The household's dynamic budget identity becomes

$$\dot{a}_t = \left[(1-\tau_r)r_t - n\right] a_t + w_t + x_t - c_t, \qquad a_0 \text{ given.}$$

As above, x_t is the per capita lump-sum transfer. In view of a balanced budget, we have at the aggregate level $x_t L_t = \tau_r r_t K_t$. So $x_t = \tau_r r_t k_t$. The No-Ponzi-Game condition is changed to

$$\lim_{t\to\infty} a_t e^{-\int_0^t [(1-\tau_r)r_s - n]ds} \geq 0,$$

and the Keynes-Ramsey rule becomes

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}[(1-\tau_r)r_t - \rho].$$

In general equilibrium we get

$$\dot{\tilde{c}}_t = \frac{1}{\theta}\left[(1-\tau_r)(f'(\tilde{k}_t) - \delta) - \rho - \theta g\right]\tilde{c}_t. \tag{11.11}$$

The differential equation for $\tilde{k}$ is again (11.7).

In steady state we get $(f'(\tilde{k}^*) - \delta)(1-\tau_r) = \rho + \theta g$, that is,

$$f'(\tilde{k}^*) - \delta = \frac{\rho + \theta g}{1-\tau_r} > \rho + \theta g > g + n,$$

where the last inequality comes from the parameter condition (A1). Because $f'' < 0$, $\tilde{k}^*$ is lower than if $\tau_r = 0$. Consequently, consumption in the long term becomes lower as well.[3] The resulting resource allocation is not Pareto optimal. There exist an alternative technically feasible resource allocation that makes everyone in society better off. This is because the capital income tax implies a wedge between the marginal transformation rate in production over time, $f'(\tilde{k}_t) - \delta$, and the marginal transformation rate over time, $(1 - \tau_r)(f'(\tilde{k}_t) - \delta)$, which consumers adapt to.

11.1.3 Effects of shifts in tax policy

We will now analyze effects of a rise in capital income taxation and focus on how these effects depend on whether the change is anticipated in advance or not and whether the change is permanent or only temporary.

An unanticipated permanent rise in τ_r

Assume that up until time t_0 the economy has been in steady state with a tax-transfer scheme based on low taxation, τ_r, of capital income. At time t_0 a new tax-transfer scheme is unexpectedly introduced, involving a higher tax rate, τ_r', on capital income and correspondingly higher lump-sum transfers. Thus, the real after-tax interest rate is now $(1 - \tau_r')r_t$. Suppose it is credibly announced that the new tax-transfer scheme will be adhered to forever.

For $t < t_0$ the dynamics are governed by (11.7) and (11.11) with $0 < \tau_r < 1$ and $x_t > 0$. The corresponding steady state, E, has $\tilde{k} = \tilde{k}^*$ and $\tilde{c} = \tilde{c}^*$ as indicated in the phase diagram in Fig. 11.2. The new tax-transfer scheme ruling after time t_0 shifts the steady state point to E' with $\tilde{k} = \tilde{k}^{*\prime}$ and $\tilde{c} = \tilde{c}^{*\prime}$. The new $\dot{\tilde{c}} = 0$ line and the new saddle path are to the left of the old, i.e., $\tilde{k}^{*\prime} < \tilde{k}^*$. Until time t_0 the economy is at the point E. Immediately after the shift in fiscal policy, equilibrium requires that the economy is on the new saddle path. So there will be a jump from point E to point A in Fig. 11.2. This upward jump in consumption is induced by the lower after-tax rate of return after time t_0.

[3] In the Diamond OLG model a capital income tax, which finances lump-sum transfers to the old generation, has an ambiguous effect on capital accumulation, depending on whether $\theta < 1$ or $\theta > 1$, cf. Exercise 5.?? in Chapter 5. In the Solow growth model a capital income tax, which finances lump-sum transfers, will have no effect on consumption and saving. This is because saving does not respond to incentives in that model.

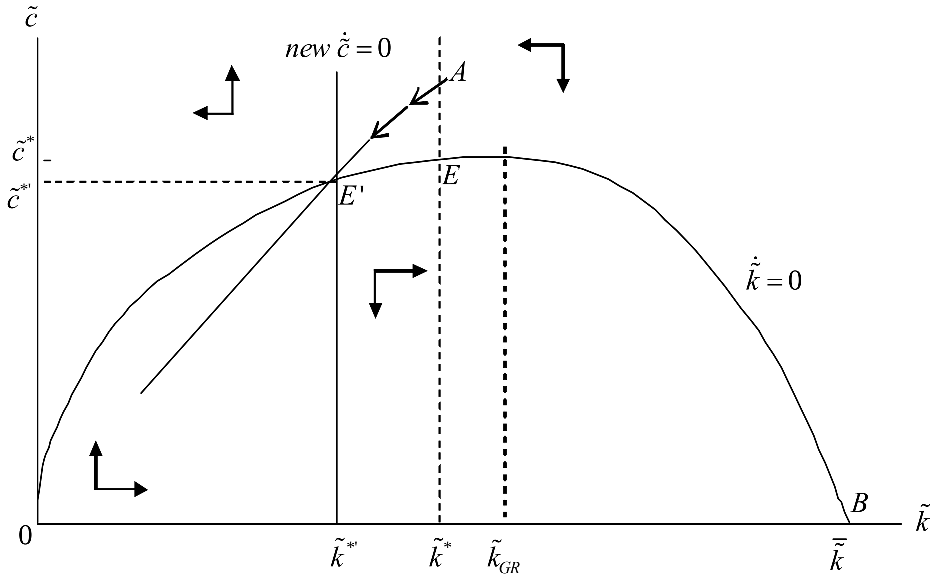

Figure 11.2: Phase portrait of an unanticipated permanent rise in τ_r.

Indeed, consider consumption immediately after the policy shock,

$$\begin{aligned} c_{t_0} &= \beta_{t_0}(a_{t_0} + h_{t_0}), \qquad \text{where} \qquad (11.12) \\ h_{t_0} &= \int_{t_0}^{\infty} (w_t + \tau_r r_t k_t) e^{-\int_{t_0}^{t}((1-\tau_r')r_z - n)dz} dt, \qquad \text{and} \\ \beta_{t_0} &= \frac{1}{\int_{t_0}^{\infty} e^{\int_{t_0}^{t}(\frac{(1-\theta)(1-\tau_r')r_z-\rho}{\theta}+n)dz} dt}. \end{aligned}$$

Two effects are present. First, both the transfers and the lower after-tax rate of return after time t_0 contribute to a higher h_{t_0}. Second, the propensity to consume, β_{t_0}, will generally be affected. If $\theta > 1$ (< 1), the reduction of the after-tax rate of return will have a negative (positive) effect on β_{t_0}, cf. (11.12). Anyway, as shown by the phase diagram, in general equilibrium there will necessarily be an *upward* jump in c_{t_0}. That we get this result even if θ is much higher than 1 is due to the assumption that all the extra tax revenue obtained is immediately transferred back to the households lump sum, thereby strengthening the positive wealth effect on current consumption through a higher h_{t_0}.

Since now $\tilde{c}_{t_0} > f(\tilde{k}_{t_0}) - (\delta + g + n)\tilde{k}_{t_0}$, saving is too low to sustain $\tilde{k}$, which thus begins to fall.[4] As indicated by the arrows in the figure, the economy

[4]Already at time t_0 this raises expected future *before-tax* interest rates, thus partly

moves along the new saddle path towards the new steady state E'. Because $\tilde{k}$ is lower in the new steady state than in the old, so is $\tilde{c}$. The development of the technology level, $\mathcal{T}$, is exogenous, by assumption; thus, also actual per capita consumption, $c \equiv \tilde{c}^* \mathcal{T}$, is lower in the new steady state.

Instead of all the extra tax revenue obtained being transferred back lump sum to the households, we may alternatively assume that a major part of it is used to finance a rise in government consumption so that $G_t = \tilde{\gamma}' \mathcal{T}_t L_t$, where $\tilde{\gamma}' > \tilde{\gamma}$.[5] In addition to the leftward shift of the $\dot{\tilde{c}} = 0$ locus this will result in a downward shift of the $\dot{\tilde{k}} = 0$ locus. The phase diagram would look like a convex combination of Fig. 11.1 and Fig. 11.2. Then it is possible that the jump in consumption at time t_0 becomes downward instead of upward.

An anticipated permanent rise in τ_r

Returning to the case where the extra tax revenue is fully transferred, we now split the change in taxation policy into two events. At time t_0, it is unexpectedly announced that a new tax-transfer policy with $\tau_r' > \tau_r$ is to be implemented at time $t_1 > 0$. We assume people believe in this announcement and that the new policy is implemented as announced. The shock to the economy is now not the event of a higher tax being implemented at time t_1; this event is as expected after time t_0. The shock occurs at time t_0 in the form of the unexpected announcement.

The phase diagram in Fig. 11.3 illustrates the evolution of the economy for $t \geq t_0$. There are two time intervals to consider. For $t \in [t_1, \infty)$, the dynamics are governed by (11.7) and (11.11) with τ_r replaced by τ_r', starting from whatever value obtained by $\tilde{k}$ at time t_1.

In the time interval $[t_0, t_1)$, however, the "old dynamics", with the lower tax rate, τ_r, in a sense still hold. Yet the path the economy follows immediately after time t_0 is different from what it would be without the information that capital income will be taxed heavily from time t_1, where also transfers will become higher. Indeed, the expectation of a lower after-tax rate of return combined with higher transfers from time t_1 implies higher present value of future labor and transfer income. Already at time t_0 this induces an upward jump in consumption to the point C in Fig. 11.3 because people have become more wealthy.

counteracting the negative effect of the higher tax on expected future *after-tax* interest rates.

[5] We assume that also $\tilde{\gamma}'$ is not larger than what allows a steady state to exist. It is understood that the government budget is still balanced for all t so that any temporary surplus or shortage of tax revenue, $\tau_r r_t K_t - G_t$, is immediately transferred or collected

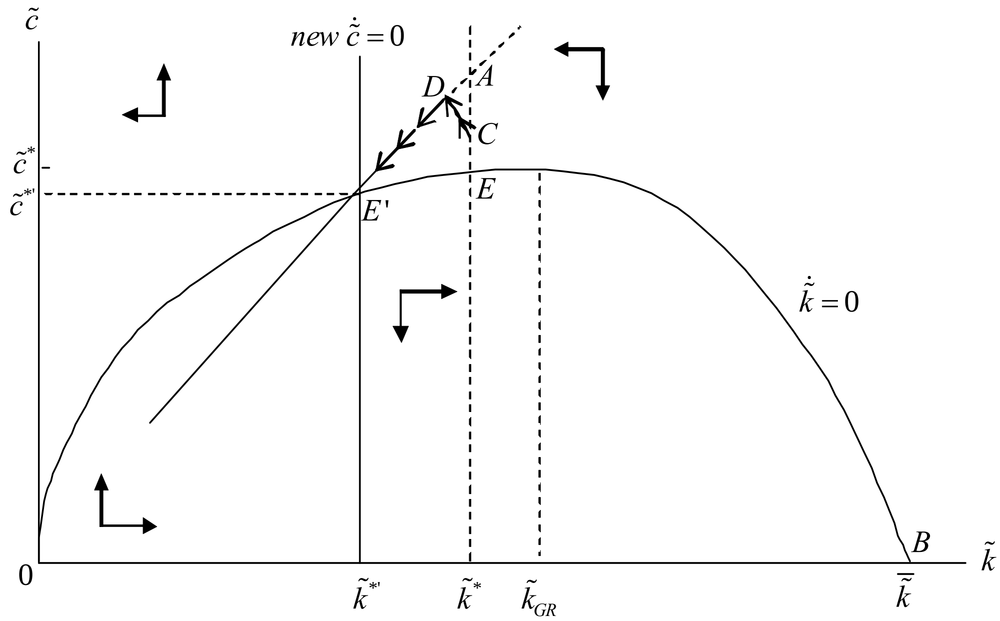

Figure 11.3: Phase portrait of an anticipated permanent rise in τ_r.

Since the low τ_r rules until time t_1, the point C is below the point A, which is the same as that in Fig. 11.2. How far below? The answer follows from the fact that there cannot be an *expected* discontinuity of marginal utility at time t_1, since that would violate the principle of consumption smoothing implied by the Keynes-Ramsey rule. To put it differently: even though the shift to τ_r' does not occur at time t_0, as in (11.12), but later, as long as it is known to occur, there cannot be jumps in the relevant integrals (these are analogue to those in (11.12)).

The intuition behind this is that a consumer will never *plan* a jump in consumption because the strict concavity of the utility function offer gains to be obtained by smoothing out a known discontinuity in consumption. Recalling the optimality condition $u'(c_{t_1}) = \lambda_{t_1}$, we could also say that along an optimal path there can be no *expected* discontinuity in the shadow price of financial wealth. This is analogue to the fact that in an asset market, arbitrage rules out the existence of a generally expected jump in the price of the asset. If we imagine an upward jump, an infinite positive rate of return could be obtained by buying the asset immediately before the jump. This generates excess demand of the asset before time t_1 and drives its price up in

lump-sum.

advance thus preventing an expected upward jump. And if we on the other hand imagine a downward jump, an infinite negative rate of return could be avoided by selling the asset immediately before the jump. This generates excess supply of the asset before time t_1 and drives its price down in advance thus preventing an expected downward jump.

To avoid existence of an expected discontinuity in consumption, the point C on the vertical line $\tilde{k} = \tilde{k}^*$ in Fig. 11.3 must be such that, following the "old dynamics", it takes exactly $t_1 - t_0$ time units to reach the new saddle path.

Immediately after time t_0, $\tilde{k}$ will be decreasing (because saving is smaller than what is required to sustain a constant $\tilde{k}$); and $\tilde{c}$ will be *increasing* in view of the Keynes-Ramsey rule, since the rate of return on saving is above $\rho + \theta g$ as long as $\tilde{k} < \tilde{k}^*$ and τ_r low. Precisely at time t_1 the economy reaches the new saddle path, the high taxation of capital income begins, and the after-tax rate of return becomes lower than $\rho + \theta g$. Hence, per-capita consumption begins to fall and the economy gradually approaches the new steady state E'.

This analysis illustrates that when economic agents' behavior depend on forward-looking expectations, a credible announcement of a future change in policy has an effect already before the new policy is implemented. Such effects are known as *announcement effects* or *anticipation effects*.

An unanticipated temporary rise in τ_r

Once again we change the scenario. The economy with low taxation has been in steady state up until time t_0. Then the new tax-transfer scheme is unexpectedly introduced. At the same time it is credibly announced that the high taxation of capital income and the corresponding transfers will cease at time $t_1 > t_0$. The phase diagram in Fig. 11.4 illustrates the evolution of the economy for $t \geq t_0$. For $t \geq t_1$, the dynamics are governed by (11.7) and (11.11) with the old τ_r again, starting from whatever value obtained by $\tilde{k}$ at time t_1.

In the time interval $[t_0, t_1)$ the "new, temporary dynamics" with the high τ_r' and high transfers rule. Yet the path that the economy takes immediately after time t_0 is different from what it would be without the information that the new tax-transfers scheme is only temporary. Indeed, the expectation of the future shift to a higher after-tax rate of return and cease of high transfers implies lower present value of expected future labor and transfer earnings than without this information. Hence, the upward jump in consumption at time t_0 is smaller than in Fig. 11.2. How much smaller? Again, the answer follows from the fact that there can not be an *expected* discontinuity of mar-

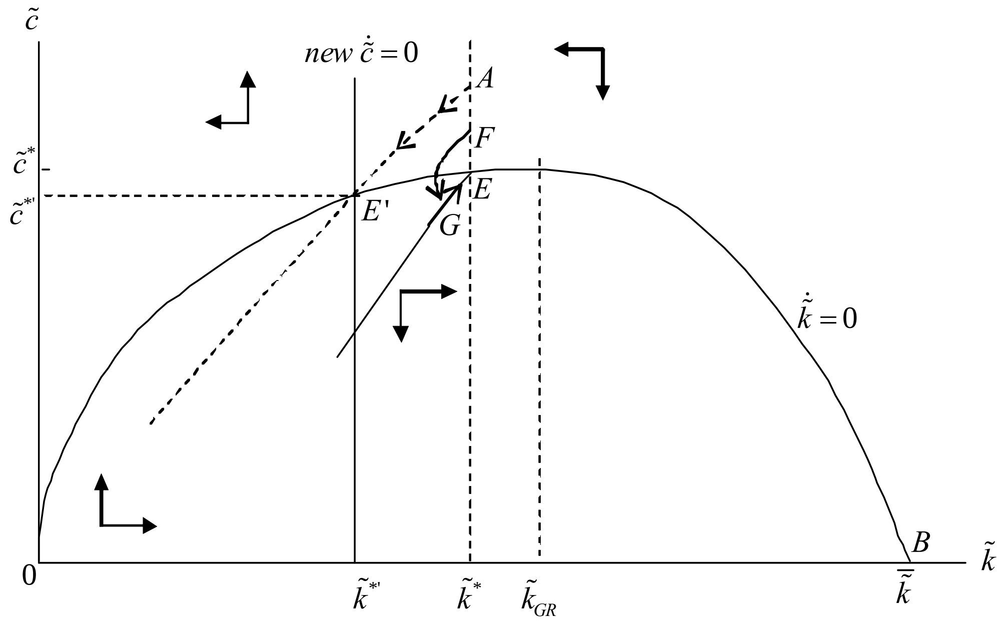

Figure 11.4: Phase portrait of an unanticipated temporary rise τ_r.

ginal utility at time t_1, since that would violate the principle of consumption smoothing. Thus the point F on the vertical line $\tilde{k} = \tilde{k}^*$ in Fig. 11.4 must be such that, following the "new, temporary dynamics", it takes exactly t_1 time units to reach the solid saddle path in Fig. 11.4 (which is in fact the same as the saddle path before time t_0). The implied position of the economy at time t_1 is indicated by the point G in the figure.

Immediately after time t_0, $\tilde{k}$ will be decreasing (because saving is smaller than what is required to sustain a constant $\tilde{k}$) and $\tilde{c}$ will be *decreasing* in view of the Keynes-Ramsey rule in a situation with after-tax rate of return lower than $\rho + \theta g$. Precisely at time t_1, when the temporary tax-transfers scheme based on τ_r' is abolished (as announced and expected), the economy reaches the solid saddle path. From that time the return on saving is high both because of the abolition of the high capital income tax and because $\tilde{k}$ is relatively low. The general equilibrium effect of this is higher saving, and so the economy moves along the solid saddle path back to the original steady-state point E.

There is a last case to consider, namely an anticipated temporary rise in τ_r. We leave that for an exercise, see Exercise 11.??

11.1.4 Ricardian equivalence

We shall now allow public spending to be financed partly by issuing government bonds and partly by lump-sum taxation. Transfers and gross tax revenue as of time t are called X_t and $\tilde{T}_t$, respectively, while the real value of government debt is called B_t. For simplicity, we assume all public debt is short-term. Ignoring any money-financing of the spending, the increase per time unit in government debt is identical to the government budget deficit:

$$\dot{B}_t = r_t B_t + G_t + X_t - \tilde{T}_t. \tag{11.13}$$

As the model ignores uncertainty, on its debt the government has to pay the same interest rate, r_t, as other borrowers.

Along an equilibrium path in the Ramsey model the long-run interest rate necessarily exceeds the long-run GDP growth rate. As we saw in Chapter 6, the government must then, as a debtor, fulfil a solvency requirement analogous to that of the households:

$$\lim_{t\to\infty} B_t e^{-\int_0^t r_s ds} \leq 0. \tag{11.14}$$

This No-Ponzi-Game condition says that the debt is in the long run allowed to grow only at a rate less than the interest rate. As in discrete time, given the accounting relationship (11.13), we have that (11.14) is equivalent to the intertemporal budget constraint

$$\int_0^\infty (G_t + X_t) e^{-\int_0^t r_s ds} dt \leq \int_0^\infty \tilde{T}_t e^{-\int_0^t r_s ds} dt - B_0. \tag{11.15}$$

This says that the present value of the planned public expenditure cannot exceed government net wealth consisting of the present value of the expected future tax revenues minus initial government debt, i.e., assets minus liabilities.

Assuming that the government does not collect more taxes than is necessary to satisfy (11.15), we replace "$\leq$" by "$=$" and rearrange to obtain

$$\int_0^\infty \tilde{T}_t e^{-\int_0^t r_s ds} dt = \int_0^\infty (G_t + X_t) e^{-\int_0^t r_s ds} dt + B_0. \tag{11.16}$$

Thus for a given evolution of G_t and X_t, the time profile of the expected tax revenue must be such that the present value of taxes equals the present value of total liabilities on the right-hand-side of (11.16). A temporary budget deficit leads to more debt and therefore also higher taxes in the future. A budget deficit merely implies a deferment of tax payments. The condition

(11.16) can be reformulated as $\int_0^\infty(\tilde{T}_t - G_t - X_t)e^{-\int_0^t r_s ds}dt = B_0$, showing that if debt is positive today, then the government has to run a positive primary budget surplus (that is, $\tilde{T}_t - G_t - X_t > 0$) in a sufficiently long time interval in the future.

We will now show that if taxes and transfers are lump sum, then *Ricardian equivalence* holds in this model. That is, a temporary tax cut will have no consequences for consumption and capital accumulation – the time profile of taxes does not matter.

Consider the intertemporal budget constraint of the representative household,

$$\int_0^\infty c_t L_t e^{-\int_0^t r_s ds}dt \leq A_0 + H_0 = K_0 + B_0 + H_0, \tag{11.17}$$

where H_0 is human wealth of the household. This says, that the present value of the planned consumption stream can not exceed the total wealth of the household. In the optimal plan of the household, we have strict equality in (11.17).

Let τ_t denote the lump-sum per capita *net* tax. Then, $\tilde{T}_t - X_t = \tau_t L_t$ and

$$\begin{aligned} H_0 &= h_0 L_0 = \int_0^\infty (w_t - \tau_t)L_t e^{-\int_0^t r_s ds}dt = \int_0^\infty (w_t L_t + X_t - \tilde{T}_t)e^{-\int_0^t r_s ds}dt \\ &= \int_0^\infty (w_t L_t - G_t)e^{-\int_0^t r_s ds}dt - B_0, \end{aligned} \tag{11.18}$$

where the last equality comes from (11.16). It follows that

$$B_0 + H_0 = \int_0^\infty (w_t L_t - G_t)e^{-\int_0^t r_s ds}dt.$$

We see that the time profiles of transfers and taxes have fallen out. Total wealth of the household is independent of the time profile of transfers and taxes. And a higher initial debt has no effect on the sum, B_0+H_0, because H_0, which incorporates transfers and taxes, becomes equally much lower. Total private wealth is thus unaffected by government debt. So is therefore private consumption. A temporary tax cut will not make people feel wealthier and induce them to consume more. Instead they will increase their saving by the same amount as taxes have been reduced, thereby preparing for the higher taxes in the future.

This is the *Ricardian equivalence* result, which we encountered also in Barro's discrete time dynasty model of Chapter 7:

> In a representative agent equilibrium model, if taxes are lump sum, then for a given evolution of public expenditure, the resource

> allocation is independent of whether current public expenditure is initially financed by taxes or by issuing bonds. The latter method merely implies a deferment of tax payments and since it is only the present value of taxes that matters, the "timing" is irrelevant.

The assumption of a representative agent is of key importance. As we saw in Chapter 6, Ricardian equivalence breaks down in OLG models without an operative Barro-style bequest motive which is implicit in the present model. In OLG models taxes levied at different times are levied on different sets of agents. In the future there are newcomers and they will bear part of the higher tax burden. Therefore, a current tax cut makes current generations feel wealthier and this leads to an increase in consumption, implying a decrease in aggregate saving as a result of the temporary deficit finance. The present generations benefit, but future generations bear the cost in the form of smaller national wealth than otherwise.

11.2 Learning by investing and first-best policy

In *new growth theory*, also called *endogenous growth theory,* the Ramsey framework has been applied extensively as a simplifying description of the household sector. In endogenous growth theory the focus is on mechanisms that generate and shape technological change. Different hypotheses about the generation of new technologies are then often combined with a simplified picture of the household sector as in the Ramsey model. Since this results in a simple determination of the long-run interest rate (the modified golden rule), the analyst can in a first approach concentrate on the main issue, technological change, without being disturbed by aspects secondary to this issue.

As an example, let us consider one of the basic endogenous growth models, the *learning-by-investing model,* sometimes called the *learning-by doing model.* Learning from investment experience and diffusion across firms of the resulting new technical knowledge (positive externalities) play an important role.

There are two popular alternative versions of the model. The distinguishing feature is whether the learning parameter, to be defined below, is less than one or equal to one. The first case corresponds to (a simplified version of) a famous model by Nobel laureate Kenneth Arrow (1962). The second case has been drawn attention to by Paul Romer (1986) where the learning parameter is assumed equal to one. The two contributions start out from a common framework which we now present.

11.2.1 The common framework

We consider a closed economy with firms and households interacting under conditions of perfect competition. Later, a government attempting to internalize the positive investment externality is introduced.

Let there be N firms in the economy (N "large"). Suppose they all have the same neoclassical production function, F, satisfying the Inada conditions and having CRS. Firm no. i faces the technology

$$Y_{it} = F(K_{it}, T_t L_{it}), \qquad i = 1, 2, ..., N, \tag{11.19}$$

where the economy-wide technology level, T_t, is an increasing function of society's previous experience, proxied by cumulative aggregate net investment:

$$T_t = \left(\int_{-\infty}^{t} I_s^n ds \right)^{\lambda} = K_t^{\lambda}, \quad 0 < \lambda \leq 1, \tag{11.20}$$

where I_s^n is aggregate net investment and $K_t = \sum_i K_{it}$.[6]

The idea is that investment − the production of capital goods − as an unintended *by-product* results in *experience* or what we may call on-the-job *learning*. This adds to the knowledge about how to produce the capital goods in a cost-efficient way and how to design them so that in combination with labor they are more productive and better satisfy the needs of the users. Moreover, as emphasized by Arrow, "each new machine produced and put into use is capable of changing the environment in which production takes place, so that learning is taking place with continually new stimuli" (Arrow, 1962).

The learning is assumed to benefit essentially all firms in the economy. There are knowledge spillovers across firms and these spillovers are reasonably fast relative to the time horizon relevant for growth theory. In our macroeconomic approach both F and T are in fact assumed to be exactly the same for all firms in the economy. That is, in this specification the firms producing consumption-goods benefit from the learning just as much as the firms producing capital-goods.

The parameter λ indicates the elasticity of the general technology level, T, with respect to cumulative aggregate net investment and is named the

[6]For arbitrary units of measurement for labor and output the hypothesis is $T_t = BK_t^{\lambda}$, $B > 0$. In (11.20) measurement units are chosen such that $B = 1$.

"learning parameter". Whereas Arrow assumes $\lambda < 1$, Romer focuses on the case $\lambda = 1$. The case of $\lambda > 1$ is ruled out since it would lead to explosive growth (infinite output in finite time) and is therefore not plausible.[7]

The individual firm

From now we suppress the time index when not needed for clarity. Consider firm i. Its maximization of profits, $\Pi_i = F(K_i, TL_i) - (r+\delta)K_i - wL_i$, leads to the first-order conditions

$$\begin{aligned} \partial \Pi_i/\partial K_i &= F_1(K_i, TL_i) - (r+\delta) = 0, \qquad (11.21)\\ \partial \Pi_i/\partial L_i &= F_2(K_i, TL_i)T - w = 0. \end{aligned}$$

Behind (11.21) is the presumption that each firm is small relative to the economy as a whole, so that each firm's investment has a negligible effect on the economy-wide technology level T_t. Since F is homogeneous of degree one, by Euler's theorem, the first-order partial derivatives, F_1 and F_2, are homogeneous of degree zero. Thus, we can write (11.21) as

$$F_1(k_i, T) = r + \delta, \qquad (11.22)$$

where $k_i \equiv K_i/L_i$. Since F is neoclassical, $F_{11} < 0$. Therefore (11.22) determines k_i uniquely.

The individual household

The household sector is described by our standard Ramsey framework with inelastic labor supply and population growth rate n. The households have CRRA instantaneous utility with parameter $\theta > 0$, and the pure rate of time preference is a constant $\rho > 0$. The flow budget identity in per capita terms is

$$\dot{a}_t = (r_t - n)a_t + w_t - c_t, \qquad a_0 \text{ given},$$

where a is per capita financial wealth. The NPG condition is

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} \geq 0.$$

The resulting consumption-saving plan implies that per capita consumption follows the Keynes-Ramsey rule,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho),$$

and that the NPG condition is satisfied with strict equality.

[7] Empirical evidence of learning-by-doing and learning-by-investing is briefly discussed in Bibliographic notes.

Equilibrium in factor markets

From (11.22) follows that the chosen k_i will be the same for all firms, say $\bar{k}$. In equilibrium $\sum_i K_i = K$ and $\sum_i L_i = L$, where K and L are the available amounts of capital and labor, respectively (both pre-determined). Since $K = \sum_i K_i = \sum_i k_i L_i = \sum_i \bar{k} L_i = \bar{k} L$, the chosen capital-labor ratio, k_i, satisfies

$$k_i = \bar{k} = \frac{K}{L} \equiv k, \qquad i = 1, 2, ..., N. \tag{11.23}$$

As a consequence we can use (11.22) to *determine* the equilibrium interest rate:

$$r_t = F_1(k_t, T_t) - \delta. \tag{11.24}$$

The implied aggregate production function is

$$\begin{aligned} Y &= \sum_i Y_i \equiv \sum_i y_i L_i = \sum_i F(k_i, T) L_i = \sum_i F(k, T) L_i \quad \text{(by (11.19) and (11.23))} \\ &= F(k, T) \sum_i L_i = F(k, T) L = F(K, TL) = F(K, K^\lambda L) \quad \text{(by (11.20))}, \quad (11.25) \end{aligned}$$

where we have several times used that F is homogeneous of degree one.

11.2.2 The arrow case: $\lambda < 1$

The Arrow case is the robust case where the learning parameter satisfies $0 < \lambda < 1$. The method for analyzing the Arrow case is analogue to that used in the study of the Ramsey model with exogenous technological progress. In particular, aggregate capital per unit of effective labor, $\tilde{k} \equiv K/(TL)$, is a key variable. Let $\tilde{y} \equiv Y/(TL)$. Then

$$\tilde{y} = \frac{F(K, TL)}{TL} = F(\tilde{k}, 1) \equiv f(\tilde{k}), \qquad f' > 0, f'' < 0. \tag{11.26}$$

We can now write (11.24) as

$$r_t = f'(\tilde{k}_t) - \delta, \tag{11.27}$$

where $\tilde{k}_t$ is pre-determined.

Dynamics

From the definition $\tilde{k} \equiv K/(TL)$ follows

$$\begin{aligned} \frac{\dot{\tilde{k}}}{\tilde{k}} &= \frac{\dot{K}}{K} - \frac{\dot{T}}{T} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - \lambda \frac{\dot{K}}{K} - n \quad \text{(by (11.20))} \\ &= (1 - \lambda) \frac{Y - C - \delta K}{K} - n = (1 - \lambda) \frac{\tilde{y} - \tilde{c} - \delta \tilde{k}}{\tilde{k}} - n, \quad \text{where } \tilde{c} \equiv \frac{C}{TL} \equiv \frac{c}{T}. \end{aligned}$$

Multiplying through by $\tilde{k}$ we have

$$\dot{\tilde{k}} = (1-\lambda)(f(\tilde{k}) - \tilde{c}) - [(1-\lambda)\delta + n]\,\tilde{k}. \tag{11.28}$$

In view of (11.27), the Keynes-Ramsey rule implies

$$g_c \equiv \frac{\dot{c}}{c} = \frac{1}{\theta}(r-\rho) = \frac{1}{\theta}\left(f'(\tilde{k}) - \delta - \rho\right). \tag{11.29}$$

Defining $\tilde{c} \equiv c/T$, now follows

$$\begin{aligned}\frac{\dot{\tilde{c}}}{\tilde{c}} &= \frac{\dot{c}}{c} - \frac{\dot{T}}{T} = \frac{\dot{c}}{c} - \lambda\frac{\dot{K}}{K} = \frac{\dot{c}}{c} - \lambda\frac{Y - cL - \delta K}{K} = \frac{\dot{c}}{c} - \frac{\lambda}{\tilde{k}}(\tilde{y} - \tilde{c} - \delta\tilde{k}) \\ &= \frac{1}{\theta}(f'(\tilde{k}) - \delta - \rho) - \frac{\lambda}{\tilde{k}}(\tilde{y} - \tilde{c} - \delta\tilde{k}).\end{aligned}$$

Multiplying through by $\tilde{c}$ we have

$$\dot{\tilde{c}} = \left[\frac{1}{\theta}(f'(\tilde{k}) - \delta - \rho) - \frac{\lambda}{\tilde{k}}(f(\tilde{k}) - \tilde{c} - \delta\tilde{k})\right]\tilde{c}. \tag{11.30}$$

The two coupled differential equations, (11.28) and (11.30), determine the evolution over time of the economy.

Phase portrait

Fig. 11.5 depicts the phase diagram. The $\dot{\tilde{k}} = 0$ locus comes from (11.28), which gives

$$\dot{\tilde{k}} = 0 \text{ for } \tilde{c} = f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k}, \tag{11.31}$$

where we realistically may assume that $\delta + n/(1-\lambda) > 0$. As to the $\dot{\tilde{c}} = 0$ locus, we have

$$\begin{aligned}\dot{\tilde{c}} &= 0 \text{ for } \tilde{c} = f(\tilde{k}) - \delta\tilde{k} - \frac{\tilde{k}}{\lambda\theta}(f'(\tilde{k}) - \delta - \rho) \\ &= f(\tilde{k}) - \delta\tilde{k} - \frac{\tilde{k}}{\lambda}g_c \equiv c(\tilde{k}) \quad \text{(from (11.29))}.\end{aligned} \tag{11.32}$$

Before determining the slope of the $\dot{\tilde{c}} = 0$ locus it is convenient to consider the steady state, $(\tilde{k}^*, \tilde{c}^*)$.

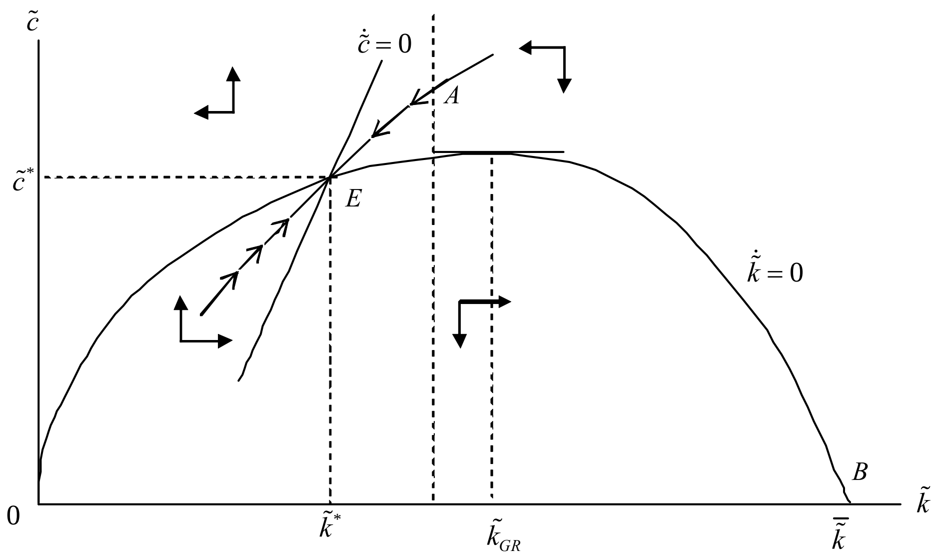

Figure 11.5:

Steady state

In a steady state $\tilde{c}$ and $\tilde{k}$ are constant so that the growth rate of C as well as K equals $\dot{T}/T + n$, i.e.,

$$\frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{\dot{T}}{T} + n = \lambda\frac{\dot{K}}{K} + n.$$

Solving gives

$$\frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{n}{1-\lambda}.$$

Thence, in a steady state

$$g_c = \frac{\dot{C}}{C} - n = \frac{n}{1-\lambda} - n = \frac{\lambda n}{1-\lambda} \equiv g_c^*, \quad \text{and} \tag{11.33}$$

$$\frac{\dot{T}}{T} = \lambda\frac{\dot{K}}{K} = \frac{\lambda n}{1-\lambda} = g_c^*. \tag{11.34}$$

The steady-state values of r and $\tilde{k}$, respectively, will therefore satisfy, by (11.29),

$$r^* = f'(\tilde{k}^*) - \delta = \rho + \theta g_c^* = \rho + \theta\frac{\lambda n}{1-\lambda}. \tag{11.35}$$

The transversality condition of the representative household is that $\lim_{t\rightarrow\infty} a_t e^{-\int_0^t (r_s - n)ds} = 0$, where a_t is per capita financial wealth. In general equilibrium $a_t = k_t \equiv \tilde{k}_t T_t$, where T in steady state grows according to (11.34). Thus, in steady state the transversality condition can be written

$$\lim_{t\rightarrow\infty} \tilde{k}^* e^{(g_c^* - r^* + n)t} = 0.$$

For this to hold we need

$$r^* > g_c^* + n = \frac{n}{1-\lambda}, \tag{11.36}$$

by (11.33). In view of (11.35), this is equivalent to

$$\rho - n > (1-\theta)\frac{\lambda n}{1-\lambda}, \tag{A1}$$

which we assume satisfied.

As to the slope of the $\dot{\tilde{c}} = 0$ locus we have from (11.32)

$$c'(\tilde{k}) = f'(\tilde{k}) - \delta - \frac{1}{\lambda}(\tilde{k}\frac{f''(\tilde{k})}{\theta} + g_c) > f'(\tilde{k}) - \delta - \frac{1}{\lambda}g_c, \tag{11.37}$$

since $f'' < 0$. At least in a small neighborhood of the steady state we can sign the right-hand side of this expression:

$$f'(\tilde{k}^*) - \delta - \frac{1}{\lambda}g_c^* = \rho + \theta g_c^* - \frac{1}{\lambda}g_c^* = \rho + \theta\frac{\lambda n}{1-\lambda} - \frac{n}{1-\lambda} > 0, \tag{11.38}$$

by (11.33) and (A1). So, combing with (11.37), we conclude that $c'(\tilde{k}^*) > 0$. By continuity, in a small neighborhood of the steady state, $c'(\tilde{k}) \approx c'(\tilde{k}^*) > 0$. Therefore, close to the steady state, the $\dot{\tilde{c}} = 0$ locus is positively sloped, as indicated in Fig. 11.5.

Still, we have to check the following question: in a neighborhood of the steady state, which is steeper, the $\dot{\tilde{c}} = 0$ locus or the $\dot{\tilde{k}} = 0$ locus? The slope of the latter is $f'(\tilde{k}) - \delta - n/(1-\lambda)$, from (11.31). At the steady state this slope is

$$f'(\tilde{k}^*) - \delta - \frac{n}{1-\lambda} = \rho + \theta\frac{\lambda n}{1-\lambda} - \frac{n}{1-\lambda} \in (0, c'(\tilde{k}^*)),$$

in view of (11.38). The $\dot{\tilde{c}} = 0$ locus is thus steeper. So, the $\dot{\tilde{c}} = 0$ locus crosses the $\dot{\tilde{k}} = 0$ locus from below and can only cross once.

Note also that in a golden rule steady state we have

$$f'(\tilde{k}_{GR}) - \delta = (\frac{\dot{Y}}{Y})^* = \frac{\dot{T}}{T} + \frac{\dot{L}}{L} = \frac{\lambda n}{1-\lambda} + n = \frac{n}{1-\lambda}, \tag{11.39}$$

by (11.34). Thus, the tangent to the $\dot{\tilde{k}} = 0$ locus at the golden rule capital intensity, $\tilde{k}_{GR}$, is horizontal and, by (11.36), $\tilde{k}^* < \tilde{k}_{GR}$, as displayed in Fig. 11.5.

Stability

The arrows in Fig. 11.5 indicate the direction of movement, as determined by (11.28) and (11.30)). We see that the steady state is a saddle point. The dynamic system has one pre-determined variable, $\tilde{k}$, and one jump variable, $\tilde{c}$. Thus the system is saddle-point stable. The divergent paths in Fig. 1 can be ruled out as equilibrium paths because they violate the transversality condition of the household. For a given initial value $\tilde{k}_0 > 0$, the economy moves along the saddle path and converges to the steady state.

In the long run c and $y \equiv Y/L \equiv \tilde{y}T = f(\tilde{k}^*)T$ grow at the rate $\lambda n/(1-\lambda)$, which is positive if and only if $n > 0$. This is an example of *endogenous growth* in the sense that the source of a positive long-run per capita growth rate is an internal mechanism (learning) in the model (in contrast to exogenous technology growth as in the Ramsey model with exogenous technological progress).

Two types of endogenous growth

One may distinguish between two types of endogenous growth. One is called *fully endogenous* growth which occurs when the long-run growth rate of c is positive without the support by growth in any exogenous factor (for example exogenous growth in the labor force). The other type is called *semi-endogenous growth* and is present if growth is endogenous but a positive per capita growth rate can not be maintained in the long run without the support by growth in some exogenous factor (for example exogenous growth in the labor force). Clearly, here in the Arrow model of learning by investing, growth is "only" semi-endogenous. The technical reason for this is the assumption that the learning parameter $\lambda < 1$, which implies diminishing returns to capital at the aggregate level. If and only if $n > 0$, do we have $\dot{c}/c > 0$ in the long run.

The key role of population growth derives from the fact that although there are diminishing marginal returns to capital at the aggregate level, there are increasing returns to scale w.r.t. capital *and* labor. For the increasing returns to be exploited, growth in the labor force is needed. To put it differently: when there are increasing returns to K and L together, growth in the labor force not only counterbalances the falling marginal product of aggregate capital (this counter-balancing role reflects the complementarity between K and L), but also upholds sustained productivity growth.

In the semi-endogenous growth case, $\partial g_y^*/\partial\lambda = n/(1-\lambda)^2 > 0$ for $n > 0$. That is, a higher value of the learning parameter implies higher per capita growth in the long run, when $n > 0$. Note also that $\partial g_y^*/\partial\rho = 0 = \partial g_y^*/\partial\theta$, that is, in the semi-endogenous growth case preference parameters do not matter for long-run growth. As indicated by (11.33), the long-run growth rate is tied down by the learning parameter, λ, and the rate of population growth, n. But, like in the simple Ramsey model, it can be shown that preference parameters matter for the *level* of the growth path. This suggests that taxes and subsidies do not have long-run growth effects, but "only" *level* effects.

11.2.3 Romer's limiting case: $\lambda = 1$, $n = 0$

We now consider the limiting case $\lambda = 1$. We should think of it as a thought experiment because, by most observers, the value 1 is considered an unrealistically high value for the learning parameter. To avoid a forever rising growth rate we have to add the restriction $n = 0$.

The resulting model turns out to be extremely simple and at the same time it gives striking results (both circumstances have probably contributed to its popularity).

First, with $\lambda = 1$ we get $T = K$ and so the equilibrium interest rate is, by (11.24),

$$r = F_1(k, K) - \delta = F_1(1, L) - \delta \equiv \bar{r},$$

where we have divided the two arguments of $F_1(k, K)$ by $k \equiv K/L$ and again used Euler's theorem. The interest rate is constant "from the beginning". The aggregate production function is now

$$Y = F(K, KL) = F(1, L)K, \qquad L \text{ constant},$$

and is thus *linear* in the aggregate capital stock. In this way the general neoclassical presumption of diminishing returns to capital has been suspended and replaced by exactly constant returns to capital. So the Romer model belongs to the class of *AK models*, that is, models where in general equilibrium the interest rate and the output-capital ratio are necessarily constant over time whatever the initial conditions.

The method for analyzing an AK model is different from the one used for a diminishing returns model as above.

Dynamics

The Keynes-Ramsey rule now takes the form

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(\bar{r} - \rho) = \frac{1}{\theta}(F_1(1, L) - \delta - \rho) \equiv \gamma, \tag{11.40}$$

which is also constant "from the beginning". To ensure positive growth we assume

$$F_1(1, L) - \delta > \rho. \tag{A2}$$

And to ensure bounded intertemporal utility (and existence of equilibrium) it is assumed that

$$\rho > (1 - \theta)\gamma \text{ and therefore } \gamma < \theta\gamma + \rho = \bar{r}. \tag{A1'}$$

Solving the linear differential equation (11.40) gives

$$c_t = c_0 e^{\gamma t}, \tag{11.41}$$

where c_0 is unknown so far (because c is not a predetermined variable). We shall find c_0 by applying the households' transversality condition

$$\lim_{t\to\infty} a_t e^{-\bar{r}t} = \lim_{t\to\infty} k_t e^{-\bar{r}t} = 0. \tag{TVC}$$

First, the dynamic resource constraint for the economy is

$$\dot{K} = Y - cL - \delta K = F(1, L)K - cL - \delta K,$$

or, in per-capita terms,

$$\dot{k} = [F(1, L) - \delta]\, k - c_0 e^{\gamma t}. \tag{11.42}$$

In this equation it is important that $F(1, L) - \delta - \gamma > 0$. To understand this inequality, note that, by (A1'), $F(1, L) - \delta - \gamma > F(1, L) - \delta - \bar{r} = F(1, L) - F_1(1, L) = F_2(1, L)L > 0$, where the first equality is due to $\bar{r} = F_1(1, L) - \delta$ and the second is due to the fact that since F is homogeneous of degree 1, we have, by Euler's theorem, $F(1, L) = F_1(1, L) \cdot 1 + F_2(1, L)L > F_1(1, L) > \delta$, in view of (A2). The key property $F(1, L) - F_1(1, L) > 0$ is illustrated in Fig. 11.6.

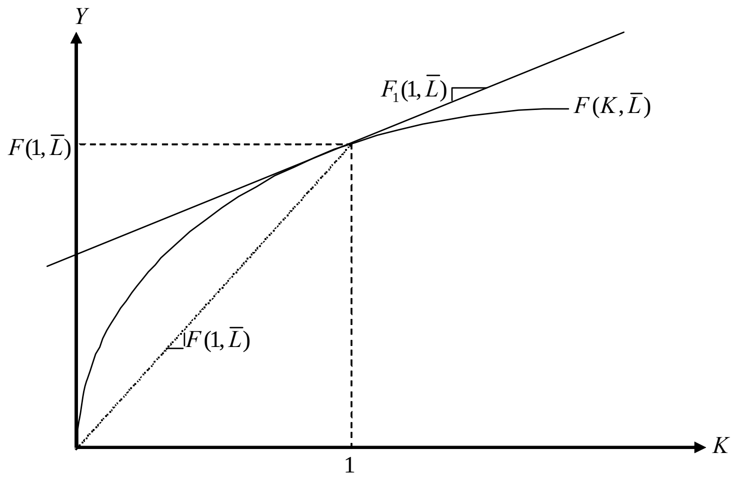

Figure 11.6: Illustration of the fact that $F(1, \bar{L}) > F_1(1, \bar{L})$.

The solution of a general linear differential equation of the form $\dot{x}(t) + ax(t) = ce^{ht}$, with $h \neq -a$, is

$$x(t) = (x(0) - \frac{c}{a+h})e^{-at} + \frac{c}{a+h}e^{ht}. \tag{11.43}$$

Thus the solution to (11.42) is

$$k_t = (k_0 - \frac{c_0}{F(1,L) - \delta - \gamma})e^{(F(1,L)-\delta)t} + \frac{c_0}{F(1,L) - \delta - \gamma}e^{\gamma t}. \tag{11.44}$$

To check whether (TVC) is satisfied we consider

$$\begin{aligned} k_t e^{-\bar{r}t} &= (k_0 - \frac{c_0}{F(1,L) - \delta - \gamma})e^{(F(1,L)-\delta-\bar{r})t} + \frac{c_0}{F(1,L) - \delta - \gamma}e^{(\gamma-\bar{r})t} \\ &\to (k_0 - \frac{c_0}{F(1,L) - \delta - \gamma})e^{(F(1,L)-\delta-\bar{r})t} \text{ for } t \to \infty, \end{aligned}$$

since $\bar{r} > \gamma$, by (A1'). But $\bar{r} = F_1(1, L) - \delta < F(1, L) - \delta$, and so (TVC) is only satisfied if

$$c_0 = (F(1, L) - \delta - \gamma)k_0. \tag{11.45}$$

If c_0 is less than this, there will be over-saving and (TVC) is violated ($a_t e^{-\bar{r}t} \to \infty$ for $t \to \infty$, since $a_t = k_t$). If c_0 is higher than this, both (TVC) and the NPG are violated ($a_t e^{-\bar{r}t} \to -\infty$ for $t \to \infty$).

Inserting the solution for c_0 into (11.44), we get

$$k_t = \frac{c_0}{F(1,L) - \delta - \gamma}e^{\gamma t} = k_0 e^{\gamma t},$$

that is, k grows at the same constant rate as c "from the beginning". Since $y \equiv Y/L = F(1, L)k$, the same is true for y. Hence, from start the system is in balanced growth (there is no transitional dynamics).

This is a case of *fully endogenous growth* in the sense that the long-run growth rate of c is positive without the support by growth in any exogenous factor. This outcome is due to the absence of diminishing returns to aggregate capital, which is implied by the assumed high value of the learning parameter. But the empirical foundation for this high value is weak, to say the least, cf. Bibliographic notes. A further drawback of this special version of the learning model is that the results are non-robust. With λ slightly less than 1, we are back in the Arrow case and growth peters out, since $n = 0$. With λ slightly above 1, it can be shown that growth becomes explosive (infinite output in finite time).[8]

The Rome case, $\lambda = 1$, is thus a *knife-edge* case in a double sense. First, it imposes a particular value for a parameter which apriori can take any value within an interval. Second, the imposed value leads to non-robust results; values in a hair's breadth distance result in qualitatively different behavior of the dynamic system.

The *causal structure* in the diminishing returns case is different than in the AK-case of Romer. In the first case the steady-state growth rate is determined first, as γ_c^*, then r^* is determined through the Keynes-Ramsey rule and, finally, Y/K is determined by the technology, given r^*. In contrast, the Romer model has Y/K and r directly given as $F(1, L)$ and $\bar{r}$, respectively. In turn, $\bar{r}$ determines the growth rate through the Keynes-Ramsey rule.

Economic policy in the Romer case

In the AK case, that is, the fully endogenous growth case, $\partial\gamma/\partial\rho < 0$ and $\partial\gamma/\partial\theta < 0$. Thus, preference parameters *matter* for the long-run growth rate and not "only" for the *level* of the growth path. This suggests that taxes and subsidies can have *long-run* growth effects. In any case, in this model there is a motivation for government intervention due to the positive externality of private investment. This motivation is present whether $\lambda < 1$ or $\lambda = 1$. Here we concentrate on the latter case, which is the simplest. We first find the social planner's solution.

The social planner The social planner faces the aggregate production function $Y_t = F(1, L)K_t$ or, in per capita terms, $y_t = F(1, L)k_t$. The social

[8] See Solow (1997).

planner's problem is to choose $(c_t)_{=0}^{\infty}$ to maximize

$$
\begin{aligned}
U_0 &= \int_0^{\infty} \frac{c_t^{1-\theta}}{1-\theta} e^{-\rho t} dt \quad \text{s.t.} \\
c_t &\geq 0, \\
\dot{k}_t &= F(1,L)k_t - c_t - \delta k_t, \qquad k_0 > 0 \text{ given,} \qquad (11.46) \\
k_t &\geq 0 \text{ for all } t > 0. \qquad (11.47)
\end{aligned}
$$

The current-value Hamiltonian is

$$
H(k, c, \eta, t) = \frac{c^{1-\theta}}{1-\theta} + \eta \left(F(1,L)k - c - \delta k \right),
$$

where $\eta = \eta_t$ is the adjoint variable associated with the state variable, which is capital per unit of labor. Necessary first-order conditions for an interior optimal solution are

$$
\frac{\partial H}{\partial c} = c^{-\theta} - \eta = 0, \text{ i.e., } c^{-\theta} = \eta, \qquad (11.48)
$$

$$
\frac{\partial H}{\partial k} = \eta(F(1,L) - \delta) = -\dot{\eta} + \rho\eta. \qquad (11.49)
$$

We guess that also the transversality condition,

$$
\lim_{t\to\infty} k_t \eta_t e^{-\rho t} = 0, \qquad (11.50)
$$

must be satisfied by an optimal solution. This guess will be of help in finding a candidate solution. Having found a candidate solution, we shall invoke a theorem on *sufficient* conditions to ensure that our candidate solution *is* really a solution.

Log-differentiating w.r.t. t in (11.48) and combining with (11.49) gives the social planner's Keynes-Ramsey rule,

$$
\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(F(1,L) - \delta - \rho) \equiv \gamma_{SP}. \qquad (11.51)
$$

We see that $\gamma_{SP} > \gamma$. This is because the social planner internalizes the economy-wide learning effect associated with capital investment, that is, the social planner takes into account that the "social" marginal product of capital is $\partial y_t / \partial k_t = F(1,L) > F_1(1,L)$. To ensure bounded intertemporal utility we sharpen (A1') to

$$
\rho > (1-\theta)\gamma_{SP}. \qquad \text{(A1")}
$$

To find the time path of k_t, note that the dynamic resource constraint (11.46) can be written

$$\dot{k}_t = (F(1, L) - \delta)k_t - c_0 e^{\gamma_{SP} t},$$

in view of (11.51). By the general solution formula (11.43) this has the solution

$$k_t = (k_0 - \frac{c_0}{F(1, L) - \delta - \gamma_{SP}})e^{(F(1,L)-\delta)t} + \frac{c_0}{F(1, L) - \delta - \gamma_{SP}} e^{\gamma_{SP} t}. \quad (11.52)$$

In view of (11.49), in an interior optimal solution the time path of the adjoint variable η is

$$\eta_t = \eta_0 e^{-[(F(1,L)-\delta-\rho]t},$$

where $\eta_0 = c_0^{-\theta} > 0$, by (11.48). Thus, the conjectured transversality condition (11.50) implies

$$\lim_{t\to\infty} k_t e^{-(F(1,L)-\delta)t} = 0, \quad (11.53)$$

where we have eliminated η_0. To ensure that this is satisfied, we multiply k_t from (11.52) by $e^{-(F(1,L)-\delta)t}$ to get

$$\begin{aligned} k_t e^{-(F(1,L)-\delta)t} &= k_0 - \frac{c_0}{F(1, L) - \delta - \gamma_{SP}} + \frac{c_0}{F(1, L) - \delta - \gamma_{SP}} e^{[\gamma_{SP}-(F(1,L)-\delta)]t} \\ &\to k_0 - \frac{c_0}{F(1, L) - \delta - \gamma_{SP}} \text{ for } t \to \infty, \end{aligned}$$

since, by (A1"), $\gamma_{SP} < \rho + \theta\gamma_{SP} = F(1, L) - \delta$ in view of (11.51). Thus, (11.53) is only satisfied if

$$c_0 = (F(1, L) - \delta - \gamma_{SP})k_0. \quad (11.54)$$

If c_0 is less than this, there will be over-saving and (TVC) is violated; if c_0 is higher than this, the NPG is violated (hence also the (TVC)).

Inserting the solution for c_0 into (11.52), we get

$$k_t = \frac{c_0}{F(1, L) - \delta - \gamma_{SP}} e^{\gamma_{SP} t} = k_0 e^{\gamma_{SP} t},$$

that is, k grows at the same constant rate as c "from the beginning". Since $y \equiv Y/L = F(1, L)k$, the same is true for y. Hence, our candidate for the social planner's solution is from start on in balanced growth (there is no transitional dynamics). Otherwise the evolution would violate the transversality condition.

The next step is to check whether our candidate solution satisfies a set of *sufficient* conditions for an optimal solution. Here we can use *Mangasarian's theorem* which, applied to a problem like this, with one control variable and one state variable, says that the following conditions are sufficient:

(a) The Hamiltonian is jointly concave in the control and state variables, here c and k.

(b) The co-state variable, η, is non-negative for all $t \geq 0$, and there is for all $t \geq 0$ a non-negativity constraint on the state variable.

(c) The candidate solution satisfies the transversality condition $\lim_{t\to\infty} k_t\eta_t e^{-\rho t} = 0$, where $\eta_t e^{-\rho t}$ is the discounted co-state variable.

In the present case we see that the Hamiltonian is a sum of concave functions and therefore is itself concave in (k, c). Further, from (11.47) we see that condition (b) is satisfied. Finally, our candidate solution is constructed so as to satisfy condition (c). The conclusion is that our candidate solution *is* an optimal solution. We call it the SP allocation.

Implementing the SP allocation in the market economy Returning to the market economy, we assume there is a policy maker, say the government, with only two activities. These are (i) paying an investment subsidy, s, to the firms so that their capital costs are reduced to

$$(1 - s)(r + \delta)$$

per unit of capital per time unit; (ii) financing this subsidy by a constant consumption tax rate τ. The government budget is always balanced.

Let us first find the size of s needed to establish the SP allocation. Firm i now chooses K_i such that

$$\frac{\partial Y_i}{\partial K_i}|_{K \text{ fixed}} = F_1(K_i, KL_i) = (1 - s)(r + \delta).$$

By Euler's theorem this implies

$$F_1(k_i, K) = (1 - s)(r + \delta) \quad \text{for all } i,$$

so that in equilibrium we must have

$$F_1(k, K) = (1 - s)(r + \delta),$$

where $k \equiv K/L$, which is pre-determined from the supply side. Thus, the equilibrium interest rate must satisfy

$$r = \frac{F_1(k, K)}{1 - s} - \delta = \frac{F_1(1, L)}{1 - s} - \delta, \tag{11.55}$$

again using Euler's theorem.

It follows that s should be chosen such that the "right" r arises. What is the "right" r? It is that net rate of return which is implied by the production technology at the aggregate level, namely $\partial Y/\partial K - \delta = F(1, L) - \delta$. If we can obtain $r = F(1, L) - \delta$, then there is no wedge between the intertemporal rate of transformation faced by the consumer and that implied by the technology. The required s thus satisfies

$$r = \frac{F_1(1, L)}{1 - s} - \delta = F(1, L) - \delta,$$

so that

$$s = 1 - \frac{F_1(1, L)}{F(1, L)} = \frac{F(1, L) - F_1(1, L)}{F(1, L)} = \frac{F_1(1, L)}{F(1, L)}.$$

It remains to find the required consumption tax rate τ. The tax revenue will be τcL, and the *required* tax revenue is

$$\mathcal{T} = s(r + \delta)K = (F(1, L) - F_1(1, L))\, K = \tau cL.$$

Thus, with a balanced budget the required tax rate is

$$\tau = \frac{\mathcal{T}}{cL} = \frac{F(1, L) - F_1(1, L)}{c/k} = \frac{F(1, L) - F_1(1, L)}{F(1, L) - \delta - \gamma_{SP}} > 0, \tag{11.56}$$

where we have used that the proportionality in (11.54) between c and k holds for all $t \geq 0$. Substituting (11.51) into (11.56), the solution for τ can be written

$$\tau = \frac{\theta\,[F(1, L) - F_1(1, L)]}{(\theta - 1)(F(1, L) - \delta) + \rho}.$$

The required tax rate on consumption is thus a constant. It therefore does not distort the consumption/saving decision on the margin.

We see that the allocation obtained by this subsidy-tax policy *is* the SP allocation. A policy, here the policy (s, τ), which in a decentralized system induces the SP allocation is called a *first-best policy.*

11.3 Concluding remarks

11.4 Literature notes

11.5 Exercises

Chapter 14

Fixed capital investment and Tobins q

The models considered so far (the OLG models as well as the representative agent models) have ignored capital adjustment costs. In the closed-economy version of the models aggregate investment is merely a reflection of aggregate saving and appears in a "passive" way as just the residual of national income after households have chosen their consumption. We can describe what is going on by telling a story in which firms just rent capital goods owned by the households and households save by purchasing additional capital goods. In these models only households solve intertemporal decision problems. Firms merely maximize current profits. This may be a legitimate abstraction in some contexts, but not if *fixed capital investment* is of central importance for the problems studied.

In the real world the capital goods used by a production firm are usually owned by the firm itself rather than rented for single periods on rental markets. This is because inside the specific plant in which these capital goods are an integrated part, they are generally worth much more than outside. So in practice firms acquire and install fixed capital equipment with a view on maximizing discounted expected earnings in the future.

Tobin's *q-theory of investment* (after the American Nobel laureate James Tobin, 1918-2002) is an attempt to model these features. In this theory, (a) *firms* make the *investment decisions* and *install* the purchased capital goods in their own business; and, (b) there are certain *adjustment costs* associated with investment in capital goods, that is, in addition to the direct cost of buying new capital goods there are costs of installation, costs of reorganizing the plant, costs of retraining workers to operate the new machines etc.; and finally, (c) the adjustment costs are *strictly convex* so that marginal adjustment costa are increasing in the level of investment.

This strict convexity of adjustment costs is the crucial constituent of the q-theory of investment; it is that element which assigns investment decisions an *active* role in the model. There will be both a well-defined saving decision and a well-defined investment decision, separate from each other. Households decide the saving, firms the physical capital investment; households accumulate financial assets, firms accumulate physical capital. As a result, in a closed economy interest rates have to adjust for aggregate demand for goods (consumption plus investment) to equal aggregate supply of goods. Interest rates no longer adjust to clear a rental market for capital. Such a market will not be active in the model we are going to consider.

To fix the terminology, from now the adjustment costs of setting up new fixed capital equipment in the firm and the associated costs of reorganizing work processes will be subsumed under the term *capital installation costs*. When faced with strictly convex installation costs, the optimizing firm has to take the *future* into account. Therefore, firms' *expectations* become important. The firm will adjust its capital stock only *gradually* when new information arises. We avoid the counterfactual outcome from earlier chapters that the capital stock in a small open economy with perfect mobility of goods and financial capital is *instantaneously* adjusted when the interest rate in the world financial market changes. Sluggishness in investment, however, is what the data show. Some empirical studies conclude that only a third of the difference between the current and the "desired" capital stock tends to be covered within a year (Clark 1979).

The q-theory of investment is one theoretical approach to the explanation of this sluggishness in investment (Tobin 1969). Under certain conditions, to be described below, the theory gives a remarkably simple operational macroeconomic investment function, in which the key variable explaining aggregate investment is the valuation of the firms by the stock market relative to the book value of the firms' physical capital. This link between asset markets and firms' aggregate investment is an appealing feature of Tobin's q-theory.

14.1 Convex capital installation costs

The technology of the representative firm is

$$Y = F(K, L),$$

where Y, K, and L are output, capital input, and labor input per time unit, respectively, while F is a neoclassical production function which we assume

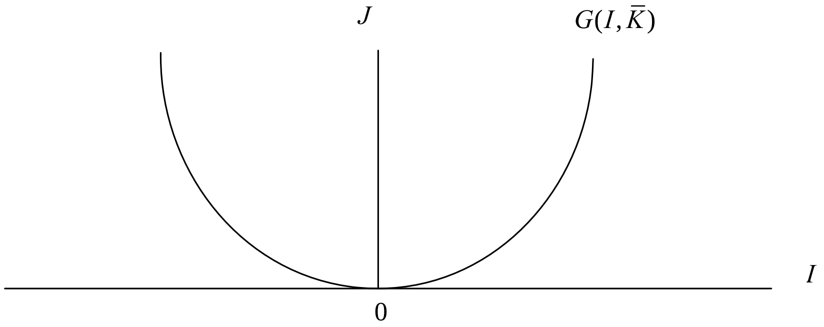

Figure 14.1: Installation costs as a function of gross investment when $K = \bar{K}$.

is concave (non-increasing returns to scale). Until further notice technological change is for simplicity ignored. Time is continuous, the dating of the variables is implicit. The increase per time unit in the firm's capital stock is given by

$$\dot{K} = I - \delta K, \qquad \delta \geq 0, \tag{14.1}$$

where I is gross fixed capital investment per time unit and δ is the rate of wearing down of capital (physical capital depreciation).

Let J denote the firm's capital installation costs (measured in units of output) per time unit. Assuming the price of investment goods is one (the same as that of output goods), then total investment costs per time unit are $I + J$, i.e., the direct purchase costs, $1 \cdot I$, plus the indirect cost associated with installation etc., J. The q-theory of investment assumes that the capital installation cost, J, is a strictly convex function of gross investment and generally also depends negatively on the current capital stock. Thus,

$$J = G(I, K),$$

where the installation cost function G satisfies

$$G(0, K) = 0,\; G_I(0, K) = 0,\; G_{II}(I, K) > 0, \quad \text{and} \quad G_K(I, K) \leq 0, \tag{14.2}$$

for all K and all (I, K), respectively. For fixed $K = \bar{K}$ the graph is as shown in Fig. 14.1. Also negative investment, i.e., sell off of capital equipment, involves costs (to dismantling, reorganization etc.). Therefore $G_I < 0$ for $I < 0$. The important assumption, however, is that $G_{II} > 0$ (strict convexity in I), implying that the marginal installation cost is increasing in the level of investment. If the firm wants to accomplish a given installation project in only half the time, then the installation costs are more than doubled (the risk of mistakes is larger, the problems with reorganizing work routines are larger etc.). Think of building a new plant in a month instead of a year.

The strictly convex graph in Fig. 14.1 illustrates the essence of the matter. Assume the current capital stock in the firm is $\bar{K}$ and that the firm wants to increase it by a given amount $\overline{\Delta K}$. If the firm chooses the investment level $\bar{I} > 0$ per time unit in the time interval $[t, t + \Delta t)$, then, in view of (14.1), $\Delta K \approx (\bar{I} - \delta\bar{K})\Delta t$. So it takes $\Delta t \approx \overline{\Delta K}/(\bar{I} - \delta\bar{K})$ units of time to accomplish the desired increase $\overline{\Delta K}$. If, however, the firm slows down the adjustment and invests only half of $\bar{I}$ per time unit, then it takes approximately twice as long time to accomplish $\overline{\Delta K}$. Total costs of the two alternative courses of action are approximately $G(\bar{I}, \bar{K})\Delta t$ and $G(\frac{1}{2}\bar{I}, \bar{K})2\Delta t$, respectively (ignoring discounting and assuming the initial increase in capital is small in relation to $\bar{K}$). By drawing a few straight line segments in Fig. 14.1 the reader will be convinced that the last-mentioned cost is smaller than the first-mentioned due to strict convexity of installation costs (see Exercise 14.1). On the other hand, there are of course limits to how slow the adjustment to the desired capital stock should be. Slower adjustment means postponement of the benefits of a higher capital stock.

In addition to the strict convexity, (14.2) imposes the condition $G_K(I, K) \leq 0$. Indeed, it seems often realistic to assume that $G_K(I, K) < 0$ for $I \neq 0$. A given amount of investment may require more reorganization in a small firm than in a large firm (size here being measured by K). When installing a new machine, a small firm has to stop production altogether, whereas a large firm can to some extent continue its production by shifting some workers to another production line. A further argument is that the more a firm has invested historically, the more experienced it is now. So, for a given I today, the associated installation costs are lower, given a larger accumulated K.

14.1.1 The decision problem of the firm

In the absence of tax distortions, asymmetric information, and problems with enforceability of financial contracts, the Modigliani-Miller theorem (Modigliani and Miller, 1958) says that the financial structure of the firm is both indeterminate and irrelevant for production decisions (see Appendix A). Although the conditions required for this theorem are very idealized, the q-theory of investment accept them in a first approach in order to concentrate on the production aspects.

With the output good as unit of account, let cash flow (the net payment stream to the firm before interest payments on debt, if any) at time t be denoted R_t (for "receipts"). Then

$$R_t \equiv F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t, \tag{14.3}$$

where w_t is the wage per unit of labor at time t. The installation cost $G(I_t, K_t)$ implies that a part of production, $F(K_t, L_t)$, is used up in transforming investment goods into installed capital so that less output is available for sale.

Assume the firm is a price taker and ignore uncertainty. The interest rate is r_t, which we assume to be positive, at least in the long run. The decision problem, as seen from time 0, is to choose a plan $(L_t, I_t)_{t=0}^{\infty}$ so as to maximize the firm's *market value*, i.e., the present value of the future stream of expected cash flows:

$$\max_{(L_t, I_t)_{t=0}^{\infty}} V_0 = \int_0^{\infty} R_t e^{-\int_0^t r_s ds} dt \quad \text{s.t.} \tag{14.4}$$

$$L_t \geq 0, I_t \text{ free (i.e., no restriction on } I_t), \tag{14.5}$$

$$\dot{K}_t = I_t - \delta K_t, \qquad K_0 > 0 \text{ given}, \tag{14.6}$$

$$K_t \geq 0 \text{ for all } t. \tag{14.7}$$

There is no specific terminal condition but we have posited the feasibility condition (14.7) that the firm can never have a negative capital stock.[1]

In the previous chapters the firm was described as solving a series of static profit maximization problems. But such a description is no longer valid, when there is dependence across time, as there is here. When installation costs are present, current decisions depend on the expected future circumstances. The firm makes a plan for the whole future so as to maximize the value of the firm, which is what matters for the owners. This is the general neoclassical hypothesis about firms' behavior. As shown in Appendix A, when strictly convex installation costs or similar dependencies across time are absent, then value maximization is equivalent to solving a series of static profit maximization problems, and we are back in the previous chapters' description.

To solve the problem (14.4) − (14.7), where R_t is given by (14.3), we use the Maximum Principle. The problem has two control variables, L and I, and one state variable, K. We set up the current-value Hamiltonian:

$$H(K, L, I, q, t) \equiv F(K, L) - G(I, K) - wL - I + q(I - \delta K), \tag{14.8}$$

[1] To fix ideas, at points of discontinuity (if any) in investment, we will generally consider investment to be a *right-continuous* function of time. That is, $I_{t_0} = \lim_{t \to t_0^+} I_t$. Likewise, at such points of discontinuity, the time derivative of the corresponding state variable, K, should be interpreted as the *right-hand* time derivative, i.e., $\dot{K}_{t_0} = \lim_{t \to t_0^+} (K_t - K_{t_0})/(t - t_0)$.

where q (to be interpreted economically below) is the adjoint variable associated with the dynamic constraint (14.6). For each $t \geq 0$ we maximize H w.r.t. the control variables. Thus, $\partial H/\partial L = F_L(K, L) - w = 0$, i.e.,

$$F_L(K, L) = w; \tag{14.9}$$

and $\partial H/\partial I = -1 - G_I(I, K) + q = 0$, i.e.,

$$1 + G_I(I, K) = q. \tag{14.10}$$

Next, we partially differentiate H w.r.t. the state variable and set the result equal to $rq - \dot{q}$, where r is the discount rate in (14.4):

$$\frac{\partial H}{\partial K} = F_K(K, L) - G_K(I, K) - \delta q = rq - \dot{q}. \tag{14.11}$$

Then, the Maximum Principle says that for an interior optimal path (K_t, L_t, I_t), there exists an adjoint variable q_t such that for all $t \geq 0$ the conditions (14.9), (14.10), and (14.11) hold and the transversality condition

$$\lim_{t\to\infty} K_t q_t e^{-\int_0^t r_s ds} = 0 \tag{14.12}$$

is satisfied.

The optimality condition (14.9) is the usual employment condition equalizing the marginal product of labor to the real wage. In the present context with strictly convex capital installation costs, this condition attains a distinct role as labor will in the short run be the only variable input. This is because the strictly convex capital installation costs imply that the firm's installed capital in the short run is a quasi-fixed production factor. So, effectively there are diminishing returns (equivalent with rising marginal costs) in the short run even though the production function might have CRS.

The left-hand side of (14.10) gives the cost of acquiring one extra unit of installed capital at time t (the sum of the cost of buying the marginal investment good and the cost of its installation). That is, the left-hand side is the marginal cost, MC, of increasing the capital stock in the firm. Since (14.10) is a necessary condition for optimality, the right-hand side of (14.10) must be the marginal benefit, MB, of increasing the capital stock. Hence, q_t represents the value to the firm of having one more unit of (installed) capital at time t. To put it differently: the adjoint variable q_t can be interpreted as the shadow price (measured in current output units) of capital along the optimal path.[2]

[2]Recall that a *shadow price,* measured in some unit of account, of a good is the maximum number of units of account that the optimizing agent is willing to offer for one extra unit of the good.

This interpretation of q_t is confirmed when we solve the differential equation (14.11). Before doing that, however, it is convenient to consider the last equality sign in (14.11). A condition for optimality must be that the firm acquires capital up to the point where the "marginal productivity of capital", $F_K - G_K$, equals "capital costs", $r_t q_t + (\delta q_t - \dot{q}_t)$; here the first term represents interest costs and the second economic depreciation. The "marginal productivity of capital" appears as $F_K - G_K$, because we should take into account the reduction, $-G_K$, of installation costs in the next instant brought about by the marginal unit of installed capital. The shadow price q_t appears as the "overall" price at which the firm can buy and sell the marginal unit of installed capital. In fact, in view of $q_t = 1 + G_I(K_t, L_t)$ along the optimal path (from (14.10)), q_t *measures*, approximately, *both* the "overall" cost associated with *increasing* investment by one unit and the "overall" cost saving associated with *decreasing* investment by one unit. In the first case the firm not only has to pay one extra unit of account in the investment goods market but must also bear an installation cost equal to $G_I(K_t, L_t)$, thereby investing "overall" q_t units of account. And in the second case the firm recovers q_t by saving both on installation costs and purchases in the investment goods market. Continuing along this line of thought, by reordering in (14.11) we get the "no-arbitrage" condition

$$\frac{F_K - G_K - \delta q + \dot{q}}{q} = r, \tag{14.13}$$

saying that along the optimal path the rate of return on the marginal unit of capital must equal the interest rate.

The transversality condition (14.12) says that the present value of the capital stock "left over" at infinity must be zero. That is, the capital stock should not grow too fast compared with its discounted shadow price. The necessity of the transversality condition is demonstrated in Appendix B, where it is also shown that the discounted shadow price itself in the far future must be asymptotically zero along an optimal path, i.e.,

$$\lim_{t\to\infty} q_t e^{-\int_0^t r_s ds} = 0. \tag{14.14}$$

If along the optimal path, K_t grows without bound, then not only must (14.14) hold but, in view of (14.12), the discounted shadow price must in the long run approach zero *faster* than K_t grows. Intuitively, otherwise the firm would be "over-accumulating". The firm would gain by reducing the capital stock "left over" for eternity, since saving on the ultimate investment and installation costs would raise the present value of the expected future cash flow.

Multiplying by $e^{-\int_0^t (r_s+\delta)ds}$ on both sides of (14.11), we get by integration and application of (14.14),[3]

$$q_t = \int_t^\infty \left[F_K(K_\tau, L_\tau) - G_K(I_\tau, K_\tau)\right] e^{-\int_t^\tau (r_s+\delta)ds} d\tau. \tag{14.15}$$

The right-hand side of (14.15) is the present value, as seen from time t, of expected future increases of the firm's cash-flow that would result if one extra unit of capital were installed at time t; indeed, $F_K(K_\tau, L_\tau)$ is the direct contribution to output of one extra unit of capital, while $-G_K(I_\tau, K_\tau) \geq 0$ represents the reduction of installation costs in the next instant brought about by the marginal unit of installed capital. However, future increases of cash-flow should be discounted at a rate equal to the interest rate *plus* the capital depreciation rate; from one extra unit of capital at time t there are only $e^{-\delta(\tau-t)}$ units left at time τ.

To concretize our interpretation of q_t as representing the value to the firm at time t of having one extra unit of installed capital, let us make a thought experiment. Assume that a extra units of installed capital at time t drops down from the sky. At time $\tau > t$ there are $a \cdot e^{-\delta(\tau-t)}$ units of these still in operation so that the stock of installed capital is

$$K'_\tau = K_\tau + a \cdot e^{-\delta(\tau-t)}, \tag{14.16}$$

where K_τ denotes the stock of installed capital as it would have been without this "injection". Now, in (14.3) replace t by τ and consider the firm's cash-flow R_τ as a function of $(K_\tau, L_\tau, I_\tau, \tau, t, a)$. Taking the partial derivative of R_τ w.r.t. a at the point $(K_\tau, L_\tau, I_\tau, \tau, t, 0)$, we get

$$\frac{\partial R_\tau}{\partial a}_{|a=0} = \left[F_K(K_\tau, L_\tau) - G_K(I_\tau, K_\tau)\right] e^{-\delta(\tau-t)}.$$

Let the value of the optimizing firm as seen from time t be denoted V_t. Then at any point where V_t is differentiable, we have

$$\begin{aligned}\frac{\partial V_t}{\partial a}_{|a=0} &= \int_t^\infty \left(\frac{\partial R_\tau}{\partial a}_{|a=0}\right) e^{-\int_t^\tau r_s ds} d\tau \\ &= \int_t^\infty [F_K(K_\tau, L_\tau) - G_K(I_\tau, K_\tau)] e^{-\int_t^\tau (r_s+\delta)ds} d\tau = q_t \end{aligned} \tag{14.17}$$

when the firm moves along the optimal path. The last equality sign is implied by (14.15). So the value of the adjoint variable, q, at time t equals the

[3] For details, see Appendix B.

marginal contribution to the firm's market value of a hypothetical "injection" of installed capital at time t. This result confirms that q_t is the shadow price of installed capital at time t and that q_t represents the benefit of the marginal unit of installed capital.

There is a convention within economics to write the derivative on the very left side of (14.17) as "$\partial V_t/\partial K_t$". Henceforth, with some hesitation, we will follow this casual convention; so when we write that

$$\partial V_t/\partial K_t = q_t \tag{14.18}$$

holds along the optimal path, (14.17) is meant. (Our hesitation is due to the fact that the short-hand expression $\partial V_t/\partial K_t$ in (14.18) can be misunderstood as indicating the effect on V_t in response to a unit increase in K at time t, keeping K unchanged at all other points in time; as V_t is an integral, there would be no effect at all on V_t from such an increase in K at a single point in time.)

On this background it becomes understandable that the control variables at any point in time should be chosen so that the Hamiltonian function is maximized. Thereby one maximizes the sum of the current direct contribution to the criterion function and the indirect contribution, which is the benefit of having a higher capital stock in the future (as measured approximately by $q_t \Delta K_t$).

As we know, the Maximum Principle gives only necessary conditions for an optimal path, not sufficient conditions. We use the principle as a tool for finding candidates for a solution. Having found in this way a candidate, one way to proceed is to check whether Mangasarian's sufficient conditions are satisfied. Given the transversality condition (14.12) and the non-negativity of the state variable, K, the only additional condition to check is whether the Hamiltonian function is jointly concave in the endogenous variables (here I, K, and L). If it is jointly concave in these variables, then the candidate *is* an optimal solution. Owing to concavity of $F(K, L)$, inspection of (14.8) reveals that the Hamiltonian function is jointly concave in (I, K, L) if $-G(I, K)$ is jointly concave in (I, K). This condition is equivalent to $G(I, K)$ being jointly convex in (I, K), an assumption allowed within the confines of (14.2); for example, $G(I, K) = (\frac{1}{2})\beta I^2/K$ as well as the simpler $G(I, K) = (\frac{1}{2})\beta I^2$ (where in both cases $\beta > 0$) will do. Thus, assuming joint convexity of $G(I, K)$, the first-order conditions and the transversality condition are not only necessary, but also sufficient for an optimal solution.

14.1.2 The implied investment function

From condition (14.10) we can derive an investment function. Rewriting (14.10) we have that an optimal path satisfies

$$G_I(I_t, K_t) = q_t - 1. \tag{14.19}$$

Combining this with the assumption (14.2) on the installation cost function, we see that

$$I_t \gtreqless 0 \text{ for } q_t \gtreqless 1, \text{ respectively,} \tag{14.20}$$

cf. Fig. 14.2.[4] In view of $G_{II} > 0$, (14.19) implicitly defines optimal investment, I_t, as a function of the shadow price, q_t, and K_t :

$$I_t = \mathcal{M}(q_t, K_t). \tag{14.21}$$

By implicit differentiation w.r.t. q_t in (14.19), we find

$$\frac{\partial I_t}{\partial q_t} = \frac{1}{G_{II}(I_t, K_t)} > 0, \qquad \text{and} \qquad \frac{\partial I_t}{\partial K_t} = -\frac{G_{IK}(I_t, K_t)}{G_{II}(I_t, K_t)},$$

where the latter cannot be signed without further specification.

It follows that optimal investment is an increasing function of the shadow price of installed capital. In view of (14.20), $M(1, K) = 0$. Not surprisingly, the investment rule is: invest now, if and only if the value to the firm of the marginal unit of installed capital is larger than the price of the capital good (which is 1, excluding installation costs). At the same time, the rule says that, because of the convex installation costs, invest only up to the point where the marginal installation cost, $G_I(I_t, K_t)$, equals $q_t - 1$, cf. (14.19).

Condition (14.21) shows the remarkable information content that the shadow price q_t has. As soon as q_t is known (along with the current capital stock K_t), the firm can decide the optimal level of investment through knowledge of the installation cost function G alone (since, when G is known, so is in principle the inverse of G_I w.r.t. I, the investment function M). All the information about the production function, input prices, and interest rates now and in the future that is relevant to the investment decision is summarized in one number, q_t. The form of the investment function, M, depends only on the installation cost function G. These are very useful properties in theoretical and empirical analysis.

[4] From the assumptions made in (14.2), we only know that the graph of $G_I(I, \bar{K})$ is an upward-sloping curve going through the origin. Fig. 14.2 shows the special case where this curve happens to be linear.

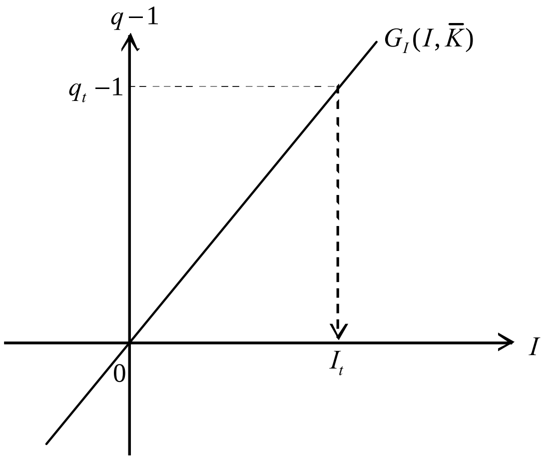

Figure 14.2: Marginal installation costs as a function of the gross investment level, I, for a given amount, $\bar{K}$, of installed capital. The optimal gross investment, I_t, when $q = q_t$ is indicated.

14.1.3 A not implausible special case

We now introduce the convenient case where the installation function G is homogeneous of degree one w.r.t. I and K so that we can, for $K > 0$, write

$$\begin{aligned} J &= G(I,K) = G(\frac{I}{K},1)K \equiv g(\frac{I}{K})K, \qquad \text{or} \qquad (14.22)\\ \frac{J}{K} &= g(\frac{I}{K}), \end{aligned}$$

where $g(\cdot)$ represents the installation cost-capital ratio and $g(0) \equiv G(0,1) = 0$, by (14.2).

LEMMA 1 The function $g(\cdot)$ has the following properties:
(i) $g'(I/K) = G_I(I,K)$;
(ii) $g''(I/K) = KG_{II}(I,K) > 0$ for $K > 0$; and
(iii) $g(I/K) - g'(I/K)I/K = G_K(I,K) < 0$ for $I \neq 0$.

Proof. (i) $G_I = Kg'/K = g'$; (ii) $G_{II} = g''/K$; (iii) $G_K = \partial(g(I/K)K)/\partial K = g(I/K) - g'(I/K)I/K < 0$ for $I \neq 0$ since, in view of $g'' > 0$ and $g(0) = 0$, we have $g(x)/x < g'(x)$ for all $x \neq 0$.[5] $\square$

The graph of $g(I/K)$ is qualitatively the same as that in Fig. 14.1 (imagine we have $\bar{K} = 1$ in that graph). The installation cost relative to the

[5] The graph of a strictly convex differentiable function lies above the tangent in any point.

existing capital stock is now a strictly convex function of the investment-capital ratio, I/K.

EXAMPLE 1 Let $J = G(I, K) = \frac{1}{2}\beta I^2/K$, where $\beta > 0$. Then G is homogeneous of degree one w.r.t. I and K and gives $J/K = \frac{1}{2}\beta(I/K)^2 \equiv g(I/K)$. □

A further important property of (14.22) is that the cash-flow function in (14.3) becomes homogeneous of degree one w.r.t. K, L, and I in the "normal" case where the production function has CRS. This has two implications. First, Hayashi's theorem applies (see below). Second, the q-theory can easily be incorporated in a model of economic growth.[6]

Does the hypothesis of linear homogeneity of the cash flow in K, L, and I make economic sense? According to the replication argument it does. Suppose a given firm has K units of installed capital and produces Y units of output with L units of labor. When at the same time the firm invests I units of account in new capital, it obtains the cash flow R after deducting the installation costs, $G(I, K)$. Then it makes sense to assume that the firm could do the same thing at another place, hereby doubling its cash-flow. (Of course, owing to the possibility of indivisibilities this reasoning does not take us all the way to linear homogeneity. Moreover, the argument ignores that also land is a necessary input. But as contended in Chapter 2, in manufacturing and service industries this source of departure from linear homogeneity seems minor and is perhaps offset by synergy effects from larger size.)

In view of (i) of Lemma 1, the linear homogeneity assumption for G allows us to write (14.19) as

$$g'(I/K) = q - 1. \tag{14.23}$$

This equation defines the investment-capital ratio, I/K , as an implicit function, m, of q :

$$\frac{I_t}{K_t} = m(q_t), \quad \text{where} \quad m(1) = 0 \quad \text{and} \quad m' = \frac{1}{g''} > 0, \tag{14.24}$$

by implicit differentiation in (14.23). In this case q encompasses all information that is of relevance to the decision about the investment-capital ratio.

In Example 1 above we have $g(I/K) = \frac{1}{2}\beta(I/K)^2$, in which case (14.23) gives $I/K = (q-1)/\beta$. So in this case we have $m(q) = q/\beta - 1/\beta$, a linear investment function, as illustrated in Fig. 14.3. The parameter β can be interpreted as the degree of sluggishness in the capital adjustment. The

[6] The relationship between our function g and other ways of formulating the theory is commented on in Appendix C.

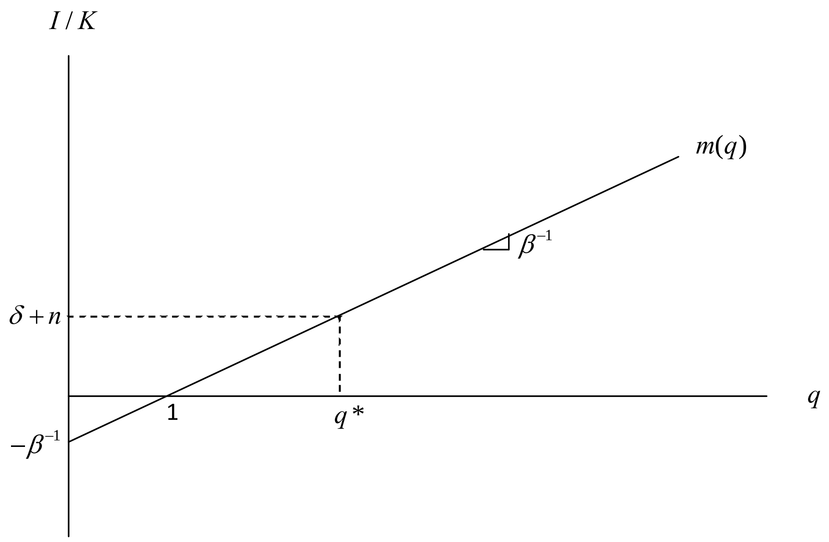

Figure 14.3: Optimal investment-capital ratio as a function of the shadow price of installed capital when $g(I/K) = \frac{1}{2}\beta(I/K)^2$.

degree of sluggishness reflects the degree of convexity of installation costs.[7] The stippled lines in Fig. 14.3 are explained below. Generally the investment function is positively sloped, but not necessarily linear.

To see how the shadow price q changes over time along the optimal path, we rewrite (14.11) as

$$\dot{q}_t = (r_t + \delta)q_t - F_K(K_t, L_t) + G_K(I_t, K_t). \tag{14.25}$$

Recall that $-G_K(I_t, K_t)$ indicates how much *lower* the installation costs are as a result of the marginal unit of installed capital. In the special case (14.22) we have from Lemma 1

$$G_K(I, K) = g(\frac{I}{K}) - \frac{I}{K}g'(\frac{I}{K}) = g(m(q)) - m(q)(q-1)$$

using (14.24) and (14.23).

Inserting this into (14.25) gives

$$\dot{q}_t = (r_t + \delta)q_t - F_K(K_t, L_t) + g(m(q_t)) - m(q_t)(q_t - 1). \tag{14.26}$$

[7] Indeed, for given q, the degree of sluggishness is proportional to the degree of convexity of adjustment costs: the degree of convexity of $g(\cdot)$ is $g''/g' = (I/K)^{-1} = \beta(q-1)^{-1}$, which gives $\beta = (q-1)g''/g'$.

This differential equation is very useful in macroeconomic analysis, as we will soon see, cf. Fig. 14.4 below.

In a macroeconomic context, for steady state to obtain, gross investment must be large enough to compensate not only for capital depreciation, but also for growth in the labor input. Otherwise a constant capital-labor ratio can not be sustained. That is, the investment-capital ratio, I/K, must be equal to the sum of the depreciation rate and the growth rate of the labor force, i.e., $\delta + n$. The level of q which is required to motivate such an investment-capital ratio is called q^* in Fig. 14.3.

14.2 Marginal q and average q

Our q above, determining investment, should be distinguished from what is usually called Tobin's q or average q. In a more general context, let p_{It} denote the purchase price (in terms of output units) per unit of the investment good (before installment). Then *Tobin's q* or *average q*, q_t^a, is defined as $q_t^a \equiv V_t/(p_{It}K_t)$, that is, Tobin's q is the ratio of the market value of the firm to the book value of the firm, sometimes called the "replacement value of the capital goods before installment" (the top index "a" stands for "average"). In our simplified context we have $p_{It} \equiv 1$ (the price of the investment good is the same as that of the output good). Therefore Tobin's q can be written

$$q_t^a \equiv \frac{V_t}{K_t}, \tag{14.27}$$

Conceptually this is different from the firm's internal shadow price on capital, our q above. In the language of the q-theory of investment our q is called *marginal* q, representing the value to the firms of one *extra* unit of installed capital relative to the value of uninstalled capital equipment. Indeed, the term marginal q is natural since along the optimal path we must have $q_t = (\partial V_t/\partial K_t)/p_{It} \equiv q_t^m$ ("m" for "marginal"). In our model above we consider the special case $p_{It} \equiv 1$ and so

$$q_t^m = q_t = \frac{\partial V_t}{\partial K_t}, \tag{14.28}$$

cf. (14.18).

The two concepts, average q and marginal q, have not always been clearly distinguished in the literature. What is relevant to the investment decision is marginal q. Indeed, the analysis above showed that optimal investment is an increasing function of q^m. Further, the analysis showed that a "critical" value of q^m is 1 and that only if $q^m > 1$, is positive gross investment warranted.

The importance of q^a is that it can be measured empirically as the ratio of the sum of the share market value of the firm and its debt to the replacement value of its total capital before installment. Since q^m is much harder to measure than q^a, it is important to know the relationship between q^m and q^a. Fortunately, we have a simple theorem giving conditions under which $q^m = q^a$.

THEOREM (Hayashi, 1982) Assume the firm is a price taker, that the production function F is jointly concave in (K, L), and that the installation cost function G is jointly convex in (I, K).[8] Then, along an optimal path we have:

(i) $q_t^m = q_t^a$ for all $t \geq 0$, if F and G are homogeneous of degree 1.

(ii) $q_t^m < q_t^a$ for all t, if F is strictly concave in (K, L) and/or G is strictly convex in (I, K).

Proof. See Appendix D.

The assumption that the firm is a price taker may, of course, seem critical. The Hayashi theorem has been generalized, however. Also a monopolistic firm, facing a downward-sloping demand curve and setting its own price, may have a cash flow which is homogeneous of degree one in the three variables K, L, and I. If so, then the condition $q_t^m = q_t^a$ for all $t \geq 0$ still holds (Abel 1990). Abel and Eberly (1994) present further generalizations.

In any case, when q^m is approximately equal to (or just proportional to) q^a, the theory gives a remarkably simple operational investment function, $I = m(q^a)K$, cf. (14.24). At the macro level we interpret q^a as the market valuation of the firms relative to the book value of their total capital stock. This market valuation is an indicator of the expected future earnings potential of the firms. Under the conditions in (i) of the Hayashi theorem the market valuation also indicates the marginal earnings potential of the firms, hence, it becomes a determinant of their investment. This establishment of a relationship between the stock market and firms' aggregate investment is the basic point in Tobin (1969).

14.3 Applications

Capital installation costs in a closed economy

Allowing for convex capital installation costs in the economy has far-reaching implications for the causal structure of a model of a closed economy. Investment decisions attain an active role in the economy and forward-looking

[8] That is, in addition to (14.2), we assume $G_{KK} \geq 0$ and $G_{II}G_{KK} - G_{IK}^2 \geq 0$. The specification in Example 1 above satisfies this.

expectations become important for these decisions. Expected future market conditions and announced future changes in corporate taxes and depreciation allowance will affect firms' investment already today.

The essence of the matter is that current and expected future interest rates have to adjust for aggregate saving to equal aggregate investment or, what amounts to the same, so that the output market and asset markets clear. Given full employment ($L_t = \bar{L}_t$), the output market clears when

$$F(K_t, \bar{L}_t) - G(I_t, K_t) = \text{value added} \equiv GDP_t = C_t + I_t,$$

where C_t is determined by the intertemporal utility maximization of the forward-looking households, and I_t is determined by the intertemporal value maximization of the forward-looking firms. This is the first time in this book that *clearing in the output market* is assigned an active role. In the earlier models investment was just a passive reflection of household saving. Desired investment was automatically equal to the residual of national income left over after consumption decisions had taken place. Nothing had to adjust to clear the output market, neither the interest rate nor output. In contrast, in the present model the adjustments needed for the continuous clearing in the output market are decisive for the determination of the macroeconomic dynamics.

In actual economies there may of course exist "secondary markets" for used capital goods and markets for renting capital goods owned by others. In view of the installation costs, however, shifting capital goods from one plant to another is costly. Therefore the turnover in that kind of markets tends to be limited.

Anyhow, contrary to the earlier models, in the present model the current interest rate is not tied down by a requirement that such markets clear. Instead, current and expected future (short-term) interest rates have to adjust so that the output market clears. We could just as well say that an adjustment of the whole structure of interest rates (the yield curve) takes place and constitutes the equilibrating mechanism in the output and asset markets. By having output market equilibrium playing this role in the model, a first step is taken towards medium- and short-run macroeconomic theory. We take further steps in later chapters, by allowing different forms of imperfect competition and nominal price rigidities to enter the picture. Then the demand side gets an active role both in the determination of q (and thereby investment) and in the determination of aggregate output and employment. This is what Keynesian theory (old and new) deals with. But for now we still assume perfect competition in all markets including the labor market; indeed, by instantaneous adjustment of the real wage, labor demand continuously matches labor supply. In this sense the present model framework remains a member of the neoclassical family (supply-dominated models).

A Ramsey model extended by strictly convex capital installation costs results in a three-dimensional dynamic system. We get a simpler set-up (as usual) by shifting the focus to a small open economy. This is what the next two subsections do.

A small open economy with capital installation costs

By introducing convex capital installation costs in a model of a small open economy (SOE), we avoid the counterfactual outcome that the capital stock adjusts *instantaneously* when the interest rate in the world financial market changes. In the standard neoclassical growth model for a small open economy, without convex capital installation costs, a rise in the interest rate leads immediately to a complete adjustment of the capital stock so as to equalize the net marginal productivity of capital to the new higher interest rate. Moreover, in that model expected *future* changes in the interest rate or in corporate taxes and depreciation allowances do *not* trigger an investment response until these changes actually happen. In contrast, when convex installation costs are present, expected future changes tend to have at least some influence on firms' investment already today.

We assume:

1. Perfect mobility across borders of goods and financial capital.

2. Domestic and foreign financial claims are perfect substitutes.

3. No mobility across borders of labor.

4. Labor supply is inelastic and constant and there is no technological progress.

5. The capital installation cost function $G(I, K)$ is homogeneous of degree 1.

In this setting the SOE faces an exogenous positive interest rate, r, which is given from the world financial market and which we assume constant. With $\bar{L} > 0$ denoting the constant labor supply in our SOE, continuous clearing in the labor market under perfect competition gives $L_t = \bar{L}$ for all $t \geq 0$ and

$$w_t = F_L(K_t, \bar{L}) \equiv w(K_t). \tag{14.29}$$

At any time t, K_t is predetermined in the sense that due to the convex installation costs, changes in K take time. Thus (14.29) *determines* the market real wage w_t.

Since the capital installation cost function $G(I, K)$ is assumed to be homogeneous of degree 1, point (iii) of Lemma 1 applies and we can write (14.26) as

$$\dot{q}_t = (r + \delta)q_t - F_K(K_t, \bar{L}) + g(m(q_t)) - (q_t - 1)m(q_t). \tag{14.30}$$

Here r and $\bar{L}$ are exogenous so that the capital stock, K, and its shadow price, q, are the only endogenous variables in this differential equation. Another differential equation with these two variables can be obtained by inserting (14.24) into (14.6) to get

$$\dot{K}_t = (m(q_t) - \delta)K_t, \qquad K_0 > 0 \text{ given.} \tag{14.31}$$

In addition, the transversality condition,

$$\lim_{t\to\infty} K_t q_t e^{-rt} = 0, \tag{14.32}$$

must hold.

Fig. 14.4 shows the phase diagram for these two coupled differential equations. Let q^* be defined as the value of q satisfying the equation $m(q) = \delta$. Since $m' > 0$, q^* is unique. Suppressing for convenience the explicit time subscripts, we then have

$$\dot{K} = 0 \ \text{ for } \ m(q) = \delta, \ \text{i.e., for } q = q^*.$$

When $\delta > 0$, we have $q^* > 1$. This is so because also mere reinvestment to offset capital depreciation requires an incentive, namely that the marginal value to the firm of replacing worn-out capital is larger than the purchase price of the investment good (since the installation cost must also be compensated). From (14.31) is seen that

$$\dot{K} \gtreqless 0 \text{ for } m(q) \gtreqless \delta, \ \text{respectively, i.e., for } q \gtreqless q^*, \ \text{respectively,}$$

cf. the horizontal arrows in Fig. 14.4.

From (14.30) we have

$$\dot{q} = 0 \text{ for } 0 = (r + \delta)q - F_K(K, \bar{L}) + g(m(q)) - m(q)(q - 1). \tag{14.33}$$

If, in addition $\dot{K} = 0$ (hence, $q = q^*$ and $m(q) = m(q^*) = \delta$), this gives

$$0 = (r + \delta)q^* - F_K(K, \bar{L}) + g(\delta) - \delta(q^* - 1), \tag{14.34}$$

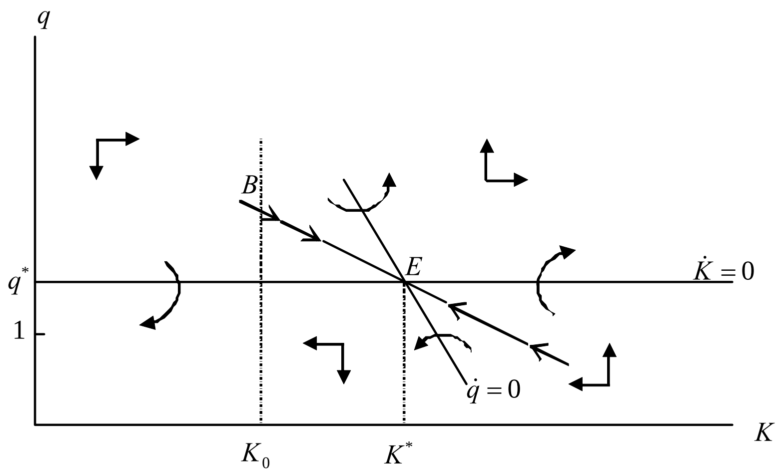

Figure 14.4: Phase diagram for investment dynamics in a small open economy (a case where $\delta > 0$).

where the right-hand-side is increasing in K, in view of $F_{KK} < 0$. Hence, there exists at most one value of K such that the steady state condition (14.34) is satisfied;[9] this value is denoted K^*, corresponding to the steady state point E in Fig. 14.4. The question is now: what is the slope of the $\dot{q} = 0$ locus? In Appendix E it is shown that at least in a neighborhood of the steady state point E this slope is negative in view of the assumption $r > 0$ and $F_{KK} < 0$. From (14.30) we see that

$\dot{q} \lesseqgtr 0$ for points to the left and to the right, respectively, of the $\dot{q} = 0$ locus,

since $F_{KK}(K_t, \bar{L}) < 0$. The vertical arrows in Fig. 14.4 show these directions of movement.

Altogether the phase diagram shows that the steady state E is a saddle point, and since there is one predetermined variable, K, and one jump variable, q, and the saddle path is not parallel to the jump variable axis, the steady state is saddle-point stable. At time 0 the economy will be at the point B in Fig. 14.4 where the vertical line $K = K_0$ crosses the saddle path. Then the economy will move along the saddle path towards the steady state. This solution satisfies the transversality condition (14.32) and is the unique solution to the model (for details, see Appendix F).

[9] And assuming that F satisfies the Inada conditions, (14.34) shows that such a value *does* exist.

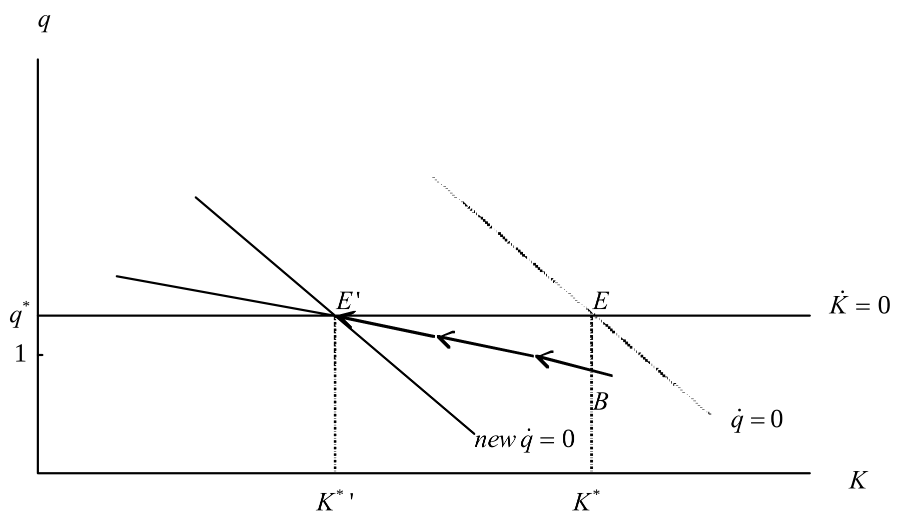

Figure 14.5:

The effect of a rise in the interest rate Assume that until time 0 the economy has been in the steady state E in Fig. 14.4. Then, an unexpected shift in the interest rate occurs so that the new interest rate is a constant $r' > r$ (and this interest rate is rightly expected to remain at this level forever in the future). From (14.31) we see that q^* is not affected by this shift, hence, the $\dot{K} = 0$ locus is not affected. However, (14.33) implies that the $\dot{q} = 0$ locus and K^* shift to the left, in view of $F_{KK}(K, \bar{L}) < 0$.

Fig. 14.5 illustrates the situation for $t > 0$. At time $t = 0$ the shadow price q jumps down to a level corresponding to the point B in Fig. 14.5. There is now a more heavy discounting of the future benefits that the marginal unit of capital can provide. As a result the incentive to invest is diminished and gross investment will not even compensate for the depreciation of capital. Hence, the capital stock decreases gradually. This is where we see a crucial role of convex capital installation costs in an open economy. For now, the installation costs are the costs associated with disinvestment (disassembling and selling out of machines). If these convex costs were not present, we would get the same counterfactual prediction as from the previous open-economy models in this book, namely that the new steady state is attained immediately after the shift in the interest rate.

As the capital stock is diminished, the marginal productivity of capital rises and so does q. The economy moves along the new saddle path and approaches the new steady state E' as time goes by.

Suppose that for some reason such a decrease in the capital stock is not desirable from a social point of view; this could be because of positive external effects of capital and investment, e.g., a kind of "learning by doing". Then the government could decide to implement an investment subsidy σ, $0 < \sigma < 1$, so that to attain an investment level I, purchasing the investment goods involves a cost of $(1-\sigma)I$. Assuming the subsidy is financed by some tax not affecting firms' behavior (for example a constant tax on households' consumption), investment is increased again and the economy may in the long run end up at the old steady-state level of K (but the new q^* will be lower than the old).

A growing small open economy with capital installation costs

The basic assumptions are the same as in the previous section except that now labor supply, $\bar{L}$, grows at the constant rate $n \geq 0$, while the technology level, T, grows at the constant rate $\gamma \geq 0$ (both rates exogenous and constant). Still assuming full employment, $L = \bar{L} = \bar{L}_0 e^{nt}$.

In this setting the production function on intensive form is useful:

$$Y = F(K, T\bar{L}) = F(\frac{K}{T\bar{L}}, 1)T\bar{L} \equiv f(\tilde{k})T\bar{L},$$

where $\tilde{k} \equiv K/(T\bar{L})$ and f satisfies $f' > 0$ and $f'' < 0$. Still assuming perfect competition, the market-clearing real wage at time t is determined as

$$w_t = F_2(K_t, T_t\bar{L}_t)T_t = \left[f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t)\right] T_t \equiv \tilde{w}(\tilde{k}_t)T_t,$$

where both $\tilde{k}_t$ and T_t are predetermined. By log-differentiation of $\tilde{k} \equiv K/(T\bar{L})$ w.r.t. time we get $\dot{\tilde{k}}_t/\tilde{k}_t = \dot{K}_t/K_t - (\gamma + n)$. Substituting (14.31), we get

$$\dot{\tilde{k}}_t = [m(q_t) - (\delta + \gamma + n)]\,\tilde{k}_t. \tag{14.35}$$

The change in the shadow price of capital is now described by

$$\dot{q}_t = (r+\delta)q_t - f'(\tilde{k}_t) + g(m(q_t)) - (q_t - 1)m(q_t), \tag{14.36}$$

from (14.26). In addition, the transversality condition,

$$\lim_{t\to\infty} \tilde{k}_t q_t e^{-(r-\gamma-n)t} = 0, \tag{14.37}$$

must hold.

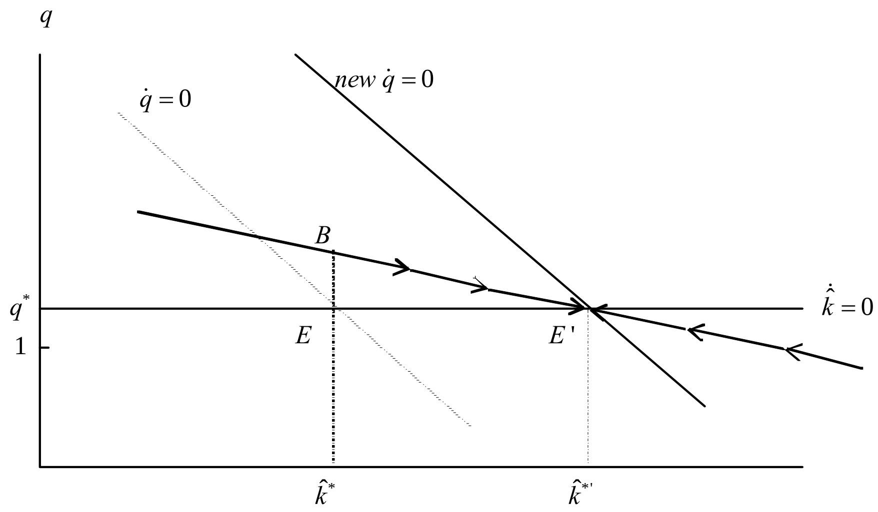

Figure 14.6:

The differential equations (14.35) and (14.36) constitute our new dynamic system. Fig. 14.6 shows the phase diagram, which is qualitatively similar to that in Fig. 14.4. We have

$$\dot{\tilde{k}} = 0 \quad \text{for} \;\; m(q) = \delta + \gamma + n, \text{ i.e., for } q = q^*,$$

where q^* now is defined by the requirement $m(q^*) = \delta + \gamma + n$. Notice, that when $\gamma + n > 0$, we get a larger steady state value q^* than in the previous section. This is so because now a higher investment-capital ratio is required for a steady state to be possible. Moreover, the transversality condition (14.12) is satisfied in the steady state.

From (14.36) we see that $\dot{q} = 0$ now requires

$$0 = (r + \delta)q - f'(\tilde{k}) + g(m(q)) - m(q)(q - 1).$$

If, in addition $\dot{\tilde{k}} = 0$ (hence, $q = q^*$ and $m(q) = m(q^*) = \delta + \gamma + n$), this gives

$$0 = (r + \delta)q^* - f'(\tilde{k}) + g(\delta + \gamma + n) - (\delta + \gamma + n)(q^* - 1).$$

Here, the right-hand-side is increasing in $\tilde{k}$ (in view of $f''(\tilde{k}) < 0$). Hence, the steady state value $\tilde{k}^*$ of the effective capital-labor ratio is unique, cf. the steady state point E in Fig. 14.6.

Assuming $r > \gamma + n$ we have, at least in a neighborhood of E in Fig. 14.6, that the $\dot{q} = 0$ locus is negatively sloped (see Appendix E).[10] Again the steady state is a saddle point, and the economy moves along the saddle path towards the steady state.

In Fig. 14.6 it is assumed that until time 0, the economy has been in the steady state E. Then, an unexpected shift in the interest rate to a *lower* constant level, r', takes place. The $\dot{q} = 0$ locus is shifted to the right, in view of $f'' < 0$. The shadow price, q, immediately jumps up to a level corresponding to the point B in Fig. 14.6. The economy moves along the new saddle path and approaches the new steady state E*'* with a higher effective capital-labor ratio as time goes by. In Exercise 14.2 the reader is asked to examine the analogue situation where an unanticipated downward shift in the rate of technological progress takes place.

14.4 Concluding remarks

Tobin's q-theory of investment gives a remarkably simple operational macroeconomic investment function, in which the key variable explaining aggregate investment is the valuation of the firms by the stock market relative to the book value of the firms' physical capital. This link between asset markets and firms' aggregate investment is an appealing feature of Tobin's q-theory.

When faced with strictly convex installation costs, the firm has to take the *future* into account to invest optimally. Therefore, the firm's *expectations* become important. The firm will adjust its capital stock only *gradually* when new information arises. By incorporating these features, Tobin's q-theory helps explaining the sluggishness in investment we see in the empirical data. And the theory avoids the counterfactual outcome from earlier chapters that the capital stock in a small open economy with perfect mobility of goods and financial capital is *instantaneously* adjusted when the interest rate in the world market changes.

Many econometric tests of the q theory of investment have been made, often with quite critical implications. Movements in q^a, even taking account of changes in taxation, seem capable of explaining only a minor fraction of the movements in investment. And the estimated equations relating fixed capital investment to q^a typically give strong auto-correlation in the residuals. Other variables, in particular availability of current corporate profits for internal financing, seem to have explanatory power independently of q^a

[10] In our perfect foresight model we in fact *have* to assume $r > \gamma + n$ for the firm's maximization problem to be well-defined. If instead $r \leq \gamma + n$, the market value of the representative firm would be infinite, and maximization would loose its meaning.

(see Abel 1990, Chirinko 1993, Gilchrist and Himmelberg, 1995). So there is reason to be somewhat sceptical towards the notion that all information of relevance for the investment decision is reflected by the market valuation of firms. This throws doubt on the basic assumption in Hayashi's theorem or its generalization, the assumption that firms' cash flow tends to be homogeneous of degree one w.r.t. K, L, and I.

Going outside the model, there are further circumstances relaxing the link between q^a and investment. In the real world with many production sectors, physical capital is heterogeneous. If for example a sharp unexpected rise in the price of energy takes place, a firm with energy-intensive technology will fall in market value. At the same time it has an incentive to invest in energy-saving capital equipment. Hence, we might observe a fall in q^a at the same time as investment increases.

Imperfections in credit markets are ignored by the model. Their presence further loosens the relationship between q^a and investment and may help explain the observed positive correlation between investment and corporate profits.

We might also question that capital installation costs really have the hypothesized *convex* form. It is one thing that there are costs associated with installation, reorganizing and retraining etc., when new capital equipment is procured. But should we expect these costs to be strictly convex in the volume of investment? To think about this, let us for a moment ignore the role of the existing capital stock. Hence, we write total installation costs $J = G(I)$ with $G(0) = 0$. It does not seem problematic to assume $G'(I) > 0$ for $I > 0$. The question concerns the assumption $G''(I) > 0$. According to this assumption the average installation cost $G(I)/I$ must be increasing in I.[11] But against this speaks the fact that capital installation may involve indivisibilities, fixed costs, acquisition of new information etc. All these features tend to imply *decreasing* average costs. In any case, at least at the microeconomic level one should expect unevenness in the capital adjustment process rather than the above smooth adjustment.

Because of the mixed empirical success of the convex installation cost hypothesis other theoretical approaches that can account for sluggish and sometimes non-smooth and lumpy capital adjustment have been considered: uncertainty, investment irreversibility, indivisibility, or financial problems due to bankruptcy costs (Nickell 1978, Zeira 1987, Dixit and Pindyck 1994, Caballero 1999, Adda and Cooper 2003). These approaches notwithstanding, it turns out that the q-theory of investment has recently been somewhat re-

[11] Indeed, for $I \neq 0$ we have $d[G(I)/I]/dI = [IG'(I) - G(I)]/I^2 > 0$, when G is strictly convex ($G'' > 0$) and $G(0) = 0$.

habilitated from both a theoretical and an empirical point of view. At the theoretical level Wang and Wen (2010) show that financial frictions in the form of collateralized borrowing at the firm level can give rise to strictly convex adjustment costs at the aggregate level yet at the same time generate lumpiness in plant-level investment. For large firms, unlikely to be much affected by financial frictions, Eberly et al. (2008) find that the theory does a good job in explaining investment behavior.

In any case, the q-theory of investment is in different versions widely used in macroeconomics because of its simplicity and the appealing link it establishes between asset markets and firms' investment. And the q-theory has also had an important role in studies of the housing market and the role of housing prices for household wealth and consumption, a theme to which we return in the next chapter.

14.5 Literature notes

Both the Ramsey model and the Blanchard OLG model for a closed market economy may be extended by adding strictly convex capital installation costs, see Abel and Blanchard (1983), Lim and Weil (2003), and Groth and Madsen (2010). Adding a public sector, such a framework is useful for the study of how different subsidies, taxes, and depreciation allowance schemes affect investment in physical capital as well as housing, see, e.g., Summers (1981).

14.6 Appendix

A. When value maximization is - and is not - equivalent with continuous static profit maximization

For the idealized case where tax distortions, asymmetric information, and problems with enforceability of financial contracts are absent, the Modigliani-Miller theorem (Modigliani and Miller, 1958) says that the financial structure of the firm is both indeterminate and irrelevant for production outcomes. Considering the firm described in Section 14.1, we may illustrate the implied separation of the financing decision from the production and investment decision in the following way.

A simple version of the Modigliani-Miller theorem Although the theorem includes conditions of risk, we here ignore risk. Let the real debt of the firm be denoted B_t and the real dividends, X_t. We then have the accounting relationship

$$\dot{B}_t = X_t - (F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t - r_t B_t).$$

A positive X_t represents dividends in the usual meaning (payout to the owners of the firm), whereas a negative X_t can be interpreted as emission of new shares of stock. Since we assume perfect competition, the time path of w_t and r_t is exogenous to the firm.

We first consider the firm's combined financing and production-investment problem, which we call *Problem I.* We assume that those who own the firm at time 0 want it to maximize its net worth, i.e., the present value of the dividends:

$$\begin{aligned}
\max_{(L_t, I_t, X_t)_{t=0}^{\infty}} \tilde{V}_0 &= \int_0^{\infty} X_t e^{-\int_0^t r_s ds} dt \quad \text{s.t.} \\
L_t &\geq 0, I_t \text{ free}, \\
\dot{K}_t &= I_t - \delta K_t, \qquad K_0 > 0 \text{ given}, \quad K_t \geq 0 \text{ for all } t, \\
\dot{B}_t &= X_t - (F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t - r_t B_t), \\
&\quad \text{where } B_0 \text{ is given}, \qquad (14.38) \\
\lim_{t \to \infty} B_t e^{-\int_0^t r_s ds} &\leq 0. \qquad \text{(NPG)}
\end{aligned}$$

The last constraint is a No-Ponzi-Game condition, saying that a positive debt should in the long run at most grow at a rate *less* than the interest rate.

In Section 14.1 we considered another problem, namely a separate investment-production problem:

$$\begin{aligned}
\max_{(L_t, I_t)_{t=0}^{\infty}} V_0 &= \int_0^{\infty} R_t e^{-\int_0^t r_s ds} dt \quad \text{s.t.,} \\
R_t &\equiv F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t, \\
L_t &\geq 0, I_t \text{ free}, \\
\dot{K}_t &= I_t - \delta K_t, \qquad K_0 > 0 \text{ given}, \quad K_t \geq 0 \text{ for all } t.
\end{aligned}$$

Let this problem, where the financing aspects are ignored, be called *Problem II.* When considering the relationship between Problem I and Problem II, the following mathematical fact is useful.

LEMMA A1 Consider a continuous function $a(t)$ and a differentiable function $f(t)$. Then

$$\int_{t_0}^{t_1} (f'(t) - a(t)f(t)) e^{-\int_{t_0}^{t} a(s)ds} dt = f(t_1) e^{-\int_{t_0}^{t_1} a(s)ds} - f(t_0).$$

Proof. Integration by parts from time t_0 to time t_1 yields

$$\int_{t_0}^{t_1} f'(t)e^{-\int_{t_0}^{t} a(s)ds}dt = f(t)e^{-\int_{t_0}^{t} a(s)ds}\big|_{t_0}^{t_1} + \int_{t_0}^{t_1} f(t)a(t)e^{-\int_{t_0}^{t} a(s)ds}dt.$$

Hence,

$$\begin{aligned} &\int_{t_0}^{t_1} (f'(t) - a(t)f(t))e^{-\int_{t_0}^{t} a(s)ds}dt \\ = \;& f(t_1)e^{-\int_{t_0}^{t_1} a(s)ds} - f(t_0). \qquad \square \end{aligned}$$

CLAIM 1 If $(K_t^*, B_t^*, L_t^*, I_t^*, X_t^*)_{t=0}^{\infty}$ is a solution to Problem I, then $(K_t^*, L_t^*, I_t^*)_{t=0}^{\infty}$ is a solution to Problem II.

Proof. By (14.38) and the definition of R_t, $X_t = R_t + \dot{B}_t - r_t B_t$ so that

$$\tilde{V}_0 = \int_0^{\infty} X_t e^{-\int_0^t r_s ds}dt = V_0 + \int_0^{\infty} (\dot{B}_t - r_t B_t)e^{-\int_0^t r_s ds}dt. \qquad (14.39)$$

In Lemma A1, let $f(t) = B_t$, $a(t) = r_t$, $t_0 = 0$, $t_1 = T$ and consider $T \to \infty$. Then

$$\lim_{T\to\infty} \int_0^T (\dot{B}_t - r_t B_t)e^{-\int_0^t r_s ds}dt = \lim_{T\to\infty} B_T e^{-\int_0^T r_s ds} - B_0 \le -B_0,$$

where the weak inequality is due to (NPG). Substituting this into (14.39), we see that maximum of net worth $\tilde{V}_0$ is obtained by maximizing V_0 and ensuring $\lim_{T\to\infty} B_T e^{-\int_0^T r_s ds} = 0$, in which case net worth equals (maximized V_0)$-B_0$, where B_0 is given. So a plan that maximizes net worth of the firm must also maximize V_0 in Problem II. $\square$

Thus it does not matter for the firm's production and investment behavior whether the firm's investment is financed by issuing new debt or by issuing shares of stock. Moreover, if we assume investors do not care about whether they receive the firm's earnings in the form of dividends or valuation gains on the shares, the firm's dividend policy is also irrelevant. Hence, from now on we concentrate on the investment-production problem, Problem II above.

The case with no capital installation costs Suppose the firm has no capital installation costs. Then the cash flow reduces to $R_t = F(K_t, L_t) - w_t L_t - I_t$.

CLAIM 2 When there are no capital installation costs, Problem II can be reduced to a series of static profit maximization problems.

Proof. Current (pure) profit is defined as

$$\Pi_t = F(K_t, L_t) - w_t L_t - (r_t + \delta)K_t \equiv \Pi(K_t, L_t).$$

It follows that R_t can be written

$$R_t = F(K_t, L_t) - w_t L_t - (\dot{K}_t + \delta K_t) = \Pi_t + (r_t + \delta)K_t - (\dot{K}_t + \delta K_t). \quad (14.40)$$

Hence,

$$V_0 = \int_0^\infty \Pi_t e^{-\int_0^t r_s ds} dt + \int_0^\infty (r_t K_t - \dot{K}_t) e^{-\int_0^t r_s ds} dt. \quad (14.41)$$

The first integral on the right-hand side of this expression is independent of the second. Indeed, the firm can maximize the first integral by renting capital and labor, K_t and L_t, at the going factor prices, $r_t + \delta$ and w_t, respectively, such that $\Pi_t = \Pi(K_t, L_t)$ is maximized at each t. The factor costs are accounted for in the definition of Π_t.

The second integral on the right-hand side of (14.41) is the present value of net revenue from renting capital out to others. In Lemma A1, let $f(t) = K_t$, $a(t) = r_t$, $t_0 = 0$, $t_1 = T$ and consider $T \to \infty$. Then

$$\lim_{T\to\infty} \int_0^T (r_t K_t - \dot{K}_t) e^{-\int_0^t r_s ds} dt = K_0 - \lim_{T\to\infty} K_T e^{-\int_0^T r_s ds} = K_0, \quad (14.42)$$

where the last equality comes from the fact that maximization of V_0 requires maximization of the left-hand side of (14.42) which in turn, since K_0 is given, requires minimization of $\lim_{T\to\infty} K_T e^{-\int_0^T r_s ds}$. The latter expression is always non-negative and can be made zero by choosing any time path for K_t such that $\lim_{T\to\infty} K_T = 0$. (We may alternatively put it this way: it never pays the firm to accumulate costly capital so fast in the long run that $\lim_{T\to\infty} K_T e^{-\int_0^T r_s ds} > 0$, that is, to maintain accumulation of capital at a rate equal to or higher than the interest rate.) Substituting (14.42) into (14.41), we get $V_0 = \int_0^\infty \Pi_t e^{-\int_0^t r_s ds} dt + K_0$.

The conclusion is that, given K_0,[12] V_0 is maximized if and only if K_t and L_t are at each t chosen such that $\Pi_t = \Pi(K_t, L_t)$ is maximized. □

[12] Note that the historically given K_0 is no more "given" than the firm may instantly let it jump to a lower or higher level. In the first case the firm would immediately sell a bunch of its machines, and in the latter case it would immediately buy a bunch of machines. Indeed, without convex capital installation costs nothing rules out jumps in the capital stock. But such jumps just reflect an immediate jump, in the opposite direction, in another asset item in the balance sheet.

The case with strictly convex capital installation costs Now we reintroduce the capital installation cost function $G(I_t, K_t)$, satisfying in particular the condition $G_{II}(I, K) > 0$ for all (I, K). Then, as shown in the text, the firm adjusts to a change in its environment, say a downward shift in r, by a *gradual* adjustment of K, in this case upward, rather than instantaneous maximization of $\Pi(K_t, L_t)$. The latter would entail an instantaneous upward jump in K_t of size $\Delta K_t = a > 0$, requiring $I_t \cdot \Delta t = a$ for $\Delta t = 0$. This would require $I_t = \infty$, which requires $G(I_t, K_t) = \infty$, telling us that such a jump is impossible.

Derivation of (14.15) Rearranging (14.11) and multiplying through by the integrating factor $e^{-\int_0^t (r_s+\delta)ds}$, we get

$$[(r_t + \delta)q_t - \dot{q}_t]\, e^{-\int_0^t (r_s+\delta)ds} = (F_{Kt} - G_{Kt})\, e^{-\int_0^t (r_s+\delta)ds}, \tag{14.43}$$

where $F_{Kt} \equiv F_K(K_t, L_t)$ and $G_{Kt} \equiv G_K(I_t, K_t)$. In Lemma A1, let $f(t) = q_t$, $a(t) = r_t + \delta$, $t_0 = 0$, $t_1 = T$. Then

$$\begin{aligned} \int_0^T [(r_t + \delta)q_t - \dot{q}_t]\, e^{-\int_0^t (r_s+\delta)ds} dt &= q_0 - q_T e^{-\int_0^T (r_s+\delta)ds} \\ &= \int_0^T (F_{Kt} - G_{Kt})\, e^{-\int_0^t (r_s+\delta)ds} dt, \end{aligned}$$

where the last equality comes from (14.43). Letting $T \to \infty$, we get

$$q_0 - \lim_{T\to\infty} q_T e^{-\int_0^T (r_s+\delta)ds} = q_0 = \int_0^\infty (F_{Kt} - G_{Kt})\, e^{-\int_0^t (r_s+\delta)ds} dt, \tag{14.44}$$

where the first equality follows from the transversality condition (14.14), which we repeat here:

$$\lim_{t\to\infty} q_t e^{-\int_0^t r_s ds} = 0. \tag{14.14}$$

Indeed, since $\delta \geq 0$, $\lim_{T\to\infty}(e^{-\int_0^T r_s ds} e^{-\delta T}) = 0$, by (14.14). Initial time is arbitrary, and so we may replace 0 and t in (14.44) by t and τ, respectively. This proves (14.15).[13]

[13] An equivalent approach, leading to the same result, can be based on applying the transversality condition (14.14) to the general solution formula for linear inhomogeneous first-order differential equations. Indeed, the first-order condition (14.11) provides such a differential equation in q_t.

B. Transversality conditions

Proof of necessity of (14.14) In view of (14.44), a qualified conjecture is that the condition $\lim_{t\to\infty} q_t e^{-\int_0^t (r_s+\delta)ds} = 0$ is necessary for optimality. This is indeed true, since this condition follows from the stronger transversality condition, (14.14), the necessity of which along an optimal path we will now prove.

Rearranging (14.11) and multiplying through by the integrating factor $e^{-\int_0^t r_s ds}$, we have

$$(r_t q_t - \dot{q}_t) e^{-\int_0^t r_s ds} = (F_{Kt} - G_{Kt} - \delta q_t)\, e^{-\int_0^t r_s ds}.$$

In Lemma A1, let $f(t) = q_t$, $a(t) = r_t$, $t_0 = 0$, $t_1 = T$. Then

$$\int_0^T (r_t q_t - \dot{q}_t) e^{-\int_0^t r_s ds} dt = q_0 - q_T e^{-\int_0^T r_s ds} = \int_0^T (F_{Kt} - G_{Kt} - \delta q_t)\, e^{-\int_0^t r_s ds} dt.$$

Rearranging and letting $T \to \infty$, we see that

$$q_0 = \int_0^\infty (F_{Kt} - G_{Kt} - \delta q_t)\, e^{-\int_0^t r_s ds} dt + \lim_{T\to\infty} q_T e^{-\int_0^T r_s ds}. \tag{14.45}$$

If, contrary to (14.14), $\lim_{T\to\infty} q_T e^{-\int_0^T r_s ds} > 0$ along the optimal path, then (14.45) shows that the firm is over-investing. By reducing initial investment by one unit the firm would save approximately $q_0 = 1 + G_I(I_0, K_0)$, which would be more than the present value of the stream of potential net gains coming from this marginal unit of installed capital (the first term on the right-hand side of (14.45)).

Suppose instead that $\lim_{T\to\infty} q_T e^{-\int_0^T r_s ds} < 0$. Then, by a symmetric argument, the firm has under-invested initially.

Necessity of (14.12) In cases where along an optimal path, K_t remains bounded for $t \to \infty$, the transversality condition (14.12) is implied by (14.14). In cases where along an optimal path, K_t is unbounded for $t \to \infty$, the transversality condition (14.12) is stronger than (14.14).

A proof of the necessity of (14.12) in this case can be based on Weitzman (2003).

C. On different specifications of the q-theory

The simple relationship we have found between I and q can easily be generalized to the case where the purchase price on the investment good, p_{It}, is allowed to differ from 1 (its value above) and the capital installation cost is $p_{It}G(I_t, K_t)$. In this case it is convenient to replace q in the Hamiltonian function by, say, λ. Then the first-order condition (14.10) becomes $p_{It} + p_{It}G_I(I_t, K_t) = \lambda_t$, implying

$$G_I(I_t, K_t) = \frac{\lambda_t}{p_{It}} - 1,$$

and we can proceed, defining as before q_t by $q_t \equiv \lambda_t/p_{It}$.

Sometimes in the literature installation costs, J, appear in a slightly different form compared to the above exposition. For example, Romer (2001, p. 371 ff.) assumes the capital installation costs, J, depend only on I so that $G_K \equiv 0$. But applied to a model with economic growth this will result in installation costs that rise faster than output and ultimately swallow the total produce.

Abel and Blanchard (1983), followed by Barro and Sala-i-Martin (2004, p. 152-160), introduce a function, ϕ, representing capital installation costs *per unit of investment* as a function of the investment-capital ratio. That is, total installation cost is $J = \phi(I/K)I$, where $\phi(0) = 0, \phi' > 0$. This implies that $J/K = \phi(I/K)(I/K)$. The right-hand side of this equation may be called $g(I/K)$, and then we are back at the formulation in Section 14.1. Indeed, defining $x \equiv I/K$, we have installation costs per unit of capital equal to $g(x) = \phi(x)x$, and assuming $\phi(0) = 0, \phi' > 0$, it holds that

$$\begin{aligned} g(x) &= 0 \text{ for } x = 0,\ g(x) > 0 \text{ for } x \neq 0, \\ g'(x) &= \phi(x) + x\phi'(x) \gtreqless 0 \text{ for } x \gtreqless 0, \text{ respectively, and} \\ g''(x) &= 2\phi'(x) + x\phi''(x). \end{aligned}$$

Here $g''(x)$ must be positive for the theory to work. But the assumptions $\phi(0) = 0, \phi' > 0$, and $\phi'' \geq 0$, imposed in p. 153 and again in p. 154 in the Barro and Sala-i-Martin book, are *not* sufficient for this (since $x < 0$ is possible). This is why we prefer the $g(\cdot)$ formulation rather than the $\phi(\cdot)$ formulation.

It is sometimes convenient to let the capital installation cost $G(I,\ K)$ appear, not as a reduction in output, but as a reduction in capital formation so that

$$\dot{K} = I - \delta K - G(I, K). \tag{14.46}$$

This approach is used in Hayashi (1982) and Heijdra and Ploeg (2002, p. 573 ff.). For example, Heijdra and Ploeg write the rate of capital accumulation as $\dot{K}/K = \varphi(I/K) - \delta$, where the "capital installation function" $\varphi(I/K)$ can be interpreted as $\varphi(I/K) \equiv [I - G(I, K)]/K = I/K - g(I/K)$; the latter equality comes from assuming G is homogeneous of degree 1. In one-sector models, as we usually consider in this text, this changes nothing of importance. In more general models this installation function approach may have some analytical advantages. What gives the best fit empirically is an open question. In our housing market model in the next chapter we apply a specification analogue to (14.46), interpreting $\dot{K}$ as the number of new houses per time unit.

D. Proof of Hayashi's theorem

For convenience we repeat:

THEOREM (Hayashi) Assume the firm is a price taker, that the production function F is jointly concave in (K, L), and that the installation cost function G is jointly convex in (I, K). Then, along the optimal path we have:

(i) $q_t^m = q_t^a$ for all $t \geq 0$, if F and G are homogeneous of degree 1.

(ii) $q_t^m < q_t^a$ for all t, if F is strictly concave in (K, L) and/or G is strictly convex in (I, K).

Proof. The value of the firm as seen from time t is

$$V_t = \int_t^\infty (F(K_\tau, L_\tau) - G(I_\tau, K_\tau) - w_\tau L_\tau - I_\tau)e^{-\int_t^\tau r_s ds} d\tau \tag{14.47}$$

We introduce the functions

$$\begin{aligned} A &= A(K, L) \equiv F(K, L) - F_K(K, L)K - F_L(K, L)L, \text{ and} && (14.48) \\ B &= B(I, K) \equiv G_I(I, K)I + G_K(I, K)K - G(I, K). && (14.49) \end{aligned}$$

Then the cash-flow of the firm at time τ can be written

$$\begin{aligned} R_\tau &= F(K_\tau, L_\tau) - F_{L\tau}L_\tau - G(I_\tau, K_\tau) - I_\tau \\ &= A(K_\tau, L_\tau) + F_{K\tau}K_\tau + B(I_\tau, K_\tau) - G_{I\tau}I_\tau - G_{K\tau}K_\tau - I_\tau, \end{aligned}$$

where we have used first $F_{L_\tau} = w$ and then the definitions of A and B above. Consequently

$$\begin{aligned} V_t &= \int_t^\infty (A(K_\tau, L_\tau) + B(I_\tau, K_\tau))\, e^{-\int_t^\tau r_s ds} d\tau && (14.50) \\ &\quad + \int_t^\infty [(F_{K\tau} - G_{K\tau})K_\tau - (1 + G_{I\tau})I_\tau]e^{-\int_t^\tau r_s ds} d\tau \\ &= \int_t^\infty (A(K_\tau, L_\tau) + B(I_\tau, K_\tau))e^{-\int_t^\tau r_s ds} d\tau + q_t K_t, \end{aligned}$$

when moving along the optimal path, cf. Lemma D1 below. Isolating q_t, it follows that

$$q_t^m \equiv q_t = \frac{V_t}{K_t} - \frac{1}{K_t}\int_t^\infty [A(K_\tau, L_\tau) + B(I_\tau, K_\tau)]e^{-\int_t^\tau r_s ds}d\tau. \qquad (14.51)$$

Since F is concave and $F(0,0) = 0$, we have for all K and L, $A(K,L) \geq 0$ with equality sign, if and only if F is homogeneous of degree one. Similarly, since G is convex and $G(0,0) = 0$, we have for all I and K, $B(I,K) \geq 0$ with equality sign, if and only if G is homogeneous of degree one. Now the conclusions (i) and (ii) follow from (14.51) and the definition of q^a in (14.27). □

LEMMA D1 The last integral on the right-hand side of (14.50) equals $q_t K_t$, when investment follows the optimal path.

Proof. We want to characterize a given optimal path $(K_\tau, I_\tau, L_\tau)_{\tau=t}^\infty$. Keeping t fixed and using z as our varying time variable, we have

$$(F_{Kz} - G_{Kz})K_z - (1+G_{Iz})I_z = [(r_z+\delta)q_z - \dot{q}_z]K_z - (1+G_{Iz})I_z$$

$$= [(r_z+\delta)q_z - \dot{q}_z]K_z - q_z(\dot{K}_z + \delta K_z) = r_z q_z K_z - (\dot{q}_z K_z + q_z \dot{K}_z) = r_z u_z - \dot{u}_z,$$

where we have used (14.11), (14.10), (14.6), and the definition $u_z \equiv q_z K_z$. We look at this as a differential equation: $\dot{u}_z - r_z u_z = \varphi_z$, where $\varphi_z \equiv -[(F_{Kz} - G_{Kz})K_z - (1+G_{Iz})I_z]$ is considered as some given function of z. The solution of this linear differential equation is

$$u_z = u_t e^{\int_t^z r_s ds} + \int_t^z \varphi_\tau e^{\int_\tau^z r_s ds} d\tau,$$

implying, by multiplying through by $e^{-\int_t^z r_s ds}$, reordering, and inserting the definitions of u and φ,

$$\begin{aligned} &\int_t^z [(F_{K\tau} - G_{K\tau})K_\tau - (1+G_{I\tau})I_\tau]e^{-\int_t^\tau r_s ds}d\tau \\ =\ & q_t K_t - q_z K_z e^{-\int_t^z r_s ds} \to q_t K_t \quad \text{for} \quad z \to \infty, \end{aligned}$$

from the transversality condition (14.12) with t replaced by z and 0 replaced by t. □

A different – and perhaps more illuminating – way of understanding (i) in Hayashi's theorem is the following.

Suppose F and G are homogeneous of degree one. Then $A = B = 0$, $G_I I + G_K K = G = g(I/K)K$, and $F_K = f'(k)$, where f is the production function in intensive form. Consider an optimal path $(K_\tau, I_\tau, L_\tau)_{\tau=t}^{\infty}$ and let $k_\tau \equiv K_\tau/L_\tau$ and $x_\tau \equiv I_\tau/K_\tau$. We want to characterize the optimal path $(K_\tau, I_\tau, L_\tau)_{\tau=t}^{\infty}$. From (14.47),

$$V_t = \int_t^\infty [f'(k_\tau) - g(x_\tau) - x_\tau]K_\tau e^{-\int_t^\tau r_s ds} d\tau. \tag{14.52}$$

From $\dot{K}_t = (x_t - \delta)K_t$ follows $K_\tau = K_t e^{-\int_t^\tau (x_s - \delta)ds}$. Substituting this into (14.52) yields

$$V_t = K_t \int_t^\infty [f'(k_\tau) - g(x_\tau) - x_\tau]e^{-\int_t^\tau (r_s - x_s + \delta)ds} d\tau,$$

along the optimal path. In view of (14.24), with t replaced by τ, the optimal investment *ratio* x_τ depends, for all τ, only on q_τ, not on K_τ, hence not on K_t. Therefore,

$$\partial V_t/\partial K_t = \int_t^\infty [f'(k_\tau) - g(x_\tau) - x_\tau]e^{-\int_t^\tau (r_s - x_s + \delta)ds} d\tau = V_t/K_t,$$

and the conclusion $q_t^m = q_t^a$ follows from (14.28) and (14.27).

Remark. We have assumed throughout that G is strictly convex in I. This does not imply that G is jointly strictly convex in (I, K). For example, the function $G(I, K) = I^2/K$ is strictly convex in I (since $G_{II} = 2/K > 0$). But at the same time this function has $B(I, K) = 0$ and is therefore homogeneous of degree one. Hence, it is not jointly strictly convex in (I, K).

E. The slope of the $\dot{q} = 0$ locus

First, we shall determine the sign of the slope of the $\dot{q} = 0$ locus in the case $g + n = 0$, considered in Fig. 14.4. Taking the total differential in (14.33) w.r.t. K and q gives

$$\begin{aligned} 0 &= -F_{KK}(K, \bar{L})dK + \{r + \delta + g'(m(q))m'(q) - [m(q) + (q-1)m'(q)]\} dq \\ &= -F_{KK}(K, \bar{L})dK + [r + \delta - m(q)] dq, \end{aligned}$$

since $g'(m(q)) = q - 1$, by (14.23). Therefore

$$\frac{dq}{dK}_{|\dot{q}=0} = \frac{F_{KK}(K, \bar{L})}{r + \delta - m(q)} \quad \text{for } r + \delta \neq m(q).$$

From this it is not possible to sign dq/dK at all points along the $\dot{q} = 0$ locus. But in a neighborhood of the steady state we have $m(q) \approx \delta$, hence

$r + \delta - m(q) \approx r > 0$. And since $F_{KK} < 0$, this implies that at least in a neighborhood of E in Fig. 14.4 the $\dot{q} = 0$ locus is negatively sloped.

Second, consider the case $g + n > 0$, illustrated in Fig. 14.6. Here we get in a similar way

$$\frac{dq}{d\tilde{k}}_{|\dot{q}=0} = \frac{f''(\tilde{k}^*)}{r + \delta - m(q)} \quad \text{for } r + \delta \neq m(q).$$

From this it is not possible to sign $dq/d\tilde{k}$ at all points along the $\dot{q} = 0$ locus. But in a neighborhood of the steady state we have $m(q) \approx \delta + \gamma + n$, hence $r + \delta - m(q) \approx r - \gamma - n$. Since $f'' < 0$, then, at least in a neighborhood of E in Fig. 14.6, the $\dot{q} = 0$ locus is negatively sloped, when $r > \gamma + n$.

F. Divergent paths

Text not available.

14.7 Exercises

14.1 *(induced sluggish capital adjustment).* Consider a firm with capital installation costs $J = G(I, K)$, satisfying

$$G(0, K) = 0, \;\; G_I(0, K) = 0, \;\; G_{II}(I, K) > 0, \quad \text{and} \quad G_K(I, K) \leq 0.$$

a) Can we from this conclude anything as to strict concavity or strict convexity of the function G? If yes, with respect to what argument or arguments?

b) For two values of K, $\underline{K}$ and $\bar{K}$, illustrate graphically the capital installation costs J in the (I, J) plane. Comment.

c) By drawing a few straight line segments in the diagram, illustrate that $G(\frac{1}{2}I, \bar{K})2 < G(I, \bar{K})$ for any given $I > 0$.

14.2 (see end of Section 14.3)

Macroeconomics 2 Short Note 3

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Uncertainty, expectations, and speculative bubbles

This lecture note provides a framework for addressing themes where expectations in *uncertain* situations are important elements. Our previous models have not taken seriously the problem of uncertainty. Where expectations were involved and these were assumed to be "rational", we only considered a special form: perfect foresight. Shocks were treated in a peculiar (almost self-contradictory) way: they might occur, but only as a complete surprise, a once-for-all event. Agents' expectations and actions never incorporated that new shocks could arrive.

We will now allow recurrent shocks to take place. The environment in which the economic agents act will be considered as inherently uncertain. How can this be modeled and how can we solve the resultant models? Since it is easier to model uncertainty in discrete time (period analysis) than in continuous time, we examine uncertainty and expectations in a discrete time framework.

Our emphasis will be on rational expectations or, with a better term, model consistent expectations. But we begin with simple mechanistic expectations formulas that have sometimes been used to describe day-to-day expectations of people who do not at all think about the stochastic properties of the economic environment.

1 Simple expectations formulas

One simple supposition is that expectations change gradually to correct past expectation errors. Let P_t denote the general price level in period t and $\pi_t \equiv (P_t - P_{t-1})/P_{t-1}$ the corresponding inflation rate. Further, let $\pi^e_{t-1,t}$ denote the "subjective" expectation (the expected value in the subjective conditional probability distribution), formed at the end of period $t-1$, of π_t, the inflation rate from period $t-1$ to period t. Then the hypothesis of *adaptive expectations* (the AE hypothesis) says that the expectation is revised in proportion to the past expectation error,

$$\pi^e_{t-1,t} = \pi^e_{t-2,t-1} + \lambda(\pi_{t-1} - \pi^e_{t-2,t-1}),\ 0 < \lambda \leq 1, \tag{1}$$

where the parameter λ is called the adjustment speed. If $\lambda = 1$, the formula reduces to

$$\pi^e_{t-1,t} = \pi_{t-1}. \tag{2}$$

This limiting case is called *static expectations;* in this case the subjective expectation is that the inflation rate will remain the same or at least that it is not more likely to go up than down.

We may write (1) on the alternative form

$$\pi^e_{t-1,t} = \lambda\pi_{t-1} + (1-\lambda)\pi^e_{t-2,t-1}. \tag{3}$$

This says that the expected value for this period is a weighted average of the actual value for the last period and the expected value for the last period. By backward substitution we find

$$\begin{aligned}
\pi^e_{t-1,t} &= \lambda\pi_{t-1} + (1-\lambda)[\lambda\pi_{t-2} + (1-\lambda)\pi^e_{t-3,t-2}] \\
&= \lambda\pi_{t-1} + (1-\lambda)\lambda\pi_{t-2} + (1-\lambda)^2[\lambda\pi_{t-3} + (1-\lambda)\pi^e_{t-4,t-3}] \\
&= \lambda\sum_{i=1}^{n}(1-\lambda)^{i-1}\pi_{t-i} + (1-\lambda)^n\pi^e_{t-n-1,t-n}.
\end{aligned}$$

Since $(1-\lambda)^n \to 0$ for $n \to \infty$, we have (for $\pi^e_{t-n-1,t-n}$ bounded as $n \to \infty$),

$$\pi^e_{t-1,t} = \lambda\sum_{i=1}^{\infty}(1-\lambda)^{i-1}\pi_{t-i}. \tag{4}$$

Thus, according to the AE hypothesis with $0 < \lambda < 1$, the expected inflation rate is a weighted average of the historical inflation rates back in time. The weights are geometrically declining with increasing time distance from the current period. And the weights sum to one (in that $\sum_{i=1}^{\infty}\lambda(1-\lambda)^{i-1} = \lambda(1-(1-\lambda))^{-1} = 1$).

The formula (4) can be generalized to the *general backward-looking expectations* formula,

$$\pi^e_{t-1,t} = \sum_{i=1}^{\infty} w_i\pi_{t-1-i}, \qquad \text{where } \sum_{i=1}^{\infty} w_i = 1. \tag{5}$$

If the weights w_i in (5) satisfy $w_i = \lambda(1-\lambda)^{i-1}$, $i = 1, 2, \ldots$, we get the AE formula (4). If the weights are

$$w_1 = 1+\beta,\ w_2 = -\beta,\ w_i = 0 \text{ for } i = 3, 4, \ldots,$$

we get

$$\pi^e_{t-1,t} = (1+\beta)\pi_{t-1} - \beta\pi_{t-2}, = \pi_{t-1} + \beta(\pi_{t-1} - \pi_{t-2}). \tag{6}$$

This is called the hypothesis of *extrapolative expectations* and says:

$$
\begin{aligned}
\text{if } \beta &> 0, \text{ then the recent direction of change in } \pi \text{ is expected to continue;} \\
\text{if } \beta &< 0, \text{ then the recent direction of change in } \pi \text{ is expected to be reversed;} \\
\text{if } \beta &= 0, \text{ then expectations are static as in (2).}
\end{aligned}
$$

There *are* cases where for instance static expectations are rational. Exercise 1 provides an example. But in many cases backward-looking formulas are too rigid, too mechanistic. They will often lead to systematic expectation errors to one side or the other. It is implausible that people should not then respond to their experience and revise their expectations formula. And when expectations are about things that really matter for people, they will most likely listen to professional forecasters who build their forecasting on statistical or econometric *models*. Such models take the interaction between different variables into account and will include new information about events that are likely to happen in the future.

2 Rational expectations

Unsatisfied with the above mechanistic formulas the American economist John F. Muth (1961) introduced a radically different approach, the hypothesis of *rational expectations*. Muth applied this hypothesis to simple microeconomic problems. But a decade later the hypothesis was extended to general equilibrium theory and macroeconomics by the leader of the New Classical Macroeconomics school, Nobel laureate Robert E. Lucas from the University of Chicago.[1]

Before we go into detail with the concept of rational expectations, it is useful to make a distinction between two ways in which expectations may enter a model.

2.1 Two model classes

Models with past expectations of current endogenous variables Assume that a given macroeconomic model can be reduced to the following linear equation with constant coefficients:

$$Y_t = a\, Y^e_{t-1,t} + c\, X_t, \qquad t = 0, 1, 2, ..., \tag{7}$$

[1] Lucas (1972, 1973).

where a and c are the given constant coefficients and X_t is a stochastic (also called random) variable which follows a specified exogenous stochastic process. In line with the above notation, $Y^e_{t-1,t}$ is the subjective expectation formed at the end of period $t-1$, of the value of the variable Y in period t; the agents in the economy are assumed to have the same expectations. What the equation (7) claims is that the endogenous variable, Y_t, depends, in the specified linear way, on the "generally held" expectation of Y_t, formed in the previous period.

So this is the case where agents' *past expectations of current endogenous variables* are in focus. Suppose nominal wages for period t are negotiated in period $t-1$ so as to achieve, in expected value, a target real wage in period t. Then Y_t could be the actual price level in period t. This price level depends positively on the negotiated nominal wage for period t. In turn the negotiated nominal wage for period t depends on what was in the previous period expected about the price level in period t. A realistic approach would recognize that expectations, in our complex world, are more or less heterogeneous. To take this heterogeneity into account in formal models is complicated. Here we leave the issue aside and take a simplified approach (quite common within macroeconomics): we assume homogeneity. An interpretation of the analysis is that it is concerned with what would happen *if* agents agreed in their expectations.

Equation (7) can also be interpreted as a vector equation (such that Y_t and $Y^e_{t-1,t}$ are n-vectors, a is an $n \times n$ matrix, c an $n \times m$ matrix, and X an m-vector). The crucial feature is that the endogenous variable(s) dated t *only* depend(s) on expectations of date-t values of this (these) variable(s) and on the exogenous variables.

Models with forward-looking expectations Another way in which agents' expectations may enter is exemplified by

$$Y_t = a\, Y^e_{t,t+1} + c\, X_t, \qquad t = 0, 1, 2, ... \tag{8}$$

Here $Y^e_{t,t+1}$ is the subjective expectation, formed in period t, of the value of Y in period $t+1$. For example: the equity price today depends on what the equity price is expected to be tomorrow. Or more generally: current expectations of a future value of an endogenous variable influence the current value of this variable. We name this the case of *forward-looking expectations*. Of course, in "everyday language" also $Y^e_{t-1,t}$ can be said to be a forward-looking variable as seen from period $t-1$. But the dividing line between the two model classes, (7) and (8), is whether current expectations of future values of the endogenous variables do or do not influence the current values of these.

The case of forward-looking expectations is important in connection with many topics within macroeconomics, including the issue of asset price bubbles. This case will be dealt with in sections 3 and 4. The simpler case, (7), will serve as a convenient point of departure for the introduction of the concept of rational expectations.

In passing we note that in both model classes, (7) and (8), it is the mean of the random variable(s) that enter. This is typical of simple macroeconomic models which often ignore other measures such as the median, mode, or higher-order moments. The latter, say the variance of X_t, may be included in more advanced models where for instance behavior towards risk is an important element.

2.2 The concept of rational expectations

Expectations are called *Muth-rational* or just *rational* if they are *model consistent.* This means, in relation to the model (7), that the subjective expectation, $Y^e_{t-1,t}$, equals the forecast that can be calculated from the model structure and relevant information available at the end of period $t-1$. Thus, the subjective expectation is assumed to coincide with the "objective" expectation (the mathematical conditional expectation based on the economic model) of Y_t conditional on the information available at the time expectations are formed. That is, the subjective expectation, $Y^e_{t-1,t}$, is named a *rational expectation*, or an *RE forecast*, if

$$Y^e_{t-1,t} = E(Y_t|I_{t-1}), \tag{9}$$

where E is the expectation operator and I_{t-1} denotes relevant information available at the end of period $t-1$.[2] The meaning of (9) is that the true conditional expectation is calculated on the basis of I_{t-1} and the economic model on hand (including knowledge of the true parameter values a and c as well as the stochastic process which X_t follows).[3]

Often the information I_{t-1} is assumed to comprise knowledge of the realized values of Y and X up until and including period $t-1$. So we may instead write (9) as

$$Y^e_{t-1,t} = E(Y_t|Y_{t-1} = y_{t-1}, \ldots, Y_{t-n} = y_{t-n}; X_{t-1} = x_{t-1}, \ldots, X_{t-n} = x_{t-n}).$$

Here information goes back to a given initial period, say period 0, in which case n equals t. Alternatively it may be assumed that information goes back to Adam and Eve, represented

[2] One may refer to I_{t-1} as the information *set.* But as indicated at the end of Appendix B, the concept of an information set is tricky and its precise definition would take us too far.

[3] We later introduce the simpler notation $E_{t-1}Y_t$ for the conditional expectation $E(Y_t|I_{t-1})$. In this introductory section, however, we prefer to let the conditioning information be explicit. As to terminology, when there is no risk of confusion, we use "expected value" and "expectation" as synonymous with "mathematical expectation".

by $n = \infty$. As time proceeds, more and more realizations of the exogenous and endogenous variables become known and so the information I_t *expands* with rising t. This should not be confused with the fact that the space of possible time paths of (X, Y) *shrinks* as time proceeds. Indeed, this space shrinks precisely because more and more realizations of the variables take place and become known.

The assumption of rational expectations is an abstraction and should be regarded as a simplifying assumption allowing a first approximation. First, the assumption entails that all agents have the *same* model of the functioning of the economic system (in this chapter also the same information), which is already a big mouthful. Second, this model is assumed to be the *true* model of the economic process, with the true parameter values. The agents being modeled, the analyst, and the government (policy maker) all share the same model. In practice it is impossible to attain such a model unless the considered economic process has reached a kind of steady state. Nevertheless, a theoretical model with rational expectations can be seen as a useful cultivation of a problem. The results that emerge in the model cannot depend on *systematic* expectation errors from the economic agents' side. In this sense the assumption of rational expectations will in many contexts make up a theoretically interesting *benchmark case*.

A more satisfactory model of expectations would give an account of the mechanisms through which agents *learn* about the economic environment. An important ingredient here would be how the agents contemplate the costs associated with further information gathering needed to avoid systematic expectation errors. For a significant subset of the agents these costs may in fact be prohibitive. Moreover, a complicating factor involved in the process of learning is that when the agents have obtained *some* knowledge about the statistical properties of the economic variables, the resulting behavior of the agents may *change* these properties. For now, these problems are set aside. We simply assume that the structure of the economy remains unchanged and that the learning process about it has already been completed.

The equations (7) and (9) constitute a simple *rational expectations model* (henceforth an RE model). We may write the model in a more compact way as

$$Y_t = aE(Y_t|I_{t-1}) + c\, X_t, \qquad t = 0, 1, 2, ... \tag{10}$$

To solve the model means to find the implied stochastic process followed by Y_t, given the stochastic process followed by X_t.

2.3 Solving a simple model

A solution procedure for a linear RE model with past expectations of current endogenous variables is the following:

1. Cast the model (or the relevant part of the model) into *reduced form*, like in (10). That is, isolate on the left-hand side of the equation(s) the endogenous variable(s) the expectation(s) of which enter the model.[4]

2. Take the conditional expectation on both sides of the equation(s) and find from this the conditional expectation of the endogenous variable(s).

3. Insert into the reduced form and reorder.

In practice there is often a fourth step, namely to express *other* endogenous variables in the model in terms of those found in step 3.

Let us see how the procedure works. The first step has already been done in writing (10). Performing the second step on (10) gives

$$E(Y_t \,|I_{t-1}) = aE(Y_t \,|I_{t-1}) + cE(X_t \,|I_{t-1}) \,. \tag{11}$$

Solving for the expected value of Y_t gives

$$E(Y_t \,|I_{t-1}) = c\frac{E(X_t \,|I_{t-1})}{1-a}, \tag{12}$$

when $a \neq 1$. The third step is to insert this into (10) to get

$$Y_t = c\frac{aE(X_t \,|I_{t-1}) + (1-a)X_t}{1-a}. \tag{13}$$

This is the solution of the model in the sense of a specification of the stochastic process for Y_t which fits the model.[5]

If for example X_t follows the process $X_t = \rho X_{t-1} + \varepsilon_t$, where ε_t has zero expected value, given all past values of X and Y, then the final solution is

$$Y_t = c\frac{a\rho X_{t-1} + (1-a)X_t}{1-a}.$$

[4] A model is cast into its *reduced form* when the current endogenous variables are expressed as functions of exogenous variables and the lagged endogenous variables.

[5] If $a = 1$, the model is inconsistent unless $E(X_t \,|I_{t-1})) = 0$ in which case there are multiple solutions. Indeed, for any number $k \in (-\infty, +\infty)$, the process $Y_t = k + cX_t$ solves the model when $E(X_t \,|I_{t-1}))$ $= 0$.

In Exercise 2 the reader is asked to solve a simple Keynesian model of this kind.

In economic models and time series analysis one often uses the name *white noise* for a sequence of stochastic variables indexed by time (a stochastic process) with zero expected value, constant variance, and zero covariance.[6]

EXAMPLE Let $Y_t = aE(Y_t \,|I_{t-1}) + cX_t$, with $X_t = \bar{x} + \varepsilon_t$, where $\bar{x}$ is a constant and ε_t is white noise with variance σ^2. Then (13) applies, so that

$$Y_t = \frac{c\bar{x}}{1-a} + c\varepsilon_t, \qquad t = 0, 1, ...,$$

with variance $c^2\sigma^2$. The RE forecast error is $e_t = Y_t - E(Y_t \,|I_{t-1}) = c\varepsilon_t$ with conditional expectation equal to $E(c\varepsilon_t \,|I_{t-1}) = 0$. The forecast error itself is white noise and is therefore clearly uncorrelated with the information on which the forecast is based. □

The notion of *perfect foresight* corresponds to the limiting case where the variance of the exogenous variable(s) is zero so that with probability one $X_t = E(X_t \,|I_{t-1})$ for all t. Then we have a deterministic (in the sense of non-stochastic) model where agents' ex post forecast error with respect to Y_t is zero.[7] To put it differently: rational expectations in a deterministic model is equivalent to perfect foresight. Note, however, that perfect foresight necessitates the exogenous variable X_t to be known in advance. Real world situations are usually not like that. If we want to allow for uncertainty, the model should be formulated in an explicit stochastic framework.

3 Models with rational forward-looking expectations

We now turn to models where current expectations of a future value of an endogenous variable have an influence on the current value of this variable, that is, the case exemplified by equation (8). At the same time we introduce a slightly changed notation in order to comply with the standard notation in macroeconomics. Instead of using capital letters to denote the stochastic variables (as we did above and as is common in mathematical statistics), we use lower case letters. So the reader should be aware that from now a lower case letter may denote a stochastic variable or a specific value of this variable, depending on the context. This notation is common in macroeconomics and is also more

[6] The expression white noise derives from electrotechnics. In electrotechnical systems signals will often be subject to noise. If this noise is arbitrary and has no dominating frequence, it looks like white light. The various colours correspond to a certain wave length, but white light is light which has all frequences (no dominating frequence).

[7] Here we disregard zero probability events.

convenient (lower case letters easier to write).[8] So an equation like (8) will from now read $y_t = a\ y^e_{t,t+1} + c\ x_t$. Under rational expectations it takes the form

$$y_t = aE(y_{t+1}\,|I_t) + c\ x_t, \qquad t = 0, 1, 2, \ldots. \tag{14}$$

A second notational change is that from now on the expected value of any stochastic variable, z_{t+i}, conditional on the information I_t, will be denoted $E_t z_{t+i}$. Thus, we write (14) as

$$y_t = aE_t y_{t+1} + c\ x_t, \quad t = 0, 1, 2, \ldots. \tag{15}$$

A stochastic difference equation of this form is called a linear *expectation difference equation of first order* with constant coefficient a.[9] A *solution* is a specified stochastic process $\{y_t\}$ which satisfies (15), given the stochastic process followed by x_t. In the economic applications usually no initial value, y_0, is given. On the contrary, the interpretation is that y_t depends, for all t, on expectations about the future. Indeed, y_t is considered to be a *jump variable* and can immediately shift its value in response to the emergence of new information about the future x's. For example, a share price may immediately jump to a new value when the accounts of the firm become publicly known (often even before, due to rumors).[10]

Due to the lack of an initial condition for y_t, there can easily be infinitely many processes for y_t satisfying our expectation difference equation. We have an infinite forward-looking "regress", where a variable's value today depends on its expected value tomorrow, this value depending on the expected value the day after tomorrow and so on. Then usually there are infinitely many expected sequences which can be self-fulfilling in the sense that if only the agents expect a particular sequence, then the aggregate outcome of their behavior will be that the sequence is realized. It "bites its own tail" so to speak. Yet, when an equation like (15) is part of a larger model, there will often (but not always) be conditions that allow us to select *one* of the many solutions to (15) as the only *economically* relevant one. For example, an economy-wide transversality condition and general

[8] In macroeconomics it is common to use capital letters for aggregate variables (whether stochastic or non-stochastic) measured in *natural units* and the corresponding lower case letter for the *logarithm* of the same variables (though sometimes also for a *ratio* of aggregate variables). Since the equations in simple stochastic macroeconomic models are often assumed *linear in the logarithm* of the variables (the reason is explained in Appendix A), the stochastic variables in expectational difference equations are usually denoted by lower case letters.

[9] To keep things simple, we let the coefficients a and c be constants, but a generalization to time-dependent coefficients is straightforward.

[10] We said that y_t "depends on" the expectation of y_{t+1}. It would be inaccurate to say that y_t is *determined* (in a one-way-sense) by expectations about the future. Rather there is *mutual dependence*. In view of y_t being an element in the information I_t, the expectation of y_{t+1} in (15) may depend on y_t just as much as y_t depends on the expectation of y_{t+1}.

equilibrium conditions may rule out divergent solutions and leave a unique convergent solution as the final solution.

We assume $a \neq 0$, since otherwise (15) itself already is the unique solution. It turns out that the set of solutions to (15) takes a different form depending on whether $|a| < 1$ or $|a| > 1$:[11]

The case $|a| < 1$. Generally, there is a unique *fundamental solution* and infinitely many explosive *bubble solutions.*

The case $|a| > 1$. Generally, there is no fundamental solution but infinitely many non-explosive solutions. (The case $|a| = 1$ to some extent resembles this.)

In the case $|a| < 1$ the expected future has modest influence on the present. Here we will concentrate on this case, since it is the case most frequently appearing in macroeconomic models with rational expectations.

4 Solutions when $|a| < 1$

Various solution methods are available. *Repeated forward substitution* is the most easily understood method.

4.1 Repeated forward substitution

The method of repeated forward substitution is based on the *law of iterated expectations,*s which says that

$$E_t(E_{t+1}y_{t+2}) = E_t y_{t+2}. \tag{16}$$

That is, as seen from period t events in period $t+1$ are stochastic and so $E_{t+1}y_{t+2}$ (the expectation conditional on these events) is a stochastic variable. Then (16) says that, given that expectations are rational, then an earlier expectation of a later expectation of y is just the earlier expectation of y. Put differently: my best forecast today of how I am going to forecast tomorrow a share price the day after tomorrow, will be the same as my best forecast today of the share price the day after tomorrow. If, beforehand, we have good reasons to expect that we will revise our expectations upward, say, when next

[11] The following is much in debt to the exposition in Blanchard and Fischer (1989, Ch. 5, Section 1).

period's additional information arrives, the original expectation would be biased, hence not rational.[12]

Repeated forward substitution consists of the following steps. We first shift (15) one period ahead:

$$y_{t+1} = a\ E_{t+1}y_{t+2} + c\ x_{t+1}.$$

Then we take the conditional expectation on both sides to get

$$E_t y_{t+1} = a\ E_t(E_{t+1}y_{t+2}) + c\ E_t x_{t+1} = a\ E_t y_{t+2} + c\ E_t x_{t+1}, \tag{17}$$

by the law of iterated expectations. Inserting (17) into (15) gives

$$y_t = a^2 E_t y_{t+2} + ac\ E_t x_{t+1} + c\ x_t. \tag{18}$$

The procedure is repeated by forwarding (15) two periods ahead; then taking the conditional expectation and inserting into (18), we get

$$y_t = a^3 E_t y_{t+3} + a^2 c\ E_t x_{t+2} + ac\ E_t x_{t+1} + c\ x_t.$$

We continue in this way and the general form (for $n = 0, 1, 2, ...$) becomes

$$y_{t+n} = a\ E_{t+n}(y_{t+n+1}) + c\ x_{t+n},$$

$$E_t y_{t+n} = a\ E_t y_{t+n+1} + c\ E_t x_{t+n},$$

$$y_t = a^{n+1} E_t y_{t+n+1} + c x_t + c \sum_{i=1}^{n} a^i E_t x_{t+i}. \tag{19}$$

4.2 The fundamental solution

PROPOSITION 1 If

$$\lim_{n\to\infty} \sum_{i=1}^{n} a^i E_t x_{t+i} \text{ exists,} \tag{20}$$

then

$$y_t = c \sum_{i=0}^{\infty} a^i E_t x_{t+i} = c x_t + c \sum_{i=1}^{\infty} a^i E_t x_{t+i}, \qquad t = 0, 1, 2, ..., \tag{21}$$

is a solution to the expectation difference equation (15).

[12] A formal account of conditional expectations and the law of iterated expectations is contained in Appendix B.

Proof Assume (20). Then the formula (21) is meaningful. In view of (19), it satisfies (15) if and only if $\lim_{n\to\infty} a^{n+1}E_t y_{t+n+1} = 0$. Hence, it is enough to show that the process (21) satisfies this latter condition.

In (21), replace t by $t+n+1$ to get $y_{t+n+1} = c\sum_{i=0}^{\infty} a^i E_{t+n+1}x_{t+n+1+i}$. Using the law of iterated expectations, this yields

$$\begin{aligned} E_t y_{t+n+1} &= c\sum_{i=0}^{\infty} a^i E_t x_{t+n+1+i} \qquad \text{so that} \\ a^{n+1}E_t y_{t+n+1} &= c\, a^{n+1}\sum_{i=0}^{\infty} a^i E_t x_{t+n+1+i} = c\sum_{j=n+1}^{\infty} a^j E_t x_{t+j}. \end{aligned}$$

It remains to show that $\lim_{n\to\infty}\sum_{j=n+1}^{\infty} a^j E_t x_{t+j} = 0$. From the identity

$$\sum_{j=1}^{\infty} a^j E_t x_{t+j} = \sum_{j=1}^{n} a^j E_t x_{t+j} + \sum_{j=n+1}^{\infty} a^j E_t x_{t+j}$$

follows

$$\sum_{j=n+1}^{\infty} a^j E_t x_{t+j} = \sum_{j=1}^{\infty} a^j E_t x_{t+j} - \sum_{j=1}^{n} a^j E_t x_{t+j}.$$

Letting $n\to\infty$, this gives

$$\lim_{n\to\infty}\sum_{j=n+1}^{\infty} a^j E_t x_{t+j} = \sum_{j=1}^{\infty} a^j E_t x_{t+j} - \sum_{j=1}^{\infty} a^j E_t x_{t+j} = 0,$$

which was to be proved. $\square$

The solution (21) is called the *fundamental solution* of (15). It is (for $c \neq 0$) defined only when the condition (20) holds. In practice this condition requires that $|a| < 1$ and that the absolute value of the expectation of the exogenous variable does not increase "too fast". More precisely, the requirement is that $|E_t x_{t+i}|$, when $i\to\infty$, has a growth factor less than $|a|^{-1}$. As an example, let $0 < a < 1$ and $g > 0$, and suppose that $E_t x_{t+i} > 0$ for $i = 0, 1, 2, ...$, and that $1+g$ is an upper bound for the growth factor of $E_t x_{t+i}$. Then

$$E_t x_{t+i} \le (1+g)E_t x_{t+i-1} \le (1+g)^i E_t x_t = (1+g)^i x_t,$$

so that $a^i E_t x_{t+i} \le a^i(1+g)^i x_t$. By summing from $i = 1$ to n,

$$\sum_{i=1}^{n} a^i E_t x_{t+i} \le x_t \sum_{i=1}^{n} [a(1+g)]^i.$$

Letting $n\to\infty$, we get

$$\lim_{n\to\infty}\sum_{i=1}^{n} a^i E_t x_{t+i} \le x_t \lim_{n\to\infty}\sum_{i=1}^{n}[a(1+g)]^i = x_t\frac{a(1+g)}{1-a(1+g)} < \infty,$$

if $1+g<a^{-1}$, using the sum rule for an infinite geometric series.

As noted in the proof of Proposition 1, the fundamental solution (21) has the property that

$$\lim_{n\to\infty} a^n E_t y_{t+n} = 0. \tag{22}$$

That is, the expected value of y is not "explosive": its absolute value has a growth factor less than $|a|^{-1}$. Given $|a|<1$, the fundamental solution is the only solution of (15) with this property. Indeed, it is seen from (19) that whenever (22) holds, (21) must also hold. In Example 1 below, y_t is interpreted as the market price of a share and x_t as dividends. Then the fundamental solution gives the share price as the present value of the expected future flow of dividends.

EXAMPLE 1 *(the fundamental value of an equity share)* Consider arbitrage between shares of stock and a riskless asset paying the constant rate of return $r>0$. Let period t be the current period. Let p_{t+i} be the market price of the share at the beginning of period $t+i$ and d_{t+i} the dividend paid out at the end of that period, $t+i$, $i=0,1,2,....$[13] As seen from period t there is uncertainty about p_{t+i} and d_{t+i} for $i=1,2,....$ Suppose agents have rational expectations and care only about expected return (risk neutrality). Then the no-arbitrage condition reads

$$\frac{d_t + E_t p_{t+1} - p_t}{p_t} = r. \tag{23}$$

This can be written

$$p_t = \frac{1}{1+r} E_t p_{t+1} + \frac{1}{1+r} d_t, \tag{24}$$

which is of the same form as (15) with $a=c=1/(1+r)\in(0,1)$. Assuming dividends do not grow "too fast", we find the fundamental solution, denoted p_t^*, as

$$p_t^* = \frac{1}{1+r} d_t + \frac{1}{1+r}\sum_{i=1}^{\infty}\frac{1}{(1+r)^i} E_t d_{t+i} = \sum_{i=0}^{\infty}\frac{1}{(1+r)^{i+1}} E_t d_{t+i}. \tag{25}$$

The fundamental solution is simply the present value of expected future dividends, in finance theory denoted the fundamental value.

[13] So an investor who buys n_t shares at time t (the beginning of period t) invests $V_t \equiv p_t n_t$ units of account at time t. At the end of the period the return comes out as the dividend d_t and the potential sales value of the share at time $t+1$. This is slightly unlike standard *finance* notation, where V_t would not be the-beginning-of-period-t market value but the-end-of-period-t market value of the stock that begins to yield dividends in period $t+1$. Also in comparison with the way we in Chapter 6 priced assets, there is ostensibly a difference. A flow of goods in period t was payed for at the end of the period at a price P_t in monetary terms. We obtain consistency if we interpret the price, p_t, of the share (a stock) at time t as an asset price, in *real* terms, defined by $p_t \equiv 1/P_{t-1}$; in this way, r in (24) is a *real* interest rate from the end of period $t-1$ to the end of period t.

If the dividend process is $d_{t+1} = d_t + \varepsilon_{t+1}$, where ε_{t+1} is white noise, then the process is known as a *random walk* and $E_t d_{t+i} = d_t$ for $i = 1, 2, \ldots$. Thus $p_t^* = d_t/r$, by the sum rule for an infinite geometric series. In this case the fundamental value is itself a random walk. More generally, the dividend process could be a *martingale,* that is, a sequence of stochastic variables with the property that the expected value next period exists and equals the current actual value, where the expectation is conditional on all information up to and including the current period. Again $E_t d_{t+1} = d_t$; but in a martingale ε_{t+1} need not be white noise; it is enough that $E_t \varepsilon_{t+1} = 0$.[14] Given the constant required return r, we still have $p_t^* = d_t/r$. So the fundamental value itself is in this case a martingale. □

As noted in the example, in finance theory the present value of the expected future flow of dividends on an equity share is referred to as the *fundamental value* of the share. It is by analogy with this that the general designation *fundamental solution* has been introduced for solutions of form (21). We could also think of p_t as the market price of a house rented out and d_t as the rent. Or p_t could be the market price of a mine and d_t the revenue (net of extraction costs) from the extracted oil in period t.

4.3 Bubble solutions

Other than the fundamental solution, the expectation difference equation (15) has infinitely many *bubble solutions.* In view of $|a| < 1$, these are characterized by violating the condition (22). That is, they are solutions whose expected value explodes over time.

It is convenient to first consider the *homogenous* expectation equation associated with (15). This is defined as that equation which emerges by setting $c = 0$ in (15):

$$y_t = a E_t y_{t+1}. \tag{26}$$

Every stochastic process $\{b_t\}$ of the form

$$b_{t+1} = a^{-1} b_t + u_{t+1}, \quad \text{where } E_t u_{t+1} = 0, \tag{27}$$

has the property that

$$b_t = a E_t b_{t+1}, \tag{28}$$

and is thus a solution to (26). The "disturbance" u_{t+1} represents "new information" which may be related to unexpected movements in "fundamentals", x_{t+1}. But it does not have to. In fact, u_{t+1} may be related to conditions that *per se* have no economic relevance whatsoever.

[14] A random walk is thus a special case of a martingale.

For ease of notation, from now we just write b_t even if we think of the whole process $\{b_t\}$ rather than the value taken by b in the specific period t. The meaning should be clear from the context. A solution to (26) is referred to as a *homogenous solution* associated with (15). Let b_t be a given homogenous solution and let K be an arbitrary constant. Then $B_t = Kb_t$ is also a homogenous solution (try it out for yourself). Conversely, any homogenous solution b_t associated with (15) can be written in the form (27). To see this, let b_t be a given homogenous solution, that is, $b_t = aE_tb_{t+1}$. Let $u_{t+1} = b_{t+1} - E_tb_{t+1}$. Then

$$b_{t+1} = E_tb_{t+1} + u_{t+1} = a^{-1}b_t + u_{t+1},$$

where $E_tu_{t+1} = E_tb_{t+1} - E_tb_{t+1} = 0$. Thus, b_t is of the form (27).

Returning to our original expectation difference equation (15), we have:

PROPOSITION 2 Let $\tilde{y}_t$ be a particular solution to (15). Then:

(i) every stochastic process of the form

$$y_t = \tilde{y}_t + b_t, \tag{29}$$

where b_t satisfies (27), is a solution to (15);

(ii) every solution to (15) can be written in the form (29) with b_t being an appropriately chosen homogenous solution associated with (15).

Proof. Let the particular solution $\tilde{y}_t$ be given. (i) Let $y_t = \tilde{y}_t + b_t$. Then $y_t = a\ E_t\tilde{y}_{t+1} + c\ x_t + b_t$, since $\tilde{y}_t$ satisfies (15). Consequently, by (26),

$$y_t = a\ E_t\tilde{y}_{t+1} + c\ x_t + a\ E_tb_{t+1} = a\ E_t(\tilde{y}_{t+1} + b_{t+1}) + c\ x_t = a\ E_tY_{t+1} + c\ x_t,$$

saying that (29) satisfies (15). (ii) Let Y_t be an arbitrary solution to (15). Define $b_t = Y_t - \tilde{y}_t$. Then we have

$$\begin{aligned} b_t &= Y_t - \tilde{y}_t = aE_tY_{t+1} + cx_t - (aE_t\tilde{y}_{t+1} + cx_t) \\ &= aE_t(Y_{t+1} - \tilde{y}_{t+1}) = aE_tb_{t+1}, \end{aligned}$$

where the second equality follows from the fact that both Y_t and $\tilde{y}_t$ are solutions to (15). This shows that b_t is a solution to the homogenous equation (26) associated with (15). Since $Y_t = \tilde{y}_t + b_t$, the proposition is hereby proved. $\square$

Proposition 2 holds for any $a \neq 0$. In case $|a| < 1$ and the fundamental solution (21) exists, it is convenient to choose this solution as the particular solution in (29). Thus, referring to the right-hand side of (21) as y_t^*, we can use the particular form,

$$y_t = y_t^* + b_t. \tag{30}$$

When the component b_t is different from zero, the solution (30) is called a *bubble solution* and b_t is called the *bubble component*. In the typical economic interpretation the bubble component shows up only because it is expected to show up next period, cf. (28). The name bubble springs from the fact that the expected value conditional on the information available in period t explodes over time when $|a| < 1$. To see this, as an example, let $0 < a < 1$. Then, from (26), by repeated forward substitution we get

$$b_t = a\ E_t(aE_{t+1}b_{t+2}) = a^2 E_t b_{t+2} = ... = a^i E_t b_{t+i}, \qquad i = 1, 2, ...$$

It follows that $E_t b_{t+i} = a^{-i} b_t$, that is, not only does it hold that

$$\lim_{i\to\infty} E_t b_{t+i} = \begin{cases} \infty, \text{ if } b_t > 0 \\ -\infty, \text{ if } b_t < 0 \end{cases},$$

but the absolute value of $E_t b_{t+i}$ will for rising i grow *geometrically* towards infinity with a growth factor equal to $1/a > 1$.

Let us consider a special case of (15) that allows a simple graphical illustration of both the fundamental solution and some bubble solutions.

4.3.1 When x_t has constant mean

Suppose the stochastic process x_t (the "fundamentals") takes the form $x_t = \bar{x} + \varepsilon_t$, where $\bar{x}$ is a constant and ε_t is white noise. Then

$$y_t = a\ E_t y_{t+1} + c(\bar{x} + \varepsilon_t), \qquad 0 < |a| < 1. \tag{31}$$

The fundamental solution is

$$y_t^* = c\ x_t + c\sum_{i=1}^{\infty} a^i \bar{x} = c\bar{x} + c\varepsilon_t + c\frac{a\bar{x}}{1-a} = \frac{c\bar{x}}{1-a} + c\varepsilon_t.$$

Referring to (i) of Proposition 2,

$$y_t = \frac{c\bar{x}}{1-a} + c\varepsilon_t + b_t \tag{32}$$

is thus also a solution of (31) if b_t is of the form (27).

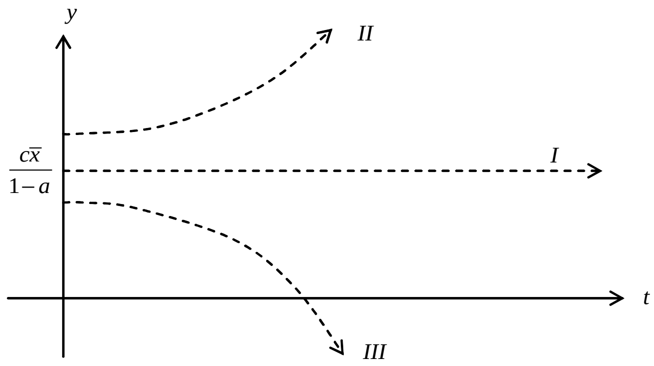

Figure 1: Deterministic bubbles (the case $0 < a < 1$, $c > 0$, and $x_t = \bar{x}$).

It may be instructive to consider the case where all stochastic features are eliminated. So we assume $u_t \equiv \varepsilon_t \equiv 0$. Then we have a model with perfect foresight; the solution (32) simplifies to

$$y_t = \frac{c\bar{x}}{1-a} + b_0 a^{-t}, \tag{33}$$

where we have used repeated *backward* substitution in (27). By setting $t = 0$ we see that $y_0 = \frac{c\bar{x}}{1-a} + b_0$. Inserting this into (33) gives

$$y_t = \frac{c\bar{x}}{1-a} + (y_0 - \frac{c\bar{x}}{1-a})a^{-t}. \tag{34}$$

In Fig. 1 we have drawn three trajectories for the case $0 < a < 1$, $c > 0$. Trajectory I has $y_0 = c\bar{x}/(1-a)$ and represents the fundamental solution, whereas trajectory II, with $y_0 > c\bar{x}/(1-a)$, and trajectory III, with $y_0 < c\bar{x}/(1-a)$, are bubble solutions. Since we have imposed no apriori boundary condition, one y_0 is as good as any other. The interpretation is that there are infinitely many trajectories with the property that if only the economic agents expect the economy will follow that particular trajectory, the aggregate outcome of their behavior will be that this trajectory is realized. This is the potential indeterminacy arising when y_t is not a predetermined variable. However, as alluded to above, in a complete economic model there will often be restrictions on the endogenous variable(s) *in addition* to the basic expectation difference equation(s), here (31). It may be that the economic meaning of y_t precludes negative values and, hence, no-one can rationally expect a path such as III in Fig. 1. And/or it may be that for some reason there is an upper bound on y_t (which could be the full-employment ceiling for output in a situation where the natural growth rate of output is smaller than $a^{-1} - 1$). Then no one can rationally expect a trajectory like II in the figure.

Our conclusion so far is that in order for a solution of a first-order linear expectation difference equation with constant coefficient a, where $|a| < 1$, to differ from the funda-

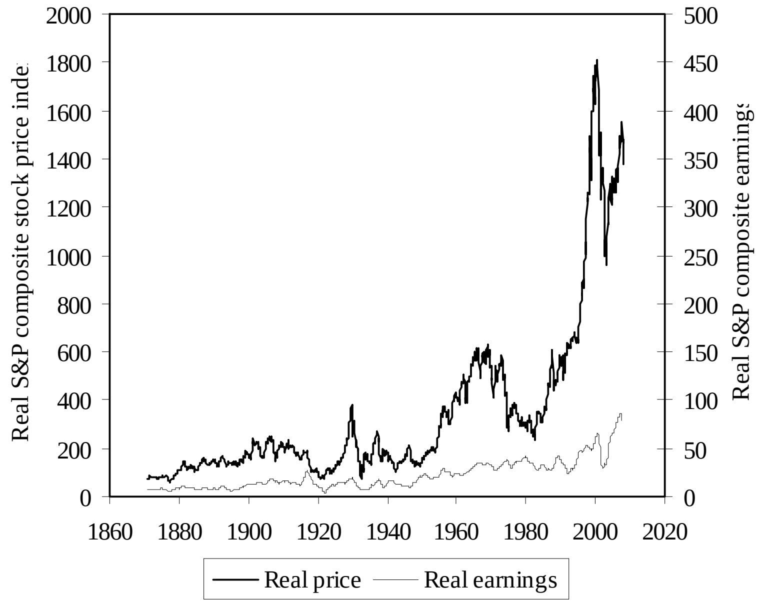

Figure 2: Monthly real S&P composite stock prices from January 1871 to January 2008 (left) and monthly real S&P composite earnings from January 1871 to September 2007 (right). Source: http://www.econ.yale.edu/~shiller/data.htm.

mental solution, the solution must have the form (30) where b_t has the form described in (27). This provides a clue as to what asset price bubbles might look like.

4.3.2 Asset price bubbles

Robert Shiller (1981, 2003) from Yale University, and others, have pointed to the phenomenon of "excess volatility". The view is that asset prices tend to fluctuate more than can be rationalized by shifts in information about fundamentals (present values of dividends). Although in no way a verification, graphs like those in Fig. 2 and Fig. 3 are certainly suggestive. Fig. 2 shows the monthly real S&P composite stock prices and real S&P composite earnings for the period 1871-2008. The unusually large increase in real stock prices since the mid-90's, which ended with the collapse in 2000, is known as the "dot-com bubble". Fig. 3 shows, on a monthly basis, the ratio of real S&P stock prices to an average of the previous ten years' real S&P earnings along with the long-term real interest rate. It is seen that this ratio reached an all-time high in 2000, by many observers considered as "the year the dot-com bubble burst".

Blanchard (1979) and Blanchard and Watson (1982) argued that bubble phenomena need not be due to irrational behavior and absence of rational expectations. This lead to the theory of *rational bubbles*, the idea that excess volatility can be explained as speculative

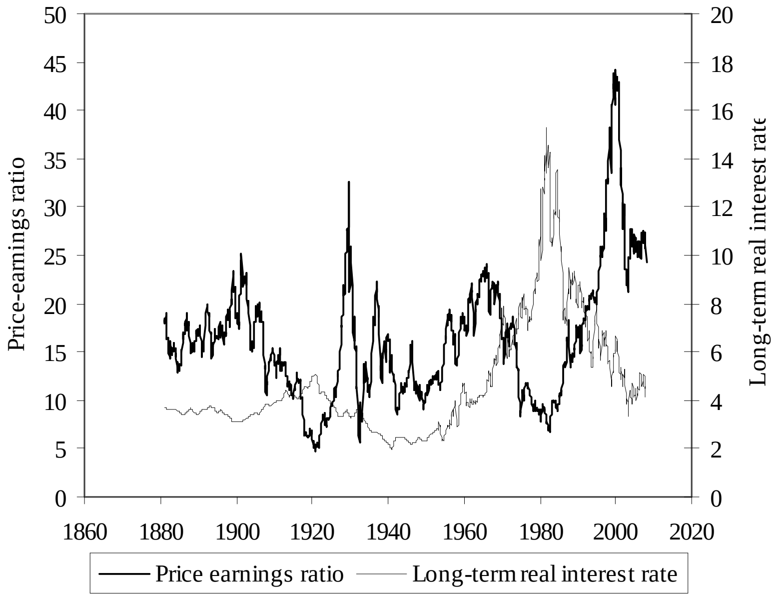

Figure 3: S&P price-earnings ratio and long-term real interest rates from January 1881 to January 2008. The earnings are calculated as a moving average over the preceding ten years. The long-term real interest rate is the 10-year Treasury rate from 1953 and government bond yields from Sidney Homer, "A History of Interest Rates" from before 1953. Source: http://www.econ.yale.edu/~shiller/data.htm.

bubbles arising solely from self-fulfilling *rational* expectations.

Consider an asset which yields either dividends or services in production or consumption in every period in the future. The fundamental value of the asset is, at the theoretical level, defined as the present value of the future flow of dividends or services.[15] An *asset price bubble* (or a *speculative bubble)* is then defined as the deviation of the market price of the asset from its fundamental value. An asset price bubble that emerges in a setting where the no-arbitrage condition (23) holds and agents have rational expectations, is called a *rational bubble.*

EXAMPLE 2 *(an ever-expanding rational bubble)* Consider again an equity share for which the no-arbitrage condition is

$$\frac{d_t + E_t p_{t+1} - p_t}{p_t} = r, \tag{35}$$

as in Example 1. Let the price of the share be $p_t = p_t^* + b_t$, where the bubble component follows the deterministic process, $b_{t+1} = (1 + r)b_t$, $b_0 > 0$. This is called a *deterministic rational bubble.* Agents may be ready to pay a price over and above the fundamental

[15] In practice there are many ambiguities involved in this definition of the fundamental value, because it relates to an unknown future.

value if they expect they can sell at a higher price later; trading with such motivation is called *speculative behavior*. If generally held, this expectation may be self-fulfilling. Yet we are not acquainted with such *ever*-expanding incidents in real world situations. Hence, a deterministic rational bubble seem implausible. □

A *stochastic* rational bubble which sooner or later *bursts* seems less implausible.

EXAMPLE 3 *(a bursting bubble)* The no-arbitrage condition is once more (35) where for simplicity we still assume the required rate of return is constant, though possibly including a risk premium. The implied expectation difference equation is $p_t = aE_t p_{t+1} + cd_t$, with $a = c = 1/(1+r) \in (0,1)$. Following Blanchard, (1979) we assume that the market price, p_t, of the share contains a stochastic bubble of the following form:

$$b_{t+1} = \begin{cases} \frac{1+r}{q_t} b_t & \text{with probability } q_t, \\ 0 & \text{with probability } 1 - q_t, \end{cases} \tag{36}$$

where $t = 0, 1, 2, ...$ and $b_0 > 0$. In addition we may assume that $q_t = f(b_t, p^*_{t+1})$, $f_b \leq 0$, $f_p \geq 0$. If $f_b < 0$, the probability that the bubble persists at least one period ahead is less, the greater the bubble has already become. If $f_p > 0$, the probability that the bubble persists at least one period ahead is higher the greater the fundamental value has become.

In this way the probability of a crash becomes greater and greater as the share price comes further and further away from fundamentals. Note also that in this case, the longer time the bubble has lasted, the higher is the expected growth rate of the bubble in the absence of a collapse.

In Exercise 3 you are asked to show that this bubble satisfies the criterion for a rational bubble. □

In the above example the bubble did not have the implausible ever-expanding form considered in Example 2. Yet, under certain conditions even a bursting rational bubble of this kind can be ruled out. The arguments (e) and (f) in the list below are cases in point.

4.4 When rational bubbles in asset prices can or can not be ruled out

We here consider different cases where rational asset price bubbles seem unlikely to arise. We concentrate on assets whose services are valued independently of the price.[16] Let p_t

[16] This is in contrast to assets that serve as means of payment.

be the market price and p_t^* the fundamental value of the asset as of time t. Even if the asset does not pay dividends but instead services, p_t^* is in principle the same for all. This is because a user who, in a given period, values the service flow of the asset relatively low can hire it out to the one who values it highest (the one with the highest willingness to pay).

(a) Assets which can freely be disposed of ("free disposal") In a market with self-interested rational agents, an object which can freely be disposed of can never have a negative price. Nobody will be willing to pay for getting rid of the object if it can just be thrown away. Consequently such assets (share certificates for instance) can not have *negative* rational bubbles; if they had, the expected asset price at some point in the future would be negative, which can not be a rational expectation. In fact, if $p_t < p_t^*$, then everyone will *buy* the asset and hold it forever, which by own use or by hiring out will imply a discounted value equal to p_t^*. Hence, there is excess demand until p_t has risen to p_t^*.

When a negative rational bubble can be ruled out, then, if at the first date of trading of the asset there were no positive bubble, neither can a positive bubble arise later. Let us make this precise:

PROPOSITION 3 Assume free disposal of a given asset. Then, if a rational bubble in the asset price is present today, it must be positive and must have been present also yesterday and so on back to the first date of trading the asset. And if a rational bubble bursts, it will not restart later.

Proof In view of free disposal, a negative rational bubble in the asset price can be ruled out. It follows that $b_t = p_t - p_t^* \geq 0$ for $t = 0, 1, 2, ...$, where $t = 0$ is the first date of trading the asset. That is, any rational bubble in the asset price must be a positive bubble. We now show by contradiction that if for $t = 1, 2, ...$, $b_t > 0$, then $b_{t-1} > 0$. Let $b_t > 0$. Then, if $b_{t-1} = 0$, we have $E_{t-1}b_t = E_{t-1}u_t = 0$ (from (27) with t replaced by $t-1$), implying, since $b_t < 0$ is not possible, that $b_t = 0$ with probability *one* as seen from period $t-1$. Ignoring zero probability events, this rules out $b_t > 0$ and we have arrived at a contradiction. Thus $b_{t-1} > 0$. Replacing t by $t-1$ and so on backward in time, we end up with $b_0 > 0$. The argument also implies that if a bubble bursts in period t, it can not restart in period $t+1$, nor, by extension, in any subsequent period. □

This proposition (due to Diba and Grossman, 1988) informs us that a rational bubble in an asset price must have been there since trading of the asset began. Yet such a conclusion is not without ambiguities. If radically new information or new technology comes up at some point in time, is a share in the firm then the same asset as before? Even if an earlier bubble has crashed, cannot a new rational bubble arise later in case of an utterly new situation?

(b) Bonds with finite maturity The finite maturity ensures that the value of the bond is given at some finite future date. Therefore, if there were a positive bubble in the market price of the bond, no rational agent would buy just before that date. Anticipating this, no one would buy the date before, and so on ... nobody will buy in the first place. By this backward induction argument follows that a positive bubble cannot get started. And since there is also "free disposal", *all* rational bubbles can be precluded. This argument in itself does not, however, rule out positive bubbles on *perpetuities* ("consols") of unique historical origin available in a limited amount.

In the remaining cases we assume negative rational bubbles are ruled out. So, the discussion is about whether *positive* rational asset price bubbles may exist or not.

(c) Assets whose supply is elastic Real capital goods (including buildings) can be reproduced and have clearly defined costs of reproduction. This precludes rational bubbles on this kind of assets, since a potential buyer can avoid the overcharge by producing instead. Notice, however, that building sites with specific amenity value and apartments in attractive quarters of a city are not easily reproducible. Therefore, rational bubbles on such assets are more difficult to rule out.

What about shares of stock in firms? The price evolution of these will be limited to the extent that market participants expect that firms issue more shares if there is a bubble. On the other hand, it is not obvious that the firm would always do this. The firm might anticipate that the bubble would burst. To "fool" the market the firm continues to behave as if no bubble is present. Thus, it is hard to completely rule out rational bubbles on shares of stock by this kind of argument.

(d) Assets for which there exists a "backstop-technology" For some commodities there exists substitutes in elastic supply which will be demanded if the price of the commodity becomes sufficiently high. Such a substitute is called a "backstop-technology". For example oil and other fossil fuels will, when their prices become sufficiently high, be subject to intense competition from substitutes (say, renewable energy sources). This precludes the unbounded bubble process in the price of oil.

On account of the arguments (c) and (d) bubbles seem more probable when it comes to assets which are not reproducible or substitutable and whose "fundamentals" are difficult to ascertain. By fundamentals we mean any information relating to the earnings capacity of an asset: a firm's technology, resources, market conditions etc. For some assets fundamentals are not easily ascertained. Examples are paintings of past great artists, rare stamps, diamonds, gold etc. Also new firms that introduce novel products and technologies are potential candidates (cf. the IT bubble in the late 1990s). Sometimes foreign exchange is added to the list; for a collection of theoretical and empirical studies of this candidate, see Flood and Garber (1994).

The above considerations are of a partial equilibrium nature. On top of this, *general equilibrium* arguments can be put forward to limit the possibility of rational bubbles. We may briefly give a flavour of two of such general equilibrium arguments. As in (a) above, we consider assets that can freely be disposed of and whose services are valued independently of the price. A house, a machine or a share in a firm yields a service in consumption or production or in the form of a dividend stream. Since such an asset has an intrinsic value, equal to the present value of the flow of services, one might believe that positive rational bubbles on such assets can be ruled out in general equilibrium. As we shall see, this is indeed true for an economy with a finite number of "neoclassical" households (to be defined below), but not necessarily in an overlapping generations model. Yet even there, rational bubbles can under certain conditions be ruled out.

(e) An economy with a finite number of infinitely-lived households Assume that the economy consists of a finite number of infinitely-lived agents − here called households − indexed $i = 1, 2, ..., N$. The households are "neoclassical" in the sense that they save only with a view to future consumption.

Under point (a) we saw that $p_t < p_t^*$ can not be an equilibrium.

We now consider the case of a positive bubble, i.e., $p_t > p_t^*$. All owners of the bubble asset who are users will in this case prefer to *sell* and then *rent*; this would imply excess supply and could thus not be an equilibrium. Hence, we turn to households that are not users, but speculators. These may pursue "short selling", that is, rent the asset and immediately sell it at p_t. This results in excess supply and so the asset price falls to p_t^*. Then the speculators buy the asset back and return it to the original owner in accordance with the loan accord. So $p_t > p_t^*$ can not be an equilibrium.

Even ruling out "short selling" (which is sometimes outright forbidden), we can exclude positive bubbles. Presuming that owners who are not users would want to hold the bubble asset for ever as a permanent investment will contradict that these owners are "neoclassical". Indeed, their transversality condition would be violated because the value of their wealth would grow at a rate equal to the rate of interest. This would allow them to increase their consumption now without decreasing it later and without violating their No-Ponzi-Game condition.

Instead we have to imagine that the households owning the bubble asset hold it against future sale. This could on the face of it seem rational enough if there were some probability that not only would the bubble continue to exist, but it would also grow so that the return would be at least as high as that yielded on an alternative investment. Owners holding the asset against subsequently expecting a capital gain will thus plan to sell at some later point in time.

Let t_i be the point in time where household i wishes to sell and let

$$T = \max\{t_1, t_2, ..., t_N\}.$$

Then nobody will plan to hold the asset after T. The household speculator having $t_i = T$ will thus not have anyone to sell to (other than people who will only pay p_T^*). Anticipating this, no-one would buy or hold the asset the period before, and so on. So no-one will want to buy or hold the asset in the first place.

The conclusion is that $p_t > p_t^*$ cannot be a rational expectations equilibrium in a setup with a finite number of "neoclassical" households.

The same line of reasoning does not, however, go through in an OLG model where *new* households – that is, new traders – enter the economy every period.

(f) An economy with interest rate above the growth rate In an OLG model with an infinite sequence of new decision makers, rational bubbles are under certain conditions theoretically possible. The argument is that with $N \to \infty$, T as defined above is not bounded. Although this unboundedness is a necessary condition for rational bubbles, it is not sufficient, however.

To see why, let us return to the arbitrage examples 1, 2, and 3 where we have $a^{-1} = 1 + r$ so that a hypothetical rational bubble has the form $b_{t+1} = (1 + r)b_t + u_{t+1}$, where $E_t u_{t+1} = 0$. So in expected value the hypothetical bubble is growing at a rate equal to the interest rate, r. If at the same time r is higher than the long-run output growth rate, the value of the bubble asset would sooner or later be larger than the economy and so nobody would be able to buy it. The agents anticipate this and so the bubble never gets started.

This point is valid when credit markets are perfect and the interest rate in the OLG economy is higher than the growth rate of the economy − which is normally considered the realistic case. Yet, the opposite case *is* possible and in that situation it is less easy to rule out rational asset price bubbles. It turns out that also the presence of segmented financial markets or externalities that create a wedge between private and social returns on productive investment may increase the scope for rational bubbles (Blanchard, 2008).

The empirical evidence concerning asset price bubbles in general and rational asset price bubbles in particular seems inconclusive. It is very difficult to statistically distinguish between bubbles and misspecified fundamentals. Rational bubbles can also have more complicated forms than the bursting bubble in Example 3 above. For example Evans (1991) and Hall et al. (1999) study "regime-switching" rational bubbles. For surveys on the theory of rational bubbles and econometric bubble tests, see Salge (1997) and Gürkaynak (2008). For discussions of famous historical bubble episodes, see the symposium in *Journal of Economic Perspectives* 4, No. 2, 1990, and Shiller (2005).

Whatever the possible limits to the emergence of rational bubbles in asset prices, it is useful to be aware of their logical structure and the variety of forms they can take as logical possibilities. Rational bubbles may serve as a benchmark for the analytically harder cases of "irrational asset price bubbles", i.e., bubbles arising when a significant fraction of the market participants are not fully rational. This takes us to *behavioral finance* theory, see, e.g., Shiller (2003).

5 Appendix

A. The log-linear specification

In macroeconomic models with rational expectations the equations are often specified as log-linear (although our specific examples in this brief lecture note do not bear this out, since they are based on a simple no-arbitrage condition). The motivation for log-linearity can be:

(a) Linearity is convenient because of the simple rule for the expected value of a sum:

$E(\alpha+\beta x+\gamma z)=\alpha+\beta E(x)+\gamma E(z)$, where x and z are stochastic variables and E is the expectation operator. Indeed, for a non-linear function, $f(x,z)$, we generally have $E(f(x,z)) \neq f(E(x),E(z))$.

(b) Linearity in logs may often seem a more realistic assumption than linearity in anything else.

(c) In time series models a logarithmic transformation of the variables followed by formation of first differences can be the road to eliminating a trend in the mean and variance.

As to point (b) we have:

CLAIM To assume linearity in logs is equivalent to assuming constant elasticities.

Proof Let the positive variables Y, X and Z be related by $Y = F(X,\ Z)$, where F is a continuous function with continuous partial derivatives. Taking the total differential on both sides of $\ln\ Y = \ln F(X,Z)$, we get

$$\begin{aligned} d\ln Y &= \frac{1}{F(X,Z)}\frac{\partial F}{\partial X}dX + \frac{1}{F(X,Z)}\frac{\partial F}{\partial X}dZ \qquad (37) \\ &= \frac{X}{Y}\frac{\partial Y}{\partial X}\frac{dX}{X} + \frac{Z}{Y}\frac{\partial Y}{\partial Z}\frac{dZ}{Z} = \eta_{YX}\frac{dX}{X} + \eta_{YZ}\frac{dZ}{Z} = \eta_{YX}d\ln X + \eta_{YZ}d\ln Z, \end{aligned}$$

where η_{YX} and η_{YZ} are the partial elasticities of Y w.r.t. X and Z, respectively. Thus, defining $y = \ln Y$, $x = \ln X$ and $z = \ln Z$, gives

$$dy = \eta_{YX}dx + \eta_{YZ}dz. \qquad (38)$$

Assuming constant elasticities amounts to putting $\eta_{YX} = \beta$ and $\eta_{YZ} = \gamma$, where β and γ are constants. Then we can write (38) as $dy = \beta dx + \gamma dz$. By integration,

$$y = \alpha + \beta x + \gamma z, \qquad (39)$$

where α is an arbitrary constant. Hereby we have shown that constant elasticities imply a log-linear relationship between the variables.

Now, let us instead start by assuming a log-linear relationship, (39). Then,

$$\frac{\partial y}{\partial x} = \beta, \frac{\partial y}{\partial z} = \gamma. \qquad (40)$$

But we also have

$$\begin{aligned} \frac{\partial y}{\partial x} &\equiv \frac{\partial \ln Y}{\partial \ln X} = \frac{\partial Y/Y}{\partial X/X} = \frac{X}{Y}\frac{\partial Y}{\partial X} \equiv \eta_{YX}, \\ \frac{\partial y}{\partial z} &\equiv \frac{\partial \ln Y}{\partial \ln Z} = \frac{\partial Y/\partial Y}{\partial Z/Z} = \frac{Z}{Y}\frac{\partial Y}{\partial Z} \equiv \eta_{YZ}. \end{aligned}$$

Comparing with (40) we see that $\eta_{YX} = \beta$ and $\eta_{YZ} = \gamma$, that is, the partial elasticities are constant. $\square$

So, when the variables are in logs, then the coefficients in the linear expressions are the elasticities. Note, however, that the interest rate is normally an exception. It is usually regarded as more realistic to let the interest rate itself and not its logarithm enter linearly. Then the associated coefficient indicates the *semi-elasticity* with respect to the interest rate.

B. Conditional expectations and the law of iterated expectations

Intuitively, the conditional expectation is a weighted sum of the possible values of the stochastic variable with weights equal to the corresponding conditional probabilities. Let Y and X be two *discrete* stochastic variables with joint probability function $j(y, x)$ and marginal probability functions $f(y)$ and $g(x)$, respectively. If the conditional probability function for Y given $X = x_0$ is denoted $h(y\,|x_0)$, we have $h(y\,|x_0) = j(y, x_0)/g(x_0)$, assuming $g(x_0) > 0$. The conditional expectation of Y given $X = x_0$, denoted $E(Y|X = x_0)$, is then

$$E(Y|X = x_0) = \sum_y y\frac{j(y, x_0)}{g(x_0)}, \tag{41}$$

where the summation is over all the possible values of y.

This conditional expectation is a function of x_0. Since x_0 is just one possible value of the stochastic variable X, we may want to interpret the conditional expectation itself as a stochastic variable. Then we write it as $E(Y|X)$. Generally, for a function of the discrete stochastic variable X, say $k(X)$, the expected value is

$$E(k(X)) = \sum_x k(x)g(x).$$

When we here let the conditional expectation $E(Y|X)$ play the role of $k(X)$ and sum over all x for which $g(x) > 0$, we get

$$\begin{aligned} E(E(Y|X)) &= \sum_x E(Y|x)g(x) = \sum_x \left(\sum_y y\frac{j(y, x)}{g(x)}\right) g(x) \qquad \text{(by (41))} \\ &= \sum_y y\left(\sum_x j(y, x)\right) = \sum_y yf(y) = E(Y). \end{aligned}$$

This result is a manifestation of the *law of iterated expectations:* the unconditional expectation of the conditional expectation of Y is given by the unconditional expectation of Y.

Now consider the case where Y and X are *continuous* stochastic variables with joint probability *density* function $j(y,x)$ and marginal density functions $f(y)$ and $g(x)$, respectively. If the conditional density function for Y given $X = x_0$ is denoted $h(y\,|x_0)$, we have $h(y\,|x_0) = j(y,x_0)/g(x_0)$, assuming $g(x_0) > 0$. The conditional expectation of Y given $X = x_0$ is now

$$E(Y|X = x_0) = \int_{-\infty}^{\infty} y\frac{j(y,x_0)}{g(x_0)}dy, \tag{42}$$

where we have assumed that the range of Y is $(-\infty, \infty)$. Again, we may view the conditional expectation itself as a stochastic variable. Then we write it as $E(Y|X)$. Generally, for a function of the continuous stochastic variable X, say $k(X)$, the expected value is

$$E(k(X)) = \int_R k(x)g(x)dx,$$

where R stands for the range of X. When we let the conditional expectation $E(Y|X)$ play the role of $k(X)$, we get

$$\begin{aligned} E(E(Y|X)) &= \int_R E(Y|x)g(x)dx = \int_R \left(\int_{-\infty}^{\infty} y\frac{j(y,x)}{g(x)}dy\right) g(x)dx \text{ (by (42))} \\ &= \int_{-\infty}^{\infty} y\left(\int_R j(y,x)dx\right) dy = \int_{-\infty}^{\infty} yf(y)dy = E(Y). \end{aligned} \tag{43}$$

Here we see the *law of iterated expectations* in action for continuous stochastic variables*:* the unconditional expectation of the conditional expectation of Y is given by the unconditional expectation of Y.

EXAMPLE Let the two stochastic variables, X and Y, follow a two-dimensional normal distribution. Then, from mathematical statistics we know that the conditional expectation of Y given X satisfies

$$E(Y|X) = E(Y) + \frac{\text{Cov}(Y,X)}{\text{Var}(X)}(X - E(X)).$$

Taking expectations on both sides gives

$$E(E(Y|X)) = E(Y) + \frac{\text{Cov}(Y,X)}{\text{Var}(X)}(E(X) - E(X)) = E(Y). \qquad \square$$

We may also express the law of iterated expectations in terms of subsets of the original outcome space of a stochastic variable. Let $\mathcal{A}$ be a subset of the outcome space behind the stochastic variable Y and let $\mathcal{B}$ be a subset of $\mathcal{A}$. Then the law of iterated expectations takes the form

$$E(E(Y|\mathcal{B})|\mathcal{A}) = E(Y|\mathcal{A}). \tag{44}$$

That is, when $\mathcal{B} \subseteq \mathcal{A}$, the expectation, conditional on $\mathcal{A}$, of the expectation of Y, conditional on $\mathcal{B}$, is the same as the expectation, conditional on $\mathcal{A}$, of Y.

In the text of this chapter we consider expectations conditional on dated information I_{t-i} $(i = 1, 2, ...)$. By informal analogy with (43) we then express the law of iterated expectations this way:

$$E(E(Y_t|I_{t-i})) = E(Y_t), \qquad \text{for } i = 1, 2, ... \tag{45}$$

In words: the unconditional expectation of the conditional expectation of Y_t, given the information up to time $t - i$ equals the unconditional expectation of Y. Similarly, by informal analogy with (44) we may write

$$E(E(Y_{t+2}|I_{t+1})|I_t) = E(Y_{t+2}|I_t). \tag{46}$$

That is, the expectation today of the expectation tomorrow, when more may be known, of a variable the day after tomorrow is the same as the expectation today of the variable the day after tomorrow. Intuitively: you ask a stockbroker in which direction she expects to revise her expectations upon the arrival of more information. If the broker answers "upward", say, then another broker is recommended.

Admittedly, the notation used in the transition from (44) to (46) is casual (but common in economics). That is why we talk of an "informal analogy". Whereas $\mathcal{A}$ and $\mathcal{B}$ are subsets of the outcome space and $\mathcal{B} \subseteq \mathcal{A}$, the intuitive meaning of an "information set" is different and does *not* fit the inclusion $I_{t+1} \subseteq I_t$. Intuitively "information" rather dictates the opposite inclusion: information is seen as a set which *expands* over time $-$ more and more information is revealed as time proceeds. We leave the matter at this informal level. A rigorous definition of information would involve the vocabulary of advanced probability theory which is outside the scope of this text.

6 Exercises

1. Suppose that $Y_t = X_t + e_t$, where $\{X_t\}$ is a random walk and e_t is white noise.

a) What is the rational expectation of Y_t conditional on all relevant information up to and including period $t - 1$?

b) Compare with the subjective expectation of Y_t based om the adaptive expectations formula with adjustment speed equal to one.

2. Consider a simple Keynesian model of a closed economy with constant wages and prices (behind the scene), excess capacity, and output determined by demand:

$$
\begin{aligned}
Y_t &= C_t + I_t, && (1)\\
C_t &= a + bY^e_{t-1,t}, \qquad a > 0, 0 < b < 1, && (2)\\
I_t &= \bar{I}_t + \varepsilon_t, \qquad \bar{I}_t > 0, && (3)
\end{aligned}
$$

where the endogenous variables are Y_t = output (= income), C_t = consumption, I_t = investment, and $Y^e_{t-1,t}$ = expected output in period t as seen from period $t-1$. The variable ε_t is white noise. To simplify, $\bar{I}_t$ is exogenous (predetermined) and known by the public in advance.

Suppose expectations are "static" in the sense that expected income in period t equals actual income in the previous period.

a) Solve for Y_t.

b) Find the income multiplier with respect to a change in $\bar{I}_t$ and ε_t, respectively.

Suppose instead that expectations are rational.

c) Explain what this means.

d) Solve for Y_t.

e) Find the income multiplier with respect to a change in $\bar{I}_t$ and ε_t, respectively.

f) Compare the result under e) with that under b). Comment.

g) Let us compare the aggregate consumption function in this simple short-run model with the aggregate consumption function in the Ramsey model with logarithmic instantaneous utility. Write down the latter consumption function.

h) Comment on the difference between this and (2).

i) Although (2) is too simple, could there be a good reason for letting the role of expected output in period t enter in a short-run model as it does here? Comment.

3. Consider arbitrage between equity shares and a riskless asset paying the constant rate of return $r > 0$. Let p_t denote the price at the beginning of period t of a share that at the end of period t yields the dividend d_t. As seen from period t there is uncertainty about p_{t+i} and d_{t+i} for $i = 1, 2, \ldots$. That is, $E_t d_t = d_t$, $E_t d_{t+i} = d_{t+i} + u_{t+i}$, where u_{t+i} is a random variable. Suppose agents have rational expectations and care only about expected return (risk neutrality).

a) Write down the no-arbitrage condition.

Suppose dividends follow the process $d_t = \bar{d} + \varepsilon_t$, where $\bar{d}$ is a positive constant and ε_t is white noise, observable in period t, but not known in advance.

b) Find the fundamental solution for p_t and let it be denoted p_t^*. *Hint:* given $y_t = aE_t y_{t+1} + c\, x_t$, the fundamental solution is $y_t = cx_t + c\sum_{i=1}^{\infty} a^i E_t x_{t+i}$.

Suppose someone claims that the share price follows the process

$$p_t = p_t^* + b_t,$$

with a given $b_0 > 0$ and, for $t = 0, 1, 2, \ldots$,

$$b_{t+1} = \begin{cases} \frac{1+r}{q_t} b_t & \text{with probability } q_t, \\ 0 & \text{with probability } 1 - q_t, \end{cases}$$

where $q_t = f(b_t)$, $f' < 0$.

c) What is an asset price bubble and what is a rational asset price bubble?

d) Can the described b_t process be a rational asset price bubble? *Hint:* a bubble solution to the inhomogenous equation $y_t = aE_t y_{t+1} + c\, x_t$ is a solution to the homogeneous equation, $y_t = aE_t y_{t+1}$.

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Chapter 16

Money in macroeconomics

> Money buys goods and goods buy money; but goods do not buy goods.
>
> –Robert W. Clower (1967).

Up to now we have put monetary issues aside. The implicit assumption has been that the exchange of goods and services in the market economy can be carried out without friction as mere barter. This is, of course, not realistic. At best it can be an acceptable approximation to reality only for a limited set of macroeconomic issues. We now turn to models in which there is a demand for money. We thus turn to *monetary theory*, that is, the study of causes and consequences of the fact that a large part of the exchange of goods and services in the real world is mediated through the use of money.

16.1 What is money?

16.1.1 The concept of money

In economics *money* is defined as an asset (a store of value) which functions as a generally accepted medium of exchange, i.e., it can be used directly to buy *any* good offered for sale. A note of IOU (a bill of exchange) may also be a medium of exchange, but it is not *generally* accepted and is therefore not money.[1] From the definition of money follows that money is characterized by being a *fully liquid asset.* An asset is *fully liquid* if it can be used directly, instantly, and without any extra costs or restrictions to make payments.

[1] Generally accepted mediums of exchange are also called *means of payment.*

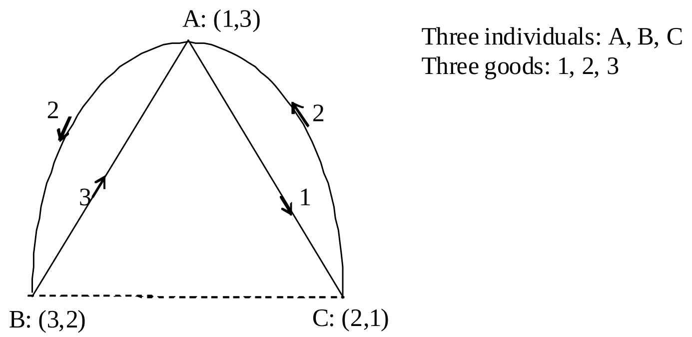

Figure 16.1: No direct exchange possible. A medium of exchange, here good 2, solves the problem (for details, see text).

Generally, liquidity should be conceived as a matter of degree so that an asset has a higher or lower degree of liquidity depending on the extent to which it can easily be exchanged for money. By "easily" we mean "immediately, conveniently, and cheaply". So an asset's *liquidity* is the *ease* with which the asset can be *converted into money or be used directly for making payments.* Where to draw the line between "money" and "non-money assets" depends on what is appropriate for the problem at hand. In the list below of different monetary aggregates (Section 16.2), M_1 corresponds most closely to the traditional definition of money. Defined as currency in circulation plus demand deposits held by the non-bank public in commercial banks, M_1 embraces all under "normal circumstances" fully liquid assets in the hands of the non-bank public.

The reason that a market economy uses money is that money facilitates trade enormously, thereby reducing transaction costs. Money helps an economy to avoid the need for a "double coincidence of wants". The classical way of illustrating this is by the *exchange triangle* in Fig. 161. The individuals A, B, and C are endowed with one unit of the goods 1, 3, and 2, respectively. But A, B, and C want to consume 3, 2, and 1, respectively. Thus, no direct exchange is possible between two individuals each wanting to consume the other's good. There is a *lack of double coincidence of wants.* The problem can be solved by indirect exchange where A exchanges good 1 for good 2 with C and then, in the next step, uses good 2 in an exchange for good 3 with B. Here good 2 serves as a medium of exchange. If good 2 becomes widely used and accepted as a medium of exchange, it is money. Extending the example to a situation with n goods, we have that exchange without money (i.e., barter) requires $n(n-1)/2$ markets ("trading spots"). Exchange with money, in the form of modern "paper money", requires only n markets.

16.1.2 Historical remarks

In the past, ordinary commodities, such as seashells, rice, cocoa, precious metals etc., served as money. That is, commodities that were easily divisible, handy to carry, immutable, and involved low costs of storage and transportation could end up being used as money. This form of money is called *commodity money*. Applying ordinary goods as a medium of exchange is costly, however, because these goods have alternative uses. A more efficient way to trade is by using currency, i.e., coins and notes in circulation with little or no intrinsic value, or pieces of paper, checks, representing claims on such currency. Regulation by a central authority (the state or the central bank) has been of key importance in bringing about this transition into the modern payment system.

Coins, notes, and pieces of paper, with no intrinsic value, that are generally accepted media of exchange, are called *paper money*. By having these pieces of paper circulating and the real goods moving only once, from initial owner to final consumer, the trading costs in terms of time and effort are minimized.

In the industrialized countries these paper monies were in the last third of the nineteenth century and until the outbreak of the First World War *backed* through the gold standard. And under the Bretton-Woods agreement, 1947-71, the currencies of the developed Western countries outside the United States were convertible into US dollars at a fixed exchange rate (or rather an exchange rate which is adjustable only under specific circumstances); and US dollar reserves of these countries were (in principle) convertible into gold by the United States at a fixed price (though in practice with some discouragement from the United States).

This indirect gold-exchange standard broke down in 1971-73, and nowadays money in most countries is *unbacked* paper money (including electronic entries in banks' accounts). This feature of modern money makes its valuation very different from that of other assets. A piece of paper money in a modern payments system has no worth at all to an individual unless she *expects* other economic agents to value it in the next instant. There is an *inherent circularity* in the acceptance of money. Hence the viability of such a paper money system is very much dependent on adequate juridical institutions as well as confidence in the ability and willingness of the government and central bank to conduct policies that sustain the purchasing power of the currency. One elementary juridical institution is that of "legal tender", a status which is conferred to certain kinds of money. An example is the law that a money debt can always be settled by currency and a tax always be paid by currency. A medium of exchange whose market value derives entirely from its legal tender status is called *fiat money* (because it exists by "fiat", a ruler's declaration). In the absence of intrinsic value, maintaining the exchange value of fiat money over time, that is, avoiding high or fluctuating inflation, is one of the central tasks of monetary policy.

16.1.3 The functions of money

The following three functions are sometimes considered to be the definitional characteristics of money:

1. It is a generally accepted medium of exchange.
2. It is a store of value.
3. It serves as a unit of account in which prices are quoted and books kept (the *numeraire*).

On can argue, however, that the last function is on a different footing compared to the two others. Thus, we should make a distinction between the functions that money *necessarily* performs, according to its definition, and the functions that money *usually* performs. Property 1 and 2 certainly belong to the essential characteristics of money. By its role as a device for making transactions money helps an economy to avoid the need for a double coincidence of wants. In order to perform this role, money *must* be a store of value, i.e., a device that transfers and maintains value over time. The reason that people are willing to exchange their goods for pieces of paper is exactly that these can later be used to purchase other goods. As a store of value, however, money is *dominated* by other stores of value such as bonds and shares that pay a higher rate of return. When nevertheless there is a demand for money, it is due to the *liquidity* of this store of value, that is, its service as a medium of exchange.

Property 3, however, is not an indispensable function of money as we have defined it. Though the money unit is usually used as the unit of account in which prices are quoted, this function of money is conceptually distinct from the other two functions and has sometimes been distinct in practice. During times of high inflation, foreign currency has been used as a unit of account, whereas the local money continued to be used as the medium of exchange. During the German hyperinflation of 1922-23 US dollars were the unit of account used in parts of the economy, whereas the mark was the medium of exchange; ad during the Russian hyperinflation in the middle of the 1990s again US dollars were often the unit of account, but the rouble was still the medium of exchange.

This is not to say that it is of little importance that money *usually* serves as numeraire. Indeed, this function of money plays an important role for the short-run macroeconomic effects of changes in the money supply. These effects are due to *nominal rigidities*, that is, the fact that prices, usually denominated in money, of many goods and services generally adjust only sluggishly.

16.2 The money supply

The money supply is the total amount of money available in an economy at a particular point in time (a stock). As noted above, where to draw the line between assets that should be counted as money and those that should not, depends on the context.

16.2.1 Different measures of the money supply

Usually the money supply in an economy is measured as one of the following alternative *monetary aggregates*:

- M_0, i.e., the *monetary base*, alternatively called *base money, central bank money,* or *high-powered money.* The monetary base is defined as fully liquid claims on the central bank held by the private sector, that is, currency (coins and notes) in circulation plus demand deposits held by the commercial banks *in the central bank.*[2] This monetary aggregate is under the direct control of the central bank and is changed by *open-market operations*, that is, by the central bank trading bonds, usually short-term government bonds, with the private sector. But clearly the monetary base is an imperfect measure of the liquidity in the private sector.

- M_1, defined as currency in circulation plus demand deposits held by the non-bank public *in commercial banks.* These deposits are also called checking accounts because they are deposits on which checks can be written and payment cards (debit cards) be used. Note that M_1 does not include currency held by commercial banks and demand deposits

[2] The commercial banks are part of the private sector and by law it is usually only the commercial banks that are allowed to have demand deposits in the central bank − the "banks' bank".

held by commercial banks in the central bank. Thus, although M_1 is a "broader" measure of money supply and generally includes by far the most of M_0, it does *not* include *all* of M_0. The measure M_1 is intended to reflect the quantity of assets which serve as media of exchange in the hands of the non-bank public.

Broader categories of money include:

- $M_2 = M_1$ plus savings deposits with unrestricted access and small-denomination time deposits (say under € 100,000). These claims may not be instantly liquid.
- $M_3 = M_2$ plus large-denomination (say above € 100,000) time-deposits.[3]

As we move down the list, the liquidity of the added assets decreases, while their interest yield increases.[4] Currency earns zero interest. When in macroeconomic texts the term "money stock" is used, most often M_1 or M_2 is meant. Along with currency, the demand deposits in the commercial banks are normally fully liquid, at least as long as they are guaranteed by a governmental deposit insurance (although normally only up to a certain maximum per account). The interest earned on these demand deposits is usually low (at least for "small" depositors) and in fact often ignored in simple theoretical models.

A related and theoretically important simple classification of money types is the following:

1. *Outside money* = money that on net is an asset of the private sector.
2. *Inside money* = money that on net is not net wealth of the private sector.

Clearly M_0 is outside money. However, most money in modern economies is inside money. Deposits at the commercial banks is an example of inside money. These deposits are an asset to their holders, but a liability of the banks. Even broader aggregates of money (or "near-money") than M_3 are sometimes considered. For instance, it has been argued that the amounts that people are allowed to charge by using their *credit cards* should be included in the concept of "broad money". But this would involve *double counting*. Actually *you* do not pay when you use a credit card at the store. It is the company issuing the credit card that pays to the store (shortly after you made

[3]So, whereas (in casual notation) $M_0 \nsubseteq M_1$, $M_1 \subset M_2 \subset M_3$.

[4]This could be an argument for weighing the different components of a monetary aggregate by their degree of liquidity (see Barnett, 1980, and Spindt, 1985).

your purchases). You postpone your payment until you receive your monthly bill from the credit card company. That is, the credit card company does the payment for you and gives credit *to you*. It is otherwise with a *payment card* where the amount for which you buy is almost instantly charged your account in the bank.

16.2.2 The money multiplier

Bank lending is the channel through which the monetary base expands to an effective money supply, the "money stock", considerably larger than the monetary base. The excess of the deposits of the public over bank reserves ("vault cash" and demand deposits in the central bank) is lent out in the form of bank loans, purchases of bonds etc. The public then deposits a fraction of these loans on checking accounts. Next, the banks lend out a fraction of these and so on. This process is named the *money multiplier process*. And the ratio of the "money stock", usually measured as M_1, to the monetary base is called the *money multiplier*.

Let

$$\begin{aligned} CUR &= \text{ currency held by the non-bank public,} \\ DEP &= \text{ demand deposits held by the non-bank public,} \\ \frac{CUR}{DEP} &= cd, \text{ the desired currency-deposit ratio,} \\ RES &= \text{ bank reserves } = \text{ currency held by the commercial banks} \\ & \quad (\text{“vault cash”}) \text{ plus their demand deposits in the central bank,} \\ \frac{RES}{DEP} &= rd, \text{ the desired reserve-deposit ratio.} \end{aligned}$$

Notice that the currency-deposit ratio, cd, is chosen by the non-bank public, whereas the reserve-deposit ratio, rd, refers to the behavior of commercial banks. In many countries there is a minimum reserve-deposit ratio required by law. On the top of that the banks may hold excess reserves depending on their assessment of lending risks etc. To find the money multiplier, note that

$$M_1 = CUR + DEP = (cd + 1)DEP, \tag{16.1}$$

where DEP is related to the monetary base, M_0, through

$$M_0 = CUR + RES = cdDEP + rdDEP = (cd + rd)DEP.$$

Substituting into (16.1) gives

$$M_1 = \frac{cd + 1}{cd + rd} M_0 = mmM_0, \tag{16.2}$$

where $mm = (cd + 1)/(cd + rd)$ is the *money multiplier*.

As a not unrealistic example consider $cd \approx 0.7$ and $rd \approx 0.07$. Then we get $mm \approx 2.2$. When broader measures of money supply are considered, then, of course, a larger money multiplier arises. It should be kept in mind that both cd and rd, and therefore also mm, are neither constant nor exogenous from the point of view of monetary models. They are highly endogenous and depend on interest rates on alternative assets (including bank loans), risks, the degree of sophistication of payment technologies etc. To some extent it is therefore a simple matter of identities and not particularly informative, when we say that, given M_0 and the currency-deposit ratio, the money supply is smaller, the larger is the reserve-deposit ratio. Similarly, since the latter ratio is usually considerably smaller than one, the money supply is also smaller the larger is the currency-deposit ratio. Nevertheless, the money multiplier turns out to fairly stable under "normal circumstances". But not always. During 1929-33, in the early part of the Great Depression, the money multiplier in the US fell sharply. Although M_0 increased by 15% during the four-year period, liquidity (M_1) declined by 27%.[5] Depositors became nervous about their bank's health and began to withdraw their deposits (thereby increasing cd) and this forced the banks to hold more reserves (thereby increasing rd). There is general agreement that this banking panic contributed to the depression and the ensuing deflation.

There is another way of interpreting the money multiplier. Let cm denote the non-bank public's desired *currency-money ratio*, i.e., $cm = CUR/M_1$. Suppose cm is a constant. Then

$$\begin{aligned} CUR &= cmM_1 = cm(cd+1)DEP \qquad &&\text{(by (16.1))} \\ &= cdDEP &&\text{(by definition of } cd\text{)}, \end{aligned}$$

so that $cm = cd/(cd + 1)$ and $1 - cm = 1/(cd + 1)$. Combining this with (16.2) yields

$$M_1 = \frac{1}{\frac{cd}{cd+1} + rd\frac{1}{cd+1}} = \frac{1}{cm + rd(1-cm)} = \frac{1}{1-(1-rd)(1-cm)}M_0 = mmM_0. \tag{16.3}$$

The way the central bank controls the monetary base is through *open-market operations*, that is, by buying or selling bonds (typically short-term government bonds) in the amount needed to sustain a desired level of the monetary base. In the next stage the aim could be to obtain a desired level of M_1 or a desired level of the short-term interest rate or (in an open economy) a desired exchange rate vis-a-vis other currencies.

[5] Blanchard (2003).

A perhaps more intuitive understanding of the money multiplier and the way commercial banks "create" money can be attained from a dynamic angle. Suppose the central bank increases M_0 by the amount ΔM_0 through an open-market operation, thus purchasing bonds. This is the first round. The seller of the bonds deposits the fraction $1 - cm$ on a checking account in her bank and keeps the rest as cash. The bank keeps the fraction rd of $(1 - cm)\Delta M_0$ as reserves and provides bank loans or buys bonds with the rest. This is the second round. Thus, in the first round money supply is increased by ΔM_0; in the second round it is further increased by $(1 - rd)(1 - cm)\Delta M_0$; in the third round further by $(1 - rd)^2(1 - cm)^2\Delta M_0$, etc.[6] In the end, the total increase in money supply is

$$\begin{aligned}\Delta M_1 &= \Delta M_0 + (1 - rd)(1 - cm)\Delta M_0 + (1 - rd)^2(1 - cm)^2\Delta M_0 + ... \\ &= \frac{1}{1 - (1 - rd)(1 - cm)}\Delta M_0 = mm\Delta M_0.\end{aligned}$$

The second last equality comes from the rule for the sum of an infinite geometric series with quotient in absolute value less than one. The conclusion is that the money supply is increased mm times the increase in the monetary base.

16.3 Money demand

Explaining in a precise way how paper money gets purchasing power and how the demand for money is determined, is a difficult task and not our endeavour here. Suffice it to say that:

- In the presence of sequential trades and the absence of complete information and complete markets, there is a need for a generally accepted medium of exchange − *money.*
- The demand for money should be seen as part of a broader *portfolio decision* through which economic agents allocate their financial wealth to different existing assets, including money. The portfolio decision involves a balance of considerations of *expected return after tax*, *risk*, and *liquidity.*

Money is demanded primarily because of its liquidity service in transactions. The demand for money therefore depends on the *amount of transactions* expected to be carried out with money in the near future. The demand

[6] For simplicity, we assume here that cm and rd are constant.

for money also depends on the need for flexibility in spending when there is uncertainty: it is convenient to have ready liquidity in case favorable opportunities should turn up. Generally money earns no interest at all or at least less interest than other assets. Therefore the demand for money involves a trade-off between the need for liquidity and the wish for a high interest yield.

The incorporation of a micro-founded money demand in macromodels is often based on one or another kind of short-cut:

- The *cash-in-advance constraint* (also called the *Clower constraint*).[7] Here it is simply postulated that money *must* be used in many or all transactions. The households' purchases of most consumption goods in a short period must be paid for by money held at the beginning of the period. In continuous time models the assumption is that the needed stock of money is given by some function increasing in the desired consumption per time unit and decreasing in the opportunity cost of holding money.

- The *shopping-costs* approach. Here the liquidity services of money are modelled as reducing shopping time or other kinds of non-pecuniary or pecuniary shopping costs.

- The *money-in-the-utility function* approach. Here, the indirect utility that money provides through reducing non-pecuniary as well as pecuniary transaction costs is modelled as if the economic agents obtain utility directly from holding money. This will be our approach in the next chapter.

- The *money-in-the-production-function* approach. Here money facilitates the firms' transactions, making the provision of the necessary inputs easier. After all, typically around a third of the aggregate money stock is held by firms.[8]

16.4 What is then the "money market"?

In macroeconomic theory, by the "money market" is usually meant an abstract market place (not a physical location) where at any particular moment the aggregate demand for money "meets" the aggregate supply of money. Suppose the aggregate demand for real money balances can be approximated

[7] After the American monetary theorist Robert Clower (1967).

[8] Mankiw and Summers (1986). For a detailed account of the different ways of modelling money demand in macroeconomics, the reader is referred to, e.g., Walsh (2003).

by the function $L(Y, i)$, where $L_Y > 0$ and $L_i < 0$ (“L” for liquidity demand). The level of aggregate economic activity, Y, enters as an argument because it is an (approximate) indicator of the volume of transactions in the near future for which money is needed. The short-term nominal interest rate, i, enters because it is the opportunity cost of holding cash instead of interest-bearing short-term securities, for instance government bonds that mature in one year or less. [9] The latter constitute a close substitute to money because they have a high degree of liquidity. They are extensively traded and under “normal circumstances” safe. Because of the short term to maturity, their market value is less volatile than longer-term securities.

Let the monetary aggregate in focus be M_1 and let P be the general price level in the economy (say the GDP deflator). Then money market equilibrium is present if

$$M_1 = PL(Y, i), \tag{16.4}$$

that is, the available amount of money equals nominal money demand. Note that supply and demand are in terms of stocks (amounts at a given point in time), not flows. One of the issues in monetary theory is to account for how this stock equilibrium is brought about at any instant. Which of the variables M_1, P, Y, and i is the equilibrating variable? In classical (pre-Keynesian) monetary theory it was P, in Keynesian monetary theory it is primarily i.[10]

One may alternatively think of the “money market” in a more narrow sense, however. We may translate (16.4) into a description of demand and supply for base money:

$$M_0 = \frac{P}{mm} L(Y, i), \tag{16.5}$$

where mm is the money multiplier. The right-hand side of this equation reflects that the demand for M_1 via the actions of commercial banks is translated into a demand for base money.[11] If the public needs more cash, the demand for bank loans increases and when granted, banks' reserves are reduced. When in the next round the deposits in the banks increase, then

[9] To simplify, we assume that none of the components in the monetary aggregate considered earns interest. In practice demand deposits in the central bank and commercial banks may earn a small nominal interest.

[10] If the economy has ended up in a “liquidity trap” with i at its lower bound, 0, an increase in M_1 will not generate further reductions in i. Agents would prefer holding cash at zero interest rather than short-term bonds at negative interest. That is, the “=” in the equilibrium condition (16.4) should be replaced by “$\geq$” or, equivalently, $L(Y, i)$ should at $i = 0$ be interpreted as a “set-valued function”. The implications of this are taken up later in this book.

[11] Although the money multiplier tends to depend positively on i as well as other interest rates, this aspect is unimportant for the discussion below and is ignored in the notation in (16.5).

generally also the banks' reserves have to increase. To maintain the required reserve-deposit ratio, banks which for a few days have too little liquidity, borrow from other banks or other institutions which have too much.

This narrowly defined money market is closely related to what is by the practitioners and in the financial market statistics called the "money market", namely the trade in debt-instruments, commercial paper, and government bonds with maturity of less than one year. The agents trading in this market not only include the central bank and the commercial banks but also the mortgage credit institutions and sometimes also life insurance companies and other financial institutions. What is in the theoretical models called the "short-term nominal interest rate" can normally be identified with what is in the financial market statistics called the *money market rate* or the *interbank rate.* This is the interest rate (measured as a per year rate) at which the commercial banks provide unsecured loans ("signature loans") to each other, often on a day-to-day basis.

Open market operations The commercial banks may under certain conditions borrow (on a secured basis) from the central bank at a rate usually called the *discount rate.* This central bank lending rate will be somewhat above the central bank *deposit rate*, that is, the interest rate, possibly nil, earned by the commercial banks on their deposits in the central bank. The interval between the discount rate and the deposit rate constitutes the *interest rate corridor*, within which, under "normal circumstances", the money market rate, i, fluctuates. The central bank deposit rate acts as a floor for the money market rate and the central bank lending rate as a ceiling. Sometimes, however, the money market rate exceeds the central bank lending rate. This may happen in a financial crisis where the potential lenders are hesitant because of the risk that the borrowing bank goes bankrupt and because there are constraints on how much and when, a commercial bank in need of cash can borrow from the central bank.

If the money market rate, i, tends to deviate from what the central bank aims at (the "target rate", also called the "policy rate"), the central bank will typically through open-market operations provide liquidity to the money market or withhold liquidity from it. The mechanism is as follows. Consider a *one-period government bond* with a secured payoff equal to 1 euro at the end of the period and no payoffs during the period (known as a zero-coupon bond or discount bond). To fix ideas, let the period length be one month. In the financial market jargon the *maturity date* is then one month after the *issue date.* Let Q be the market price (in euros) of the bond at the beginning of the month. The implicit monthly interest rate, x, is then the solution to the equation $Q = (1+x)^{-1}$, i.e.,

$$x = \frac{1-Q}{Q} = Q^{-1} - 1.$$

Translated into an annual interest rate, with monthly compounding, this amounts to $i = (1+x)^{12} - 1 = Q^{-12} - 1$ per year. With $Q = 0.9975$, we get $i = 0.03049$ per year.[12]

Suppose the central bank finds that i is too high and buys a bunch of these bonds. Then less of them are available for the private sector, which on the other hand now has a larger money stock at its disposal. According to the Keynesian monetary theory (which is by now quite commonly accepted), under normal circumstances the general price level for goods and services is sticky in the short run. It will be the bond price, Q, which responds. In the present case it moves up, thus lowering i, until the available stocks of bonds and money are willingly held. In practice this adjustment of Q, and hence i, to a new equilibrium level takes place rapidly.

In recent decades the short-term interest rate has been the *main* monetary policy tool when trying to stimulate or dampen the general level of economic activity and control inflation. Under normal circumstances the open market operations give the central bank a narrow control over the short-term interest rate. Central banks typically *announce* their *target level* for the short-term interest rate and then adjust the monetary base such that the actual money market rate ends up close to the announced interest rate. This is what the European Central Bank (the ECB) does when it announces the target for EONIA (euro overnight index average) and what the U.S. central bank, the Federal Reserve, does when it announces the target for the *federal funds rate,* which in spite of its name is not an interest rate charged by the U.S. central bank but a weighted average of the interest rates commercial banks in the U.S. charge each other, usually overnight.

In the narrowly defined "money market" close substitutes to money are traded. From a logical point of view a more appropriate name for this market would be the "short-term bond market" or the "near-money market". This would entail using the term "market" in its general meaning as a "market place" where a certain type of goods or assets are traded *for money.* Moreover, speaking of a "short-term bond market" would be in line with the standard name for market(s) for financial assets with maturity of more than one year, namely *market(s) for longer-term bonds and equity*; by practitioners these markets are also called the *capital markets.* Anyway, in this book

[12] With continuous compounding we have $Q = e^{-i/12}$ so that $i = 12 \ln Q^{-1} = 0.03004$ when $Q = 0.9975$.

we shall use the term "money market" in its broad theoretical meaning as an abstract market place where the aggregate demand for money "meets" the aggregate supply of money. As to what kind of money, "narrow" or "broad", further specification is always to be added.

The open-market operations by the central bank affect directly or indirectly all the equilibrating prices in the financial markets as well as expectations about the future path of these prices. This influence derives from the direct control over the monetary base, M_0. The central bank has no direct control, however, over the money supply in the broader sense of M_1, M_2, or M_3. These broader monetary aggregates are also affected by the behavior of the commercial banks and the non-bank public. The money supply in this broad sense can at most be an intermediate target for monetary policy, that is, a target that can be reached in some average-sense in the medium run.

16.5 Key questions in monetary macroeconomics

Some of the central questions in monetary macroeconomics are:

1. How does the *level* and the *growth rate* of the money supply (in the M_0 sense) affect:

   (a) the real variables in the economy (resource allocation),

   (b) the price level and the rate of inflation?

2. How can monetary policy be designed to stabilize the purchasing power of money and optimize the liquidity services to the inhabitants?

3. How can monetary policy be designed to stabilize business cycle fluctuations?

4. Do rational expectations rule out persistent real effects of changes in the money supply?

5. What kind of regulation of commercial banks is conducive to a smooth functioning of the credit system and reduced risk of a financial crisis?

6. Is hyperinflation always the result of an immense growth in the money supply or can hyperinflation be generated by self-fulfilling expectations?

As an approach to answering long-run monetary issues, we will in the next chapter consider a kind of classical monetary model by Sidrauski (1967). In this model money enters as a separate argument in the utility function. The model has been applied to the study of long-run aspects like the issues 1, 2, and 6 above. The model is less appropriate, however, for short- and medium-run issues such as 3, 4, and 5 in the list. These issues are dealt with in later chapters.

16.6 Literature notes

In the *Arrow-Debreu model*, the basic microeconomic general equilibrium model, there is assumed to exist a *complete set of markets.* That is, there is a market for each "contingent commodity", by which is meant that there are as many markets as there are possible combinations of physical characteristics of goods, dates of delivery, and "states of nature" that may prevail. In such an idealized world any agent knows for sure the consequences of the choices made. All trades can be made once for all and there will thus be no need for any money holding.

16.7 Exercises

Macroeconomics 2 Short Note 4

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Back to short-run macroeconomics

In this lecture note we shift the focus from long-run macroeconomics to short-run macroeconomics. The *long-run* models concentrated on mechanisms that are important for the economic evolution over a time horizon above, say, 10 years. With such a horizon it is the development on the supply side (capital accumulation, population growth, and technical progress) that is the primary determinant of cumulative changes in output and consumption − the trend. The demand side and monetary factors are important for the fluctuations around the trend. In a long-run perspective these fluctuations have limited quantitative importance. But within a short horizon, say up to four years, the demand-side, monetary factors, nominal rigidities, and expectation errors are quantitatively important. The present note re-introduces these *short-run* factors and aims at suggesting how short-run and long-run theory are linked. In fact this means hinting a little on theory dealing with the *medium run,* say 3 to 10 years.[1] The purpose of medium-run theory is to explain the regularities in the fluctuations (business cycles) around the trend and to study what can be accomplished by stabilization policy. In that context the *dynamic interaction* between demand and supply factors and the time-consuming adjustment in relative prices play an important role. In this way medium-run theory bridges the gap between the long run and the short run.

1 Stylized facts for the short run

The Keynesian idea that prices of goods and services are sticky in the short run rests on the empirical observation that in the short run firms in the manufacturing and service industries typically let output do the adjustment to changes in demand while keeping prices unchanged. In industrialized societies firms are able to do that because under "normal circumstances" there is excess capacity in the economy. Indeed, three of the most salient short-run features that arise from macroeconomic time series analysis of industrialized

[1] These number-of-years declarations should not be understood as more than a rough indication. Their appropriateness will depend on the specific historical circumstances and on the problem at hand.

market economies are the following (cf. Blanchard and Fischer, 1989, Nishimura, 1992, Christiano et al., 1999):

1) Shifts in aggregate demand (induced by sudden changes in the state of confidence, exports, fiscal or monetary policy, or other events) are largely accommodated by changes in quantities rather than changes in nominal prices − *nominal price insensitiveness.*

2) Large movements in quantities are often associated with little or no movement in relative prices, including real wages − *real price insensitiveness.*

3) By contrast, nominal prices are sensitive to general changes in input *costs.*

These stylized facts pertain to final goods and services. The claim is not that *all* nominal prices are insensitive vis-a-vis demand changes in the short run. One must distinguish between production of final goods and services on the one hand and production of primary foodstuff and raw materials on the other. This leads to the associated distinction between "cost-determined" and "demand- determined" prices.

Final goods and services are typically differentiated (imperfect substitutes) and their production takes place under conditions of imperfect competition. As a result of existing reserves of productive capacity, generally speaking, the *supply is elastic.* Increases in demand are mainly met by increases in production rather than in prices. The price changes which do occur result mostly from changes in costs of production. Hence the name "cost-determined" prices.

For *primary foodstuff* and many *raw materials* the situation is different. To increase the supply of agricultural products requires considerable time. This is also true (though not to the same extent) with respect to mining of raw materials and extraction and transport of crude oil. When supply is *inelastic* in the short run, an increase in demand results in a diminution of stocks and a rise in price. Hence the name "demand-determined prices". The price rise may be enhanced by a speculative element: temporary hoarding in the expectation of further price increases . The price of oil and coffee − two of the most traded commodities in the world market − fluctuate a lot. Through the channel of *costs* the changes in these demand-determined prices spill over to the prices of final goods.

As Keynesian macroeconomics deals with industrialized economies, where manufacturing and services are the main sectors, the general price level is regarded as essentially cost-determined rather than demand determined. Modern Keynesian theory emphasizes two aspects on top of this. First, many wages and prices are set in nominal terms by *price setting agents*, say, craft unions and firms operating in imperfectly competitive markets. Second, these wages and prices are typically kept unchanged for some time even if changes in the environment of the agent occurs; this aspect, interpreted as reflecting the presence of costs associated with changing prices, is called *nominal price rigidity.* Both aspects have vast consequences for the functioning of the economy as a whole compared with a regime of perfect competition.

2 A simple short-run model

Here we consider a very simple model which by Krugman (1999) was named *The World's Smallest Macroeconomic Model.* Krugman tells he "learned this model from Robert Hall back in 1975". The model is useful in at least two ways:

- the model displays the fundamental difference in the functioning of a flex price economy (where prices are fully flexible) and a fix price economy (where prices are sticky);
- the model is a suitable point of departure for a definition of the important Keynesian concept of *effective demand.*

2.1 The model

We consider a monetary closed economy which produces a consumption good. There are three sectors in the economy, a production sector, a household sector, and a public sector with a consolidated government/central bank. Time is discrete. There is a current period and "the future", compressing the second period and onward. Labor is the only input in production.

The production function has CRS,

$$Y = \alpha N, \qquad \alpha > 0, \tag{1}$$

where Y is aggregate output of a consumption good and N is aggregate employment, both in the current period. The price of the consumption good in terms of money, i.e., the "nominal" price, is P and the nominal wage is W. Let both prices be taken as given by the representative firm (price setting agents are considered later). The nominal profit is

$$\Pi = PY - WN. \tag{2}$$

There is free exit from the production sector in the sense that the representative firm can decide to produce nothing. Hence, an equilibrium with positive production requires that profits are non-negative.

There is a representative household supplying labor inelastically in the amount $\bar{N}$ and receiving the profit obtained by the firm, if any. The household demands the consumption good in the amount C in the current period. Current income not consumed is saved for the future. Only one asset is available for saving, namely money. There is no private banking sector in the economy. So "money" means base money (currency in circulation) and is on net an asset in the private sector as a whole. The preferences of the household are given by the intertemporal utility function,

$$U = \ln C + \beta \ln \frac{M'}{P^e}, \qquad 0 < \beta < 1, \tag{3}$$

where β is the utility discount factor (so the utility discount rate, ρ, is given by $(1+\rho)^{-1} = \beta$), M' is the money stock transferred to "the future", and P^e is the expected future price level.

The consolidated government/central bank has full control over the monetary base. There are no other non-human assets than money. Let M be the aggregate stock of money held by the representative household at the beginning of the current period.

Consider the household's choice problem. Given the expected future price level, P^e, and facing given P and W, the household chooses C and M' to maximize U s.t.

$$PC + M' = M + WN + \Pi \equiv B, \tag{4}$$

where M is the predetermined stock of money held at the beginning of the current period. The sum of initial financial wealth, M, and nominal income, $WN + \Pi$, constitutes the budget, B.[2] Nominal financial wealth at the beginning of the next period is $M' = M + WN + \Pi - PC$, i.e., the sum of initial financial wealth and saving, $WN + \Pi - PC$. The benefit obtained by transferring M' depends on the expected purchasing power of M', hence it is M'/P^e that enters the utility function. This minimalist way of taking the future into account is of course a shortcut.

[2] As time is discrete, expressions like $M + WN + \Pi$ are legitimate. Although it is meaningless to add a stock and a flow since they have different denominations, the sum $M + WN + \Pi$ should be interpreted as $M + (WN + \Pi)\Delta t$, where Δt is the period length, equal to one time unit.

Substituting $M' = B - PC$ into (3), we get the first-order condition

$$\frac{dU}{dC} = \frac{1}{C} + \beta \frac{P^e}{B - PC}(-\frac{P}{P^e}) = 0,$$

which gives

$$PC = \frac{1}{1+\beta}B. \tag{5}$$

We see that the marginal (= average) propensity to consume is $(1+\beta)^{-1}$ and thus inversely related to the patience parameter β. The resulting stock of money held at the end of the period is

$$M' = (1 - \frac{1}{1+\beta})B = \frac{\beta}{1+\beta}B.$$

We see that the expected price level, P^e, in the future does not affect the demands, C and M'. This is a special feature and due to the additive logarithmic specification of the utility function in (3).

By (5) there is equilibrium in the output market if and only if

$$Y = C = \frac{M + WN + \Pi}{P(1+\beta)} = \frac{\frac{M}{P} + Y}{1+\beta},$$

where we have inserted (2). We may express Y in terms of M/P :

$$Y = \beta^{-1}\frac{M}{P}. \tag{6}$$

We have assumed that agents take prices as given when making their decisions. But we have said nothing about whether nominal prices are flexible or rigid.

2.2 The case of fully flexible W and P

Classical macroeconomics, sometimes called Walrasian macroeconomics, assumes that both W and P are fully flexible so as to make market clearing under perfect competition possible. Indeed, both W and P are thought to immediately adjust to clear the labor and output markets (like in centralized auction markets). Clearing in the labor market requires that labor demand, N, equals labor supply, $\bar{N}$:

$$N = \bar{N}. \tag{7}$$

In view of CRS, this requires that firms have zero profit. In turn, by (1) and (2), zero profit requires that the real wage equals labor productivity:

$$\frac{W}{P} = \alpha. \tag{8}$$

Given clearing in the labor market, output equals full-employment output,

$$Y = \alpha \bar{N} \equiv Y^*.$$

Output is here said to be *supply-determined*, i.e., determined by the supply of labor. Profits are $\Pi = PY^* - W\bar{N} = (Y^* - \alpha\bar{N})P = 0$, in view of (8). The output market equilibrium condition (6) can then be solved for the equilibrium price level

$$P = \beta^{-1}\frac{M}{Y^*} \equiv P^*. \tag{9}$$

This, together with (8), gives the equilibrium wage

$$W = \alpha P^*. \tag{10}$$

General equilibrium also requires that desired money holding at the end of the period equals money supply. By Walras' law this equality follows automatically from the household's Walrasian budget constraint and equilibrium in the output and labor markets. Indeed, the household's Walrasian budget constraint is given by (4) with actual employment, N, replaced by labor supply, $\bar{N}$. The value of excess demands adds to

$$\begin{aligned} W(N - \bar{N}) + P(C - Y) + M' - M &= -\Pi - W\bar{N} + PC + M' - M \\ &= -\Pi - W\bar{N} + PC + M + W\bar{N} + \Pi - PC - M = 0, \end{aligned}$$

where the first equality comes from (2) and the second from the Walrasian budget constraint. So the condition $M' = M$ is ensured as soon as there is clearing in the output and labor markets.

The intuitive adjustment process leading to general equilibrium is the following *classical* mechanism. Imagine that in an ultra-short sub-period $W/P - \alpha \neq 0$. In case $W/P - \alpha > 0$ (< 0), there will be excess supply (demand) in the labor market. This lowers W (raises W). Only when $W/P = \alpha$ and full employment obtains, will the system be at rest. Next imagine that $P - P^* \neq 0$. In case $P - P^* > 0$ (< 0), there is pure excess supply (demand) in the output market. This lowers P (raises P). Again, only when $W/P = \alpha$ *and* $P = P^*$, so that the output market clears under full employment, will the system be at rest.

By comparative statics based on (9) and (10) we see that in the classical regime P and W are lower the lower is M. Output is at the unchanged full-employment level whatever the level of M. This is the *neutrality of money* result of classical macroeconomics.

The neutrality result also holds in a quasi-dynamic context where we consider an actual change in the money supply. Since by assumption money is the only asset in the economy, changes in the money supply can not be brought about through open market operations where the central bank buys or sells another financial asset. Instead we imagine that a sudden once for all change in M occurs at the beginning of the current period in the form of a lump-sum "helicopter drop" of size ΔM. So we replace (4) by

$$PC + M' = M + \Delta M + WN + \Pi \equiv B.$$

Owing to the absence of any taxes these transfers implies the presence of a budget deficit of size ΔM, being financed by money issue. If we replace M in the previous formulas by $\hat{M} \equiv M + \Delta M$, we see that money neutrality still holds.

2.3 The case of rigid W and P

Keynesian macroeconomics assumes that the nominal wages and prices are given in the short run, fixed in advance by workers (or workers' unions) and firms, respectively. Let P be given at the level $\bar{P}$ and W at the level $\bar{W}$. Because of free exit, an equilibrium with positive production requires that the real wage does not exceed labor productivity:

$$\frac{\bar{W}}{\bar{P}} \leq \alpha. \tag{11}$$

Otherwise the firms would choose zero output.

Suppose (11) holds so that $\bar{P} > P^*$, where P^* is given in (9). From the output-market equilibrium condition (6) we then find equilibrium output to be

$$Y = \beta^{-1}\frac{M}{\bar{P}} < \beta^{-1}\frac{M}{P^*} = Y^*. \tag{12}$$

Output is here said to be *demand-determined*. From (1) follows that employment is

$$N = \frac{Y}{\alpha} = \frac{M}{\alpha\beta\bar{P}} < \bar{N},$$

and profits are $\Pi = \bar{P}Y - \bar{W}N = (\alpha - \frac{\bar{W}}{\bar{P}})\bar{P}N = (\alpha - \frac{\bar{W}}{\bar{P}})\frac{M}{\alpha\beta} \geq 0$.

This solution for (Y, N) is known as an equilibrium with "Keynesian unemployment". We see the vicious circle. Output is below the full-employment level because of low consumption demand, and consumption demand is low because of the low employment. Even if $\Pi > 0$, firms have no incentive to increase production because there would not be sufficient demand for the extra output. We also see that here money is *not neutral*. Given $P = \bar{P}$, a higher M results in higher output and higher employment.

To illustrate the situation the reader is invited to draw the standard 45° Keynesian diagram in the (Y, Y^d) plane, where $Y^d = C = (1+\beta)^{-1}(Y + M/\bar{P})$.

Note that in this Keynesian setup the households' consumption demand depends *not* on how much labor the households would *prefer* to sell at the going wage rate, but on how much they are *able* to sell, that is, on a *quantity signal* received from the market. Indeed, it is the *actual* employment, N, that enters the operative budget constraint, (4), not the desired employment as in classical or Walrasian theory. Similarly, the output supplied by the firms not only depends on the sales price and costs, but also on how much they are *able* to sell. That is, again in contrast to Walrasian theory, a *quantity signal* from the market matters.

On this background Keynesian theory defines an agent's *effective demand* as the demand chosen on the basis of not only price signals, but also the quantity signals received from the markets. Similarly, an agent's *effective supply* is defined as the supply chosen on the basis of not price signals as well as quantity signals received from the markets. Finally, an agent's *effective budget constraint* is defined as the budget constraint that is operative under the given price signals as well as quantity signals received from the markets.

In contrast, classical or Walrasian theory is based on the notions of *Walrasian demand, Walrasian supply,* and *Walrasian budget constraint*, respectively, all of which depends only on price signals and initial resources.

Of course the above model has many limitations:

a) The wage and price setting should be modelled and there should be an explanation of the wage and price rigidity.

b) To incorporate better the key role of financial markets, at least one extra asset should enter, an interest-bearing asset.

c) The model should be truly dynamic with forward-looking expectations and gradual wage and price changes via some kind of an expectations-augmented Phillips curve.

We now comment briefly on a), b), and c).

3 Price adjustment costs

The classical theory of perfectly flexible wages and prices and neutrality of money seems contradicted by overwhelming empirical evidence. At the theoretical level the theory ignores that the dominant market form is not perfect competition. Wages and prices are usually set by agents with market power. And there may be costs associated with changing prices and wages. Here we consider such costs.

The literature has modelled price adjustment costs in two different ways. *Menu costs* refer to the case where there are *fixed costs* of changing price. Another case considered in the literature is the case of *strictly convex adjustment costs*, where the marginal price adjustment cost is increasing in the size of the price change.

The most obvious examples of *menu costs* are of course costs associated with

1. remarking commodities with new price labels,
2. reprinting price lists ("menu cards") and catalogues.

But the term menu costs should be interpreted in a broader sense, including pecuniary as well non-pecuniary costs of:

3. information-gathering,
4. recomputing optimal prices,
5. conveying the new directives to the sales force,
6. the risk of offending customers by frequent price changes,
7. search for new customers willing to pay a higher price,
8. renegotiating contracts.

Menu costs induce firms to change prices less often than if no such costs were present. And some of the points mentioned in the list above, in particular point 7 and 8, may be relevant also in the different labor markets.

The menu cost theory is one of the microfoundations provided by modern Keynesian economics for the presumption that nominal prices and wages are sticky in the short run. The main theoretical insight of the menu cost theory is the following. There are menu costs associated with changing prices. Even *small* menu costs can be enough to prevent firms from changing their price. This is because the opportunity cost of not changing price is only of second order, i.e., "small"; this is a reflection of the *envelope theorem* (see

Appendix). But owing to imperfect competition (price $>$ MC), the effect on aggregate output, employment, and welfare of not changing prices is of first order, i.e., "large".

The menu cost theory provides the more popular explanation of nominal price rigidity. Another explanation rests on the presumption of *strictly convex price adjustment costs*. In this theory the price change cost for firm i is assumed to be $k_{it} = \alpha_i(P_{it} - P_{it-1})^2$, $\alpha_i > 0$. Under this assumption the firm is induced to avoid large price changes, which means that it tends to make frequent, but small price adjustments. This theory is related to the customer market theory. Customers search less frequently than they purchase. A large upward price change may be provocative to customers and lead them to do search in the market, thereby perhaps becoming aware of attractive offers from other stores. The implied "kinked" demand curve can explain that firms are reluctant to suddenly increase their price.

4 Adding interest-bearing assets

To incorporate the key role of financial markets for the performance of the macroeconomy, at least one extra asset should enter in a short-run model, an interest-bearing asset. This gives rise to the IS-LM model that should be familiar from Blanchard, *Macroeconomics*, 3rd edition, 2003, in particular Chapter 5.

An extended IS-LM model is presented in Blanchard, *Macroeconomics: A European Perspective*, 2010, Chapter 20. The advantage of this extended version is that the commercial banking sector is introduced more explicitly so that the model incorporates both a centralized bond market and decentralized markets for bank loans.

5 Adding dynamics and a Phillips curve

Adding dynamics, expectations formation, and a Phillips curve leads to a *medium-run model*. An introduction is provided in the first-mentioned Blanchard textbook, chapters 8 and 14. The medium-run model describes fluctuations around trend GDP, the latter being given as output at the "natural rate of unemployment". Adding capital accumulation, technical progress, and growth in the labor force, GDP gets a rising trend.

Roughly speaking, this course, Macroeconomics 2, is about an economy moving along this trend. We have more or less ignored the fluctuations, simply by assuming flexible prices and perfect competition. In a realistic economy with imperfect competition and price stickiness in both output and labor markets the natural rate of unemployment is likely to be higher than in an economy with perfect competition. And hump-shaped deviations from trend GDP - business cycles - are likely to arise when the economy is hit by large shocks.

The third macro course, Macroeconomics 3, deals with short and medium run theory and emphasizes issues related to monetary policy.

6 Appendix

ENVELOPE THEOREM Let $y = f(a, x)$ be a continuously differentiable function of two variables, of which one, a, is conceived as a parameter and the other, x, as a control variable. Let $g(a)$ be a value of x at which $\frac{\partial f}{\partial x}(a, x) = 0$, i.e., $\frac{\partial f}{\partial x}(a, g(a)) = 0$. Let $F(a) \equiv f(a, g(a))$. Provided $F(a)$ is differentiable,

$$F'(a) = \frac{\partial f}{\partial a}(a, g(a)),$$

where $\partial f / \partial a$ denotes the partial derivative of $f(\cdot)$ w.r.t. the first argument.

Proof $F'(a) = \frac{\partial f}{\partial a}(a, g(a)) + \frac{\partial f}{\partial x}(a, g(a))g'(a) = \frac{\partial f}{\partial a}(a, g(a))$, since $\frac{\partial f}{\partial x}(a, g(a)) = 0$ by definition of $g(a)$. $\square$

That is, when calculating the total derivative of a function w.r.t. a parameter and evaluating this derivative at an interior maximum w.r.t. a control variable, the envelope theorem allows us to ignore the terms that arise from the chain rule. This is also the case if we calculate the total derivative at an interior minimum.[3]

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[3] For extensions and more rigorous framing of the envelope theorem, see for example Sydsaeter et al. (2006).