1

1. $$2(\sqrt{k+1}-\sqrt{k}) = \frac{2(\sqrt{k+1}-\sqrt{k})(\sqrt{k+1}+\sqrt{k})}{(\sqrt{k+1}+\sqrt{k})} = \frac{2}{(\sqrt{k+1}+\sqrt{k})} < \frac{2}{2\sqrt{k}} = \frac{1}{\sqrt{k}}$$

$$2(\sqrt{k}-\sqrt{k-1}) = \frac{2(\sqrt{k}-\sqrt{k-1})(\sqrt{k}+\sqrt{k-1})}{(\sqrt{k}+\sqrt{k-1})} = \frac{2}{(\sqrt{k}+\sqrt{k-1})} > \frac{2}{2\sqrt{k}} = \frac{1}{\sqrt{k}}$$

$$2(\sqrt{k+1}-\sqrt{k}) < \frac{1}{\sqrt{k}} < 2(\sqrt{k}-\sqrt{k-1})$$

Now putting $k = 2, 3, \ldots n$ we get

$$2\sqrt{3}-2\sqrt{2} < \frac{1}{\sqrt{2}} < 2\sqrt{2}-2\sqrt{1}$$

$$2\sqrt{4}-2\sqrt{3} < \frac{1}{\sqrt{3}} < 2\sqrt{3}-2\sqrt{2}$$

$$\cdots\cdots$$

$$2\sqrt{(n+1)}-2\sqrt{n} < \frac{1}{\sqrt{n}} < 2\sqrt{n}-2\sqrt{n-1}$$

$$2\sqrt{(n+1)}-2\sqrt{2} < \sum \frac{1}{\sqrt{r}} < 2\sqrt{n}-2\sqrt{1}$$

but we have, $(2\sqrt{n}-2)-(2\sqrt{n+1}-2\sqrt{2}+1)$

$$= 2(\sqrt{n}-\sqrt{n+1})-3+2\sqrt{2}$$

$$= 2(\sqrt{n}-\sqrt{n+1})+(2\sqrt{2}-\sqrt{3}) < 0$$

as $\sqrt{n} < \sqrt{n+1}$ and $2\sqrt{2} < \sqrt{3}$

hence, $2\sqrt{n}-2 < 2\sqrt{n+1}-2\sqrt{2}+1$

$$\Rightarrow 2\sqrt{n}-3 < 2\sqrt{n+1}-2\sqrt{2}$$

hence, $2\sqrt{n}-3 < \sum \frac{1}{\sqrt{r}} < 2\sqrt{n}-2$

from definition of GIF $\left[\sum \frac{1}{\sqrt{r}}\right] = 2\sqrt{n}-3$

$$\therefore \left[\sum_{r=2}^{10000} \frac{1}{\sqrt{r}}\right] = 2\sqrt{10000}-3 = 200-3 = \cancel{199}197$$

2

2. $$P_n = \left(1+\frac{1}{a_1}\right)\left(1+\frac{1}{a_2}\right)\cdots\cdots\left(1+\frac{1}{a_n}\right)$$

$$= \frac{(a_1+1)(a_2+1)\cdots\cdots(a_n+1)}{a_1 \cdot a_2 \cdot a_3 \cdots\cdots a_n}$$

$$= \frac{a_2 a_3 a_4 \cdots\cdots a_{n+1}}{2.3.4 \cdots (n+1)\, a_1 a_2 a_3 \cdots a_n} = \frac{a_{n+1}}{(n+1)!} = \frac{1+a_n}{n!}$$

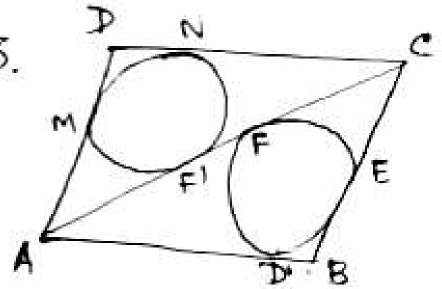
$$= \left(\frac{1}{n!} + \frac{a_n}{n!}\right)$$

$$= \left(\frac{1}{n!} + \frac{1}{(n-1)!} + \frac{a_{n-1}}{(n-1)!}\right)$$

$$= \left(\frac{1}{n!} + \frac{1}{(n-1)!} + \frac{1}{(n-2)!} + \cdots + \frac{1}{1!} + \frac{a_1}{1!}\right)$$

$$\lim_{n\to\alpha} P_n = \lim_{n\to\alpha}\left(1 + 1/1! + 1/2! + \cdots + 1/n!\right) = e$$

3.



$AD' = AF \quad BE = BD \quad CE = CF$

so, $AF - FC = AD' - CE = AB - BC$

similarly,

$AF' - F'C = \cancel{AD-BCCE}\ AM - CN$

$= \cancel{AB-BC}\ AD - DC$

but from $AB + CD = AD + BC \Rightarrow AB - BC = AD - DC$

hence, $AF - FC = AF' - F'C$

which implies $FF' = 0$

and hence the problem.

4 $\quad 0 \le f_i \le e_i \Rightarrow 0 \le \sum_{i=1}^{k} f_i \le \sum_{i=1}^{k} e_i = S$ (given)

number of tuples $= (f_1 f_2 \ldots f_n)$

$f_1 + f_2 + f_3 + \cdots + f_k = t$

coefficient of x^t in the expansion of

$$\prod_{i=1}^{k} (1 + x + x^2 + \cdots + x^{e_i}) = \prod_{i=1}^{k} (1 + x + x^2 + \cdots + x^{e_i})$$

$$= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

$|A_k| = a_0 + a_2 + a_4 + a_6 + a_8 + \cdots$

$|B_k| = a_1 + a_3 + a_5 + \cdots$

$|A_k| - |B_k| = a_0 - a_1 + a_2 - a_3 + \cdots$

$\prod_{i=1}^{k} (1 - 1 + \cdots + (-1)^{e_i}) = 1$ if e_i is odd for any i

$= 0$ if e_i is even for any i

5. Domain of $D=\{(x,y) \mid x^2+y^2\le 1\}$ allows us to let $x=r\cos\theta$; $y=r\sin\theta$; $0\le r\le 1$

$$f(x,y) = r(\sin\theta+\cos\theta) = r/\sqrt{2}\,\sin(\theta+\pi/4) \le r/\sqrt{2} \le 1/\sqrt{2}$$

$$(x,y) \in (0, 1/\sqrt{2})$$

6 If $P(t)$ does not have a root in (a_j, a_{j+1}) then since $p(t)$ is continous over $\mathbb{R}$, either $P(t)>0$ for all $t\in(a_j,a_{j+1})$ or $P(t)<0$ for $t\in(a_j,a_{j+1})$

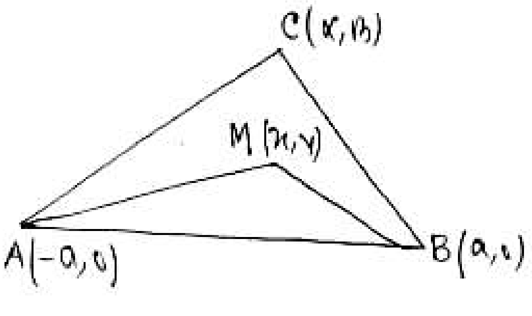
If $P(t)>0$ for all $t\in(a_j,a_{j+1})$ then $\int_{a_j}^{a_{j+1}} P(t)\,dt>0$, a contradiction.

If $P(t)<0$ for all $t\,(a_j,a_{j+1})$ then $\int_{a_j}^{a_{j+1}} P(t)\,dt<0$ a contradiction

Thus for each $0\le j\le(n-1)$, $p(t)$ has at least one root in (a_j,a_{j+1}). This makes a totall of n roots since n degree polynominal can attain at most n roots, if any interval thus can't attain more than 1 root else it violates the fundamental theorem of algebra.

Hence the problem.

7.



from shoelace theorem,

$$\begin{vmatrix} \cancel{\alpha} & \cancel{\beta} \\ -a & 0 \\ \cancel{x} & \cancel{0} \\ \alpha & \beta \end{vmatrix} = 2 \begin{vmatrix} \cancel{a} & \cancel{0} \\ x & y \\ -a & 0 \\ a & 0 \end{vmatrix}$$

$$\begin{vmatrix} \alpha & \beta \\ -a & 0 \\ a & 0 \\ \alpha & \beta \end{vmatrix} = 2 \begin{vmatrix} a & 0 \\ x & y \\ -a & 0 \\ a & 0 \end{vmatrix}$$

$\Rightarrow a\beta + \alpha\beta = 2(ay+ya)$

$\Rightarrow (a+\alpha)\beta = 4ay$

which denotes a st line.

8 let's consider one extremum case of $m \times n$ matrix in which all the product of the rows and columns is -1. Then $(-1)^m \equiv (-1)^n$

which means $m \equiv n \pmod 2$

so only in this case we can make a valid matrix

For each matrix $(m-1)(n-1)$ where $n \equiv m \pmod 2$ we can make a valid matrix by putting one elements on the front of each row and column and nothing on the last row and columns they will be considered in such way

We can fill up then only $(n-1)(m-1)$ boxes in $2^{(n-1)(m-1)}$ ways