B.MATH 2009 Admission Test

1. If we consider a triangle with vertices α, β, γ then the barycentric coordinates let the circumcentre of the triangle denoted as $\alpha x+\beta y+\gamma z$ which is given equal to 0. unless Δ is degenerate it is impossible and hence the problem.

method 2:

$$\alpha x+\beta y+\gamma z=0 \Rightarrow \alpha x+\beta y=-\gamma z=\gamma(x+z) \quad \text{as } (x+y+z)=0$$

$$\Rightarrow \alpha\left(\frac{x}{x+y}\right)+\beta\left(\frac{y}{x+y}\right)=\gamma$$

$$\Rightarrow \left|\alpha\left(\frac{x}{x+y}\right)+\beta\left(\frac{y}{x+y}\right)\right|=|\gamma|=1$$

$$\Rightarrow \alpha^2x^2+\beta^2y^2=(x+y)^2-2\alpha\beta xy$$

$$\Rightarrow x^2+y^2=(x^2+y^2+2xy)-2\alpha\beta xy$$

as $|\alpha|=|\beta|=1$

$$\Rightarrow xy=xy\alpha\beta$$

$$\Rightarrow \alpha\beta=1$$

as similarly we can get $\alpha\beta=\beta\gamma=\alpha\gamma=1$

hence $\alpha=\gamma=\beta$

2. consider the function

$$f(x)=x(x+1)(x+2)(x+3)\cdots(x+2009)$$

let α be a root to the equation in the question then $f(\alpha)=c$

assume α is a root of multiplicity 2 then $f'(\alpha)=c$

Its easy to watch

$$f'(x)=f(x)\left(1/x+1/x+1+\cdots+1/x+2009\right)$$

Put $x=\alpha$

$$c=c\left(1/\alpha+1/\alpha+1+1/\alpha+2+\cdots+1/\alpha+2009\right)$$

$$\Rightarrow 1=1\left(1/\alpha+1/\alpha+1+\cdots+1/\alpha+2009\right)$$

Now for the root to be multiplicity of 3 $f''(\alpha)=c$

$$f''(x)=f'(x)\left(1/x+1/x+1+\cdots+1/x+2009\right)$$

now putting $x=\alpha$, and $f(x)=f'(x)=c$

and $1=\left(\frac{1}{\alpha}+\frac{1}{\alpha+1}+\cdots+\frac{1}{\alpha+2009}\right)$

we get,

$$f''(\alpha)=c-c\left(\frac{1}{\alpha^2}+\frac{1}{(\alpha+1)^2}+\cdots+\frac{1}{(\alpha+2009)^2}\right)$$

which can't be equal to c for any value of α

QED

3. let us take a n digit number. Total possible numbers $=9\times10^{n-1}$

now consider the number with zeros

$$\binom{n-1}{1}9\times9^{n-2}+\binom{n-1}{2}9\times9^{n-3}+\binom{n-1}{3}9\times9^{n-4}+\cdots$$

$$=9\times(1+9)^{n-1}-9^n$$

now total number of without zeroes $=9\times10^{n-1}-9(10)^{n-1}+9^n$
$=9^n$

method 2:

each number (without zero) has 9 possibilites

hence,

total number of without zeroes $=9^n$

now Σ of number having n digits

$$\frac{1}{10^{n-1}}+\frac{1}{10^{n-1}+1}+\cdots$$

number of numbers we add $=9^n$

each number $<\frac{1}{10^{n-1}}$

so we have sum $<9^n\times\frac{1}{10^{n-1}}$

now number of digits varies from 1 to α

$$\text{sum}\leq 9+9\times\left(\frac{9}{10}\right)+9\times\left(\frac{9}{10}\right)^2+\cdots$$

$$\leq 9\left(\frac{1}{1-\frac{9}{10}}\right)=90$$

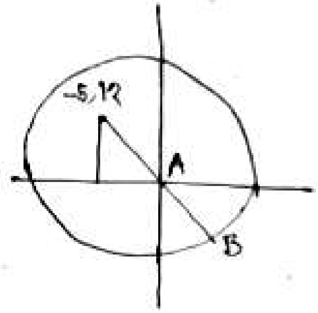
QED

4. Projecting into polar coordinate we get

$$x = 14\cos\alpha - 9$$
$$y = 14\sin\alpha + 12$$

$(x^2+y^2)_{min}$ is easy to find

method 2:



It is clear from the picture that AB is the least solution

$= OB - OA = 14 - 13 = 1$

$(x,y) \in \left(5/13, -12/13\right)$

5. $\frac{10^r}{p} - \frac{1}{p} = a_1a_2a_3 \dots a_r$

$\Rightarrow 10^r - 1 = A.P$ (where $A = a_1a_2a_3 \dots a_r$)

$=$ a natural number

$\Rightarrow 10^r \equiv 1 \pmod{p}$

QED

6. $y = \frac{b_2a - b_1c}{ad - bc}$ now $b_2 b_1$ are multiples of $(ad-bc)$

$\therefore y$ is similar to x

7. let, h be oneside of rectangle with base along BC

area $= h\left(a - h(\cot B + \cot C)\right)$

using $a = 2R\sin A$

and complete the square

max area $= R^2 \sin A \sin B \sin C = M/R$

8 opposite pairs for paints are chosen in $\binom{6}{2}\binom{4}{2} = 90$ ways

which can be arranged in 31 unique ways in total

$\frac{90}{3!} = 15$ way

as the three pairs are indistinguishable

9. Write the polynominal in the form $u(x-1)(x+1)+vx+w$

then $f(-1)=-v+w$, $f(1)=v+w$, $f(0)=w-u$

there is a formula describing $f(x)$ in terms of $f(-1), f(0), f(1)$

because a quadratic polynominal is fixed by giving any ~~of~~ three of its values (at different points)

— lagrange interpolation.

method 2:

If $f(-1), f(0), f(1)$ are known then one can apply a linear system of equations on a, b, c so it remains to prove that if u, v, w are in $[-1, 1]$

$$u = f(-1);\ v = f(0);\ w = f(1) \text{ then}$$

$$\left|v(1-x^2) + \tfrac{u}{2}(x^2-x) + \tfrac{w}{2}(x^2+x)\right| \le 3/2$$

apply triangle inequality on LHS

remains to show that

$$|1-x^2| + \frac{|x^2-x|}{2} + |x^2+x| \le 3/2$$

which is easy to prove.

10. rational root $\Rightarrow$ D must be perfect square

$$D = b^2 - 4ac = \text{odd} \ (\text{odd} - \text{even})$$

and every odd perfect square is of form $8n+1$; $n \in N$

This gives

$$8m+1 - 4ac = 8n+1$$

or, $ac = \text{odd} = (m-n)$ contradicts!

which shows the roots must be rational