1. How many zeros are at the end of 1000! ?
   (a) 240                   (b) 248                  (c) 249                 (d) None

   Ans:-(c) The number of two’s is enough to match each 5 to get a 10.

   So,
   
   \[ \begin{align*}
   5^1 & \rightarrow 200 \\
   5^2 & \rightarrow 40 \\
   5^3 & \rightarrow 8 \\
   5^4 & \rightarrow 1
   \end{align*} \]

   [Theorem: (de Polinac's formula)
   Statement: Let p be a prime and e be the largest exponent of p such that \( p^e \) divides \( n! \), then
   \( e = \sum \left\lfloor \frac{n}{p^i} \right\rfloor \), where \( i \) is running from 1 to infinity.]

   So, \( \left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{25} \right\rfloor + \left\lfloor \frac{1000}{125} \right\rfloor + \left\lfloor \frac{1000}{625} \right\rfloor = 249 \).

   Thus, 1000! ends with 249 zeros.

2. The product of the first 100 positive integers ends with
   (a) 21 zeros    (b) 22 zeros     (c) 23 zeros     (d) 24 zeros.

   Ans:-  \( 5^1 \rightarrow 20 \)  (d) 24 zeros.

   \( 5^2 \rightarrow 4 \)  

   Alternatively, put \( p=5,n=100 \), thus from above theorem we have \( \left\lfloor \frac{100}{5} \right\rfloor + \left\lfloor \frac{100}{25} \right\rfloor = 24 \) zeros as the answer.

3. Let \( P(x) \) be a polynomial of degree 11 such that \( P(x) = \frac{1}{x+1} \) for \( x = 0 \) (1)11.

   Then \( P(12) = ? \)

   (a) 0    (b) 1    (c) \( \frac{1}{13} \)    (d) none of these
Ans: (a) \( P(x) = \frac{1}{x+1} \)

\[ (x+1)[P(x)]-1 = c (x-0)(x-1)\ldots(x-11) \]

Putting \( x = -1, 0-1 = c (-1)(-2)\ldots(-12) \)

\[ \Rightarrow c = -\frac{1}{12!} \]

\[ \therefore [P(x)](x+1)-1 = -\frac{1}{12!}(x-0)(x-1)\ldots(x-11) \]

\[ \Rightarrow P(12) 13-1 = -\frac{1}{12!} 12.11.\ldots2.1 \]

\[ \Rightarrow P(12) 13-1 = -1 \]

\[ \Rightarrow P(12) = 0. \]

4. Let \( s = \{(x_1, x_2, x_3) | 0 \leq x_i \leq 9 \ and \ x_1 + x_2 + x_3 \ is \ divisible \ by \ 3 \}. \)

Then the number of elements in \( s \) is

(a) 334 (b) 333 (c) 327 (d) 336

Ans: (a) with each \( (x_1, x_2, x_3) \) identify a three digit code, where reading zeros are allowed. We have a bijection between \( s \) and the set of all non-negative integers less than or equal to 999 divisible by 3. The no. of numbers between 1 and 999, inclusive, divisible by 3 is \( \left(\frac{999}{3}\right) = 333 \)

Also, ‘0’ is divisible by 3. Hence, the number of elements in \( s \) is \( = 333 + 1 = 334. \)

5. Let \( x \) and \( y \) be positive real number with \( x < y. \) Also \( 0 < b < a < 1. \)

Define \( E = \log_{a} \left( \frac{y}{x} \right) + \log_{b} \left( \frac{x}{y} \right) . \) Then \( E \) can’t take the value

(a) -2 (b) -1 (c) \( -\sqrt{2} \) (d) 2

Ans: (d) \( E = \log_{a} \left( \frac{y}{x} \right) + \log_{b} \left( \frac{x}{y} \right) = \frac{\log_{a} y}{\log_{a} x} - \frac{\log_{b} y}{\log_{b} x} \)

\[ = \log_{a} \left( \frac{y}{x} \right) \left( \frac{1}{\log_{a} a} - \frac{1}{\log_{b} b} \right) = \log_{a} \left( \frac{y}{x} \right) \left( \frac{\log_{b} b - \log_{a} a}{(\log_{a} a)(\log_{b} b)} \right) \]

\[ = \log_{a} \left( \frac{y}{x} \right) \frac{\log_{b} \left( \frac{a}{b} \right)}{(\log_{a} a)(\log_{b} b)} \]

\[ = - \log_{a} \left( \frac{y}{x} \right) \frac{\log_{b} \left( \frac{a}{b} \right)}{(\log_{a} a)(\log_{b} b)} \]

Log \( 0 < a < 1, 0 < b < 1 \) \( \therefore \log_{a} \) and \( \log_{b} \) are both negative.
Also \( \frac{y}{x} > 1 \) and \( \frac{a}{b} > 1 \). Thus \( \log \left( \frac{y}{x} \right) \) and \( \log \left( \frac{a}{b} \right) \) are both positive. Finally E turns out to be a negative value. So, E can’t take the value ‘2’.

6. Let \( S \) be the set of all 3- digits numbers. Such that
   (i) The digits in each number are all from the set \{1, 2, 3, ..., 9\}
   (ii) Exactly one digit in each number is even

The sum of all number in \( S \) is

(a) 96100     (b) 133200     (c) 66600     (d) 99800

Ans:- (b) The sum of the digits in unit place of all the numbers in \( s \) will be same as the sum in tens or hundreds place. The only even digit can have any of the three positions,

i.e. \( ^3C_1 \) ways.

And the digit itself has 4 choices (2, 4, 6 or 8). The other two digits can be filled in \( 5 \times 4 = 20 \) ways.

Then the number of numbers in \( S = 240 \).

Number of numbers containing the even digits in units place = \( 4 \times 5 \times 4 = 80 \)

The other 160 numbers have digits 1, 3, 5, 7 or 9 in unit place, with each digit appearing

\[
\frac{160}{5} = 32 \text{ times. Sum in units place} = 32 \times (1+3+5+7+9) + 20 (2+4+6+8)
\]

\[
= 32 \times 25 + 20 \times 20 = 1200
\]

\[\therefore \text{The sum of all numbers= } 1200 (1+10+10^2) = 1200 \times 111 = 133200.\]

7. Let \( y = \frac{x}{x^2+1} \), Then \( y^4 (1) \) is equals
   (a) 4     (b) -3     (c) 3     (d) -4

Ans:- (b) Simply differentiating would be tedious,

So we take advantage of ‘\( i \) the square root of ’-1’

\[
y = \frac{x}{x^2+1} = \frac{1}{2} \left( \frac{1}{x-i} + \frac{1}{x+i} \right)
\]

\[
\frac{d^4 y}{dx^4} = \frac{1}{2} \left\{ \frac{4!}{(x-i)^5} + \frac{4!}{(x+i)^5} \right\}
\]

Note that, \( \frac{d^n}{dx^n} \left\{ \frac{1}{x+a} \right\} = \frac{(-1)^n n!}{(x+a)^{n+1}} \)
So, \( y^4(x) = \frac{4!}{2} \left\{ \frac{11}{(x-i)^5} + \frac{11}{(x+i)^5} \right\} \) Then

\[
y^4(1) = 12 \left\{ \frac{11}{(1-i)^5} + \frac{11}{(1+i)^5} \right\} = 12 \left\{ \frac{1-i}{(-2i)^3} + \frac{1-i}{(2i)^3} \right\} = 12 \left\{ \frac{1-i}{8i} + \frac{1-i}{8i} \right\} = 12 \left\{ \frac{1}{8} - \frac{1}{8} \right\} = -3.
\]

8. A real \( 2 \times 2 \) matrix. \( M \) such that \( M^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 - \varepsilon \end{pmatrix} \)

(a) exists for all \( \varepsilon > 0 \)  
(b) does not exist for any \( \varepsilon > 0 \)

(c) exists for same \( \varepsilon > 0 \)  
(d) none of the above

Ans: (b) since \( M^2 \) is an diagonal matrix, so \( M = \begin{bmatrix} i & 0 \\ 0 & \sqrt{1 - \varepsilon} \end{bmatrix} \).

So, \( M \) is not a real matrix, for any values of \( \varepsilon \)

\( M \) is a non – real matrix.

9. The value of \( \left( \frac{1+i\sqrt{3}}{2} \right)^{2008} \) is

(a) \( \frac{1+i\sqrt{3}}{2} \)  
(b) \( \frac{1-i\sqrt{3}}{2} \)  
(c) \( \frac{-1+i\sqrt{3}}{2} \)  
(d) \( \frac{-1+i\sqrt{3}}{2} \)

Ans: (c) \( A = \left( \frac{1+i\sqrt{3}}{2} \right), A^2 = \frac{-1+i\sqrt{3}}{2}, A^4 = \frac{-1-i\sqrt{3}}{2} = -A \)

\( \therefore A^{2008} = (A^4)^{502} = A^4 = \frac{-1-i\sqrt{3}}{2}. \)

10. Let \( f(x) \) be the function \( f(x) = \begin{cases} \frac{x^p}{(\sin x)^q} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases} \)

Then \( f(x) \) is continuous at \( x = 0 \) if

(a) \( p > q \)  
(b) \( p > 0 \)  
(c) \( q > 0 \)  
(d) \( p < q \)

Ans: (b) \( |f(x) - f(0)| = \left| \frac{x^p}{(\sin x)^q} - 0 \right| \leq |x^p| < \varepsilon \)

Whenever \( |x-0| < \frac{1}{\varepsilon} = \delta \) if \( p > 0 \).

So, \( f(x) \) is continuous for \( p > 0 \) at \( x = 0 \).

11. The limit \( \lim_{x \to 0} \log(x) \left( 1 - \frac{1}{n^2} \right)^n = \)

(a) \( e^{-1} \)  
(b) \( e^{-\frac{1}{2}} \)  
(c) \( e^{-2} \)  
(d) 1
Ans:- (d) \( L = (1 - \frac{1}{n^2})^n \)

\[ \Rightarrow \log L = n \log (1 - \frac{1}{n^2}) \]

\[ \Rightarrow \lim_{x \to \infty} \log L = \lim_{x \to \infty} \left[ -n \left( \frac{1}{n^2} + \frac{1}{2n^4} + \cdots \infty \right) \right] = 0 \]

\[ \therefore L = e^0 = 1. \]

12. The minimum value of the function \( f(x, y) = 4x^2 + 9y^2 - 12x - 12y + 14 \) is

(a) 1  (b) 3  (c) 14  (d) none

Ans:- (a) \( f(x, y) = 4x^2 + 9y^2 - 12x - 12y + 14 \)

\[ = (4x^2 - 12x + 9) + (9y^2 - 12y + 4) + 1 \]

\[ = (2x - 3)^2 + (3y - 2)^2 + 1 \geq 1 \]

So, minimum value of \( f(x, y) \) is 1.

13. From a group of 20 persons, belonging to an association, A president, a secretary and there members are to be elected for the executive committee. The number of ways this can be done is

(a) 30000  (b) 310080  (c) 300080  (d) none

Ans:- (b) \( \binom{20}{1} \times \binom{19}{1} \times \binom{18}{3} \) or \( \frac{20!}{1!1!1!3!15!} = 310080 \)

14. The \( \lim_{x \to 0} \frac{\cos x - \sec x}{x^2(1+x)} \) is

(a) -1  (b) 1  (c) 0  (d) does not exist

Ans:- (a) \( \lim_{x \to 0} \frac{\cos x - \sec x}{x^2(1+x)} = \lim_{x \to 0} \frac{-\sin^2 x}{\cos x (x^2)(x+1)} \)

\[ = - \lim_{x \to 0} \frac{1}{\cos x} \left( \frac{\sin x}{x} \right)^2 \cdot \frac{1}{(x+1)} = -1.1.1 = -1. \]

15. Let \( R = \frac{49^{52} - 46^{52}}{96^{26} + 92^{26}} \). Then \( R \) satisfies

(a) \( R < 1 \)  (b) \( 23^{26} < R < 24^{26} \)  (c) \( 1 < R < 23^{26} \)  (d) \( R > 24^{26} \)

Ans:- (b) \( R = \frac{(2.24)^{52} - (2.23)^{52}}{(4.24)^{26} + (4.23)^{26}} = \frac{2^{52}(24^{52} - 23^{52})}{4^{26}(24^{26} + 23^{26})} = \frac{2^{52}}{2^{252}} \cdot \frac{(24^{26} + 23^{26})(24^{26} - 23^{26})}{24^{26} + 23^{26}} \)

\[ = 24^{26} - 23^{26} < 24^{26} \]

Also, \( R = 24^{26} - 23^{26} = (1 + 23)^{26} - 23^{26} \)

\[ = 23^{26} + 26c_1 \cdot 23^{25} + 26c_2 \cdot 23^{24} + \ldots + 1 - 23^{26} \]
= 26. 23^{25} + 26c_2 \cdot 23^{24} + \ldots + 1 > 26. 23^{25} > 23. 23^{25} = 23^{26}

\therefore 23^{26} < R < 24^{26}

16. A function f is said to be odd if f (-x) = -f (x) \ \forall x. Which of the following is not odd?

(a) f (x + y) = f(x) + f(y) \ \forall x, y

(b) f (x) = \frac{xe^{x/2}}{1 + e^x}

(c) f (x) = x - [x]

(d) f (x) = x^2 \sin x + x^3 \cos x

Ans:- (c) f (x + y) = f(x) + f(y) \ \forall x, y

Let x = y = 0

\Rightarrow f (0) = f (0) + f (0)

\therefore f (0) = 0

Replacing y with -x, we have

f (x - x) = f(x) + f (-x)

\Rightarrow f (0) = f(x) + f (-x)

\Rightarrow f(x) + f (-x) = 0

\Rightarrow f (-x) = -f(x)

Thus f is odd.

Again for f (x) = \frac{xe^{x/2}}{1 + e^x}

f(-x) = \frac{(-x)(e^{-x/2})}{1 + e^{-x}} = \frac{(-x)(e^{-x/2})e^x}{1 + e^x} = \frac{xe^{x/2}}{1 + e^x} = -f (x)

\therefore f is odd.

f (x) = x - [x] is not odd.

Counter example:-

f (-2.3) = -2.3 – [-2.3] = -2.3 – (-3) = 3 - 2.3 = 0.7

f (2.3) = 2.3 – [2.3] = 2.3 -2 = 0.3

\therefore f(2.3) \neq f(-2.3)
Thus f is not odd

\[ f(x) = x^2 \sin x + x^3 \cos x \]

\[ f(-x) = -x^2 \sin x - x^3 \cos x = -f(x) \]

\[ \therefore f \text{ is odd here.} \]

17. Consider the polynomial \( x^5 + ax^4 + bx^3 + cx^2 + dx + 4 \). If (1+2i) and (3-2i) are two roots of this polynomial then the value of a is

(a) \(-524/65\)  \hspace{1cm} (b) \(524/65\)  \hspace{1cm} (c) \(-1/65\)  \hspace{1cm} (d) \(1/65\)

Ans:- (a) The polynomial has 5 roots. Since complex root occur in pairs, so there is one real root taking it as m.

So, m, 1+2i, 1-2i, 3+2i, 3-2i are the five roots.

Sum of the roots= \(-a\) = 8 + m.

Product of the roots= \((1+4)(9+4)m = 65 \cdot m = \frac{4}{65}\)

\( \therefore m = \frac{4}{65} \).

\( \therefore a = -8 - \frac{4}{65} = -\frac{524}{65}. \)

18. In a special version of chess, a rook moves either horizontally or vertically on the chess board. The number of ways to place 8 rooks of different colors on a 8×8 chess board such that no rook lies on the path of the other rook at the start of the game is

(a) \(8 \times 8\)  \hspace{1cm} (b) \(8 \times \binom{8}{8}\)  \hspace{1cm} (c) \(2^8 \times 8\)  \hspace{1cm} (d) \(2^8 \times \binom{64}{8}\)

Ans:- The first rook can be placed in any row in 8 ways & in any column in 8 ways. So, it has 8² ways to be disposed off. Since no other rook can be placed in the path of the first rook, a second rook can be placed in 7² ways for there now remains only 7 rows and 7 columns.

Counting in this manner, the number of ways = 8². 7². 6² ... 1² = (8!)²

19. The differential equation of all the ellipses centered at the origin is

(a) \(y^2 + x(y')^2 - yy' = 0\)  \hspace{1cm} (b) \(x y + x(y')^2 - yy' = 0\)

(c) \(y y'' + x(y')^2 - xy' = 0\)  \hspace{1cm} (d) none

Ans:- (d) \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\), after differentiating w.r.t x, we get

\(\Rightarrow \frac{2x}{a^2} + \frac{2yy'}{b^2} = 0 \Rightarrow \frac{yy'}{b^2} = -\frac{x}{a^2}.\)
\[ (y')^2 + \frac{y''}{b^2} = -\frac{1}{a^2} \]

\[ (y')^2 + \frac{y''}{a^2} = -\frac{b^2}{a^2}. \]

20. If \( f(x) = x + \sin x \), then find \( \frac{2}{\pi^2} \cdot \int_{\pi}^{2\pi} (f^{-1}(x) + \sin x)\,dx \)

(a) 2  (b) 3  (c) 6  (d) 9

Ans:- (b) Let \( x = f(t) \Rightarrow \frac{dx}{dt} = f'(t) \)

\[ \int_{\pi}^{2\pi} f^{-1}(x)\,dx = \int_{\pi}^{2\pi} t \cdot f'(t)\,dt = (t [f(t)])_{\pi}^{2\pi} - \int_{\pi}^{2\pi} f(t)\,dt = \left(4\pi^2 - \pi^2\right) - \int_{\pi}^{2\pi} f(t)\,dt \]

\[ I = \int_{\pi}^{2\pi} (f^{-1}(x) + \sin x)\,dx = \int_{\pi}^{2\pi} f^{-1}(x)\,dx + \int_{\pi}^{2\pi} \sin x\,dx \]

\[ I = 3\pi^2 - \int_{\pi}^{2\pi} f(t)\,dt + \int_{\pi}^{2\pi} \sin x\,dx \]

\[ I = 3\pi^2 - \int_{\pi}^{2\pi} (f(x) - \sin x)\,dx \]

\[ I = 3\pi^2 - \int_{\pi}^{2\pi} x\,dx = 3\pi^2 - \frac{1}{2} (4\pi^2 - \pi^2) \]

\[ \frac{3}{2} \pi^2 \]

\[ \Rightarrow \frac{2}{\pi^2} I = 3. \]

21. Let \( P = (a, b), Q = (c, d) \) and \( 0 < a < b < c < d \), \( L \equiv (a, 0), M \equiv (c, 0) \), \( R \) lies on \( x \)-axis such that \( PR + RQ \) is minimum, then \( R \) divides \( LM \)

(a) Internally in the ratio \( a: b \)  
(b) Internally in the ratio \( b: c \)

(c) Internally in the ratio \( b: d \)  
(d) Internally in the ratio \( d: b \)

Ans:- (c) Let \( R = (\alpha, 0) \). \( PR + RQ \) is least

\[ \Rightarrow PQR \text{ should be the path of light} \]

\[ \Rightarrow \Delta PRL \text{ and } QRM \text{ are similar} \]

\[ \frac{LR}{RM} = \frac{PL}{QM} \implies \frac{a - \alpha}{c - \alpha} = \frac{b}{d} \]

\[ \Rightarrow \alpha d - \alpha d = bc - \alpha b \]

\[ \Rightarrow \alpha = \frac{ad + bc}{b + d} \]
⇒ R divides LM internally in the ratio b : d (as $\frac{b}{d} > 0$)

22. A point (1, 1) undergoes reflection in the x-axis and then the co-ordinate axes are rotated through an angle of $\frac{\pi}{4}$ in anticlockwise direction. The final position of the point in the new co-ordinate system is-

(a) $(0, \sqrt{2})$  
(b) $(0, -\sqrt{2})$  
(c) $-\sqrt{2}, 0$  
(d) none of these

Ans:- (b) Image of (1, 1) in the x-axis is (1, -1). If (x, y) be the co-ordinates of any point and (x’, y’) be its new co-ordinates, then $x' = x \cos \theta + y \sin \theta$, $y' = y \cos \theta - x \sin \theta$, where $\theta$ is the angle through which the axes have been rotated.

Here $\theta=\frac{\pi}{4}$, x= 1, y= -1

∴ x’ = 0, y’ = $-\sqrt{2}$

23. If $a, x_1, x_2, \ldots, x_k$ and $b, y_1, y_2, \ldots, y_k$ from two A.P. with common difference m and n respectively, then the locus of point $(x, y)$ where $x=\frac{\sum_{i=1}^{k}x_i}{k}$ is and $y=\frac{\sum_{i=1}^{k}y_i}{k}$ is

(a) $(x-a)m= (y-b)n$  
(b) $(x-m) a= (y-n) b$  
(c)$(x-n)a = (y-m)b$  
(d) $(x-a) n-(y-b) m$

Ans:- (d)

$X= \frac{k(x_1+x_k)}{2} = \frac{x_1+x_k}{2} = \frac{a+m+a mk}{2}$

or, $x = a + \frac{(k+1)m}{2}$

or, $2(x-a)= (k+1)m$ ……………..(1)

Similarly,

$2(y-b)= (k+ 1)n$ ……………..(2)

We have to eliminate k

From (1) and (2)

$\frac{x-a}{y-b} = \frac{m}{n}$
or, (x- a)n = (y -b)m

24. The remainder on dividing \(1234^{567} + 89^{1011}\) by 12 is

(a) 1  (b) 7  (c) 9  (d) none

Ans:- (c) \(1234 \equiv 1 \pmod{3} \Rightarrow 1234^{567} \equiv 1 \pmod{3}\) \(\text{and } 89 \equiv -1 \pmod{3}\)

\(\Rightarrow 89^{1011} \equiv -1 \pmod{3}\)

\(\therefore 1234^{567} + 89^{1011} \equiv 0 \pmod{3}\)

Here 1234 is even, so \(1234^{567} \equiv 0 \pmod{4}\) \(\text{and } 89 \equiv 1 \pmod{4}\)

\(\Rightarrow 89^{1011} \equiv 1 \pmod{4}\)

Thus \(1234^{567} + 89^{1011} \equiv 1 \pmod{4}\)

Hence it is 9 (mod 12)

25. The sum of the series \(1 + \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \cdots\) is

(a) \(e^2\)  (b) 3  (c) \(\sqrt{5}\)  (d) \(\sqrt{8}\)

Ans. (d) \(\sqrt{8} = 2^2 = \left(\frac{1}{2}\right)^{-2} = 1 - \left(\frac{1}{2}\right)^{3}\)

\[= 1 + \left(\frac{1}{2}\right)\left(\frac{3}{2}\right) + \left(\frac{3}{2} - 1\right)\left(-\frac{1}{2}\right)^2 + \cdots\]

\[= 1 + \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \cdots\]

26. If \(f(x) = \cos x + \cos ax\) is a periodic function, then a is necessarily

(a) an integer  (b) a rational number  (c) an irrational number  (d) an event number

Ans. (b) Period of \(\cos x = 2\pi\) and period of \(\cos ax = \frac{2\pi}{|a|}\)

Period of \(f(x) = \text{L.C.M. of } \frac{2\pi}{1} \text{ and } \frac{2\pi}{|a|} = \frac{\text{L.C.M. of } 2\pi \text{ and } 2\pi}{\text{H.C.F. of } 1 \text{ and } |a|}\)

Since \(k = \text{H.C.F. of } 1 \text{ and } |a|\)

\(\therefore \frac{1}{k} = \text{an integer}= m \text{ (say) and } \frac{|a|}{k} = \text{an integer} = n \text{ (say)}\)

\(\therefore |a| = \frac{n}{m} \Rightarrow a = \pm \frac{n}{m} = \text{a rational number.}\)
27. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(x) = x^3 + x^2 + 100x + 5 \sin x \), then \( f \) is

(a) many-one onto  
(b) many-one into  
(c) one-one onto  
(d) one-one into

Ans. (c)

\[
f(x) = x^3 + x^2 + 100x + 5 \sin x
\]

\[
\Rightarrow f'(x) = 3x^2 + 2x + 100 + 5 \cos x
\]

\[
= 3x^2 + 2x + 94 + (6 + 5 \cos x) > 0
\]

\[
\Rightarrow f \text{ is an increasing function and consequently a one-one function.}
\]

Clearly \( f(-\infty) = -\infty \), \( f(\infty) = \infty \) and \( f(x) \) is continuous, therefore range \( f = \mathbb{R} = \text{co domain } f \). Hence \( f \) is onto.

28. Let \( f(x) = \frac{\sin 101x}{[\frac{x}{\pi}]+\frac{1}{2}} \), where \([x]\) denotes the integral part of \( x \) is

(a) an odd function  
(b) an even function  
(c) neither odd nor even function  
(d) both odd and even function

Ans. (a) when \( x = n\pi, n \in \mathbb{I} \), \( \sin x = 0 \) and \( \left[\frac{x}{\pi}\right] + \frac{1}{2} \neq 0 \)

\[
\Rightarrow f(x) = 0
\]

\[
\Rightarrow \text{ when } x = n\pi, f(x) = 0 \text{ and } f(-x) = 0
\]

\[
\Rightarrow f(-x) = f(x)
\]

When \( x \neq n\pi, n \in \mathbb{I} \), \( \frac{x}{\pi} \neq \text{ an integer} \)

\[
\Rightarrow \left[\frac{x}{\pi}\right] + \frac{1}{2} = -\left[\frac{x}{\pi}\right] - 1 = -\left(\left[\frac{x}{\pi}\right] + \frac{1}{2}\right)
\]

Now \( f(-x) = \frac{\sin 101(-x)}{[\frac{-x}{\pi}]+\frac{1}{2}} = -\frac{\sin x}{[\frac{x}{\pi}]+\frac{1}{2}} = \frac{\sin x}{[\frac{x}{\pi}]+\frac{1}{2}} = f(x) \)

Hence in all cases \( f(-x) = f(x) \)
29. If k be the value of x at which the function 

\[ f(x) = \int_{-1}^{x} t(e^t - 1)(t - 1)(t - 2)^3(t - 3)^5 \, dt \]

has maximum value and \( \sin x + \csc x = k \), then for \( n \in \mathbb{N} \), \( \sin^n x + \csc^n x = … \)

(a) 2  (b) -2  (c) \( \frac{\pi}{2} \)  (d) \( \pi \)

Ans. (a) \( f'(x) = x(e^x - 1)(x - 1)(x - 2)^3(x - 3)^5 \)

By Sign Rule we get

\( f(x) \) has max. at \( x = 2 \)

\( \therefore k = 2 \)

Now \( \sin x + \csc x = k \Rightarrow \sin x + \csc x = 2 \)

\( \Rightarrow (\sin x - 1)^2 = 0 \Rightarrow \sin x = 1 \)

\( \therefore \csc x = 1 \)

Hence \( \sin^n x + \csc^n x = 2 \)

30. If \( f(x + y) = f(x) + f(y) - xy - 1 \) for all \( x, y \in \mathbb{R} \) and \( f(1) = 1 \), then the number of solutions of \( f(n) = n \), \( n \in \mathbb{N} \) is

(a) 0  (b) 1  (c) 2  (d) more than 2

Ans. (b)

Given \( f(x + y) = f(x) + f(y) - xy - 1 \) \( \forall x, y \in \mathbb{R} \) .................(1)

\( f(1) = 1 \) ...................(2)

\( f(2) = f(1+1) = f(1) + f(-1) - 1 - 1 = 0 \)

\( f(3) = f(2+1) = f(2) + f(1) - 2.1 - 1 = -2 \)

\( f(n+1) = f(n) + f(1) - n - 1 = f(n) - n < f(n) \)

Thus \( f(1) > f(2) > f(3) > … \) and \( f(1) = 1 \)

\( \therefore f(1) = 1 \) and \( f(n) < 1 \), for \( n > 1 \)

Hence \( f(n) = n, n \in \mathbb{N} \) has only one solution \( n = 1 \)