ISI B.STAT & B.MATH SUBJECTIVE QUESTIONS & SOLUTIONS

SET – 1

Q1. How many natural numbers less than 10^8 are there, whose sum of digits equals 7?

Solution:-

We need to count the no. of solutions of $x_1 + x_2 + \cdots + x_8 = 7$

Which satisfies $0 \leq x_i \leq 7$, i= 1, 2, 3, …, 8 ……………….(1)

The number of solution of (1) is= coefficient of x^7 in $(1 + x + x^2 + \cdots + x^7)^8$

$$= coefficient\ of\ x^7\ in\ (1 - x^8)^8\ (1 - x)^8$$

= coefficient of x^7 in $(1 - 8x^8)(1 + 8c_1 x + 4c_2 x^2 + \ 10c_3 k + \cdots)$

$$= 14c_7$$

$$= 3432.\ \ (Ans)$$

Q2. Find the number of positive integers less than or equal to 6300 which are not divisible by 3, 5 and 7.

Solution:- S= {1, 2, 3, …, 6300}

Let A: Set of integers divisible by 3

B: Set of integers divisible by 5

C: Set of integers divisible by 7

We are to find:- n(s) – n(A∪ B∪ C) = n(S) – [n(A)+n(B) +n(C) – n(A ∩ B)- n(B ∩ C)- n(A ∩ C)+ n(A∩ B∩ C)]

$$= 6300 - \left\{\left[\frac{6300}{3}\right] + \left[\frac{6300}{5}\right] + \left[\frac{6300}{7}\right] - \left[\frac{6300}{3\times5}\right] - \left[\frac{6300}{5\times7}\right] - \left[\frac{6300}{3\times7}\right] + \left[\frac{6300}{3\times5\times7}\right]\right\}$$

$$i.e., n(A \cup B \cup C)^c = 2880.$$

Q3. If c is a real number with 0 < c < 1, then show that the values taken by the function

$y = \frac{x^2+2x+c}{x^2+4x+3c}$, as x varies over real numbers, range over all real numbers.

Solution:-

$$y = \frac{x^2+2x+c}{x^2+4x+3c};$$

$$\Longrightarrow x^2y + 4xy + 3cy = x^2 + 2x + c$$

$$\Longrightarrow (y-1)x^2 + 2x(2y-1) + c(3y-1) = 0 \quad [\because x \, is \, real]$$

$$\therefore \{2(2y-1)\}^2 - 4(y-1).c(3y-1) \geq 0$$

$$\Longrightarrow c \leq \frac{(2y-1)^2}{(y-1)(3y-1)} \quad \because 0 < c < 1,$$

$$So, \frac{1}{3} < y < 1.$$

Q4. Let X = {0, 1, 2, 3, …, 99}. For a, b in X, we define a * b to be the remainder obtained by dividing the product ab by 100. For example, 9*18 = 62 and 7*5 = 35. Let x be an element in X. An element y in X is called the inverse and write down their inverses.

Solution:- x*y = 1, ⟹ xy = 100k +1 for x = {0, 1, 2, …, 99}

(1) For x = 1, y = 1, x * y = 100k + where x = c

∴ Inverse of 1 is 1.

(2) There is no integral multiple of 2, 4, 5, 6 having 1 at unit place, ⟹ 2, 4, 5, 6 have no inverse.
(3) 3 and 7 can have inverses
(i) For x= 3 , 3 y = 1 i.e. 3y = 100k +1

The least k satisfying is 2, i.e. 3y = 201, y = 67 and the next k satisfying is 5, i.e. 3y = 167 but 167 ∉ X.

∴ 3 has only inverse = 67.

(ii) For x = 7, y = 1, i.e. 7y = 100k +1

The least k satisfying is Ʒ, i.e. 7y = 301, y 43

The next k satisfying is ID, i.e. 7y = 1001, y = 143 but 143 ∉ X.

∴ 7 has only inverse = 43.

Q5. Evaluate $\lim_{n\to\infty}\left\{\frac{1}{n+1}+\frac{1}{n+2}+\ ...+\frac{1}{n+n}\right\}$.

Solution:-

$$\underset{n\to\infty}{\text{Lt}}\ \frac{1}{n}\left[\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\cdots+\frac{1}{1+\frac{n}{n}}\right]$$

$$=\underset{n\to\infty}{\text{lt}}\ \frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\frac{r}{n}}=\int_0^1\frac{dx}{1+x}=[\log_e(1+x)]_0^1=\ \log_e 2.$$

Q6. Tangents are drawn to a given circle from a point on a given straight line, which does not meet the given circle. Prove that the locus of the mid-point of the chord joining the two points of contact of the tangents with circle is a circle.

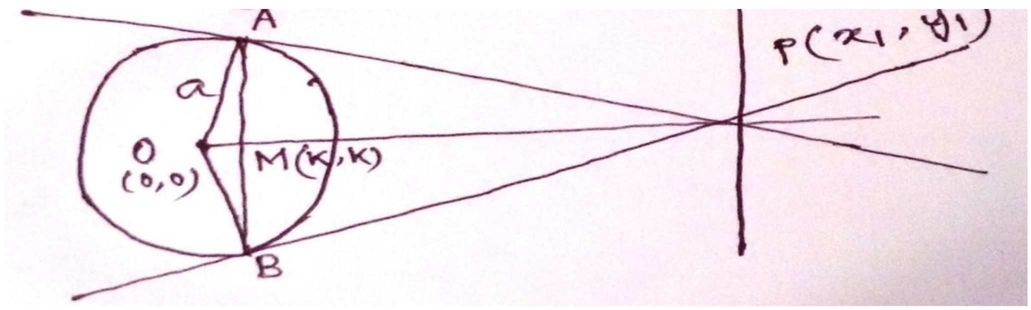
Solution:- Slope of OM = k/h [taking centre (0, 0)]

∴Slope of AB = - h/k . [∵ AB ⊥ OM]

∴ Equation of AB, whose slope is –h/k and which passes through the point (h, k) is y –k = $\frac{h}{k}(x-h)$

or, $hx+ky=\ h^2+k^2$ ………… (1)

And equation of AP, the tangent is $xx_1+yy_1=\ a^2$ ………… (2)



∴ From (1) and (2) , we have,

$$\frac{x_1}{h}=\frac{y_1}{k}=\frac{a^2}{h^2+k^2}$$

∴ $x_1=\frac{ha^2}{h^2+k^2}, y_1=\ \frac{ka^2}{h^2+k^2}$.

∴ Put these values of x_1 *and* y_1 *in* $lx_1+my_1+n=0$

We get, $l.\frac{ha^2}{h^2+k^2}+m.\frac{ka^2}{h^2+k^2}+n=0$

$$\Rightarrow lha^2+mka^2+n(h^2+k^2)=0$$

$\Longrightarrow h^2+k^2+\frac{la^2}{n}h+\frac{ma^2}{n}k=0$, i.e. the required focus of M.

So, the equation of the circle is $x^2+y^2+\frac{la^2}{n}x+\frac{ma^2}{n}y=0$.

Q7. Draw the graph (on plain paper) of f(x) = min {|x| - 1, |x -1|- 1, |x- 2| - 1}.

Solution:- y = |x| -1

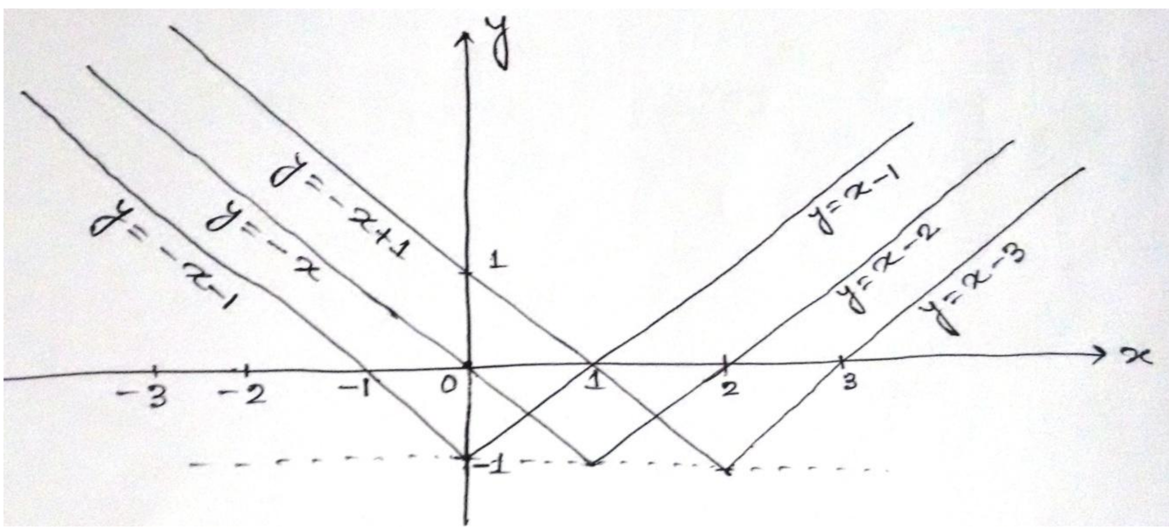
$$=\begin{cases} x-1, when\ x\ \geq 0 \\ -x-1, when\ x<0 \end{cases}$$

Z = |x -1| -1

$$=\begin{cases} x-1-1, when\ x\ \geq 1 \\ -x+1-1, when\ x<1 \end{cases}$$

W = |x- 2| -1

$$=\begin{cases} x-2-1, when\ x\ \geq 2 \\ -x+2-1, when\ x<2 \end{cases}$$



Q8. Let $\{C_n\}$ be an infinite sequence of circles lying in the positive quadrant of the XY –plane, with strictly decreasing radii and satisfying the following conditions. Each C_n touches both X-axis and the Y-axis. Further, for all n ≥ 1, the circle C_{n+1} touches the circle C_n externally. If C_1 has radius 10cm, then show that the sum of the areas of all these circles is $\frac{25\pi}{3\sqrt{2}-4}$sq. cm.

Solution:- $OO_1=R_1\sqrt{2}\ \therefore OP_2=R_1\sqrt{2}-R_1$

$$\therefore OQ=R_1\sqrt{2}+R_1=R_1(\sqrt{2}+1)$$

$$\therefore\ R_1 = \frac{OQ}{\sqrt{2}+1}, now, OP = R_2(\sqrt{2}+1), R_2 = \frac{OP}{\sqrt{2}} = R_1\frac{\sqrt{2}-1}{\sqrt{2}+1}$$

$\therefore R_3 = R_2\frac{\sqrt{2}-1}{\sqrt{2}+1} = R_1\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2$

$$\therefore Area = \pi({R_1}^2 + {R_2}^2 + \cdots + \propto)$$

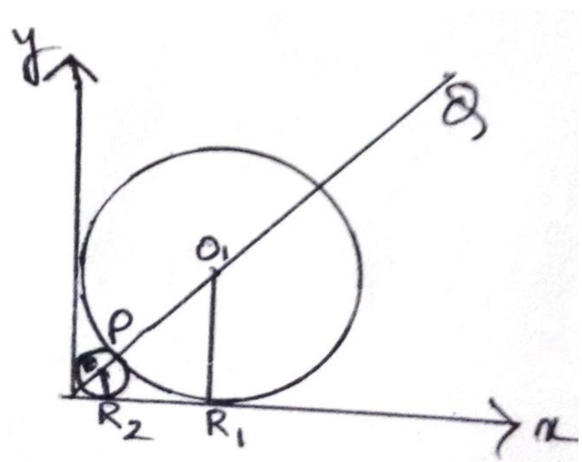
$$= \pi\left\{{R_1}^2 + {R_1}^2\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2 + {R_1}^2\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^4 + \cdots + \propto\right\}$$

$$= \pi{R_1}^2\left\{1 + \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2 + \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^4 + \cdots + \propto\right\}$$

$$= \pi{R_1}^2\left\{\frac{1}{1-\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2}\right\} = \pi{R_1}^2\left(\frac{3+2\sqrt{2}}{4\sqrt{2}}\right) = \pi{R_1}^2\left(\frac{3\sqrt{2}+4}{8}\right)$$

$$= \frac{\pi}{8}{R_1}^2\left(\frac{18-16}{3\sqrt{2}-4}\right) = \frac{\pi}{4}.100.\frac{1}{3\sqrt{2}-4}\ \ sq.cm\ \ [\because R_1 = 10cm.]$$

$$= \frac{25\pi}{3\sqrt{2}-4} sq.cm\ [proved]$$



Q9. Consider the system of equations x + y = 2, ax + y = b. Find conditions on a and b under which

(i) the system has exactly one solution;

(ii) the system has no solution;

(iii) The system has more than one solution.

Solution:-

$\Delta= \begin{vmatrix} 1 & 1 \\ a & 1 \end{vmatrix} = 1 - a;\ \Delta_1 = \begin{vmatrix} 2 & 1 \\ b & a \end{vmatrix} = 2a - b\ ;\ \Delta_2 = \begin{vmatrix} 1 & 2 \\ a & b \end{vmatrix} = b - 2a.$

(i) For exactly one solution, $\Delta \neq 0$ i.e. $1 - a \neq 0 \Longrightarrow a \neq 1$.

(ii) For no solution, $\Delta = 0$, i.e. a = 1, $\Delta_1 \neq 0, \Delta_2 \neq 0$. i.e. $2a \neq b$.

(iii) For more than one solution, $\Delta = \Delta_1 = \Delta_2 = 0$, a = 1, b = 2.

Q10. Let $\{x_n\}$ b e a sequence such that $x_1 = 2,\ x_2 = 1\ and\ 2x_n - 3x_{n-1} + x_{n-2} = 0$

For n > 2. Find an expression for x_n .

Solution:- $x_1 = 2, x_2 = 1, 2x_n - 3x_{n-1} + x_{n-2} = 0$.

Let, $x_n = ka^n, \therefore 2ka^n - 3ka^{n-1} + ka^{n-2} = 0$

or, $2a^2 - 3a + 1 = 0$

or, (2a- 1)(a- 1) = 0

or, $a_1 = \frac{1}{2}, a_2 = 1$.

$\therefore x_1 = k_1 {a_1}^n + k_2 {a_2}^n = k_1 \left(\frac{1}{2}\right)^n + k_2(1)^n$.

Again, $x_1 = 2 = k_1 \left(\frac{1}{2}\right) \neq k_2(1)^1 = \frac{k_1}{2} + k_2$ …………… (1)

And $x_2 = 1 = k_2 \left(\frac{1}{2}\right)^2 + k_2(1)^1 = \frac{k_1}{4} + k_2$ ………… (2)

From (1) And (2), we get $k_1 = 4, k_2 = 0$. $\therefore x_n = 4\left(\frac{1}{2}\right)^n = \frac{1}{2^{n-2}}$.