B.Math Admission 2007

1. $n = (a\times b)\times(c\times d)\times(e\times f)$

let $a<b<c<d<e<f$ then it must be $(a\times f) * (b\times e) = (c\times d) = n$

To be odd number of divisors either $(a=f)$ or $(b=e)$ or $(c=d)$.

Hence the problem.

2. let (x_0, y_0) be a non-zero solution to $ax^2+by^2=0$ then $x_t = \left(\frac{t-b}{2by_0}\right)x$ and $y_t = \left(\frac{t-b}{2bx_0}\right)y_0+1$ is a solution to $ax^2+by^2=1$

3 The question is a bit wrong there →

either, a) show that sum of the distances of these points from 1 is $2n$

or, b) Show that the product of the distances of these points from 1 is $2n$

Denote $\omega = e^{2\pi i/n}$; $e^{\frac{2\pi i k}{n}} = \omega^k$

a) Then the distance to 1 ~~from~~ are $|1-\omega^k|$

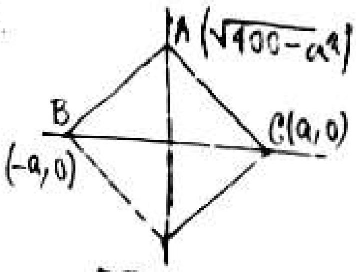
$$|1-\omega^k|^2 = (1-\omega^k)(\overline{1-\omega^k}) = (1-\omega^k)(1-\omega^{n-k}) = 2-\omega^k-\omega^{n-k}$$

Thus $\sum_{k=1}^{n-1} |1-\omega^k|^2 = 2(n-1) - 2\sum_{k=1}^{n-1} \omega_k = 2n$

b) $P(x) = x^{n-1} + \cdots + x + 1 = \prod_{k=1}^{n-1} (x-\omega^k)$

hence $|P(x)| = \prod_{k=1}^{n-1} |x-\omega^k|$; therefore $\prod_{k=1}^{n-1} |1-\omega^k| = |P(1)| = 1$

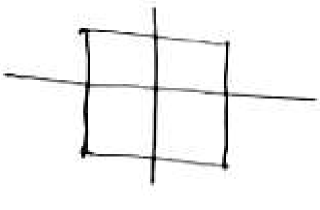
4.



$PD + PM = 1$

$x/a + y/\sqrt{400-a^2} = 1$

$$PM = \left|\frac{h/a + k/\sqrt{400-a^2} - 1}{\sqrt{(1/a)^2 + (1/400-a^2)}}\right| \quad PD = \left|\frac{-h/a + k/\sqrt{400-a^2} - 1}{\sqrt{(1/a)^2 + (1/400-a^2)}}\right|$$



The locus will be like this. hence, is a st line

5. a) Assume that there are more than $(d+2)$ distint integer roots of $P(x^2)-1=0$.

Then both of $P(x)\pm1=0$ have atleast 3 of them as roots

let $a_1, b_1, c_1, a_2, b_2, c_2$ are distinct integer such that

$a_1<b_1<c_1$ $P(a_1)-1=P(b_1)-1=P(c_1)-1=0$

$a_2>b_2>c_2$ $P(a_2)+1=P(b_2)+1=P(c_2)+1=0$

by the way,

When an integer α is a root of $P(x)-1=0$ and an integer $\beta(\neq\alpha)$ is a root of $P(x)+1=0$ then $\alpha-\beta \mid P(\alpha)-P(\beta)=2$

b) So we get $(a_1-a_2),(b_1-a_2),(c_1-a_2),(c_1-b_2),(c_1-c_2)\mid 2$

$\Rightarrow (a_1-a_2),(b_1-a_2),(c_1-a_2),(c_1-b_2),(c_1-c_2)\mid 2$

$\Rightarrow (a_1-a_2),(b_1-a_2),(c_1-a_2),(c_1-b_2),(c_1-c_2)=\pm1,\pm2$

Thus 2 of the 5 integers must be equal

but $(a_1-a_2)<b_1-a_2<c_1-a_2<c_1-b_2<c_1-c_2$

a contradiction

6. $(4+3+2+1)=10$ members belong to ISI club.

7. $f(\theta)=\sin\theta/\theta$; $f'(\theta)=\dfrac{\theta\cos\theta-\sin\theta}{\theta^2}=-\dfrac{\tan\theta-\theta}{\theta^2}\cos\theta$

$\varphi(\theta)=\tan\theta-\theta$; $\varphi'(\theta)=\sec^2\theta-1=\tan^2\theta>0$

$\varphi(\theta)>\varphi(0)=0$

$f'(\theta)<0$ $\theta<\pi/2$

$f(\theta)$ is a decreasing function

$\sin\theta/\theta \geq \dfrac{\sin\pi/2}{\pi/2}=2/\pi$

$\sin\theta\geq 2\theta/\pi$

8. $P(x)=x$ has noreal roots, then either $P(x)>x$ for all x or $P(x)<x$ for all x. Thus, either $P(P(x))>P(x)>x$ or $P(P(x))<P(x)<x$ for all x

9. It can be said, that everyone has atleast one friend. Let A,B,C,D,E are five men $\curvearrowright$ means friend, $\curvearrowleft$ means enemy

ABCDE as ABC thereby ABC must occurs, hence
ABCDE as ABCDE thereby AE must occurs

now, ABCDE = ADE = ADE

For A, ABCDE   For B, ABCDE

~~For C~~, ~~ABCDE~~ as, ABE = BE ∴ ABCDE

It can be generalised easily.

10. 1^{2}_{3} 1's sides are filled up.

Now on the side of 2 only 1 can sitted as 3 already placed on and proceeding in that sequence.

$$1^{2}_{3} \longrightarrow 1^{2\ 4}_{3} \longrightarrow 1^{2\,4}_{3\,5} \longrightarrow 1^{2\,4\,6\,8\,10}_{3\,5\,7\,9\,11}$$