CHALLENGING MATHEMATICAL PROBLEMS

1000 Solved Problems for Bachelor's Entrances

By Tanujit Chakraborty

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PREFACE

“This book is useful for the BSTAT and BMATH Entrances conducted by Indian Statistical Institute and Chennai Mathematical Institute. Readers are also suggested to go through www.ctanujit.in and www.ctanujit.org for other online resources. Since 2013, I am involved with teaching for different entrance exams at ISI. I tried my best to cover several previous year’s papers questions asked in **ISI CMI BSTAT / BMATH** Entrance exams. If you have any queries you can mail me at tanujitisi@gmail.com.”

With Best Wishes,

- Tanujit Chakraborty,

Founder of Ctanujit Classes.

Research Scholar,

Indian Statistical Institute, Kolkata.

My Blog: www.ctanujit.org

Content of the Book:

ISI B.STAT/B.MATH OBJECTIVE QUESTIONS & SOLUTIONS

SET – 1

1. **How many zeros are at the end of 1000! ?**

   (a) 240 (b) 248
   (c) 249 (d) None

   Ans:- (c) The number of two's is enough to match each 5 to get a 10.

   So,

   $5^1 \longrightarrow 200$

   $5^2 \longrightarrow 40$

   $5^3 \longrightarrow 8$

   $5^4 \longrightarrow 1$

   ∴Thus, 1000! Ends with 249 zero's.

   [Theorem: (de Polinac's formula) Statement: Let p be a prime and e be the largest exponent of p such that p^e divides n! , then e=$\sum$ [n/p^i], where i is running from 1 to infinity.]

2. **The product of the first 100 positive integers ends with**

   (a) 21 zeros (b) 22 zeros (c) 23 zeros (d) 24 zeros.

   Ans:- $5^1 \longrightarrow 20$
   (d) 24 zeros .

   $5^2 \longrightarrow 4$

   Alternatively, put p=5,n=100,thus from above theorem we have [100/5]+[100/25]=24 zeros as the answer.

3. **Let P (x) be a polynomial of degree 11 such that P (x) = $\frac{1}{x+1}$ $for\ x = 0\ (1)11.$**

Then P (12) = ?

   (a) 0 (b) 1 (c) $\frac{1}{13}$ (d) none of these

Ans:- (a) P (x)= $\frac{1}{x+1}$

⇒ (x+1)[P (x)]-1 = c (x-0)(x-1)….(x-11)

Putting x= -1, 0- 1= c (-1)(-2)….(-12)

$\Rightarrow c = -\frac{1}{12!}$

∴ [P (x)](x+1)-1= - $\frac{1}{12!}$(x-0)(x-1)….(x-11)

⇒ P (12) 13-1 = - $\frac{1}{12!}$ 12 .11. ….2.1

⇒ P (12) 13-1 = -1

⇒P (12) = 0.

4. **Let s= {(x_1, x_2, x_3)| 0≤ x_i ≤ 9 and $x_1 + x_2 + x_3$ is divisible by 3}.**

Then the number of elements in s is

   (a) 334 (b) 333 (c) 327 (d) 336

Ans:- (a) with each (x_1, x_2, x_3) identify a three digit code, where reading zeros are allowed. We have a bijection between s and the set of all non-negative integers less than or equal to 999 divisible by 3. The no. of numbers between 1 and 999, inclusive, divisible by 3 is $\left(\frac{999}{3}\right) = 333$

Also, '0' is divisible by 3. Hence, the number of elements in s is = 333 + 1= 334.

5. **Let x and y be positive real number with x< y. Also 0 < b< a < 1.**

Define E = $\log_a\left(\frac{y}{x}\right) + \log_b\left(\frac{x}{y}\right)$. Then E can't take the value

(a) -2 (b) -1 (c) -$\sqrt{2}$ (d) 2

Ans :- (d) E =$\log_a\left(\frac{y}{x}\right) + \log_b\left(\frac{x}{y}\right)$

$= \frac{\log\frac{y}{x}}{\log_a} - \frac{\log\frac{y}{x}}{\log_b} = \log\left(\frac{y}{x}\right)\left\{\frac{1}{\log_a} - \frac{1}{\log_b}\right\}$

$= \log\left(\frac{y}{x}\right)\left\{\frac{\log_b - \log_a}{(\log_a)(\log_b)}\right\}$

$= \log\left(\frac{y}{x}\right) . \frac{\log(\frac{b}{a})}{(\log_a)(\log_b)}$

$= -\log\left(\frac{y}{x}\right) . \frac{\log(\frac{a}{b})}{(\log_a)(\log_b)}$

Log 0< a < 1, 0< b <1 $\therefore \log_a$ and $\log_b$ are both negative.

Also $\frac{y}{x} > 1$ and $\frac{a}{b} > 1$. Thus $\log\left(\frac{y}{x}\right)$ and $\log\left(\frac{a}{b}\right)$ are both positive. Finally E turns out to be a negative value. So, E can't take the value '2'.

6. **Let S be the set of all 3- digits numbers. Such that**
   (i) The digits in each number are all from the set {1, 2, 3, …., 9}
   (ii) Exactly one digit in each number is even

The sum of all number in S is

(a) 96100 (b) 133200 (c) 66600 (d) 99800

Ans:- (b) The sum of the digits in unit place of all the numbers in s will be same as the sum in tens or hundreds place. The only even digit can have any of the three positions,

i.e. ${}^3c_1$ ways.

And the digit itself has 4 choices (2, 4, 6 or 8). The other two digits can be filled in 5× 4 = 20 ways.

Then the number of numbers in S = 240.

Number of numbers containing the even digits in units place = $4 \times 5 \times 4 = 80$

The other 160 numbers have digits 1, 3, 5, 7 or 9 in unit place, with each digit appearing

$\frac{160}{5}$ = 32 times. Sum in units place = 32 (1+ 3+ 5+ 7+ 9) + 20 (2+ 4+ 6+ 8)

$= 32.5^2 + 20 \times 2 \times \frac{4\times5}{2} = 32\times 25 + 20 \times 20 = 1200$

$\therefore$ Sum of all numbers= 1200 (1+ 10 +10^2) = 1200× 111 = 133200.

7. **Let y = $\frac{x}{x^2+1}$, Then $y^4(1)$is equals**
   (a) 4 (b) -3 (c) 3 (d) -4

Ans:- (b) Simply differentiating would be tedious, So we take advantage of 'i' the square root of '-1'

y = $\frac{x}{x^2+1} = \frac{1}{2}\{\frac{1}{(x-i)} + \frac{1}{(x+i)}\}$

$\frac{d^4y}{dx^4} = \frac{1}{2}\{\frac{4!}{(x-i)^5} + \frac{4!}{(x-i)^5}\}$

Note that, $\frac{d^n}{dx^n}\left\{\frac{1}{x+a}\right\} = \frac{(-1)^n\, n!}{(x+a)^{n+1}}$

So, $y^4(x) = \frac{4!}{2}\{\frac{1!}{(x-i)^5} + \frac{1!}{(x-i)^5}\}$ Then

$y^4(1) = 12\,\{\frac{1!}{(x-i)^5} + \frac{1!}{(x-i)^5}\}$ = 12 $\{\frac{1-i}{(-2i)^3} + \frac{1-i}{(2i)^3}\}$ = 12 $\{\frac{1-i}{8i} + \frac{1-i}{8i}\}$ = 12 $(-\frac{1}{8} - \frac{1}{8})$ = -3.

8. A real 2× 2 matrix. M such that $M^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1-\epsilon \end{pmatrix}$

(a) exists for all ∊ > 0

(b) does not exist for any ∊> 0

(c) exists for same ∊> 0

(d) none of the above

Ans:- (b) since M^2 is an diagonal matrix, so M= $\begin{bmatrix} i & 0 \\ 0 & \sqrt{1-\epsilon} \end{bmatrix}$,

So, M is not a real matrix, for any values of ∊. M is a non –real matrix.

9. The value of $\left(\frac{1+i\sqrt{3}}{2}\right)^{2008}$ is

(a) $\frac{1+i\sqrt{3}}{2}$ **(b)** $\frac{1-i\sqrt{3}}{2}$

(c) $\frac{-1-i\sqrt{3}}{2}$ **(d)** $\frac{-1+i\sqrt{3}}{2}$

Ans:- (c) A = $(\frac{1+i\sqrt{3}}{2})$, $A^2 = \frac{-1+i\sqrt{3}}{2}, A^4 = \frac{-1-i\sqrt{3}}{2} = -A$

$\therefore A^{2008} = (A^4)^{502} = A^4 = \frac{-1-i\sqrt{3}}{2}$.

10. Let f(x) be the function f(x)= $\begin{cases} \frac{x^P}{(sinx)^q} & if\ x > 0 \\ 0 & if\ x = 0 \end{cases}$

Then f(x) is continuous at x= 0 if

(a) p > q (b) p > 0 (c) q > 0 (d) p < q

Ans:- (b) |f(x) - f(0)| = $|\frac{x^P}{(sinx)^q} - 0| \le |x^P| < \epsilon$

Whenever |x-0| < $\epsilon^{\frac{1}{p}} = \delta$ if p > 0.

So, f(x)is continuous for p > 0 at x= 0.

11. The limit $\lim_{x\to\infty} \log(1 - \frac{1}{n^2})^n$ equals

(a) e^{-1} (b) $e^{-\frac{1}{2}}$ (c) e^{-2} (d) 1

Ans:- (d) L = $(1 - \frac{1}{n^2})^n$

⇨ $logL = nlog(1 - \frac{1}{n^2})$

⇨ $\lim_{x\to\infty} logL = \lim_{x\to\infty}[-n\{\frac{1}{n^2} + \frac{1}{2n^4} + \cdots\infty\}] = 0$

∴L = $e^0 = 1$.

12. The minimum value of the function f(x, y)= $4x^2 + 9y^2 - 12x - 12y + 14$ is

(a) 1 (b) 3 (c) 14 (d) none

Ans:- (a) f(x, y) = $4x^2 + 9y^2 - 12x - 12y + 14$

= $(4x^2 - 12x + 9)$+$(9y^2 - 12y + 4)$+1

= $(2x-3)^2 + (3y-2)^2 + 1 \geq 1$

So, minimum value of f(x, y) is 1.

13. From a group of 20 persons, belonging to an association, A president, a secretary and there members are to be elected for the executive committee. The number of ways this can be done is

(a) 30000 (b) 310080

(c) 300080 (d) none

Ans:- (b) $20_{c_1} \times 19_{c_1} \times 18_{c_3}$ *or* $\frac{20!}{1!1!3!15!}$ = 310080

14. The $\lim\limits_{x\to 0} \frac{\cos x - \sec x}{x^2(1+x)}$ is

(a) -1 (b) 1 (c) 0

(d) does not exist

Ans:- (a) $\lim\limits_{x\to 0} \frac{\cos x - \sec x}{x^2(1+x)} = \lim\limits_{x\to 0} \frac{-\sin^2 x}{\cos x\,(x^2)(x+1)}$

= - $\lim\limits_{x\to 0} \frac{1}{\cos x}\left(\frac{\sin x}{x}\right)^2 . \frac{1}{(x+1)}$ = -1. 1. 1 = - 1.

15. Let R = $\frac{48^{52} - 46^{52}}{96^{26} + 92^{26}}$. Then R satisfies

(a) R < 1 (b) 23^{26}< R < 24^{26}

(c) 1 < R < 23^{26} (d) R > 24^{26}

Ans:- (b) R= $\frac{(2.24)^{52} - (2.23)^{52}}{(4.24)^{26} + (4.23)^{26}} = \frac{2^{52}(24^{52} - 23^{52})}{4^{26}(24^{26} + 23^{26})}$

= $\frac{2^{52}}{2^{52}} . \frac{(24^{26} + 23^{26})(24^{26} - 23^{26})}{24^{26} + 23^{26}}$

= $24^{26} - 23^{26}$

< 24^{26}

Also, R= $24^{26} - 23^{26} = (1+23)^{26} - 23^{26}$

= $23^{26} + 26_{c_1} . 23^{25} + 26_{c_2} . 23^{24} + \ldots + 1 - 23^{26}$

= 26. 23^{25}+$26_{c_2} . 23^{24} + \ldots + 1$ >26. 23^{25} > 23. 23^{25} = 23^{26}

∴ 23^{26} < R < 24^{26}

16. A function f is said to be odd if f (-x)= -f (x) ∀ x. Which of the following is not odd?

(a) f (x+ y)= f(x)+ f(y) ∀ x, y

(b) f (x)= $\frac{xe^{x/2}}{1+e^x}$

(c) f (x) = x - [x]

(d) f (x) = $x^2 \sin x + x^3 \cos x$

Ans:- (c) f (x+ y)= f(x)+ f(y) ∀ x, y

Let x = y = 0

⇒ f (0) = f (0) + f (0)

∴ f (0)= 0

Replacing y with –x , we have

f (x- x) = f(x) + f (-x)

⇒ f (0) = f(x) + f (-x)

⇒ f(x) + f (-x) = 0

⇒ f (-x) = -f(x)

Thus f is odd.

Again for f (x) = $\frac{xe^{x/2}}{1+e^x}$

f(-x)= $\frac{(-x)(e^{-x/2})}{1+e^{-x}}$ = $\frac{(-x)\left(e^{-x/2}\right).e^{x}}{1+e^{x}}$ = - $\frac{xe^{x/2}}{1+e^{x}}$ = -f (x)

∴ f is odd.

f (x) = x- [x] is not odd.

Counter example:-

f (-2.3) = -2.3 – [-2.3] =-2.3 – (-3) = 3- 2.3 = 0.7

f (2.3) = 2.3 – [2.3] = 2.3 -2 =0.3

∴ f(2.3) ≠ f(-2.3)

Thus f is not odd

f (x) = $x^2 \sin x + x^3 \cos x$

f(-x) = -$x^2 \sin x - x^3 \cos x$ = -f(x)

∴ f is odd here.

17. Consider the polynomial $x^5 + ax^4 + bx^3 + cx^2 + dx + 4$. If (1+2i) and (3-2i) are two roots of this polynomial then the value of a is

(a) -524/65 (b) 524/65 (c) -1/65 (d) 1/65

Ans:- (a) The polynomial has 5 roots. Since complex root occur in pairs, so there is one real root taking it as m.

So, m, 1+2i, 1-2i, 3+2i, 3-2i are the five roots.

Sum of the roots= $-\frac{a}{1} = 8 + m.$

Product of the roots= (1+4)(9+4)m= 65

m=$\frac{4}{65}$

∴ m = $\frac{4}{65}$.

∴ a= $-8 - \frac{4}{65} = -\frac{524}{65}$.

18. In a special version of chess, a rook moves either horizontally or vertically on the chess board. The number of ways to place 8 rooks of different colors on a 8×8 chess board such that no rook lies on the path of the other rook at the start of the game is

(a) 8× ⌊8 (b) ⌊8 × ⌊8 (c) 2^8 × ⌊8 (d) $2^8 \times \binom{64}{8}$

Ans:- The first rook can be placed in any row in 8 ways & in any column in 8 ways. So, it has 8^2 ways to be disposed off. Since no other rook can be placed in the path of the first rook, a second rook can be placed in 7^2 ways for there now remains only 7 rows and 7 columns. Counting in this manner, the number of ways = $8^2.7^2.6^2 \ldots 1^2 = (8!)^2$

19. The differential equation of all the ellipses centered at the origin is

(a) $y^2 + x(y')^2 - yy' = 0$

(b) x y y″+ $x(y')^2 - yy' = 0$

(c) y y″+ $x(y')^2 - xy' = 0$

(d) none

Ans:- (d) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, after differentiating w.r.t x, we get

$\Rightarrow \frac{2x}{a^2} + \frac{2yy'}{b^2} = 0 \Rightarrow \frac{yy'}{b^2} = -\frac{x}{a^2}$

$\Rightarrow \frac{(y')^2}{b^2} + \frac{y(y'')}{b^2} = -\frac{1}{a^2}$

$\Rightarrow (y')^2 + y(y'')^2 = -\frac{b^2}{a^2}$.

20. If f(x)= x+ sinx, then find
$\frac{2}{\pi^2}.\int_{\pi}^{2\pi}(f^{-1}(x)+sinx)dx$
(a) 2 **(b) 3**
(c) 6 **(d) 9**

Ans:- (b) Let x= f(t) ⇒ dx= f′(t)dt

$\Rightarrow \int_{\pi}^{2\pi} f^{-1}(x)dx = \int_{\pi}^{2\pi} t\ f'(t)dt =$
$(t\,[f(t)])^{2\pi}{}_{\pi} - \int_{\pi}^{2\pi} f(t)dt = (4\pi^2 - \pi^2) - \int_{\pi}^{2\pi} f(t)dt$

$I= \int_{\pi}^{2\pi}(f^{-1}(x) + sinx)dx = \int_{\pi}^{2\pi} f^{-1}(x)dx + \int_{\pi}^{2\pi} sinxdx$

$= 3\pi^2 - \int_{\pi}^{2\pi} f(t)dt + \int_{\pi}^{2\pi} sinxdx$

$= 3\pi^2 - \int_{\pi}^{2\pi}(f(x) - sinx)dx$

$= 3\pi^2 - \int_{\pi}^{2\pi} xdx = 3\pi^2 - \frac{1}{2}(4\pi^2 - \pi^2)$

$=\frac{3}{2}\pi^2$

$\Rightarrow \frac{2}{\pi^2} I = 3.$

21. Let P= (a, b), Q= (c, d) and 0 < a < b < c < d, L≡(a, 0), M≡(c, 0), R lies on x-axis such that PR + RQ is minimum, then R divides LM

(a) Internally in the ratio a: b
(b) internally in the ratio b: c

(c) internally in the ratio b: d
(d) internally in the ratio d: b

Ans:- (c) Let R = (α, 0). PR+RQ is least

⟹ PQR should be the path of light

⟹ Δ PRL and QRM are similar

$\Longrightarrow \frac{LR}{RM} = \frac{PL}{QM} \Longrightarrow \frac{\alpha-a}{c-\alpha} = \frac{b}{d}$

⟹ αd- αd= bc –αb

$\Longrightarrow \alpha= \frac{ad+bc}{b+d}$

⟹ R divides LM internally in the ratio b : d (as $\frac{b}{d} > 0$)

22. A point (1, 1) undergoes reflection in the x-axis and then the co-ordinate axes are roated through an angle of $\frac{\pi}{4}$ in anticlockwise direction. The final position of the point in the new co-ordinate system is-

(a) $(0, \sqrt{2})$ **(b) $(0, -\sqrt{2})$**
(c) $-\sqrt{2}, 0$ **(d) none of these**

Ans:- . (b) Image of (1, 1) in the x-axis is (1, -1). If (x, y) be the co-ordinates of any point and (x’ , y’) be its new co-ordinates, then x’ = x cos θ+ y sin θ,

y’= y cosθ – x sin θ, where θ is the angle through which the axes have been roated.

Here $\theta = \frac{\pi}{4}$, x= 1, y= -1

∴ x’= 0, y’= -$\sqrt{2}$

23. If a, $x_1, x_2, \dots, x_k$ and b, $y_1, y_2, \dots, y_k$ from two A.P. with common difference m and n respectively, then the locus of point (x, y) where x= $\frac{\sum_{i=1}^{k} x_1}{k}$ $is\ and\ y = \frac{\sum_{i=1}^{k} y_1}{k}$ is

(a) (x-a)m= (y-b)n
(b) (x-m) a= (y-n) b

(c)(x-n)a = (y-m)b
(d) (x-a) n-(y-b) m

Ans:- (d)

X= $\frac{\frac{k}{2}(x_1+x_k)}{k} = \frac{x_1+x_k}{2} = \frac{a+m+a\,mk}{2}$

$or,\ x = a + \frac{(k+1)m}{2}$

or, 2(x-a)= (k+1)m ……………..(1)

Similarly,

2(y-b)= (k+ 1)n ……………….(2)

We have to eliminate k

From (1) and (2)

$\frac{x-a}{y-b} = \frac{m}{n}$

or, (x- a)n = (y -b)m

24. The remainder on dividing $1234^{567} + 89^{1011}$ by 12 is

(a) 1 **(b) 7**
(c) 9 **(d) none**

Ans:- (c) 1234≡1 (mod 3)⇒ $1234^{567} \equiv 1\ (mod\ 3) and\ 89 \equiv\ -1(mod 3)$

$\Rightarrow 89^{1011} \equiv -1\ (mod\ 3)$

$\therefore 1234^{567} + 89^{1011} \equiv 0\ (mod\ 3)$

Here 1234 is even, so $1234^{567} \equiv 0(mod\ 4) and\ 89 \equiv 1\ (mod\ 4)$

$\Rightarrow 89^{1011} \equiv 1\ (mod\ 4)$

Thus $1234^{567} + 89^{1011} \equiv 1\ (mod\ 4)$

Hence it is 9 (mod 12)

25. The sum of the series 1+$\frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \cdots$ is

(a) e^2 **(b) 3**
(c) $\sqrt{5}$ **(d) $\sqrt{8}$**

Ans. (d) $\sqrt{8} = 2^{\frac{3}{2}} = (\frac{1}{2})^{-\frac{3}{2}} = (1 - \frac{1}{2})^{-\frac{3}{2}}$

= 1+ $(\frac{1}{2})\left(\frac{3}{2}\right) + \frac{\left(-\frac{3}{2}\right)(-\frac{3}{2}-1)}{2!}(-\frac{1}{2})^2 + \cdots$

= 1+$\frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \cdots$

26. If f(x) = cos x+ cos ax is a periodic function, then a is necessarily

(a) an integer (b) a rational number
(c) an irrational number (d) an event number

Ans. (b) Period of cos x= 2π and period of cos ax= $\frac{2\pi}{|a|}$

Period of f(x) = L.C.M. of $\frac{2\pi}{1}\ and\ \frac{2\pi}{|a|} = \frac{L.C.M. of\ 2\pi\ and\ 2\pi}{H.C.F. of\ 1\ and\ |a|}$

Since k= H.C.F. of 1 and |a|

∴ $\frac{1}{k}$= an integer= m (say) and $\frac{|a|}{k}$= an integer = n (say)

∴ |a|=$\frac{n}{m} \Longrightarrow a = \pm\frac{n}{m}$ = a rational number.

27. Let f : R⟶ R defined by f(x)= $x^3 + x^2 + 100x + 5\sin x$, then f is

(a) many-one onto (b) many-one into
(c) one-one onto (d) one-one into

Ans. (c)

$f(x) = x^3 + x^2 + 100x + 5$ sin x

∴ f'(x)= $3x^2 + 2x + 100 + 5\cos x$

= $3x^2 + 2x + 94 + (6 + 5\,cosx) > 0$

∴ f is an increasing function and consequently a one –one function.

Clearly f($-\infty$)=$-\infty$, f(∞)=∞ and f(x) is continuous, therefore range f= R= co domain f. Hence f is onto.

28. Let f(x) = $\frac{\sin^{101}x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}$, where [x] denotes the integral part of x is

(a) an odd function
(b) an even function

(c) neither odd nor even function
(d) both odd and even function

Ans. (a) when x= nπ, n ϵ I, sin x = 0 and $\left[\frac{x}{\pi}\right]+\frac{1}{2}\neq 0$

∴ f(x) = 0

∴ when x = nπ, f(x)= 0 and f(-x)= 0

∴ f(-x)= f(x)

When x $\neq n\pi, n\,\epsilon\, I, \frac{x}{\pi} \neq an\ integer$

$$\therefore \left[\frac{x}{\pi}\right]+\left[-\frac{x}{\pi}\right] = -1 \ \therefore \left[-\frac{x}{\pi}\right] = -1-\left[\frac{x}{\pi}\right]$$

$$\Rightarrow \left[-\frac{x}{\pi}\right]+\frac{1}{2} = -\left[\frac{x}{\pi}\right]-\frac{1}{2} = -\left(\left[\frac{x}{\pi}\right]+\frac{1}{2}\right)$$

Now $f(-x) = \frac{\sin^{101}(-x)}{\left[-\frac{x}{\pi}\right]+\frac{1}{2}} = \frac{-\sin x}{-(\left[\frac{x}{\pi}\right]+\frac{1}{2})} = \frac{\sin x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}} = f(x)$

Hence in all cases f(-x)= f(x)

29 . If k be the value of x at which the function

f(x) =$\int_{-1}^{x} t(e^t-1)(t-1)(t-2)^3(t-3)^5\, dt$ has maximum value and sinx + cosecx = k, then for n ∈N, $\sin^n x + \mathrm{cosec}^n x = \ldots$

(a) 2 **(b) -2**
(c) $\frac{\pi}{2}$ **(d) π**

Ans. (a) f'(x) = $x(e^x-1)(x-1)(x-2)^3(x-3)^5$

By Sign Rule we get

f(x) has max. at x = 2

∴ k = 2

Now sin x + cosec x = k ⟹ sin x + cosec x = 2

$\Longrightarrow(\sin x-1)^2 = 0 \ \Rightarrow \sin x \ = 1$

∴ cosec x = 1

Hence $\sin^n x + \mathrm{cosec}^n x = 2$

30. If f(x+ y) = f(x) + f(y) – xy – 1for all x, y ∈R and f(1)=1, then the number of solutions of f(n)= n, n ∈N is

(a) 0 **(b) 1** **(c) 2** **(d) more than 2**

Ans. (b)

Given f(x+ y)= f(x)+f(y) - xy - 1 $\forall\ x, y, \epsilon\ R$ ……………(1)

f(1)= 1 ……………..(2)

f(2) = f(1+1)= f(1)+f(-1)-1-1= 0

f(3)= f(2+1)= f(2)+f(1)-2.1-1= -2

f(n+1) = f(n) +f(1) – n – 1 = f(n)- n< f(n)

Thus f(1) > f(2) > f(3)> …and f(1)= 1

∴f(1)= 1 and f(n)< 1, for n> 1

Hence f(n)= n, n ϵ N has only one solution n= 1

ISI B.STAT/B.MATH OBJECTIVE QUESTIONS & SOLUTIONS

SET – 2

1. $\boldsymbol{a_1 = a_2 = 1, a_3 = -1, a_n = a_{n-1} . a_{n-3}}$**. The value of** $\boldsymbol{a_{1964}}$ **is**
(a) 1 **(b) -1**
(c) 0 **(d) none**

Ans:- (b) $a_1 = a_2 = 1, a_3 = a_4 = a_5 = -1, a_6 = 1, a_7 = -1$

$\underbrace{+1, +1, -1, -1, -1, 1, -1}$,

$\underbrace{1, 1, -1, -1, -1, 1, -1}$, …………

Since 1964= (7×280)+4= 7×280 +4. Thus we have a_{1964}= -1.

2. If a, b are positive real variables whose sum is a constant λ, then the minimum value of $\sqrt{\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)}$ **is**
(a) $\boldsymbol{\lambda - \frac{1}{\lambda}}$ **(b)** $\boldsymbol{\lambda + \frac{2}{\lambda}}$
(c) $\boldsymbol{1+ \frac{2}{\lambda}}$ **(d) none**

Ans:- (c) $E^2 = 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{ab} = \frac{a+b+1}{ab} + 1 = 1 + \frac{\lambda+1}{ab}$, it will be minimum when ab is maximum. Now we know that if sum of two quantities is constant, then their product is maximum when the quantities are equal.

∴ a+ b = λ ⇒a = b = $\frac{\lambda}{2}$

∴ $E^2 = 1 + \frac{\lambda+1}{\frac{\lambda^2}{4}} = (\frac{\lambda+2}{\lambda})^2$ ⇒ E = 1+ $\frac{2}{\lambda}$, which is the required result.

Alternative: (c) $\sqrt{\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)}$ will minimum when a and b will take the maximum value.

a+b = λ, then the max. Value of a and b is a= b=$\frac{\lambda}{2}$,

Putting these, we get, $\sqrt{\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)}$ min = $\sqrt{\left(1+\frac{2}{\lambda}\right)(1+\frac{2}{\lambda})}$ = 1+ $\frac{2}{\lambda}$.

3. The number of pairs of integers (m, n) satisfying $\boldsymbol{m^2 + mn + n^2}$ **= 1 is**
(a) 8 **(b) 6**
(c) 4 **(d) 2**

Ans:- (b) Consider $m^2 + mn + n^2 - 1$

The equation is symmetric in m and n, we make the substitution

u = m + n and v= m- n

So that $u^2 + v^2 = 2(m^2 + n^2), u^2 - v^2 = 4mn$

Multiplying the given equation by 4, we have

$4m^2 + 4mn + 4n^2 = 4$

$\Rightarrow 4\,(m^2 + n^2) + 4mn = 4$

$\Rightarrow 2\,(u^2 + v^2) + u^2 - v^2 = 4$

$\Rightarrow 3u^2 + v^2 = 4$

Set $u^2 = x, v^2 = y$ with x, y ≥ 0, then we get 3x+ y= 4

The ordered pairs (x, y) satisfying the above equation in integers are (0, 4) and (1, 1).

We have,

u^2= 0 and $u^2 = 1$ & $v^2 = 4$, and $v^2 = 1$

I.e. u= 0, v= 2; u=0, v= -2;

u=1, v= 1; u =1, v= -1;

u =-1, v =1; u = -1, v= -1;

Giving 6 ordered pair solutions (m, n) viz (1, -1), (-1, 1), (1, 0), (0, 1), (0, -1), (-1, 0)

4. **The sum of the digits of the number $\mathbf{100^{13} - 26}$, written in decimal notation is**
   (a) 227 **(b) 218**
   (c)228 **(d) 219**

Ans:- (a) $10^{26} - 26 = \underbrace{100 \ldots..0} - 26$

$= \underbrace{999 \ldots.974}$

with 24 9's

$\therefore$ The sum of the digits = $24 \times 9 + 7 + 4 =$ 227.

5. **The great common divisor (gcd) of $\mathbf{2^{2^{22}} + 1}$ *and* $\mathbf{2^{2^{222}} + 1}$ is**
   (a) 1 **(b) $\mathbf{2^{2^{22}} + 1}$**
   (c) $\mathbf{2^{2^{11}} - 1}$ **(d) $\mathbf{2^{2^{21}} - 1}$**

Ans:- (a) let $F_n = 2^{2^n} + 1$, with m > n

$F_n - 2 = 2^{2^n} + 1 - 2 = 2^{2^n} - 1 =$ $(2^{2^{n-1}})^2 - 1 = (2^{2^{n-1}} + 1)(2^{2^{n-1}} - 1)$

$= (2^{2^{n-1}} + 1)(2^{2^{n-2}} - 1)(2^{2^{n-2}} + 1) = (2^{2^m} + 1)(2^{2^m} - 1)(2^{2^{m-n-1}} + 1)$

$= \lambda\, F_m$; Now, $F_n - \lambda\, F_m$=2

Let d | F_n and d |F_m then d|2. Then d= 1 or 2. But F_m & F_n are both odd, hence gcd = 1.

6. **The number of real roots of the equation $\mathbf{1+ \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^7}{7} = 0}$**

(without factorial) is

(a) 7 **(b) 5**
(c) 3 **(d) 1**

Ans:- (d) let f has a minimum at x = x_0 , where then f'(x_0) = 0

f (x)= $1+ \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^6}{6}$;

$\Rightarrow 1+ x_0 + {x_0}^2 + {x_0}^3 + {x_0}^4 + {x_0}^5 = 0$

$\Rightarrow \frac{{x_0}^6 - 1}{x_0 - 1} = 0$

$\Rightarrow \frac{({x_0}^3 - 1)({x_0}^3 + 1)}{x_0 - 1} = 0$

$\Rightarrow ({x_0}^2 + x_0 + 1)({x_0}^2 - x_0 + 1)(x_0 + 1) = 0$

Which has a real root x_0 = -1

But, f (-1)= 1-1 + ($\frac{1}{2}-\frac{1}{3}$)+ ($\frac{1}{4}-\frac{1}{5}$) +$\frac{1}{6}$ > 0

The f (x)> 0 and hence f has no real zeros.

Now let, g (x)= 1+ $\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^7}{7}$

An odd degree polynomial has at least one real root.

If our polynomial g has more than one zero, say x_1, x_2

Then by Role's theorem in (x_1, x_2) we have 'x_3' such that g′ (x_3) = 0

⇒ 1+ $x_3 + {x_3}^2 + \cdots + {x_3}^6 = 0$

But this has no real zeros. Hence the given polynomial has exactly one real zero.

7. Number of roots between –π and π of the equation $\frac{2}{3}\sin x . x$= 1 is
(a) 1 (b) 2 (c) 3 (d) 4

Ans:- (d) $\sin x = \frac{3}{2x}$

Now, draw the curve of y= $\sin x$ and y = $\frac{3}{2x}$ or xy = $^3/_2$

∴ there are 4 real roots. (Draw the graph yourself)

8. The number a^{7389b}, a, b are digits, is divisible by 72 , Then a+ b equals
(a) 10 (b) 9 (c) 11 (d) 12

Ans:- (b) 72 = 8× 9, and 8 and 9 are co prime. As the number a^{7389b} is divisible by 72, it is divisible by 9 and 8 both. For divisibility by 8, the last three digits must be divisible by 8.

i.e. 800+ 90+b |8.

so (b+2)| 8 ; ∴ b = 6

For divisibility by 9, the sum of the digits A + 7 +3 +8+ 9 +b should be divisible by 9.

i.e. a+ 7+ 3+ 8+ 9+ 6≡ 0 (mod 9)

⇒ a+ 6 ≡ 0 (mod 9)

⇒ a ≡ -6(mod 9)

⇒ a ≡ 3 (mod 9)

∴ a= 3 only. Hence a+ b= 9

9. 3 balls are distributed to 3 boxes at random. Number of way in which we set at most 1 box empty is (a) 20 (b) 6 (c) 24 (d) none

Ans:- (c) zero box empty + 1 box empty

= 3 balls in 3 boxes + {$^3C_1$ × 3 balls in 2 boxes}

= 3! + 3× $(3)_2 = 24$.

10. The value of $\max\limits_a I(a)$, *where* $I(a) = \int_{a-1}^{a+1} e^{-|x|}dx$ is
(a) e^a (b) 2- $e^{a-1} - e^{a+1}$ (c) $e^{a-1} - e^{a+1}$ (d) none

Ans:- (b) $I(a) = \int_{a-1}^{a} e^{-|x|}dx + \int_{a}^{a+1} e^{-|x|}dx$

Let 0< a< 1 , then $\int_{a-1}^{a} e^{-|x|}dx = \int_{a-1}^{0} e^{x}dx + \int_{0}^{a-1} e^{-x}dx$

= 1- $e^{a-1} - (e^{-a} - 1)$

And, $\int_a^{a+1} e^{-|x|}dx = e^{-a} - e^{-a-1}$

$\therefore$ I (a) = 2 - ($e^{a-1} + e^{-a-1}$)

$\therefore \frac{d}{dx} I\,(a) = 0$

$\Rightarrow e^{a-1} = e^{-a+1}$

$\Rightarrow$ a= 0

Also, -1 < a < 0 ; $\int_{a-1}^{a} e^x dx = e^a - e^{a-1}$

And $\int_a^{a+1} e^{-|x|}dx = \int_a^0 e^x dx + \int_0^{a+1} e^{-x}dx$

= (1- e^a) – ($e^{a+1} - 1$)

= 2 -$e^a - e^{a+1}$

$\therefore$ I (a) = 2- $e^{a-1} - e^{a+1}$

$\therefore$ I(a) is maximum at a = 0.

11. The value of $\prod_{n=2}^{\infty}(1 - \frac{1}{n^2})$ is

(a) 1 **(b) 0** **(c) ½** **(d) none**

Ans:- (c) $\prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right) = \prod_{n=2}^{\infty}\left(1 + \frac{1}{n}\right)(1 - \frac{1}{n}) = \prod_{n=2}^{\infty}\frac{n+1}{n} . \frac{n-1}{n}$

= $(\frac{2+1}{2} . \frac{2-1}{2})(\frac{3+1}{3} . \frac{3-1}{3})$……… = $\frac{3}{2}.\frac{1}{2}.\frac{4}{3}.\frac{2}{3}$……. = $\frac{1}{2}$.

12. $cos^8\theta - sin^8\theta$= 1. Number of roots are there in between [0, 2π] is

(a) 1 **(b) 2** **(c) 3** **(d) 0**

Ans:- (c) Note that , $cos^8\theta = 1 + sin^8\theta$ is possible only if,

$cos^8\theta = 1 \; and \; sin^8\theta$ =0

$\therefore \theta$= 0, π, 2π

Hence 3 roots are three between [0, 2π]

13. If $u_n = \frac{1}{1.n} + \frac{1}{2(n-1)} + \frac{1}{3\,(n-2)} + \ldots\ldots + \frac{1}{(n-1)}$; Then $\lim_{n\to\infty} u_n$ equals

(a) 0 **(b) 1** **(c) ∞** **(d) π**

Ans:- (a) $u_n = \frac{1}{(n+1)}\left[\left(1 + \frac{1}{n}\right) + \left(\frac{1}{2} + \frac{1}{n-1}\right) + \left(\frac{1}{3} + \frac{1}{n-2}\right) + \cdots + (\frac{1}{n} + 1)\right]$

$$= \frac{1}{(n+1)}\, 2\,(1 + \frac{1}{2} + \cdots + \frac{1}{n})$$

$\therefore \lim_{n\to\infty} u_n = 2 \lim_{n\to\infty} \frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{n} . \lim_{n\to\infty}\frac{n}{n+1}$ = 2. 0.1= 0.

14. If x+ $\left(\frac{1}{x}\right)$ = -1, The value of $x^{99} + \left(\frac{1}{x^{99}}\right)$ is

(a) 1 **(b) 2** **(c) 0** **(d) none**

Ans:- (b) If $a_n = x^n + \frac{1}{x^n}$

Then, $a_{n+1} = a_n . a_1 - a_{n-1} \; for \; n \geq 1$

$a_0 = 2\,, a_2 = -a_1 - a_0 = -1, a_3 = 2, a_4 = -1, a_5 = 2, a_1 = -1\,;\; a_n = -1, a_{n-1} = -1.$

$$so\,, a_{99} = 2$$

Or , $\therefore a_{n+1} = x^{99} + \frac{1}{x^{99}} = \left(x^{98} + \frac{1}{x^{98}}\right)\left(x + \frac{1}{x}\right) - (x^{97} + \frac{1}{x^{97}})$

= - $a_n - a_{n-1} = +1 + 1 = 2$

15. Consider the equation of the form $x^2 + bx + c = 0$. The number of such, equations that have real roots and

have coefficients b and c in the set {1, 2, 3, 4, 5,6} , (b may be equal to c) is
(a) 16 **(b) 19**
(c) 21 **(d) none**

Ans:- (b) Let $x^2 + bx + c = 0$ has real roots, then $b^2 - 4c \geq 0$, and also , s= {1, 2, 3, 4, 5, 6}.

Now $s_1 = \{4, 8, 12, 16, 20, 24\}$= set of possible values of 4c.

Thus the number of equations will be same as the number of pairs of elements (a_1, a_2),

$a_1 \in$ s, $a_2 \in s_1$ such that

${a_1}^2 - 4a_2 \geq 0$, i.e. 1+ 2+ 4+ 6+6 = 19

16. If f: R ⟶R, satisfies f(x +y)= f(x)+f(y)∀ $\boldsymbol{x, y} \in \mathbb{R}$ and f(1)=7, then$\sum_{r=1}^{n} \boldsymbol{f(r)}$ is
(a) $\frac{7\,(n+1)}{2}$ **(b) 7n (n+1)**
(c) $\frac{7\,n(n+1)}{2}$ **(d)none**

Ans:- (c) putting x= 1, y=0, then f(1)= f(1)+f(0)

⇒f(0)=0, ⇒ f(1)=7

Again , putting x=1, y=1, then f(2)= 2f(1)= 14, similarly,

f(3)=21 and so on.

$\sum_{r=1}^{n} f(r)$= 7 {1+ 2+ 3+…..+ n}= $\frac{7\,n(n+1)}{2}$.

17. Let f(0)= 1, $\lim\limits_{x\to\infty} \boldsymbol{f''(x) = 4 \text{ and } f(x) \geq f(1)}$. Let f(x) is polynomial ∀ x∈ℝ. The value of f(2) is
(a) 4 **(b) 0**
(c) 1 **(d) none**

Ans:- (c) $f''(x) = 4 = constant$

⇒ f(x)= $2x^2 + ax + b$

f(0)= 1 ⇒ b= 1

f(1)= 3+ a

f(x) ≥f(1) ⇒ f′(1)=0

⇒ 4+ a= 0

⇒ a = -4

∴ f(x)= $2x^2 + 4x + 1$

∴ f(2)= 1.

18. Let $\frac{1-3P}{2}, \frac{1+4P}{3}, \frac{1+P}{6}$ are the probabilities of 3 mutually exclusive and exhaustive events, then the set of all values of P is
(a) [-1/4, 1/3] **(b) (0, 1)**
(c) (0, ∞) **(d) none**

Ans:- (a) $\frac{1-3P}{2} \geq 0, \frac{1+4P}{3} \geq 0, \frac{1+P}{6} \geq 0$ and

$$\frac{1-3P}{2} + \frac{1+4P}{3} + \frac{1+P}{6} = 1$$

$\Rightarrow -\frac{1}{4} \leq P \leq \frac{1}{3} \Rightarrow P \in [-\frac{1}{4}, \frac{1}{3}]$

19. If $\sqrt{5x - x^2 - 6} + \frac{\pi}{2}\int_0^x dt >$ $x\int_0^{\pi} \sin^2 t dt$, then x ∈
(a) (2, 3) **(b) (-∞, 2) ∪ (3, ∞)**
(c) (5/2 , 3) **(d) none**

Ans:- (a) $\sqrt{5x - x^2 - 6} + \frac{\pi x}{2} >$ $x\,\{\frac{1}{2}\int_0^{\pi}(1 - cos2t)dt\}$

$\Rightarrow \sqrt{5x - x^2 - 6} + \frac{\pi x}{2} > x\,\{\frac{1}{2}(t - \frac{1}{2}\sin 2t)\,_0^{\pi}\}$

$\Rightarrow \sqrt{5x - x^2 - 6} + \frac{\pi x}{2} > \frac{\pi x}{2}$

$\Rightarrow \sqrt{5x - x^2 - 6} > 0$

$\Rightarrow x^2 - 5x + 6 < 0,$

$\Rightarrow$ (x-2)(x-3)< 0, i.e. , x ∈ (2, 3).

20. If f(x)= $(4+x)^n$, n ∈ N and $f^r(0)$ represents the r^{th} derivative of f(x) at x= o, then the value of $\sum_{r=0}^{\infty} \frac{f^r(0)}{r!} =$

(a) 2^n (b) e^n (c) 5^n (d) none

Ans:- (c) f′(x)= n $(4+x)^{n-1}$

f″(x)= n (n-1) $(4+x)^{n-2}$

$f^r(x)$= n (n-1)….. (n- r+1) . $(4+x)^{n-r}$, r $\leq n$

$f^r(0) = \frac{n!}{(n-r)!} . 4^{n-r}$, r $\leq n$

= 0, r $>n$

$\therefore \sum_{r=0}^{\infty} \frac{f^r(0)}{r!} = \sum_{r=0}^{n} \binom{n}{r} . 4^{n-r} = (1+4)^n = 5^n$.

21. The two lines $\vec{r} = \vec{a} + \lambda(\vec{b} + \vec{c})$ and $\vec{r} = \vec{b} + \mu(\vec{c} + \vec{a})$ intersects at a point, where λ and μ are scalars, then

(a) $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar
(b) $|\vec{a}| = |\vec{b}| = |\vec{c}|$

(c) $\vec{a}.\vec{c} = \vec{b}.\vec{c}$
(d) $\lambda(\vec{b}\times\vec{c}) + \mu(\vec{c}\times\vec{a}) = \vec{c}$

Ans. (c)

The two lines intersect

$\therefore \vec{a} + \lambda(\vec{b} \times \vec{c}) = \vec{b} + \mu(\vec{c} \times \vec{a})$

Taking dot product with $\vec{c}$ on both sides, we get

$\vec{a} . \vec{c} = \vec{b} . \vec{c}$

22. Let f(x)=

$$\begin{cases} \boldsymbol{x|x|}; & \boldsymbol{x \leq -1} \\ [\boldsymbol{x+1}] + [\boldsymbol{1-x}]; & \boldsymbol{-1} < x < 1 \\ \boldsymbol{-x\,|x|}; & \boldsymbol{x \geq 1} \end{cases}$$

Then the value of $\int_{-2}^{2} f(x)dx$ is

(a) - $\frac{8}{3}$ (b) - $\frac{7}{3}$ (c) $\frac{7}{3}$ (d) none

Ans:- (a) f(x) = $\begin{cases} -x^2, & x \leq -1 \\ 1, & -1 < x < 0 \\ 2, & x = 0 \\ 1, & 0 < x < 1 \\ x^2, & x \geq 1 \end{cases}$

$\therefore$ f(x) is an even function, i.e. $\int_{-2}^{2} f(x)dx = 2\int_{0}^{2} f(x)dx$

$= 2\,\{\int_0^1 f(x)dx + \int_1^2 f(x)dx\}$

$= 2\,(1 - \frac{x^3}{3})_1^2 = -\frac{8}{3}$.

23. Area bounded by y = g(x), x-axis and the lines x= - 2,

Where g (x)=

$$\begin{cases} \textbf{max}\{\boldsymbol{f(t)}: \boldsymbol{-2} \leq \boldsymbol{t} \leq \boldsymbol{x}\}, & \boldsymbol{where -2 \leq x} < 0; \\ \textbf{min}\{\boldsymbol{f(t)}: \boldsymbol{0} \leq \boldsymbol{t} \leq \boldsymbol{x}\}, & \boldsymbol{where\ 0 \leq x \leq 3} \end{cases}$$

And f(x)= $x^2 - |x|$, is equal to

(a) $\frac{113}{24}$ (b) $\frac{111}{24}$ (c) $\frac{117}{24}$ (d) none

Ans:- (a) g(x)=

$$\begin{cases} 2\ ; & -2 \le x < 0 \\ x^2 - x\ ; & 0 \le x \le \frac{1}{2} \\ -\frac{1}{4}\ ; & \frac{1}{2} < x \le 3 \end{cases}$$

$\therefore$ Required area = $\int_{-2}^{0} 2dx + \int_{0}^{\frac{1}{2}}(x - x^2)dx + \int_{\frac{1}{2}}^{3}\left(\frac{1}{4}\right)dx = \frac{113}{24} unit^2$

24. Total number of positive integral values of n such that the equations $\cos^{-1} x + (\sin^{-1} y)^2 = \frac{n\pi^2}{4}$ and $(\sin^{-1} y)^2 - \cos^{-1} x = \frac{\pi^2}{16}$ are constant, is equal to

(a) 1 (b) 2 (c) 3 (d) none

Ans:- (a) Here 2 $(\sin^{-1} y)^2 = \frac{4n+1}{16}\pi^2$

$\Rightarrow 0 \le \frac{4n+1}{16}\pi^2 \le \frac{\pi^2}{4}$,

$\Rightarrow -\frac{1}{4} \le n \le \frac{7}{4}$.

Also. $2(\cos^{-1} x) = \frac{4n-1}{16}\pi^2$

$\Rightarrow 0 \le \frac{4n-1}{16}\pi^2 \le \pi$,

$\Rightarrow \frac{1}{4} \le n \le \frac{8}{\pi} + 1$.

Hence, the least positive integral value of n is 1.

25. Radius of bigger circle touching the circle $x^2 + y^2 - 4x - 4y + 4 = 0$ and both the co-ordinate axis is

(a) $3+2\sqrt{2}$ (b) $2(3+2\sqrt{2})$ (c) $3- 2\sqrt{2}$ (d) none

Ans:- (b) Let (h, h) be the centre of the required circle.

$\therefore \angle COD = \angle CBE = \frac{\pi}{4}$, CB= h+ 2 AND BD= h- 2.

$\therefore \frac{h-2}{h+2} = cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$,

$\Rightarrow$ h= $\frac{2(\sqrt{2}+1)}{(\sqrt{2}-1)} = 2(3 + 2\sqrt{2})$.

26. Tangents and normal drawn to parabola at A $(at^2, 2at), t \ne 0$ meet the X- axis at point B and D, respectively. If the rectangle ABCD is

(a) y-2a= 0 (b) y+ 2a= 0 (c) x-2a= 0 (d) none

Ans:- (c) Evolution of tangent & normal at A are $y^t = X + at^2, y = -tx + 2at + at^3$.

$\therefore$ B = $(-at^2, 0)$ and D= $(2a + at^2, 0)$

Suppose ABCD is rectangle,

Then midpoints of BD and AC will be coincident,

$\therefore$ h+$at^2 = 2a + at^2 - at^2$ $and\ k + 2at = 0$

i.e. h= 2a, k= -2at.

Hence, the locus is X= 2a, i.e. X-2a=0.

27. The series $\sum_{k=2}^{\infty}\left(\frac{1}{k(k-1)}\right)$ converges to

(a) -1 (b) 1 (c) 0 (d) does not converges

Ans:- (b) $s_n = \sum_{k=2}^{\infty}\frac{1}{k(k-1)} = \sum_{k=2}^{n}(\frac{1}{(k-1)} - \frac{1}{k}) = \left(1 - \frac{1}{n}\right)$

$\therefore \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1-\frac{1}{n}\right) = 1.$

28. The limit $\lim_{x\to\infty}\left(\frac{3x-1}{3x+1}\right)^{4x}$ equaqls

(a) 1 **(b) 0** **(c) $e^{-\frac{8}{3}}$** **(d) $e^{\frac{4}{9}}$**

Ans:- (c) $\lim_{x\to\infty}\{(\frac{1-\frac{1}{3x}}{1+\frac{1}{3x}})^x\}^4 = (\frac{e^{-\frac{1}{3}}}{e^{\frac{1}{3}}})^4 = e^{-\frac{8}{3}}$

[since $\lim_{x\to\infty}(1+\frac{k}{x})^x = e^k$].

29. $\lim_{x\to\infty}\frac{1}{n}\left(\frac{n}{n+1}+\frac{n}{n+2}+\cdots+\frac{n}{2n}\right)$ equals

(a) ∞ **(b) 0** **(c) $\log_e 2$** **(d) 1**

Ans:- (c) $\lim_{x\to\infty}\frac{1}{n}(\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\cdots+\frac{1}{1+\frac{n}{n}})$

$= \int_0^1 \frac{1}{1+x}dx = [\log(1+x)]_0^1 = \log_e 2.$

30. Let k be an integer greater than 1. Then $\lim_{n\to\infty}\left[\frac{n}{n+1}+\frac{n}{n+2}+\cdots\right]$ is

(a) $\log_e k$ **(b) (k-1) $\log_e k$**
(c) 0 **(d) ∞**

Ans:- (a) $\lim_{n\to\infty}\left[\sum_{r=1}^{n(k-1)}\frac{1}{n+r}\right] = \int_0^{k-1}\frac{dx}{1+x} = \log(1+x)]_0^{k-1} = \log_e k$.

ISI B.STAT/B.MATH OBJECTIVE QUESTIONS & SOLUTIONS

SET – 3

1. Number of solutions are possible in $0\le x \le 99$ for the equation

$|3-3^x| + |1-3^x| = 1-3^x-\frac{3^{-x}}{4}$ is

(a) 1 **(b) 0** **(c) 2** **(d) none**

Ans:- (b) LHS= $|3-3^x| + |3^x-1| \ge |(3-3^x)+(3^x-1)| \ge 2$

But RHS= 1 - $(3^x + \frac{3^{-x}}{4})$

=1- $\{(3^{\frac{x}{2}} + \frac{3^{-\frac{x}{2}}}{2})$ - 2. $3^{\frac{x}{2}}.\frac{3^{-\frac{x}{2}}}{2}$ }

= 2 - $(3^{\frac{x}{2}} + \frac{3^{-\frac{x}{2}}}{2})^2 < 2$

$\therefore$ given equation has no solution for any real x.

2. If f(x) = $\log_e(6-|x^2+x-6|)$, then domain of f(x) has how many integral values of x?

(a) 5 **(b) 4** **(c) infinite** **(d) none of these**

Ans:- (b) f(x) is defined only when $6 - |x^2+x-6| > 0$

i.e. $|x^2+x-6| < 6$

$\Rightarrow$ -6 < $x^2+x-6 < 6$

$\Rightarrow x^2 + x > 0$ and $x^2 + x - 12 < 0$

$\Rightarrow$ x (x+1)>0 and (x+4)(x-3)< 0

$\Rightarrow$ (x< -1 or x > 0) and (-4 < x < 3)

$\Rightarrow x \in (-4, -1) \cup (0, 3) \Rightarrow$ x= -3, -2, 1, 2 as integral values.

3. The sum of the real solution of $2|x|^2+ 51= |1+ 20x|$ is

(a) 5 **(b) 0**

(c) 24 **(d) none of these**

Ans:- (d) $2x^2 + 51 = \pm(1 + 20x)$

$\Rightarrow x^2 - 10x + 25 = 0$ or $x^2 + 10x + 26 = 0$

$\Rightarrow (x - 5)^2 = 0$ $or (x + 5)^2 + 1 = 0$(impossible)

$\Rightarrow$ x= 5, 5

$\therefore$ Sum of the real solution = 5+ 5= 10.

4. The solution set of $||x- 1|-1| + x \leq 2$ is

(a) $(-\infty, 2]$ **(b) [0, 1)**

(c) [0, 2) **(d) [1, 2)**

Ans:- (a) (i) If x < 0, then $|1- x- 1|+x \leq 2$

$\Rightarrow |x|+x \leq 2$

$\Rightarrow -x +x \leq 2$

$\Rightarrow 0 \leq 2$ (true)

$\therefore$ x < 0

(ii) If $0 \leq x \leq 1$, then $|1- x- 1|+x \leq 2$

$\Rightarrow |x|+x \leq 2$

$\Rightarrow 2x \leq 2$

$\Rightarrow x \leq 1, \therefore 0 \leq x < 1$

(iii) If $1 \leq x < 2$, then $|x-1 -1|+x \leq 2$

$\Rightarrow |x-2|+x \leq 2$

$\Rightarrow 2- x +x \leq 2$

$\Rightarrow 2 \leq 2$ (true)

(iv) If $x \geq 2$, then $|x-1 -1|+x \leq 2$

$\Rightarrow x-2 +x \leq 2$

$\Rightarrow x \leq 2$

$\therefore$ x=2 [$\because x\geq 2$]

$\therefore$ Required solution set is $(-\infty, 2]$

5. If domain of f(x)= $\sqrt{\frac{1}{|x-1|+[x]}}$ be (a, b), then ([.] denotes greatest integer function)

(a) a= 1, b=∞ **(b) a= -∞, b= 0**

(c) a= -∞, b= 1 **(d) none of these**

Ans:- (c) we must have, |x-1|> [x].......(1)

$\therefore$ x-1 < [x]$\leq$ x, i.e. [x]> x – 1..........(2)

$\therefore$ on combining (1) and (2), we have |x -1|> x -1

This is true only if x-1< 0, i.e. if x < 1, i.e. if $x \in (-\infty, 1)$

$\therefore D_f = (-\infty, 1) \Rightarrow$ a = $-\infty$, b= 1

6. If there are 4 distinct solutions of $||x - 2012| + \log_2 a| = 3$, then a $\in$

(a) $(-\infty, -6)$ **(b) $(-\infty, \frac{1}{8})$**

(c) $(-\infty, -\frac{1}{8})$ **(d) none of these**

Ans:- (b) we have $| x -2012|+\log_2 a = \pm 3$

$\Rightarrow | x -2012| = -\log_2 a + 3, -\log_2 a - 3$

$\therefore$ If there are 4 distinct solutions of the above equation, then we must have

$-\log_2 a + 3 > 0$ and $-\log_2 a - 3 > 0$

i.e. $\log_2 a < 3$ and $\log_2 a < -3 \Rightarrow a < 2^{-3}$

$\therefore a \in (-\infty, \frac{1}{8})$

7. The number of value of k for which the equation $x^3 - 3x + k = 0$ has two distinct roots lying in the interval (0, 1) are

(a) 3 (b) 2 (c) infinitely many (d) no value of k satisfies the requirement

Ans:- (d) Let there be a value of k for which $x^3 - 3x + k = 0$ has two distinct roots between 0 and 1. Let, a, b are two distinct roots of $x^3 - 3x + k = 0$ lying between 0 and 1 such that a < b

Let f (a)= f(b)= 0. Since between any two roots of a polynomial f(x) there exist at least one roots of its derivative f′(x).

Therefore, f′(x) = $3x^3 - 3$ has at least one root between a and b

But f′(x) =0 has two roots equal to $\pm$ 1 which don't lie between 0 and 1 for any value of k.

8. If $\frac{dy}{dx} = f(x) + \int_0^1 f(x)dx$ then the equation of the curve y=f(x) passing through (0, 1) is

(a) f(x)= $\frac{2e^x - e + 1}{3 - e}$ (b) f(x)= $\frac{3e^x - 2e + 1}{2(x - e)}$ (c) f(x)= $\frac{e^x - 2e + 1}{e + 1}$ (d) none of these

Ans:- (a) f″(x)= f′(x)

$\Rightarrow \frac{f''(x)}{f'(x)} = 1$

On integrating f′(x) = c e^x

Which gives f(x)= c e^x+D

But f(0) = 1 ⇒ c+ D = 1

$\therefore$ f(x)= c $e^x + 1 - c$

So, f′(x)= c e^x putting it in f′(x)= f(x)+ $\int_0^1 f(x)dx$

$\Rightarrow$ c e^x= c $e^x + 1 - c + \int_0^1 (c\, e^x + 1 - c)dx$

$\Rightarrow$ c = $\frac{2}{3-e}$. So, f(x)= $\frac{2e^x - e + 1}{3 - e}$

9. A staircase has 10 steps, a person can go up the steps one at a time, or any combination of 1's and 2's . The number of ways in which the person can go up the stairs is

(a) 89 (b) 144 (c) 132 (d) 211

Ans:- (a)

x+ 2y= 10, where x is the number of times he takes single steps, and y is the number times he takes two steps

	Case	Total no. of ways
1	X=0, y=5	5! /5!=1
2	X=2, y= 4	6! /2! 4! =15
3	X=4, y=3	7! /3!4!= 35
4	X=6, y=2	8! /2! 6!= 28
5	X=8, y=1	9! /8! = 9
6	X=10, y=0	10! /10! =1

$\therefore$ P= 89

10. The remainder when $1690^{2608} + 2608^{1690}$ is divided by 7 is

(a) 1 **(b) 2** **(c) 3** **(d) none**

Ans:- (a) 1690= 7$\times$ 241 + 3;
2608= 7 $\times$372 +4

Let s =$1690^{2608} + 2608^{1690}$

= $(7 \times 241 + 3)^{2608} + (7 \times 372 + 4)^{1690}$

= a number multiple of 7+$3^{2608} + 4^{1690}$

Let s′= $3^{2608} + 4^{1690}$

Clearly remainder in s and s′ will be same when divided by 7.

s′= $3\times 3^{3\times867} + 4 \times 4^{3\times563}$

= $3\times 27^{867} + 4 \times 64^{563}$

= $3(28 - 1)^{867} + 4\,(63 + 1)^{563}$

= 3[multiple of 7- 1]+ 4[multiple of 7+ 1]

= multiple of 7+ 1

∴ Hence remainder is 1.

11. The value of $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\underset{(i\neq j\neq k)}{\frac{1}{3^i3^j3^k}}$ is

(a) 80/207 **(b) 81/208** **(c) 1/208** **(d) none**

Ans:- (b) Let us first of all find the sum without any restriction i, j, k.

$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i3^j3^k} = \left(\sum_{i=0}^{\infty}\frac{1}{3^i}\right)^3 = \frac{27}{8}$

For the requirement sum we have to remove the cases when i= j= k or when any two of them are equal and not equal to other variable (say, i= j$\neq k$).

Case –I:- when i= j= k

In this case $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i3^j3^k} = \sum_{i=0}^{\infty}\frac{1}{3^i} = \frac{27}{16}$

Case – II:- i= j$\neq k$

In this case, $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i3^j3^k} = \left(\sum_{i=0}^{\infty}\frac{1}{3^{2i}}\right)\left(\sum_{k=0}^{\infty}\frac{1}{3^k}\right)$

= $\sum_{i=0}^{\infty}\frac{1}{3^{2i}}\left(\frac{3}{2} - \frac{1}{3^i}\right)$

= $\frac{3}{2}.\frac{9}{8} - \frac{27}{26} = \frac{135}{8.26}$

Hence required sum =$\frac{27}{8} - \frac{27}{26} - \left(\frac{135}{8.26}\right).3 = \frac{81}{208}$

12. The solution of the differential equation f(x) $\frac{dy}{dx} + f'(x)y = 1$ is given by f(x) =

(a) yx + c **(b) $\frac{x+c}{y}$** **(c) yc** **(d) none**

Ans:- (b) f(x)dy +f′(x)ydx = dx

i.e. d (f(x), y) = d (x)

Integrating we get, y. f(x) = x+ c

or, f(x) = $\frac{x+c}{y}$

13. If $\int_0^x f(x)sintdt$ = constant, 0 < x < 2π and f(π)= 2 Then find the value of f$\left(\frac{\pi}{2}\right)$

(a) 2 **(b) 4** **(c) 6** **(d) 8**

Ans:- (b) Differentiable both sides, we get

f′(x) (1- cosx)+ f(x)sinx= 0

$\Rightarrow \int \frac{f'(x)}{f(x)} dx = \int \frac{sinx}{1-cosx} dx$

$\Rightarrow$ ln(f (x)| = -2ln sin$\frac{x}{2} + lnc$

$\Rightarrow$ f (x)= $\frac{c}{(\sin\frac{x}{2})^2}$ $\Rightarrow$ f(π)=2 $\Rightarrow$ c=2:   f($\frac{\pi}{2}$)=4

14. For a ∊ R if |x+ a-3| + |x- 2a|= |2x –a - 3| is three for all x ∊R, then exhaustive set of a is

(a) a ∊ [-4, 4] (b) a ∊ [-3, 2] (c) a ∊ {-2, 2} (d) a ∊ {1}

Ans:- (d) |x|+ |y| = |x+y|

$\Rightarrow$ xy ≥ 0, therefore (x- (3- a)) (x- 2a)≥ 0 $\forall\, x \in R$

$\Rightarrow x^2 - x(3+a) + 2a(3-a) \geq 0\ \forall\, x \in R$

$\Rightarrow (a+3)^2 - 8a(3-a) \leq 0 \Rightarrow$
$(a-1)^2 \leq 0 \Rightarrow a = 1$ which is true $\forall\, x \in R$

15. If A is skew–symmetric matrix, then B = (I- A) $(\boldsymbol{I}+\boldsymbol{A})^{-1}$ is (where I is the identity matrix of the same order as A)

(a) idempotent matrix (b) symmetric matrix (c)orthogonal matrix (d) none

Ans:- (c) B= (I- A) $(I+A)^{-1}$

$\Rightarrow B^T = (I+A^T)^{-1}(I+A^T)$ = $(I - A)^{-1}$(I+ A)

$B\ B^t$= I as (I- A) (I+ A)= (I+ A) (I-A)

16. If f(x)= max ($\frac{1}{\pi}\cos^{-1}(\boldsymbol{cos\pi x}), \{\boldsymbol{x}\}$) and g (x) min ($\frac{1}{\pi}\cos^{-1}(\boldsymbol{cos\pi x})$, $\{\boldsymbol{x}\}$) (where { .} represents fractional part of x). Then find the value of $\int_1^2 \boldsymbol{f(x)dx}/$ $\int_1^2 \boldsymbol{g(x)dx}$ is

(a) 1 (b) 3 (c) 5 (d) 7

Ans:- (b)

$\Rightarrow \int_1^2 f(x)dx = \frac{3}{4}\ \ and\ \ \int_1^2 g(x)dx = \frac{1}{4}$

$\Rightarrow$ Ratio = 3

17. If sin (sinx +cosx)= cos (cosx- sinx) and largest possible value of sinx is $\frac{\pi}{k}$, then the value of k is

(a) 2 (b) 3 (c) 4 (d) none

Ans:- (c) sin (sinx +cosx)= cos (cosx- sinx)

cos (cosx- sinx) = cos ($\frac{\pi}{2} - (sinx + cosx)$)

$\therefore cosx - sinx = 2n\pi \pm (\frac{\pi}{2} - (sinx + cosx))$

Taking + ve sign

$cosx - sinx = 2n\pi + \frac{\pi}{2} - sinx - cosx$

$cosx$= $n\pi + \frac{\pi}{4}$, for n= 0, $cosx = \frac{\pi}{4}$, which is the only possible value

$\Rightarrow sinx = \frac{\sqrt{16-\pi^2}}{4}$ ………………..(i)

Taking –ve sign

$sinx = \frac{\pi}{4}$ ……………..(ii)

From (i) & (ii) , we get $\frac{\pi}{4}$ as the largest value. Hence k= 4.

18. The number of solution(s) of the equation $z^2 - z - |z|^2 - \frac{64}{|z|^5} = 0$ is / are

(a) 0 (b) 1 (c) 2 (d) 3

Ans:- (b) z = 2 is the only solution.

So there is only one solution of the given equation.

19. If function f(x) = cos(nx)×sin($\frac{5x}{n}$), satisfies f(x+ 3π)= f(x), then find the number of integral value of n

(a) 8 (b) 9 (c) 10 (d) 11

Ans:- (a) f(x+ λ) = f(x)

$\Rightarrow$ cosn(x+ λ) sin$\left(\frac{5(x+\lambda)}{n}\right) = \cos(nx)\sin(\frac{5x}{n})$

At x = 0, cos(nλ)sin ($\frac{5\lambda}{n}$)= 0

If cos(nλ)= 0, nλ= rπ+$\frac{\pi}{2}$, r ∈ I

n (3π)= rπ+$\frac{\pi}{2}$ (∵λ= 3π)

(3n- r)= ½ [not possible]

∴ cosn$\lambda \neq 0$ ∴ sin ($\frac{5\lambda}{n}$)= 0 ⇒ $\frac{5\lambda}{n} = P\pi (P \in I) \Rightarrow n = \frac{15}{P}$

For P= $\pm 1, \pm 3, \pm 5, \pm 15$

n=$\pm 15, \pm 5, \pm 3, \pm 1$

20. Let a, b, c be any real numbers such that $a^2 + b^2 + c^2 = 1$ then the quantity ab +bc+ ca satisfies the conditions

(a) ab+ bc+ ac = constant

(b) - ½ ≤ $ab + bc + ca \le 1$

(c) - ¼ ≤ $ab + bc + ca \le 1$

(d) -1 ≤ $ab + bc + ca \le \frac{1}{2}$

Ans:- (a) $(a+b+c)^2 \geq 0$

$\Rightarrow a^2 + b^2 + c^2 \geq -2(ab + bc + ca)$

$\Rightarrow \frac{1}{2} \geq -(ab + bc + ca) \Rightarrow (ab + bc + ca) \geq -\frac{1}{2}$

21. The maximum value of xyz for +ve x, y, z subject to condition that xy + yz+ zx= 12 is

(a) 9 (b) 6 (c) 8 (d) none

Ans:- (c) $\frac{xy + yz + zx}{3} \geq (xy . yz. zx)^{1/3}$

⇒ (xyz)≤ 8

22. Let a, b, c are 3 positive real numbers such that a+ b+ c= 2, then the value of $\frac{a}{1-a}.\frac{b}{1-b}.\frac{c}{1-c}$ is always

(a) > 8 (b) < 8 (c) 8 (d) none

Ans:- (a) Let 1- a= x, 1- b= y, 1- c = z

3- (a+ b+ c) = x+ y+ z = 1(∵ a+ b+ c= 2)

Now, $\frac{1-x}{x}.\frac{1-y}{y}.\frac{1-z}{z}$

$= \frac{y+z}{x}.\frac{z+x}{y}.\frac{y+x}{z}$

$= \left(\frac{y+z}{2}\right)\left(\frac{z+x}{2}\right)\left(\frac{y+x}{2}\right).\frac{8}{xyz} >$ $\sqrt{yz}\,\sqrt{zx}\,\sqrt{zy}\,.\frac{8}{xyz}$ (By AM> GM inequality)

$\Rightarrow \frac{a}{1-a}.\frac{b}{1-b}.\frac{c}{1-c} > 8$

23. Let a+ b +c = 1 then the value of the quantity is always $\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1}$
(a) equals 21 **(b) ≤ 21**
(c) > 21 **(d) none**

Ans:- (b) 4a+ 4b+ 4c = 4

⇒(4a+ 1)+ (4b+ 1)+ (4c +1) = 7

Applying c-s inequality:- $a_1 = \sqrt{4a+1}$, $a_2 = \sqrt{4b+1}$, $a_3 = \sqrt{4c+1}$ & b_i =1

∴ $(\sum_{i=1}^{3} a_i\,.1)^2 \leq (\sum_{i=1}^{3} a_i^2)(\sum_{i=1}^{3} 1)$; where a_i=a_1, a_2, a_3

$\Rightarrow (\sqrt{4a+1}+\sqrt{4b+1}+\sqrt{4c+1})^2 \leq (4a+1+4b+1+4c+1)\times(1+1+1)$

$= 3 \times 7 = 21$

24. If f(x)is a polynomial function satisfying f(x)f$\left(\frac{1}{x}\right)$= f(x)+ f$\left(\frac{1}{x}\right)$ and f(3)=28 then f(4) is
(a) 28 **(b) 65**
(c) 78 **(d) none**

Ans:- (c) The given functional equation is satisfied by f(x)= $\pm\, x^n + 1$

f(3)= $+3^3 + 1 = 28$

Hence, n= 3

So, f(4)= $4^3 + 1$ = 65.

25. If 2x+ 4y= 1, then prove that the quantity $x^2 + y^2$ is always greater than equal to
(a) 1/20 **(b) 5/64**
(c) 1 **(d) none**

Ans:- (a) Maximize $x^2 + y^2$ subject to 2x+ 4y -1=0 by

Method of Lagrange multiplier⟶

F= $x^2 + y^2 + \lambda(2x + 4y - 1)$

$\frac{\partial F}{\partial x} = 2x + 2\lambda = 0$; $\frac{\partial F}{\partial y} = 2y + 2\lambda = 0$

∴ x= -λ ∴ y= -2λ

2x+ 4y = 1 $x_{max} = +\frac{1}{10}$, $y_{max} = \frac{1}{5}$

$\Rightarrow \lambda = \frac{-1}{10}$;

$\therefore x^2 + y^2 \geq \frac{1}{100} + \frac{4}{100} = \frac{5}{100} = \frac{1}{20}$.

26. If a, b, c are positive real numbers ∋ a+ b+ c= 1. The value of $a^2 + b^2 + c^2$ is always
(a) ½ **(b) 1/3**
(c) ¼ **(d) none**

Ans:- (b) Using C-S inequality,

$(\sum_{i=1}^{3} x_i y_i)^2 \leq \left(\sum x_i^2\right)\left(\sum y_i^2\right)$ Taking $y_i = 1$ & x_i = a, b, c.

$\Rightarrow (a+b+c)^2 \leq (a^2+b^2+c^2).3$

$\Rightarrow a^2 + b^2 + c^2 \geq \frac{1}{3}$

27. If a, b, c , x are real numbers such that abc≠ 0 *and* $\frac{xb+(1-x)c}{a} = \frac{xc+(1-x)a}{b} = \frac{xa+(1-x)b}{c}$

Then prove that a+ b+ c equals to

(a) 1 **(b) 2**
(c) 0 **(d) none**

Ans:- (c) $\frac{xb+(1-x)c}{a} = \frac{xc+(1-x)a}{b} = \frac{xa+(1-x)b}{c} = 1$

$\therefore$ x= $\frac{a-c}{b-c}, x = \frac{b-a}{c-a}, x = \frac{c-b}{a-b}$

The solutions are : a= b= c or a+ b+ c= 0.

28. If f : R ⟶R is given by f(x)= $\frac{4^x}{4^x+2}$ $\forall\, \boldsymbol{x} \in \boldsymbol{R}$, check f(x)+f(1-x)= 1.

Hence the value of f$\left(\frac{1}{1997}\right) + f\left(\frac{2}{1997}\right) + \cdots + f\left(\frac{1996}{1997}\right)$ is

(a) 998 **(b) 1996**
(c) 1997 **(d) none**

Ans:- (a) f(1- x)= $\frac{4^{1-x}}{4^{-x+1}+2} = \frac{4/4^x}{4/4^x+2}$ = f(x)+ f(1- x)= 1.

Now, putting x = $\frac{1}{1997}, \frac{2}{1997}, \frac{3}{1997}, \ldots, \frac{998}{1997}$

So, f$\left(\frac{1}{1997}\right) + f\left(\frac{2}{1997}\right) + \cdots + f\left(\frac{1996}{1997}\right)$

= $\underbrace{(1+1+\cdots+1)}_{\text{998 terms}}$

= 998

29. If gcd (a, b)=1, then gcd (a+b, a-b) is
(a) a or b **(b) 1 or 2**
(c) 1 or 3 **(d) none**

Ans:- (b) let d = gcd (a+ b, a- b)then d | (a+ b) and d |(a -b).

$\therefore$ d | (a +b+ a -b) , ⇒ d| 2a and

$\therefore$ d | (a +b- a +b) , ⇒ d| 2b

Thus d |(2a, 2b), ⇒ d|2(a,b)

Hence d= 1 or 2, because gcd(a, b)= 1

30. The number of solution (positive integers) of the equation 3x+ 5y = 1008 is
(a) 61 **(b) 67**
(c)79 **(d) none**

Ans:- (b) x, y ∈ ℕ, then 3 |5y ⇒ 3|y, y = 3k $\forall\, k \in \mathbb{N}$

Thus 3x + 15k = 1008

⇒ x + 5k= 336 ⇒ 5k ≤ 335 ⇒ k ≤ 67

ISI B.STAT/B.MATH OBJECTIVE QUESTIONS & SOLUTIONS

SET – 4

1. If S_n denotes the sum of first n terms of an A.P. whose

(a) P$\sum_{r=1}^{n} r$ **(b) n$\sum_{p=1}^{n} p$**
(c) a$\sum_{r=1}^{p} r$ **(d) none of these**

Ans. (d)

$\frac{S_{nx}}{S_x} = \frac{\frac{nx}{2}[2a+(nx-1)d]}{\frac{x}{2}[2a+(x-1)d]} = \frac{n[2a-1]+nxd}{(2a-d)+xd}$

For $\frac{S_{nx}}{S_x}$ to be independent of x

2a - d= 0

∴ 2a= d

Now, $S_p = \frac{p}{2}[2a + (p-1)d] = p^2 a$

2. if $a_{n=} \int_0^{\pi} \frac{\sin(2n-1)}{sinx}$ dx, then $a_1, a_2, a_3,$........are in

(a) A.P. and H.P.
(b) A.P. and G.P. but not in H.P.

(c) G.P. and H.P.
(d) A.P., G.P. and H.P.

Ans. (b)

$$a_{n+1} - a_n = \int_0^{\pi} \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin x} dx = \int_0^{\pi} \frac{2\cos 2nx . sinx}{sinx} dx = \left[\frac{2\sin 2nx}{2n}\right]_0^{\pi} = 0$$

$\therefore a_{n+1} = a_n \Rightarrow a_1 = a_2 = a_3 = \cdots$
Also $a_1 = \pi \neq 0$

Hence $a_1, a_2, ... a_n$ are in A.P. and G.P. but not in H.P. (Equal numbers cannot be in H.P)

3. If a, b, c are proper fractions and are in H.P. and x=$\sum_{n=1}^{\infty} a^n$, y=$\sum_{n=1}^{\infty} b^n$, z=$\sum_{n=1}^{\infty} c^n$,

then x, y, z are in

(a) A.P. (b) G.P. (c) H.P. (d) none of these

Ans. (c)

X = $\frac{a}{1-a} \Rightarrow a = \frac{x}{1-x}$

Similarly, b = $\frac{y}{1-y}, c = \frac{z}{1-z}$

Now, a, b, c are in H.P.

$\Rightarrow \frac{1+x}{x}, \frac{1+y}{y}, \frac{1+z}{z}$ are in A.P.

$\Rightarrow \frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P.

⟹ x, y, z are in H.P.

4. If a, b, c be the p^{th},q^{th} and r^{th} terms respectively of an A.P. and G.P. both, then the product of the roots of equation $a^b\, b^c c^a x^2 - abcx + a^c b^a c^b$=0 is equal to

(a) -1 (b) 1 (c) 2 (d) (b-c)(c-a)(a-b)

Ans. (b)

a= x+ (p- 1)d, b= x+ (q-1)d, c= x+ (r-1)d

a=$mn^{p-1}, b = mn^{q-1}, c = mn^{r-1}$

∴ Product of roots = $(mn^{p-1})^{(r-q)d}, (mn^{q-1})^{(p-r)d}, (mn^{r-1})^{(q-p)d} = m^0 . n^0$= 1.

5. If a, b, c, be the p^{th},q^{th} and r^{th} terms respectively of a G.P. then the equation-

$a^q b^r c^p x^2$ + pqrx + $a^r b^n c^q$ = 0 has

(a) both roots zero
(b) at least one root zero
(c) no root zero
(d) both roots unity

Ans. (c)

Product of roots= $a^{r-q}b^{p-r}c^{q-p} = 1 \neq 0$

$\Longrightarrow$ no root is equal to zero.

6. If $(r)_n$denotes the number rrr….. (n digits), where r=1, 2, 3,…9 and a=$(6)_n$, b=$(8)_n$, c=$(4)_{2n}$, then

(a) a^2+b+c=0
(b) a^2+b-c=0

(c) a^2+b-2c=0
(d) a^2+b-9c=0

Ans. (b)

A = $(6)_n = 6\,6\,6 \ldots 6(n\ digits) = 6 \times 1 + 6 \times 10 + 6 \times 10^2 + \cdots 6 \times 10^{n-1}$

$= \frac{6}{9}(10^n - 1) = \frac{2}{3}(10^n - 1)$

b=$\frac{8}{9}(10^n - 1), c = \frac{4}{9}(10^{2n} - 1)$

Now $a^2 + b$=$\frac{4}{9}(10^n - 1)^2 + \frac{8}{9}(10^n - 1) = \frac{4}{9}(10^n - 1)^2(10^n - 1 + 2) = \frac{4}{9}(10^{2n} - 1) = c$

7. Let a=1 1 1.....1(55 digits), b=1+10+10^2+…10^4, c=1+10^5+10^{10}+10^{15}+…+10^{50}, then

(a) a=b+c **(b) a=bc**
(c) b=ac **(d) c=ab**

Ans. (b)

a= 1+ 10+ $10^2 + \cdots + 10^{54} = \frac{10^{55}}{10-1} = \frac{10^{55}-1}{10^5-1}.\frac{10^5-1}{10-1} = bc$

8. If $\sum_{r=1}^{n} t_r = \sum_{k=1}^{n} \sum_{j=1}^{k} \sum_{i=1}^{j} 2$, then $\sum_{r=1}^{n} \frac{1}{t_r} =$

(a) $\frac{n+1}{n}$ **(b) $\frac{n}{n+1}$**
(c) $\frac{n-1}{n}$ **(d) $\frac{n}{n-1}$**

Ans. (b)

$\sum_{k=1}^{n} \sum_{j=1}^{h} \sum_{i=1}^{j} 2 = \sum_{k=1}^{n} \sum_{j=1}^{k} 2j$= $2\sum_{k=1}^{n} \sum_{j=1}^{k} j = 2\sum_{k=1}^{n} \frac{k(k+1)}{2} = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$

$=\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{3}$

$\therefore S_n = \frac{n(n+1)(n+2)}{3}$

$\Longrightarrow t_r = S_r - S_{r-1} = \frac{r(r+1)(r+2)}{3} - \frac{(r-1)r(r+1)}{3} = r(r+1)$

$\frac{1}{t_r} = \frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1} \quad \therefore \sum_{r=1}^{n} \frac{1}{t_r} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$

9. If a=$\sum_{r=1}^{\infty} \frac{1}{r^4}$, then $\sum_{r=1}^{\infty} \frac{1}{(2r-1)^4}$ =

(a) $\frac{16}{15}$a **(b) $\frac{a}{2}$**
(c) $\frac{15}{16}$a **(d) $\frac{14}{15}$a**

Ans. (c)

$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^4} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots + to\ \infty$

$= \left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + to\ \infty\right) - \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \cdots + to\ \infty\right)$

= a-$\frac{1}{2^4}\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots+to\ \infty\right)=a-\frac{a}{16}=\frac{15}{16}a$

10. If a_1, a_2, a_3,…are in G.P. having common ratio r such that $\sum_{k=1}^{n} a_{2k-1}$=$\sum_{k=1}^{n} a_{2k+2} \neq 0$, then number of possible values of r is

(a) 1 **(b) 2**
(c) 3 **(d) none of these**

Ans. (c)

Given $a_1+a_3+a_5+\cdots+a_{2n-1}=a^4+a^6+a^8+\cdots+a_{2n+2}$

= $r^3(a_1+a_3+a_5+\cdots+a_{2n-1})$

$\Rightarrow r^3=1 \Rightarrow r=1,\omega,\omega^2$

11. If x^2- x + a - 3= 0 has at least one negative value of x,then complete set of values of 'a' is

(a) $(-\infty, \mathbf{1})$ **(b)** $(-\infty, \mathbf{2})$
(c) $(-\infty, \mathbf{3})$ **(d) none**

Ans. (c) x^2- x +a - 3= 0 has at least one negative root and for real roots,

1- 4(a - 3) ≥ 0

⇒ a ≤ $\frac{13}{4}$

⇒ a ∈ $(-\infty, \frac{13}{4})$

Now, both root will be non-negative of D ≥ $0, \Rightarrow a-3 \geq 0 \Rightarrow a \geq 3$

∴ a ∈ $(3, \frac{13}{4})$

∴ a ∈ $(-\infty, \frac{13}{4})$∪ a ∈ $(3, \frac{13}{4})$

∴$(-\infty, 3)$

12. Let α, β are the roots of the equation x^2+ax +b=0, then maximum value of the expression - (x^2+ax +b) - $(\frac{\alpha-\beta}{2})^2$ will be

(a) $\frac{1}{4}(\boldsymbol{a^2-4b})$ **(b) 0**
(c) 1 **(d) none**

Ans. (b) let z= - (x^2-ax +b)

Now, $z_{max.}=-\frac{D}{4a}=-\frac{a^2-4b}{4}=\frac{4b-a^2}{4}=+(\frac{\alpha-\beta}{2})^2$

∴Thus the maximum value of the given equation is 0.

13. Let P (x) = x^2+bx +c, where b and c are integers and P(x) is a factor of

both x^4+6x^2+25 *and* $3x^4+4x^2+28x+5$, then P(1) is

(a) 4 **(b) 8**
(c) 24 **(d) none**

Ans. (a) ∴ P(x) is a factor of 3 (x^4+6x^2+25)- $(3x^4+4x^2+28x+5)$= $14(x^2-2x+5)$

∴ P(x) = x^2-2x+5

⇒ P(1)= 4.

14. The value of a for which $(a^2-1)x^2+2(a-1)x+2>0\ \forall\ x$ are

(a) a≥ 1 **(b) a≤ 1**
(c) a > - 3 **(d) none**

Ans. (d) we know, $Px^2+qx+c>0$ if P > 0, and $q^2-4Pc<0$,

∴ $(a^2-1)x^2+2(a-1)x+2>0\ \forall\ x$

Now, $a^2 - 1 > 0$ and 4 $(a-1)^2 - 8(a^2-1) \leq 0$

$\Rightarrow a^2 - 1 \geq 0$ and -4(a-1)(a+3) ≤ 0

$\Rightarrow$ a ≤ -1 or a ≥ 1 $and\ a \leq -3\ or\ a\ \geq 1$

i.e., $a \leq -3\ or\ a\ \geq 1$.

15. The sum of real roots of the equation $\boldsymbol{x^2 - 2^{2007}.x + |x - 2^{2006}| + 2(2^{4011} - 1) = 0}$ is

(a) $\boldsymbol{2^{2006}}$ **(b) $\boldsymbol{2^{2007}}$** **(c) $\boldsymbol{2^{2006}+2^{2007}}$** **(d) none**

Ans. (b) $\therefore (x - 2^{2006})^2 + |x - 2^{2006}| - 2 = 0$

$\Rightarrow |x - 2^{2006}|^2 + |x - 2^{2006}| - 2 = 0 \Rightarrow$ x= 2^{2006}+1, 2^{2006} -1.

$\therefore$The sum of real roots are =2^{2007}

16. Consider an expression $\boldsymbol{x^2 + y^2 + 2x + y}$= constant. If for two constants $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, the conditions x> $\boldsymbol{\alpha}$ and x > $\boldsymbol{\beta}$ imply the same limits for the value of y, then $\boldsymbol{\alpha + \beta}$ is

(a) -2 **(b) -4** **(c) 1** **(d) none**

Ans. (a) $x^2 + y^2 + 2x + y = k$

$\Rightarrow (x+1)^2 + (y + \frac{1}{2})^2 = k + \frac{5}{4}$

$\Rightarrow$ x= $-1 \pm \sqrt{(k + \frac{5}{4})^2 - (y + \frac{1}{2})^2}$

Now, the two values of x corresponds to α and β as y takes the same limits of values.

Hence $\alpha + \beta$= -2.

17. $\boldsymbol{\frac{a^4+b^4}{a^2+b^2} + \frac{b^4+c^4}{b^2+c^2} + \frac{c^4+a^4}{c^2+a^2} \geq}$

(a) a+ b+ c **(b) $\boldsymbol{a^2 + b^2 + c^2}$** **(c) ab+ bc+ ca** **(d) none**

Ans. (b) $(a^2 - b^2)^2 \geq 0$

$\Rightarrow a^4 + b^4 \geq 2a^2.b^2$

$\Rightarrow 2a^4 + 2b^4 \geq a^4 + b^4 + 2a^2.b^2 = (a^2+b^2)^2$

$\Rightarrow \frac{a^4+b^4}{a^2+b^2} \geq \frac{a^2+b^2}{2}$ …………….(1)

Similarly, $\frac{b^4+c^4}{b^2+c^2} \geq \frac{b^2+c^2}{2}$ ……………(2)

And, $\frac{c^4+a^4}{c^2+a^2} \geq \frac{c^2+a^2}{2}$ ……………………(3)

(1)+(2)+(3) implies

$\frac{a^4+b^4}{a^2+b^2} + \frac{b^4+c^4}{b^2+c^2} + \frac{c^4+a^4}{c^2+a^2} \geq a^2 + b^2 + c^2$.

18. Let m > 1, n ∈ℕ, then $\boldsymbol{1^m + 2^m + 2^{2m} + 2^{3m} + \cdots + 2^{nm-m}}$>

(a) $\boldsymbol{n^{1-m}}$ **(b) $\boldsymbol{(1-m)^n}$** **(c) $\boldsymbol{n^{1-m}(2^{n-1})}$** **(d) none**

Ans. (c) $\frac{1^m + 2^m + 2^{2m} + 2^{3m} + \cdots + (2^{n-1})^m}{n} > (\frac{1+2+4+\cdots+2^{n-1}}{n})^m$

[∵ m> 0 and AM of mth power > mth power of AM]

$\Rightarrow 1^m + 2^m + 2^{2m} + 2^{3m} + \cdots + (2^{n-1})^m > n(\frac{2^{n-1}}{n})^m > n^{1-m}(2^{n-1})^m$

19. Let $x^2+y^2=c^2$, then the least value of $x^{-2}+y^{-2}$ is

(a) c **(b) c^2**
(c) c^3 **(d) none**

Ans. (d) Let z= $x^{-2}+y^{-2} = \frac{x^2+y^2}{x^2y^2} = \frac{c^2}{x^2y^2}$ and

It will be minimum when x^2y^2 will be maximum.

As $x^2+y^2=c^2$, then x^2y^2 is maximum when $x^2=y^2=\frac{c^2}{2}$

$\therefore z_{min.} = \frac{c^2}{\frac{c^4}{4}} = \frac{4}{c^2}$.

20. $n^n(\frac{n+1}{2})^{2n} >$

(a) n! **(b) $(n!)^2$**
(c) $(n!)^3$ **(d) none.**

Ans. (c) $\frac{1^3+2^3+\cdots+n^3}{n} > (1^3.2^3.\ldots.n^3)^{\frac{1}{n}}$ [∵AM> GM]

$\Rightarrow \frac{n(n+1)^2}{4} > \{(n!)^3\}^{\frac{1}{n}}$

$\Rightarrow n^n(\frac{n+1}{2})^{2n} > \{(n!)^3\}$.

21. If $a_1, a_2, a_3, \ldots, a_n$ are non- negative and $a_1, a_2, a_3, \ldots, a_n = 1$,

then (1+a_1)(1+a_2) …(1+a_n)≥

(a) 2^n **(b) 3^n**
(c) 4^n **(d) none**

Ans. (a) $\left(\frac{1+a_i}{2}\right) \geq \sqrt{ai}$, where i= 1(1)n. (AM ≥ GM)

Putting the all I value and then multiplies the in equations,

(1+a_1)(1+a_2) … (1+a_n)≥ $2^n\sqrt{a_1, a_2, a_3, \ldots, a_n}$

⇒ (1+a_1)(1+a_2) … (1+a_n)≥ 2^n(∵$a_1, a_2, a_3, \ldots, a_n = 1$)

22. If $a_1, \ldots, a_n$ are positive real nos. whose product is a fixed number c, then the minimum value of $a_1 + a_2 + \cdots + a_{n-1} + 2_{an}$ is

(a) $n(2c)^{\frac{1}{n}}$ **(b) $(n+1)c^{\frac{1}{n}}$**
(c) $2nc^{\frac{1}{n}}$ **(d) done**

Ans. (a) AM ≥ GM

So, LHS ≥ $n(a_1 \ldots 2_{an})^{\frac{1}{n}}$ = $n(2c)^{\frac{1}{n}}$

23. If f(x) = $\int_0^x \frac{e^{\cos t}}{e^{\cos t}+e^{-\cos t}} dt$, then 2f($\pi$) =

(a) 0 **(b) π** **(c) –π**
(d) none of these

Ans. (b)

$f(\pi) = \int_0^{\pi} \frac{e^{\cos t}}{e^{\cos t}+e^{-\cos t}} dt$……….(1)

$f(\pi) = \int_0^{\pi} \frac{e^{-\cos t}}{e^{-\cos t}+e^{\cos t}} dt$ …………(2)

[since cos (π-t)= - cost]

∴ 2f(π)= $\int_0^{\pi} dt = \pi$

24. Let [x] denotes the greatest integer less than or equal to x, then $\int_0^{\frac{\pi}{4}} \boldsymbol{sinx\, d(x-[x])}$=

(a) ½ (b) 1 - $\frac{1}{\sqrt{2}}$
(c) 1 (d) none of these

Ans. (b) $\int_0^{\pi/4} \sin x \, d(x-[x]) =$ $\int_0^{\pi/4} \sin x \, dx = -[\cos x]_0^{\frac{\pi}{4}} = -\left[\frac{1}{\sqrt{2}} - 1\right] =$ $1 - \frac{1}{\sqrt{2}}$

[$\because$ 0 < x <$\frac{\pi}{4}$ $\therefore [x] = 0$]

25. Let g(x) = $\int_0^x \boldsymbol{f(t)dt}$, $\boldsymbol{where\, \frac{1}{2} \le f(t) \le 1, t \in [0,1] and 0 \le f(t) \le \frac{1}{2}\, for\, t \in (1,2]}$. Then (a) $-\frac{3}{2} \le g(2) < \frac{1}{2}$ (b) $0 \le g(2) < 2$ (c) $\frac{3}{2} < g(2) \le \frac{5}{2}$ (d) 2 < g(2)< 4

Ans. (b) $g(2) = \int_0^2 f(t)\, dt = \int_0^1 f(t)\, dt + \int_1^2 f(t)\, dt\; as \frac{1}{2} \le f(t) \le 1\; for\; 0 \le t \le 1,$

$$\therefore \int_0^1 \frac{1}{2} dt \le \int_0^1 f(t)dt \le \int_0^1 1\, dt$$

$or, \frac{1}{2} \le \int_0^1 f(t)dt \le 1$ ………. (1)

$$as\; 0 \le f(t) \le \frac{1}{2}\; for\; 1 < t \le 2,$$

$$\therefore \int_1^2 0\; dt \le \int_1^2 f(t)\, dt \le \int_1^2 \frac{1}{2} dt$$

$or,\; 0 \le \int_1^2 f(t)\, dt \le \frac{1}{2}$ ……………. (2)

(1) + (2) $\Rightarrow \frac{1}{2} \le g(2) \le \frac{3}{2}$

$\therefore$ g(2) satisfies the inequality 0$\le g(2) < 2$.

26. The tangent at point P of a curve meets the y- axis at B, the line through P parallel to y-axis meets the x-axis at A. If the area of ΔAOB is constant, the curve is a

(a) parabola (b) hyperbola
(c) ellipse (d) circle

Ans. (b)

Let P= (x, y)

Equation of tangent to the curve at P(x, y) is

Y- y = $\frac{dy}{dx}(X - x)$

When X= 0, Y= y – x $\frac{dy}{dx}$

$\therefore$B$\equiv \left(0, y - x\frac{dy}{dx}\right)$

Area of ΔAOB ant=k

$\therefore \frac{1}{2}x\left(y - x\frac{dy}{dx}\right) \Rightarrow xy - \frac{x^2dy}{dx} = \pm 2k \Rightarrow \frac{x^2dy}{dx} - xy = \pm 2k = c \Rightarrow \frac{dy}{dx} + y\left(-\frac{1}{x}\right) = \frac{2}{x^2}$

$$I.F. = e^{-\log x} = \frac{1}{x}$$

$$solution\; is\; given\; by\; y, \frac{1}{x} = \int \frac{C}{x^3} dx + a$$

$$or\; y = x\left(-\frac{C}{2x^2}\right) + ax$$

$$or\; 2xy = -C + 2ax^2$$

$or\ 2ax^2 - 2xy - C = 0$ …………(1)

Here h= -1, a= a, b= 0

$\therefore h^2 >$ ab. Hence curve (1) is a hyperbola

27. The function f(k) = $\frac{d}{dk}\int_0^k \frac{dx}{1-\cos k.\cos k}$ satisfies the differentiable equation

(a) $\frac{df}{dk} + 2f(k).\cot k = 0$
(b) $\frac{df}{dk} + 2f(x).\cos k = 0$

(c) $\frac{df}{dk} - 2f(k).cos^2 k = 0$
(d) none of these

Ans. (a) $f(x) = \frac{1}{1-\cos k \cot k} = cosec^2 k$

$$\frac{df}{dk} = 2\ cosec\ k\ (-cosec\ k \cot k)$$
$$= -2f(k)\cot k$$

$$or \frac{df}{dk} + 2f(k)cotk = 0$$

28. The largest value of 'c' such that there exists a differentiable function f(x) for –c< x < c that satisfies the equation $y_1 = 1 + y^2$ with f(0)= 0 is

(a) 1 **(b) π**
(c) $\frac{\pi}{3}$ **(d) $\frac{\pi}{2}$**

Ans. (d) $\frac{dy}{dx} = 1 + y^2 \Rightarrow \tan^{-1} y = x + k$

∵ f(x) satisfies the equation

$\therefore \tan^{-1} f(x) = x + k$

Now, f(0)= 0= k= 0

$\Rightarrow$ x= $\tan^{-1} f(x)$ $\therefore -\frac{\pi}{2} < x < \frac{\pi}{2}$

29. If y= (x) and $\frac{2+\sin x}{y+1}\left(\frac{dy}{dx}\right) = -\cos x$, $y(0) = 1$, $y\left(\frac{\pi}{2}\right)$ equals:

(a) 1/3 **(b) 2/3** **(c) -1/3** **(d) 1**

Ans. (a) Given, $\frac{2+\sin x}{y+1}\frac{dy}{dx} = -\cos x$ ………..(1) & $y(0) =$ 1 …………(2)

$$(1) \Rightarrow \int \frac{dy}{y+1} = -\int \frac{\cos x}{2+\sin x}dx$$

$$\Rightarrow \log(y+1) = -\log(2+\sin x) + \log c$$
$$\Rightarrow y + 1 = \frac{c}{2+\sin x}$$

$\Rightarrow y = \frac{c}{2+\sin x} -$ 1 …………..(3)

$$Also\ given\ y(0) = 1 \Rightarrow 1 = \frac{c}{2} - 1 \Rightarrow c = 4$$

$$\therefore from\ (3), y(x) = \frac{2-\sin x}{2+\sin x}$$

$$\therefore y = \left(\frac{\pi}{2}\right) = \frac{2-1}{2+1} = \frac{1}{3}$$

30. If l_1 and l_2 are the side length of two variables squares s_1 and s_2, respectively. If $l_1 = l_2 + {l_2}^3 + 6$, then the rate of change of the area of s_2 with respect to rate of change of the area of s_1 when $l_2 = 1$ is

(a) 3/2 **(b) 2/3**
(c) 4/3 **(d) none**

Ans. (d) Let Δ_1 *and* Δ_2 be the area of the sequences s_1 *and* s_2,

$\Delta_1 = {l_1}^2$ *and* $\Delta_2 = {l_2}^2$

$\therefore \frac{d\Delta_1}{dl_1} = 2l_1$ *and* $\frac{d\Delta_2}{dl_2} = 2l_2$.

$\Rightarrow \frac{d\Delta_2}{d\Delta_1} = \frac{l_2}{l_1} . \frac{dl_2}{dl_1} = \frac{l_2}{l_1} . \frac{1}{1+3l_2}$

When $l_2 = 1, l_1 = 8$, *then* $\frac{d\Delta_2}{d\Delta_1} = \frac{1}{32}$.

ISI B.STAT/B.MATH OBJECTIVE QUESTIONS & SOLUTIONS

SET – 5

1. If $a_n = \int_0^{\frac{\pi}{2}} \frac{\sin^2 nx}{\sin^2 x} dx$, then

$$\begin{bmatrix} a_1 & a_{51} & a_{101} \\ a_2 & a_{52} & a_{102} \\ a_3 & a_{53} & a_{103} \end{bmatrix} =$$

(a) 1 **(b) 0**
(c) -1 **(d) none of these**

Ans. (b) $a_{n+2} + a_n - 2a_{n+1} = 0$

$\Longrightarrow a_1, a_2, a_3, \ldots$ *are in A.P.*

$\therefore a_1 + a_{101} = 2a_1 + 2a_1 + 100d = 2(a_1 + 50d) = 2a_{51}$

$a_2 + a_{102} = 2a_{52}, a_3 + a_{103} = 2a_{53}$

2. If $t_r = 2^{r/3} + 2^{-r/3}$, then $\sum_{r=1}^{100} {t_r}^3 - 3\sum_{r=1}^{100} t_r + 1 =$

(a) $\frac{2^{101+1}}{2^{100}}$ **(b) $\frac{2^{101-1}}{2^{100}}$**
(c) $\frac{2^{201-1}}{2^{100}}$ **(d) None of these**

Ans. (c)

${t_r}^3 = 2^r + 2^{-r} + 3t_r$

$\therefore \sum_{r=1}^{100} {t_r}^3 = \sum_{r=1}^{100} 2^r + \sum_{r=1}^{100} \frac{1}{2^r} + 3\sum_{r=1}^{100} t_r$

$= 2(2^{100} - 1 + \frac{\frac{1}{2}\left(1 - \frac{1}{2^{100}}\right)}{1 - \frac{1}{2}} + 3\sum_{r=1}^{100} t_r =$

$2^{101} - 2 + 1 - \frac{1}{2^{100}} + 3\sum_{r=1}^{100} t_r$

$= \frac{2^{201}-1}{2^{100}} - 1 + 3\sum_{r=1}^{100} t_r$

3. If $\sum_{r=1}^{n} r$.r! = 100! - 1, then n equals

(a) 100 **(b) 101**
(c) 99 **(d) none of these**

Ans. (c)

t_r= r. r != (r+1 -1)r!= (r+1)! –r!

$\therefore \sum_{r=1}^{n} t_r = (n+1)! - 1! = (n+1)! - 1$

4. If m = $\sum_{r=0}^{\infty} a^r$, n=$\sum_{r=0}^{\infty} b^r$ where 0<a<1,0<b<1,then the quadratic equation whose

roots are a and b is

(a) mnx^2+(m+n-2mn)x+mn-m-n+1=0
(b) mnx^2+(2mn-m-n)x+mn-m-n+1=0

(c) mnx^2+(2mn+m+n)x+mn+m+n+1=0
(d) mnx^2-(2mn+m+n)x+mn+m+n+1=0

Ans. (a)

m = $\frac{1}{1-a} \Longrightarrow a = \frac{m-1}{m}$, *similarly* $b = \frac{n-1}{n}$.

Required quadratic equation is

$x^2-(a+b)x+ab=0$

or, $x^2-\left(\frac{m-1}{m}+\frac{n-1}{n}\right)x+\frac{(m-1)(n-1)}{mn}=0$

or, mn$x^2-(2mn-m-n)x+mn-m-n+1=0$

5. If $\sum_{r=1}^{n} r^4 = a_n$, then $\sum_{r=1}^{n} r^4\,(2r-1)^4=$

(a) $a_{2n}+a_n$ **(b) $a_{2n}-a_n$**
(c) $a_{2n}-16a_n$ **(d) $a_{2n}+16b_n$**

Ans. (c) $\sum_{r=1}^{n}(2r-1)^4=1^4+3^4+5^4+\cdots+(2n-1)^4$

$=[1^4+2^4+3^4+\cdots+(2n)^4]-[2^4+4^4+6^4+\cdots+(2n)^4]$

$= a_{2n}-2^4(1^4+2^4+3^4+\cdots+n^4)=a_{2n}-16a_n.$

6. If positive numbers a, b, c be in H.P., then equation $x^2-kx+2b^{101}-a^{101}=0$ $(k\in R)$ has

(a) both roots positive
(b) both roots negative

(c) one positive and one negative root
(d) both roots imaginary.

Ans. (c)

a, b, c are in H.P.

⟹H.M. of a and c= b⟹$\sqrt{ac}$ > b (∵G.M. > H.M.)

Since A .M. > G.M.

$\therefore \frac{a^{101}+c^{101}}{2} > (\sqrt{ac})^{101} > b^{101}$ [$\because \sqrt{ac} > b$]

$\Rightarrow 2b^{101}-a^{101}-c^{101}<0$

Let f(x)= $x^2-kx+2b^{101}-a^{101}-c^{101}$

Then $f(-\infty)=\infty>0, f(0)=2b^{101}-a^{101}-c^{101}<0, f(\infty)=\infty>0.$

Hence equation f(x)= 0 has one root in $(-\infty,0)$and other in$(0,\infty)$.

7. If the sum of the series $\sum_{n=0}^{\infty} r^n$, |r|<1, is s, then sum of the series $\sum_{n=0}^{\infty} r^{2n}$ is

(a) s^2 **(b) $\frac{2s}{s^2-1}$**
(c)$\frac{s^2}{2s+1}$ **(d)$\frac{s^2}{2s-1}$**

Ans. (d)

s= $\sum_{n=0}^{\infty} r^n=1+r+r^2+r^3+\cdots to\ \infty=\frac{1}{1-r}$

∴ r=1- $\frac{1}{s}=\frac{s-1}{s}$

$\sum_{n=0}^{\infty} r^{2n}=\frac{1}{1-r^2}=\frac{1}{1-\frac{(s-1)^2}{s^2}}=\frac{s^2}{2s-1}$

8. The limit of the product $\sqrt[2]{5},\sqrt[4]{5},\ldots\sqrt[2n]{5}$ as n→∞ is

(a) $\frac{1}{5}$ **(b) $\log_{10}5$**
(c) 1 **(d) 5**

Ans. (d) Required limit=

$\underset{n\to\infty}{\text{Lt}}\ 5^{\frac{1}{2}}.5^{\frac{1}{4}}.5^{\frac{1}{8}}\ldots.5^{\frac{1}{2^n}}=\underset{n\to\infty}{\text{Lt}}\ 5^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}}=5^{\frac{\frac{1}{2}}{1-\frac{1}{2}}}=5$

9. If numbers p, q, r are in A.P. , then m^{7p}, m^{7q}, m^{7r} (m>0) are in

(a) A.P. **(b) G.P.** **(c) H.P.** **(d)none of these**

Ans. (b) $\frac{m^{7q}}{m^{7p}} = m^{7(q-p)}, \frac{m^{7r}}{m^{7q}} = m^{7(r-q)}$

∴ q- p= r- q

∴ m^{7p}, m^{7q}, m^{7r} are in G.P.

10. Let n be a positive integer and $(1 + x + x^2)^n = a_0 + a_1x + \cdots + a_{2n}x^{2n}$, then the value of $a_0{}^2 - a_1{}^2 + a_2{}^2 \ldots\ldots + a_{2n}{}^2$ is

(a) 0 **(b) a_0** **(c) a_n** **(d) a_{2n}**

Ans. (c) Replacing x by (- 1/x), we get

$$(1 - \frac{1}{x} + \frac{1}{x^2})^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots - a_{2n-1}.\frac{1}{x^{2n-1}} + \frac{a_{2n}}{x^{2n}}$$

or, $(1 - x + x^2)^n = a_0x^{2n} - a_1x^{2n-1} + a_2x^{2n-2} + \cdots + a_{2n}$ ……………….(1)

And given $(1 + x + x^2)^n = a_0 + a_1x + \cdots + a_{2n}x^{2n}$…………………(2)

Multiplying corresponding sides of (1) and (2), we have

$(1 + x^2 + x^4)^n = (a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}) \times (a_0x^{2n} - a_1x^{2n-1} + a_2x^{2n-2} + \cdots + a_{2n})$ …….. (3)

$(1 + x^2 + x^4)^n = (a_0 + a_1x^2 + a_2x^4 + \cdots + a_nx^n + \cdots + a_{2n}x^{4n})$ ……………..(4)

Equating coefficient of x^{2n} on both sides of (3) and (4)

$a_0{}^2 - a_1{}^2 + a_2{}^2 \ldots\ldots + a_{2n}{}^2 = a_n$.

11. The set of all real number x such that ||3-x|-|x+2||=5 is

(a) [3, ∞) **(b) (-∞, −2]** **(c) (-∞, −2] ∪[3, ∞)** **(d) (-∞, −3] ∪[2, ∞)**

Ans. (c) $(||3 - x| - |x + 2||)^2 = 25$

⇒$(3 - x)^2 + (x + 2)^2 - 2|3-x||x+2| = 25$

⇒ $x^2 - x - |-x^2 + x + 6| = 6$

So, it is clear that $-x^2 + x + 6 < 0$,

i.e. $-x^2 + x + 6 \geq 0$

(x-3)(x+2)≥ 0. So, $x \leq -2$ & $x \geq 3$

∴x ∈ (-∞, −2] ∪[3, ∞).

12. The differential equation of the system of circle touch the y – axis at the origin is

(a) $x^2 + y^2 - 2xy\frac{dy}{dx} = 0$ **(b) $x^2 + y^2 + 2xy\frac{dy}{dx} = 0$**

(c) $x^2 - y^2 - 2xy\frac{dy}{dx} = 0$ **(d) $x^2 - y^2 + 2xy\frac{dy}{dx} = 0$**

Ans. (d) $x^2 + y^2 - 2ax = 0$

$2x + 2y\frac{dy}{dx} - 2a = 0$

⇒$2(x + y\frac{dy}{dx}) = 2\left(\frac{x^2+y^2}{2x}\right)$

⇒$2x^2 + 2xy\frac{dy}{dx} = x^2 + y^2$

⇒$x^2 - y^2 + 2xy\frac{dy}{dx} = 0$.

13. Let y(x) be a non-trivial solution of the second order liner differential equation

$\frac{d^2y}{dx^2} + 2c\frac{dy}{dx} + ky = 0, where\ c < 0, k > 0, c^2 - k$. **Then**

(a) |y(x)|⟶∞ $as\ x \to \infty$
(b) |y(x)|⟶0 $as\ x \to \infty$

(c) $\lim_{x\to\pm\infty}$ |y(x)| exists & is finite
(d) none

Ans. (a) $m^2 + 2cm + k = 0$

$\therefore$ m = $\frac{-2c\pm\sqrt{4c^2-4k}}{2} = \frac{-2c\pm\sqrt{4\,(c^2-k)}}{2} = \frac{-2c\pm2a}{2}$ $[\because c^2 - k = a^2 = c^2 - k]$

= $\frac{-c-a}{2}, \frac{-c+a}{2}$

The general solution of the given L.D.E. is y $= c_1e^{m_1x} + c_2e^{m_2x} = c_1e^{-(\frac{c+a}{2})x} + c_2e^{-(\frac{c-a}{2})x}$

So, |y(x)|⟶∞ $as\ x \to \infty$

14. Let y be a function of x satisfying $\frac{dy}{dx} = 2x^3\sqrt{y} - 4xy$. If y(0)= o and then y(1)equals

(a) $\frac{1}{4e^2}$ **(b) 1/e**
(c) $e^{1/2}$ **(d) $e^{3/2}$**

Ans. (a) $\frac{dy}{dx}(4x)y$=2$\sqrt{y}x^3$ (Bernoulli's Equation)

Putting$\sqrt{y} = z$, the equation reduces to

$\frac{dz}{dx} + (2x)z = x^3$(linear in z)

$\therefore$ I. F.= e$\int 2xdx = e^{x^2}$

Multiplying and integrating

z$e^{x^2} = \int x^3e^{x^2}dx$ (put $x^2 = u$)

=$\frac{1}{2}(x^2 - 1)e^{x^2} + c$

$\therefore$ General solution is given by:- $\sqrt{y} = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$

Since y(0)=0, so, c= $\frac{1}{2}$

$\therefore$y(1)= $(\frac{1}{2e})^2 = \frac{1}{4e^2}$.

15. Let x_i are non -ve reals and s= $x_1 + x_2 + \cdots + x_n$, $then\ x_1x_2 + x_2x_3 + \ ... + x_{n-1}x_n \leq$

(a) $\frac{s^2}{2}$ **(b) $\frac{s^2}{3}$**
(c) $\frac{s^2}{4}$ **(d) none**

Ans. (c) $(x_1 + x_3 + x_5 + \cdots)(x_2 + x_4 + x_6 + \cdots) \geq x_1x_2 + x_2x_3 + \ ... + x_{n-1}x_n$

As when expanding LHS, we must get RHS and many additional non- negative terms since $x_i \neq 0$.

Thus maximum achieved by taking $x_1 = x, x_2 = s - a$ and all other terms 0, but

x(s-x) ≤ $\frac{s^2}{4}$ with equality when x = $\frac{s}{2}$ (using AM ≥ GM)

16. For any positive reals x, y, z and a is the arithmetic mean of x, y, z then $x^xy^yz^z$ is

(a) ≥ $(xyz)^a$ **(b) < $(xyz)^a$**
(c) >$(xyz)^a$ **(d) none**

Ans. (a) Let ≥ $y \geq z$, then $x^xy^y \geq x^yy^x$, as $\left(\frac{x}{y}\right)^x \geq \left(\frac{x}{y}\right)^y$ is obviously true.

Similarly, $y^y z^z \geq y^z z^y and z^z x^x \geq \ z^x x^z$

Multiplying all these, $(x^x y^y z^z)^2 \geq x^{y+z}. y^{z+x}. z^{x+y}$

$\Rightarrow x^x y^y z^z \times (x^x y^y z^z)^2 \geq x^{x+y+z}. y^{x+y+z}. z^{x+y+z}$

$\Rightarrow (x^x y^y z^z)^3 \geq (xyz)^{3a}$

$\Rightarrow x^x y^y z^z \geq (xyz)^a$

17. The number of integers between 1 and 567 are divisible by either 3 or 5, is

(a) 200 **(b) 250**
(c) 300 **(d) none**

Ans. (d) Let z= {1, 2, 3, …., 566, 567}

P = {x ∈ $\frac{z}{3}$ $devides\ x$} and

Q = { x ∈ $\frac{z}{5}$ $devides\ x$ }

Here, |P|= 189 [∵ 567= 189× 3]

And |Q|= 113 [∵ 567= 113× 5 + 2]

P ∩ Q= set of multiple of both 3 and 5,

| P ∩ Q |=37; |P ∪ Q|= 189+ 113- 37= 265.

18. Sets A and B have 3 and 6 elements respectively. The minimum number of elements in A ∪B is

(a) 3 **(b) 6**
(c) 9 **(d) none**

Ans. (b) n(A ∪ B)≥ max{$n\ (A), n\ (B)$}

Thus n (A ∪ B)≥ max {3, 6} = 6.

19. A has n elements. How many (B, C) are such that ≠ $\boldsymbol{B} \subseteq \boldsymbol{C} \subseteq \boldsymbol{A}$?

(a)2^n **(b) 3^n**

(c) 4^n **(d) none**

Ans. (b) There are $\binom{n}{m}$ choices for a subject B with m elements.

Then each of the remaining n-m elements can be in C or not, so there are 2^{n-m} choices for C

Thus the total no of pairs (B, C) is $\sum 2^{n-m} . n_{c_m} = \sum 2^m . n_{c_m}$= $(1+2)^n = 3^n$ (from binomial theorem) [∵$n_{c_m} = n_{c_{n-m}}$]

20. The value of the integral $\int_{-10}^{0} \frac{\left|\frac{2[x]}{3x-[x]}\right|}{\frac{2[x]}{3x-[x]}}$ dx, where [.] denotes greatest integer function is

(a) 0 **(b) 10**
(c) -10 **(d) none of these**

Ans. (d)

Let $f(x) = \frac{\left|\frac{2[x]}{3x-[x]}\right|}{\frac{2[x]}{3x-[x]}}$

Clearly f is not defined if x= 0 and when 3x= [x]

So in (-10, 0), f is not defined at x= $-\frac{1}{3}$.

When x∈$(-10, -\frac{1}{3})$

[x] < 0 and 3x- [x]< 0

So, $\frac{[x]}{3x-[x]} > 0 \implies f(x) = 1$

When x ∈ $(-\frac{1}{3}, 0)$

[x] < 0 and 3x- [y] > 0⟹ f(x)= -1

$$\int_{-10}^{0} f(x)\,dx = \int_{-10}^{-1/3} dx + \int_{-1/3}^{0} (-1)\,dx$$

$$= [x]\begin{matrix}-\frac{1}{3}\\-10\end{matrix} - (x)\begin{matrix}0\\-\frac{1}{3}\end{matrix}$$

$$= \left(-\frac{1}{3} + 10\right) - \left(0 + \frac{1}{3}\right)$$

$$= 10 - \frac{2}{3}$$

21. The equation $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(a|\sin x| + \frac{b\sin x}{1+\cos x} + c\right) dx = 0$ gives a relation between

(a) a, b and c **(b) a and b**
(c) b and c **(d) a and c**

Ans. (d) I = 2a$\int_0^{\frac{\pi}{4}} |\sin x|\,dx + 0 + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} c\,dx = 2a\int_0^{\frac{\pi}{4}} sinx\,dx + c.\frac{\pi}{2}$

$$= -2a[cosx]\begin{matrix}\frac{\pi}{4}\\0\end{matrix} + \frac{\pi}{2}c$$

$$= -2a\left(\frac{1}{\sqrt{2}} - 1\right) + \frac{\pi}{2}c$$

22. Let f(x) = max. {2- x, 2, 1+ x} then $\int_{-1}^{1} f(x)\,dx$=

(a) 0 **(b) 2** **(c) 9/2** **(d) none of these**

Ans. (c) ∴ f(x) = 2-x, x≤ 0

= 2, 0≤x≤1

= 1+ x, x ≥1

I= $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)dx + \int_0^1 f(x)dx = \int_{-1}^{0} (2-x)dx + \int_0^1 2\,dx$

$$= \left[2x - \frac{x^2}{2}\right]\begin{matrix}0\\-1\end{matrix} + 2[x]\begin{matrix}1\\0\end{matrix}$$

$$= 0 - \left(-2 - \frac{1}{2}\right) + 2(1-0)$$

$$= \frac{9}{2}$$

23. Let f(x) be a continuous function such that f(a-x)+f(x)=0 for all x ∈[0, a].

Then $\int_0^a \frac{dx}{1+e^{f(x)}}$ equals

(a) a **(b) a/2** **(c) ½ f(a)** **(d) none of these**

Ans. (b) Given, f(a- x)= - f(x)

Now 2I= $\int_0^a \frac{dx}{1+e^{f(x)}} + \int_0^a \frac{dx}{1+e^{f(a-x)}} = \int_0^a \frac{dx}{1+e^{f(x)}} + \int_0^a \frac{dx}{1+e^{-f(x)}} = \int_0^a dx = a$

$$\therefore I = \frac{a}{2}$$

24. Let f(x) be an integrable odd function in [-5, 5] such that f(10+ x)= f(x),

then $\int_x^{10+x} f(t)dt$ equals

(a) 0 **(b) 2$\int_x^5 f(x)dx$**
(c) > 0 **(d) none of these**

Ans. (a) Let y= $\int_x^{x+10} f(t)\,dt$…… (1)

Then, $\frac{dy}{dx} = f(x+10).1 - f(x) = 0$ [∵ f(10+x)= f(x)]

∴ y is independent of x.

Putting x= -5 in (1), we get

y= $\int_{-5}^{5} f(t)\,dt = 0$ ………… (2)

Since y is independent of x, therefore y has same value for all x.

$\therefore \int_{x}^{x+10} f(x)\, dx = 0$

25. If $\int_0^1 xe^{x^2}\, dx = k\int_0^1 e^{x^2}\, dx$, then

(a) k > 1 **(b) 0 < k < 1**
(c) k=1 **(d) none**

Ans. (b) Here 0 < x < 1

$\Rightarrow 0 < xe^{x^2} < e^{x^2} \Rightarrow 0 < \int_0^1 xe^{x^2} dx < \int_0^1 e^{x^2} dx \Rightarrow 0 < k\int_0^1 e^{x^2} dx < \int_0^1 e^{x^2} dx$

$\Rightarrow 0 < k < 1$ $[dividing\ by\ \int_0^1 e^{x^2} dx]$

26. Consider the parabola $3y^2 - 4y - 6x + 8 = 0$. The points on the axis of this parabola from where 3 distinct normals can be drawn are given by

(a) $\left(\frac{2}{3}, h\right), where\ h > \frac{29}{18}$
(b) $\left(h, \frac{1}{3}\right), where\ h > \frac{19}{18}$

(c) $\left(h, \frac{2}{3}\right), where\ h > \frac{29}{18}$
(d) none of these

Ans. (c)

Given parabola is $\left(y - \frac{2}{3}\right)^2 = 2\left(x - \frac{10}{9}\right)$

Let X= $x - \frac{10}{9}, Y = y - \frac{2}{3}$

$\therefore Y^2 = 2x$ becomes the equation of parabola with reference to the new origin.

Hence equation of normal will be

Y= mX – m- $x - \frac{m^3}{2}$

[$\because$ three normals are drawn from point on the axis (H, 0) (say)]

$\therefore$ H= $1 + \frac{m^2}{2} \Rightarrow$ m= $\pm\sqrt{2H - 1}$

For m to be real, H > ½

$\Rightarrow h - \frac{10}{9} > \frac{1}{2} \Rightarrow h > \frac{29}{18}$

[where h is the abscissa w.r.t. the previous co-ordinate system]

Hence the points are given by $\left(h, \frac{2}{3}\right), where\ h > \frac{29}{18}$.

27. A (x_1, y_1) and B (x_2, y_2) are any two points on the parabola y= $cx^2 + bx + a$. If P(x_3, y_3) be the point on the arc AB where the tangent is parallel to the chord AB, then

(a) x_2 is the A.M. between $x_1 and\ x_3$
(b) x_2 is the G.M. between $x_1 and\ x_3$

(c) x_2 is the H.M. between $x_1 and\ x_3$
(d) none of these

Ans. (d) Slope of tangent at p= $\frac{dy}{dx} at\ (x_3, y_3) = 2ax_3 + b = \frac{y_2 - y_1}{x_2 - x_1}$ [given]……….(A)

$\because$ A and B lie on the parabola,

$\therefore y_1 = ax_1{}^2 + bx_1 + c$ ………………(1)

And $y_2 = ax_2{}^2 + bx_2 + c$ ……………..(2)

$\therefore y_1 - y_2 = [a(x_1 + x_2)(x_1 - x_2) + b](x_1 - x_2)$

$\therefore \frac{y_2 - y_1}{x_2 - x_1} = a\,(x_1 + x_2) + b$

$\therefore$ From (A), $a(x_1 + x_2) + b = 2ax_3 + b$

$\Rightarrow \frac{x_1+x_2}{2} = x_3$

28. Let P (α,β) be any point on parabola $y^2 = 4x(0 \le \beta \le 2)$. M is the foot of perpendicular from the focus S to the tangent at P, then the maximum value of area of

(a) 1 **(b) 2** **(c) $\frac{\pi}{3}$** **(d) $\frac{\pi}{6}$**

Ans. (a) Let $\alpha = t^2, \beta = 2t$

$\therefore$ 0≤ 2t ≤ 2 ⟹ 0≤ t ≤ 1

Equation of tangent at $(t^2, 2t)$ *is* $yt = x + t^2$

If S be the focus, then S ≡(1, 0)

SM= $\frac{|1+t^2|}{\sqrt{1+t^2}} = \sqrt{1+t^2}$

PS= $\sqrt{(t^2-1)^2 + 4t^2} = (t^2 + 1)$

PM= $\sqrt{PS^2 - SM^2} = t\sqrt{t^2+1}$

Area of Δ PMS= ½ .PM.SM= ½ .t$\sqrt{t^2+1}.\sqrt{t^2+1}$

$= \frac{t(t^2+1)}{2}$

Which is an increasing function hence its maximum value occurs at t= 1

$\therefore$ Maximum area= 1 sq. unit.

29. The point A on the parabola $y^2 = 4x$ for which |AC-AB | is maximum, where

B≡ $(\mathbf{0}, \boldsymbol{a})$ *and* $\boldsymbol{C} \equiv (-\boldsymbol{a}, \mathbf{0})$ is

(a) (a, 2a) **(b) (4a, 4a)**
(c) (a- 2a) **(d) none of these**

Ans. (a)

For any three points A, B, and C

|AC - AB|≤ BC

$\therefore$ required point A will be on the intersection of BC and the parabola.

$\therefore$ A≡ (a, 2a) [∵ AB ia tangent to the parabola]

30. Let f : R ⟶R be a function defined by f(x)=$\frac{e^{|x|}-e^{-x}}{e^x+e^{-x}}$, then

(a) 'f' is one-one and onto
(b) 'f' is one-one but not onto

(c) 'f' is not one-one but onto
(d) 'f' is neither one-one nor onto

Ans. (d)

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}, x \ge 0 = 0, x \le 0$$

Since f(x)= 0, for all x ≤ 0

$\therefore$ f(x) is a many –one function

Let y= $\frac{e^x - e^{-x}}{e^x+e^{-x}}, x \ge 0$

$\Rightarrow \frac{e^x - e^{-x}}{e^x+e^{-x}} = \frac{1}{y}$

$\Rightarrow \frac{e^x}{e^{-x}} = \frac{1+y}{1-y}$

$\Rightarrow e^{2x} = \frac{1+y}{1-y} \Rightarrow x = \frac{1}{2}\log\frac{1+y}{1-y}$

$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1}, x \geq 0$$

Clearly $e^{2x} \geq 1 \; for \; x \geq 0$

∴y≥ 0 for x ≥ 0

∴ Range f = [0, ∞) ≠ co domain f.

Hence f is not onto.

Thus f is a many-one into mapping.

ISI B.STAT/B.MATH OBJECTIVE QUESTIONS & SOLUTIONS

SET – 6

1. If x ∈={1, 2, 3,……, 9}and f_n(x)=xxx……x (n digits),then f_n^2(3)+f_n(2) =

(a) 2f_{2n}(1)
(b) f_n^2(1)

(c) f_{2n}(1)
(d)$-f_{2n}$(4)

Ans. (c)

$f_n(x) = x.1 + x.10^2 + x.10^3 + \cdots x.10^{n-1} = x\frac{(10^n-1)}{10-1} = \frac{x}{9}(10^n - 1)$

$\therefore f_n{}^2(3) + f_n(2) = \left[\frac{3}{9}(10^n-1)\right]^2 + \frac{2}{9}(10^n - 1) = \frac{1}{9}(10^n-1)(10^n - 1 + 2) = \frac{10^n-1}{9} = f_{2n}(1)$

2. If a_i∈R-{0}, i=1, 2, 3, 4 and x ∈ R and ($\sum_{i=1}^{3} a_i^2)x^2$ - 2x($\sum_{i=1}^{3} a_i\, a_i$+1) + $\sum_{i=2}^{4} a_i^2 \geq$ 0,

Then a_1,a_2,a_3,a_4 are in

(a) A.P. **(b) G.P.**
(c) H.P. **(d) none of these**

Ans. (b) Given quadratic expression≥ 0 ∴ D ≤ 0

$\Rightarrow(\sum_{i=1}^{3} a_i a_i + 1)^2 - (\sum_{i=1}^{3} a_i{}^2)(\sum_{i=1}^{4} a_i{}^2) \leq 0$

$\Rightarrow(a_1a_2 + a_2a_3 + a_3a_4)^2 - (a_1{}^2 + a_2{}^2 + a_3{}^2)(a_2{}^2 + a_3{}^2 + a_4{}^2) \leq 0$

$\Rightarrow(a_2{}^2 - a_1a_3)^2 + (a_3{}^2 - a_2a_4)^2 + (a_2a_3 - a_1a_4)^2 = 0$

$\Rightarrow(a_2{}^2 - a_1a_3)^2 = 0, (a_3{}^2 - a_2a_4)^2 = 0, (a_2a_3 - a_1a_4)^2 = 0$

$\Rightarrow\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3}$

3. Let a = $\frac{1}{n!}$ + $\sum_{r=1}^{n-1} \frac{r}{(r+1)!}$, b = $\frac{1}{m!}$ + $\sum_{r=1}^{m-1} \frac{r}{(r+1)!}$ then a+b equaals

(a) 0 **(b) 1** **(c)2**
(d) none of these

Ans. (c)

$\frac{r}{(r+1)!} = \frac{r+1-1}{(r+1)!} = \frac{1}{r!} - \frac{1}{(r+1)!}$

$\therefore \sum_{r=1}^{n-1} \frac{r}{(r+1)!} = 1 - \frac{1}{n!} \Rightarrow a = 1$

Similarly, $\frac{1}{m!} + \sum_{r=1}^{m} \frac{r}{(r+1)!} = 1$

∴ a= 1, b=1 ⟹a+ b= 2

4. If $\sum_{n=1}^{k}[\frac{1}{3}+\frac{n}{90}]$ =21, where [x] denotes the integral part of x, then k=

(a) 84 **(b) 80**
(c) 85 **(d) none of these**

Ans. (b)

21= $\sum_{n=1}^{k}\left[\frac{1}{3}+\frac{n}{90}\right], where\ m=k!$

$= \left[\frac{1}{3}+\frac{1}{90}\right]+\left[\frac{1}{3}+\frac{2}{90}\right]+\cdots+\left[\frac{1}{3}+\frac{59}{90}\right]+\left[\frac{1}{3}+\frac{60}{90}\right]+\left[\frac{1}{3}+\frac{61}{90}\right]+\cdots+\left[\frac{1}{3}+\frac{k}{90}\right]$

$=(0+0+\cdots to\ 59\ terms).+(1+1+\cdots to\ (k-59) terms)$

∴ 21= k- 59⟹ k= 80.

5. Let f: R→R such that f(x) is continuous and attains only rational value at all real x and f(3)=4. If a_1,a_2,a_3,a_4,a_5 are in H.P., then $\sum_{r=1}^{4} a_r\, a_{r+1}$=

(a) f(5).a_1a_5 **(b) f(3).a_4a_5**
(c) f(3).a_1a_2 **(d) f(2).a_1a_3**

Ans. (a)

Since f(x) is continuous and attains only rational values

∴ f(x)= constant= 4

∴ f(2) = f(3) = f(5) = 4

Since a_1, a_2, a_3, a_4, a_5 are in H.P.

∴ $a_1\, a_2+a_2a_3+a_3\, a_4+a_4\, a_5=4a_1\, a_5=f(5).\, a_1\, a_5$

6. If three successive terms of a G.P. with common ratio r >1 from the sides of a triangle and [r] denotes the integral part of x, then [r] + [-r]=

(a) 0 **(b) 1**
(c) -1 **(d) none**

Ans. (b)

Since root of equation

F(x)= $x^2+2(a-3)x+9=0$ lie between-6 and 1

∴(i) D≥ 0 (ii) f(-6)> 0 (iii) f(1) > 0 (iv) -6 <$\frac{\alpha+\beta}{2}$ (v) 1 > $\frac{\alpha+\beta}{2}$

Hence 6 ≤ $\alpha<\frac{27}{4}$

∴|a| =6

$a_3=2+3d=2+3.\ \frac{6-2}{21}=2+\frac{4}{7}=\frac{18}{7}$

$\frac{1}{h_{18}}=\frac{1}{2}+18.\left(\frac{\frac{1}{6}-\frac{1}{2}}{21}\right)=\frac{1}{2}-\frac{2}{7}=\frac{3}{14}$

∴ $a_3h_{18}=\frac{18}{7}.\frac{14}{3}=12$

7. If x_1, x_2, x_3, x_4, x_5 are in H.P. then $\frac{1}{x_1x_5}(\sum_{k=1}^{4} x_k\ \ x_{k+1})$ is a root of equation

(a) $x^2-3x+2=0$
(b) $x^2-5x-4=0$

(c) $x^2-9x+20=0$
(d) $x^2-6x-8=0$

Ans.(c)

x_1, x_2, x_3, x_4, x_5 are in H.P.

∴$\sum_{k=1}^{4} x_k x_{k+1}=x_1x_2+x_2\, x_3+x_3x_4+x_4\, x_5=4x_1x_5$

$\therefore \frac{1}{x_1 x_5}\sum_{k=1}^{4} x_k x_{k+1} = 4$

Clearly, 4 is a root of equation

$x^2 - 9x + 20$ =0.

8. Let f : (0, ∞)⟶R and F(x)= $\int_0^x f(t)\,dt$
If F$(x^2) = x^2(1+x)$, then $f(4) =$

(a) 5/4 **(b) 7**
(c) 4 **(d) 2**

Ans. (c)

Given, F(x)= $\int_0^x f(t)\,dt$ ………(1)

$F(x^2) = x^2(1+x)$ ………..(2)

From (1), F'(x)= f(x)

∴ f(4)= F'(4)…………..(3)

From (1),

F'(x^2).2x= 2x+ 3x^2

⟹ F'$(x^2) = \frac{2+3x}{2}$ [$\because 0 < x < \infty \therefore x \neq 0$]

⟹ $F'(4) = \frac{2+6}{2} = 4$ [Putting x= 2]

∴ from (3), f(4)= 4

9. If n > 1 then $\int_0^\infty \frac{dx}{(x+\sqrt{1+x^2})^n} =$

(a) $\frac{n}{n^2-1}$ **(b) $\frac{n^2-1}{n}$**
(c) $-\frac{n}{n^2-1}$ **(d) $\frac{1-n^2}{n}$**

Ans. (a)

Put z= x+$\sqrt{1+x^2}$

∴ z- x= $\sqrt{1+x^2}$

=> $z^2 + x^2 - 2zx = 1 + x^2$ => $x = \frac{z^2-1}{2z}$

$$\therefore dx = \frac{1}{2}\frac{[z.2z - (z^2-1).1]}{z^2}dz$$
$$= \frac{z^2+1}{2z^2}dz$$

When x= 0, z= 1 and when x= ∞, $z = \infty$

$\therefore$ I= $\int_1^\infty \frac{1}{z^n}\frac{z^2+1}{2z^2}dz = \frac{1}{2}\int_1^\infty (z^{-n} + z^{-n-2})\,dz = \frac{1}{2}\left[\frac{z^{-n+1}}{-n+1} + \frac{z^{-n-1}}{-n-1}\right]_1^\infty$

$$= \frac{1}{2}\left[0 - \left(\frac{1}{1-n} - \frac{1}{1+n}\right)\right]$$

$$= \frac{1}{2}\left(-\frac{2n}{1+n^2}\right) = \frac{n}{n^2-1}$$

10. If f(x)= ae^{2x}+ be^{x} +cx satisfies the conditions f(0)= -1, f'(log 2)= 28,

$\int_0^{\log 4}[f(x) - cx]dx = \frac{39}{2}$, then

(a) a= 5, b=6, c= 3 **(b) a= 5, b= - 6, c= 0** **(c) a= -5, b=6, c= 3** **(d) none**

Ans. (b)

Given f(x)= $ae^{2x} + be^{x} + cx$ ……(1)

$given, f(0) = -1 \Longrightarrow a + b = -1$…..(2)

f'(x)= 2 $ae^{2x} + be^{x} + cx$

∴f' (log 2)= $ae^{\log_e 4} + be^{\log_e 2} + c$

Given 8a+ 2b+c= 28……….(3)

Given, $\int_0^{\log 4}(ae^{2x} + be^{x})\,dx = \frac{39}{2}$

$$\Longrightarrow \left[\frac{a}{2}e^{2x} + be^{x}\right]_0^{\log 4} = \frac{39}{2}$$

$\Rightarrow \frac{a}{2}e^{\log 16} + be^{\log 4} - \left(\frac{a}{2}+b\right) = \frac{39}{2}$

$\Rightarrow$ 15a + 6b= 39……….(4)

Thus a= 5, b= -6, c= 0

11. Let $\frac{d}{dx}f(x) = \frac{e^{\sin x}}{x}, x > 0$. $If \int_1^4 \frac{2e^{\sin x^2}}{x} dx = f(k) - f(1)$, then one of the possible value of k is

(a) -4 **(b) 0**
(c) 2 **(d) 16**

Ans. (d)

Given, $\frac{d}{dx}\left(f(x)\right) = \frac{e^{\sin x}}{x}, x > 0$

$now\ I = \int_1^4 \frac{2e^{sinx^2}}{x} dx$ [put z= $x^2, dz = 2x\ dx$]

$\therefore$ I= $\int_1^4 \frac{2e^{sinx^2}}{x^2} dx = \int_1^{16} \frac{e^{\sin z}}{z} dz = [f(z)]_1^{16} = f(16) - f(1)$

$\therefore$ f(k)= f(16)

$\therefore$ one possible value of k= 16

12. All the values of a for which $\int_1^2 [a^2 + (4-4a)x + 4x^3]\ dx \le 12$ are given by

(a) a= 3 **(b) a ≤ 4**
(c) 0≤ $a \le 3$ **(d) none of these**

Ans. (a)

$$\int_1^2 [a^2 + (4-4a)x + 4x^3]\, dx = a^2[x]_1^2 + (2-2a)[x^2]_1^2 + [x^4]_1^2$$

$= a^2 + (2-2a)3 + 15$, $Given, a^2 - 6a + 21 \le 12$

$\Rightarrow a^2 - 6a + 9 \le 0 \Rightarrow (a-3)^2 \le 0 \Rightarrow (a-3)^2 = 0 \Rightarrow a = 3$

13. $\underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{n} \frac{(2r)^k}{n^{k+1}}, k \ne -1$, is equal to

(a) $\frac{2^k}{k-1}$ **(b) $\frac{2^k}{k}$**
(c) $\frac{1}{k-1}$ **(d) $\frac{2^k}{k+1}$**

Ans. (d) Reqd. limit = $\underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{n} \frac{(2r)^k}{n^{k+1}}$

$$= \underset{n\to\infty}{\text{Lt}}\, 2^k \sum_{r=1}^{n} \frac{r^k}{n^k.n} = \underset{n\to\infty}{\text{Lt}}\, 2^k \sum_{r=1}^{n} \left(\frac{r}{n}\right)^k = \frac{1}{n}$$

$$= 2^k \int_0^1 x^k dx$$

$$= 2^k.\left[\frac{x^{k+1}}{k+1}\right]_0^1 = \frac{2^k}{k+1}$$

14. $\underset{n\to\infty}{\text{Lt}} \{\frac{n!}{(kn)^n}\}^{\frac{1}{n}}, k \ne 0$, is equal to

(a) $\frac{k}{e}$ **(b) $\frac{e}{k}$**
(c) $\frac{1}{ke}$ **(d) none of these**

Ans. (c)

Let P= $\underset{n\to\infty}{\text{Lt}} \frac{1}{k}\left(\frac{n!}{n^n}\right)^{1/n} = \frac{1}{k} \underset{n\to\infty}{\text{Lt}} \left(\frac{n!}{n^n}\right)^{1/n} = \frac{1}{ke}$

15. $\underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{n} \frac{\sqrt{n}}{(\sqrt{r}\,(3\sqrt{r}+4\sqrt{n})^2}$=

(a) $\frac{1}{7}$ **(b)** $\frac{1}{10}$
(c) $\frac{1}{14}$ **(d) none of these**

Ans. (c)

Required limit

$= \underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{n} \frac{\sqrt{n}}{\sqrt{r}.n\left(3\sqrt{\frac{r}{n}}+4\right)^2} =$

$\underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{n} \frac{1}{\sqrt{\frac{r}{n}}\left(3+\sqrt{\frac{r}{n}}+4\right)^2}.\frac{1}{n}$

$$= \int_0^1 \frac{1}{\sqrt{x}\,(3\sqrt{x}+4)^2}\,dx$$

Put z= $3\sqrt{x}+4$, then dz= $\frac{3}{2\sqrt{x}}\ dx$

When x= 0, z= 4, when x= 1, z= 7

∴Reqd. limit= $\frac{2}{3}\int_4^7 \frac{dz}{z^2} = \frac{2}{3}\left[-\frac{1}{z}\right]_4^7 = -\frac{2}{3}\left[\frac{1}{7}-\frac{1}{4}\right] = -\frac{2}{3}\left(-\frac{3}{28}\right) = \frac{1}{14}$

16. If f(x) = e^{x}cosx.sin x, |x|≤ $\boldsymbol{2 = 2}$, $\boldsymbol{otherwise\ then\ \int_2^3 f(x)dx}$ is equal to

(a) 0 **(b) 1**
(c) 2 **(d) 3**

Ans. (c)

$$\int_{-2}^{3} f(x)\,dx = \int_{-2}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx$$
$$= \int_{-2}^{2} e^{\cos x}.\sin x\ dx + \int_2^3 2\,dx = 0 + 2(3-2),$$
$$= 2$$

$[\because e^{\cos x}.\sin x\ \ is\ an\ odd\ function = 2]$

18. The area of the region enclosed by the curves y= xe^x and y= xe $^{-x}$ and the line x= 1, is

(a) 1/e **(b) 1- 1/e**
(c) 2/e **(d) 1- 2/e**

Ans. (c)

y=xe^x …….(1)

y= xe^{-x}……(2)

equating y from (1) and (2) we get

x$e^x = xe^{-x} \Rightarrow x(e^x - e^{-x}) = 0$

⟹x= 0

∴ Required area=

$$\int_0^1 (y_1 - y_2)\,dx = \int_0^1 (xe^x - xe^{-x})\,dx$$
$$= [xe^x - e^x — xe^{-x} - e^{-x})\begin{matrix}1\\0\end{matrix}$$

$$= (e-e) - (0 - e^0) + [(e^{-1}+e^{-1}) - (0+1)] = \frac{2}{e}$$

19. The area bounded by y = xe$^{|x|}$ and the lines |x|=1, y= 0 is

(a) 1 **(b) 2**
(c) 4 **(d) 6**

Ans. (b)

For x ≥ 0, curve is y= xe^x……..(1)

For curve (1), $\frac{dy}{dx} = e^x(1+x) > 0$

∴ y is increasing.

$\frac{d^2y}{dx^2} = e^x(2+x) > 0$

∴ curve is convex downward.

For x ≤ 0, y= xe^{-x}

∴$\frac{dy}{dx} = e^{-x}(1-x) > \ 0$

∴ y is increasing

$$\frac{d^2y}{dx^2} = e^{-x} - e^{-x}(1-x) > 0$$
$$= \ -e^{-x}(2-x) < 0$$

∴ curve is concave downward.

Required area = $2\int_0^1 xe^x x\, dx =$ $2[xe^x - e^x]_0^1 = 2[(e-e)-(0-e^0)] = 2$

20. A bag contains unlimited number of white, red, black, and blue balls. The number of ways of selecting 10 balls so that there is at least one ball of each color is

(a) 180 (b0 270 (c) 192 (d) none

Ans. (d) Number of ways= coefficient of $X^{10} in\ (X + X^2 + X^3 + \cdots)^4$

= coefficient of $X^{10} in\ X^4(1-X)^{-4}$

= coefficient of $X^6 in\ \ (1-X)^{-4}$

= $\frac{(6+1)(6+2)(6+3)}{1.2.3}$ [∵coefficient of $x^r in (1-X)^{-4}$ =$\frac{(r+1)(r+2)(r+3)}{1.2.3}$]

= $\frac{7\times8\times9}{1\times2\times3}$ = 84.

21. The number of ways of selecting r balls with replacement out of n balls numbered 1, 2, 3, …., 100 such that the largest numbered selected is 10 is 271, then r=

(a) 3 (b) 4 (c) 5 (d) none

Ans. (a) from the given condition, we can write

$10^r - 9^r$= 271,

Applying Trial and error method:-

r= 1, 10-9= 1

r =2, $10^2 - 9^2$= 19

r = 3, $10^3 - 9^3$ = 271

∴ r= 3.

22. N men and n women sit along a line alternatively in x ways and along a circle in y ways such that x= 10y, then the number of ways in which n men can sit at a round table so that all shall not have same neighbors is

(a) 6 (b) 12 (c) 36 (d) none

Ans. (b) $\frac{x}{y} = \frac{2.\lfloor n \lfloor n}{\lfloor n-1 \ \lfloor n} = 2n$

⇒ x = 2ny = 10y ⇒ n = 5

Hence the required number = $\frac{1}{2}\times$ ⌊4= 12.

23. A contest consists of predicting the result (win, draw or defeat) of 10 matches. The number of ways in which one entry contains at least 6 incorrect results is

(a) $\sum_{r=6}^{10} 10_{c_r}.3^r$ (b) $\sum_{r=1}^{5} 10_{c_r}.2^r$ (c) $\sum_{r=6}^{10} 10_{c_r}$ (d) none

Ans. (d) Since total number of ways predicting the results of one match is 3 , so results of 10 match is 3^{10}, now number of ways that the result of one match is correct is 1 and also number of ways to predict wrongly of one match is 2 .

No. of ways to predict wrongly exactly r matches $=10_{c_r}.2^r\ 1^{10-r}$

$\therefore$ The required number is $3^{10}-\sum_{r=1}^{4} 10_{c_r}.2^r$

24. Let 1 to 20 are placed in any around a circle. Then the sum of some 3 consecutive numbers must be at least

(a) 30 (b) 31 (c) 32 (d) none

Ans. (c) Suppose $x_1, x_2, \ldots, x_{20}$ be the numbers placed around the circle. Now the mean of the 20 sums of 3 consecutive numbers such as $(x_1+x_2+x_3)$, $(x_2+x_3+x_4)$, …..,

$(x_{19}+x_{20}+x_{21})$, $(x_{20}+x_1+x_2)$ is $\frac{1}{20}\{3(x_1+x_2+\cdots+x_{20})\}=\frac{3\times 20\times 21}{2\times 20}=31.5$

Thus from Pigon hole principle that at least one of the sums must be ≥ 32.

25. The number of different seven–digit numbers can be written using only there digits 1, 2, 3 under the condition that the digit 2 occurs twice in each number is

(a) 512 (b) 640 (c) 672 (d) none

Ans. (c) We have to put 2 twice in each numbers, so any 2 out of the 7 places can be chosen in $7c_2$ ways. The remaining 5 places can be filled with the other two numbers in 2^5 ways.

The required numbers of numbers are $7c_2\times 2^5 = 672$.

26. The value of $\{\sum_{i=0}^{100}\binom{k}{i}\binom{M-k}{100-i}(\frac{M-k}{100-i})\}/\binom{M}{100}$, where M - k > 100, k >100, is

(a) $\frac{k}{M}$ (b) $\frac{M}{k}$ (c) $\frac{k}{M^2}$ (d) none

Ans. (a) $\{\sum_{i=0}^{100}\binom{k}{i}\binom{M-k}{100-i}(\frac{M-k}{100-i})\}/\binom{M}{100}$

$=(\frac{k}{M-100})\sum_{i=0}^{100}[\frac{\binom{k}{i}\binom{M-k}{100-i}}{\binom{M}{100}}-\sum_{i=0}^{100}\frac{i\binom{k}{i}\binom{M-k}{100-i}}{(M-100)\binom{M}{100}}]$

$=\frac{k}{M-100}.\frac{\binom{M}{100}}{\binom{M}{100}}-\frac{\frac{k}{M}.100\binom{M}{100}}{(M-100)\binom{M}{100}}=\frac{k}{M}.$

27. Let n be an odd positive integer. If $i_1, i_2, \ldots, i_n$ is a permutation of 1, 2, 3, …., n.

Then (1-i_1)(2-i_2)….(n-i_n)is

(a) Odd (b) even (c) prime (d) none

Ans. (b) since n is odd, let n= 2m+ 1, where m is a non-negative integer.

Then set s ={1, 2, …, n} contains m+ 1 odd nos, namely 2, 4, …, 2m.

This is also true for the permutation $i_1, i_2, \dots, i_n$ of s.

Consider m+ 1 numbers$1 - i_1$, $3 - i_3$, …. n - i_n which are of the from r - i_r, where r is odd.

Since i_5 is even for only m values of s, by P.H.P. , one of the m+ 1 , numbers, $i_1, i_2, \dots, i_n$, say it is odd, where t is also odd. Hence t-i_t is even and the product (1-i_1)(2-i_2)….(n-i_n) is even.

28. The value of $\sum_{n=1}^{\infty} \sin^{-1} \frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}} =$

(a) $\frac{\pi}{4}$ **(b)** $\frac{\pi}{2}$
(c) - $\frac{\pi}{2}$ **(d)** $\frac{\pi}{3}$

Ans. (b) $t_n = \sin^{-1}\left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}}\right)$

$\therefore t_n = \sin^{-1}\frac{1}{\sqrt{n}} - \sin^{-1}\frac{1}{\sqrt{n+1}}$

$\therefore S_n = \sin^{-1}(1) - \sin^{-1}\frac{1}{\sqrt{n+1}}$

$\therefore S_\infty = \underset{n\to\infty}{\text{Lt}} S_n = \sin^{-1}(1) - \sin^{-1}(0)$

29. The number of ways to give 16 different things to 3 persons, according as A< B < C so that B gets 1 more than A and C get 2 more than B, is

(a) 4!5!7! **(b)** $\frac{4!5!7!}{16!}$
(c) $\frac{16!}{4!5!7!}$ **(d) none**

Ans. (c) Here x+ y +z=16, x= y+1, y= z+2

∴x=4, y=5, z= 7

∴ Required number of ways = $16c_4 \times 12c_5 \times 7c_7 = \frac{16!}{4!5!7!}$

30. For how many positive integers n less than 17, ⌊n+ ⌊n+1+ ⌊n+2 is an integral multiple of 49? (a) 4
(b) 5 **(c) 6**
(d) none

Ans. (b) ⌊n+ ⌊n+1+ ⌊n+2= ⌊n{1+(n+1)+(n+2)(n+1)}= ⌊n$(n+2)^2$

Since 49 divides $(n+2)^2$ ⌊n, so either 7 devides (n+2) or 49 divides ⌊n. Thus n=5, 12, 14, 15, 16, i.e. number of integers are 5.

ISI B.STAT/B.MATH OBJECTIVE QUESTIONS & SOLUTIONS

SET – 7

1. Let x, y, z be different from 1 satisfying x+ y +z = 2007,

Then the value of $\frac{1}{1-x} + \frac{1}{1-y} + \frac{1}{1-z}$ **is**

(a) 0 **(b) 1** **(c) 2008** **(d)** $\frac{1}{2008}$

Ans:- (a) $\frac{1}{1-x} + \frac{1}{1-y} + \frac{1}{1-z}$

$= \frac{3-2(x+y+z)+(xy+yz+zx)}{(1-x)(1-y)(1-z)} = \frac{3-2\times2007+4011}{(1-x)(1-y)(1-z)} =$ 0

2. In a Δ**ABC, if r=** $r_2 + r_3 + r_1$, $and\ angle(A) > \frac{\pi}{3}$ $then\ the\ range\ of\ \frac{s}{a}$ **is equal to**

(a) (½, 2) **(b) (½, ∞)**
(c) (½, 3) **(d) (3, ∞)**

Ans. (a)

$r = r_2 + r_3 - r_1$

$\frac{\Delta}{s} = \frac{\Delta}{s-b} + \frac{\Delta}{s-c} - \frac{\Delta}{s-a}$

$\Rightarrow \frac{1}{s} + \frac{1}{s-a} = \frac{1}{s-b} + \frac{1}{s-c}$

$\Rightarrow \frac{2s-a}{2s-b-c} = \frac{s(s-a)}{(s-b)(s-c)}$

$\Rightarrow \frac{2s-a}{a} = \cot^2\frac{A}{2} \Rightarrow \frac{s}{a} = \frac{1}{2}\left(\cot^2\frac{A}{2} + 1\right) \Rightarrow \frac{s}{a} \in \left(\frac{1}{2}, 2\right)$

3. If $a_1, a_2, \dots, a_n$ are positive real nos, then $\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$ is always

i) $\geq n$ ii) $\leq n$ iii) $n^{1/n}$ iv) none of these.

Ans:- AM ≥ GM gives

$\frac{a_1}{a_2} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} \geq n\sqrt[n]{\frac{a_1}{a_2}.\dots.\frac{a_{n-1}}{a_n}.\frac{a_n}{a_1}} = 1$

$\therefore \frac{a_1}{a_2} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} \geq n$

4. The maximum possible value of xy^2z^3 subject to the condition $xyz \geq 0$ and

x+y+z = 3 is

i) 1 ii) $1/8$ iii) $1/4$ iv) $27/16$

Ans:- x+ y+ z = 3

$\Rightarrow x.\frac{2y}{2} + 3.\frac{z}{3} = 3$

Applying AM ≥ GM,

So, $\frac{x+2.\frac{2y}{2}+3.\frac{z}{3}}{1+2+3} \geq 6\sqrt{x(\frac{y}{2})^2(\frac{z}{3})^3}$

$\Rightarrow (\frac{3}{6})^6 \geq \frac{xy^2z^3}{2^2.3^3}$

$\Rightarrow xy^2z^3 \leq \frac{27}{16}$.

5. If y(t) is a solution of $(1+t)\frac{dy}{dt} - ty = 1$ and y(t) then y(1) equals

(a) ½ **(b) e + ½** **(c) e+ ½** **(d)- ½**

Ans:- (d) $\frac{dy}{dt} - \frac{t}{1+t}y = \frac{1}{1+t}$

$\therefore$I.F. $= e^{-\int\frac{t}{1+t}dt} = e^{-(t-\log(1+t))} = e^{-t}.(1+t)$

Multiplying and integrating

$ye^{-t}.(1+t) = \int e^{-t}.(1+t)\frac{dt}{(1+t)} = e^{-t} + c$

When y(0) = -1, ⇨ c= 0.

$\therefore ye^{-t}.(1+t) = -e^{-t}$

$\therefore y = -\frac{1}{1+t}$ $\therefore$y(1)= - ½.

6. If the quadratic equation $x^2 + ax + b + 1 = 0$ has non- zero

Integer solutions, then

a) **$a^2 + b^2$ is a prime number**
b) **ab is prime number**
c) **Both a) and b)**

d) Neither a) nor b)

Ans:- (d) $\alpha+\beta$= -a, $\alpha\beta$= (b+ 1)

$\therefore a^2 + b^2 = (\alpha + \beta)^2 + (\alpha\beta - 1)^2$

= $(\alpha^2 + 1)(\beta^2 + 1)$

7. Let u = $(\sqrt{5} - 2)^{1/3} - (\sqrt{5} + 2)^{1/3}$ **and v=** $(\sqrt{189} - 8)^{1/3} - (\sqrt{189} + 8)^{1/3}$**,**

Then for each positive integer n, $u^n + v^{n+1}$ = ?

(a) -1 **(b) 0**
(c) 1 **(d) 2**

Ans:- (b) $u^3 = (\sqrt{5} - 2) - (\sqrt{5} + 2) - 3(\sqrt{5} - 2)^{1/3}(\sqrt{5} + 2)^{1/3}.(u)$

i.e. $u^3 = -4 - 3u$

⇒ (u-1)($u^2 - u + 4$)= 0

$u^2 - u + 4$ is always +ve. So, u= 1

Similarly $v^3 + 15v + 16 = 0$

⇒ (v +1)($v^2 - v + 16$)= 0

⇒ v= -1

So, for each n, $u^n + v^{n+1} = 0$

8. The number of real values of x satisfying the equation

$x.2^{1/x} + \frac{1}{x}.2^x = 4$ **is / are**

(a) 1 **(b) 2**
(c) 3 **(d)4**

Ans:- (a) if x < 0, LHS = -ve but RHS= +ve

If x = 0, LHS= not defined.

If x > 0, use AM ≥ GM inequality

$x.2^{1/x} + \frac{1}{x}.2^x \geq 2\sqrt{2^{1/x+x}} \geq 2.\sqrt{2^2}$ =4

⇒ $x.2^{1/x} = \frac{1}{x}.2^x$; so, x= 1.

9. Let f (x) and g (x) be functions, which take integers as arguments. Let f (x+ y) = f (x) + f (y) + 8 for all integers x and y. Let f (x) = x for all negative numbers x and let g(8)=17, then f (0)=?

(a) 8 **(b) 9** **(c) 17** **(d) 72**

Ans:- (c) put x = -8, y= 8 in the given functional equation.

10. Let x = $\left[\frac{2007.2006.2004.2003}{\frac{1}{3}\times(2005)^4}\right]$**, where [x] denotes the greatest integer integer less than or equal to x. then** $\frac{((x+1).x^2)+1}{(x^2+1)}$ **is**

(a) 80 **(b) 80.2**
(c) 80.5 **(d) 81**

Ans:- (b) x= $[3.\frac{2007}{2005}.\frac{2006}{2005}.\frac{2004}{2005}.\frac{2003}{2005}]$

= $[3(1+\frac{2}{2005})(1+\frac{1}{2005})(1-\frac{1}{2005})(1-\frac{2}{2005})]$

= $[3(1-\frac{4}{(2005)^2})(1-\frac{1}{(2005)^2})]$

⇒ x=2.

11. A graph defined in polar co – ordinates by r (θ) = cos θ+$\frac{1}{2}$. The smallest x –co- ordinates of any point on this graph is

(a) 1/16 **(b) -1/16**
(c) 1/8 **(d) -1/8**

Ans:- (b) x =r cosθ

= $\cos^2\theta + \frac{1}{2}cos\theta$

= $(cos\theta + \frac{1}{4})^2$ = 1/16

12. A monic polynomial is one in which the coefficient of the highest order term is 1. The monic polynomial P(x) (with integer coefficient) of least degree that satisfies P ($\sqrt{2} + \sqrt{5}$)= 0 is

(a) $x^4 - x^3 - 14x^2 + 9$=0
(b) $x^4 - 14x^2 + 9$= 0

(c) $x^4 + x^3 - 14x^2 + 9$=0
(d) $x^4 + 14x^2 - 9$

ANS:- (b) Let x=$\sqrt{2} + \sqrt{5}$. Squaring, $x^2 = 7 + 2\sqrt{10}$

$\Rightarrow x^2 - 7 = 2\sqrt{10}$. Squaring again, $x^4 - 14x^2 + 9$= 0

13. The number of distinct real roots of the equation $x^4 + 8x^2 + 16 = 4x^2 - 12x + 9$ is

(a) 1 **(b) 2**
(c) 3 **(d) 4**

Ans:- (a) $(x^2 + 4)^2 = (2x - 3)^2 \quad \Rightarrow x^2 + 4 = \pm(2x - 3)$

Giving $x^2 - 2x + 7 = 0$ and $x^2 + 2x + 1 = 0$.

Solving x = - 1 only one real root.

14. If in an isosceles triangle with base 'a', vertical angle 20° and lateral side of each wih length 'b' is given then the value of $a^3 + b^3$ equals

(a) 3ab **(b) 3ab^2**
(c) $3a^2b$ **(d) 3**

Ans:- (b) sin 10° = $\frac{a}{2b}$ $\Rightarrow$ sin 30° = $3\,sin10° - 4sin^3 10°$

$\Rightarrow \frac{1}{2} = \frac{3a}{2b} - \frac{4a^3}{8b^3}$

$\Rightarrow 1 = \frac{3a}{b} - \frac{4a^3}{8b^3}$

$\Rightarrow a^3 + b^3 = 3ab^2$.

15. If $a^2 + b^2 + c^2 - 2ab = 0$, then the point of concurrency of family of lies ax+ by+ c= 0 lies on the line

(a) y= x **(b) y = x+ 1**
(c) y = -x **(d)3x= y**

Ans:- (c) $(a - b)^2 - c^2 = 0$

$\Rightarrow$ (a-b -c)(a- b+ c)=0

If a- b= c $\Rightarrow$ ax + by+ (a-b)=0

$\Rightarrow$ a (x+1)+b(y-1)=0$\Rightarrow$ x=-1 , y= 1

If a-b = -c $\Rightarrow$ ax+ by+ (b-a) =0

$\Rightarrow$ a (x- 1)+ b(y +1)=0

$\Rightarrow$ x=1, y= -1.

16. The value of k for which the inequality k$\cos^2 x - kcosx + 1 \geq 0\ \forall\, x \in (-\infty, \infty)$holds is

(a) k < -$\frac{1}{2}$ **(b) k > 4** **(c) $-\frac{1}{2} \le k \le 4$** **(d) $\frac{1}{2} \le k \le 5$**

Ans:- (c) $k\cos^2 x - k\cos x + 1 \ge 0 \;\forall\, x \in (-\infty, \infty)$

$\Rightarrow$ k $(\cos^2 x - \cos x)$ +1 ≥ 0…………(i)

But $\cos^2 x - \cos x = \left(\cos x - \frac{1}{2}\right)^2 - \frac{1}{4}$

$\Rightarrow -\frac{1}{4} \le \cos^2 x - \cos x \le 2$

From (i) we get 2k+1 $\ge 0 \Rightarrow k \ge -\frac{1}{2}$

$\Rightarrow -\frac{k}{4} + 1 \ge 0$

$\Rightarrow$ k ≤ 4

$\Rightarrow -\frac{1}{2} \le k \le 4$

17. The remainder obtained when 1! +2! +3!+….+ 95! is divided by 15, is

(a) 3 **(b) 5** **(c) 7** **(d) none**

Ans. (a) here 1! +2! +3! +4! = 33 and n! is divisible by 15 where n ≥ 5.

The remainder is same as the remainder obtained by dividing 33 with 15, i.e., 3.

18. The value of $\underset{x\to 0}{\text{Lt}} \frac{e^{\tan x} - e^x}{\tan x - x}$ is

(a) 0 **(b) 1** **(c) e** **(d) none**

Ans. (b) $\underset{x\to 0}{\text{Lt}} \frac{e^{\tan x} - e^x}{\tan x - x} = \underset{x\to 0}{\text{Lt}}\, e^x \left(\frac{e^{\tan x} - 1}{\tan x - x}\right) = e^0 . \log_e e = 1$

19. Total number of solutions of sinx = $\frac{|x|}{10}$ is

(a) 0 ; (b) 3 ; (c) 4 ; (d) none

Ans. (d) Two graphs meet exactly 6 times, hence, it has 6 solutions. Draw graph yourself.

20. A rigid body is spinning about a fixed point (3, -2, -1) with angular velocity of 4 rad/sec., the axis of rotation being in the direction of (1, 2, -2), then the velocity of the particle at the point (4, 1, 1) is

(a) 4/3 (1, -4, 10) **(b) 4/3 (4, -10, 1)** **(c) 4/3 (10, -4, 1)** **(d) 4/3(10, 4, 1)**

Ans:- (c)

$\vec{\omega} = 4\left(\frac{\hat{\imath} + 2\hat{\jmath} - 2\hat{k}}{\sqrt{1+4+4}}\right) = \frac{4}{3}\left(\hat{\imath} + 2\hat{\jmath} - 2\hat{k}\right)$

$\vec{r} = \vec{OP} - \vec{OA}$

$= (4\hat{\imath}+\hat{\jmath}+\hat{k})-(3\hat{\imath}-2\hat{\jmath}-\hat{k}) = \hat{\imath}+3\hat{\jmath}+2\hat{k}$

$\hat{v} = \hat{\omega} \times \vec{r} = \frac{4}{3}\left(\hat{\imath} + 2\hat{\jmath} - 2\hat{k}\right) \times \left(\hat{\imath} + 3\hat{\jmath} + 2\hat{k}\right) = \frac{4}{3}(10\hat{\imath} - 4\hat{\jmath} + \hat{k})$

21. A particle has an angular speed of 3 rad /sec and the axis of rotation passes through the point (1, 2, 2) and (1, 2, -2), then the velocity of the particle at the point P(3, 6, 4) is

(a) $\frac{3}{\sqrt{17}}$(22, 8, -2) **(b) $\frac{3}{\sqrt{17}}(22, 0, 2)$** **(c) $\frac{3}{\sqrt{17}}(22, -8, -2)$** **(d) $\frac{3}{\sqrt{17}}(22, -8, 2)$**

Ans:- (c)

$\vec{OA} = \hat{i} + \hat{j} + 2\hat{k}$

$\vec{OB} = \hat{i} + 2\hat{j} - 2\hat{k}$

$\therefore \vec{AB} = \hat{j} - 4\hat{k}$

$\Rightarrow |\vec{AB}| = \sqrt{17}$

$\vec{AP} = (3\hat{i}+6\hat{j}+4\hat{k}) - (\hat{i}+\hat{j}+2\hat{k})$

$= 2\hat{i}+5\hat{j}+2\hat{k}$

$\therefore \vec{\omega} = \frac{3}{\sqrt{17}}(\hat{j}-4\hat{k})$

$\vec{v} = \omega \times \vec{r} = \frac{3}{\sqrt{17}}(\hat{j}-4\hat{k}) \times (2\hat{i} + 5\hat{j} + 2\hat{k}) = \frac{3}{\sqrt{17}}(22\hat{i} - 8\hat{j} - 2\hat{k})$

22. In a group of equal number of boys and girls, 20% girls and 35% boys are graduate. If a member of the group is selected at random, then the probability of this member not being a graduate is

(a) $\frac{11}{40}$ **(b)** $\frac{9}{20}$ **(c)** $\frac{11}{20}$ **(d)** $\frac{29}{40}$

Ans. (d) Let A and B denotes the events that the member selected at random is a boy and a girl respectively. Let E denotes the event that the member selected is a graduate. Reqd. prob.

=1- [P(A).P(E/A)+P(B).P(E/B)]

$=1- \left[\frac{1}{2}.\frac{35}{100} + \frac{1}{2}.\frac{20}{100} = \frac{55}{200}\right] = \left(1 - \frac{11}{40}\right) = \frac{29}{40}$

23. for any two events A and B in a sample space

(a) P (A/B)≥ $\frac{P(A)+P(B)-1}{P(B)}, P(B) \neq 0$ is always true.

(b) P (A∩$\bar{B}$) = P (A)- P(A∩B) does not hold

(c) P (A∪B) =1-P ($\bar{A}$).P($\bar{B}$) if A and B are independent

(d) P (A∪B) =1-P ($\bar{A}$).P($\bar{B}$) if A and B are disjoint

Ans. (c)

$P(A/\bar{B}) + P(\bar{A}/\bar{B}) = 1$,

$\therefore P(\bar{A}+\bar{B}) = 1 - P(A/\bar{B})$

$$\frac{1 - P(A \cup B)}{P(B')} = \frac{P(A \cup B)'}{P(B')}$$
$$= \frac{P(A' \cap B')}{P(B')} = P\left(\frac{A'}{B'}\right)$$

24. one hundred identical coins, each with probability P, of showing up heads are tossed. If 0 < P < 1 and the probability of heads showing on 50 coins is equal to that of the heads showing on 51 coins, then p=

(a) $\frac{1}{2}$ **(b)** $\frac{49}{101}$

(c) $\frac{50}{101}$ **(d)** $\frac{51}{101}$

Ans. (d)

Here n= 100, p= p, q= 1-p

Given, p(50) = p(51)

$\Rightarrow 100_{C_{50}} p^{50}(1-p)^{50} = 100_{C_{51}} p^{51}(1-p)^{49}$

$$\Rightarrow \frac{100!}{50!\,50!}(1-p) = \frac{100!}{51!\,49!}p$$
$$\Rightarrow 51(1-p) = 50p \Rightarrow p = \frac{51}{101}$$

25. A box contains 24 identical balls of which 12 are white and 12 are black. The balls are drawn at random from the box one at a time with replacement. The probability that a white ball is drawn for the 4th time on the 7th draw is

(a) $\frac{5}{64}$ **(b)** $\frac{27}{32}$
(c) $\frac{5}{32}$ **(d)** $\frac{1}{2}$

Ans. (c) Probability of drawing a white ball in any draw=$\frac{12}{24} = \frac{1}{2}$

A white ball will be drawn for the 4th time on the 7th draw ball is drawn in the 7th draw and 3 white balls are drawn in the first 6 draws.

∴ Required probability = $6_{C_3} p^3 q^3 . p = 20. \left(\frac{1}{2}\right)^3 . \left(\frac{1}{2}\right)^3 . \frac{1}{2} = \frac{5}{32}$

26. If [x]denotes the integral part of x, then the domain of the function

f(x)= $\sin^{-1}[2x^2 - 3] + \log_2\{\log_{1/2}(x^2 - 5x + 5)\}$ **is**

(a) $\left(-\sqrt{\frac{5}{2}}, -1\right)$ **(b)** $\left(1, \sqrt{\frac{5}{2}}\right)$
(c) $\left(-\sqrt{\frac{5}{2}}, -1\right) \cup \left(1, \sqrt{\frac{5}{2}}\right)$ **(d) none of these**

Ans. (d)

For f(x) to be defined

(i) $[2x^2 - 3]$= -1, 0, 1

$\Rightarrow -1 \le 2x^2 - 3 < 2 \Rightarrow 2 \le 2x^2 < 5$

$\Rightarrow 1 \le x^2 < \frac{5}{2}$

$$\Rightarrow \begin{cases} 1 \le x^2 \Rightarrow x \le -1 \text{ or } x \ge 1 \\ x^2 < \frac{5}{2} \Rightarrow -\sqrt{\frac{5}{2}} < x < \sqrt{\frac{5}{2}} \end{cases}$$

$\Rightarrow -\sqrt{\frac{5}{2}} < x \le -1 \text{ or } 1 \le x < \sqrt{\frac{5}{2}}$ ………..(A)

(ii) $x^2 - 5x + 5 > 0 \Rightarrow x < \frac{5-\sqrt{5}}{2} \text{ or } x > \frac{5+\sqrt{5}}{2}$ …….(B)

(iii) $\log_{\frac{1}{2}}(x^2 - 5x + 5) > 0$

$\Rightarrow x^2 - 5x + 5 < \left(\frac{1}{2}\right)^0$

$\Rightarrow x^2 - 5x + 5 < 1 \Rightarrow x^2 - 5x + 4 < 0$

$\Rightarrow$ 1< x< 4 ……………(C)

From (A), (B) and (C), $1 \le x < \frac{5-\sqrt{5}}{2}$

27. If f(x)= $\underset{m\to\infty}{\text{Lt}}\ \underset{n\to\infty}{\text{Lt}}\ \cos^{2m} n!\,\pi x$**, then range of f(x)is**

(a) [0, 1] **(b) [0, 1]**
(c) (0, 1) **(d) {0}**

Ans. (b)

When x is rational say p/q , then n! $x\pi$ is a multiple of π and $\cos^2 n!\, x\pi = 1$

∴ $\underset{m\to\infty}{\text{Lt}} \cos^{2m} n!\, x\pi = \underset{m\to\infty}{\text{Lt}} 1^m = 1$ ∴ f(x)= 1

When x is irrational,

n! $x\pi \ne$a multiple of π

$\therefore \cos^2 n!\, x\pi \ne 1$

$\therefore 0 \leq cos^2 n!\, x\pi < 1$

$\therefore \underset{m\to\infty}{\text{Lt}}\ cos^{2m}\, n!\, x\pi =$
$\underset{m\to\infty}{\text{Lt}} (cos^2 n!\, x\pi)^m = 0$

Thus f(x) = 0, when x is rational

= 1, when x is irrational

∴ Range f= {0, 1}

28. The normal at any point P $(t^2, 2t)$ on the parabola $y^2 = 4x$ meets the curve again at Q, the area of ΔPOQ, O being the origin is $\frac{k}{|t|}(1+t^2)(2+t^2)$ then

(a) k > 2 **(b) k=2**
(c) k < 2 **(d) k= 1**

Ans. (b) Given P ≡ $(t^2, 2t)$

Given parabola is $y^2 = 4x$ ……….. (1)

Here a= 1.

Let Q=($t_1^2, 2t_1$)

Since normal at P meet the curve again at Q.

$\therefore t_1 = -t - \frac{2}{t} = \frac{t^2+2}{t}$ ………… (2)

Now O≡ (0, 0), P≡ $(t^2, 2t)$, Q=($t_1^2, 2t_1$)

Given,

$\frac{k}{|t|}(1+t^2)(2+t^2) = area\ of\ \Delta POQ$

$= \frac{1}{2}|t^2.2t_1 - 2t.t_1^2| = |t^2 t_1 - t t_1^2| =$
$\left|-t^2\left(\frac{t^2+2}{2}\right) - t\frac{(t^2+2)^2}{t^2}\right|$

$= (t^2+2)\left|t + \frac{(t^2+2)}{t}\right| = (t^2+2)\left|t + \frac{(t^2+2)}{t}\right| = (t^2+2)2\ \frac{(1+t)^2}{|t|}$

∴ k= 2

29. If {x} denotes the fractional part of x, then $\left\{\frac{3^{200}}{8}\right\} =$

(a) $\frac{1}{8}$ **(b) $\frac{3}{8}$**
(c) $\frac{5}{8}$ **(d) none**

Ans. (d)

$$\frac{3^{200}}{8} = \frac{9^{100}}{8} = \frac{(1+8)^{100}}{8}$$
$$= \frac{1 + 100_{C_1}8 + 100_{C_1}8^2 + \cdots + 8^{100}}{8}$$
$$= \frac{1}{8} + an\ integer$$

$\therefore \left\{\frac{3^{200}}{8}\right\} = \frac{1}{8}$

30. Which of the following function does not obey mean value theorem in [0, 1]

(a) f(x)= $\frac{1}{2}$ - x, x < ½; f(x)=$\left(\frac{1}{2} - x\right)^2, x \geq \frac{1}{2}$
(b) f(x) = $\frac{\sin x}{x}, x \neq 0; f(x) = 1, x = 0$

(c) f(x)= x |x|
(d) f(x)= |x|

Ans. (a) Let f(x) = $x^3 - 3x + k$

Then f'(x) = $3(x^2 - 1) < 0\ in\ (0, 1)$

⟹ f'(x) has no root in (0, 1)

But f(x) = 0 has two distinct roots $\alpha$and β in (0, 1)

⟹ f'(x)= 0 has at least one root in (α, β).

ISI B.STAT/B.MATH OBJECTIVE QUESTIONS & SOLUTIONS

SET – 8

1. If [x] denotes the integral part of x, then $\underset{x\to 0}{\mathrm{Lt}} \frac{\sin[\cos x]}{1+[\cos x]} =$

(a) 0 **(b) 1** **(c) $\frac{\sin 1}{2}$**

(d) does not exist

Ans. (a) $\underset{x\to 0-0}{\mathrm{Lt}} [\cos x] = 0$

[∵ when x ⟶0- 0, 0 < cosx < 1] and

$\underset{x\to 0+0}{\mathrm{Lt}} [\cos x] = 0$

[∵ when x ⟶0+ 0, 0 < cosx < 1]

$\therefore \underset{x\to 0-0}{\mathrm{Lt}} \frac{\sin[\cos x]}{1+[\cos x]} = \frac{\sin 0}{1+0} =$
$0 \underset{x\to 0+0}{\mathrm{Lt}} \frac{\sin[\cos x]}{1+[\cos x]} = \frac{\sin 0}{1+0} = 0$

∴Required limit = 0

2. Let f(x)= $\underset{n\to\infty}{\mathrm{Lt}} \sum_{r=0}^{n-1} \frac{x}{(rx+1)\{(r+1)x+1\}}$ then

(a) f(x) is continuous but not differentiable at x= 0

(b) f(x) is both continuous and differentiable at x= 0

(c) f(x) is neither continuous nor differentiable at x=0

(d) f(x) is a periodic function

Ans. (c)

$$t_{r+1} = \frac{x}{(rx+1)\{(r+1)x+1\}}$$
$$= \frac{(r+1)x+1-(rx+1)}{(rx+1)[(r+1)x+1]}$$

$$= \frac{1}{(rx+1)} - \frac{1}{(r+1)x+1}$$

$$\therefore S_n = \sum_{r=0}^{n-1} t_{r+1} \frac{1}{nx+1} = 1, x \neq 0 = 0, x$$
$$= 0$$

$$\therefore \underset{n\to\infty}{\mathrm{Lt}} S_n = \underset{n\to\infty}{\mathrm{Lt}} \left(1 - \frac{1}{nx+1}\right)$$

Thus, f(x)= $\begin{cases} 1, x \neq 0 \\ 0, x = 0 \end{cases}$

$\therefore \underset{x\to 0}{\mathrm{Lt}} f(x) = 1 \; and \; f(0) = 0$

Hence f(x) is neither continuous nor differentiable at x= 0

Clearly f(x) is not a periodic function.

3. Let f(x)= $\underset{n\to\infty}{\mathrm{Lt}} \frac{\log(2+x)-x^{2n}\sin x}{1+x^{2n}}$ then f(x) is discontinuous at

(a) x= 1 only **(b) x=-1 only**
(c) x= -1, 1 only **(d) no point**

Ans. (c)

$$\underset{n\to\infty}{\mathrm{Lt}} X^{2n} = \underset{n\to\infty}{\mathrm{Lt}} (x^2)^n$$
$$= \begin{cases} \infty, & x^2 > 1 \\ 0, & 0 \leq x^2 < 1 \\ 1, & x^2 = 1 \end{cases}$$
$$= \begin{cases} \infty, & x < -1 \; or \; x > 1 \\ 0, & -1 < x < 1 \\ 1, & x = \pm 1 \end{cases}$$

$\therefore f(x)$

$$= \begin{cases} -sinx, & x < -1 \; or \; x > 1 \\ \log(2+x), & -1 < x < 1 \\ \dfrac{\log(2+x) - sinx}{2}, & x = \pm 1 \end{cases}$$

$$\underset{x\to 1+0}{\text{Lt}} f(x) = -sin1, \underset{x\to 1-0}{\text{Lt}} f(x)$$
$$= log3, \underset{x\to 1+0}{\text{Lt}} f(x)$$
$$= \underset{x\to -1}{\text{Lt}} \log(2+x) = 0,$$

$$\underset{x\to 1-0}{\text{Lt}} f(x) = \underset{x\to -1}{\text{Lt}} (-\sin x) = sin1$$

Clearly f(x) is discontinuous only at two points x= -1, 1

4. The function f(x) = max {(1-x), (1+x), 2} is, where x ∈(-∞, ∞)

(a) discontinuous at all points
(b) differentiable at all points

(c) differentiable at all points except -1 and 1 (d) continuous at all points except -1 and 1

Ans. (c)

We draw the graph of y= 1- x, y= 1+ x and y = 2

f(x)= max.{1-x, 1+x, 2}

∴ f(x)= 1- x, x ≤ -1= 2, -1≤ x≤ 2 = 1+x, x ≥ 2

From graph it is clear that f(x) is continuous at all x and differentiable at all x except x= -1 and x= 1

5. If f(x) = p |sin x|+$qe^{|x|}$ + $r|x|^3$ *and* $f(x)$ is differentiable at x=0, then

(a) p= q= r=0
(b) p=0, q=0, r== any real number

(c) q=0, r=0, p is any real number
(d) r=0, p=0, q is any real number

Ans. (b)

At x= 0,

L.H. derivative of p |sin x|= - p

R.H. derivative of p |sin x|= p

∴ For p |sin x| to be differentiable at x= 0, p= - p or p= 0

At x= 0, L.H. derivative of q$e^{|x|}$= q

For q$e^{|x|}$to be differentiable at x= 0, -q = q or q= 0

d.c. of r $|x|^3$ at x= 0 is 0

∴ for f(x) to be differentiable at x= 0, p= 0, q= 0and r may be any real number.

<u>Second method:</u>

f'(0- 0)= $\underset{h\to 0-0}{\text{Lt}} \frac{f(h)-f(0)}{h} =$
$\underset{h\to 0-0}{\text{Lt}} \frac{p|\sin h|+qe^{|h|}+r|h|^3-q}{h}$

$$= \underset{h\to 0-0}{\text{Lt}} \frac{-p\sin h + qe^{h} - rh^3 - q}{h}$$
$$= \underset{h\to 0-0}{\text{Lt}} \left\{ -p\frac{sinh}{h} - \frac{q(e^{h}-1)}{-h} - rh^2 \right\}$$
$$= -p - q$$

Similarly, f' (0+0)= p+q

Since f(x) is differentiable at x= 0

∴ f'(0- 0) = f'(0+ 0)⟹ - p- q= p+ q

⟹ p+ q= 0

Here r may be any real number.

6. Let f(x)= $\boldsymbol{x^3 - x^2 + x + 1}$,

$$\boldsymbol{g(x) = max.\{f(t), 0 \le t \le x\}, 0 \le x \le 1}$$

$$= \boldsymbol{3 - x, 1} < x \ \le 2$$

then in [0, 2] the points where g(x) is not differentiable is (are)

(a) 1 (2) 2
(c) 1 and 2 (d) none of these

Ans. (a) $f(t) = t^3 - t^2 + t + 1$

∴ f' (t)= $3t^2 - 2t + 1 > 0$

∴ f(t) is an increasing function.

Since 0≤ t ≤ x

∴ max f(t)= f(x)= $x^3 - x^2 + x + 1$

Thus g(x)= $x^3 - x^2 + x + 1, 0 \le x \le 1$= 3-x, 1 < x ≤ 2

The only doubtful point for differentiability of g(x) in [0, 2] is x = 1

Clearly, $\underset{x\to1+0}{\text{Lt}} g(x) = 1^3 - 1^2 + 1 + 1 = 2$

$\underset{x\to1+0}{\text{Lt}} g(x) = \underset{x\to1}{\text{Lt}} (3 - x) = 2 \ and \ g(1) = 2$

∴ g(x) is continuous at x= 1

Also g' (x)= $3x^2 - 2x + 1, 0 \le x < 1 = -1, 1 < x \le 2$

∴ g'(1-0)= 3. $1^2 - 2.1 + 1 = 2$ and g'(1+0)= -1

Hence g(x) is not differentiable at x= 1.

7. If [x] denotes the integral part of x and f(x)= [x]$\boldsymbol{\left\{\frac{sin\frac{\pi}{[x+1]}+\sin\pi[x+1]}{1+[x]}\right\}}$, then

(a) f(x) is continuous in R
(b) f(x) is continuous but not differentiable in R

(c) f"(x) exists for all x in R
(d) f(x) is discontinuous at all integer points in R

Ans. (d)

Sin π [x+ 1]= 0

Also [x+ 1]= [x]+ 1

∴ f(x)= $\frac{[x]}{1+[x]}\sin\frac{\pi}{[x]+1} \ at \ x = n, n \ \in I, f(x) = \frac{n}{1+n}\sin\frac{\pi}{n+1} \ for \ n - 1 < x < n, [x] = n - 1$

∴ f(x)= $\frac{n-1}{n}\sin\frac{\pi}{4}$

$$Hence \ \underset{x\to n-0}{\text{Lt}} f(x)\frac{n-1}{n}\sin\frac{\pi}{4},$$

∴ f(n)= $\frac{n}{1+n}\sin\frac{\pi}{n+1}$

∴ f(x) is discontinuous at all n ∈ I

8. Let f(x) = $\frac{[tan^2x]-1}{tan^2x-1}$, $x \neq n\pi \pm \frac{\pi}{4} = 0, x = n\pi \pm \frac{\pi}{4}$ then f(x) is

(a) continuous at all x (b) continuous at x= $\frac{\pi}{4}$ (c) discontinuous at x= $\frac{\pi}{4}$ (d) none

Ans. (c)

Since tan x is not defined at

X= (2n+1) $\frac{\pi}{2}, n \in I$, $therefore\ f(x) is\ discontinuous\ at\ x = (2n+1)\frac{\pi}{2}, n \in I$

Now f $\left(\frac{\pi}{4}\right) = 0$

$$\underset{x\to\frac{\pi}{4}-0}{\text{Lt}} f(x) = \underset{x\to\frac{\pi}{4}-0}{\text{Lt}} \frac{[tan^2x]-1}{tan^2x-1}$$
$$= \underset{x\to\frac{\pi}{4}-0}{\text{Lt}} \frac{0-1}{tan^2x-1} = \infty$$

Hence f(x) is discontinuous at x= $\frac{\pi}{4}$

9. Let f(x)= $\int_0^x t\, sin\frac{1}{t}\, dt$, then the number of points of discontinuity of f(x) in (0, π) is

(a) 0 **(b) 1**
(c) 2 **(d) more than 2**

Ans. (a)

$$f(x) = \int_0^x t \sin\frac{1}{t}\, dt$$

∴ f'(x)= x sin $\frac{1}{x}$

Clearly f'(x) is a finite number at all x in (0, π).

∴ f(x) is differentiable and hence continuous at all x in (0, π)

10. if [x] denotes the integral part of x and in (0, π), we define

f(x)= $\left[\frac{2(sinx-sin^nx)+|sinx-sin^nx|}{2(sinx-sin^nx)-|sinx-sin^nx|}\right] = 3, x = \frac{\pi}{2}$ $x \neq \frac{\pi}{2}$ then for n > 1

(a) f(x) is continuous but not differentiable at x= $\frac{\pi}{2}$
(b) both continuous and differentiable at x= $\frac{\pi}{2}$
(c) (c) neither continuous nor differentiable at x= $\frac{\pi}{2}$
(d) $\underset{x\to\frac{\pi}{2}}{\text{Lt}} f(x)\ exist\ but\ \underset{x\to\frac{\pi}{2}}{\text{Lt}} f(x) \neq f(\frac{\pi}{2})$

Ans. (b)

For $0 < x < \frac{\pi}{2}$ or $\frac{\pi}{2} < x < \pi$ $0 < \sin x < 1$

∴ for n > 1, sin x > sin^4x

∴ f(x)=$\left[\frac{3(\sin x - sin^4x)}{\sin x - sin^nx}\right] = 3, x \neq \frac{\pi}{2} = 3, x = \frac{\pi}{2}$

Thus in (0, π) , f(x) = 3

Hence f(x)is continuous and differentiable at x= $\frac{\pi}{2}$

11. If[x] denotes the integral part of x and f(x) = [n+ psin x], 0< x < π, n∈I and p is a prime number, then the number of points where f(x) is not differentiable is

(a) p-1 **(b) p**
(c) 2p-1 **(d) 2p +1**

Ans. (c) [x] is not differentiable at integral points.

Also [n+ p sin x]= n+ [p sin x]

∴ [p sin x] is not differentiable, where p sin x is an integer. But p is prime and 0 < sin x ≤1 [∵ 0 < x < π]

∴ p sin x is an integer only when

Sinx = $\frac{r}{p}$, where 0< r ≤ p and r ∈ N

For r= p, sin x= 1 ⟹ x= $\frac{\pi}{2}$ in (0, π)

For 0 < r < p, sin x= $\frac{r}{p}$

∴ x= $\sin^{-1}\frac{r}{p}$ $or\ \pi - \sin^{-1}\frac{r}{p}$

Number of such values of x= p- 1+p- 1= 2p -2

∴ Total number of points where f(x) is not differentiable

= 1+ 2p – 2= 2p – 1

12. If $\frac{sec^4\theta}{a} + \frac{tan^4\theta}{b} = \frac{1}{a+b}$, *then*

(a) |b|=|a| **(b) |b|≤ $|a|$**
(c) |b|≥ $|a|$ **(d) none of these**

Ans. (b)

$$\frac{sec^4\theta}{a} + \frac{tan^4\theta}{b} = \frac{1}{a+b} = \frac{sec^2\theta - tan^2\theta}{a+b}$$

$\Rightarrow \frac{sec^2\theta}{a(a+b)}[(a+b)sec^2\theta - a] + \frac{tan^2\theta}{(a+b)b}[(a+b)tan^2\theta + b] = 0$

$\Rightarrow a tan^2\theta + b sec^2\theta = 0$

$\Rightarrow sin^2\theta = -\frac{b}{a}$ *is non*-negative and ≤ 1⟹ $\left|\frac{b}{a}\right| \leq 1$

13. If c be a positive constant and |f(y)- f(x)|≤ $c(y-x)^2$ for all real x and y, then

(a) f(x)= 0 for all x **(b) f(x)= x for all x**
(c) f'(x)= 0 for all x **(d) f'(x)= c for all x**

Ans. (c)

Given, |f(y)- f(x)| ≤$c(y-x)^2, c > 0$

⟹|f(y)- f(x)| ≤$c|y-x|^2 \Rightarrow \left|\frac{f(y)-f(x)}{y-x}\right| \leq c|y-x| \Rightarrow \underset{y\to x}{\text{Lt}}\left|\frac{f(y)-f(x)}{y-x}\right| \leq \underset{y\to x}{\text{Lt}}\ c|y-x|$

$\Rightarrow |f'(x)| \leq 0$

⟹$|f'(x)| \leq 0$ for all real x

⟹ f'(x)= 0∀x∈ R

14. Let f(t)in t. then $\frac{d}{dx}\left\{\int_{x^2}^{x^3} f(t)dt\right\}$

(a) has a value 0 when x= 0
(b) has a value 0 when x=1 , x= 4/9

(c) has a value $9e^2 - 4e$ when x=e
(d) has a differential coefficient 27e -8 when x= e

Ans. (c) $\frac{d}{dx}\int_{x^2}^{x^3} f(t)\,dt = f(x^3).3x^2 - f(x^2).2x$

$= logx^3.3x^2 - \log x^2.2x$
$= 9x^2\,logx - 4x\,logx$
$= x\,logx(9x-4)$

$let\ z = x\log x\,(9x-4) then \dfrac{dz}{dx}$
$= (1+\log x)(9x-4)$
$+ 9x\log x$

$at\ x = e, \quad \dfrac{dz}{dx} = 2(9e-4) + 9e$
$= 27e - 8$

15. If a, $\alpha_1, \alpha_2, \ldots \alpha_{2n-1}, b$ are in $A.P.$,

$a, \beta_1, \beta_2, \ldots \beta_{2n-1}, b$ are in G.P. and a, $\gamma_1, \gamma_2, \ldots \gamma_{2n-1}, b$ are in H.P., where a, b are positive, then the equation

$\alpha_n x^2 - \beta_n x + \gamma_n = 0$ has

(a) real and equal roots
(b) real and unequal roots

(c) imaginary roots
(d) roots which are in A.P.

Ans. (c)

The middle terms of the A.P., G.P. and H.P. are α_n, β_n and γ_n respectively

$\therefore \alpha_n$= A.M. of a and b,

β_n= G.M. of a and b,

γ_n= H.M. of a and b,

$\because$ AH=G^2

$\therefore \alpha_n\gamma_n = {\beta_n}^2$

Now, discreminant of given equation =${\beta_n}^2 - 4\alpha_n\gamma_n = -3\alpha_n\gamma_n < 0$ ($\because \alpha_n, \gamma_n$ are positive)

16. If a_n = the digit at units place in the number 1! +2! +3! +…+n! for n≥ 4,

then $a_4, a_5, a_6, \ldots$ are in

(a) A.P. only (b) G.P. only (c) A.P. and G.P. only (d) A.P., G.P., and H.P.

Ans.(c)

1 ! +2 ! +3 ! +4! = 33

The digits at units place in each of 5 !, 6 !,… is 0

$\therefore a_4 = a_5 = a_6 = \cdots = 3$

Clearly $a_4, a_5, a_6, \ldots$ are in A.P. and G.P. but not in H.P. as they are equal.

17. Let p, q, r ∊ R^+ and 27pqr ≥ $(p+q+r)^3$ and 3p +4q +5r = 12 then $p^3 + q^4 + r^5$ is equal to

(a) 3 (b) 6 (c) 2 (d) none of these.

Ans. (c)

1 ! +2 ! +3 ! +4! = 33

The digits at units place in each of 5 !, 6 !,… is 0

$\therefore a_4 = a_5 = a_6 = \cdots = 3$

Clearly $a_4, a_5, a_6, \ldots$ are in A.P. and G.P. but not in H.P. as they are equal.

18. If (2+ x)(2+x^2)$(2+x^3)\ldots(2+x^{100})$ = $\sum_{r=0}^{n} x^r$, then n equals

(a) 2550 (b) 5050
(c) 2^8 (d) none of these.

Ans. (b) $x^n = x^{1+2+3+\cdots+100} = x^{\frac{100\times101}{2}} = x^{5050}$

$\Longrightarrow$ n= 5050

19. If p, q, r, s ∊ R, then equation $(x^2 + px + 3q)(- x^2 + rx + q)(- x^2 + sx - 2q)$= 0 has

(a) 6 real roots
(b) at least two real roots

(c) 2 real and 4 imaginary roots
(d) 4 real and 2 imaginary roots.

Ans. (b)

$D_1 + D_2 + D_3 = p^2 - 12q + r^2 + 4q + s^2 + 8q = p^2 + r^2 + s^2 \geq 0$

$\Longrightarrow$ at least one of $D_1, D_2, D_3 \geq 0$

20. If a, b, c, d, are four non-zero real numbers such that $(d + a - b)^2 + (d + b - c)^2$= 0 and roots of the equation a(b-c) x^2 + b(c-a)x + c(a-b)= 0 and real and equal, then a, b, c

(a) are equal (b) are not equal (c) are zero (d) none of the above

Ans. (a) Equation $a(b - c)x^2 + b(c - a)x + c(a - b)$= 0 has equal roots

$\Longrightarrow$ b= $\frac{2ac}{a+c}$ …….(1)

$(d + a - b)^2 + (d + b - c)^2$ =0

$\Longrightarrow$ a-b = b- c = -d $\Longrightarrow$ 2b= a+ c ……(2)

$\Longrightarrow \frac{4ac}{a+c}$ = a +c $\Longrightarrow (a - c)^2 = 0 \Longrightarrow$ a= c

From (2), b= a

Thus a= b= c.

21. If p, q be non zero real numbers and f(x)≠ 0 in [0, 2] and $\int_0^1 f(x).(x^2 + px + q)dx = \int_0^2 f(x).(x^2 + px + q)dx = 0$ then equation $x^2 + px + q$= 0 has

(a) two imaginary roots
(b) no root in (0, 2)

(c) one root in (0,1) and other in (1,2)
(d) one root in (-∞, 0) and other in (2,∞)

Ans. (c)

Let F(x) = $\int f(x)(x^2 + px + q)dx,$

Then according to question

F(1) –F(0) = 0, F(2) – F(1) = 0

∴ F(0) = F(1) and F(1)= F(2)

Hence, equation F'(x)= 0 i.e. equation f(x). $(x^2 + px + q) = 0$

i.e., equation $x^2 + px + q$ = 0 has at least one root (here exactly one root) in (0, 1) and exactly one root in (1, 2).

22. If a, b, c, ∊ R, a ≠ 0 and $(b - 1)^2 < 4ac$, then the number of roots of the system of equation (in three unknowns x_1, x_2, x_3)

$$ax_1{}^2 + bx_1 + c = x_2$$

$$ax_2{}^2 + bx_2 + c = x_3$$

$$ax_3{}^2 + bx_3 + c = 1 \quad \textbf{is}$$

(a) 0 (b) 1 (c) 2 (d) 3

Ans. (a) Let f(x) = $ax^2 + (b - 1)x$+c

Given system of equation is equivalent

to $\left.\begin{matrix} f(x_1)=x_2-x_1 \\ f(x_2)=x_3-x_2 \\ f(x_3)=x_1-x_3 \end{matrix}\right\}$

$\Rightarrow f(x_1)+f(x_2)+f(x_3)=0$

$\therefore af(x_1)+af(x_2)+af(x_3)=0$ (not possible)

As $(b-1)^2-4ac<0$.

$\therefore af(x_1), af(x_2), af(x_3)>0$.

Hence given system of equation has no real root.

23. If α, β are the roots of the equation x^2-ax +b= 0 and $A_n=\alpha^n+\beta^n$ then which of the following is true?

(a) $A_{n+1}=aA_n+bA_{n-1}$
(b) $A_{n+1}=bA_n+aA_{n-1}$

(c) $A_{n+1}=aA_n-bA_{n-1}$
(d) $A_{n+1}=bA_n-aA_{n-1}$

Ans.(a)

$\alpha+\beta$= a, $\alpha\beta$= b

Given, $A_n=\alpha^n+\beta^n$

Now, $A_{n+1}=\alpha^{n+1}+\beta^{n+1}$

= $(\alpha^n+\beta^n)(\alpha+\beta)-\alpha\beta(\alpha^{n-1}+\beta^{n-1})$

= aA_n+bA_{n-1}

24. If x satisfies |x-1|+|x-2|+|x-3|≥ 6, then

(a) $0\le x\le 4$ (b) $x\le -2$ or $x\ge 4$
(c) $x\le 0$ or $x\ge 4$ (d) $x\ge 0$

Ans. (c)

For $x\le 1, -3x+6\ge 6 \Rightarrow x\le 0$ ……(A)

For $1\le x\le 2, -x+4\ge 6 \Rightarrow x\le -2$

(not acceptable as $1\le x\le 2$)

For $x\ge 3, 3x-6\ge 6 \Rightarrow x\ge 4$ ………(B)

From (A) and (B) all positive value of x are given by $x\le 0$ or $x\ge 4$

25. $2^{sinx}+2^{cosx}\ge 2^{1-\frac{1}{\sqrt{2}}}$

(a) only for x ≥0 (b) only for x≤ 0
(c) for all real x (d) only for x ≠0

Ans. (c)

Since A.M. ≥ G.M

$\therefore \frac{2^{\sin x}+2^{\cos x}}{2}\ge\sqrt{2^{\sin x}.2^{\cos x}}=$
$2^{\left(\frac{1}{2}\right)(\sin x+\cos x)}=2^{\left(\frac{1}{\sqrt{2}}\right)\sin\left(x+\frac{\pi}{4}\right)}$

$\Rightarrow 2^{\sin x}+2^{\cos x}\ge 2^{1+\frac{1}{\sqrt{2}}\sin(x+\frac{\pi}{4})}\ge 2^{1-\frac{1}{\sqrt{2}}}$

[$\because$ least value of $\sin\left(x+\frac{\pi}{4}\right)=-1$]

26. How many different nine digit numbers can be formed from the number 223355888 by rearranging its digits so that the odd digits occupy even positions?

(a) 16 (b) 36
(c) 60 (d) 180

Ans. (c)

Number of digits= 9

Number of odd digits = 4, number of even digits= 5

Number of even places= 4

Odd digits can be arranged in even paces in $\frac{\underline{|4}}{\underline{|2}\underline{|2}}$ ways. Even digits can be arranged in remaining 5 places in $\frac{\underline{|5}}{\underline{|2}\underline{|3}}$ ways

∴ Required number = $\frac{\underline{|4}}{\underline{|2}\underline{|2}}.\frac{\underline{|5}}{\underline{|2}\underline{|3}} = 60$

27. For 2≤ $\boldsymbol{r \leq n}$, $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} =$

(a) $\binom{n+1}{r-1}$ (b) $2\binom{n+1}{r+1}$
(c) $2\binom{n+2}{r}$ (d) $\binom{n+2}{r}$

Ans. (d)

$\binom{n}{r}$ *stands for* n_{C_r}

Now $n_{C_r} + 2n_{C_{r-1}} + n_{C_{r-2}}$

$= \left(n_{C_r} + n_{C_{r-1}}\right) + \left(n_{C_{r-1}} + n_{C_{r-2}}\right)$

$= n + 1_{C_r} + n + 1_{C_{r-1}} = n + 2_{C_r}$

28. If $\sum_{i=1}^{10} \sin^{-1} X_i = 5\pi, then \sum_{i=1}^{10} {X_i}^2 =$

(a) 0 (b) 5
(c) 10 (d) none of these

Ans. (c)

$$\sum_{i=1}^{10} \sin^{-1} x_i = 5\pi = 10.\frac{\pi}{2}$$

$\Rightarrow \sin^{-1} x_i = \frac{\pi}{2}, \forall i \Rightarrow x_i = 1 \quad \forall \quad i \quad \Rightarrow \sum_{i=1}^{10} {x_i}^2 = 1$

29. Range of f(x) = $\boldsymbol{sin^{20}x + cos^{48}x}$ is

(a) [0, 1] (b) (0, 1)
(c) (0, ∞) (d) none of these

Ans. (b)

$0 \leq sin^2x \leq 1 \implies sin^{20}x \leq sin^2x$

Thus $0 \leq sin^{20}x \leq sin^2x$ ………. (1)

Again $0 \leq cos^{48}x \leq cos^2x$ ………..(2)

[∵ $sin^{20}x\ and\ cos^{48}x\ cannot\ be\ zero\ at\ a\ time$]

⟹ 0 < f(x) ≤ 1. Hence range of f(x) = (0, 1)

30. Let x, y, z = 105, where x, y, z ∈N. Then number of ordered triplets (x, y, z) satisfying the given equation is:

(a) 15 (b) 27
(c) 6 (d) none of these

Ans. (b)

105= 3× 5 × 7

When no 1 is taken as a solution, number of solutions= $\underline{|3} = 6$

When only 1' s taken, number of solutions= $3_{C_2}.\underline{|3} = 18$

When two 1's are taken, number of solutions = $3_{C_1}.\frac{\underline{|3}}{\underline{|2}} = 3$

∴ Reqd. number= 6+18+3 =27

ISI B.STAT/B.MATH OBJECTIVE QUESTIONS & SOLUTIONS

SET – 9

1. If f(x) = $(p\alpha-\alpha^2-2)x-\int_0^x(\cos^4 t+\sin^2 t-2)\,dt$ is a decreasing function of x for all x ∈R and α∈ R, where α being independent of x, then

(a) p∈ (-∞, 1) **(b) p∈ (-1, $\sqrt{3}$)**
(c) p∈ (1, ∞) **(d) none of these**

Ans. (b) Given, $f(x)=(p\alpha-\alpha^2-2)x-\int_0^x(\cos^4 t+\sin^2 t-2)\,dt$ ……….(1)

$\therefore f'(x)=p\alpha-\alpha^2-2-(\cos^4 x+\sin^2 x-2)=-\alpha^2+p\alpha-(\cos^4 x+\sin^2 x)$

$=-\alpha^2+p\alpha-(\cos^4 x+\cos^2 x+1)$

$$=-\alpha^2+p\alpha-\left[\left(\cos^2 x-\frac{1}{2}\right)^2+\frac{3}{4}\right]$$
$$=-\alpha^2+p\alpha-\frac{3}{4}-\left(\cos^2 x-\frac{1}{2}\right)^2$$

Clearly f'(x) ≤ $-\alpha^2+p\alpha-\frac{3}{4}$ [∵ $Min.value\ of\ \left(\cos^2 x-\frac{1}{2}\right)^2=0$]

For f(x) to be decreasing for all real x, i.e., f'(x) ≤ 0

$\therefore -\alpha^2+p\alpha-\frac{3}{4}\le 0 \Rightarrow 4\alpha^2-4p\alpha+3\ge 0, \forall\ \alpha\in R$

$\therefore$ D ≤ 0$\Rightarrow 16p^2-48\le 0 \Rightarrow -\sqrt{3}\le p\le \sqrt{3}$

2. Consider the following statements S and R. S: both sinx and cosx are decreasing function in $(\frac{\pi}{2},\pi)$ & R: If a differentiable function decreases in (a, b) then its derivative also decreases in (a, b). Which of the following are true?

(a) both S and R are wrong
(b) S is correct and R is wrong

(c) both S and R are correct but R is not the correct expiation for S

(d) S is correct and R is the correct explanation for S

Ans. (b) From the trend of value of sin x and cos x we know sin x and cosx decrease in $\frac{\pi}{2}<x<\pi$. So, the statement S is correct.

The statement R is incorrect cos x is a differentiable function which decreases in $\left(\frac{\pi}{2},\pi\right)$ but its d.c. – sin x is increasing in $\left(\frac{\pi}{2},\pi\right)$

3. If f(x)= $\int_{x^2}^{x^2+1}e^{-t^2}\,dt$, then the interval in which f(x) is increasing is

(a) (0, ∞) **(b) (-∞, 0)**
(c) [-2, 2] **(d) none of these**

Ans. (b)

$$f(x) = \int_{x^2}^{x^2+1} e^{-t^2}\,dt\ f'(x)$$

$$= e^{-(x^2+1)^2}.2x - e^{-x^4}.2x$$

$$= \frac{2x}{e^{(x^2+1)^2}}\left[1 - e^{-x^4+(x^2+1)^2}\right]$$

$$= \frac{2x}{e^{(x^2+1)^2}}\left[1 - e^{2x^2+1}\right]$$

$$= \frac{2\left(e^{2x^2+1}-1\right)}{e^{(x^2+1)^2}}(-x)$$

But $e^{2x^2+1} > 1$

∴ f'(x)> 0 in (-∞, 0) and hence f(x) is increasing in (-∞, 0)

4. The value of $\int_0^x \frac{(t-|t|)^2}{1+t^2}\,dt$ is equal to

(a) 4(x-$\tan^{-1} x$) if x < 0 **(b) 0 if > 0**
(c) $\log(1+x^2)$ if x > 0 **(d) none of these**

Ans. (a)

$$I = \int_0^x \frac{(t-|t|)^2}{1+t^2}\,dt$$

$Case\ I: x > 0\ , then\ 0 < t < x, |t| = t$

∴ I= $\int_0^x \frac{(t-t)^2}{1+t^2}\,dt = 0$

Case II: x < 0, then x < t< 0⟹ |t|= -t

∴ I= $\int_0^x \frac{(t+t)^2}{1+t^2}\,dt = \int_0^x \frac{4t^2}{1+t^2}\,dt =$ $4\int_0^x \left(1 - \frac{1}{1+t^2}\right)dt = 4[t - \tan^{-1} t]_0^x$

$= 4(x - \tan^{-1} x)$

5. If $\int \frac{x^2-2}{(x^4+5x^2+4)\tan^{-1}\left(\frac{x^2+2}{x}\right)^{dx}} =$ $log|f(z)| + c$, then

(a) f(z) = $\tan^{-1} z\,, where\ z = \sqrt{x+2}$
(b) f(z) = $\tan^{-1} z\,, where\ z = x + \frac{2}{x}$
(c) f(z)= $\sin^{-1} z\,, where\ z = \frac{x+2}{x}$
(d) none of these

Ans. (b)

I= $\int \frac{x^2-2}{(x^4+5x^2+4)\tan^{-1}(\frac{x^2+2}{x})}\,dx$

Dividing numerator & denominator by x^2 we have

$= \int \frac{1-\frac{2}{x^2}}{\left(x^2+5+\frac{4}{x^2}\right)\tan^{-1}\left(x+\frac{2}{x}\right)}\,dx =$

$\int \frac{1-\frac{2}{x^2}}{[\left(x+\frac{2}{x}\right)^2+1]\tan^{-1}\left(x+\frac{2}{x}\right)}\,dx$

$$let\ \tan^{-1}\left(x + \frac{2}{x}\right) = u$$

$$\Longrightarrow \frac{1}{1+(x+\frac{2}{x})^2}.\left(1 - \frac{2}{x^2}\right) = dx = du$$

$$Now\ I = \int \frac{1}{u}\,du = \log|u| + c$$

$$= log\left|\tan^{-1}\left(x + \frac{2}{x}\right)\right|$$

$$= \tan^{-1} z, where\ z = \left(x + \frac{2}{x}\right)$$

6. $\int x \log\left(1 + \frac{1}{x}\right)dx = f(x)\log(x+1) + g(x)x^2 + Lx + c$, then

(a) L= 1 **(b) f(x) = $\frac{1}{2}x^2$**
(c) g(x) = log x **(d) none of these**

Ans. (d)

I= $\int x \log\left(1+\frac{1}{x}\right)\ dx = \int x\ log\ (x+1)dx - \int x\ log\ x\ dx$

$= \frac{x^2}{2}\log(x+1) - \frac{1}{2}\int \frac{x^2}{1+x}dx - \frac{x^2}{2}logx + \frac{1}{2}\int x\ dx = \frac{x^2}{2}\log(x+1) - \frac{x^2}{2}logx - \frac{1}{2}\int\left(x-1+\frac{1}{x+1}\right)dx + \frac{1}{2}\int x\ dx$

$$= \frac{x^2}{2}\log(x+1) - \frac{x^2}{2}\log x - \frac{1}{2}\log(x+1) + \frac{x}{2} + c$$

$$f(x) = \frac{x^2}{2} - \frac{1}{2}, g(x) = -\frac{1}{2}\log x$$

$$L = \frac{1}{2}.$$

7. $\int \frac{dx}{(x-1)^{\frac{3}{4}}(x+2)^{\frac{5}{4}}} =$

(a) $\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{\frac{1}{4}} + c$ **(b)** $\frac{4}{3}\sqrt{\frac{x-1}{x+2}} + c$

(c) $\left(\frac{x+2}{x-1}\right)^{\frac{1}{4}} + c$ **(d) none of these**

Ans. (a)

$$I = \int \frac{dx}{(x-1)^2\left(\frac{x+2}{x-1}\right)^{\frac{5}{4}}}$$

$$put\ z = \frac{x+2}{x-1}, then$$

$$dx = \frac{(x-1).1-(x-2).1}{(x-1)^2}dx = -\frac{3}{(x-1)^2}dx$$

$$Now\ I = \frac{1}{3}\int z^{-5/4}\,dz = \frac{4}{3}z^{-1/4} + C = \frac{4}{3}.\left(\frac{x-1}{x+2}\right)^{1/4} + C$$

8. $\int e^x \frac{1+n.x^{n-1}-x^{2n}}{(1-x^n)\sqrt{1-x^{2n}}}\ dx$=

(a) $\frac{e^x\sqrt{1-x^{2n}}}{1-x^{2n}} + c$ **(b)** $\frac{e^x\sqrt{1-x^n}}{1-x^n} + c$

(c) $\frac{e^x\sqrt{1-x^{2n}}}{1-x^n} + c$ **(d) none of these**

Ans. (c)

I= $\int e^x\left[\frac{1-x^{2n}+nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}}\right]dx = \int e^x\left[\frac{\sqrt{1-x^{2n}}}{1-x^n} + \frac{nx^{n-1}}{(1-x^n)^2}\sqrt{\frac{1-x^n}{1+x^n}}\right]dx = \int e^x\{f(x) + f'(x)]dx,$

$$where\ f(x) = \sqrt{\frac{1-x^{2n}}{1-x^n}} = e^x f(x) + C = e^x\frac{1-x^{2n}}{1-x^n} + C$$

9. $\int \frac{(x+1)}{x(1+xe^x)^2}dx = log|-f(x)| + f(x) + c\ then\ f(x) =$

(a) $\frac{1}{x+e^x}$ **(b)** $\frac{1}{x+xe^x}$

(c) $\frac{1}{(1+xe^x)^2}$ **(d) none of these**

Ans. (b)

Put z= xe^x, then dz = ($e^x + xe^x$) dx

I= $\int \frac{dz}{z(1+z)^2} = \int\left[\frac{1}{z} - \frac{1}{1+z} - \frac{1}{(1+z)^2}\right]dz = \log\frac{z}{1+z} + \frac{1}{1+z} + C = log\left|\frac{xe^x}{1+xe^x}\right| + \frac{1}{1+xe^x} + c$

$= \log\left|1 - \frac{1}{1+xe^x}\right| + \frac{1}{1+xe^x} + C$

10. If $I_n = \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x}{sinx} dx, and\ a_n = \int_0^{\frac{\pi}{2}} (\frac{\sin n\theta}{\sin\theta})^2 d\theta, then\ a_{n+1} - a_n$=

(a) I_n **(b) $2I_n$**
(c) $I_n + 1$ **(d) 0**

Ans. (c) $a_{n+1} - a_n = \int_0^{\frac{\pi}{2}} \frac{sin^2(n+1)x - sin^2 nx}{sin^2 x} dx$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)\, x sin\, x}{sin^2 x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x} dx$$

$$= I_{n+1}$$

11. If n $\neq 1$, $\int_0^{\frac{\pi}{4}} (tan^n x + tan^{n-2} x)\, d(x - [x])$=

(a) $\frac{1}{n-1}$ **(b) $\frac{1}{n+1}$**
(c) $\frac{1}{n}$ **(d) $\frac{2}{n-1}$**

Ans. (a)

Let $I_n = \int_0^{\frac{\pi}{4}} (tan^n x + tan^{n-2} x)\, d(x - [x])$

$here\ 0 < x < \frac{\pi}{4} \therefore [x] = 0\ \therefore x - [x] = x$

$$now\ I_n = \int_0^{\frac{\pi}{4}} tan^{n-2} x sec^2 x\, dx$$

$$= \int_0^1 z^{n-2}\, dz, putting\ z$$
$$= \tan x$$

$= \left[\frac{z^{n-1}}{n-1}\right]_0^1 = \frac{1}{n-1}$

12. If f(α)= f(β) and n ∈N, then the value of $\int_\alpha^\beta \left(g(f(x))\right)^n g'(f(x)).f'(x)\, dx$ =

(a) 1 **(b) 0**
(c) $\frac{\beta^{n+1} - \alpha^{n+1}}{n+1}$ **(d) none of these**

Ans. (b)

Put z = g(f(x)), then dz = g'(f(x)) f'(x) dx

= $\int z^n dz = \frac{z^{n+1}}{n+1}$

$I = \frac{1}{n+1} [\{g(f(x))]^{n+1}]\ _\alpha^\beta =$
$\frac{1}{n+1}\left[[\{g(f(\alpha))]^{n+1} - [\{g(f(\beta))]^{n+1}\right] = 0$
[$\because$ f(α)= f(β)]

13. Let [x] denotes the integral part of a real number x and {x} = x- [x], then solution of 4{x}= x+ [x] are

(a) $\pm\frac{2}{3}, 0$ **(b) $\pm\frac{4}{3}, 0$**
(c) 0, $\frac{5}{3}$ **(d) $\pm 2, 0$**

Ans. (c)

4{x}= x+ [x]= [x]+ {x} +[x]

$\Longrightarrow$ {x}= $\frac{2}{3}[x]$ …….(1)

Since $0 \leq \{x\} < 1$

$\therefore 0 \leq \frac{2}{3}[x] < 1 \Longrightarrow 0 \leq [x] < \frac{3}{2}$

Hence [x]= 0, 1

$\therefore${x}= 0, $\frac{2}{3}$ [from (1)]

$\therefore$ x= [x]+ {x}= 0, $\frac{5}{3}$

14. The maximum number of real roots of the equation $x^{2n} - 1 = 0$ (n ∈N) is

(a) 2 **(b) 3**
(c) n **(d) 2n**

Ans. (a)

$x^{2n} - 1 = 0 \Rightarrow x^{2n} = 1 = \cos 0 + i\sin 0$

$\therefore x = \cos\frac{2r\pi}{2n} + i\sin\frac{2r\pi}{2n} = \cos\frac{r\pi}{n} + i\sin\frac{r\pi}{n}, r = 0, 1, \ldots, (2n-1)$

x will be real only when $\sin\frac{r\pi}{n} = 0$

or $\frac{r\pi}{n} = m\pi$

or r = mn = a multiple of n

But, r= 0, 1, 2,, …, 2n- 1

∴ r = 0, n

$\therefore x^{2n} - 1 = 0$ has only two real root 1, -1.

Second method: Let f(x) = $x^{2n} - 1$

Then, f'(x) = $2nx^{2n-1}$

Sign scheme for f'(x) is

Hence graph, of y= f(x) will either intersect x –axis at two points or touch x-axis or will not interest x-axis or will not interest x-axis. Therefore eqn. f(x)= 0 has two distinct real roots or two equal real roots or no real root.

15. The roots of equation $7^{\log_7(x^2-4x+5)}$ are

(a) 4, 5 **(b) 2, -3**
(c) 2, 3 **(d) 3, 5**

Ans. (c)

Given, $x^2 - 4x + 5 = x - 1$

$\Rightarrow x^2 - 5x + 6 = 0 \Rightarrow$ x= 2, 3

16. Equation $\frac{a^2}{x-\alpha} + \frac{b^2}{x-\beta} + \frac{c^2}{x-\gamma} = m - n^2x$ (a, b, c, m, n ∈ r) has necessarily

(a) all the roots real
(b) all the roots imaginary

(c) two real and two imaginary roots
(d) two rational and two irrational roots

Ans.(a)

Let p + iq be a root of given equation, then

$\frac{a^2}{p-\alpha+iq} + \frac{b^2}{p-\beta+iq} + \frac{c^2}{p-\gamma+iq} = m - n^2(p + iq)$

$\Rightarrow \frac{a^2[p-\alpha-iq]}{(p-\alpha)^2+q^2} + \frac{b^2[(p-\beta)-iq]}{(p-\beta)^2+q^2} + \frac{c^2[(p-\gamma)-iq]}{(p-\gamma)^2+q^2} = m - n^2p - in^2q$

Equating imaginary parts we get

$q\left[\left\{\frac{a^2}{(p-\alpha)^2+q^2} + \frac{b^2}{(p-\beta)^2+q^2} + \frac{c^2}{(p-\gamma)^2+q^2}\right\} + n^2\right] = 0$

∴ q= 0.

Hence p +iq= p= a real number.

17. If a, b, c ∈ { 1, 2, 3, 4, 5}, the number of equations of the form $ax^2 + bx + c = 0$ which have real roots is

(a) 25 **(b) 26**
(c) 207 **(d) 24**

Ans. (d)

For real roots ac $\leq \frac{b^2}{4}$

B	$\frac{b^2}{4}$	Possible value of ac such that ac $\le \frac{b^2}{4}$	No. of possible pairs (a, c)
2	1	1	1
3	2.25	1.2	3
4	4	1, 2, 3, 4	8
5	6.25	1, 2, 3, 4, 5, 6	12
		Total	24

Value of ac	Possible pairs (a, c)
1	(1, 1)
2	(1, 2), (2, 1)
3	(1, 3), (3, 1)
4	(1, 4), (4, 1), (2, 2)
5	(1, 5), (5, 1)
6	(2, 3), (3, 2)

Hence number of quadratic equations having real roots = 24

18. If x, $a_1, a_2, a_3, \dots, a_n \in R$ and $(x - a_1 + a_2)2 + (x - a_2 + a_3)^2 + \cdots$

$+(x - a_{n-1} + a_n)^2 = 0$,
then $a_1, a_2, a_3, \dots, a_n$ are in

(a) A.P. (b) G.P. (c) H.P. (d) none of these.

Ans. (a) $(x - a_1 + a_2)^2 + (x - a_2 + a_3)^2 + \cdots + (x - a_{n-1} + a_n)^2 = 0$

$\Rightarrow a_1 - a_2 = a_2 - a_3 = \cdots = a_{n-1} - a_n = x$

$\Rightarrow a_1, a_2, a_3, \dots . a_n$ are in A.P. with common difference x.

19. Let f(x) = $ax^2 + bx + c$ and g(x) = af(x) + bf′(x) + cf″(x) If f(x) > 0 for all x , then the sufficient condition for g(x) to be > 0 v x is

(a) c > 0 **(b) b > 0**
(c) b< 0 **(d) c < 0**

Ans. (d)

g(x)= $a(ax^2 + bx + c) + b(2ax + b) + c + 2a = a^2x^2 + 3abx + b^2 + 3ac$

discriminant of its corresponding equation ,

D = $9a^2b^2 - 12a^3c$

$= 9a^2b^2 - 36a^3c + 24a^3c$

$= 9a^2(b^2 - 4ac) + 24a^3c$ ……(1)

Since f(x)> 0, $\forall\, x \in R$

∴ a > 0 and $b^2 - 4ac$< 0

For g(x)> 0 $\forall\, x \in R, a^2 > 0$ *and* $D < 0$

But from (1), D < 0 when c < 0

20. The constant term of the quadratic expression $\sum_{k=1}^{n}\left(x - \frac{1}{k+1}\right)\left(x - \frac{1}{k}\right)$ *as* $n \to \infty$ is

(a) -1 **(b) 0**
(c) 1 **(d) none of these**

Ans. (c) Constant term

c = $\frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$

$$\lim_{n\to\infty} c = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$$

21. If $\theta_i \in \left[0, \frac{\pi}{6}\right], i = 1, 2, 3, 4, 5$ and $sin\theta_1 z^4 + sin\theta_2 z^3 + sin\theta_3 z^2 + sin\theta_4 z + sin\theta_5 = 2$ then z satisfies

(a) |z|> $\frac{3}{4}$ (b) |z|< $\frac{1}{2}$ (c) $\frac{1}{2}$ < |z|< $\frac{3}{4}$ (d) none of these

Ans.(a)

Since $0 \leq \theta_i, \leq \frac{\pi}{6}$

$\therefore 0 \leq sin\theta_i, \leq \frac{1}{2}$

From given condition

$|2| = |sin\theta_5 + zsin\theta_4 + z^2 sin\theta_3 + z^3 sin\theta_2 + z^4 sin\theta_1|$

$\Rightarrow 2 \leq |sin\theta_5| + |z||sin\theta_4| + |z|^2|sin\theta_3| + |z|^3|sin\theta_2| + |z|^4|sin\theta_1| \leq \frac{1}{2} + \frac{1}{2}|z| + \frac{1}{2}|z|^2 + \frac{1}{2}|z|^3 + \frac{1}{2}|z|^4| < \frac{1}{2}|z| + |z|^2 + \cdots \ to\ \infty$ ….(1)

When $|z| < 1$, from (1),

$2 < \frac{1}{2} . \frac{1}{1-|z|}$

$\therefore 1 - |z| < \frac{1}{4} \Rightarrow |z| > \frac{3}{4}$

When |z|> 1, clearly $|z| > \frac{3}{4}$

Thus $|z| > 1, clearly\ |z| > \frac{3}{4}$

22. Number of solutions of $3^{|x|} = |2 - |x||$ is

(a) 0 **(b) 2**
(c) 4 **(d) infinite**

Ans. (b)

Given equation is

$\left(\frac{1}{3}\right)^x = 2 - x, \quad -\infty < x \leq -2$

$= 2 + x, \quad -2 \leq x \leq 0$

$3^x = 2 - x, 0 \leq x \leq 2$

$= x - 2, \ 2 \leq x < \infty$

At x = 2, $3^x - x + 2 = 9$

For x >2, $3^x - x + 2 > 9$

(As $3^x - x + 2$ is an increasing function for x > 2)

For x= -2, $\left(\frac{1}{3}\right)^x + 2 + x = 9$

For x < -2, $\left(\frac{1}{3}\right)^x + 2 + x < 9$

[as $\left(\frac{1}{3}\right)^x + 2 + x\ is\ decreasing$]

Hence given equation has only two solutions -2 and 2.

23. The number of real roots of the equation $(9 + sinx)^{\frac{1}{1-x}} + (10 + sinx)^{\frac{1}{1-x}} = (11 + sinx)^{\frac{1}{1-x}}$ for x ∈ (0, 1) is

(a) exactly one **(b) at least one** **(c) at most one** **(d) none of these**

Ans. (a)

Given eqn. is f(x) = 1,

where f(x) = $\left(1 + \frac{1}{10+\sin x}\right)^{\frac{1}{1-x}} - \left(1 - \frac{1}{10+\sin x}\right)^{\frac{1}{1-x}}$

Clearly

f(0)= $\frac{1}{5} < 1\ and\ f(1-0) = \infty$

Also f(x) is an increasing function

∴ f(x)= 1 only for one value of x.

24. If 0 < αr < 1 for r= 1, 2, 3, …, k and m be the number of real solutions of equation

$\sum_{r=1}^{k}(a_r)^x = 1$ & n be the number of real solution of equation $\sum_{r=1}^{k}(x-a_r)^{101} = 0$, then

(a) m= n **(b) m ≤ n**
(c) m ≥ n **(d) m > n**

Ans. (b)

Let α be a root of eqn.

$a_1{}^x + a_2{}^x + \cdots + a_k{}^x = 1$ …………….(1)

Then when x < α, L.H.S. of (1)> 1

And when x > α, L.H.S. of (1)< 1

Hence, eqn. (1) cannot have more than one root.

∴ $m \le 1$

Let f(x) = $(x-a_1)^{101} + (x-a_2)^{101} + \cdots + (x-a_n)^{101}$

∴ f'(x) > 0 ⟹ f(x) is an increasing function

Also f$(-\infty) = -\infty < 0$ and $f(\infty) = \infty > 0$

∴ f(x) = 0 has exactly one real root

∴ n= 1

Hence m $\le n$.

25. If m be number of integral solutions of equation $2x^2 - 3xy - 9y^2 - 11 = 0$ and n be the number of real solutions of equation $x^3 - [x] - 3$= 0, then m =

(a) n **(b) 2n**
(c) n/2 **(d) 3n**

Ans. (b)

Given, $2x^2 - 3xy - 9y^2 - 11 = 0$

⟹ (2x +3y)(x- 3y)= 11

$\therefore \left.\begin{matrix} 2x+3y=1 \\ x-3y=11 \end{matrix}\right\}, \left.\begin{matrix} 2x+3y=11 \\ x-3y=1 \end{matrix}\right\}$

$\left.\begin{matrix} 2x+3y=-1 \\ x-3y=-11 \end{matrix}\right\}, \left.\begin{matrix} 2x+3y=-11 \\ x-3y=-1 \end{matrix}\right\}$

∴ x= 4, y= 1, x= -4, y= -1

∴ m= 2

Again, given

$x^3 - [x] - 3 = 0 \Rightarrow x^3 - (x-\alpha) - 3 = 0,$

Where α= {x}= x-[x]

⟹$x^3 - x = 3 - \alpha$. But 0≤ $\alpha < 1$

∴ 2 < $x^3 - x \le 3$

For x ≥ 2,

$x^3 - x = x(x^2-1) \ge 2(2^2-1) = 6$

For x≤ $-1, x^3 - x = x(x^2-1) < 0$

For -1 < x< 0, $x^3 - x < 1 < 2$

For 0 < x ≤ $1, x^3 - x < x^3 < 1 < 2$

For x= 0, $x^3 - x = 0 < 2$

∴1 < x< 2 ∴ [x]=1

∴Given equation becomes

x^3 − 4= 0⟹x= $4^{\frac{1}{3}}$

∴ n= 1

Thus m= 2, n=1

26. If [x] denotes the integral part of x and k = $sin^{-1}\frac{1+t^2}{2t} > 0$, then integral value of α for which the equation (x-[k])(x+α) - 1 = 0 has integral roots is

(a) 1 **(b) 2**
(c) 4 **(d) none of these**

Ans. (d)

For $\sin^{-1}\frac{1+t^2}{2t}$ $to\ be\ defined, \left|\frac{1+t^2}{2t}\right| \leq 1$

$\Rightarrow \frac{1+t^2}{2t} < 1$

$\Rightarrow$ 1+ $|t|^2 \leq 2|t|$

$\Rightarrow$ (1+ $|t|)^2 \leq 0$

$\Rightarrow$ (1+ $|t|)^2 = 0 \Rightarrow\ |t| = 1$

$\Rightarrow$ t= ± 1

$\therefore$ k= $\sin^{-1} 1 = \frac{\pi}{2}$ ($\because$k> 0)

$\therefore$ [k] = [$\frac{\pi}{2}$]= 1

Given equation is (x-1)(x-α)-1= 0

$\Rightarrow$ (x- 1)(x+ α)= 1 ……. (1)

We have to find integral value of α for which equation (1) has integral roots.

$\therefore$ x and α are integers.

From (1), (i) x- 1 = 1$\Rightarrow$ x= 2

X+ α= 1$\Rightarrow$ α= 1- x= -1

(ii) x- 1 = -1$\Rightarrow$ x= 0

X+ α= -1 $\Rightarrow$ α= -1

27. If [x] denotes the integral part of x and m= $\left[\frac{|x|}{1+x^2}\right], n =$ $integral\ values\ of\ \frac{1}{2-sin3x}$, then

(a) m≠ n **(b) m > n**
(c) m + n = 0 **(d) $n^m = 0$**

Ans. (a)

$0 \leq \frac{|x|}{1+x^2} < 1$;

$\therefore$ m= $\left[\frac{|x|}{1+x^2}\right] =\ 0$

Again 1$\leq 2 - \sin 3x\ \leq 3$

$\therefore \frac{1}{3} \leq \frac{1}{2-\sin 3x} \leq 1$

$\therefore$ n= integral value of $\frac{1}{2-\sin 3x}$= 1

$\therefore$m≠ n is the correct choice.

28. If 1 lies between the roots of equation $y^2 - my + 1 = 0$ and [x] denotes the integral part of x, then $\left[\left(\frac{4|x|}{x^2+16}\right)^m\right]$ =

(a) 1 **(b) 0** **(c) undefined** **(c) 2**

Ans. (b)

Since 1 lies between the roots of equation

$y^2 - my + 1 = 0,$

$\therefore$ f(1)< 0

$\Rightarrow$ 2-m < 0 $\Rightarrow$ m > 2 …… (1)

Let y= $\frac{4|x|}{x^2+16} = \frac{4|x|}{|x|^2+16}$=$\frac{4z}{z^2+16}$, where z= |x|

$\therefore yz^2 - 4z + 16y = 0$

Since z is real,

$\therefore 16 - 64y^2 \geq 0 \Rightarrow -\frac{1}{2} \leq y \leq \frac{1}{2}$

$\therefore 0 \leq y \leq \frac{1}{2}$ [$\because y > 0$]

$\therefore 0 \leq y^m \leq \frac{1}{2^m} < 1$

$\therefore [y^m] = 0$ …….. (2)

29. Equation sin x + 2sin 2x +3 sin 3x = $\frac{8}{\pi}$ has at least one root in

(a) $\left(\pi, \frac{3\pi}{2}\right)$ **(b) $\left(0, \frac{\pi}{2}\right)$**
(c) $\left(\frac{\pi}{2}, \pi\right)$ **(d) none of these**

Ans. (b)

Let f(x) = $\frac{8}{\pi}x + \cos x + \cos 2x + \cos 3x$

$$then\ f'(x) = \frac{8}{\pi}x - sinx - 2\sin 2x - 3\sin 3x$$

f(x) is continuous and differentiable at every point

Also f(0)= f$\left(\frac{\pi}{2}\right)$ $\therefore$ By Rolle's theorem

f'(c)= 0 for at least one c in $\left(0, \frac{\pi}{2}\right)$

30. Let f(x) and g(x) be differentiable functions for 0$\leq x \leq$ 1 such that f(0) = 2, g(0)= 0, f(1)= 6.Let there exist a real number c in (0, 1) such that f'(c)= 2g'(c), then g(1)=

(a) 1 **(b) 2**
(c) -2 **(d) -1**

Ans. (b)

Let ϕ(x)= f(x)+Ag(x)

Then ϕ'(x)= f'(x)+ Ag'(x)

Choosing A such that φ(0)= ϕ(1),

We have A= - $\frac{f(1)-f(0)}{g(1)-g(0)}$

for this value of A using Rolle's theorem for ϕ(x) in (0, 1), we have

ϕ'(c)= 0 for some c ∊ (0, 1)

$$\Rightarrow \frac{f'(c)}{g'(c)} = -A = \frac{f(1)-f(0)}{g(1)-g(0)}$$
$$= \frac{6-2}{g(1)-0} \Rightarrow 2 = \frac{4}{g(1)}$$
$$\Rightarrow g(1) = 2$$

ISI B.STAT/B.MATH OBJECTIVE QUESTIONS & SOLUTIONS

SET – 10

1. Let
f(x) = cosx(sinx + $\sqrt{sin^2x + sin^2\theta}$),

where 'θ' is a given constant,

then maximum value of f(x) is

(a) $\sqrt{1+\cos^2\theta}$ **(b) $\sqrt{1+\sin^2\theta}$**
(c) |cosθ| **(d) none**

Ans. (b) $\{f(x)secx - sinx\}^2 = sin^2x + sin^2\theta$,

$\Rightarrow f^2(x)(1 + tan^2x) - 2f(xtanx) = sin^2\theta$

$\Rightarrow f^2(x)tan^2x - 2f(xtanx) + f^2(x) - sin^2\theta = 0$

$\Rightarrow 4f^2(x) \geq 4f^2(x)\{f^2(x) - sin^2\theta\}$

$\Rightarrow f^2(x) \leq 1 + sin^2\theta$

i.e. $|f(x)| \leq \sqrt{1 + sin^2\theta}$.

2. $\lim\limits_{n\to\infty} \frac{1+\sqrt{2}+3\sqrt{3}+\cdots+n\sqrt{n}}{n}$

(a) equals 0 **(b) equals 1**
(c) equals ∞ **(d) none**

Ans. (b) Cauchy's First limit theorem:-
If $\lim\limits_{n\to\infty} u_n = l, then \lim\limits_{n\to\infty} \frac{u_1+u_2+\cdots+u_n}{n} = l$.

Here $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1, so, by$ Cauchy's first limit theorem

$\lim\limits_{n\to\infty} \frac{u_1+u_2+\cdots+u_n}{n} = \lim\limits_{n\to\infty} \frac{1+\sqrt{2}+3\sqrt{3}+\cdots+n\sqrt{n}}{n} = 1.$

3. If 0 < x < 1, then the sum of the infinite series $\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \cdots$ **is**

(a) $\log\frac{1+x}{1-x}$ **(b)** $\frac{x}{1-x} + \log(1+x)$
(c) $\frac{1}{1-x} + \log(1-x)$ **(d)** $\frac{x}{1-x} + \log(1-x)$

Ans. (b) $\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \cdots$

$= (1-\frac{1}{2})\, x^2 + (1-\frac{1}{3})\, x^3 + (1-\frac{1}{4})x^4 + \ldots.$

$= \{x^2+x^3+x^4+\ldots.\ \} - \{\frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots\}$

$= \{1+x+x^2+\ldots.\} - \{x+\frac{1}{2}x^2 + \frac{1}{3}x^3+\ldots\} - 1$

$= \frac{1}{1-x} + \log(1-x) - 1$

$= \frac{x}{1-x} + \log(1-x)$

4. The polar equation r= acosθ represents

(a) a spiral **(b) a parabola**
(c) a circle **(d) none**

Ans. (c) $r^2 = arcos\theta$

$\therefore x^2 + y^2 = ax$ (since r = $x^2 + y^2, x = acos\theta$)

$\therefore x^2 + ax + y^2 = 0$

$\therefore (x+\frac{a}{2})^2 + y^2 = \frac{a^2}{4}$

This is a circle of radius $\frac{a}{2}$ and centre $(-\frac{a}{2}, 0)$.

5. If f(x) = $\mathop{\mathrm{Lt}}\limits_{i\to\infty} \frac{(1+sin\pi x)^t-1}{(1+sin\pi x)^t+1}$**, then range of f(x) is**

(a) {-1, 1} **(b) {0, 1}**
(c) {-1, 1} **(d) {-1, 0, 1}**

Ans. (d)

$$f(x) = \mathop{\mathrm{Lt}}\limits_{t\to\infty} \frac{(1+\sin\pi x)^t-1}{(1+\sin\pi x)^t+1} = \begin{cases} \dfrac{1-\frac{1}{(1+\sin\pi x)^t}}{1+\left(\frac{1}{1+\sin\pi x}\right)^t}, & \sin\pi x > 0 \\ \dfrac{0-1}{0+1}, & \sin\pi x < 0 \\ \dfrac{1-1}{1+1}, & \sin\pi x = 0 \end{cases} = \begin{cases} 1, & \sin\pi x > 0 \\ -1, & sin\pi x < 0 \\ 0, & sin\pi x = 0 \end{cases}$$

$\therefore$ Range f= {-1, 0, 1}

6. If f: $\left(0, \frac{\pi}{n}\right) \to R, defined\ by\ f(x) = \sum_{k=1}^{n}[1 + \sin kx]$, where [x] denotes the integral part of x, then range of f(x) is

(a) {n-1, n+1} **(b) {n-1, n, n+1}**
(c) {n, n+1} **(d) none of these**

Ans. (c)

f(x)= $\sum_{k=1}^{n}(1 + [\sin kx]) = n + [\sin x] + [\sin 2x] + \ ... + [\sin nx]$ …….(1)

case 1: when kx $\neq \frac{\pi}{2}$ for k= 1, 2, 3, …, n

since 0 < kx < π and kx $\neq \frac{\pi}{2}$

∴ 0 < sin kx < 1, for k= 1, 2, …., n

∴ [sin kx]= 0, for k= 1, 2, 3, …., n

∴ from (1), f(x)= n

When exactly one of x, 2x, 3x, …, nx is $\frac{\pi}{2}$. Here not more than one of x, 2x, 3x, …, nx can be $\frac{\pi}{2}$.

In this case one of sin x, sin2x, …, sinmnx is 1 and other lie between 0 and 1

∴ From (1), f(x)= n+1

Hence range of f= {n, n+ 1}

7. If f(x) = $\underset{n\to\infty}{\text{Lt}} \frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \frac{x}{(2x+1)(3x+1)} + \cdots +$ to n terms, then range of f(x) is

(a) {0, 1} **(b) {-1, 0}** **(c) {-1, 1}** **(d) none of these**

Ans. (a)

$$S_n = \left(1 - \frac{1}{1+x}\right) + \left(\frac{1}{1+x} - \frac{1}{1+2x}\right) + \left(\frac{1}{1+2x} - \frac{1}{1+3x}\right) + \cdots + \left(\frac{1}{1+(n-1)x} - \frac{1}{1+nx}\right)$$

$$= 1 - \frac{1}{1 + nx}$$

$but \underset{n\to\infty}{\text{Lt}}\ nx = \infty, x > 0 = -\infty, x < 0 = 0, x = 0$

$$\therefore f(x) = \underset{n\to\infty}{\text{Lt}}\ S_n = 1, when\ x \neq 0$$
$$= 0, \qquad when\ x = 0$$

Hence range f= {0, 1}

8. Period of f(x) = sin $\frac{\pi x}{(n-1)!} + \cos\frac{\pi x}{n!}$ is

(a) n ! **(b) 2 (n!)**
(c) 2 (n-1)! **(d) none of these**

Ans. (b) sin $\frac{\pi x}{(n-1)!}$ is a periodic function with period 2π+ $\frac{\pi}{(n-1)!} = 2(n-1)!$

$$Period\ of\ \cos\frac{\pi x}{n!} = 2\pi + \frac{\pi}{n!} = 2n!$$

$L.C.M.of\ 2(n-1)!\ and\ 2(n!) is\ 2(n!)$

$$\therefore Period\ of\ f(x) is\ 2(n!)$$

9. Period of the function cos {(x+3)-[x+3]}, where [x] denotes the integral part of x is

(a) 1 **(b) 2** **(c) π** **(d) 2π**

Ans. (a)

x- [x] is a periodic function with period 1.

∴ (x+ 3)- [x+ 3] is a periodic function with period 1.

∴ cos {(x+ 3)}- [x+ 3]] is a periodic function with period 1.

10. If f(x) = $2^{sin^3\pi x+x-[x]}$, where [x] denotes the integral part of x is a periodic function with period

(a) 1 (b) 2 (c) π (d) none of these

Ans. (b)

Period of x- [x] is 1 and period of $sin^3\pi x$ is 2.

L.C.M. of 1 and 2 is 2

∴ f(x) is a periodic function with period 2.

11. If f(x)= cos x+ cos ax is a periodic function, then a is necessarily

(a) an integer (b) a rational number
(c) an irrational number (d) an event number

Ans. (b) Period of cos x= 2π and period of cos ax= $\frac{2\pi}{|a|}$

Period of f(x) = L.C.M. of $\frac{2\pi}{1}$ and $\frac{2\pi}{|a|}$ = $\frac{L.C.M.of\ 2\pi\ and\ 2\pi}{H.C.F.of\ 1\ and\ |a|}$

Since k= H.C.F. of 1 and |a|

∴ $\frac{1}{k}$= an integer= m (say) and $\frac{|a|}{k}$= an integer = n (say)

∴ |a|=$\frac{n}{m} \Rightarrow a = \pm\frac{n}{m}$ = a rational number.

12. If f is an increasing function and g is a decreasing function such that g(f(x)) exists, then

(a) g(f(x)) is an increasing function
(b) g(f(x)) is an decreasing

(c) nothing can be said
(d) g(f(x)) is a constant function

Ans. (b)

f' (x)> 0 (∵ f(x) is an increasing function)

g'(x)< 0

(∵g(x) is a decreasing function)

(g(f(x)))'= g' (f(x)).f'(x) < 0

∴g(f(x)) is a decreasing function.

13. f: R⟶ R, f(x)=x|x| is

(a) one-one and onto
(b) one-one but not onto

(c) not one-one but onto
(d) neither one-one nor onto

Ans. (a)

$$f(x) = \begin{cases} -x^2, x \leq 0 \\ x^2, x \geq 0 \end{cases}$$

$$\therefore\ f'(x) = \begin{cases} -2x^2, x \leq 0 \\ 2x, x \geq 0 \end{cases}$$

∴ f'(x)=>0 ∴ f(x) is an increasing function and consequently it is a one –one function.

Also f($-\infty$)= -∞, f(∞)=∞, Hence range f= R

14. Let f(r) = 1+ $\frac{1}{2}$ +$\frac{1}{3}$ $+\cdots+$ $\frac{1}{r}$, *then* $\sum_{i=1}^{n} f(i)$=

(a) (n+1) f(n)-(n- 1) (b) (n+ 1) f(n)- n
(c) n f(n)- (n-1) (d) (n-1) f(n)

Ans. (b)

f(1)+ f(2)+ …+f(n)= 1+ $\left(1+\frac{1}{2}\right)+\left(1+\frac{1}{2}+\frac{1}{3}\right)+\cdots\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$

$$= n+\frac{(n-1)}{2}+\frac{(n-2)}{3}+\cdots+\frac{[n-(n-1)]}{n}$$

$$= n\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-\left(\frac{1}{2}+\frac{2}{3}+\cdots+\frac{n-1}{n}\right)$$

$$= nf(n)-\left[\left(1-\frac{1}{2}\right)+\left(1-\frac{1}{3}\right)+\cdots+\left(1-\frac{1}{n}\right)\right]$$

$$= nf(n)-(n-1)+f(n)-1 = (n+1)f(n)-n$$

15. The period of f(x) = $e^{\sin\{x\}}+\sin\left(\frac{\pi}{2}[x]\right)$ is ([.]) and {.} are the greatest integer function and fraction function

(a) 1 (b) 4
(c) 2 (d) not periodic

Ans. (b)

Period of {x} i.e. x- [x] is 1 and period of $\sin\left(\frac{\pi}{2}[x]\right)$ *is* 4

L.C.M. of 1 and 4 is 4

∴ Period of f(x) is 4

16. If f(x) = $(a-x^n)^{1/n}$, x > 0 and g(x)> x ∀x ∈R, then for all x > 0

(a) g(g(x))= f(f(x)) (b) g(g(x))>2 f(f(x)) (c) g(g(x))< f(f(x)) (d) g(g(x))> f(f(x))

Ans. (d)

f(x)= $(a-x^n)^{1/n}, x>0$

∴ f(f(x))= f(y), where y= f(x)= $(1-y^n)^{\frac{1}{n}}=\{1-(a-x^n)\}^{\frac{1}{n}}= x, x> 0$

Given, g(x)-> 0 ∀ x ϵ R

∴ g(g(x))-g(x)> 0 ∀ x ϵ R

[Putting g(x) in place of x]

Adding we get, g(g(x))-x> 0

⟹ g(g(x))> x ⟹g(g(x))> f(f(x)), x > 0

[∵ f(f(x))= x, x > 0]

17. Given, y= sgn(x), then

(a) |x|= x sgn(x)
(b) sgn(sgn(x)) = sgn(x)

(c) x= |x|sgn(x)
(d) all of (a), (b), (c)

Ans. (d)

$$f(x)=sgn(x)=\begin{cases}1, x>0\\0, x=0\\-1, x<0\end{cases}$$

$$x\ sgn\ (x)=\begin{cases}x, x>0\\0, x=0\\-x, x<0\end{cases} = |x|$$

$$sgn\left(sgn(x)\right) = \begin{cases} sgn(1), x > 0 \\ sgn(0), x = 0 \\ sgn(-1), x < 0 \end{cases}$$

$$= \begin{cases} 1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{cases} = sgn(x)$$

$$|x|sgn(x) = \begin{cases} x, x > 0 \\ 0, x = 0 \\ x, x < 0 \end{cases} = x, \forall\, x \in R$$

18. For positive real numbers $a_1, a_2, \dots \dots a_{100}$, let P= $\sum_{i=1}^{100} a_i$ and $q = \sum_{1 \le i \le j \le 100} a_i a_j$, then

(a) q =$\frac{P^2}{2}$ **(b) $q^2 \le \frac{P^2}{2}$**
(c) q < $\frac{P^2}{2}$ **(d) none**

Ans. (c) $a_1 + a_2 + \cdots \dots + a_{100}$ = P,

P^2= $(a_1 + a_2 + \cdots \dots + a_{100})^2$= $\sum_{i=1}^{100} {a_i}^2 + 2\sum_{i<j}^{100} a_i a_j$

∴ P^2- 2q ≥ 0 [∵$\sum_{i=1}^{100} {a_i}^2 \ge 0$]

∴ q ≤ $\frac{P^2}{2}$.

19. Number of integral terms in the expansion of $(\sqrt{6} + \sqrt{7})^{32}$=

(a) 15 **(b) 17**
(c) 19 **(d) none**

Ans. (b) $(\sqrt{6} + \sqrt{7})^{32} = \sum_{r=0}^{32} 32_{c_r} . 6^{\frac{r}{2}} . 7^{\frac{32-r}{2}}$

For integral terms $\frac{r}{2}$ and $\frac{32-r}{2}$ both are integers and w is in turn possible if $\frac{r}{2}$ is an integer.

∴ r = 0, 2, 4,…, 32 means r can take 17 different values.

20. Let P is an odd prime and n= 1+p!, then total number of prime in the list n+1, n+2, n+3, ……, n+p-1 is equal to

(a) P- 3 **(b) P- 5**
(c) 0 **(d) none**

Ans. (c) ∵ n= 1+p!

∴ n+ r= (r+1)+p!

If $1 \le r \le p - 1, then\ 2 \le r + 1 \le p$ and clearly,

(n+ r) is divisible by r+1. ∴ n+ r can't be a prime

Hence, there is no prime in the given list.

21. Let f : (0, +∞) → R and $F(x) = \int_0^x f(t)dt$ if $F(x^2) = x^2(1 + x)$, then f(4) equals

(a) 5/4 **(b) 7**
(c) 4 **(d) 2**

Ans. (c)

We have, f(x^2)= $\int_0^{x^2} f(t)\ dt = x^2 + x^3$

Differentiating both sides, we get;

f(x^2).$2x = 2x + 3x^2$

⟹ f(x^2)= 1+(3/2)x

⟹ f(4)= 1+3/2(2)= 4

22. The equation of a curve is y= f(x). The tangents at (α, f(α)), (β,f(β)) and (γ, f(γ))

make angles $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}$ respectively with the positive direction of the x- axis. Then the value of

$\int_{\beta}^{\gamma} f'(x). f''(x)dx + \int_{\alpha}^{\gamma} f''(x)dx$ is equal to

(a) $-\frac{1}{\sqrt{3}}$ **(b) $\frac{1}{\sqrt{3}}$**
(c) 0 **(d) none of these**

Ans. (a)

Given, $f'(\alpha) = \frac{1}{\sqrt{3}}, f'(\beta) = \sqrt{3}, f'(\gamma) = 1$

Now $\int_{0}^{\gamma} f'(x)f''(x)\, dx + \int_{\alpha}^{\gamma} f''(x)\ dx = \left[\frac{1}{2}\left(f'(x)\right)^2\right]_{\beta}^{\gamma} + [f'(x)]_{\alpha}^{\gamma} = \frac{1}{2}\left(f'(y)\right)^2 - \frac{1}{2}\{f'(\beta)\}^2 + f'(\gamma) - f'(\alpha) = \frac{1}{2}(1-3) + 1 - \frac{1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$

23. A rod of length 10ft sides with ends on the co-ordinates axes. If the end on x-axis moves with constant velocity of 2ft/ minute, then the magnitude of the velocity of the middle point at the instant the rod makes an angle of 30° with x-axis is

(a) 2ft / sec **(b) 3 ft / sec**
(c) $\sqrt{3}$ ft / sec **(d) none of these**

Ans. (a)

Let AB be the position of rod at any time t and p be its middle point.

Let OA= x, OB= y, then P ≡ $\left(\frac{x}{2}, \frac{y}{2}\right)$

$$x^2 + y^2 = 10^2$$

$$\therefore 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{x}{y}.2$$

$$when\ \theta = 30°, \frac{x}{y} = \cot 30° = \sqrt{3}$$

$$\therefore \frac{dy}{dt} = -2\sqrt{3}ft/sec$$

$$now\ V = \sqrt{\left(\frac{1}{2}\frac{dx}{dt}\right)^2 + \left(\frac{1}{2}\frac{dy}{dt}\right)^2}$$
$$= \frac{1}{2}\sqrt{2^2 + 12} = 2\ ft/sec$$

24. Two persons are moving on the curve $x^3 + y^3 = a^3$. When the position of first and second persons are (α, β) and (γ, δ) the second persons is in the direction of the instantaneous motion, then

(a) $\frac{\gamma}{\alpha} + \frac{\delta}{\beta} + 1 = 0$ **(b) $\frac{\alpha}{\gamma} + \frac{\beta}{\delta} - 1 = 0$** **(c) $\alpha\gamma + \beta\delta = 1$** **(d) none of these**

Ans. (a)

Given curve is $x^3 + y^3 = a^3$……….(1)

Let P≡ $(\alpha, \beta), Q \equiv (\gamma, \delta)$

Since P and Q lie on(1)

$\therefore \alpha^3 + \beta^3 = a^2\ and\ \gamma^3 + \delta^3 = a^2$

$\alpha^3 - \gamma^3 = \delta^3 - \beta^3$……………(2)

From(1), $\frac{dy}{dx} = -\frac{x^2}{y^2}$

Equation of tangent at P(α, β) $is\ y - \beta = \frac{\alpha^2}{\beta^2}(x - \alpha)$………..(3)

According to question, (3) passes through

Q $\therefore \alpha^2(\alpha - \gamma) = \beta^2(\delta - \beta)$ ……(4)

$$\frac{\alpha^2 + \gamma^2 + \square\gamma}{\alpha^2} = \frac{\beta^2 + \delta^2 + \beta\beta}{\beta^2}$$

$$\Rightarrow 1 + \frac{\gamma^2}{\alpha^2} + \frac{\gamma}{\alpha}$$

$$= 1 + \frac{\delta^2}{\beta^2} + \frac{\delta}{\beta}$$

$$\Rightarrow \left(\frac{\gamma}{\alpha}\right)^2 - \left(\frac{\delta}{\beta}\right)^2 = \left(\frac{\gamma}{\alpha} - \frac{\delta}{\beta}\right) \Rightarrow \frac{\gamma}{\alpha} - \frac{\delta}{\beta} + 1 = 0[\because \frac{\gamma}{\alpha} \neq \frac{\delta}{\beta}]$$

25. The triangle formed by the tangents to the curve f(x) = $x^2 + bx - b$ at the point (1, 1) and the co-ordinate on the first quadrant. If its area is 2 then the value of b is

(a) -1 **(b) 3**
(c) -3 **(d) 1**

Ans. (c) $\frac{dy}{dx} = 2x + b$

$\therefore$ The equation of the tangent at (1, 1) is

y-1= (2+b)(x-1)

or (2+b)x- y= 1+ b

$\therefore$ OA= $\frac{1+b}{2+b}$ $and\ OB = -(1 + b)$

Since Δ AOB lies in the first quadrant,

$\therefore \frac{1+b}{2+b} > 0 and\ 1 + b < 0$

$\therefore$1+b < 0, 2+b < 0$\Rightarrow$ b < -2………..(1)

Now, area (ΔAOB)= 2

$\therefore$ 2= $\frac{1}{2} . \frac{1+b}{2+b}\{-(1 + b)\}$

or, 4(2+b)+ $(1 + b)^2 = 0$

or, $b^2 + 6b + 9 = 0$

Or $(b + 3)^2 = 0$

$\therefore$ b= -3 > 1

26. If 2a+ 3b +6c = 0, then equation $ax^2 + bx + c = 0$ has at least one root in

(a) (-1, 1) **(b) (1, 2)**
(c) (-1, 0) **(d) (2, 3)**

Ans. (a)

Let f'(x) = a$x^2 + bx + c,$ then

f(x) = $\frac{ax^3}{3} + \frac{bx^2}{2} + cx = \frac{2ax^3 + 3bx^2 + 6cx}{6}$

$$f(1) = \frac{2a + 3b + 6c}{6} = 0, f(0) = 0$$

$\therefore$ f(0)= f(1) $\therefore$ there exists α, 0< α < 1 such that f' (α)= 0

i.e., equation a$x^2 + bx + c = 0$ has at least one root in (0, 1).

27. If u_r denotes the number of one–one functions from $\{x_1, x_2, \dots, x_r\}$ to $\{y_1, y_2, \dots, y_r\}$ such that f$(x_i) \neq y_i$, for i= 1, 2, 3, …, r then u_4 =

(a) 9 **(b) 44**
(c) 265 **(d) none of these**

Ans. (a)

u_r = number of ways of putting $x_1, x_2, \dots, x_r$ in r corresponding place so that no x_1 is put in the corresponding place

$$= \underline{|r}\left(\frac{1}{\underline{|2}} - \frac{1}{\underline{|3}} + \cdots + \frac{(-1)^r}{\underline{|r}}\right)$$

$$\therefore u_r = \underline{|4}\left(\frac{1}{\underline{|2}} - \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}}\right) = 12 - 4 + 1$$

28. Number of positive unequal integral solutions of equation x+ y+ z = 6 is

(a) 4! **(b) 3!**
(c) 6! **(d) 2×4!**

Ans. (b)

Given x+ y+ z = 6 …….(1)

x, y, z ∈ N and are unequal.

⟹ x, y, z ∈ {1, 2, 3} and are unequal

∴ Required number of solutions= 3! = 6

29. The plain containing the two straight lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{b} + \mu\vec{a}$ is

(a) $[\vec{r}\ \ \vec{a}\ \ \vec{b}] = 0$
(b) $[\vec{r}\ \ \vec{a}\ \ \vec{a}\times\vec{b}] = 0$

(c) $[\vec{r}\ \ \vec{b}\ \ \vec{a}\times\vec{b}] = 0$
(d) none

Ans. (a)

Given lines are $\vec{r} = \vec{a} + \lambda\vec{b}$ …………..(1)

$\vec{r} = \vec{b} + \mu\vec{a}$ ……………..(2)

lines (1) and (2) intersect at $(\vec{a} + \vec{b})$

Then the plane passes through $(\vec{a} + \vec{b})$

Also, line (1) is parallel to $\vec{b}$ and line (2) is parallel to $\vec{a}$ ⟹ $(\vec{a}\times\vec{b})$ is normal to plane containing these lines.

∴ Eqn. of reqd. plane is

$[\vec{r} - (\vec{a}+\vec{b})].(\vec{a}\times\vec{b}) = 0$

$\vec{r}(\vec{a}\times\vec{b}) - (\vec{a}+\vec{b}).(\vec{a}\times\vec{b}) = 0$

⟹ $[\vec{r}\ \vec{a}\ \vec{b}] = 0$

30. Let $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$. If $\vec{c}$ is a vector such that $\vec{a}.\vec{c} = |\vec{c}|$, $|\vec{c}-\vec{a}| = 2\sqrt{2}$ and angle between $|(\vec{a}\times\vec{b})\times\vec{c}|$ =

(a) 2/3 **(b) 1/3**
(c) 3/2 **(d) 1**

Ans. (c)

Given $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$

$\vec{b} = \hat{i} + \hat{j}$

$\vec{a}.\vec{c} = |\vec{c}|$……………(1)

$|\vec{c}-\vec{a}| = 2\sqrt{2}$ …………….(2)

Angle between $(\vec{a}\times\vec{b})$ and $\vec{c} = \frac{\pi}{6}$

Now,

$|(\vec{a}\times\vec{b})\times\vec{c}| = |\vec{a}\times\vec{b}||\vec{c}|\sin\frac{\pi}{6} = \frac{3}{2}|\vec{c}|$……(3)

From (3), $|\vec{c} - \vec{a}|^2 = 8$

$\Rightarrow(\vec{c} - \vec{a}).(\vec{c} - \vec{a})= 8$

$\Rightarrow|\vec{c}|^2 + |\vec{a}|^2 - 2\vec{a}.\vec{c} = 8$

$\Rightarrow |\vec{c}|^2 + 9 - 2|\vec{c}| = 8$ (from (1))

$\therefore\ |\vec{c}| = 1$

From (3), $|(\vec{a} \times \vec{b})\times \vec{c}\,| = \frac{3}{2}$.

ISI B.STAT & B.MATH SUBJECTIVE QUESTIONS & SOLUTIONS

SET – 1

Q1. How many natural numbers less than 10^8 are there, whose sum of digits equals 7?

Solution:-

We need to count the no. of solutions of $x_1 + x_2 + \cdots + x_8 = 7$

Which satisfies $0 \le x_i \le 7$, i= 1, 2, 3, …, 8 ……………….(1)

The number of solution of (1) is= coefficient of x^7 in $(1 + x + x^2 + \cdots + x^7)^8$

$$= coefficient\ of\ x^7\ in\ (1 - x^8)^8\ (1 - x)^8$$

= coefficient of x^7 in $(1 - 8x^8)(1 + 8c_1x + 4c_2x^2 + \ 10c_3k + \cdots)$

$$= 14c_7$$

$$= 3432.\ \ (Ans)$$

Q2. Find the number of positive integers less than or equal to 6300 which are not divisible by 3, 5 and 7.

Solution:- S= {1, 2, 3, …, 6300}

Let A: Set of integers divisible by 3

B: Set of integers divisible by 5

C: Set of integers divisible by 7

We are to find:- n(s) – n(A∪ B∪ C) = n(S) – [n(A)+n(B) +n(C) – n(A ∩ B)- n(B ∩ C)- n(A ∩ C)+ n(A∩ B∩ C)]

$$= 6300 - \left\{\left[\frac{6300}{3}\right] + \left[\frac{6300}{5}\right] + \left[\frac{6300}{7}\right] - \left[\frac{6300}{3\times5}\right] - \left[\frac{6300}{5\times7}\right] - \left[\frac{6300}{3\times7}\right] + \left[\frac{6300}{3\times5\times7}\right]\right\}$$

$i.e., n(A\cup B\cup C)^c = 2880.$

Q3. If c is a real number with 0 < c < 1, then show that the values taken by the function

$y = \frac{x^2+2x+c}{x^2+4x+3c}$, as x varies over real numbers, range over all real numbers.

Solution:-

$$y = \frac{x^2 + 2x + c}{x^2 + 4x + 3c};$$

$$\Rightarrow x^2y + 4xy + 3cy = \ x^2 + 2x + c$$

$$\Rightarrow (y-1)x^2 + 2x(2y-1) + c(3y-1) = 0\ \ [\because x\ is\ real]$$

$$\therefore \{2(2y-1)\}^2 - 4(y-1).c(3y-1) \ge 0$$

$$\Rightarrow c\ \le \frac{(2y-1)^2}{(y-1)(3y-1)}\quad \because 0 < c < 1,$$

$$So,\quad \frac{1}{3} < y < 1.$$

Q4. Let X = {0, 1, 2, 3, …, 99}. For a, b in X, we define a * b to be the remainder obtained by dividing the product ab by 100. For example, 9*18 = 62 and 7*5 = 35. Let x be an element in X. An element y in X is called the inverse and write down their inverses.

Solution:- x*y = 1, ⟹ xy = 100k +1 for x = {0, 1, 2, …, 99}

(1) For x = 1, y = 1, x * y = 100k + where x = c

∴ Inverse of 1 is 1.

(2) There is no integral multiple of 2, 4, 5, 6 having 1 at unit place, ⟹ 2, 4, 5, 6 have no inverse.
(3) 3 and 7 can have inverses
(i) For x= 3 , 3 y = 1 i.e. 3y = 100k +1

The least k satisfying is 2, i.e. 3y = 201, y = 67 and the next k satisfying is 5, i.e. 3y = 167 but 167 ∉ X.

∴ 3 has only inverse = 67.

(ii) For x = 7, y = 1, i.e. 7y = 100k +1

The least k satisfying is Ʒ, i.e. 7y = 301, y 43

The next k satisfying is ID, i.e. 7y = 1001, y = 143 but 143 ∉ X.

∴ 7 has only inverse = 43.

Q5. Evaluate $\lim\limits_{n\to\infty}\left\{\frac{1}{n+1}+\frac{1}{n+2}+\ \dots+\frac{1}{n+n}\right\}$.

Solution:-

$$\underset{n\to\infty}{\mathrm{Lt}}\ \frac{1}{n}\left[\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\cdots+\frac{1}{1+\frac{n}{n}}\right]$$

$$=\underset{n\to\infty}{\mathrm{lt}}\ \frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\frac{r}{n}}=\int_0^1\frac{dx}{1+x}$$

$$=[\log_e(1+x)]\begin{matrix}1\\0\end{matrix}=\ \log_e 2.$$

Q6. Tangents are drawn to a given circle from a point on a given straight line, which does not meet the given circle. Prove that the locus of the mid-point of the chord joining the two points of contact of the tangents with circle is a circle.

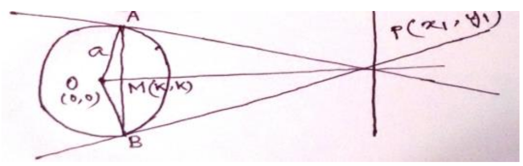
Solution:- Slope of OM = k/h [taking centre (0, 0)]

∴Slope of AB = - h/k . [∵ AB ⊥ OM]

∴ Equation of AB, whose slope is –h/k and which passes through the point (h, k) is y –k = $\frac{h}{k}(x-h)$

or, $hx+ky=\ h^2+k^2$ ………… (1)

And equation of AP, the tangent is $xx_1+yy_1=\ a^2$ ………… (2)



∴ From (1) and (2) , we have,

$$\frac{x_1}{h}=\frac{y_1}{k}=\frac{a^2}{h^2+k^2}$$

∴ $x_1=\frac{ha^2}{h^2+k^2}, y_1=\ \frac{ka^2}{h^2+k^2}$.

∴ Put these values of $x_1\ and\ y_1\ in\ lx_1+my_1+n=0$

We get, $l.\frac{ha^2}{h^2+k^2}+m.\frac{ka^2}{h^2+k^2}+n=0$

$$\Longrightarrow lha^2+mka^2+n(h^2+k^2)=0$$

$\Longrightarrow h^2 + k^2 + \frac{la^2}{n}h + \frac{ma^2}{n}k = 0$, i.e. the required focus of M.

So, the equation of the circle is $x^2 + y^2 + \frac{la^2}{n}x + \frac{ma^2}{n}y = 0.$

Q7. Draw the graph (on plain paper) of f(x) = min {|x| - 1, |x -1|- 1, |x- 2| - 1}.

Solution:- y = |x| -1

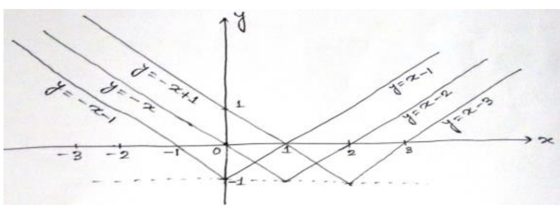
$$= \begin{cases} x - 1, when\ x \ \geq 0 \\ -x - 1, when\ x < 0 \end{cases}$$

Z = |x -1| -1

$$= \begin{cases} x - 1 - 1, when\ x \ \geq 1 \\ -x + 1 - 1, when\ x < 1 \end{cases}$$

W = |x- 2| -1

$$= \begin{cases} x - 2 - 1, when\ x \ \geq 2 \\ -x + 2 - 1, when\ x < 2 \end{cases}$$



Q8. Let $\{C_n\}$ be an infinite sequence of circles lying in the positive quadrant of the XY – plane, with strictly decreasing radii and satisfying the following conditions. Each C_n touches both X-axis and the Y-axis. Further, for all n ≥ 1, the circle C_{n+1} touches the circle C_n externally. If C_1 has radius 10cm, then show that the sum of the areas of all these circles is $\frac{25\pi}{3\sqrt{2}-4}$sq. cm.

Solution:- $OO_1 = R_1\sqrt{2}\ \therefore OP_2 = R_1\sqrt{2} - R_1$

$$\therefore OQ = R_1\sqrt{2} + R_1 = R_1(\sqrt{2}+1)$$

$$\therefore\ R_1 = \frac{OQ}{\sqrt{2}+1}, now, OP = R_2(\sqrt{2}+1), R_2 = \frac{OP}{\sqrt{2}} = R_1\frac{\sqrt{2}-1}{\sqrt{2}+1}$$

$$\therefore R_3 = R_2\frac{\sqrt{2}-1}{\sqrt{2}+1} = R_1\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2$$

$$\therefore Area = \pi({R_1}^2 + {R_2}^2 + \cdots + \propto)$$

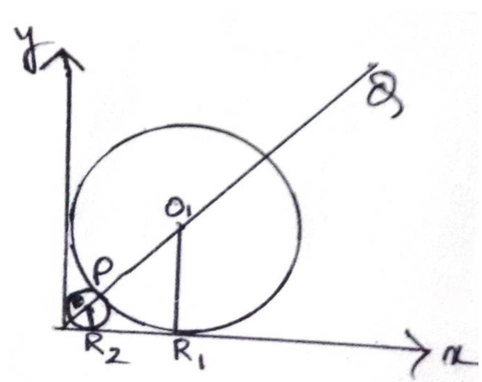
$$= \pi\left\{{R_1}^2 + {R_1}^2\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2 + {R_1}^2\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^4 + \cdots + \propto\right\}$$

$$= \pi{R_1}^2\left\{1 + \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2 + \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^4 + \cdots + \propto\right\}$$

$$= \pi{R_1}^2\left\{\frac{1}{1-\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2}\right\} = \pi{R_1}^2\left(\frac{3+2\sqrt{2}}{4\sqrt{2}}\right) = \pi{R_1}^2\left(\frac{3\sqrt{2}+4}{8}\right)$$

$$= \frac{\pi}{8}\ {R_1}^2\left(\frac{18-16}{3\sqrt{2}-4}\right) = \frac{\pi}{4}.100.\frac{1}{3\sqrt{2}-4}\ \ sq.cm\ [\because R_1 = 10cm.]$$

$$= \frac{25\pi}{3\sqrt{2}-4}sq.cm\ [proved]$$



Q9. Consider the system of equations x + y = 2, ax + y = b. Find conditions on a and b under which

(i) the system has exactly one solution;

(ii) the system has no solution;

(iii) The system has more than one solution.

Solution:-

$\Delta = \begin{vmatrix} 1 & 1 \\ a & 1 \end{vmatrix} = 1 - a$; $\Delta_1 = \begin{vmatrix} 2 & 1 \\ b & a \end{vmatrix} = 2a - b$; $\Delta_2 = \begin{vmatrix} 1 & 2 \\ a & b \end{vmatrix} = b - 2a.$

(i) For exactly one solution, $\Delta \neq 0$ i.e. $1 - a \neq 0 \Longrightarrow a \neq 1$.

(ii) For no solution, $\Delta = 0$, i.e. a = 1, $\Delta_1 \neq 0, \Delta_2 \neq 0$. i.e. $2a \neq b$.

(iii) For more than one solution, $\Delta = \Delta_1 = \Delta_2 = 0$, a = 1, b = 2.

Q10. Let $\{x_n\}$ b e a sequence such that $x_1 = 2$, $x_2 = 1$ and $2x_n - 3x_{n-1} + x_{n-2} = 0$

For n > 2. Find an expression for x_n .

Solution:- $x_1 = 2, x_2 = 1, 2x_n - 3x_{n-1} + x_{n-2} = 0.$

Let, $x_n = ka^n, \therefore 2ka^n - 3ka^{n-1} + ka^{n-2} = 0$

or, $2a^2 - 3a + 1 = 0$

or, (2a- 1)(a- 1) = 0

or, $a_1 = \frac{1}{2}, a_2 = 1.$

$\therefore x_1 = k_1 {a_1}^n + k_2 {a_2}^n = k_1 \left(\frac{1}{2}\right)^n + k_2 (1)^n.$

Again, $x_1 = 2 = k_1 \left(\frac{1}{2}\right) \neq k_2 (1)^1 = \frac{k_1}{2} + k_2$ …………… (1)

And $x_2 = 1 = k_2 \left(\frac{1}{2}\right)^2 + k_2 (1)^1 = \frac{k_1}{4} + k_2$ …………. (2)

From (1) And (2), we get $k_1 = 4, k_2 = 0.$

$\therefore x_n = 4\left(\frac{1}{2}\right)^n = \frac{1}{2^{n-2}}.$

ISI B.STAT & B.MATH SUBJECTIVE QUESTIONS & SOLUTIONS

SET – 2

Q1. A vessel contains x gallons of wine and another contains y gallons of water. From each vessel z gallons are taken out and transferred to the other. From the resulting mixture in each vessel, z gallons are again take out and transferred to the other. If after the second transfer, the quantity of wine in each vessel remains the same as it was after the first transfer, then show that z (x + y) = x y.

Hints:- $\frac{x-z}{x} = \frac{z}{y}$ ⇨ $xy - zy = zx$ ⇨$xy = z(x + y)$ [Proved]

Q2. Suppose k, n are integers ≥ 1. Show that (k. n)! is divisible by $(\boldsymbol{k}!)^{\boldsymbol{n}}$.

Solution: We write the numbers from 1 to kn in k rows of n numbers each as follows:

1, 2, 3, …., n

n+1, n+2 , n+3, ….., 2n

2n + 1, 2n +2, 2n +3,…….., 3n

⋮

kn – n+1, kn – n+2, kn –n + 3, ……, kn

Since each row has n consecutive positive integers, the product of the numbers in each row is divisible by n!. The product of all the numbers (kn)! Is divisible by $(n!)^k$.

Q3. All the permutations of the letters a, b, c, d, e are written down and arranged in alphabetical order as in a dictionary. Thus the arrangement abcde is in the first position and abced is in the second position. What is the position of the arrangement debac?

Solution:-

Words starting with 'a' ⟶ 4!

″ ″ ″ 'b' ⟶ 4!

″ ″ ″ 'c' ⟶ 4!

″ ″ ″ 'd' ⟶ 3! + 3! + 3! + 3!

(i.e. da ⟶ 3!, db ⟶ 3!, dc ⟶ 3!, de ⟶ 3!)

Total no. of words before debac including it is = $3 \times 4! + 3 \times 3! + 3 = 93$.

Q4. (i) Determine m so that the equation $\boldsymbol{x^4 - (3m+2)x^2 + m^2 = 0}$ has four real roots in arithmetic progression.

(ii)Let a and b be two real numbers. If the roots of the equation $\boldsymbol{x^2 - ax - b = 0}$ have absolute value less than one, show that each of the following conditions holds:

(i) |b| < 1, (ii) a+ b <1 and (iii) b - a <1.

Solution:-

(i) $x^4 - (3m+1)x^2 + m^2 = 0$

	A	B	C	D	E	F
A	×	0	2	0	2	2
B	2	×	2	2	0	2
C	0	0	×	$\boxed{2}$	$\boxed{2}$	0
D	2	0	0	×	2	2
E	0	2	0	0	×	2
F	0	0	2	0	0	×

Let four roots be $\alpha - 3\beta$, $\alpha - \beta$, $\alpha + \beta$, $\alpha + 3\beta$.

So, sum of roots = coefficient of x = 0

$\therefore \alpha = 0$.

So, roots are -3β, $-\beta$, β, 3β.

$\therefore -3\beta^2 + 3\beta^2 - 9\beta^2 - \beta^2 - 3\beta^2 + 3\beta^2 = -(3m+1)$

$$\Longrightarrow -10\beta^2 = -(3m+1)$$

$$\therefore \beta^2 = \frac{3m+1}{10}.$$

Also, $9\beta^4 = m^2$

$$\therefore 9\left(\frac{3m+1}{10}\right)^2 = m^2$$

$$\Longrightarrow 9m + 3 = \pm 10m \Longrightarrow m = 3, -\frac{3}{19}.$$

(ii)$x^2 - ax - b = 0$

Let roots be α, β, $|\alpha| < 1$, $|\beta| < 1$,

$\therefore |\alpha + \beta| \leq |\alpha| + |\beta| < 2$, as $|\alpha| < 2$, or $|\alpha|\,|\beta| < 1$

Or, $|\alpha\beta| < 1$

$\therefore$ |b| <1

Again, ab <2, |b| -|a| <1, and |b| -|a| ≤| b- a|

$\therefore$ b –a <1 and b+ a< 1.

Q5. Let a and b be real numbers such that the equations 2x + 3y = 4 and ax – by = 7 have exactly one solution. Then, show that the equations 12x – 8y = 9 and bx + ay = 0 also have exactly one solution.

Solution:-

$$\left.\begin{array}{r} 2x + 3y = 4 \\ ax - by = 7 \end{array}\right\} \text{—————— (i)}$$

$\therefore \Delta_1 = \begin{vmatrix} 2 & 3 \\ a & -b \end{vmatrix} \neq 0$, since it has only one solution.

Final score
6
8
4
6
4
2

$\Longmapsto$ - (3a+2b) $\neq$ 0$\Longmapsto$ (3a+2b) $\neq$0.

$\left.\begin{matrix} 12x - 8y = 7 \\ bx + ay = 0 \end{matrix}\right\}$ ———————— (ii)

$\therefore \ \Delta_2 = \begin{vmatrix} 12 & -8 \\ b & a \end{vmatrix} = 4(3a + 2b)$, since (3a+2b) $\neq$0,

So, $\Delta_2 \neq 0$; So, the equations in (ii) has only one solution.

Q6. In a competition, six teams A, B, C, D, E, F play each other in the preliminary round – called round robin tournament. Each game ends either in a win or a loss. The winner is awared two points while the loser is awared zero points. After the round robin tournament, the three teams with the highest scores move to the final round. Based on the following information, find the score of each team at the end of the round robin tournament.

(i) In the game between E and F, team E won.

(ii) After each team had played four games, team A had 6 points, team B had 8 points and team C had 4 points. The remaining matches yet to be played were

(i)Between A and D;

(ii)Between B and E; and

(iii)Between C and F.

(iv)The teams D, E and F had won their games against A, B and C respectively.

(v)Teams A, B and D had moved to the final round of the tournament.

Solution:-

Steps:-

1. First use (i) then (iii)
2. Since after 4 games, B had 8 pts. And B lost to E later, so B had won against A, C, D, F.
3. Since A had 6 pts. After 4 games and A had lost to B had D, so A won against C, E and F.
4. C had 4 pts. After 4 games. So, C won against D and E.
5. Since A, B and D moved to final round and total 4 pts, so D must have won the games against E and F.

Q7. If A = $\int_0^{\pi} \frac{\boldsymbol{cosx}}{(x+2)^2} \boldsymbol{dx}$,

$\boldsymbol{then\ show\ that}\ \int_0^{\pi/2} \frac{\boldsymbol{sinx\ cosx}}{(x+1)} \boldsymbol{dx} = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{\pi+2} - \boldsymbol{A}\right)$.

Solution:-

$$\int_0^{\frac{\pi}{2}} \frac{sinx\ cosx}{(x+1)} dx$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{sin2x}{(x+1)} dx$$

$$= \frac{1}{2} \int_0^{\pi} \frac{sinz}{\frac{z}{2}+1} . \frac{dz}{2}$$

$$= \frac{1}{2} \int_0^{\pi} \frac{sinz\ dz}{z+2} = \ -\frac{1}{2} . \int_0^{\frac{\pi}{2}} \frac{cosz}{(z+2)^2} dz$$

$$= \frac{1}{2}\left(\frac{1}{\pi+2} + \frac{1}{2}\right) - \frac{1}{2} A$$

$$= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{\pi+2} - A\right)$$

Q8. If a, b and c are the lengths of the sides of a triangle ABC and if p_1, p_2 *and* p_3 are the lengths of the perpendiculars drawn from the circumcentre onto the sides BC, CA and AB respectively, then show that

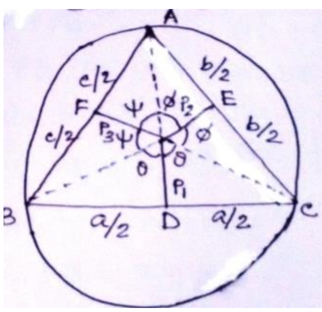
$$\frac{a}{p_1}+\frac{b}{p_2}+\frac{c}{p_3}=\frac{abc}{4p_1p_2p_3}.$$

Solution:- As P_1, P_2, P_3 are the lengths of the perpendiculars drawn from the circum centre O to the sides of length a, b, c respectively, then from the diagram. D, E, F are the mid points of BC, CA, AB respectively.

Hence , in ΔBOD and ΔCOD,

$\angle$BDO =$\angle$CDO, BD = DC & OD is common.

$\therefore$ ΔBOD ≡ ΔCOD.



Similarly, ΔCOE ≡

ΔAOE and ΔAOF ≡

ΔBOF,

$\Longrightarrow$ $\angle$BOD = $\angle$COD= θ, say

$\angle$COE = $\angle$AOE= ϕ, say

$\angle$ AOF = $\angle$BOF= ψ, say.

$\therefore$ $\angle$BOD +$\angle$COD +$\angle$COE +$\angle$AOE +$\angle$AOF +$\angle$BOF = 2(θ+ ϕ+ ψ) = 2Π

$\therefore\theta+\phi=\Pi-\psi$

$\Longrightarrow$ tan (θ+ ϕ)= tan (Π- ψ)= - tan ψ.

Hence we can show, tan θ+ tan ϕ +tan ψ = tanθtanψtanϕ

i.e., $\frac{a}{2p_1}+\frac{b}{2p_2}+\frac{c}{2p_3}=\frac{abc}{8p_1p_2p_3}$ $\quad or, \frac{a}{p_1}+\frac{b}{p_2}+\frac{c}{p_3}=\frac{abc}{4p_1p_2p_3}$.

Q9. (a) Study the derivatives of the function f(x) = $\frac{x+1}{(x-1)(x-7)}$ to make conclusions about the behavior of the function as x ranges over all possible values for which the above formula for f(x) is meaningful.

(b) Use the information obtained in (a) to draw a rough sketch of the graph of f(x) on plain paper.

Solution:- f(x) = $\frac{x+1}{(x-1)(x-7)} = -\frac{1}{3}.\frac{1}{x+1}+\frac{4}{3}.\frac{1}{x-7}$.

(a) f'(x) = $\frac{1}{3}.\frac{1}{(x+1)^2}-\frac{4}{3}.\frac{1}{(x-7)^2}$

For, $0 \le x < -1, f'(x) is\ positive$;

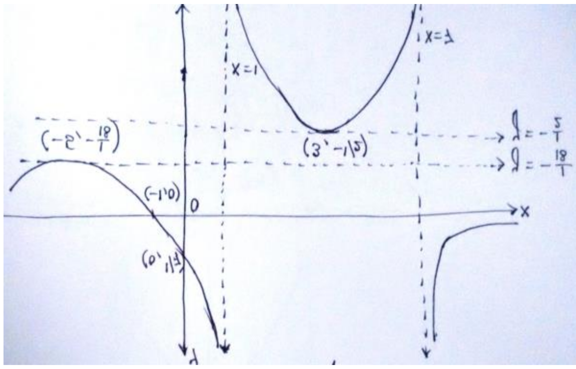
For, -∞ < x < 0, $f'(x)$ is negative;

x	0	$\frac{\pi}{2}$
z	0	π

For, 0 < x <7, $f'(x)$ is negative;

For, 7< x < ∞, $f'(x)$ is negative.

(b)



Q10. Show that there is exactly one value of x which satisfies the equation

$$\mathbf{2cos^2\left(x^3 + x\right) = \ 2^x + 2^{-x}}.$$

Solution:- $-1 \leq cos^2(x^3 + x) \leq 1$ implying $-2 \leq 2cos^2(x^3 + x) \leq 2$

By AM ≥ GM inequality we have $2^x + 2^{-x} \geq 2$

So $2cos^2(x^3 + x) = \ 2^x + 2^{-x} = 2$ satisfies when x = 0.

So there is only one value of x = 0 which satisfy the given equation.

ISI B.STAT & B.MATH SUBJECTIVE QUESTIONS & SOLUTIONS

SET – 3

Q1. Let P(x) = $x^n + \ a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + \ a_0$ be a polynomial with integer coefficients, such that, P(0) and P(1) are odd integers. Show that:

(a) P(x) does not have any even integer roots.

(b) P(x) does not have any odd integer roots.

Solution:-

P(0) = a_0= odd, P(1) = 1+ $a_{n-1} + a_{n-2} + \cdots + a_0 = odd$

(a) Case-I:-

If x = 2m, then

P(x) = $x^n + a_{n-1}x^{n-1} + \cdots + a_0$= odd, as all the term containing x will be even but $a_0 =$ odd.

⟹ x = 2m cannot be a root of the equation P(x) = 0 [proved]

(b) Case-II:-

If x = 2m+ 1

$p_x = \ x^n + a_{n-1}x^{n-1} + \cdots + a_0 = \ x^n + a_{n-1}(even + 1) + a_{n-2}(even + 1) + \cdots + a_1(even + 1)a_0 = \ x^n + a_0 + (a_{n-1}even + a_{n-2}even + \cdots + a_1even) + (a_1 + b_2 + \cdots + a_{n-1}) = \ x^n + (a_{n-1}even + a_{n-2}even + \cdots + a_1even) + (a_0 + a_1 + \cdots + a_{n-1})$

= odd ⟹ x = (2m +1) cannot be a root of the equation P(x) = 0 [proved.]

Q2. Let a_0 and b_0 be any two positive integers. Define a_n, b_n for n ≥ 1 using the relations $a_n = \ a_{n-1} + 2b_{n-1}, b_n = \ a_{n-1} + b_{n-1}$ and let $c_n = \frac{a_n}{b_n}$, for n= 0, 1, 2, …

(a) Write $\left(\sqrt{2} - c_{n-1}\right)$ in $terms$ of $\left(\sqrt{2} - c_n\right)$.

(b) Show that $|\sqrt{2} - c_{n+1}| < \frac{1}{1+\sqrt{2}} |\sqrt{2} - c_n|$.

(c) Show that $\lim\limits_{n\to\infty} c_n = \sqrt{2}$.

Solution:-

$a_n = a_{n-1} + 2b_{n-1}, b_n = a_{n-1} + b_{n-1}, c_n = \frac{a_n}{b_n}.$

(a) $\sqrt{2} - c_{n-1} = \sqrt{2} - \frac{a_{n-1}}{b_{n+1}}$

$$= \sqrt{2} - \frac{a_n + 2b_n}{a_n + b_n} = \sqrt{2} - \frac{\frac{a_n}{b_n} + 2}{\frac{a_n}{b_n} + 1}$$

$$= \sqrt{2} + \frac{c_{n+2}}{c_{n-1}}$$

$$= \frac{\sqrt{2}c_n + \sqrt{2} - c_n - 2}{c_{n+1}}$$

$$= \frac{(\sqrt{2}-1)c_n - \sqrt{2}(\sqrt{2}-1)}{c_n + 1}$$

$$= \frac{(\sqrt{2}-1)(c_n - \sqrt{2})}{(c_{n+1})}.$$

(b) $\frac{|\sqrt{2}-c_n+1|}{|\sqrt{2}-c_n|} = \left|\frac{1-\sqrt{2}}{c_n+1}\right| = \frac{1}{(1+\sqrt{2})(c_n+1)} < \frac{1}{1+\sqrt{2}}$

$\Longrightarrow |\sqrt{2} - c_n + 1| < \frac{1}{1+\sqrt{2}} |\sqrt{2} - c_n|$

(c) $\operatorname*{lt}_{n\to\infty} \frac{c_n - \sqrt{2}}{c_{n+1} + \sqrt{2}} = \operatorname*{lt}_{n\to\infty} \frac{c_{n+1}}{1-\sqrt{2}} = 1.$

Q3. Let ABC be any triangle, right –angled at A, with D any point on the side AB. The line DE is drawn parallel to BC to meet the side AC at the point E. F is the foot of the perpendicular drawn from E to BC. If AD= x_1, Db = x_2, BF = x_3, EF = x_4 and AE = x_5, then show that

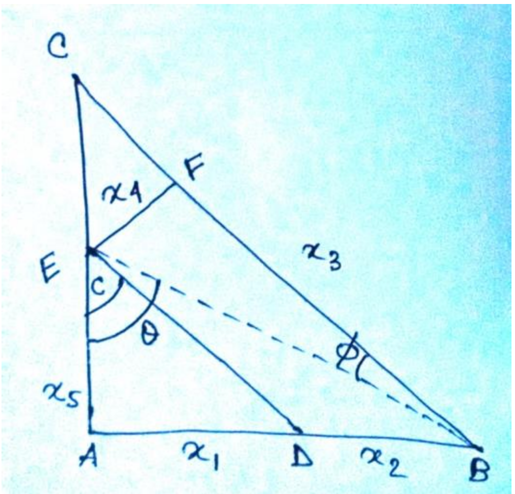
$$\frac{x_1}{x_5} + \frac{x_2}{x_5} = \frac{x_1x_3 + x_4x_5}{x_3x_5 - x_1x_4}.$$

Solution:-

LHS $= \frac{x_1}{x_5} + \frac{x_2}{x_5} = \frac{x_1+x_2}{x_5} = tan\theta = \tan(\phi + c) = \frac{tanc + tan\phi}{1 - tanc\ tan\phi}$

$$= \frac{\frac{x_1}{x_5} + \frac{x_4}{x_3}}{1 - \frac{x_1}{x_5}.\frac{x_4}{x_3}}$$

$$= \frac{x_1x_3 + x_4x_5}{x_3x_5 - x_1x_4}.$$



Q4. Let [x] denote the largest integer less than or equal to x. For example, $\left[4\frac{1}{2}\right] = 4; [4] = 4.$ Draw a rough sketch of the graphs of the following functions on plain paper:

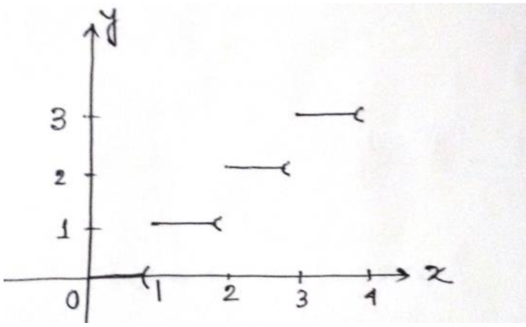
(i) f(x) = [x] ;

(ii) g(x) = x – [x];

(iii) h(x)= $\frac{1}{[x]}$.

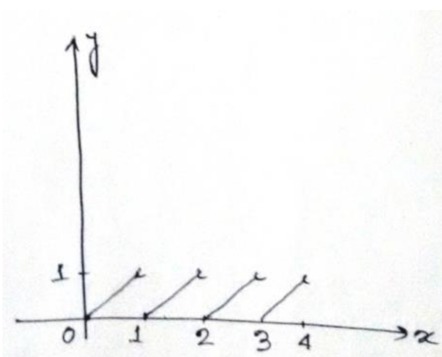
Solution:- (i)

x	y
[0, 1)	0
[1, 2)	1
[2, 3)	2
[3, 4)	3
⋮	⋮



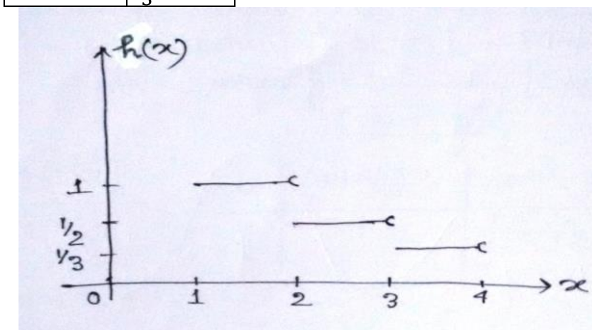
(ii) g(x) = x – [x] = {x}.

X	g(x)
0	0
$\frac{1}{4}$	$\frac{1}{4}$
$\frac{1}{2}$	$\frac{1}{2}$
$\frac{3}{4}$	$\frac{3}{4}$
1	0
$1\frac{1}{4}$	$\frac{1}{4}$
$1\frac{1}{2}$	½
⋮	⋮



(iii) h(x) = $\frac{1}{[x]}$

x	h(x)
1	1
$1\frac{1}{4}$	1
$1\frac{1}{2}$	1
$1\frac{3}{4}$	1
2	$\frac{1}{2}$
$2\frac{1}{2}$	$\frac{1}{2}$
$2\frac{3}{4}$	$\frac{1}{2}$
3	$\frac{1}{3}$



Q5. Show that the area of the bounded region enclosed between the curves

$\boldsymbol{y^3 = x^2 \text{ and } y = 2 - x^2, \text{ is } 2\frac{2}{15}}$.

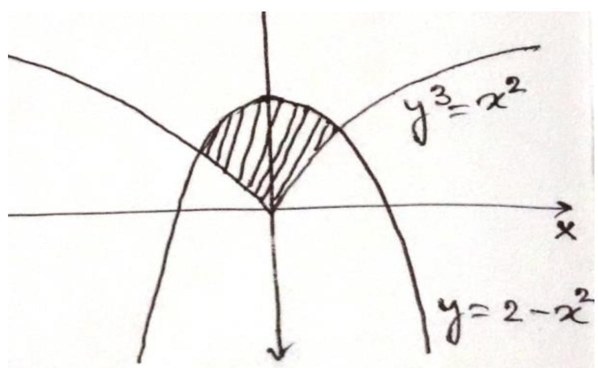
Solution:- $y^3 = x^2$ …………. (i)
y = $2 - x^2$ …………… (ii)

⟾ y = 2 - y^3

⟾(y -1) ($y^2 + y + 2$) = 0

∴y = 1, y = ½ (-1$\pm\sqrt{3}$)

∴ x = ± 1, since y is real & equal to 1. Point of intersection of the two curves are (1, 1) and (-1 , 1).



∴ Area of shaded region = $\int_{-1}^{1}(y_1 - y_2)dx =$
$\int_{-1}^{1}\left[2 - x^2 - x^{\frac{2}{3}}\right]dx = 2\frac{2}{15}\ sq.units.$

Q6. We say that a sequence $\{a_n\}$ has property P, if there exists a positive integer m such that $a_n \leq 1$ for every n ≥ m. For each of the following sequences, determine whether it has the property P or not. [Do not use any result on limits.]

$$(\boldsymbol{i}) \quad \boldsymbol{a_n} = \begin{cases} \boldsymbol{0.9 + \frac{200}{n}} & \boldsymbol{if\ n\ is\ even} \\ \boldsymbol{\frac{1}{n}} & \boldsymbol{if\ n\ is\ odd} \end{cases}$$

$$(\boldsymbol{ii}) \quad \boldsymbol{a_n} = \begin{cases} \boldsymbol{1 + \frac{\cos\frac{n\pi}{2}}{n}} & \boldsymbol{if\ n\ is\ even} \\ \boldsymbol{\frac{1}{n}} & \boldsymbol{if\ n\ is\ odd}. \end{cases}$$

Solution:- (i) For every even number n ≥ 2000, $a_n \leq 1$

And for every odd positive integer, $a_n \leq 1$.

Here, $\{a_n\}$ is a decreasing sequence and $a_{2000} = 0.9 + \frac{200}{2000} = 1$.

∴ a_n satisfies property P.

(ii) Let n = 4k (even)

i.e., $a_{4k} = 1 + \frac{1}{4k}\cos\left(\frac{4k\pi}{2}\right) = 1 + \frac{1}{4k}\cos(2k\pi) = 1 + \frac{1}{4k} > 1$.

[∵ cos (2kπ) = 1] So, here a_n does not satisfy P.

Q7. Suppose that the roots $x^2 + px + q = 0$ are rational numbers and p, q are integers. Then show that the roots are integers.

Solution:-

The roots of the equation $x^2 + px + q = 0$ are x = $\frac{-p \pm \sqrt{p^2-4q}}{2}$.

As roots are rational, hence D is a perfect square.

i.e. $p^2 - 4q = k^2$, where k ∈ I.

or, $p^2 = k^2 + 4q$.

Now, when p = even, $k^2 + 4q$ = even.

⤇ k^2= even, so k = even.

⤇ x = $\frac{-even \pm even}{2}$ = integer ………………… (i)

Again when p = odd, $k^2 + 4q$ = odd

⤇ k^2= odd, so k= odd.

⤇ x =$\frac{-odd \pm odd}{2}$ = integer …………………….. (ii)

Hence the proof is complete.

Q8. Consider the set S of all integers between and including 1000 and 99999. Call two integers x and y in S to be in the same equivalence class if the digits appearing in x and y are the same. For example, if x = 1010, y= 1000 and z = 1201, then x and y are in the same equivalence class, but y and z are not. Find the number of distinct equivalence classes that can be formed out of S.

Solution:-

If ‘a’ is a member of some equivalence class then it’s distinct digit determine the equivalence class completing. Hence, no. of equivalence classes are the number of ways in which ‘i’ integers can be selected from {1, 2, 3, …, 9} for 2 ≤ i ≤ 5 and {1, 2, 3, …, 9} for i = 1.

Now, this can be done in 9 + $\sum_{i=2}^{5}\binom{10}{i} = \sum_{i=1}^{5}\binom{10}{i} - 1$.

Q9. For x > 0, show that $\frac{x^n-1}{x-1} \geq nx^{\frac{n-1}{2}}$, where n is a positive integer.

Solution:-

Let us take the sequence of numbers as $\{x^{n-1}, x^{n-2}, \dots, x^0\}$

Applying AM ≥ GM inequality:-

$$\frac{1}{n}(x^{n-1}, x^{n-2}, \dots, x^0) \geq \sqrt[n]{x^{(n-1)+(n-2)+\cdots+0}}$$

$$or, \quad \frac{1}{n}.\frac{x^n-1}{x-1} \geq \left\{x^{\frac{n(n-1)}{2}}\right\}^{\frac{1}{n}}$$

$$or, \quad \frac{x^n-1}{x-1} \geq nx^{\frac{n-1}{2}}.$$

Q10. Show that $\frac{3}{1.2.4} + \frac{4}{2.3.5} + \frac{5}{3.4.6} + \cdots + \frac{n+2}{n(n+1)(n+3)} = \frac{1}{6}\left[\frac{29}{6} - \frac{4}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right]$.

Solution:- $t_n = \frac{n+2}{n(n+1)(n+3)}$

$$= \frac{1}{6}\left[\frac{1}{n+1} - \frac{1}{n+2}\right] + \frac{1}{6}\left[\frac{1}{n+2} - \frac{1}{n+3}\right] + \frac{2}{3}\left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$So, S = \sum_{n=1}^{n} T_n = \frac{1}{6}\sum_{n=1}^{n}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \frac{1}{6}\sum_{n=1}^{n}\left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \frac{2}{3}\sum_{n=1}^{n}\left(\frac{1}{n} - \frac{1}{n+1}\right).$

$$= \frac{1}{6}\left(\frac{1}{2} - \frac{1}{n+1}\right) + \frac{1}{6}\left(\frac{1}{3} - \frac{1}{n+3}\right) + \frac{2}{3}\left(1 - \frac{1}{n+1}\right)$$

$$= \frac{1}{6}\left[\frac{29}{6} - \frac{4}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right] \quad (proved)$$

ISI B.STAT & B.MATH SUBJECTIVE QUESTIONS & SOLUTIONS

SET – 4

Q1. If f(x) is a real-valued function of a real variable x, such that 2f(x) + 3 f(-x) = 15 – 4x for all x, find the function f(x).

Solution:-

2f(x) + 3f(-x) = 15 – 4x

Put x = -x , 2f(-x) + 3f(x) = 15 + 4x

Solving, we get, f(x) = 3 + 4x

Q2. Show that there is exactly one value of x which satisfies the equation

$$2cos^2(x^3 + x) = 2^x + 2^{-x}.$$

Solution:-

We know $cos^2(x^3 + x) \leq 1$.

$$\frac{2^x + 2^{-x}}{2} \geq \sqrt{2^x.2^{-x}}$$

$$\Longrightarrow 2^x + 2^{-x} \geq 2$$

$$2cos^2(x^3 + x) = 2^x + 2^{-x}$$

$$\therefore cos^2(x^3 + x) = 1$$

For x = 0, the equation is satisfied.

Q3. There are 1000 doors $D_1, D_2, \dots, D_{1000}$ and 1000 persons $P_1, P_2, \dots, P_{1000}$. Initially

all the doors were closed. Person P_1 goes and opens all the doors. Then person P_2 closes doors $D_2, D_4, \dots, D_{1000}$ and leaves the odd-numbered doors open. Next, P_3 changes the state of every third door, that is, $D_3, D_6, \dots, D_{999}$. (For instance, P_3 closes the open door D_3 and opens the closed door D_6, and so on.) Similarly, P_m changes the state of the doors $D_m, D_{2m}, D_{3m}, \dots, D_{nm}, \dots$ while leaving the other doors untouched. Finally, P_{1000} opens D_{1000} if it were closed and closes it if it were open. At the end, how many doors remain open?

Solution:- By the problem, the persons P_m will change the state of the door D_n, where m $|$ n i.e. m is one of the factors of n.

At first, all the doors were closed and we are to determine the no. of doors remaining open, i.e. we are to determine the no. of doors whose states are finally changed.

Now, for the door D_n, n will either have even or odd no. of factors.

It is obvious, for even no. of factors, state of doors remain same, so whenever no. of fact (n) odd, state of OrD_n changes.

$\Rightarrow$ (n) = 2k +1, k ∈ $I^+ \Rightarrow \sigma(n) = (2p+1)(2q+1) \times \dots (2r+1) \dots; \Rightarrow n = a^{2p}\, b^{2q}\, c^{2r} \dots$. For prime a, b, c …..

$\Rightarrow$Square numbered door remains open.

∴No. of doors remaining open = $\left(\sqrt{1000}\right) = 31$

Q4. Find the maximum and minimum values of the function f(x) = $x^2 - x\,sinx$, in the closed interval $\left[0, \frac{\pi}{2}\right]$.

Solution:-

f(x) = $x^2 - x\,sinx$

∴f(0) = 0 and f'(x) = 2x –sin x –x cos x

= (x –sin x) + x (1 –cos x)

In the interval [0, $\frac{\Pi}{2}$], $x - sinx \geq 0$ and $1 - cosx \geq 0$.

∴ f'(x)≥ 0 ∴ f(x) is an increasing function of x in[0, $\frac{\Pi}{2}$]

It's min value will be f(0) = 0, maximum value will be $f\left(\frac{\Pi}{2}\right) = \frac{\Pi^2}{4} - \frac{\Pi}{2}$.

Q5. Let A and B be two fixed points 3 cm apart.

(a) Let P be any point not collinear with A and B, such that PA = 2PB. The tangent at p to the circle passing through the points P, A and B meets the extended line AB at the point K. Find the lengths of the segments KB and KP.

(b) Hence or otherwise, prove that the locus of all points P in the plane such that PA = 2PB is a circle.

Solution:-

(a) AB = 2a = 3 cm, let P(h, k) be the co-ordinate of P and PA = 2PB

∴$(h+a)^2 + k^2 = 4\{(h-a)^2 + k^2\}$

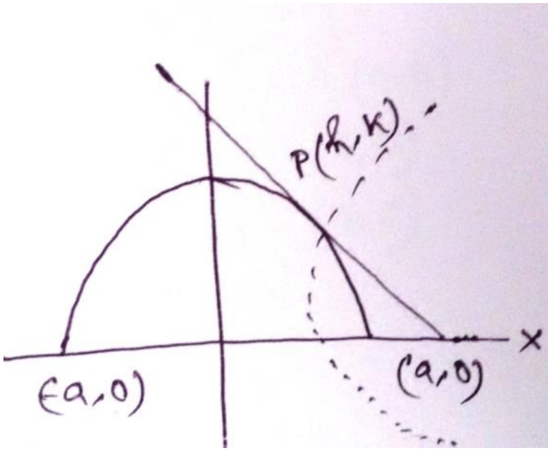
$\Longmapsto 3h^2 + 3k^2 - 10ha + 3a^2 = 0$

Locus of P is $x^2 + y^2 - \frac{10}{3}ax + a^2 = 0$

$\Longmapsto \left(x - \frac{5}{3}a\right)^2 + y^2 = \left(\frac{4a}{3}\right)^2$ which is a circle with centre at $\left(\frac{5}{3}a, 0\right)$ $and\ radius = \frac{4a}{3}$.

∴ Co-ordinates of K are $\left(\frac{5a}{3}, 0\right)$ & $KP = \frac{4a}{3}$.

$\therefore$ KB = OK - OB= $\frac{5a}{3} - a = \frac{2a}{3} = \frac{2}{3} \times \frac{3}{2} = 1cm.$



Q6. Sketch, on plain paper, the regions represented on the plane by the following:

(i) |y|= sin x;

(ii) |x| - |y| ≥ 1.

Solution:-

(i) y = sin x, when y > 0……………… [case -I]

= 0, when y = 0 …………………..[case -II]

= -sin x, when y < 0 ……………..[case -III]

[case -I]
[case -II]

X	(4n +1)$\frac{\pi}{2}$	Nπ+$(-1)^n \frac{\pi}{6}$
y	1	½

For all n = 0, $\pm1, \pm2, \pm3$,…..

(ii) | x | - | y | ≥ 1.

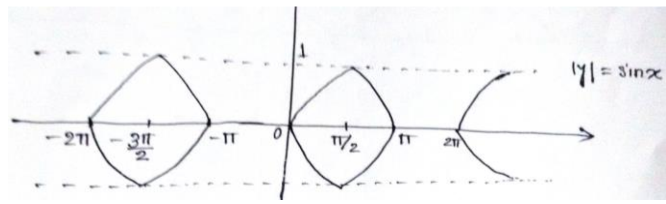
There are four cases:- (I) x – y ≥ 1 when x > 1, y > 0

(II) x + y ≥ 1 when x ≥ 1, y < 0

(III) –x –y ≥ 1 when x < 0, y > 0

(IV) –x + y ≥ 1 when x< 0, y <0

Graph of (i):-



Q7. Show that the larger of the two areas into which the circle $x^2 + y^2 = 64$ is divided by the curve $y^2 = 12x$ is $\frac{16}{3}(8\pi - \sqrt{3})$.

Solution:- $x^2 + y^2 = 64$ ……………… (i)
$y^2 = 12x$ ……………….. (ii)

$\Longmapsto$ 64 $-x^2 = 12x$ $\Longmapsto$ (x - 4) (x + 16) = 0 $\therefore$ x = - 16, 4

But x = -16 is not possible as radius of a circle is $\sqrt{64}$ units.

X	(4n +1)$\frac{\pi}{2}$	Nπ+$(-1)^n \frac{\pi}{6}$
y	-1	- ½

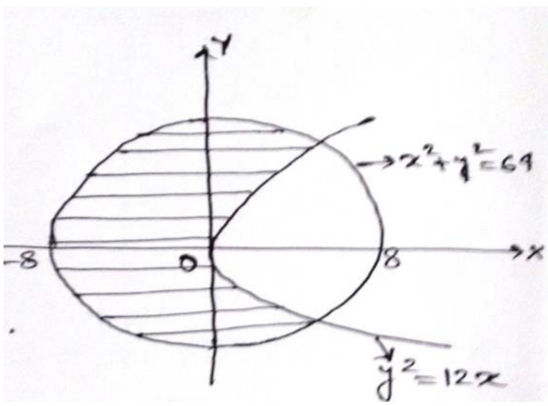
Area of the shaded region is

$= 2\int_0^4 (y_1 - y_2)dx + \frac{64\pi}{2},\ where\ y_1 = \sqrt{64-x^2}, y_2 = 2\sqrt{3x}$

$$= 2\int_0^4 \left(\sqrt{64-x^2} - 2\sqrt{3x}\right)dx + 32\pi$$

$$= 2\left[\frac{x-\sqrt{64-x^2}}{2} + \frac{64}{2}\sin^{-1}\left(\frac{x}{8}\right) - 2\sqrt{3}.\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4 + 32\pi$$

$$= \frac{16}{3}(8\pi - \sqrt{3})sq.units.$$



Q8. Let $x_n = \frac{1}{2}.\frac{3}{4}.\frac{5}{6}......\frac{2n-1}{2n}$. Then show that $x_n \le \frac{1}{\sqrt{3n+1}}, for\ all\ n = 1, 2, 3, ...$

Solution:- $x_n = \frac{1}{2}.\frac{3}{4}.\frac{5}{6}......\frac{2n-1}{2n} = \frac{1.2.3.4....(2n-1)(2n)}{(2.2....to\ 2n\ times)(1.2.3...n)^2} = \frac{(2n)!}{2^{2n}.(n!)^2}$

We need to show that $\frac{(2n)!}{2^{2n}.(n!)^2} \le \frac{1}{\sqrt{3n+1}} \forall\, n \in \mathbb{N}$.

By induction, P(1) is true.

Let P(m) is true, i.e., $\frac{(2m)!}{2^{2m}.(m!)^2} \le \frac{1}{\sqrt{3m+1}}$,

P(m+ 1) = $\frac{(2m+2)!}{2^{2m+2}.[(m+1)!]^2} = \frac{(2m)!(2m+1)(2m+2)}{4.2^{2m}\,(m!)^2(m+1)^2}$

$= \frac{(2m)!}{4.2^{2m}\,(m!)^2}.\frac{(2m+1)}{2(m+1)} \le \frac{1}{\sqrt{3m+1}}.\frac{2m+1}{2(m+1)}.\frac{1}{\sqrt{3m+4}}$; $[show\ it]$

∴ P(m+1) is true.

So, by induction $x_n \le \frac{1}{\sqrt{3n+1}} \forall\, n \in \mathbb{N}$.

Q9. Show that if n is any odd integer greater than 1, then $n^5 - n$ is divisible by 80.

Solution:-

Take n = 2k + 1,

For n = 3, $3^5 - 3 = 240 \mid 80$.

Now, P(n) $= n^5 - n = (2k+1)^5 - (2k+1)$

$= \{(2k-1)+2\}^5 - (2k-1) + 2$

$$= (2k-1)^5 + 5_{C_1}(2k-1)^4.2 + 5_{C_2}(2k-1)^3.2^2 + 5_{C_3}(2k-1)^2.2^3 + 5_{C_4}.(2k-1).2 + 2^5 - (2k-1) + 2$$

$$= \{(2k-1)^5 - (2k-1)\} + 10(2k-1)^4 + \frac{5.4}{2.1}\times 4(2k-1) + \frac{5.4}{2.1}.8(2k)^2 + 5 \times 16(2k-1) + 30$$

$\equiv Mutiple\ of\ 80 + 10(16k^4 - 24k^2 + 16k)$

$\equiv Mutiple\ of\ 80 + 80(2k^4 - 3k^2 + 2k)$

∴ $(2k-1)^5 - (2k-1)$ is divisible by 80.

∴ By induction method, for all odd integer n, $n^5 - n$ is divisible by 80.

Q10. If any one pair among the straight lines

ax + by = a + b, bx – (a+ b)y = - a, (a+ b)x –ay = b intersect, then show that the three straight lines are concurrent.

Solution:-

Let 1st two lines intersect.

ax + by = a + b ______________ (i) $\times b$

bx – (a+ b)y = - a ____________ (ii)$\times a$

$$abx + b^2y = ab + b^2$$

$$abx - a(a+b)y = -a^2$$

$$- \qquad + \qquad +$$

____

$$y = 1.$$

∴ x = $\frac{a+b-by}{a} = 1.$

In the third line, (a+ b) x – ay = LHS = b = RHS.

So, there straight lines are concurrent.

ISI B.STAT & B.MATH SUBJECTIVE QUESTIONS & SOLUTIONS

SET – 5

Q1. If a and b are positive real numbers such that a + b = 1, prove that

$$\left(\boldsymbol{a}+\frac{\mathbf{1}}{\boldsymbol{a}}\right)^{\mathbf{2}}+\left(\boldsymbol{b}+\frac{\mathbf{1}}{\boldsymbol{b}}\right)^{\mathbf{2}} \geq \frac{\mathbf{25}}{\mathbf{2}}.$$

Solution:-

Let S = $\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2$

$= a^2 + b^2 + \frac{(a^2+b^2)}{a^2b^2} + 4$

$= S_1 + S_2 + 4.$

$S_1 = a^2 + b^2 \geq \frac{1}{2}\Big[(a-b)^2 \geq 0$
$\Rightarrow a^2 + b^2 - 2ab \geq 0$
$\Rightarrow (a+b)^2 \geq 4ab \Rightarrow 4ab \leq 1 \qquad \Rightarrow ab \leq \frac{1}{4} \quad a^2 + b^2 \geq 1 - 2\left(\frac{1}{4}\right) = \frac{1}{2}\Big]$

$S_2 = \frac{a^2+b^2}{a^2b^2}, a^2b^2 \leq \frac{1}{16}$

∴ $\frac{1}{a^2b^2} \geq 16.$

∴$S_2 \geq 8$

∴$S_1 + S_2 \geq \frac{1}{2} + 8$

∴ S $\geq \frac{1}{2} + 8 + 4 = \frac{25}{2}.$

Q2. Suppose that P(x) is a polynomial of degree n such that

P(k) = $\boldsymbol{\frac{k}{k+1}}$ for k= 0, 1, …, n.

Find the value of P(n + 1).

Solution:-

Given that, P(x) is a polynomial of degree n such that

P(k) = $\frac{k}{k+1} \forall\, k = 0, 1, \ldots\ldots, n$

Let, Q(x) = (x +1) P(x) –x

The polynomial Q(x) vanishes for k = 0, 1, …., n

i.e. (x+ 1) P(x) –x = a(x) (x-1) (x- 2) … (x- n)

Putting x = -1, 1 = a(1) (-2) (-3) …. (-1 -n)

$\Rightarrow 1 = a\,(-1)^{n+1}(n+1)1 \Rightarrow a = \frac{1}{(-1)^{n+1}}.\frac{1}{(n+1)!}$

$$P(x) = \frac{ax(x-1)(x-2)\dots(x-n)+x}{(x+1)}$$

$$= \frac{(-1)^{n+1}x(x-1)(x-2)\dots(x-n)}{(n+1)!} + x$$

∴ P(n+ 1)= $(-1)^{n+1}\frac{\{(n+1)n(n-1)\dots 3.2.1\}+x+1}{\frac{(n+1)!}{(n+2)}}$

$$= (-1)^{n+1}\frac{\frac{(n+1)}{(n+1)!}+(n+1)}{(n+2)}$$

$$= \frac{(-1)^{n+1}+n+1}{(n+2)}$$

∴ P(n+ 1) = $\begin{cases} \frac{1}{n} & for\ n = odd \\ n+2 & for\ n = even \end{cases}$

Q3. Suppose $x_1 = \tan^{-1}2 > x_2 > x_3 > \cdots$ are positive real numbers satisfying $\sin(x_{n+1} - x_n) + 2^{-(n+1)}sinx_n\,sinx_{n+1} = 0\ for\ n \geq 1$. Find $cotx_n. Also, show\ that\ \lim_{n\to\infty} x_n = \frac{\pi}{4}$.

Solution:-

$$sinx_{n+1}\cos x_n - cosx_n - \cos x_{n+1}\sin x_{n+2} - (n+1)\sin x_n \sin x_{n+1} = 0$$

$$\Longmapsto \sin x_{n+1}\{\cos x_{n+2} - (n+1)\sin x_n\} = \cos x_{n+1}\sin x_n$$

$\Longmapsto \cot x_{n+1} = \cot x_{n+2} - (n-1)$ [dividing bysin $x_{n+1} + \sin x_n$]

$\therefore \cot x_n = \cot x_{n-1} + 2^{-n} = \cot x_{n-2} + 2^{-(n-1)} + 2^{-n} = \cdots$ … … … … … … ….

$= \cot x_1 + 2^{-2} + 2^{-3} + \cdots + 2^{-n}$

$= 2^{-1} + 2^{-2} + 2^{-3} + \cdots + 2^{-n}$

$= 1 - \left(\frac{1}{2}\right)^n$ [$\because \tan^{-1}2 = x_1$]

$\therefore \cot x_n = 1 - \left(\frac{1}{2}\right)^n$.

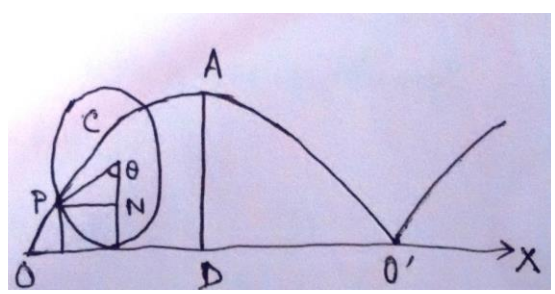
$\underset{n\to\infty}{\text{lt}} \cot x_n = 1. \Longmapsto \cot\left(\underset{n\to\infty}{\text{lt}} x_n\right) = 1$

$$\Longmapsto \underset{n\to\infty}{\text{lt}} x_n = \frac{\Pi}{4}.$$

Q4. Consider the circle of radius 1 with its centre at the point (0, 1). From this initial position, the circle is rolled along the positive x-axis without slipping. Find the locus of the point P on the circumference of the circle which is on the origin at the initial position of the circle.

Solution:-

Let 'P' be the point on the circle NP, let the line OMX on which the circle rolls is X-axis and the point 'O' is origin. Radius of circle is one unit,



Let P= (x, y) and ∠PCM= θ, where θ is the angle through which the circle turns as the point P tracts out of the locus.

$\therefore$ OM = $\overset{\frown}{PM}$ = 1.θ = θ, let PL $\perp$OX, x= OL= OM-LM = θ- sin θ.

Y = PL= NM = CM – CN = 1- cos θ.

Q5. Sketch, on plain paper, the graph of y = $\frac{x^2+1}{x^2-1}$.

Solution:-

y = $\frac{x^2+1}{x^2-1} = \frac{(x^2-1)^2+2}{x^2-1}$

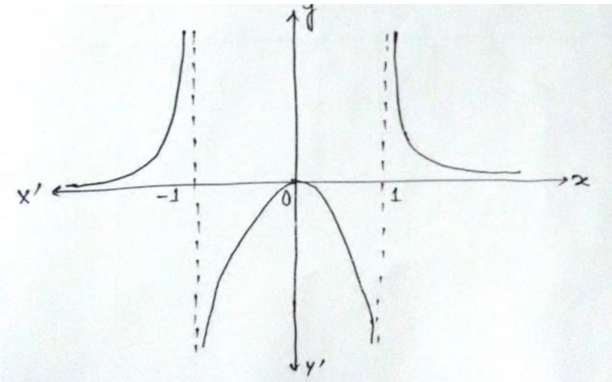
$\therefore$ y = 1+ $\frac{2}{x^2-1}$ $\therefore \frac{dy}{dx} = -\frac{4x}{(x^2-1)^2}$

For, $-\infty \le x < -1, y' is\ positive.$

For, -1 < x $\le$ 0, y' is positive

For, 0 $\le$ x <1, y' is negative

For, 1< x $\le \infty$, y' is negative.



Q6. Find the area of the region in the xy-plane, bounded by the graphs of

$y = x^2, x + y = 2\ and\ y = -\sqrt{x}$.

Solution:-

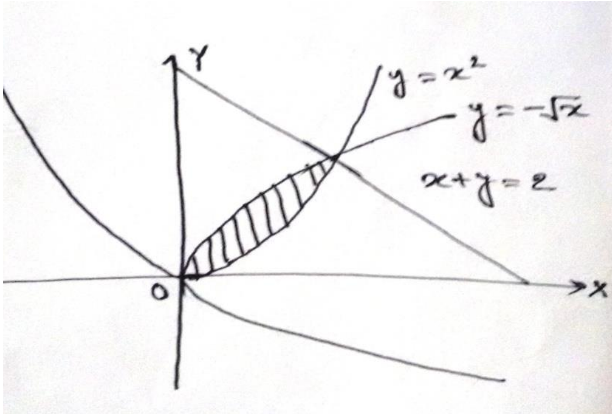
y = x^2 ……………. (i) x + y = 2 ………………… (ii)

y = $-\sqrt{x}$ ………….. (iii)

So, $x^4 = x$, or, x ($x^3 - 1$)= 0, so, x = 0, 1, from (i) & (ii)

And, $(2-x)^2 = x$, or $x^2 - 5x + 4 = 0$ or, x = 1, 4, from (ii) & (iii)

So, point of intersection is x = 1.



Area of the shaded region is = $\left| \int_0^1 (y_2 - y_1)dx \right|, where\ y_1 = -\sqrt{x},\ y_2 = x^2$

= $\left| \int_0^1 (-\sqrt{x} - x^2)dx \right| = 1\ sq.units.$

Q7. Let x and n be positive integers such that $1 + x + x^2 + \cdots + x^{n-1}$ is a prime number. Then show that n is a prime number.

Solution:-

P= $1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n-1}{x-1}$

If P is prime, then x – 1 = 1 $\Rightarrow$ x = 2.

$\therefore$ P= $\frac{2^n-1}{2-1} = 2^n - 1$ is a prime.

Let n is not a prime, then n = pq [p, q are +ve integers]

So, $2^n - 1$ is divided by both $2^p - 1$ and $2^q - 1$

i.e., $2^n - 1$ is not a prime.

But we know $2^n - 1$ is prime, so, by contradiction n is also prime.

Q8. Show that for every positive integer n, 7 divides $\boldsymbol{3^{2n+1} + 2^{n+2}}$.

Solution:- $3^{2n+1} + 2^{n+2} = 3.3^{2n} + 4.2^n = 3(2+7)^n + 4.2^n$

$$= 3\left[2^n + n_{c_1}.2^{n-1}.7 + \cdots + 7^n\right] + 4.2^n$$
$$= 7.2^n + 3.7.n_{C_1}.2^{n-1} + \cdots + 3.7^n$$

$$= 7\left(2^n + 3.n_{C_1}.2^{n-1} + \cdots + 3.7^{n-1}\right) \equiv multiple\ of\ 7.$$

Q9. If a, b, c are positive numbers, then show that

$$\boldsymbol{\frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} + \frac{a^2+b^2}{a+b} \geq a+b+c}.$$

Solution:-

$$b^2 + c^2 \geq \frac{(b+c)^2}{2}$$

$$\frac{b^2+c^2}{b+c} \geq \frac{b+c}{2}$$

$$similarly, \frac{c^2+a^2}{c+a} \geq \frac{c+a}{2}\ \&\ \frac{a^2+b^2}{a+b} \geq \frac{a+b}{2}.$$

$$Adding, we\ get\ \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} + \frac{a^2+b^2}{a+b} \geq a+b+c$$

Q10. Out of a circular sheet of paper of radius a, a sector with central angle $\boldsymbol{\theta}$ is cut out and folded into the shape of a conical funnel. Show that the volume of the funnel is maximum when $\boldsymbol{\theta}$ equals $\boldsymbol{2\pi\sqrt{\frac{2}{3}}}$.

Solution:-

Perimeter of the marked region of the circle = 2πa –a (2π -θ) = aθ

Perimeter of the base of the cone = 2πr

$\therefore$ 2πr = aθ $\Rightarrow$r = $\frac{a\theta}{2\pi}$

Volume of the cone = V = $\frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi.\frac{a^2\theta^2}{4\pi^2}\sqrt{a^2 - \frac{a^2\theta^2}{4\pi^2}}$$

$$= \frac{a^3}{24\pi^2}\theta^2\sqrt{4\pi^2 - \theta^2}$$

$$\therefore \frac{dv}{d\theta} = \frac{a^3}{24\pi^2}\left(2\theta\sqrt{4\pi^2-\theta^2} - \frac{\theta^3}{\sqrt{4\pi^2-\theta^2}}\right)$$

For V to be max or min, $\frac{dv}{d\theta} = 0$

$\therefore 2\theta = \sqrt{4\pi^2-\theta^2} - \frac{\theta^3}{\sqrt{4\pi^2-\theta^2}} = 0$

$\Rightarrow 8\pi^2 - 2\theta^2 - \theta^2 = 0$

($\because \theta \neq 0$)

$\therefore \theta = 2\pi\sqrt{\frac{2}{3}}$

$$\left[\frac{d^2v}{d\theta^2}\right]_{\theta=2\pi\sqrt{\frac{2}{3}}} = \frac{a^3}{24\pi^2}\left[2\sqrt{4\pi^2-\theta^2} - \frac{1}{2}\frac{4\theta}{\sqrt{4\pi^2-\theta^2}} - \frac{3\theta^2}{\sqrt{4\pi^2-\theta^2}} + \frac{2\theta^4}{(4\pi^2-\theta^2)^{\frac{3}{2}}}\right]_{\theta=2\pi\sqrt{\frac{2}{3}}}$$

$\therefore$ The volume of the funnel is max. when θ equals $2\pi\sqrt{\frac{2}{3}}$ [proved]

ISI B.STAT & B.MATH SUBJECTIVE QUESTIONS & SOLUTIONS

SET – 6

Q1. Show that if n >2, then $(n!)^2 > n^n$.

Solution:-

2 (n- 2) > n -2 ⤇ 2n – 4+ 2 > n ⤇ 2(n- 1) > n ………….. (i)

3 (n- 3) > n -3 ⤇ 3n – 9 > n- 3 ⤇ 3 (n- 2) > n …………… (ii)

Similarly, we have 4 (n- 3) > n ……………… (iv)

5 (n- 4) > n …………….. (v)

⋮

Multiplying all these up to (n- 2) terms, we get

$[1.2.3.\dots(n-1)]^2 > n^{n-2}$

⤇ $[(n-1)!]^2 > \frac{n^n}{n^2}$

⤇ $[n!]^2 > n^n$.

Q2. Show that for all real x, the expression $ax^2 + bx + c$ (where a, b, c are real constants with a > 0), has the minimum value $\frac{(4ac-b^2)}{4a}$. Also find the value of x for which this minimum value is attained.

Solution;-

Let P = $ax^2 + bx + c$

$= \frac{1}{4a}[4a^2x^2 + 4abx + 4ac]$

$= \frac{1}{4a}[(2ax)^2 + 4axb + b^2] + \frac{1}{4a}[4ac - b^2]$

$= \frac{(2ax+b)^2}{4a} + \frac{1}{4a}[4ac - b^2]$

P is minimum when (2ax + b) = 0, i.e. x = $-\frac{b}{2a}$,

and $P_{min} = \frac{4ac-b^2}{4a}$.

Q3. A pair of complex numbers z_1, z_2 is said to have property P if for every complex number z, we can find real numbers r and s such that z = r $z_1 + sz_2$. Show that a pair z_1, z_2 has property P if and only if the points z_1, z_2 and 0 on the complex plane are not collinear.

Solution:-

Now, $z_1 = a + ib, z_2 = x + iy, 0 = 0 + i.0$

And let $z_1, z_2, 0$ be collinear then

$$\begin{vmatrix} 0 & 0 & 1 \\ a & b & 1 \\ x & y & 1 \end{vmatrix} = 0 \Rrightarrow \frac{a\;x}{b\;y} = k(say)$$

$\therefore z_1 = bk + ib = b(k+i), z_2 = y(k+i)$

So, ∃ some real ‘r’ or‘s’ such that z = rz_1 + $sz_2 = rb(k+i) + sy(k+i) = (rb+sy)(k+i)$,

Which does not hold good.

So, z_1, z_2 and 0 should not be collinear.

Q4. In a club of 80 members, 10 members play none of the games Tennis, Badminton and Cricket. 30 members play exactly one of these three games and 30 members play exactly two of these games. 45 members play at least one of the games among Tennis and Badminton. Determine the number of Cricket playing members.

Solution:-

Let n(U) = Number of elements in universal set = 80

n(NM) = Number of non-playing members = 10

n(PM) = Number of playing members= 80-10= 70.

Given that number of members playing exactly one of the three games = $t_1 + b_1 + c_1$= 30.

Number of members playing exactly two of these three games = tb + bc + ct = 30.Number of members playing there games = 10.

∴ No. of members playing cricket=$c_1 + tc + tbc + bc$

$= 25 + 12 + 10$

$= 47$.

Q5. Each pair in a group of 20 persons is classified by the existence of kinship relation and friendship relation between them. The following table of data is obtained from such a classification.

KINSHIP AND FRIENDSHIP RELATION AMONG 20 PERSONS

Friendship→ Kinship ↓	**Yes**	**No**
Yes	**27**	**31**
No	**3**	**129**

Determine (with justifications) whether each of the following statements is supported by the above data:

(i) Most of the friends are kin.
(ii) Most of the kin are friends.

Solution:-

Friend / Kin	Yes	No.	Total
Yes	27	31	58
No.	3	129	132
Total	30	160	190

1. Most of the friends are kin because 3 of 30 friends are not kin.
2. Most of the friends are kin, which is not true.

Q6. Evaluate $\lim_{n\to\infty} \left\{\left(1+\frac{1}{2n}\right)\left(1+\frac{3}{2n}\right)\left(1+\frac{5}{2n}\right)\ldots\left(1+\frac{2n-1}{2n}\right)\right\}^{\frac{1}{2n}}$.

Solution:-

Let p = $\operatorname*{lt}_{n\to\infty} \left\{\left(1\frac{1}{2n}\right)\left(1+\frac{3}{2n}\right)\ldots.\left(1+\frac{2n-1}{2n}\right)\right\}^{\frac{1}{2n}}$

$$\log P = \operatorname*{lt}_{n\to\infty} \frac{1}{2n}\left\{log\left(1+\frac{1}{2n}\right)+\log\left(1+\frac{3}{2n}\right) + \cdots + \log\left(1+\frac{2n-1}{2n}\right)\right\}$$

$$= \frac{1}{2}\operatorname*{lt}_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \log\left(1+\frac{2r-1}{2r}\right)$$

$$= \frac{1}{2}\operatorname*{lt}_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \log\left(1+\frac{r}{n}-\frac{1}{2n}\right)$$

$$= \frac{1}{2}\operatorname*{lt}_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \log\left(1+\frac{r}{n}\right), since \operatorname*{lt}_{n\to\infty}\frac{1}{2n} = 0.$$

$$= \frac{1}{2}\int_0^1 \log(1+x)\,dx$$

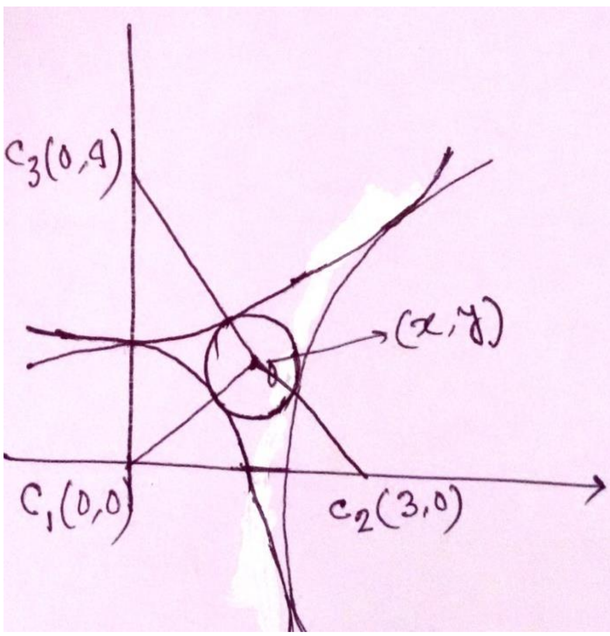
$$= \frac{1}{2}[x\log(1+x) - x + \log(1+x)]\,\begin{matrix}1\\0\end{matrix}$$

$= \log\left(\frac{4}{e}\right)^{\frac{1}{2}}$

$\therefore P = \left(\frac{4}{e}\right)^{\frac{1}{2}} = \frac{2}{\sqrt{e}}.$

Q7. The circles C_1, C_2 *and* C_3 with radii 1, 2 and 3, respectively, touch each other externally. The centres of C_1 *and* C_2 lie on the x-axis, while C_3 touches them from the top. Find the ordinate of the centre of the circle that lies in the region enclosed by the circles C_1, C_2 *and* C_3 and touches all of them.

Solution:-



For simplicity, centre of $C_1 \equiv (0,0)$; $C_2 \equiv (3,0)$; $C_3 \equiv (0,4)$

Let (x, y) be the co-ordinates of the centre of the circle touching C_1, C_2 *and* C_3 and let r be its radius.

Then, $(r+1)^2 = x^2 + y^2$………………. (1)

$(r+2)^2 = (x-3)^2 + y^2$ ………………… (2)

$(r+3)^2 = x^2 + (y-4)^2$ …………………… (3)

Solving these there equations, we will get the following equation:-

$23x^2 - 90x + 63 = 0$

$\therefore x = \frac{90 \pm \sqrt{90^2 - 4.23.63}}{2.23} = \frac{21}{23}$ $or, 3$

By the diagram, x can't be 3, so, $x = \frac{21}{23}, \therefore y = \frac{20}{23}$.

$\therefore$ Required centre of the circle is $\left(\frac{21}{23}, \frac{20}{23}\right)$.

Q8. Using calculus, sketch the graph of the following function on a plain paper: f(x) = $\frac{5-3x^2}{1-x^2}$.

Solution:- f(x) = $\frac{5-3x^2}{1-x^2} = \frac{3(1-x^2)+2}{1-x^2} = 3 + \frac{2}{1-x^2}$

$\therefore f'(x) = \frac{4x}{(1-x^2)^2},$

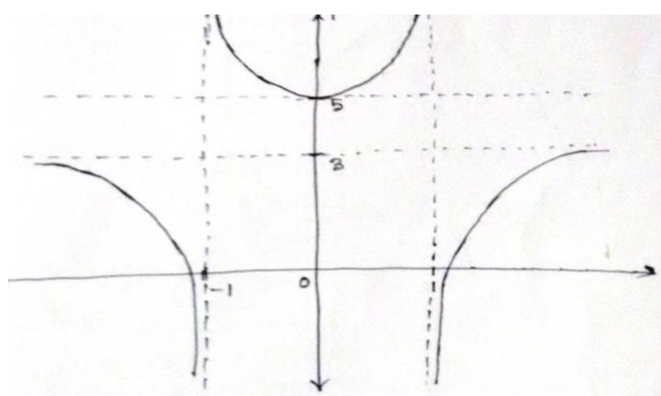
For, $-\infty < x < -1, f'(x) is\ negative;$

For, -1 < x ≤ 0, $f'(x)$ is negative;

For, 0 < x <1, $f'(x)$ is positive;

For, 1< x <∞, $f'(x)$ is positive;

x	-3	3	-2	2	0	$-\frac{1}{2}$	$\frac{1}{2}$
f(x)	$2\frac{3}{4}$	$2\frac{3}{4}$	$2\frac{1}{3}$	$2\frac{1}{3}$	5	$\frac{17}{3}$	$\frac{17}{3}$



Q9. Let ABC be an isosceles triangle with AB = BC = 1 cm and ∠A = 30°. Find the volume of the solid obtained by resolving the triangle about the line AB.

Solution:- Here AB = BC = 1

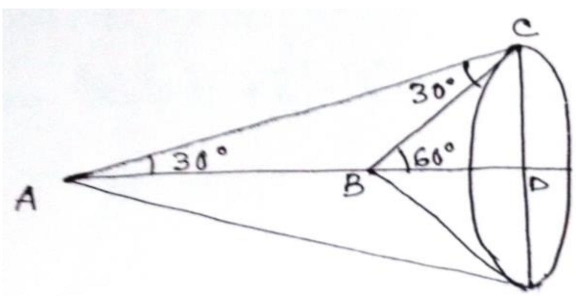
BD = BC Cos 60° $= 1.\frac{1}{2} = \frac{1}{2}$

CD = BC Sin 60° $= 1.\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

∴AD = AB+ BD = $1+\frac{1}{2} = \frac{3}{2}$

∴ Required volume $= \frac{1}{3}\pi.\frac{3}{4}$

$\therefore \left(\frac{3}{2} - \frac{1}{2}\right) sq.unit = \frac{1}{3}.\frac{3\pi}{4} sq.unit = \frac{\pi}{4} sq.unit.$



Q10. (a) Prove that, for any odd integer n, n^4 when divided by 16 always leaves remainder 1.

(b) Hence or otherwise show that we cannot find integers $n_1, n_2, \ldots, n_8$ such that $n_1{}^4 + n_2{}^4 + \cdots + n_8{}^4 = 1993$.

Solution

(a) Let n = 2a + 1

$$n^4 = (2a+1)^4$$

$$= (4a^2 + 4a + 1)^2$$

$$= [4a(a+1) + 1]^2$$

$$= 16a^2(a+1)^2 + 8a(a+1) + 1$$

$$= 16[a(a+1)]^2 + 8a(a+1) + 1$$

Now, a (a+ 1) is divisible by 2.

$\therefore n^4 \equiv 1 \ (mod\ 16)$.

(b)Solution:-

Now, $n_1{}^4 \equiv 1 (mod\ 16)$

$n_2{}^4 \equiv 1 \ (mod\ 16)$

⋮ ⋮

$n_8{}^4 \equiv 1 \ (mod\ 16)$

$n_1{}^4 + n_2{}^4 + \cdots + n_8{}^4 \equiv 8 \ (mod\ 16)$

But 1993 ≡ 9 (mod 16)

So, the value of $n_1{}^4 + n_2{}^4 + \cdots + n_8{}^4$ can't be 1993.

ISI B.STAT & B.MATH SUBJECTIVE QUESTIONS & SOLUTIONS

SET – 7

Q1. Let x be a positive number, A sequence $\{x_n\}$ of real numbers is defined as follows:

$\boldsymbol{x_1 = \frac{1}{2}\left(x + \frac{5}{x}\right), x_2 = \frac{1}{2}\left(x_1 + \frac{5}{x_1}\right)}$, **…, and in general,**

$$\boldsymbol{x_{n+1} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right) \ for\ all\ n \geq 1.}$$

(a) Show that, for all n ≥ 1, $\frac{x_{n-\sqrt{5}}}{x_n+\sqrt{5}} = \left(\frac{x-\sqrt{5}}{x+\sqrt{5}}\right)^{2^n}$.

(b) Hence find $\lim\limits_{n\to\infty} x_n$.

Solution:-

(a) $x_n = \frac{1}{2}\left(x_{n-1} + \frac{5}{x_{n-1}}\right) = \frac{\sqrt{5}}{2}\left(\frac{x_{n-1}}{\sqrt{5}} + \frac{\sqrt{5}}{x_{n-1}}\right)$

$\Longrightarrow \frac{x_n}{\sqrt{5}} = \frac{1}{2}\left(\frac{x_{n-1}}{\sqrt{5}} + \frac{\sqrt{5}}{x_{n-1}}\right).$

By componendo-dividend, we get-

$$\frac{x_n - \sqrt{5}}{x_n + \sqrt{5}} = \frac{\frac{1}{2}\left(\frac{x_{n-1}}{\sqrt{5}} + \frac{\sqrt{5}}{x_{n-1}}\right) - 1}{\frac{1}{2}\left(\frac{x_{n-1}}{\sqrt{5}} + \frac{\sqrt{5}}{x_{n-1}}\right) + 1}$$

$$= \frac{{x_{n-1}}^2 + \sqrt{5} - 2\sqrt{5}\, x_{n-1}}{{x_{n-1}}^2 + \sqrt{5} + 2\sqrt{5}\, x_{n-1}}$$

$$= \left(\frac{x_{n-1} - \sqrt{5}}{x_{n-1} + \sqrt{5}}\right)^2 = \left(\frac{x_{n-2} - \sqrt{5}}{x_{n-2} + \sqrt{5}}\right)^{2^2} = \cdots$$

$$= \left(\frac{x_1 - \sqrt{5}}{x_1 + \sqrt{5}}\right)^{2n-1}$$

$$= \left(\frac{x - \sqrt{5}}{x + \sqrt{5}}\right)^{2^n} \quad (proved)$$

(b)Since x is a positive number, $x - \sqrt{5} < x + \sqrt{5}$

$\Longrightarrow \frac{x-\sqrt{5}}{x+\sqrt{5}} < 1$

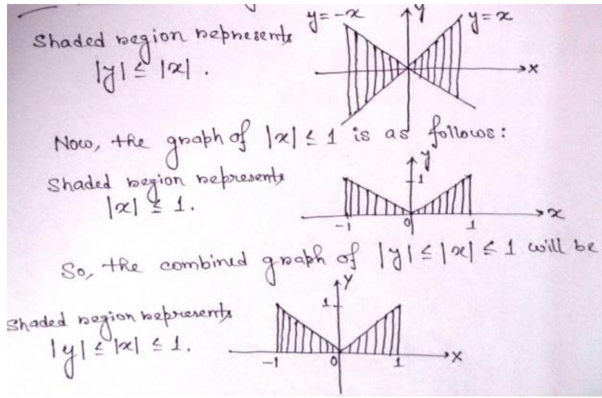
$\therefore \left(\frac{x-\sqrt{5}}{x+\sqrt{5}}\right)^{2^n} \longrightarrow 0 \; as \; n \; \rightarrow \infty.$

$\Longrightarrow \frac{x-\sqrt{5}}{x+\sqrt{5}} \longrightarrow 0 \;\Longrightarrow x - \sqrt{5} \longrightarrow 0$

$\therefore \lim\limits_{n \to \infty} x_n = \sqrt{5}.$

Q2. Draw the region of points (x, y) in the plane, which satisfy |y| ≤|x| ≤ 1.

Solution:-

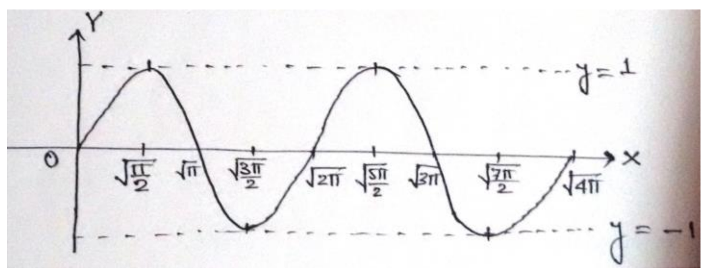


Q3. Sketch on plain paper, the graph of the function y = sin(x^2), in the range 0 ≤ x ≤ $\sqrt{4\pi}$.

Solution:-

Taking different values of x, we get different y.

x	0 $\sqrt{\pi/4}$ $\sqrt{\pi/2}$ $\sqrt{\pi}$ $\sqrt{3\pi/2}$ $\sqrt{2\pi}$ $\sqrt{5\pi/2}$ $\sqrt{3\pi}$ $\sqrt{7\pi/2}$ $\sqrt{4\pi}$
y	0 $1/\sqrt{2}$ 1 0 -1 0 1 0 -1 0



Q4. If n is a positive integer greater than 1 such that 3n + 1 is perfect square, then show that n + 1 is the sum of three perfect squares.

Solution:-

As 3n + 1 is a perfect square, so let

3n + 1 = a^2

⇛ a is not a multiple of 3.

⤇ a may be of the form either 3k + 1 or 3k + 2, k ∈ I.

Taking a = 3k + 1, 3n + 1= $(3k + 1)^2 = 9k^2 + 6k + 1$;

Or, n = $3k^2 + 2k$

Or, n + 1 = $3k^2 + 2k + 1 = k^2 + k^2 + (k+1)^2$

i.e. , sum of three perfect squares.

Taking a = 3k + 2, 3n + 1= $(3k + 2)^2 = 9k^2 + 6k + 4$;

Or, n =$3k^2 + 4k + 1$

Or, n + 1= $3k^2 + 4k + 2 = k^2 + (k+1)^2 + (k+1)^2$

i.e. sum of 3 perfect squares. [Proved]

Q5. Let x = $(x_1, x_2, \ldots, x_n)$, y = $(y_1, y_2, \ldots, y_n)$, where $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$ are real numbers. We write x > y, if for some k, 1 ≤ k ≤ (n- 1), $x_1 = y_1,\ x_2 = y_2, \ldots, x_k = y_k$, but $x_{k+1} > y_{k+1}$. Show that for u = $(u_1, \ldots, u_n)$, v=$(v_1, \ldots, v_n)$, w = $(w_1, \ldots, w_n)$ and z = $(z_1, \ldots, z_n)$, if u > v and w > z, then (u +w) > (v +z).

Solution:- u > v

& $u_1 = v_1$

$u_2 = v_2$

⋮

$u_k = v_k$

But $u_{k+1} > v_{k+1}$

Again w > z

& $w_1 = z_1$

$w_2 = z_2$

⋮

$w_k > z_k$

But $w_{k+1} = z_{k+1}$

So, $u_k + w_k = v_k + z_k$ ⤇ (u+ w)k = (v+ z)k

But $u_{k+1} + w_{k+1} > v_{k+1} + z_{k+1}$.

So, u + w > v + z.

Q6. Consider the set of points S = {(x, y): x, y are non-negative integers ≤ n}.

Find the number of squares that can be formed with vertices belonging to S and sides parallel to the axes.

Solution:-

The number of squares with sides of unit length = $n \times n = n^2$.

The number of squares with sides of length 2 units is = $(n-1) \times (n-1) = (n-1)^2$ and so on.

∴ Total number of squares = $n^2 + (n-1)^2 + \cdots + 2^2 + 1^2 = \frac{n(n+1)(2n+1)}{6}$.

Q7. If $\frac{\sin^4 x}{a} + \frac{\cos^4 x}{b} = \frac{1}{a+b}$, then show that $\frac{\sin^6 x}{a^2} + \frac{\cos^6 x}{b^2} = \frac{1}{(a+b)^2}$.

Solution:-

$$\frac{sin^4 x}{a} + \frac{cos^4 x}{b} = \frac{1}{a+b},$$

$$or, \quad \frac{sin^4x}{a} + \frac{(1-sin^2x)^2}{b} = \frac{1}{a+b}$$

$$or, \quad (a+b)^2sin^4x - 2asin^2x(a+b) + a^2 = 0$$

$$\therefore \{(a+b)sin^2x - a\}^2 = 0$$

$$\therefore sin^2x = \frac{a}{a+b} \ \therefore \ cos^2x = 1 - \frac{a}{a+b} = \frac{b}{a+b};$$

$$\therefore \frac{sin6x}{a^2} + \frac{cos6x}{b^2} = \frac{a^3}{\frac{(a+b)^3}{a^2}} + \frac{b^3}{\frac{(a+b)^3}{b^2}} = \frac{1}{(a+b)^2}.$$

Q8. Suppose there are k terms playing a round robin tournament; that is, each team plays against all the other teams and no game ends in a draw. Suppose the ith team loses l_i games and wins w_i games. Show that

$$\sum_{i=1}^{k} l_i^2 = \sum_{i=1}^{k} w_i^2.$$

Solution:-

By the problem, every team will play k -1 matches. As the i-th team loses l_i & $wins\ w_i$ matches, they play a total of $l_i + w_i$ matches, as no match ends in draw.

$\therefore l_i + w_i = k-1$ ………… (1)

Obviously, total no. of wins in the tournament

= total no. of loses in the tournament

$\Rightarrow \sum_{i=1}^{k} l_i = \sum_{i=1}^{k} w_i$ …………… (2)

Now, $\sum_{i=1}^{k} l_i^2 = \sum_{i=1}^{k} w_i^2 = \sum_{i=1}^{k}\left(l_i^2 - w_i^2\right)$

$$= \sum_{i=1}^{k}(l_i + w_i)(l_i - w_i)$$

$$= \sum_{i=1}^{k}(k-1)(l_i - w_i) \ \ [from\ (1)]$$

$= (k-1)\{\sum_{i=1}^{k} l_i -\} = (k-1).0\ [from\ (2)] = 0$

$$\Rightarrow \sum_{i=1}^{k} l_i^2 = \sum_{i=1}^{k} w_i^2 \ \ (proved)$$

Q9. Let $P_1, P_2, \dots, P_n$ be polynomials in x, each having all integer coefficients, such that $P_1 = P_1^2 + P_2^2 + \cdots + P_n^2$. Assume that P_1 is not the zero polynomial. Show that $P_1 = 1\ and\ P_2 = P_3 = \cdots = P_n = 0$.

Solution:-

According to the question, $p_1 = p_1^2 + p_2^2 + p_3^2 + \cdots + p_n^2$

This is possible when the degree of p_1 is0.

$p_1^2 \geq p_1$, $\frac{p_2^2+p_3^2+\cdots+p_n^2}{p_1^2+p_2^2+\cdots+p_n^2} \geq 0$ given that $p_1^2 + p_2^2 + p_3^2 + \cdots + p_n^2 = p_1, p_1^2 = p_1, \because p_1 \neq 0 \ \therefore \ p_1 = 1$

And $p_2^2 + p_3^2 + \cdots + p_n^2 = 0$

i.e., $p_2 = p_3 = \cdots p_n = 0$ [Proved]

Q10. Let P(x) = $x^4 + ax^3 + bx^2 + cx + d$, where a, b, c, and d are integers. The sums of the pairs of roots of P(x) are given by 1, 2, 5, 6, 9 and 10 find P(½).

Solution:-

P(x) = $x^4 + ax^3 + bx^2 + cx + d$, now sum of the roots= -a

$\Longrightarrow \alpha+ \beta+ \gamma+ \delta$= -a

Again, $\alpha + \beta$ = +1, $\gamma + \delta$ =10, $\alpha + \gamma$ = 2, $\alpha +\delta$ = 6, $\beta + \gamma$= 5, $\beta +\delta$ = 9, Adding, 3($\alpha+ \beta+ \gamma+ \delta$) = 33

$\Longrightarrow \alpha + \beta +\gamma + \delta$ = 11

$\Longrightarrow$ a = -11

Solving, the equations,

α = -1 [Note: ($\alpha+ \gamma$) and ($\alpha+ \delta$) both should be either

β = 2 even or both should be odd, else a, b, c, d

γ = 3 will not be integers.]

δ = 7

We know, $\alpha\beta +\beta\gamma + \gamma\delta +\delta\alpha +\alpha\gamma +\beta\delta$ = b

$\alpha\beta\gamma+ \beta\gamma\delta+ \gamma\delta\alpha+ \alpha\beta\delta$ = -c

$\alpha\beta\, \gamma\delta$ = d

Putting the values of $\alpha, \beta, \gamma, \delta$, b = 29, c = -1, d = -42, a = -11.

$\therefore$ a, b, c, d any integers.

$\therefore$P(x) = $x^4 - 4x^3 + 29x^2 - x - 42$ and, P(1/2)_= $\frac{585}{16}$

ISI B.STAT & B.MATH SUBJECTIVE QUESTIONS & SOLUTIONS

SET – 8

Q1. Sketch the set A ∩ B in the Argand Plane, where A = {z: $|\frac{z+1}{z-1}| \leq 1$} and B= {z: |z| - Re z ≤ 1}.

Solution:-

Let z = x + iy, then | $\frac{z+1}{z-1}$ | ≤ 1,

$\Longmapsto$ |z+1| ≤ |z- 1|,

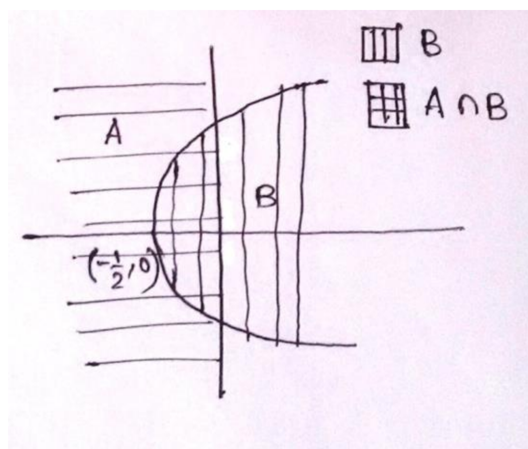
$\Longmapsto (x+1+iy)^2 \leq (x+iy-1)^2 \;\Longmapsto 4x \leq 0$ $\Longmapsto x \leq 0.$

A = {z : x ≤ 0} ………………….. (i)

Also, |z| - Re(z) ≤ 1

$\Longmapsto \sqrt{x^2+y^2} \leq x+1 \;\; \Longmapsto x^2+y^2 \leq x^2 + 2x+1 \;\; \Longmapsto y^2 \leq 2x+1 = 2\left(x+\frac{1}{2}\right)$

$\therefore y^2 = 2\left(x+\frac{1}{2}\right)$ is a parabola, having its vertex at $\left(-\frac{1}{2}, 0\right)$ and axis on X-axis.



Q2. A function f from a set A into a set B is rule which assigns to each element x in A, a unique (one and only one) element (denoted by f(x)) in B. A function f from A into B is called an onto function, if for each element y in B there is some element x in A, such that f(x) = y. now suppose that A = {1, 2, …, n}

and B = {1, 2, 3}. Determine the total number of onto functions from A into B.

Solution:-

Number of onto functions from {1, 2, 3, …, n} to {1, 2} is $2^n - 2$.

Here, A = {1, 2, 3, …, n}, B = {1, 2, 3}, for each i ∈ A have 3 possibilities, so total no. of f(n) from A to B is 3^n.

But there are $\binom{3}{2}(2^n - 2)f(n)$ image consist of 2 points and 3 f(n) whose image is singleton. Hence, total number of onto functions f(n) from A to B is $\left\{3^n - \binom{3}{2}(2^n - 2) - 3\right\}$.

Q3. Let D=$a^2 + b^2 + c^2$, where a and b are successive positive integers and c = ab. Prove that $\sqrt{D}$ is an odd positive integer.

Solution:-

Let a = 2n

b = 2n + 1

c = 2n (2n +1)

$= 4n^2 + 2n$

D = $a^2 + b^2 + c^2$

∴ D = $16n^4 + 16n^3 + 12n^2 + 4n + 1$

D- 1 = $2(8n^4 + 8n^3 + 6n^2 + 2n)$

∴ D – 1 is an even number.

∴ D is an odd number.

So, $\sqrt{D}$ is an odd number.

Also, D = $16n^4 + 16n^3 + 12n^2 + 4n + 1$

$= (4n^2 + 2n + 1)^2$

∴ $\sqrt{D} = 4n^2 + 2n + 1$

$= 2(2n^2 + n) + 1$

∴$\sqrt{D}$ is an odd positive number.

Q4. Show that a necessary and sufficient condition for the line ax + by +c = 0, where a, b, c are non-zero real numbers, to pass through the first quadrant is either ac < 0 or bc < 0.

Solution:-

ax + by +c = 0

$\Longrightarrow$ y = $-\frac{ax+c}{b}$ >0.

∴ $\frac{ax+c}{b} < 0 \Longrightarrow \frac{acx^2+c^2}{bc} < 0.$

There will be at least one point on the line for which x > 0 and y >0 [∵ If passes through the 1st quadrant]

(i) If $acx^2 + c^2 > 0$, then bc <0, now, x>0, $c^2 > 0$.
∴ ac > 0.

(ii) If $acx + c^2 < 0$, then bc >0, now, x >0, $c^2 > 0$.

∴The necessary and sufficient conditions for the line to pass through the 1st quadrant is either ac >0 or, bc >0.

Q5. The sum of squares of the digits of a three digits positive number is 146, while the sum of the two digits in the unit's and the ten's place is 4 times the digit in the hundred's place. Further, when the number is written in the reverse order, it is increased by 297. Find the number.

Solution:-

Let the no. be (xyz), i.e. N = 100x + 10y +z

Given $x^2 + y^2 + z^2 = 146$, ... (i)

4x = y + z ………………………………… (ii)

100z + 10y + x = 100x +10y + z + 297

⟾ z –x = 3 …………………. (iii)

Solving (i), (ii) and (iii), we have x = 4, y= 9, z = 7.

So, the number is 497.

Q6. Show that there is at least one real value of x for which $\sqrt[3]{x} + \sqrt{x} = 1$.

Solution:-

Let y = 1- $\sqrt{x}$ …………… (i)

And $y_0 = \sqrt[3]{x}$ ……………. (ii)

For function (i) x = 0, y= 1; x = 1, y =0; x= $\frac{1}{4}, y = \frac{1}{2}$;

This is a continuous function curve which decreases from 1 to 0

For function (ii) x = 0, $y_0 = 0$; x = 1, $y_0 = 1$; x= $\frac{1}{8}, y_0 = \frac{1}{2}$;

This is also a continuous function curve and it increases from 0 to 1 in the interval 0 ≤ x ≤ 1.

Hence, they must meet each other, i.e. their value will be some at some points between 0 ≤ x ≤ 1.

Hence, the given equation has only one real root.

Q7. Suppose S = {0, 1} with the following addition and multiplication rules:

0 + 0= 1 +1 = 0   0.0= 0.1 = 1.0 = 0

0 + 1 = 1+ 0 = 1   1. 1 = 1

A system of polynomials is defined with coefficients in S. The sum and product of two polynomials in the system are the usual sum and product, respectively, where for the addition and multiplication of coefficients the above mentioned rules apply. For example, in the system,

$$(\boldsymbol{x} + \boldsymbol{1}).(\boldsymbol{x^2} + \boldsymbol{x} + \boldsymbol{1}) = \boldsymbol{x^3} + (\boldsymbol{1}+\boldsymbol{1})\boldsymbol{x^2} + (\boldsymbol{1}+\boldsymbol{1})\boldsymbol{x} + \boldsymbol{1} = \boldsymbol{x^3} + \boldsymbol{0x^2} + \boldsymbol{0x} + \boldsymbol{1} = \boldsymbol{x^3} + \boldsymbol{1}.$$

Show that in this system $\boldsymbol{x^3 + x + 1 = (ax+b).(cx^2+dx+e)}$ $\boldsymbol{can't\ hold}$.

Where a, b, c, d and e are elements of S.

Solution:-

Let us try to write,

$$x^3 + x + 1 = (ax+b)(cx^2+dx+e) = acx^3 + (bc+ad)x^2 + (bd+ae)x + be;$$

$$\Rightarrow ac = 1, bc + ad = 0, bd + ae = 1, be = 1$$

∵ $a, b, c, d, e \in s$, as we have assumed.

∴ From the given rule, a = 1, c= 1, b = 1, e= 1

∴ bc+ ad = 0, substituting the rules,

1+ d= 0 ⟹ d = 1 (∵ 1+ 1 = 0)

Again, bd +ac = 1, substituting the values,

1+ 1= 1, but, by the rule 1+ 1 = 0, which contradicts over assumption.

⟹ $x^3 + x + 1$ cannot be factorial in this system.

Q8. Show that $\boldsymbol{\int_0^{\pi/2} \left|\frac{\sin nx}{x}\right| dx \geq \frac{2}{\pi}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)}$.

Solution:-

$\int_0^{\pi} \left| \frac{\sin hx}{x} \right| dx$, as x ranges from [0, π], so

Let us put nx = z $\therefore$ ndx = dz

I = $\int_0^{n\pi} \left| \frac{sinz}{z/n} \right| \frac{dz}{n} = \int_0^{n\pi} \left| \frac{sinz}{z} \right| dz =$
$\int_0^{\pi} \left| \frac{sinz}{z} \right| dz + \int_{\pi}^{2\pi} \left| \frac{sinz}{z} \right| dz + \cdots +$
$\int_{(n-1)\pi}^{\pi} \left| \frac{sinz}{z} \right| dz$

Now, $\int_0^{\pi} \left| \frac{sinz}{z} \right| dz = \int_0^{\pi} \frac{sinz}{z} dz \geq$
$\int_0^{\pi} \frac{sinz}{\pi} dz = \frac{1}{\pi} \int_0^{\pi} sinz\, dz$

Now, $\int_{\pi}^{2\pi} \left| \frac{sinz}{z} \right| dz = \int_0^{\pi} \left| \frac{-siny}{\pi+y} \right| dy$, where π + y = z,

= $\int_0^{\pi} \frac{siny}{\pi+y} dy \geq \frac{1}{2\pi} \int_0^{\pi} siny\, dy = \frac{2}{2\pi}$

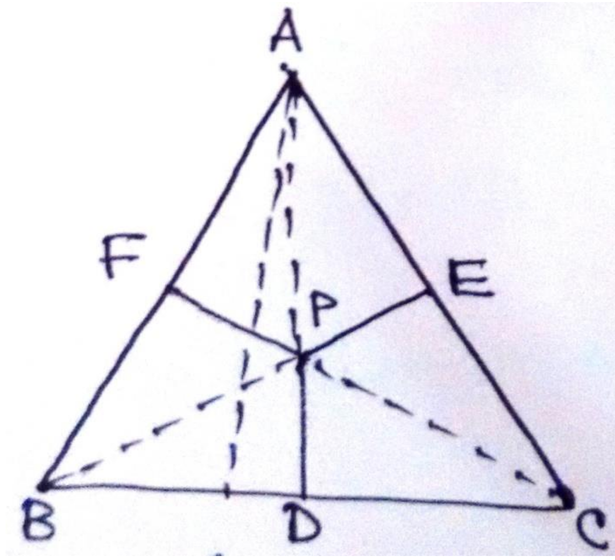
Proceeding in this way, $\int_{2\pi}^{3\pi} \left| \frac{sinz}{z} \right| dz \geq$
$\frac{2}{3\pi}, \ldots, \int_{(n-1)\pi}^{n\pi} \left| \frac{sinz}{z} \right| \geq \frac{2}{n\pi}$

$\therefore I \geq \frac{2}{\pi} + \frac{2}{2\pi} + \frac{2}{3\pi} + \cdots + \frac{2}{n\pi} = \frac{2}{\pi}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).$

Q9. Inside an equilateral triangle ABC, an arbitrary point P is taken from which the perpendiculars PD, PE and PF are dropped onto the sides BC, CA and AB, respectively. Show that the ratio $\frac{PD+PE+PF}{BD+CE+AF}$ does not depend upon the chice of the point P and find its value.

Solution:- We have (i) AB = BC = CA & ∠ B =∠C =∠A = 60°

[∵ ΔABC is equilateral]



(ii) PD⊥ BC, PE ⊥ AC, PF ⊥AB.

So, from (i) ∠PAF = ∠PAE = 30°,

∠PCE = ∠PCD = 30°, ∠PBD= ∠PBF= 30°.

∴In ΔPAF, $\frac{PF}{AF} = \tan 30° = \frac{1}{\sqrt{3}} \quad or, PF = \frac{AF}{\sqrt{3}}.$

Similarly, for ΔPBD, we get PD = $\frac{1}{\sqrt{3}} BD, and$ for ΔPCE, we get PE= $\frac{1}{\sqrt{3}} CE.$

$\therefore \frac{PD+PE+PF}{BD+CE+AF} = \frac{\frac{1}{\sqrt{3}}(BD+CE+AF)}{BD+CE+AF} = \frac{1}{\sqrt{3}}.$

As each of the PD, PE, PF can be represented w.r.t. BD, CE, AF respectively, so the specified ratio does not depend upon the choice of the point P.

Q10. AB is a chord of a circle C.

(a) Find a point P on the circumference of C such that PA. PB is the maximum.

(b) Find a point P on the circumference of C which maximizes PA+ PB.

Solution:- (a) Let PA = x and PB = y

∴ PA.PB = xy = $\left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2$

xy is maximum when x = y.

i.e. PA = PB $\Longmapsto \Delta$PAB will be an isosceles one.

Position of P:-

P will be the point of intersection of the perpendicular bisector of AB and the circles.

(b)$\frac{AP}{\sin B} = \frac{BP}{\sin A} = \frac{AB}{\sin P}$

AP = $\frac{AP}{\sin P} \times \sin B\,; BP = \frac{AB}{\sin P} \times \sin A\,;$

$\therefore$ AP + BP = $\frac{AB}{\sin P} \times (\sin A + \sin B) = \frac{AB}{\sin P} \times 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} = \frac{AB}{\sin P}.2\cos\frac{P}{2}\cos\frac{A-B}{2}.$

$\because$ AB = constant, $\measuredangle$P = constant,

$\therefore$AP + BP = maximum when $\cos\frac{A-B}{2}$= maximum =1.

$\Longmapsto \frac{A-B}{2} = 0 \Longmapsto A = B \Longmapsto PA = PB.$

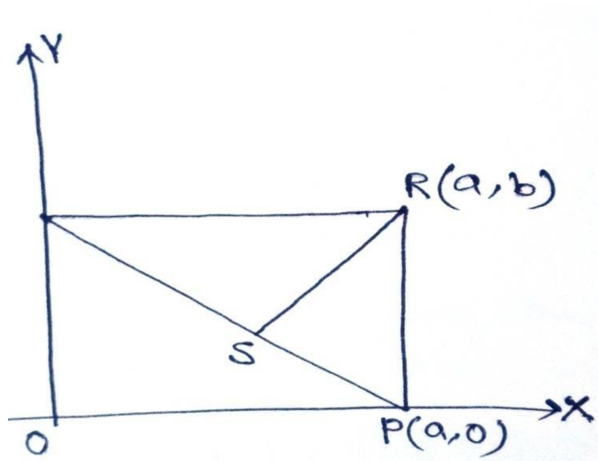
$\Longmapsto$ P lies on the point of intersection of the bisector of AB.

ISI B.STAT & B.MATH SUBJECTIVE QUESTIONS & SOLUTIONS

SET – 9

Q1. Let PQ be a line segment of a fixed length l with its two ends P and Q sliding along the X- axis an Y-axis respectively. Complete the rectangle OPRQ where O is the origin. Show that the locus of the foot of perpendicular drawn O is the origin. Show that the locus of the foot of the perpendicular drawn from R on PQ is given by $x^{2/3} + y^{2/3} = L^{2/3}$.

Solution:-



Equation of PQ: $\frac{x}{a} + \frac{y}{b} = 1.$

$\therefore$ bx +ay – ab = 0 ……………. (1)

As PQ = L, so, $a^2 + b^2 = L^2$……………. (2)

Equation of the line through R(a, b) and perpendicular to PQ is

y-b = $\frac{a}{b}(x-a)$

$or, ax - by - (a^2 - b^2) = 0$ ………….. (3)

Both the lines PQ & RS meet at point S, whose locus we are to find, the variables being a, b, which are connected by $a^2 + b^2 = L^2$.

Solving (1) and (3), we have

$$\frac{x}{-a^3 + ab^2 - ab^2} = \frac{y}{-a^2b + a^2b - b^3} = \frac{1}{-b^2 - a^2}$$

$$\Longmapsto \frac{x}{-a^3} = \frac{y}{-b^3} = -\frac{1}{b^2 + a^2} = \frac{1}{L^2};$$

$$\Longmapsto a = (L^2x)^{\frac{1}{3}} \& \, b = (L^2y)^{\frac{1}{3}};$$

$\Longmapsto a = (L^2x)^{\frac{1}{3}}$& $b = (L^2y)^{\frac{1}{3}}$;

$\because\ a^2 + b^2 = L^2$

$\therefore (L^2x)^{\frac{2}{3}} + (L^2y)^{\frac{2}{3}} = (L^2)^{\frac{2}{3}}$

$\therefore\ x^{\frac{2}{3}} + y^{\frac{2}{3}} = L^{\frac{2}{3}}$ which is the required locus of S.

Q2. Let [x] denote the largest (positive, negative or zero) less than or equal to x. Let y= f(x)= [x]+ $\sqrt{x-[x]}$ be defined for all real numbers x.

(i) Sketch on plain paper, the graph of the function f(x) in the range -5 ≤ x ≤ 5.

(ii) Show that, given any real number y_0, there is a real number x_0, such that $y_0 = f(x_0)$.

Solution:- y = f(x) = [x] + $\sqrt{x-[x]}$ = [x] + $\sqrt{\{x\}}$

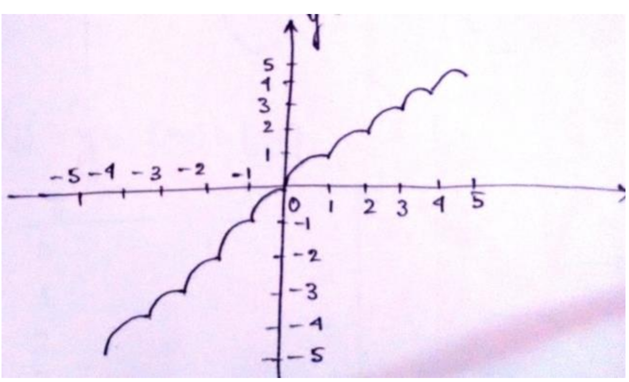
(i) We know 0 ≤ {x} < 1

$\Longmapsto \sqrt{\{x\}} \geq \{x\}$

$\Longmapsto$[x] + $\sqrt{\{x\}}$≥ [x] + {x}

$\Longmapsto$ f(x) ≥ x

'=' holds when x takes integral values.



(ii) Again, y = f(x) = [x] + $\sqrt{\{x\}}$

As 0 ≤ {x} < 1, hence $\sqrt{\{x\}}$ is always real,

$\Longmapsto$ f(x) is always real.

$\Longmapsto$ there is a x_0, $\forall\, y_0 \in \mathbb{R}$ &$x_0 \in \mathbb{R}, \ni y_0 = f(x_0)$

Q3. A troop 5 metres long starts marching. A soldier at the end of the file steps out and starts marching forward at a higher speed. On reaching the head of the column, he immediately turns around and marches back at the same speed. As soon as he reaches the end of the file, the troop stops marching, and it is found that the troop has moved by exactly 5 metres. What distance has the soldier travelled?

Solution:- Let Vel. Of troop = V_t

Vel. Of soldier = V_s

Now, distance travelled by troop, when the soldier reached that point = x m.

$\therefore$ Time taken = $\frac{x}{V_t} = \frac{distance}{velocity}$.

So, distance traveled by soldier = length of troop + x = x+ 5

$\therefore$ Time taken = $\frac{x+5}{V_s}$.

Given that $\frac{x}{V_t} = \frac{x+5}{V_s} \Longmapsto \frac{V_s}{V_t} = \frac{x+5}{x}$ …………….(i)

Again, the soldier retreated x units.

$\therefore$ Time taken by him to retreat = $\frac{x}{V_s}$

The troop travelled = {10 – (5+ x)}m = (5-x)m.

∴ Time taken by them to travel= $\frac{5-x}{V_t}$

So, by the question, we have $\frac{x}{V_s} = \frac{5-x}{V_t}$

$\Rightarrow \frac{V_s}{V_t} = \frac{x}{5-x}$ …………… (ii)

Equating (i) and (ii), we have

$\frac{x+5}{x} = \frac{x}{5-x}$

$\Rightarrow$ x = $\frac{5}{\sqrt{2}}$.

∴ The soldier travelled = 5+ x+ x= (5+ 5$\sqrt{2}$) m.

Q4. Given m identical symbols, say H's, show that the number of ways in which you can distribute them in k boxes marked 1, 2, …, k, so that no box goes empty is $\binom{m-1}{k-1}$.

Solution:-

(a) Put 1 ball in each box, so we have m-x identical balls to be distributed in k boxes.

Let, the jth box got i j balls out of (m - k) balls, where j = 1, 2, 3, …, k; 0 ≤ i, j ≤ m- k.

So, we need to find the no. of solutions of the equation $i_1 + i_2 + \cdots + i_k = m - k$, where each i j is non-negative integers.

∴ Total number f solutions = coefficient of x^{m-1} in $(1-x)^{-k}$

$$= \binom{m-1}{k-1}.$$

Q5. Show that for every positive integer n, $\sqrt{n}$ is either an integer or an irrational number.

Solution:-

When 'n' is perfect square,

Then $\sqrt{n}$ is an integer.

When 'n' is not a perfect square, then let

$\sqrt{n}$ is a rational number= $\frac{p}{q}$, where gcd (p, q)= 1.

i.e. n = $\frac{p^2}{q^2}$

∵p and q are relatively prime to each other, p^2 and q^2 should be relatively prime to each other and $\frac{p^2}{q^2}$ can't be an integer. Hence, $\sqrt{n}$ is not a rational number.

Q6. Show that $2^{2n} - 3n - 1$ is divisible by 9 for all n ≥ 1.

Solution:-

$$2^{2n} - 3n - 1 = 4^n - 3n - 1$$
$$= (1+3)^n - 3n - 1$$

$$= \left\{1 + 3n + 9.n_{c_2} + \cdots + 3^n\right\} - 3n - 1$$

$$= 9(n_{c_2} + 3.n_{c_3} + \cdots + 3^{n-2})$$

i.e., $2^{2n} - 3n - 1$ is divisible by 9 $\forall\, n \geq 1$.

Q7. Find the set of all values of m such that y = $\frac{x^2-x}{1-mx}$ can take all real values.

Solution:-

$$y = \frac{x^2 - x}{1 - mx}$$

$$\Rightarrow y - mxy = x^2 - x$$

$$\Rightarrow x^2 + (my - 1)x - y = 0$$

$$\therefore (my-1)^2 + 4y \geq 0 \quad [\because x \text{ is real}]$$

$$\Rightarrow m^2y^2 + (4 - 2m)y + 1 \geq 0$$

$$\therefore (4-2m)^2 - 4m^2 \geq 0 \quad [\because y \text{ is real}]$$

$$\Rightarrow (2-m)^2 - m^2 \geq 0$$

$$\Rightarrow 2 - 2m \geq 0 \Rightarrow m \leq 1.$$

Q8. If A, B, C are the angles of a triangle, then show that sin A + sin B - cos C ≤ $\frac{3}{2}$.

Solution:-

Sin A + sin B –sin C = Sin A + sin B-sin $\left(\frac{\Pi}{2} - c\right)$

$$= \sin A + \sin B + \sin\left(C - \frac{\Pi}{2}\right)$$

$$= \sin A + \sin B + \sin D\,, \qquad D = C - \frac{\Pi}{2}.$$

$$Now, A + B + C = \Pi;\ \therefore A + B + C - \frac{\Pi}{2}$$
$$= A + B + D = \frac{\Pi}{2}.$$

Let, f(x)= sin x, we plot its graph such that taking the abscissa A, B, D as $A + B + D = \frac{\Pi}{2}$, or, plotting in the interval $\left[0, \frac{\Pi}{2}\right]$

∴ Centroid of ΔPQR≡G≡ $\left(\frac{A+B+D}{3}, \frac{\sin A+\sin B+\sin D}{3}\right)$

We take a point f(x) = sin x, such that it is of the same abscissa that of G, but of greater ordinate.

∴ M ≡ $\left(\frac{A+B+D}{3}, sin\frac{A+B+D}{3}\right)$

∴ Ordinate of G < ordinate of M,

$\Longrightarrow \frac{\sin A+\sin B+\sin D}{3} < sin\frac{A+B+D}{3}$,

Or, $\sin A + \sin B + \sin D < \frac{3}{2}$.

When A, B, D are not distinct, i.e., A= B=D= $\frac{\Pi}{6}$, equality holds. $\Longrightarrow \sin A + \sin B + \sin D \leq \frac{3}{2}, \sin A + \sin B - \cos C \leq \frac{3}{2}$.

Q9. Let X be a point on a straight line segment AB such that AB.BX = AX^2. Let C be a point on the circle with centre at A and radius AB such that BC = AX. Show that the angle BAC= 36°.

Solution:- Let, a = radius of the circle,

AX = x = BC. Since, AB. BX = AX^2 ,

$\Longrightarrow$ a (a -x) = $x^2 \Longrightarrow a^2 - ax - x^2 = 0,$

$\Longrightarrow \left(\frac{x}{a}\right)^2 + \frac{x}{a} - 1 = 0,$

$\therefore \frac{x}{a} = \frac{-1+\sqrt{5}}{2}.$

Now, x = BC = 2a sin$\frac{\theta}{2}$ ∴ sin$\frac{\theta}{2} = \frac{x}{2a} = \frac{-1+\sqrt{5}}{4} =$ sin 18°

$\Longrightarrow \frac{\theta}{2} = 18° \Longrightarrow \theta = 36°$

$\Longrightarrow$∠BAC= 36°.

Q10. Let a, b, c, d be positive real numbers such that abcd = 1. Show that (1+ a)(1+ b)(1+ c)(1+ d)≥ 16.

Solution:- $\frac{1+a}{2} \geq \sqrt{a}, \frac{1+b}{2} \geq \sqrt{b}, \frac{1+c}{2} \geq \sqrt{c}, \frac{1+d}{2} \geq \sqrt{d}.$

Multiplying corresponding sides of the above inequalities, we have,

(1+ a) (1+ b) (1+c) (1+d) ≥ 16 $\sqrt{abcd}$ ≥ 16.

ISI B.STAT & B.MATH SUBJECTIVE QUESTIONS & SOLUTIONS

SET – 10

Q1. For a real number x, let [x] denote the largest integer less than or equal to x and <x> denote x – [x]. Find all the solutions of the equations 13[x] +25 <x> = 271.

Solution:-

13 [x] + 25 ⟨x⟩ = 271

or, ⟨x⟩ = $\frac{271-13[x]}{25}$,

We know 0 ≤ ⟨x⟩ <1,

⟾ 0 ≤ $\frac{271-13[x]}{25} < 1$

⟾ 13[x] - 271 > -25

⟾ [x] > 18. 9

The nearest integers in this interval are 19 and 20.

Putting [x] = 19, ⟨x⟩ = $\frac{271-13\times19}{25} = 0.96$

Putting [x] =n 20, ⟨x⟩ = $\frac{271-13\times20}{25} = 0.44$

∴ x = 19.96, 20.44 (Answer)

Q2. Consider the function f(t) = $e^{-\frac{1}{t}}. t > 0$. Let for each positive integer n, p_n be the polynomial such that $\frac{d^n}{dt^n} f(t) = P_n\left(\frac{1}{t}\right) e^{-\frac{1}{t}}$ for all t > 0. Show that

$$\boldsymbol{P_{n+1}(x) = x^2 \left(P_n(x) - \frac{d}{dx} P_n(x)\right).}$$

Solution:- f(t) = $e^{-1/t}, t > 0$

$$\frac{d^n}{dt^n} = \left(f(t)\right) = P_n\left(\frac{1}{t}\right) e^{-\frac{1}{t}};$$

$$\frac{d^{n+1}}{dt^{n+1}} f(t) = \frac{d}{dt}\left\{P_n\left(\frac{1}{t}\right)\right\}. e^{-\frac{1}{t}} + e^{-\frac{1}{t}}\left(\frac{1}{t^2}\right). P\left(\frac{1}{t}\right)$$

$$Now, P_{n+1}\left(\frac{1}{t}\right) = e^{\frac{1}{t}}. \frac{d^{n+1}}{dt^{n+1}}. f(t)$$

$$= e^{\frac{1}{t}}\left[\frac{d}{dt}\left\{P_n\left(\frac{1}{t}\right)\right\}. e^{-\frac{1}{t}} + e^{-\frac{1}{t}}\left(\frac{1}{t^2}\right) P_n\left(\frac{1}{t}\right)\right]$$

$$= \frac{d}{dt}\left[P_n\left(\frac{1}{t}\right)\right] + \frac{1}{t^2} P_n\left(\frac{1}{t}\right).$$

Let us put $\frac{1}{t}$ = x, $P_{n+1}(x) = \frac{d}{dx}\{P_n(x)\} / \frac{dt}{dx} + x^2$.

$\therefore P_n(x) = x^2\, P_n(x) - x^2 \frac{d}{dx}\{P_n(x)\} = x^2\left(P_n(x) - \frac{d}{dx} P_n(x)\right)$.

Q3. Study the derivative of the function f(x) = $\boldsymbol{x^3 - 3x^2 + 4}$, and roughly sketch the graph of f(x), on plain paper.

Solution:- $f(x) = x^3 - 3x^2 + 4$

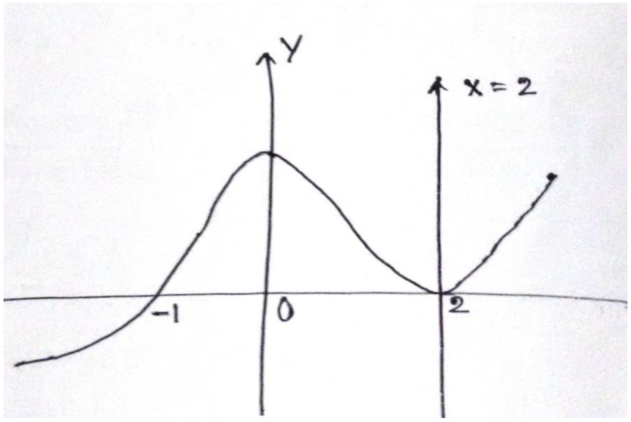
$f'(x) = 3x^2 - 6x = 3x(x-2)$

f'(x) > 0 for -∞ ,x < 0

f'(x) < 0 for 0< x < 2

f'(x) > 0 for 2 < x< ∞

x	-2	-1	0	1	2	3
f(x)	-16	0	4	2	0	4



Q4. Study the derivative of the function

f(x) = $\log_e x - (x-1), for\ x > 0$, and roughly sketch the graph of f(x), on plain paper.

Solution:-

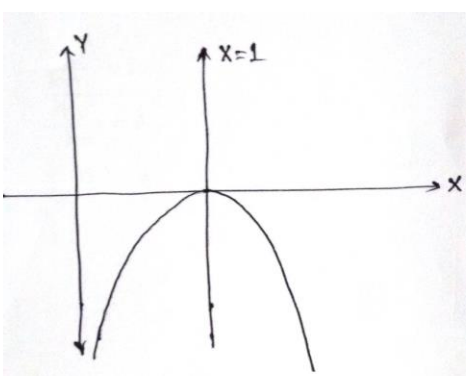
$f(x) = \log_e x - (x-1), for\ x > 0.$

$f'(x) = \frac{1}{x} - 1 = \frac{1-x}{x};$

$f'(x) > 0\ for\ 0 < x < 1$

$f'(x) < 0\ for\ 1 < x < \infty$

x	0.5	1	2	3
f(x)	0.2	0	-0.7	-1.5



Q5. (i) Find the number of all possible ordered k- tuples of non-negative integers $(n_1, n_2, \dots, n_k)$ such that $\sum_{i=1}^{k} n_i = 100.$

(ii) Show that the number of all possible ordered 4- tuples of non-negative integers (n_1, n_2, n_3, n_4) such that $\sum_{i=1}^{4} n_i \le 100\ is\ \binom{100}{4}$.

Solution:-

(i) Let S = $\{n_1, n_2, \dots, n_k\}$ be a set with k distinct elements. Given $n_i \ge 0\ and\ \sum_{i=1}^{k} n_i = 100$.

So, total number of all possible ordered k – tuples of such kinds are = $\binom{k-1+100}{100}$.

(ii)Here k = 4, and $\sum_{i=1}^{k} n_i \le 100$

So, no. of such possible cases are = $\binom{4-1+101}{4} = \binom{104}{4}$.

Q6. Let P be the fixed point (3, 4) and Q the point (x, $\sqrt{25-x^2}$). If M(x) is the slope of the line PQ, find $\lim_{x\to 3} M(x)$.

Solution:-

P(3, 4); Q (x, $\sqrt{25-x^2}$)

Slope, M(x)= $\frac{y_1 - y_2}{x_1 - x_2} = \frac{\sqrt{25-x^2}-4}{x-3}$

Now, $\lim_{x\to 3} M(x) = \lim_{x\to 3} \frac{\sqrt{25-x^2}-4}{x-3}$

$= -\frac{3}{4}$ [Do yourself applying L'Hospital Rule]

Q7. Solve $6x^2 - 25x + 12 + \frac{25}{x} + \frac{6}{x^2} = 0$.

Solution:-

$6x^2 - 25x + 12 + \frac{25}{x} + \frac{6}{x^2} = 0;$

$\Longmapsto 6x^2 + 12 + \frac{6}{x^2} - 25x + \frac{25}{x} = 0$

$\Longrightarrow 6x^2 - 12 + \frac{6}{x^2} - 25x + \frac{25}{x} + 24 = 0$

$\Longrightarrow 6\left(x - \frac{1}{x}\right)^2 - 25\left(x - \frac{1}{x}\right) + 24 = 0$

Let, $x - \frac{1}{x} = y$; so, $6y^2 - 25y + 24 = 0$

$\Longrightarrow$ y = $\frac{25 \pm \sqrt{625-576}}{2\times 6} = \frac{8}{3}\ \ or \frac{3}{2}$.

When y = $\frac{8}{3}, x = 3\ or \frac{1}{3}$,

For y = $\frac{3}{2}, x = 2\ or -\frac{1}{2}$.

Q8. (i) In the identity

$$\frac{\boldsymbol{n}!}{\boldsymbol{x(x+1)(x+2)\ldots(x+n)}} = \sum_{\boldsymbol{k=0}}^{\boldsymbol{n}} \frac{\boldsymbol{A_k}}{\boldsymbol{x+k}},$$

Prove that $\boldsymbol{A_k = (-1)^k \binom{n}{k}}$ **.**

(ii) Deduce that: $\boldsymbol{\binom{n}{0}\frac{1}{1.2} - \binom{n}{1}\frac{1}{2.3} + \binom{n}{2}\frac{1}{3.4} - \ldots + (-1)^n \binom{n}{n}\frac{1}{(n+1)(n+2)} = \frac{1}{n+2}}$

Solution:-

(i) From n! = $\sum_{k=0}^{n} A_k\,(x+1)(x+2)\ldots(x+k-1)(x+k+1)\ldots(x+n)$

Putting x = - k,

n! = $\sum_{k=0}^{n}(-1)^{-k}\ A_k.\,k!\,(n-k)!$

$\Rightarrow A_k = (-1)^k \binom{n}{k}$.

(ii)Considering $(1+x)^n = n_{c_0} + n_{c_1}x + n_{c_2}x^2 + \cdots + n_{c_n}x^n$.

Integrating w.r.t.x, we have

$$\frac{(1+x)^{n+1}}{n+1} = n_{c_0}x + \frac{n_{c_1}x^2}{2} + \cdots + \frac{n_{c_n}x^{n+1}}{n+1} + constant$$

Putting x = 0, then constant = $\frac{1}{n+1}$.

So, $\frac{(1+x)^{n+1}}{n+1} = n_{c_0}x + \frac{n_{c_1}x^2}{2} + \cdots + \frac{n_{c_n}x^{n+1}}{n+1} + \frac{1}{n+1}$.

$$\frac{(1+x)^{n+2}}{(n+1)(n+2)} = \frac{n_{c_0}x^2}{1.2} + \frac{n_{c_1}x^3}{2.3} + \ \ldots + \frac{n_{c_n}x^{n+2}}{(n+1)(n+2)} + \frac{x}{n+1} + constant\ .$$

Putting x= 0, then constant = $\frac{1}{(n+1)(n+2)}$.

So, $\frac{(1+x)^{n+2}}{(n+1)(n+2)} = \frac{n_{c_0}x^2}{1.2} + \frac{n_{c_1}x^3}{2.3} + \ \ldots + \frac{n_{c_n}x^{n+2}}{(n+1)(n+2)} + \frac{x}{n+1} + \frac{1}{(n+1)\ (n+2)}$.

Putting x= -1, we get

0 = $\frac{n_{c_0}}{1.2} - \frac{n_{c_1}}{2.3} + \frac{n_{c_2}}{3.4} \ldots + \ (-1)^n \frac{n_{c_n}}{(n+1)(n+2)} - \frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)}$.

So, $\binom{n}{0}\frac{1}{1.2} - \binom{n}{1}\frac{1}{2.3} + \binom{n}{2}\frac{1}{3.4} + \cdots + (-1)^n\binom{n}{n}\frac{1}{(n+1)(n+2)}$ = $\frac{1}{n+2}$. (Proved)

Q9. A regular five pointed star is inscribed in a circle of radius r. Show that the area of the region inside the star is $\boldsymbol{\frac{10r^2\tan\left(\frac{\pi}{10}\right)}{3-tan^2\left(\frac{\pi}{10}\right)}}$.

Solution:-

Suppose, O be the centre of the circle which lies in the star and whereas ∠MOL = θ, so, ∠LAM = $\frac{\theta}{2}$.

Here 5. R. θ = 2πr

$\therefore \theta = \frac{2\pi}{5}$

From ΔOAL,

∠AOL = $2\frac{\pi}{5} \Rightarrow \frac{1}{2} = \frac{\pi}{5}$.

∠LAO= $\frac{1}{2}\angle AOL = \frac{\pi}{10}$

∠OLA = π- (∠AOL+∠LAO)

= π- $\left(\frac{\pi}{5} + \frac{\pi}{10}\right) = \frac{7\pi}{10}$

Let, AL = a, OL = b & OA= R (given)

∴ From Δ OAL, we have, $\frac{a}{\sin\frac{\pi}{5}} = \frac{b}{\sin\frac{\pi}{10}} = \frac{R}{\sin\frac{7\pi}{10}}$

∴ a = R$\frac{\sin\frac{\pi}{5}}{\sin\frac{7\pi}{10}}$ & $b = R\,\frac{\sin\frac{\pi}{10}}{\sin\frac{7\pi}{10}}$

Thus area of ΔAOL = $\frac{1}{2}ab\sin\frac{7\pi}{10} = \frac{1}{2}R^2$

$$\frac{\sin\frac{\pi}{5}\sin\frac{\pi}{10}}{sin^2 7\frac{\pi}{10}} \times \sin 7\frac{\pi}{10}$$

$$= \frac{1}{2}R^2 2\sin\frac{\pi}{10}\cos\frac{\pi}{10}\sin\frac{\pi}{10}$$

$$\left[\because \sin\frac{7\pi}{10} = \sin 3\frac{\pi}{10}\right] = \frac{R^2 sin^2\frac{\pi}{10}\cos\frac{\pi}{10}}{3\sin\frac{\pi}{10} - 4\sin\frac{3\pi}{10}}$$

$$= \frac{R^2\tan\frac{\pi}{10}}{3 - tan^2\frac{\pi}{10}}$$

[Dividing N^r & D^r by $\sin\frac{\pi}{10}cos^2\frac{\pi}{10}$]

Hence, required area of the star = $\frac{10R^2\tan\frac{\pi}{10}}{3-tan^2\frac{\pi}{10}}$

[proved]

Q10. For the following function f study its derivatives and use them to sketch its graph on plain paper:

f(x) = $\frac{x-1}{x+1} + \frac{x+1}{x-1}$ for x ≠ -1, 1.

Solution:- f(x) = $\frac{x-1}{x+1} + \frac{x+1}{x-1}$ $[for\ x\ \neq 1, x\ \neq -1] = \frac{2(x^2+1)}{(x^2-1)} = 2 + \frac{4}{x^2-1}$. $\therefore f'(x) = \frac{-8x}{(x^2-1)^2}$.

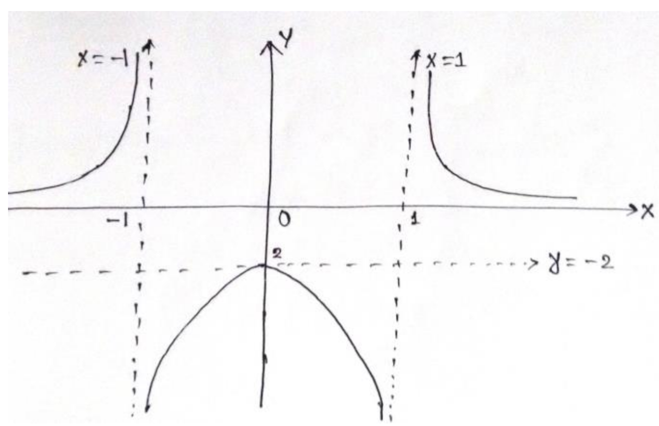
For, $-\infty < x < -1, f'(x) is\ positive$;

For, -1 < x < 0, $f'(x)$ is positive;

For, 0 < x <1, $f'(x)$ is negative;

For, 1< x < ∞, $f'(x)$ is negative.

x	-3	-2	0	- ½	½	2	3
f(x)	$2\frac{1}{2}$	$3\frac{1}{3}$	-2	$-3\frac{1}{3}$	$3\frac{1}{3}$	$3\frac{1}{3}$	2



PROBLEMS WITH SOLUTIONS FOR I.S.I. / C.M.I. ENTRANCE TESTS

1. **Prove that for all natural numbers n ≥ 3 there exist odd natural numbers x_n, y_n such that $7{x_n}^2 + {y_n}^2 = 2^n$.**

Sol: For n = 3, we have $x_3 = y_3 = 1$. Now suppose that for a given natural number n we have odd natural numbers x_n, y_n such that $7x_n^2 + y_n^2 = 2^n$ we shall exhibit a pair (X, Y) such that $7X^2 + Y^2 = 2^n$ we shall exhibit a pair (X, Y) such that $7{x_n}^2 + {y_n}^2 = 2^{n+1}$. In fact,

$$7\left(\frac{x_n \pm y_n}{2}\right)^2 + \left(\frac{7x_n \pm y_n}{2}\right)^2 = 2(7{x_n}^2 + {y_n}^2) = 2^{n+1}$$

One of $\frac{(x_n + y_n)}{2}$ and $\frac{|x_n - y_n|}{2}$ is odd (as their sum is the larger of x_n and y_n which is odd), giving the desired pair.

2. **The circles k_1 and k_2 with respective centers $O_1 and\ O_2$ are externally tangent at the point C, while the circle k with center O is externally tangent k_1 and k_2. Let l be the common tangent of k_1 and k_2 at the point C and let AB be the diameter of k perpendicular o l. Assume that O and A lie on the same side of l. Show that that the lines AO_2, BO_1, l have a common point.**

Sol.: Let r, r_1, r_2 be the respective radii of k, k_1, k_2. Also let M and N be the intersections of AC and BC with k. Since AMB is a right triangle, the triangle AMO is isosceles and

$$\angle AMO = \angle OAM = \angle O_1CM = \angle CMO_1$$

Therefore O, N, O_1 are collinear and AM / MC = OM / M$O_1 = r/r_1$

Similarly O, N, O_2 are collinear and BN / NC = OM / N$O_2 = r/r_2$.

Let P be the intersection of l with AB; the lines AN, BM, CP concur at the orthocenter of ABC, so by Ceva's theorem.

AP / PB = (AM / MC) (CN / NB)= r_2/r_1. Now let D_1 and D_2 be the intersections of l with BO_1 and AO_2. Then $\frac{CD_1}{D_1P} = \frac{O_1C}{PB} = \frac{r_1}{PB}$, $and\ similarly$ $\frac{CD_2}{D_2P} = \frac{r_2}{PA}$. Thus $\frac{CD_1}{D_1P} = \frac{CD_2}{D_2P}$ and $D_1 = D_2$, and so AO_2, BO_1, l have a common point.

3. **Let a, b, c be real numbers and let M be the maximum of the function $y = |4x^3 + ax^2 + bx + c|$ in the interval |-1, 1|. Show that M ≥ 1 and find all cases where equality occurs.**

Sol.: a = 0, b = -3, c = 0, where M = 1, with the maximum achieved at -1, -1/2, ½ , 1. On the other hand, if M < 1 for some choice of a, b, c, then

$$(4x^3 + ax^2 + bx + c) - (4x^3 + 3x)$$

Must be positive at -1, negative -1/2, positive at ½, and negative at 1, which is impossible for a quadratic function. Thus M ≥ 1, and the same argument shows that equality only occurs for (a, b, c) = (0, -3, 0). (Note: this is a particular case of the minimum deviation property of Chebyshev polynomials).

4. **The real numbers $a_1, a_2, \ldots, a_n (n \geq 3)$ from an arithmetic progression. There exists a permutation $a_{i1}, a_{i2}, \ldots, a_{in}$ of $a_1, a_2, \ldots, a_n$ which is a geometric progression. Find the numbers $a_1, a_2, \ldots a_n$ if they are all different and the largest of them as equal to 1996.**

Sol.: Let $a_1 < a_2 < -< a_n = 1996$ and let q be the ratio of the geometric progression $a_{i1} \ldots \ldots \ldots a_{in}$; clearly q$\neq 0 \ \pm 1$. By reversing the geometric progression if needed, we may assume |q| > 1, and so $|a_{i1}|$< $|a_{i2}|$ < –$|a_{in}|$. Note that either all of the terms are positive, or they alternate in sign; in the latter case, the terms of either sign form a geometric progression by themselves.

There cannot be three positive terms, or else we would have a three term geometric progression a, b, c which is also an arithmetic progression, violating the AM –GM inequality.

Similarly, there cannot be three negative terms, so there are at most two terms of each sign and n ≤4.

If n = 4, we have $a_1 < a_2 < 0 < a_3 < a_4 and\ 2a_2 = a_2 + a_3, 2a_3 = a_2 + a_4$. In this case, q < -1 and the geometric progression is either $a_3, a_2, a_4, a_1\ or\ a_2, a_3, a_1, a_4$. Suppose the former occurs (the argument in similar in the latter case): then

$2a_3q = a_3q^3 + a_3\ and\ 2a_3 + a_3q^3 + a_3q^2$, giving q =1, a contradiction.

We deduce n = 3 and consider two possibilities. If $a_1 < a_2 < 0 < a_3 = 1996, then\ 2a_2 = a_2q^2 + a_2q,$ so $q^2 + q - 2 = 0\ and\ q = -2$, yielding $(a_1, a_2, a_3) = (-3992, -998, 1996)$. If $a_1 < 0 < a_2 < a_3 = 1996$, then

$2a_2 = a_2q + a_2q^2$, so again q = -2, yielding $(a_1, a_2, a_3) = (-998, 499, 1996)$.

5. **Find all prime numbers p, q for which pq divides $(\mathbf{5^p - 2^p})(\mathbf{5^q - 2^q})$.**

Sol.: If p$\,|\,5^p - 2^p, thenp\,|$ 5 -2 by Fermat's theorem,

So p = 3, suppose p, q ≠3; then p$\,|\,5^q - 2^q$ and q$\,|\,5^p - 2^p$. Without lose of generality assume p >q, so that (p, q -1) = 1. Then if a is an integer such that 2a ≡5 (mod q), then the order of a mod q divides p as well as q -1, a contradiction.

Hence one of p, q is equal to 3. If q ≠ 3, then q$\,|\,5^3 - 2^3 = 9.13$. so q = 13, and similarly p ∈(3, 13).

Thus the solutions are (p, q) = (3, 3), (3, 13), (13, 3).

6. **Find the side length of the smallest equilateral triangle in which three dises of radii 2, 3, 4 can be placed without overlap.**

Sol.: A short computation shows that dises of radii 3 and 4 can be fit into two corners of an equilateral triangle of side $11\sqrt{3}$ so as to just touch, and that a disc of radius 2 easily fits into the third corner without overlap. On the other hand, if the discs of radii 3 and 4 fit into an equilateral triangle without overlap, there exists a line separating them (e.g. a tangent to one perpendicular to their line of centers) dividing the triangle into a triangle and a (possibly degenerate) convex quadrilateral. Within each piece, the disc can be moved into one of the corners of the original triangle. Thus the two discs fit into the corners without overlap, so the side length of the triangle must be at least $11\sqrt{3}$.

7. **The equilateral ABCD is inscribed in a circle. The lines AB and CD meet at E, while the diagonals AC and BD meet at F. The circumcircles of the triangles AFD and BFC meet again at H. Prove that ∠EHF= 90°.**

Sol.: (We use directed angles modulo π.) Let O be the circumcenter of ABCD; then ∠AHB= ∠AHF+∠FHB=∠ADF+∠FCB= 2∠ADB=

$\angle AOB$, so O lies on the circumcircle of AHB, and similarly on the circumcircle of CHD. The radical axes of the circumcircles of AHB, CHD and ABCD concur; these lines are AB, CD and HO, so E, H, O are collinear. Now note that $\angle OHF = \angle OHC + \angle CHF = \angle ODC + \angle CBF = \frac{\pi}{2} - \angle CAD + \angle CBD. So\ \angle EHF = \ \angle OHF = \ \frac{\pi}{2}$ as desired. (Compare IMO 1985/5.)

8. A 7×7 chessboard is given with its four corners deleted.

(a) What is the smallest number of squares which can be colored black so that an uncolored 5 squares (Greek) cross cannot be found?

(b) Prove that an integer can be written in each square such that the sum of the integers in each 5 squares cross is negative while the sum of the numbers in all squares of the board is positive.

Sol.: The 7 squares

(2, 5), (3, 2), (3, 3), (4, 6), (5, 4), (6, 2), (6, 5)

suffice, so we need only show that 6 or fewer will not suffice. The crosses centered at

(2, 2), (2, 6), (3, 4), (5, 2), (5, 6), (6, 4)

are disjoint, so one square must be colored in each, hence 5 or fewer squares do not suffice. Suppose exactly 6 squares are colored. Then none of the squares (1, 3), (1, 4), (7, 2) can be colored; by a series of similar arguments, no square on the perimeter can be colored. Similarly, (4, 3) and (4, 5) are not covered, and by a similar argument, neither is (3, 4) or (5, 4). Thus the center square (4, 4) must be covered.

Now the crosses centered at

(2, 6), (3, 3), (5, 2), (5, 6), (6, 4)

are disjoint and none contains the center square, so each contains one colored square. In particular, (2, 2) and (2, 4) are not colored. Replacing (3, 3) with (2, 3) in the list shows that (3, 2) and (3, 4) are not colored. Similar symmetric arguments now show that no squares beside the center square can be covered, a contradiction. Thus 7 squares are needed.

(a) Write -5 in the 7 squares listed above and 1 in the remaining squares. Then clearly each cross has a negative sum, but the total of all of the numbers is 5 (−7) + (45 -7) = 3.

9. If α, β, γ are the roots of $x^3 - x - 1 = 0$, compute $\frac{1-\alpha}{1+\alpha} + \frac{1-\beta}{1+\beta} + \frac{1-\gamma}{1+\gamma}$.

Sol.: The given quantity equals

$$2\left(\frac{1}{\alpha+1} + \frac{1}{\beta+1} + \frac{1}{\gamma+1}\right) - 3.$$

Since $P(x) = x^3 - x - 1$ has roots α, β, γ, the polynomial $P(x-1) = \ x^3 - 3x^2 + 2x - 1$ has roots α+1, β+1, γ+1.

By a standard formula, the sum of the reciprocals of the roots of $x^3 + c_2x^2 + c_1x + c_0\ is - c_1/c_0$, so the given expression equals 2(2)-3= 1.

10. Find all real solution to the following system of equations:

$$\frac{4x^2}{1+4x^2} = y$$
$$\frac{4y^2}{1+4y^2} = z$$
$$\frac{4z^2}{1+4z^2} = x.$$

Sol.: Define $f(x) = \frac{4x^2}{(1+4x^2)}$; the range of f is [0, 1), so x, y, z must lie in that interval. If one of x, y, z is zero, then all three are, so assume they are nonzero. Then $\frac{f(x)}{x} = \frac{4x}{(1+4x^2)}$ is at least 1 but the AM –GM inequality, with equality for x = ½ . Therefore x ≤y ≤ z ≤ x, and so equality holds everywhere, implying x = y = z = ½ . Thus the solutions are (x, y, z) = (0, 0, 0), (½ , ½ , ½).

11. Let f(n) be the number of permutations $a_1, \ldots, a_n$ of the integers 1, …, n such that

(i) $a_1 = 1$;

(ii) $|a_i - a_{i+1}| \leq 2, i = 1, \ldots, n-1.$

Determine whether f(1996) is divisible by 3.

Sol.: Let g(n) be the number of permutations of the desired form with $a_n = n$. Then either $a_{n-1} = n-1$ *or* $a_{n-1} = n-2$; in the latter case we must have $a_{n-2} = n-1$ *and* $a_{n-3} = n-3$. Hence g(n) = g(n-1) + g(n -3) for n ≥4. In particular, the values of g(n) modulo 3 are g(1) = 1, 1, 1, 2, 0, 1, 0, 0….. repeating with period 8.

Now let h(n) = f(n) – g(n); h(n) counts permutations of the desired from where n occurs in the middle, sandwiched between n-1 and n -2. Removing n leaves an acceptable permutation, and any acceptable permutation on n -1 symbols can be so produced except those ending in n -4, n -2, n -3, n -1. Hence h(n) = h(n -1)+ g(n -1) –g(n -4) = h(n -1)+ g(n -2); one checks that h(n) modulo 3 repeats with period 24.

Since 1996 ≡ 4 (mod 24), we have f(1996) ≡ f(4) = 4(mod 3), so f(1996) is not divisible by 3.

12. Let ΔABC be an isosceles triangles with AB = AC. Suppose that the angle bisector of ∠B meets AC at D and that BC = BD + AD. Determine ∠A.

Sol.: Let α =∠A, β=$\frac{(\pi-\alpha)}{4}$ and assume AB = 1.

Then by the Law of Sines,

$$BC = \frac{\sin\alpha}{\sin 2\beta}, BD = \frac{\sin\alpha}{\sin 3\beta}, AD = \frac{\sin\beta}{\sin 3\beta}.$$

Thus we are seeking a solution to the equation $\sin(\pi - 4\beta)\sin 3\beta = (\sin(\pi - 4\beta) + \sin\beta)\sin 2\beta$.

Using the sum-to-product formula, we rewrite this as

$$\cos\beta - \cos 7\beta = \cos 2\beta - \cos 6\beta + \cos\beta - \cos 3\beta.$$

Cancelling $\cos\beta$, we have $\cos 3\beta - \cos 7\beta = \cos 2\beta - \cos 6\beta$, which implies

$$\sin 2\beta \sin 5\beta = \sin 2\beta \sin 4\beta.$$

Now $\sin 5\beta = \sin 4\beta$, *so* $9\beta = \pi$ *and* $\beta = \frac{\pi}{9}$.

13. Let $r_1, r_2, \ldots, r_m$ be a given set of positive rational numbers whose sum is 1. Define the function f by $f(n) = n - \sum_{k=1}^{m} \lfloor [r_k n] \rfloor$ for each positive integer n. Determine the minimum and maximum values of f(n).

Sol.: Of course $\lfloor [r_k n] \rfloor \leq r_k n, so\ f(n) \geq 0$, with equality for n = 0, so 0 is the minimum value. On the other hand, we have $r_k n - \lfloor [r_k n] \rfloor < 1, so\ f(n) \leq m-1$.

Here equality holds for n = t- 1 if t is the least common denominator of the r_k.

14. Let H be the orthocenter of acute triangle ABC. The tangents from A to the circle with diameter BC touch the circle at P and Q. Prove that P, Q, H are collinear.

Sol.: The line PQ is the polar of A with respect to the circle, so it suffices to show that A lies on the pole of H.

Let D and E be the feet of the altitudes from A and B, respectively; these also lie on the circle, and H = AD ∩BE. The polar of the line AD is the intersection of the tangents AA and DD, and the polar of the line BE is the intersection of tangents BB and EE. The collinearity of these two intersections with C = AE ∩ BD follows from applying Pascal's theorem to the cyclic hexagons AABDDE and ABBDEE. (An elementary solution with vectors is also possible and not difficult.)

15. Find the smallest positive integer K such that every K-element subset of (1, 2, …, 50) contains two distinct elements a, b such that a+b divides ab.

Sol.: The minimal value is k = 39. Suppose a, b∈S are such that a + b divides ab. Let c = gcd (a, b) and put a = c$a_1, b = cb_1$, so that a_1 and $b_1$are relatively prime. Then c$(a_1 + b_1) divides\ c^2 a_1 b_1, so\ a_1 + b_1\ divides\ ca_1 b_1$.

Since a_1 and b_1 have no common factor, neither do a_1 and $a_1 + b_1, or$ $b_1 and$ $a_1 + b_1$.In short, $a_1 + b_1$ divides c.

Since S ⊆ $\{1, \dots, 50\}$, we have a +b ≤99, so c$(a_1 + b_1) \leq 99$, which implies $a_1 + b_1 \leq 9$, on the other hand, of course $a_1 + b_1 \geq 3$. An exhaustive search produces 23 pairs, a, b satisfying the conditions.

$a_1 + b_1 = 3$ (6, 3), (12, 6), (18, 9), (24, 12), (30, 15), (36, 18), (42, 21), (48, 24)

$a_1 + b_1 = 4$ (12, 4), (24, 8), (36, 12), (48, 16)

$a_1 + b_1 = 5$ (20, 5), (40, 10), (15, 10), (30, 20), (45, 30)

$a_1 + b_1 = 6$ (30, 6)

$a_1 + b_1 = 7$ (42, 7), (35, 14), (28, 21)

$a_1 + b_1 = 8$ (40, 24)

$a_1 + b_1 = 9$ (45, 36)

Let M = {6, 12, 15, 18, 20, 21, 24, 35, 40, 42, 45, 48} and T = {1, …, 50} –M. Since each pair listed above contains an element of M, T does not have the desire property. Hence we must take k ≥$|\mathrm{T}| + 1 = 39$. On the other hand, from the 23 pairs mentioned above we can select 12 pairs which are mutually disjoint:

$(6, 3), (12, 4), (20, 5), (42, 7), (24, 8), (18, 9), (40, 10), (35, 14), (30, 15), (48, 16), (28, 21), (45, 36)$.

Any 39-element subset must contain both elements of one of these pairs. We conclude the desired minimal number is k = 39.

16. Eight singers participate in an art festival where m songs are performed. Each song is performed by 4 singers, and each pair of singers performs together in the same number of songs. Find the smallest m for which this is possible.

Sol.: Let r be the number of songs each pair of singers performs together, so that

$$m\binom{4}{2} = r\binom{8}{2}$$

And so m = $\frac{14r}{3}$; in particular, m ≥14. However, m = 14 is indeed possible, using the arrangement

{1, 2, 3, 4} {5, 6, 7, 8} {1, 2, 5, 6} {3, 4, 7, 8}

{3, 4, 5, 6} {1, 3, 5, 7} {2, 4, 6, 8} {1, 3, 6, 8}

{2, 4, 5, 7} {1, 4, 5, 8} {2, 3, 6, 7} {1, 4, 6, 7}

{1, 2, 7, 8} {2, 3, 5, 8}

17. In triangle ABC, $\angle C = 90°, \angle A = 30°$ and $BC = 1$.Find the minimum of the length of the longest side of a triangle inscribed in ABC (that is, one such that each side of ABC contains a different vertex of the triangle).

Sol.: We first find the minimum side length of an equilateral triangle inscribed in ABC. Let D be a point on BC and put x = BD. Then take points E, F on CA, AB, respectively, such that CE = $\frac{\sqrt{3}x}{2}$ and $BF = 1 - \frac{x}{2}$. A calculation using the Law of Cosines shows that

$$DF^2 = DE^2 = EF^2 = \frac{7}{4}x^2 - 2x + 1$$
$$= \frac{7}{4}\left(x - \frac{4}{7}\right)^2 + \frac{3}{7}$$

Hence the triangle DEF is equilateral, and its minimum possible side length is $\sqrt{\frac{3}{7}}$.

We know argue that the minimum possible longest side must occur for some equilateral triangle. Starting with an arbitrary triangle, first suppose it is not isosceles. Then we can side one of the endpoints of the longest side so as to decrease its length; we do so until there are two longest sides, say DE and EF. We now fix D, move E so as to decrease DE and move F at the same time so as to decrease EF; we do so until all three sides become equal in length. (It is fine if the vertices move onto the extensions of the sides, since the bound above applies in that case as well.)

Hence the minimum is indeed $\sqrt{\frac{3}{7}}$, as desired.

18. Prove that if a sequence $\{G(n)\}_{n=0}^{\infty}$ of integers satisfies

G(0) = 0,

G(n) = $n - G\{G(n)\}$ (n= 1, 2, 3, ….)

then

(a) $G(k) \geq G(k-1)$ for any positive integer k;

(b) No integer k exists such that G(k - 1) = G(k) = G(k +1).

Sol.:

(a) We show by induction that $G(n) - G(n-1) \in \{0, 1\}$ for all n. If this holds up to n, then

$$G(n+1) - G(n) = 1 + G\big(G(n-1)\big) - G\big(G(n)\big).$$

If $G(n-1) = G(n), then$ $G(n+1) - G(n) = 1$; otherwise, $G(n-1) and$ $G(n)$ are consecutive integers not greater than n, so $G\big(G(n)\big) - G\big(G(n-1)\big) \in \{0, 1\}$, again completing the induction.

(b) Suppose that G(k -1)= G(k) = G(k+1)+A for some k, A. Then
A= G(k+ 1)= k +1 –G(g(k))= k+1-G(A)
And similarly A = k –G(A) (replacing k +1 with k above), a contradiction.
Note: It can be shown that G(n) = $\lfloor n\omega \rfloor$ for $\omega = \frac{(\sqrt{5}-1)}{2}$.

19. Let ABC be an acute triangle with altitudes AP, BQ, CR. Show that for any point P in the interior of the triangle PQR, there exists a tetrahedron ABCD such that P is the point of the face ABC at the greatest distance (measured along the surface of the tetrahedron) from D.

Sol.: We first note that if S is the circumcircle of an acute triangle KLM, then for any point X $\neq$S inside the triangle, we have

$$\min\{XK, XL, XM\} < SK = SL = SM,$$

Since the discs centered at K, L, M whose bounding circles pass through S cover the entire triangle.

Fix a point V in the interior of the triangle PQR, we first assume the desired tetrahedron exists and determine some of its properties. Rotate the faces ABD, BCD, CAD around their common edges with face ABC into the plane ABC, so that the images D_1, D_2, D_3 of D lie outside of triangle ABC. We shall choose D so that triangle $D_1D_2D_3$ is acute, contains triangle ABC and has circumcenter V; this suffices by the above observation.

In other words, we need a point D such that AV is the perpendicular bisector of D_1D_3, BV that of D_1D_2, and CV that of D_2D_3. We thus need $\angle D_1D_2D_3 = \pi - \angle BVC$ and so on. Since V lies inside PQR, the angle BVC is acute, and so $\angle D_1D_2D_3$ is fixed and acute. We may then construct an arbitrary triangle $D_1'D_2'D_3'$ similar to the unknown triangle $D_1D_2D_3$ let V' be its circumcenter, and construct points A', B', C' on the rays from V through the midpoints of $D_3'D_1', D_1'D_2', D_2'D_3'$,respectively, so that triangle A'B'C' and ABC are similar. We can also ensure that the entire triangle A'B'C' lies inside $D_1'D_2'D_3'$. Then folding up the hexagon $A'D_1'B'D_2'C'D_3'$ along the edges of triangle A'B'C' produces a tetrahedron similar to the required tetrahedron.

20. An acute angle XCY and points A and B on the rays CX and CY, respectively, are given such that │CX│< │CA│= │CB│< │CY│. Show how to construct a line meeting the ray CX and the segments AB, BC at the points K, L, M, respectively, such that

KA.YB = XA.MB= LA.LB ≠ 0

Sol.: Suppose K, L, M have already been constructed. The triangle ALK and BYL are similar because

∠LAK = ∠YBL and $\frac{KA}{LA} = \frac{LB}{YB}$.Hence ∠ALK = ∠BYL. Similarly, from the similar triangles ALX and BML we get ∠AXL = ∠MLB. We also have ∠MLB = ∠ALK since M, L, K are collinear, we conclude ∠LYB = ∠AXL.

Now ∠XLY = ∠XLB + ∠BLY = ∠XAL +∠AXL + ∠ABM -∠LYB = 2∠ABC

We are construct the desired line as follows draw the arc of points L such that ∠XLY = 2∠ABC, and let L be its intersection with AB. Then construct M on BC such that ∠BLM = ∠AXL, and let K be the intersection of LM with CA.

21. For which integers k does there exist a function f : N →Z such that

(a) f(1995) = 1996, and

(b) f(xy) = f(x) + f(y) + kf(gcd(x, y))for all x, y ∈ N?

Sol.: Such f exists for k = 0 and k = -1. First take x = y in (b) to get $f(x^2) = (k+2)f(x)$. Applying this twice, we get

$$f(x^4) = (k+2)f(x^2) = (k+2)^2 f(x).$$

On the other hand,

$$f(x^4) = f(x) + f(x^3) + kf(x)$$
$$= (k+1)f(x) + f(x^3)$$

$$= (k+1)f(x) + f(x) + f(x^2) + kf(x)$$

$$= (2k+2)f(x) + f(x^2) = (3k+4)f(x).$$

Setting x = 1995 so that f(x) ≠ 0, we deduce $(k+2)^2 = 3k+4$, which has roots k = 0, -1. For k = 0, an example is given by

$$f({p_1}^{e_1} \dots {p_n}^{e_n}) = e_1 g(p_1) + \cdots + e_n g(p_n).$$

Where g(5) = 1996 and g(p) = 0 for all primes p ≠5 for k = 1, as example is given by

$$f({p_1}^{e_1} \dots {p_n}^{e_n}) = g(p_1) + \cdots + g(p_n)$$

22. A triangle ABC and points K, L, M on the sides AB, BC, CA respectively, are given such that

$$\frac{AK}{AB} = \frac{BL}{BC} = \frac{CM}{CA} = \frac{1}{3}$$

Show that if the circumcircles of the triangles of the triangles AKM, BLK, CML are congruent, then so are the in circles of these triangles.

Sol.: We will show that ABC is equilateral, so that AKM, BLK, CML are congruent and hence have the same in radius.

Let R be the common circumradius; then

KL = 2R sin A, LM = 2R sin B, MK = 2R sin C,

So the triangles KLM and ABC are similar. Now we compare areas:

[AKM] = [BLK] = [CLM] = $\frac{2}{9}$[ABC],

So, [KLM] = $\frac{1}{3}$[ABC] and the coefficient of similarity between KLM and ABC must be $\sqrt{\frac{1}{3}}$.

By the law of cosines applied to ABC and AKM.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\frac{1}{3}a^2 = \left(\frac{2p}{3}\right)^2 + \left(\frac{c}{3}\right)^2 - 2\frac{2b}{3}\frac{c}{3}\cos A.$$

From these we deduce $a^2 = 2b^2 - c^2$, and similarly $b^2 = 2c^2 - a^2, a^2 = 2a^2 - b^2$. Combining these gives $a^2 = b^2 = c^2$, so ABC is equilateral, as desired.

23. Let ABC be a triangle and construct squares ABED, BCGF, ACHI externally on the sides of ABC. Show that the points D, E, F, G, H, I are concyclic if and only if ABC is equilateral or isosceles right.

Sol.: Suppose D, E, F, G, H, I are concyclic; the perpendicular bisectors of DE, FG, HI coincide with those of AB, BC, CA respectively, so the center of the circle must be the circumcenter O of ABC. By equating the distances OD and OF, we find

$$(\cos B + 2\sin B)^2 + sin^2 B = (\cos C + 2\sin C)^2 = sin^2 C$$

Expanding this end cancelling like terms, we determine $sin^2 B + \sin B\cos B = sin^2 C + \sin C\cos C$.

Now note that

$$2(sin^2\theta + \sin\theta\cos\theta) = 1 - \cos 2\theta + \sin\theta$$

$$= 1 + \sqrt{2}\sin\left(2\theta - \frac{\pi}{4}\right).$$

Thus we either have B = C or $2B - \frac{\pi}{4} + 2C - \frac{\pi}{4}$

$$= \pi, or\ B + C = \frac{3\pi}{4}.$$

In particular, two of the angles must be equal, say A and B, and we either have A = B = C, so the triangle is equilateral, or $B + (\pi - 2B) = \frac{3\pi}{4}$, in which case A = B = $\frac{\pi}{4}$ and the triangle is isosceles right.

24. Let a, b be positive integers with a odd. Define the sequence $\{u_n\}$ as follows: $u_0 = b$ & n ∈ ℕ.

$$u_{n+1} = \begin{cases} \frac{1}{2}u_n & if\ u_n\ is\ even \\ u_n + a & otherwise \end{cases}$$

(a) Show that $u_n \le a$ for some n ∈ℕ.

(b) Show that the sequence $\{u_n\}$ is periodic from some point onwards.

Sol:

(a) Suppose $u_n > a$, if u_n is even, $u_{n+1} = \frac{u_n}{2} < u_n$; if $u_n$is odd, $u_{n+2} = \frac{(u_n+a)}{2} < u_n$. Hence for each term greater than a, there is a smaller subsequent term. These form a decreasing subsequence which must eventually terminate, which only occurs once $u_n \leq a$.

(b) If $u_m \leq a$, then for all n ≥ m, either $u_n \leq a, or, u_n$ is even and $u_n \leq 2a$, by induction on n. In particular, $u_n \leq 2a \; for \; all \; m \geq n$, and so some value of u_n eventually repeats, leading to a periodic sequence.

25. (a) Find the minimum value of x^x for x a positive real number.

(b) If x and y are positive real numbers, show that $x^x + y^x > 1$.

Sol.:

(a) Since $x^x = e^{x\log x} \; and \; e^x$ is an increasing function of x, it suffices to determine the minimum of x log x. This is easily done by setting its derivative 1+ log x to zero, yielding $x = \frac{1}{e}$. The second derivative $\frac{1}{x}$ is positive for x > 0, so the function is everywhere convex, and the unique extremum is needed a global minimum. Hence x^x has minimum value $e^{-1/e}$.

(b) If x ≥ 1, then $x^y \geq 1$ for y > 0, so we may assume 0< x, y<1. Without loss of generality, assume x ≤y; now note that the function $f(x) = x^x + y^x$ has derivative $f'(x) = x^x \log x + y^{x-1}$. Since $y^x \geq x^x \geq x^y \; for \; x \leq y \; and \frac{1}{x} \geq -\log x$, we see that $f'(x) > 0 \; for \; 0 \leq x \leq y$ and so the minimum of f occurs with x = 0, in which case f(x) = 1; since x > 0, we have strict inequality.

26. Starting at (1, 1), a stone is moved in the coordinate plane according to the following rules:

(i) From any point (a, b), the stone can move to (2a, b) or (a, 2b).

(ii) From any point (a, b), the stone can move to (a –b, b) if a > b, or to (a, b –a) if a < b.

For which positive integers x, y can the stone be moved to (x, y)?

Sol.: It is necessary and sufficient that gcd(x, y) = 2^x for some nonnegative integer s. We show necessity by nothing that gcd(p, q) = gcd(p, q –p), so an odd common divisor can never be introduced, and nothing that initially gcd(1, 1)= 1.

As for sufficiency, suppose gcd(x, y) = 2^x. Of those pairs (p, q) from which (x, y) can be reached, choose one to minimize p +q. Neither p and q can be even, else one of $\left(\frac{p}{2}, q\right) \; or \; \left(p, \frac{q}{2}\right)$ is an admissible pair. If p > q, then (p, q) is reachable from $\left(\frac{(p+q)}{2}, q\right)$, a contradiction; similarly p < q is impossible. Hence p = q, but gcd(p, q) is a power of 2 and neither p nor q is even. We conclude p =q = 1, and so (x, y) is indeed reachable.

27. Suppose S is a union of finitely many disjoint subintervals of [0, 1] such that no two point in S have distance $\frac{1}{10}$. Show that the total length of the intervals comprising S is at most $\frac{1}{2}$.

Sol.: Cut the given segment into 5 segments of length $\frac{1}{5}$. Let AB be one of these segments and M its midpoint.

Translate each point of AM by the vector MB. No colored point can have a colored image, so all of the colored intervals of AB can be placed in MB without overlap, and their total length therefore does not exceed $\frac{1}{10}$. Applying this reasoning to each of the 5 segments gives the desired result.

28. Prove that every integer k > 1 has a multiple less than k^4 whose decimal expansion has at most four distinct digits.

Sol.: Let n be the integer such that $2^{n-1} \leq k \leq 2^n$. For n ≤6 the result is immediate, so assume n > 6.

Let S be the set of nonnegative integers less than 10^n whose decimal digits are all 0s or 1s. Since $|S| = 2^n > k$, we can find two elements a < b of S which are congruent modulo, k, and b − a only has the digits 8, 9, 0, 1 in its decimal representation. On the other hand,

$$b - a \leq b \leq 1 + 10 + \cdots + 10^{n-1} < 10^n < 16^{n-1} \leq k^4,$$

Hence b – a is the desired multiple.

29. Let ABC be ab acute triangle, AD, BE, CZ its altitudes and H its orthocenter. Let AI, AΘ be the internal and external bisectors of angle A. Let M, N be the midpoints of BC, AH, respectively. Prove that

(a) MN is perpendicular EZ;

(b) If MN cuts the segment AI, AΘ at the points K, L, then KZ = AH.

Sol.:

(a) The circle with diameter AH passes through Z and E, and so ZN = ZE. On the other hand, MN is a diameter of the nine-point circle of ABC, and Z and E lie on that circle, so ZN = ZE implies that ZE ⊥MN.

(b) As determined in (a), MN is the perpendicular bisector of segment ZE. The angle bisector AI of ∠EAZ passes through the midpoint of the minor arc EZ, which clearly lies on MN; therefore this midpoint is k. By similar reasoning, L is the midpoint of the major are EZ. Thus KL is also a diameter of circle EAZ, so KL = MN.

30. Given 81 natural numbers whose prime divisors belong to the set {2, 3, 5}, prove there exist 4 numbers whose product is the fourth power of an integer.

Sol.: It suffices to take 25 such numbers. To each number, associate the triple (x_2, x_3, x_5) recording the parity of the exponents of 2, 3 and 5 in its prime factorization. Two numbers have the same triple if and only if their product is a perfect square. As long as there are 9 numbers left, we can select two whose product is a square, in so doing, we obtain 9 such pairs. Repeating the process with the square roots of the products of the pairs, we obtain four numbers whose product is a fourth power.

31. Prove the following inequality for positive real numbers x, y, z:

$$\boldsymbol{(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}}.$$

Sol.: After clearing denominators, the given inequality becomes

$$\sum_{sym} 4x^5y - x^4y^2 - 3x^3y^3 + x^4yz - 2x^3y^2z + x^2y^2z^2 \geq 0$$

Where the symmetric sum runs over all six permutations of x, y, z. (In particular, this means the coefficient of x^3y^3 in the final expression is -6, and that $x^2y^2z^2$ is 6.)

Recall Schur's inequality:

$$x(x-y)(x-z)+y(y-z)(y-x) + z(z-x)(z-y) \geq 0$$

Multiplying by 2xyz and collecting symmetric terms, we get

$$\sum_{sym} x^4yz - 2x^3y^2z + x^2y^2z^2 \geq 0$$

On the other hand,

$$\sum_{sym} (x^5y - x^4y^2) + 3(x^5 - x^3y^3) \geq 0$$

By two applications of AM-GM; combining the last two displayed inequalities gives the desired result.

32. Prove that for every pair m, k of natural numbers, m has a unique representation in the from

$$\boldsymbol{m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_t}{t}}$$

where $\boldsymbol{a_k > a_{k-1} > \cdots > a_t \geq t \geq 1}$.

Sol.: We first show uniqueness. Suppose m is represented by two sequences $a_k, \ldots, a_t$ and $b_k, \ldots., b_t$. Find the first position in which they differ, without loss of generally, assume this position is k and that $a_k > b_k$. Then

$m \leq \binom{b_k}{k} + \binom{b_{k-1}}{k-1} + \cdots + \binom{b_k - k + 1}{1} < \binom{b_k+1}{1} \leq m$, a contradiction.

To show existence, apply the greedy algorithm: find the largest a_k such that $\binom{a_k}{m} \leq m$, and apply the same algorithm with m and k replaced by $m - \binom{a_k}{k}$ *and* $k-1$.

We need only make sure that the sequence obtained is indeed decreasing, but this follows because by assumption, $m < \binom{a_k+1}{m}$, *and so* $m - \binom{a_k}{k} < \binom{a_k}{k-1}$.

33. Let P(x) be a polynomial with rational coefficients such that $\boldsymbol{P^{-1}(Q) \subseteq Q}$. Show that P is linear.

Sol: By a suitable variable substitution and constant factor, we may assume P(x) is monic and has integer coefficients; let P(0)= c_0. If p is a sufficiently large prime, the equation P(x)= p $+c_0$ has a single real root, which by assumption is rational and which we may also assume is positive (since P has positive leading coefficient). However, by the rational root theorem, the only rational roots of P(x) – p - c_0 can be ± 1 *and* $\pm p$. Since the root must be positive and cannot be 1 for large p, we have P(p) –p -$c_0 = 0$ for infinitely many p, so P(x) = x $+c_0$ is linear.

34. For each positive integer n, find the greatest common divisor of n! +1 and (n+1)!.

Sol: If n + 1 is composite, then each prime divisor of (n+ 1)! is a prime less than n, which also divides n! and so does not divide n! +1. Hence f(n) = 1. If n +1 is prime, the same argument shows that f(n) is a power of n +1, and in fact n +1 $|$n! +1 by Wilson's theorem. However, $(n+1)^2$ does not divide (n +1)!, and thus f(n) = n +1.

35. For each positive integer n, let S(n) be the sum of the digits in the decimal expansion of n. Prove that for all n,
$\boldsymbol{S(2n) \leq 2S(n) \leq 10S(2n)}$ & show that there exists n such that S(n) = 1996S(3n).

Solution: It is clear that S(a +b) ≤ S(a) + S(b), with equality if and only if there are no carries in the addition of a and b. Therefore S(2n) ≤ 2S(n). Similarly S(2n) ≤ 5S(10n) = 5S(n). An example with S(n) = 1996S(3n) is 133 … 35 (with 5968 threes).

36. Let F be +the midpoint of side BC of triangle ABC. Construct isosceles right triangles ABD and ACE externally on sides AB and AC with the right angles at D and E, respectively. Show that DEF is an isosceles right triangle.

Solution: Identifying A, B, C with numbers on the complex plane, we have $F = \frac{(B+C)}{2}, D = B + (A-B)r, E = A + (C-A)r, where\ r = \frac{(1+i)}{2}. Then\ E - F = \frac{A(1-I)}{2} - \frac{B}{2} + \frac{Ci}{2}\ and\ D - F = \frac{A(1+i)}{2} - \frac{Bi}{2} - \frac{C}{2}$; in particular, $D - F = i(E-F)$ and so DEF is an isosceles right triangle.

37. Show, with proof, how to dissect a square into at most five pieces in such a way that the pieces can be reassembled to from three squares no two of which have the same area.

Solution: We dissect a 7 × 7 square into a 2 × 2 square A, a 3 × 3 square B, and three pieces C, D, E which from a 6 × 6 square, as shown below.

$$\begin{matrix} C & C & C & C & C & A & A \\ C & C & C & C & C & A & A \\ C & C & C & C & C & D & D \\ C & C & C & C & C & D & D \\ C & C & C & C & B & B & B \\ C & C & C & C & B & B & B \\ E & E & E & E & B & B & B \end{matrix}$$

38. Let F_n denote the Fibonacci sequence, so that $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for n ≥0. Prove that

(i) The statement "$F_{n+k} - F_n$ is divisible by 10 for all positive integers n" is true if k = 60 and false or any positive integer k < 60;

(ii) The statement "$F_{n+t} - F_n$ is divisible by 100 for all positive integers n" is true if t = 300 and false or any positive integer t<300.

Solution: A direct computation shows that the Fibonacci sequence has period 3 modulo 2 and 20 modulo 5(compute terms until the initial terms 0, 1 repeat, at which time the entire sequence repeats), yielding (a). As for (b), one computes that the period mod 4 is 6. The period mod 25 turns out to be 100, which is awfully many terms to compute by hand, but knowing that the period must be a multiple of 20 helps, and verifying the recurrence $F_{n+8} = tF_{n+4} + F_n$, where t is an integer congruent to 2 modulo 5, shows that the period divides 100, finally, an explicit computation shows that the period is not 20.

39. Prove that for all positive integers n, $\mathbf{2^{1/2}.4^{1/4}}\ \ldots.(\mathbf{2^n})^{1/2^n} < 4$.

Solution: It is sufficient to show

$$\sum_{n=1}^{x} \frac{n}{2^n} = 2;$$

$$\sum_{n=1}^{x} \frac{n}{2^n} = \sum_{n=1}^{x} \sum_{n=1}^{x} \frac{1}{2^k} = \sum_{n=1}^{x} \frac{1}{2^{n-1}} = 2.$$

40. Let p be a prime number and a, n positive integers.

Prove that if $2^p + 3^p = a^n$, then n = 1.

Solution: If p = 2, we have $2^2 + 3^2 = 13$ and n = 1. If p > 2, then p is odd, so 5 divides $2^p + 3^p$ and so 5 divides a. Now if n > 1, then 25 divides a^n and 5 divides $\frac{2^p+3^p}{2+3} = 2^{p-1} - 2^{p-2}.3 + \cdots + 3^{p-1} \equiv p2^{p-1}$ (mod 5), a contradiction if p ≠ 5. Finally, if p = 5, then $2^5 + 3^5 = 753$ is not a perfect power, so n = 1 again.

41. Let ABC be an acute triangle and let D, E, F be the feet of the altitudes from A, B, C respectively. Let P, Q, R be the feet of the perpendiculars from A, B, C to EF, FD, DE, respectively. Prove that the lines AP, BQ, CR are concurrent.

Solution: It is a routine exercise to show that each of AP, BQ, CR passes through the circumcenter of ABC, so they all concur.

42. On a 5 × 9 rectangular chessboard, the following game is played. Initially a number of discs are randomly placed on some of the squares, no square containing more than one disc. A turn consists of moving all of the discs subject to the following rules:

(i) Each disc may be moved one square up, down left, or right;

(ii) If a disc moves up or down on one turn, it must move left or right on the next turn, and vice versa;

(iii) At the end of each turn, no square can contain two or more discs. The game stops if it becomes impossible to complete another turn. Prove that if initially 33 discs are placed on the board, the game must eventually stop. Prove also that it is possible to place 32 discs on the board so that the game can continue forever.

Solution: If 32 discs are placed in an 8 × 4 rectangle, they can all move up, left, down, right, up, etc. To show that a game with 33 discs must stop, label the board as shown:

1 2 1 2 1 2 1 2 1
2 3 2 3 2 3 2 3 2
1 2 1 2 1 2 1 2 1
2 3 2 3 2 3 2 3 2
1 2 1 2 1 2 1 2 1

Note that a disc on 1 goes to a 3 after two moves, a disc on 2 goes to a 1 or 3 immediately. And a disc on 3 goes to a 2 immediately. Thus if k disc start on 1 and k > 8, the game stops because there are not enough 3s to accommodate these disc. Thus we assume k ≤8, in which case there are at most 16 squares on 1 or 3 at the start, and so at least 17 on 2. Of these 17, at most 8 can move onto 3 after one move, so at least 9 end up on 1; these discs will not all be able to move onto 3 two moves later, so the game will stop.

43. Among triangles with one side of a given length l and with given area S, determine all of those for which the product of the lengths of the three altitudes is maximum.

Solution: Let A, B be two fixed points with AB = l, and vary C along a line parallel to AB at distance $\frac{2S}{l}$. The product of the altitudes of ABC is $8S^3$ divided by the lengths of the three sides, so it suffices to minimize AC, BC, or equivalently to maximize $\sin C$. Let D be the intersection of the perpendicular bisector of AB with the line through C. If ∠D is not acute, the optimal triangles are clearly those with a right angle at C.

Suppose ∠D is acute and C ≠ D, and assume C is on the same side of the perpendicular bisector of AB as B: we show ∠D ≥ ∠C, and so the optimal triangle is ABD. The triangles DAC and DBC have equal base and height, so equal altitude. However, AC > BC since ∠CAB > ∠CBA, so sin $\angle DAC >$ sin $\angle DBC$, and since the former is acute, we have ∠DAC < ∠DBC. Adding ∠CAB + ∠ABD to both sides, we get ∠DAB + ∠DBA < ∠CAB + ∠CBA, and so ∠ADB > ∠ACB, as claimed.

44. Prove that the equation $a^2 + b^2 = c^2 + 3$ has infinitely many integer solutions (a, b, c).

Sol.: let a be any odd number, let b = $\frac{(a^2-5)}{2}$ $and\ c = \frac{(a^2-1)}{2}$. Then

$$c^2 - b^2 = (c+b)(c-b) = a^2 - 3.$$

45. Let A and B be opposite vertices of a cube of edge length 1. Find the radius of the sphere with center interior to the cube, tangent to the three faces meeting at A and tangent to the three edges meeting at B.

Solution: Introduce coordinates so that A = (0, 0, 0), B = (1, 1, 1) and the edges are parallel to the coordinate axes. If r is the radius of the sphere, then (r, r, r) is its center, and (r, 1, 1) is the point of tangency of one of the edges at B. Therefore $r^2 = 2(1-r)^2, giving\ r^2 - 4r + 2 = 0$ and so r = $2 - \sqrt{2}$ (the other root puts the center outside of the cube).

46. Given an alphabet with three letters a, b, c find the number of words of n letters which contain an even number of a's.

Solution: If there are 2k occurrences of a, these can occur in $\binom{n}{2k}$ places, and the remaining positions can be filled in 2^{n-2k} ways. So the answer is

$$\sum_k \binom{n}{2k} 2^{n-2k}.$$

To compute this, note that

$$(1+x)^n + (1-x)^n = 2\sum_k \binom{n}{2k} x^{2k}.$$

So the answer is

$$\frac{1}{2}2^n\left[\left(1+\frac{1}{2}\right)^n + \left(1-\frac{1}{2}\right)^n\right] = \frac{1}{2}(3^n+1).$$

47. What is the minimum number of squares that one needs to draw on a white sheet in order to obtain a complete grid with n squares on a side?

Solution: It suffices to draw 2n -1 squares: in terms of coordinates, we draw a square with opposite corners (0, 0) and (i, i) for 1 ≤ i ≤ n and a square with opposite corners (i, i) and (n, n) for 1 ≤ i≤ n -1.

To show this many squares are necessary, note that the segments from (0, i) to (1, i) and from (n -1, i) to (n, i) for 0 < i < n all must lie on different squares, so surely 2n -2 squares are needed. If it were possible to obtain the complete grid with 2n -2 squares, each of these segments would lie on one of the squares, and the same would hold for the segments from (i, 0) to (i, 1) and from (i, n-1) to (i, n) for 0 < I < n. Each of the aforementioned horizontal segments shares a square with only two of the vertical segments, so the only possible arrangements are the one we gave above without the square with corners (0, 0) and (n, n), and the 90° rotation of this arrangement, both of which are insufficient. Hence 2n -1 squares are necessary.

48. Consider a triangulation of the plane, i.e. a covering of the plane with triangles such

that no two triangles have overlapping interiors, and no vertex lies in the interior of an edge of another triangle. Let A, B, C be three vertices of the triangulation and let θ be the smallest angle of the triangle ∆ABC. Suppose no vertices of the triangulation lie inside the circumcircle of ∆ABC. Prove there is a triangle σ in the triangulation such that $\sigma \cap$ ∆ABC $\neq \theta$ and every angle of σ is greater than θ.

Sol.: We may assume $\theta = \angle$A. The case where ABC belongs to the triangulation is easy, so assume this is not the case. If BC is an edge of the triangulation, one of the two triangles bounded by BC has common interior points with ABC, and this triangle satisfies the desired condition. Otherwise, there is a triangle BEF in the triangulation whose interior intersects BC. Since EF crosses BC at an interior point, ∠BEF < ∠BAF < ∠BAC, so triangle BEF satisfies the desired condition.

49. Let m and n be positive integers with gcd(m, n) = 1. Compute gcd($5^m + 7^m$, $5^n + 7^n$).

Sol.: Let $s_n = 5^n + 7^n. If\ n \geq 2m$, note that

$$s_n = s_m s_{n-m} - 5^m 7^m s_{n-2m},$$

So gcd(s_m, s_n) = gcd(s_m, s_{n-2m}) … similarly, if m < n < 2m, we have gcd(s_m, s_n)= gcd(s_m, s_{n-2m}). Thus by the Euclidean algorithm, we conclude that if m + n is even, then gcd(s_m, s_n) = gcd(s_1, s_2) = 12, and if m + n is odd, then gcd(s_m, s_n) = gcd(s_0, s_1) = 2.

50. Let x > 1 be a real number which is not an integer. For n = 1, 2, 3, …., let $a_n = \llcorner \lfloor x^{n+1} \rfloor \lrcorner - x \llcorner \lfloor x^n \rfloor$. Prove that the sequence $\{a_n\}$ is not periodic.

Solution: Assume, on the contrary, that there exist p > 0 such that $a_{p+n} = a_n$ for every n. Since $\lfloor x_n \rfloor \longrightarrow \infty\ as\ n\ \to \infty$,we have $\llcorner \lfloor x^{n+p} \rfloor \lrcorner - \llcorner \lfloor x^n \rfloor \lrcorner > 0$ for some n; then setting $a_{n+p} = a_n$ and solving for x, we get

$$x = \frac{\lfloor x^{n+p+1} \rfloor - \lfloor x^{n+1} \rfloor}{\lfloor x^{n+p} \rfloor - \lfloor x^n \rfloor}$$

And so x is rational.

Put y = x^p and

$$b_m = \sum_{k=0}^{p-1} x^{p-k-1} a_{mp+k}$$
$$= \llcorner \lfloor x^{m+p} \rfloor - x^p \llcorner \lfloor x^m r \rfloor \lrcorner$$
$$= \llcorner \lfloor y^{m+1} \rfloor - y \llcorner \lfloor y^m \rfloor \lrcorner .$$

Since $a_{p+n} = a_p$, we have $b_{m+1} = b_m$, and y is also rational number which is not an integer. Now put $c_m = \llcorner \lfloor y^{m+1} - y^m \rfloor \lrcorner\ ; then\ c_{m+1} = y c_m = y^m c_1$. This means c_m cannot be an integer for large m, a contradiction.

51. Let θ be the maximum of the six angles between the edges of a regular tetrahedron and a given plane. Find the minimum value of θ over all positions of the plane.

Sol.: Assume the edges of the tetrahedron Γ = ABCD have length l. If we place the tetrahedron so that AC and BC are parallel to the horizontal plane H. We obtain $\theta = 45°$, and we shall show this is the minimum angle. Let a, b, c, d be the projections of A, B, C, D to the horizontal plane H, and $l_1, \ldots, l_6$ the projection of the edges $L_1, \ldots, L_6$. Since the angle between L_1 and H has cosine l, it suffices to consider the shortest l_i.

If a, b, c, d from a convex quadrilateral with largest angle at a, then one of ab or ad is at most $\frac{1}{\sqrt{2}}$ since bd ≤ 1. Otherwise, it is easily shown that one of the l_1 originating from the vertex inside the convex hull has length at most $\frac{1}{\sqrt{3}}$.

52. Let q be a real number with $\frac{(1+\sqrt{5})}{2} < q < 2$. For a number n with binary representation

$$\mathbf{n = 2^k + a_{k-1}.2^{k-1} + \cdots + a_1.2 + a_0}$$

with $a_1 \in \{0, 1\}$, we define P_n as follows

$$\boldsymbol{p^n = q^k + a_{k-1}q^{k-1} + \cdots + a_1 q + a_0}.$$

Prove that there exist infinitely many positive integers k for which there does not exist a positive integer l such that $p_{2k} < p_1 < p_{2k+1}$.

Solution: Define the sequence a_n as follows:

$$a_{2m} = \sum_{k=0}^{m} 2^{2k}, a_{2m+1} = \sum_{k=0}^{m} 2^{2k+1}.$$

We will show that k = a_n satisfies the given condition by induction on n. The case n = 0, 1 follow by noting 1 < q < q +1 <$q^2 < q^2 + 1 < q^2 + q < q^2 + q + 1$

and $p_1 \geq q^p \geq q^3 > q^2 + q = P_6 \, for \, 1 \geq 8$.

Now suppose n ≥ 2, assume the induction hypothesis, and suppose by way of contradiction that there exist l such that $p_{2a_n} < p_1 < p_{2a_{n+1}}$. The argument falls into six cases, which we summarize in a table. The first column gives the conditions of the case, the second gives a lower bound of p_{2a_n}, the third is always equal to p_1, and the fourth gives an upper bound for $p_{2a_{n+1}}$; from these a contradiction to the induction hypothesis will become evident.

n even, $l = 2r + 1qp_{2a_{n-1}} + 1qp_r + 1\, qp_{2a_{n-1}} + 1$

n even, $l = 4r \qquad q^2p_{2a_{n-2}}q^2p_r . q^2p_{2a_{n-1}} + 1$

n even, $l = 4r + 2q^2p_{2a_{n-2}} + qq^2p_r + qq^2p_{2a_{n-1}} + q$

n odd, $l = 2r\ qp_{2a_{n-1}}\ qp_r \qquad qp_{2a_{n-1}-1}$

n even, $l = 4r + 1 \qquad q^2p_{2a_{n-2}} + 1\ q^2p_r + 1\, q^2p_{2a_{n-2}+1} + 1$

n even, $l = 4r + 3\ q^2p_{2a_{n-2}} + q + 1q^2p_r + q + 1q^2p_{2a_{n-2}+1} + q + 1$

53. Find all pairs (n, r), with n a positive integer and r a real number, for which the polynomial $(x + 1)^n - r$ is divisible by $2x^2 + 2x + 1$.

Sol.: Let t = $\frac{(-1+i)}{2}$ be one of the roots of $2x^2 + 2x + 1$; then $(x + 1)^n - r$ is divisible by $2x^2 + 2x + 1$ for r real if and only if $(t + 1)^n = r$. Since the argument of t + 1 is $\frac{\pi}{4}$, this is possible if and only if n = 4m, in which case $(t + 1)^4 m = (-4)^4$. Hence $(4m, (-4)^m)$ are the only solutions.

54. Let ABC be a triangle and P a point inside it such that ∠PBC = ∠PCA < ∠PAB. The line PB cuts the circumcircle of ABC at B and E, and the line CE cuts the circumcircle of APE at E and F. Show that the ratio of the area of the quadrilateral APEF to the area of the triangle ABP does not depend on the choice of P.

Sol.: Note that ∠AEP = ∠AEB = ∠ACB = ∠CBP, so the lines AE and CP are parallel. Thus [APE] = [ACE] and [APEF][ACF]. Now note that ∠AFC = π- ∠EPA = ∠APB and ∠ACF = ∠ACE = ∠ABE. Therefore triangles ACF and ABP are similar and $\frac{[ACF]}{[AB]} = \left(\frac{AC}{AB}\right)^2$ independent of the choice of p.

55. Let ABCD be a tetrahedron with ∠BAC = ∠ACD and ∠ABD = ∠BDC. Show that edges AB and CD have the same length.

Sol.: Assume AB ≠ CD. Draw the plane through AC bisecting the dihedral angle formed by the planes ABC and ACD, then draw a line l in that plane perpendicular to AC through the midpoint O to AC. Now let B' and D' be the images of B and D, respectively, under the half-turn around the line l; by assumption, B' ≠ D and D' ≠ B; since ∠BAC = ∠ACD, B' lies on CD and D' lies on AB. Now note that the quadrilateral BB'D'D has total angular sum 2π. However, a non-polar quadrilateral always has total angular sum less than 2π (divide it into two triangles, which each have angular sum π, and apply the spherical triangle inequality) ∠ABC + ∠CBD > ∠ABD, so the lines AB and CD are coplanar. Contradicting the assumption that ABCD is a tetrahedron.

56. For a natural number k, let p(k) denote the smallest prime number which does not divide k. If p(k) > 2, define q(k) to be the product of all primes less than p(k), otherwise let q(k)= 1. Consider the sequence. $x_0 = 1, \quad x_{n+1} = \frac{x_n p(x_n)}{q(x_n)} \quad n = 0, 1, 2, \ldots$

Determine all natural numbers n such that $x_n = 111111$.

Sol.: An easy induction shows that if $p_0, p_1, \ldots$, are the primes in increasing order an n has base 2 representations $c_0 + 2c_1 + 4c_2 + \cdots$, then $x_n = p_0^{c_0} p_1^{c_1} \ldots$ in particular, 111111 = 3.7.11.13.37 = $p_1p_3p_4p_5p_{10}$, so $x_n = 111111$ if and only if n $= 2^{10} + 2^5 + 2^4 + 2^3 + 2^1 = 1082$.

57. Find the greatest positive integer n for which there exist n nonnegative integers $x_1, x_2, \ldots, x_n$, not all zero, such that for any sequence $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, of elements of {-1, 0, 1}, not all zero, n^3 does not divide $\epsilon_1 x_1 + \epsilon_2 x_2 + \cdots + \epsilon_n x_n$.

Solution: The statement holds for n = 9 by choosing 1, 2, $2^2, \ldots, 2^8$, since in that case

$$\left|\epsilon_1 + \cdots + \epsilon g 2^8\right| \leq 1 + 2 + \cdots + 2^8 < 9^3.$$

However, if n = 10, then $2^{10} > 10^3$, so by the pigeonhole principle, there are two subsets A and B of $\{x_1, \ldots, x_{10}\}$ whose sums are congruent modulo 10^3. Let $\epsilon_1 = 1$ if x_i occurs in A but not in B, -1 if x_i occurs in B but not in A, and 0 otherwise; then $\sum \epsilon_i x_i$ is divisible by n^3.

58. Let x, y be real numbers. Show that if the set

$$\{\cos(n\pi x) + \cos(n\pi y) \mid n \in N\}$$

Is finite, then x, y ∈ Q.

Sol.: Let $a_n = \cos n\pi x$ and $b_n = \sin n\pi x$. Then

$$(a_n + b_n)^2 + (a_n - b_n)^2 = 2(a_n{}^2 + b_n{}^2) = 2 + (a_{2n} + b_{2n}).$$

If $\{a_n + b_n\}$ is finite, it follows that $\{a_n - b_n\}$ is also a finite set, and hence that $\{a_n\}$ is finite, since

$$a_n = \frac{1}{2}[(a_n + b_n)(a_n - b_n)].$$

And similarly $\{b_n\}$ is finite. In particular, $a_m = a_n$ for some m < n, and so (n –m)πx is an integral multiple of π. We conclude x and y are both rational.

59. Let ABCD be a cyclic quadratilateral and let M be the set of incenters and excenters of the triangles BCD, CDA, DAB, ABC (for a total of 16 points). Show that there exist two sets of parallel lines K and L, each

consisting of four lines, such that any line of K ∪ L contains exactly four points M.

Solution: Let T be the midpoint of the arc AB of the circumcircle of ABC, I the incenter of ABC, and I_B, I_C the excenters of ABC opposite B and C, respectively. We first show TI = TA = TB = TI_c. Note that

∠TAI = ∠TAB + ∠BAI = $\frac{(\angle C+\angle A)}{2}$ = $\angle ICA + \angle IAC = \angle TAI$

So TI = TA, and similarly TI = TB. Moreover, in the right triangle

$$AI_CI, \angle AI_CT = \frac{\pi}{2} - \angle AIT = \frac{\pi}{2} - \angle TAI$$

$$= \angle TAI_C, so\ TA = TI_C also$$

We next show that the midpoint U of I_BI_C is also the midpoint of the arc BAC. Note that the line I_BI_C bisects the exterior angles of ABC at A, so the line $I_BI_C$passes through the midpoint V of the arc BAC. Considering the right triangles $I_BBI_C\ and\ I_BCI_C$, we note BU = $\frac{(I_BI_C)}{2}$ = CU, so U lies on the perpendicular bisector of BC, which suffices to show U = V. (Note that $I_B\ and\ I_C$ lie on the same side of BC as A, so the same is true of U).

Let E, F, G, H be the midpoints of the arcs AB, BC, CD, DA. Let I_A, I_B, I_C, I_D be the incenters of the triangles BCD, CDA, DAB, ABC, respectively. Let A_B, A_C, A_D be the excenters of BCD opposite B, C, D, respectively, and so on.

By the first observation, $I_CI_DC_DD_C$ is a rectangle with center E, and the diagonals, which contain the points C and D, have length 2EA = 2EB. Similarly, we obtain rectangle centered at F, G, H.

Now consider the excenters of the from X_Y where X and Y are opposite vertices in ABCD. We shall prove the claim with K = $\{B_CC_B, I_CI_B, I_DI_A, A_DD_A\}$, L = $\{A_BB_A, I_AI_B, I_CI_D, C_D\ D_C\}$.

Consider the rectangle $B_CI_DB_AP$, where P is an unknown point. From the second observation above, the midpoint K of diagonal B_AB_C is the midpoint of arc CDA, so it lies on the internal bisector BK of triangle ABC. Again by the first observation, we conclude M = D_A, soD_A lies on the lines B_CC_B and B_AA_B, and so on, proving the claim.

60. Let n ≥ 3 be an integer and $\boldsymbol{x_1, x_2, \dots, x_{n-1}}$ nonnegative integers such that

$$\boldsymbol{x_1 + x_2 + \cdots + x_{n-1} = n}$$
$$\boldsymbol{x_1 + 2x_2 + \cdots + (n-1)x_{n-1} = 2n - 2}.$$

Find the minimum of the sum

$$\boldsymbol{F(x_1, \dots, x_{n-1}) = \sum_{k=1}^{n-1} kx_k\,(2n-k)}.$$

Sol.: The desired sum can be written as

$$2n(2n-2) - \sum_{k=1}^{n-1} k^2 x_k.$$

Now note

$$\sum_{k=1}^{n-1} k^2 x_k = \sum_{k=1}^{n-1} x_k + (k-1)(k+1)x_k$$
$$\leq n + n$$

$$\sum_{k=1} n-1\,(k-1)x_k = n + n(2n-2-n)$$
$$= n^2 - n.$$

Hence the quantity in question is at most

$2n(2n-2)-(n^2-n)=3n^2-3n$, with equality for $x_1=n-1, x_2=\cdots= x_{n-2}= 0, x_{n-1}=1$.

61. Let n, r be positive integers and A a set of lattice points in the plane, such that any open disc of radius r contains a point of A. Show that for any coloring of the points of A using n colors, there exist four points of the same color which are the vertices of a rectangle.

Sol.: Consider a square of side length L = $4\mathrm{n}r^2$ with side parallel to the coordinate axes. One can draw $(2nr^2)=4n^2r^2$ disjoint disks of radius r inside the square, hence such a square contains at least $4n^2r^2$ points of A.

The lattice point in A lie on L -1 = $4\mathrm{n}r^2-1$ vertical lines; by the pigeonhole principle, some vertical line contains n +1 points of A. Again by the pigeonhole principle, two of these points are colored in the same color.

Now consider an infinite horizontal strip made of ribbons of side length L; some two of them have two points in the same position in the same color, and these four points from the vertices of a rectangle.

62. Find all prime numbers p, q for which the congruence $\boldsymbol{\alpha^{3pq}\equiv\alpha(mod\ 3pq)}$ holds for all integers $\boldsymbol{\alpha}$.

Sol.: Without loss of generality assume p ≤ q; the unique solution will be (11, 17), for which one many check the congruence using the Chinese Remainder Theorem.

We first have $2^{3pq}\equiv2(mod\ 3)$, which means p and q are odd. In addition, if α is a primitive root mod p, then $\alpha^{3pq-1}\equiv1\ (mod\ p)$ implies that p -1 divides 3pq -1 as well as 3pq -1- 3q(p -1) = 3q -1, and conversely that q -1 divides 3p -1. If p = q, we now deduce p = q = 3, but $4^{27}\equiv1(\text{mod } 27)$, so this fails. Hence p < q.

Since p and q are odd primes, q ≥ p +2, so $\frac{(3p-1)}{(q-1)}<3$. Since this quantity is an integer, and it is clearly greater than 1, it must be 2. That is, 2q = 3p +1. On the other hand, p -1 divides 3q -1= $\frac{(9p+1)}{2}$ as well as (9p +1) –(9p -9) = 10. Hence p = 11, q = 17.

63. Let n ≥ 3 be an integer and p ≥ 2n -3 a prime. Let M be a set of n points in the plane, no three collinear, and let f: M ⟶{0, 1, ..., p -1} be a function such that:

(i) Only one point of M maps to 0, and

(ii) If A, B, C are distinct points in M and k is the circumcircle of the triangle ABC, then

$$\sum_{P\in M\cap k}\boldsymbol{f(P)\equiv0(mod\ P)}.$$

Show that all of the points of M lie on a circle.

Solution: Let X be the point mapping to 0. We first show that if every circle through X and two points of M contains a third point of M, then all of the points of M lie on a circle. Indeed, consider an inversion with center at X. Then the image of M – {X} has the property that the line through any two of its points contains a third point; it is a standard result that this means the points are collinear. (Otherwise, find a triangle ABC minimizing the length of the altitude AH; there is another point N on BC, but then either ABN OR CAN has a shorter altitude than AH, contradiction).

Now suppose the points of M do not lie on a circle. By the above, there exists a circle passing through M and only two points A, B of M. Let f(A) = i, so that by the hypothesis, f(B)

= p –i. Let a be the number so circles passing through X, A and at least one other point of M, let b be the number of circles passing through X, B and at least one other point of M, and let S be the sum of f(P) over all P in M. By adding the relations obtained from the circles through X and A, we get S + (a -1) i ≡ 0 (mod p), and similarly, S + (b -1) (p –i) ≡ 0 (mod p). Therefore a + b -2 ≡ 0 (mod p) ; since a + b ≤2n +4 < p, we have a + b = 2 and so a = b = 1, contradicting the assumption that the points do not all lie on a circle.

64. Let x, y, z be real numbers. Prove that the following conditions are equivalent.

(i) x, y, z > 0 and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \le 1$.

(ii) For every quadrilateral with sides a, b, c, d, $a^2x + b^2y + c^2z > d^2$.

Sol.: To show (i) implies (ii), note that

$$a^2x + b^2y + c^2z \ge (a^2x + b^2y + c^2z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \ge (a+b+c)^2 > d^2.$$

Using Cauchy-Schwarz after the first inequality.

To show (i) implies (ii), first note that if x ≤ 0, we may take a quadrilateral of sides a = n, b = 1, c = 1, d = n and get y + z >$n^2(1-x)$, a contradiction for large n. Thus x > 0 and similarly y > 0, z > 0. Now use a quadrilateral of sides $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\ and \frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{1}{n}$, where n is large. We then get $\frac{x}{x^2}+\frac{y}{y^2}+\frac{z}{z^2} > \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{1}{n}\right)^2$

Since this holds for all n, we may take the limit as n⟶∞ and get

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \ge \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{1}{n}\right)^2$$

And hence $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \le 1$.

65. Let n be a positive integer and D a set of n concentric circles in the plane. Prove that if the function f : D ⟶D satisfies

d(f(A), f(B)) ≥ d(A, B) for all A, B, ϵ, D, then d(f(A), f(B)) = d(A, B) for every A, B, ϵD.

Sol.: Label the circles $D_1, \dots, D_n$ in increasing order of radius, and let r_1 denote the radius D_i. Clearly the maximum of d(A, B) occurs when A and B are antipodal points on D. Let ABCD be the vertices of a square inscribed in D_n; then f(A) and f(C) are antipodal, as are f(B) and f(D). In addition, each of the minor arcs f(A) f(B) and f(B) f(C) must be at least a quarter arc, thus f(B) bisects one of the semicircles bounded by f(A) and f(C), and f(D) bisects the other. Now if P is any point on the minor arc AB, then the arcs f(P) f(A) and f(P) f(B), which are at least as long as the arc PA and PB, and up to the quarter arc f(P) f(B). We conclude f is isometric on D_n.

Since f is clearly injective and is now bijective on $D_n f$ maps $D_1 U \dots . U D_{n-1}$ into itself. Thus we many repeat the argument to show that f is isometric on each D_i. To conclude, it suffices to show that distances between adjacent circles, say D_1 and D_2, are preserved. This is easy; choose a square ABCD on D_1 and A', B', C', D' be the points on D_2 closet to A, B, C, D, respectively. Then A'B'C'D' also from a square, and the distance from A to C' is the maximum between any point on D_1 and any point on D_3. Hence the eight points maintain their relative position under f, which suffices to prove isometry.

66. Let n ≥ 3 be an integer and X ⊆ {1, 2, …, n^3} a set of $3n^2$ elements. Prove that one can find the distinct numbers $a_1, \dots, a_9$ in X such that the system

$$a_1x + a_2y + a_3z = 0$$
$$a_4x + a_5y + a_6z = 0$$
$$a_7x + a_8y + a_9z = 0$$

Has a solution (x_0, y_0, z_0) in nonzero integers.

Sol.: Label the elements of X in increasing order $x_1 < \cdots < a_{3n^2}$, and put

$$X_1 = \{x_1, \dots, x_{n^2}\}, X_2 = \{x_{n^2+1}, \dots, x_{2n^2}\}, X_3 = \{x_{n^2+1}, \dots, x_{3n^2}\},$$

Define the function f: $X_1 \times X_2 \times X_3 \to X \times X$ as follows: f(a, b, c) = (b –a, c –b).

The domain of f contains n^6 elements. The range of f, on the other hand, is contained in the subset of X × X of pairs whose sum is at most n^3, a set of cardinality.

$$\sum_{k=1}^{n^3-1} k = \frac{n^3(n^3-1)}{2} < \frac{n^6}{2}.$$

By the pigeonhole principle, some three triples (a_i, b_i, c_i) (i = 1, 2, 3) map to the same pair, in which case x =$b_1 - c_1, y = c_1 - a_1, z = a_1 - b_1$ is a solution in nonzero integers. Note that a_i, cannot equal b_j since X_1 and X_2 and so on, and that $a_1 = a_2$ implies that the triple (a_1, b_1, c_1) and (a_2, b_2, c_2) are identical, a contradiction. Hence the nine numbers chosen are indeed distinct.

67. Which are there more of among the natural numbers from 1 to 1000000, inclusive: numbers that can be represented as the sum of a perfect square and a (positive) perfect cube, or numbers that cannot be?

Sol.: There are more numbers that not of this form. Let n = $k^2 + m^3$, where k, m, n ∈ N and n ≤ 1000000. Clearly k ≤ 1000 and m ≤ 100. Therefore there cannot be more numbers in the desired from than the 1000000 pairs (k, m).

68. Let x, y, p, n, k be natural numbers such that

$$x^n + y^n = p^k.$$

Prove that if n > 1 is odd, and p is an odd prime, then n is a power of p.

Sol.: Let m = gcd(x, y). Then x = m$x_1, y = my_1$ and by virtue of the given equation, $m^n({x_1}^n + {y_1}^n) = p^k$, and so m = p^α for some nonnegative integer α. It follows that ${x_1}^n + {y_1}^n = p^{k-n\alpha}$. (1)

Since n is odd,

$$\frac{{x_1}^n + {y_1}^n}{x_1 + y_1} = {x_1}^{n-1} - {x_1}^{n-2}y_1 + {x_1}^{n-3}{y_1}^2 - \cdots - x_1{y_1}^{n-2} + {y_1}^{n-1},$$

Let A denote the right side of the equation. By the condition p > 2, it follows that at least one of x_1, y_1 is greater than 1, so since n > 1. A > 1.

From (1) it follows that A$(x_1 + y_1) = p^{k-n\alpha}$, so since $x_1 + y_1 > 1$, A >1, both of these numbers are divisible by p, moreover, $x_1 + y_1 = p^\beta$ for some natural number β.

Thus

$$A = {x_1}^{n-1} - {x_1}^{n-2}(p^\beta - x_1) + \cdots - x_1(p^\beta - x_1)^{n-2} + (p^\beta - x_1)^{n-1} = n{x_1}^{n-1} + Bp.$$

Since A is divisible by p and x_1 is relatively prime to p, it follows that n is divisible by p.

Let n = pq. Then $x^{pq} + y^{pq} = p^k$ or $(x^p)^q + (y^p)^q = p^k$. If q > 1, then by the same argument, p divides q. If q =1, then n = p. Repeating this argument, we deduce that n = p^l for some natural number l.

69. In the Duma there are 1600 delegates, who have formed 16000 committees of 80 persons each. Prove that one can find two committees having no fewer than four common members.

Sol.: Suppose any two committees have at most three common members. Have two deputies count the possible ways to choose a chairman for each of three sessions of the Duma. The first deputy assumes that any deputy can chair any session, and so gets 1600^3 possible choices. The second deputy makes the additional restriction that all of the chairmen belong to a single committee. Each of the 16000 committees yields 80^3 choices, but this is an over count; each of the 16000 $\frac{(16000-1)}{2}$ pairs of committees give at most 3^3 overlapping choices. Since the first deputy counts no fewer possibilities than the second, we have the inequality

$$1600^3 \geq 16000.80^3 - \frac{16000.15999}{2}3^3.$$

However,

$$1600.80^3 - \frac{16000.15999}{2}3^3 > 16000.80^3 - \frac{16000.15999}{2}\frac{4^2}{2}$$

$$= \frac{16000.4^3}{4} + 2^{13}.10^6 - 2^{12}.10^6 > 2^{12}.10^6 = 1600^3.$$

We have a contradiction.

70. Show that in the arithmetic progression with first term 1 and ratio 729, there are infinitely many powers of 10.

Sol.: We will show that for all natural numbers n, $10^{81n} - 1$ is divisible by 729. In fact, $10^{81n} - 1 = (10^{81})^n - 1^n = (10^{81} - 1).A, and$

$$10^{81n} - 1 = \underbrace{9\ldots9}_{81}$$

$$= \underbrace{9\ldots9}_{9}\ldots\underbrace{10\ldots01}_{8}\underbrace{10\ldots01}_{8}\ldots\underbrace{10\ldots01}_{8}$$

$$= 9\underbrace{1\ldots9}_{9}\ldots\underbrace{10\ldots01}_{8}\underbrace{10\ldots01}_{8}\ldots\underbrace{10\ldots01}_{8}$$

The second and third factors are composed of 9 units, so the sum of their digits is divisible by 9, that is, each is a multiple of 9. Hence $10^{81n} - 1$ is divisible by $9^3 = 729$, as is $10^{81n} - 1$ for any n.

71. Two piles of coins lie on a table. It is known that the sum of the weights of the coins in the two piles are equal, and for any natural number k, not exceeding the number of coins in either pile, the sum of the weights of the k heaviest coins in the first pile is not more than that of the second pile. Show that for any natural number x, if each coin (in either pile) of weight not less than x is replaced by a coin of weight x, the first pile will not be lighter than the second.

Sol.: Let the first pile have n coins of weights

$x_1 \geq x_2 \geq \cdots \geq x_n$, and let the second pile have m coins of weights $y_1 \geq y_2 \geq \cdots \geq y_m, where\ x_1 \geq \cdots \geq x_s \geq x \geq x_{s+1}$ k and $y_1 \geq \cdots \geq y_t \geq x \geq y_{t+1} \geq \cdots \geq y_m$. (If there are no coins of weight greater than x, the result is clear). We need to show that $xs +$

$x_{s+1} + \cdots + x_n \geq xt + y_{t+1} + \cdots + y_m$. Since $x_1 + \cdots + x_n = y_1 + \cdots + y_m = A$, this inequality can be equivalently written $xs + (A - x_1 - \cdots - x_m) \geq xt + (A - y_1 - \cdots - y_t)$, which in turn can be rewritten

$x_1 + \cdots + x_s + x(t-s) \leq y_1 + \cdots + y_t$,

This is what we will prove,

If t ≥ s, then

$$x_1 + \cdots + x_s + x(t-s) = (x_1 + \cdots + x_s) + \underbrace{(x + \cdots + x)}_{t-s} \leq (y_1 + \cdots + y_s) + (y_{s+1} + \cdots + y_t),$$

Since $x_1 + \cdots + x_s \leq y_1 + \cdots + y_s$ (from the given condition) and $y_{s+1} \geq \cdots \geq y_t \geq x$.

If t < s, then $x_1 + \ldots + x_s + x(t-s) \leq y_1 + \cdots + y_t$ is equivalent to

$$x_1 + \cdots + x_s \leq y_1 + \cdots + y_t + \underbrace{(x + \cdots + x)}_{t-s}$$

The latter inequality follows from the fact that

$$x_1 + \cdots + x_s \leq y_1 + \cdots + y_s = (y_1 + \cdots + y_t) + (y_{t+1} + \cdots + y_s) \text{and } y_s \leq \cdots \leq y_{t+1} \leq x.$$

72. Can a 5 × 7 checkerboard be covered by L's (figures formed from a 2 × 2 square by removing one of its four 1 × 1 corners), not crossing its borders, in several layers so that each square of the board is covered by the same number of L's?

Sol.: No such covering exists. Suppose we are given a covering of a 5 × 7 checkerboard with L's such that every cell is covered by exactly k L's. Number the rows 1,…, 5 and the columns 1, …, 7, and consider the 12 squares lying at the intersections of odd numbered rows with odd numbered columns. Each of these cells is covered by k L's, so at least 12k L's must be used in total. But these cover 3. 12k > 35k cells in total, a contradiction.

73. Points E and F are given are given on side BC of convex quadrilateral ABCD (with E closer than F to B). It is known that ∠BAE = ∠CDF and ∠EAF = ∠FDE. Prove that ∠FAC = ∠EDB.

Solution: By the equality of angles EAF and FDE, the quadrilateral AEFD is cyclic. Therefore ∠AEF + ∠FDA = 180°. By the equality of angles BAE and CDF we have

∠ADC + ∠ABC = ∠FDA +∠CDF +∠AEF -∠BAE = 180°

Hence the quadrilateral ABCD is cyclic, so ∠BAC = ∠BDC. It follows that ∠FAC = ∠EBD.

74. Find all natural numbers n, such that there exist relatively prime integers x and y and an integer k >1 satisfying the equation $3^n = x^k + y^k$.

Sol.: The only solution is n = 2.

Let $3^n = x^k + y^k$, where x, y are relatively prime integers with x > y, k > 1, and n a natural number. Clearly neither x nor y is a multiple of 3. Therefore, if k is even$x^k \, and \, y^k$ are congruent to 1 mod 3, so their sum is congruent to 2 mod 3, and so is not a power of 3.

If k is odd and k > 1, then $3^n = (x + y)(x^{k-1} - \cdots + y^{k-1})$. Thus x + y = 3^m for some m ≥ 1. We will show that n ≥ 2m. Since $\frac{3}{k}$ (see the solution to Russia 3), by putting

$x_1 = x^{k/3}$ and $y_1 = y^{k/3}$ we may assume k = 3.

Then $x^3 + y^3 = 3^m$ and x + y = 3^n. To prove the inequality n ≥ 2m, it suffices to show that $x^3 + y^3 \geq (x + y)^2$, or $x^2 - xy + y^2 \geq x + y$. Since x ≥ y +1, $x^2 - x = x(x-1) \geq xy, and\ (x^2 - x + xy) + (y^2 - y) \geq y(y-1) \geq 0$, and the inequality n ≥ 2m follows.

From the identity $(x + y)^3 - (x^3 + y^3) = 3xy(x + y)$ it follows that $3^{2m-1} - 3^{n-m-1} = xy$.

But 2m -1 ≥ 1, and n –m -1 ≥n -2m ≥0. If strict inequality occurs in either place in the last inequality, then $3^{2m-1} - 3^{n-m-1}$ is divisible by 3 while xy is not. Hence n –m -1 = n -2m = 0, and so m = 1, n = 2 and $3^2 = 2^3 + 1^3$.

Note: The inequality $x^2 - xy + y^2 \geq x + y$ can alternatively be shown by nothing that

$x^2 - xy + y^2 - x - y = (x - y)^2 + (x-1)(y-1) - 1 \geq 0$,

Since $(x - y)^2 \geq 1$.

75. Show that if the integers $a_1, \ldots, a_m$ are nonzero and for each k = 0, 1, …, m(n < m -1),

$$\boldsymbol{a_1 + a_2 2^k + a_3 3^k + \cdots + a_m m^k = 0},$$

Then the sequences $a_1, \ldots, a_m$ contains at least n + 1 pairs of consecutive terms having opposite sings.

Solution: We many assume $a_m > 0$, since otherwise we may multiply each of the numbers by -1. Consider the sequence $b_1, \ldots, b_m$, where $b_i = \sum_{j=0}^{n} c_j i^j$ for an arbitrary sequence of real numbers $c_0, \ldots, c_n$. From the given condition,

$$\sum_{i=1}^{m} a_i b_i = \sum_{i=1}^{m} a_i \sum_{j=0}^{n} c_j i^j = \sum_{j=0}^{n} c_i \sum_{j=1}^{n} a_i i^j = 0.$$

Suppose now that the sequence $a_1, \ldots, a_m$ has k pairs of neighbors that differ in sign, where k < n +1, and let $i_1, \ldots, i_k$ be the indices of the first members of these pairs.

Let $b_i = f(i) = (i - x_1)(i - x_2) \ldots (i - x_k)$, where $x_k = i_k + \frac{1}{2} (i = 1, 2, \ldots, k)$. The function f changes sign only at the points $x_1, \ldots, x_k$, and so b_1 and b_{i+1} have different signs if and only one of the x_l falls between them, which means i = i_l. We deduce that the sequences $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$ have the same pairs of neighbors of opposite sign. Since a_m and b_m are positive, we have that a_i and b_i have the same sign for i = 1, …., m, so $\sum_{i=1}^{m} a_i b_i > 0$, a contradiction.

76. At the vertices of a cube are written eight pair wise district natural numbers, and on each of its edges is written the greatest common divisor of the numbers at the end points of the edge. Can the sum of the numbers written at the vertices be the same as the sum of the numbers written at the edges?

Sol.: This is not possible. Note that if a and b are natural numbers with a > b, then gcd(a, b) ≤ b and gcd(a, b) ≤$\frac{a}{2}$.It follows that if a ≠ b, then gcd(a, b)≤$\frac{(a+b)}{3}$. Adding 12 such inequalities, corresponding to the 12 edges, we find that the desired condition is only possible if gcd(a, b) = $\frac{(a+b)}{3}$ in each case. But in this case the larger of a and b is twice the smaller; suppose a = 2b. Consider the numbers c and d assigned to the vertices of the other end points of the other two edges coming out of the vertex labeled a. Each of these is either half of or twice a. If at least one is less a, it equals b; otherwise, both are equal. Either option contradicts the assumption that the numbers are distinct.

77. Three sergeants and several solders serve in a platoon. The sergeants take turns on duty. The commander has given the following orders:

(a) Each day, at least one task must be issued to a soldier.

(b) No soldier may have more than two tasks or receive more than one tasks in a single day.

(c) The lists of soldiers receiving tasks for two different days must not be the same.

(d) The first sergent violating any of these orders will be jailed.

Can at least one of the sergeants, without conspiring with the others, give tasks according to these rules and avoid being jailed?

Sol.: The sergeants who goes third can avoid going to jail. We call a sequence of duties by the first, second and third sergeants in succession a round. To avoid going to jail, the third sergeant on the last day of each round gives tasks to precisely those soldiers who received one task over the previous two days. (Such soldiers exist by the third condition). With this strategy, at the end of each cycle each soldier will have received either two tasks or none, and the number of the latter will have decreased. It will end up, at some point, that all of the soldiers have received two tasks, and the first sergeant will go to jail.

78. Can the number obtained by written the numbers from 1 to n in order (n >1) be the same when read left-to-right and right-to left?

Sol.: This is not possible. Suppose N = 123 …321 is an m digit symmetric number, formed by writing the numbers from 1 to n in succession. Clearly m >18. Also let A and B be the numbers formed from the first and last k digits, respectively, of N, where k = $\lfloor m/2 \rfloor$ ⎿ . Then if 10^p is the largest power of 10 dividing A, then n> 2. 10^{p+1}, that is, n has at most p +2 digits. Moreover, A and B must contain the fragments

$$\underbrace{99\ldots9}_{p}\,\underbrace{100\ldots01}_{p}\ \ and\ \ \underbrace{100\ldots0}_{p}\,\underbrace{199\ldots9}_{p}$$

Respectively, which is impossible.

79. Do there exist three natural numbers greater than 1, such that the square of each, minus one, is divisible by each of the others?

Sol.: Such integers do not exist. Suppose a ≥ b ≥ c satisfy the desired condition. Since $a^2 -$ 1is divisible by b, the numbers a and b are relatively prime. Hence the number $c^2 - 1$, which is divisible by a and b, must be a multiple of ab, so in particular $c^2 - 1 \geq ab. But\ a \geq c\ and\ b \geq c, so\ ab \geq\ c^2$, a contradiction.

80. In isosceles triangle ABC (AB = BC) one draws the angle bisector CD. The perpendicular to CD through the center of the circumcircle of ABC intersects BC at E. The parallel to CD through E meets AB at F. Show that BE = FD.

Solution: We use directed angles modulo π. Let O be the circumcircle of ABC, and K the intersection of BO and CD. From the equality of the acute angles BOE and DCA having perpendicular sides, it follows that ∠BOE = ∠KCE (CD being an angle bisector), which means the points K, O, E, C lie on a circle. From this it follows that ∠OKE = ∠OCE; but ∠OCE = ∠OBE, so OB = OC, and hence ∠BKE = ∠KBE, or in other words BE = KE. Moreover, ∠BKE = ∠KBE = ∠KBA, and so KE ∥ AB. Consequently, FEKD is a parallelogram and DF = KE. Therefore, DF = KE = BE as desired.

81. Does there exist a finite set M of nonzero real numbers, such that for any natural number n a polynomial of degree no less than n with coefficients in M, all of whose roots are real and belong M?

Solution: Such a set does not exist. Suppose on the contrary that M = $\{a_1, a_1, \dots, a_n\}$ satisfies the desired property. Let m = min $\{|a_1|, \dots, |a_n|\}$ and M = max $\{|a_1|, \dots, |a_n|\}$; the condition implies M ≥ m > 0.

Consider the polynomial P(x) = $b_k x^k + \cdots + b_1 x + b_0$ all of whose coefficients $b_0, \dots, b_k$ are roots $x_1, \dots, x_k$ lie in M. By Vieta's theorem.

$$-\frac{b_{k-1}}{b_k} = x_1 + \cdots x_k$$

$$x_1x_2 + x_1x_3 + \cdots + x_{k-1}x_k = \frac{b_{k-2}}{b_k}$$

And so

$$x_1{}^2 + \cdots + x_k{}^2 = \frac{b_{k-1}{}^2}{b_k{}^2} - 2\frac{b_{k-2}}{b_k}.$$

It follows that

$$km^2 \le x_1{}^2 + \cdots + x_k{}^2 = \frac{b_{k-1}{}^2}{b_k{}^2} - 2\frac{b_{k-2}}{b_k} \le \frac{M^2}{m^2} + 2\frac{M}{m}.$$

Hence $k \le \frac{M^2}{m^4} + \frac{2M}{m^3}$, contradiction the fact that P may have arbitrarily large degree.

82. The natural numbers a and b are such that $\frac{a+1}{b} + \frac{b+1}{a}$ is an integer. Show that the greatest common divisor of a and b is not greater than $\sqrt{a+b}$.

Solution: Let d = gcd(a, b) and put a = md and b = nd. Then we have $\frac{(md+1)}{nd} + \frac{(nd+1)}{md} = \frac{(m^2d+m+n^2d+n)}{mnd}$ is an integer, so that in particular, d divides $m^2d + m + n^2d + n$ and also m + n. However, this means d ≤ m + n, and so d ≤ $\sqrt{d(m+n)} = \sqrt{a+b}$.

83. Let G be the centroid of the triangle ABC. Prove that if AB + GC = AC + GB, then ABC is isosceles.

Sol.: Let a, b, c, be the lengths of sides BC, CA, AB, respectively. By Stewart's theorem and the fact that G trisects each median (on the side further from the vertex), we deduce

$$9GB^2 = 2a^2 + 2c^2 - b^2, 9GC^2 = 2a^2 + 2b^2 - c^2.$$

Now assume b > c. Assuming AB + GC = AC + GB, we have

$$3(b-c) = \sqrt{2a^2 + 2b^2 - c^2} - \sqrt{2a^2 + 2c^2 - b^2}$$

$$= \frac{3(b^2 - c^2)}{\sqrt{2a^2 + 2b^2 - c^2} + \sqrt{2a^2 + 2c^2 - b^2}}$$

$$< \frac{3(b^2 - c^2)}{\sqrt{2\,(b-c)^2 + 2b^2 - c^2} + \sqrt{2\,(b-c)^2 + 2c^2 - b^2}}$$

Since $a^2 > (b-c)^2$ by the triangle inequality. However,

$2\,(b-c)^2 + 2b^2 - c^2 = (2b-c)^2$, so we have

$3(b-c) < \frac{3(b^2-c^2)}{2b-c+|2c-b|}$.

If b ≤2c then the two sides are equal, a contradiction. If b ≤ 2c we get $9(b-c)^2 < 3(b^2-c^2)$; upon dividing off 3(b –c) and rearranging, we get 2b < 4c, again a contradiction. Thus we cannot have b > c or similarly b < c, so b = c.

84. Find all real solutions of the equation

$$\sqrt{x^2 - p} + 2\sqrt{x^2 - 1} = x$$

For each real value of p.

Sol.: Squaring both sides, we get

$$x^2 = 5x^2 - 4 - p + 4\sqrt{(x^2-p)(x^2-1}$$

Isolating the radical and squaring again, we get

$$16(x^2-p)(x^2-1) = \ (4x^2-p-4)^2,$$

Which reduces to $(16-8p)x^2 = \ p^2 - 8p + 16$. Since x ≥ 0(it is the sum of two square roots), we have x = $\frac{|p-4|}{\sqrt{16-8p}}$

If a solution exists. We need only determine when this value actually satisfies. Certainly we need p ≤ 2. In that case plugging in our claimed value of x and multiplying through by $\sqrt{16-8p}$ gives │3p -4│+2│p│ = 4 –p.

If p ≥ $\frac{4}{3}$ this becomes 6p = 8, or p = $\frac{4}{3}$; if 0 ≤ p ≤ $\frac{4}{3}$ this holds identically; if p ≤ 0 this becomes 4p = 0, or p = 0. We conclude there exists a solution if and only if 0≤ p ≤ 4/3, in which case it is the solution given above.

85. At port Aventura there are 16 secret agents. Each agent is watching one or more other agents, but no two agents are both watching each other. Moreover, any 10 agents can be ordered so that the first a watching the second, the second is watching the third, etc. , and the last is watching the first. Show that any 11 agents can also be so ordered.

Sol.: We say two agents are partners if neither watches the other. First note that each agent watches at least 7 others; if an agent were watching 6 or fewer others, we could take away 6 agents and leave a group of 10 which could not be arranged in a circle. Similarly, each agent is watched by at least 7 others. Hence each agent is allied with at most one other.

Given a group of 11 agents, there must be one agent x who is not allied with any of the others in the group (since allies come in pairs). Remove that agent and arrange the other 10 in a circle. The Removed agent watches at least one of the other 10 and is watched by at least one. Thus there exists a pair, u, v of agents with u watching v, u watching x and x watching v (move around the circle until the direction of the arrow to x changes); thus x can be spliced into the loop between u and v.

86. Let $\prod_{n=1}^{1996}\left(1 + nx^{3^n}\right) = 1 + a_1x^{k_1} + a_2x^{k_2} + \cdots + a_mx^{k_m}$,

where $a_1, a_2, \ldots, a_m$ are nonzero and $k_1 < k_2 < \cdots < k_m$. Find a_{1996}.

Sol.: Note that k_i is the number obtained by writing i in base 2 and reading the result as a number in base 3, and a_i is the sum of the exponents of the powers of 3 used. In particular, $1996 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^3 + 2^2, so\ a_{1996} = 10 + 9 + 8 + 6 + 6 + 3 + 2 = 45$.

87. In a parallelogram ABCD with ∠A <90°, the circle with diameter AC meets the lines CB and CD again at E and F, respectively, and the tangent to this circle at A meets BD at P. Show that P, F, E are collinear.

Sol.: Without loss of generality, suppose B, D, P occur in that order along BD. Let G and H be the second intersection of AD and AB with the circle. By Menelaos's theorem, it suffices to show that

$$\frac{CE.BP.DF}{EB.PD.FC} = 1$$

Find note that

$\frac{BP}{AB}\frac{AD}{DP} = \frac{\sin\angle BAP \sin\angle APD}{\sin\angle APB \sin\angle DAP} = \frac{\sin\angle BAP}{\sin\angle DAP}$

Since AP is tangent to the circle, ∠BAP = ∠HAP

= π - ∠HCA = π -∠FCA; similarly, ∠DAP = ∠GCA = ∠EAC. We conclude

$\frac{BP}{AB}\frac{AD}{DP} = \frac{\sin\angle FAC}{\sin\angle EAC} = \frac{FC}{EC}$

Finally we note that $\frac{DF}{BE} = \frac{DA}{AB}$ because the right triangles AFD and AED have the same angles at B and D and are thus similar. This prove the claim.

88. Given real number s $0 = x_1 < x_2 < \cdots < x_{2n} < x_{2n+1} = 1$ with $x_{i+1} - x_i \leq h$ for 1 ≤ i ≤ 2n, show that $\frac{1-h}{2} < \sum_{i=1}^{n} x_{2i}\,(x_{2i+1} - x_{2i-1}) < \frac{1+h}{2}$.

Sol.: The different between the middle quantity and $\frac{1}{2}$ is the difference between the sum of the areas of the rectangles bounded by the lines x = $x_{2i-1}, x = x_{2i+1}, y = 0, y = x_{2i}$ and the triangle bounded by the lines y = 0, x = 1, x = y. The area contained in the rectangles but not the triangle is a union of triangles of total base less than 1 and height at most h, as is the area contained in the triangle but not the rectangles. Hence the sum differs from $\frac{1}{2}$ but at most $\frac{h}{2}$, as desired.

89. In a convex quadrilateral ABCD, triangles ABC and ADC have the same area. Let E be the intersection of AC and BD, and let the parallels through E to the lines AD, DC, CB, BA meet AB, BC, CD, DA at K, L, M, N, respectively. Compute the ratio of the areas of the quadrilaterals KLMN and ABCD.

Solution: The triangles EKL and DAC are homothetic, so the ratio of their areas equals $\left(\frac{EK}{AD}\right)\left(\frac{EL}{CD}\right) = \left(\frac{BE}{BD}\right)^2 = \frac{1}{4}$, since B and D are equidistant from the line AC. Similarly the ratio of the areas of EMN and BCA is $\frac{1}{4}$, so the union of the triangles EKL and EMN has area $\frac{1}{4}$ that of ABCD.

As for triangle EKN, its base KN is parallel to BD and half as long, so its area is one-fourth that of ABD. Similarly EML has area one-fourth that of BCD, and so the union of the two triangles EKN and EML has area one fourth that of ABCD, and so the quadrilateral KLMN has area one-half that of ABCD.

90. Find the maximum number of pair wise disjoint sets of the from $S_{a,b} = \{n^2 + an + b: n \in Z\} with\ a, b \in Z$.

Solution: Only two such sets are possible, for example, with (a, b) = (0, 0) and (0, 2) (since 2 is not a difference of squares). There is no loss of generality in assuming a ∈ {0, 1} by a suitable shift of n, and the sets generated by (0, a) and (1, b) have the common value $(a-b)^2 + a = (a-b)^2 + (a-b) + b$.Thus we have a = 0 or a = 1 universally.

First suppose a = 0. If b –c ≠ 2 (mod 4), then (0, b) and (0, c) gives a common value because b –c is a difference of squares, clearly this precludes having three disjoint sets. Now suppose a = 1. If b –c is even, we can find x, y such that b –c = (x +y +1) (x –y), and so $x^2 + x + b = y^2 + y + c$, again, this precludes having three disjoint sets.

91. For which ordered pairs of positive real numbers (a, b) is the limit of every sequence $\{x_n\}$ satisfying the condition.

$$\lim_{n \to \alpha}(ax_{n+1} - bx_n) = 0$$

Sol.: The holds if and only if b < a, if b > a, the sequence $x_n = \left(\frac{b}{a}\right)^n$ satisfies the condition but does not go to zero, if b = a, the sequence $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ does likewise. Now suppose b < a. If L and M are the limit inferior and limit superior of the given sequence, the condition implies M ≤ $\left(\frac{b}{a}\right) L$; since L ≤ M, we have M ≤ $\left(\frac{b}{a}\right) M$, and so L, M ≥ 0. Similarly, the condition implies L ≥ $\left(\frac{b}{a}\right) M$, and since M ≥ L, we have L ≥ $\left(\frac{b}{a}\right) L$, so L, M ≤ 0; therefore L = M = 0 and the sequence converges to 0.

92. Consider the pair of four –digit positive integers (M, N) = (3600, 2500). Notice that M and N are both perfect squares, with equal digits in two places, and differing digits in the remaining two places. Moreover, when the digits differ, the digit in M is exactly one greater than the corresponding digit in N. Find all pairs of four –digit positive integers (M, N) with these properties.

Sol.: If M = m^2 $and\ N = n^2$, then (m+ n) (m –n) ∈{11, 101, 110, 1001, 1010, 1100}.

Since M and N are four-digit numbers, we must have 32 ≤ n <m ≤ 99, and so 65 ≤ m +n ≤ 197. Moreover, m +n and m –n are both odd or both even, so 11, 110 and 1010 lead to no solutions. From this we get exactly five acceptable factorizations.

101 = (m+ n) (m –n) = 101 × 1

1001 = (m +n) (m –n) = 143 × 7

1001 = (m +n) (m –n) = 91 × 11

1001 = (m+ n) (m –n) = 77 × 13

1100 = (m+ n) (m –n) = 110 × 10

Giving the solutions (M, N) = (2601, 2500), (5625, 4624), (2601, 1600), (2025, 1024), (3600, 2500).

93. A function f defined on the positive integers satisfies f(1) = 1996 and $\boldsymbol{f(1) + f(2) + \cdots + f(n) = n^2 f(n)}(\boldsymbol{n} > 1)$.

Sol.: An easy induction will show that

$$f(n) = \frac{2 \times 1996}{n(n+1)}$$

Namely,

$$f(n) = \frac{1}{n^2-1}\left(\frac{3992}{1.2} + \cdots + \frac{3992}{(n-1)n}\right)$$

$$= \frac{3992}{n^2-1}\left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n}\right)$$

$$= \frac{3992}{(n+1)(n-1)}\left(1 - \frac{1}{n}\right)$$

$$= \frac{3992}{(n+1)(n-1)}\frac{n-1}{n} = \frac{3992}{n(n+1)}$$

In particular, f(1996) = $\frac{2}{1997}$

94. Define $\boldsymbol{q(n)} = \left\lfloor \frac{\boldsymbol{n}}{\lfloor\sqrt{\boldsymbol{n}}\rfloor} \right\rfloor$ $(\boldsymbol{n} = \mathbf{1}, \mathbf{2}, \ldots)$.

Determine all positive integers n for which q(n) > q(n +1).

Sol.: We have q(n) > q(n+1) if and only if n +1 is a perfect square. Indeed, if n +1 = m^2, then

$$q(n) = \left\lfloor \frac{m^2-1}{m-1} \right\rfloor = m+1, q(n+1) = \left\lfloor \frac{m^2}{m} \right\rfloor = m$$

On the other hand, for n = $m^2 + d$ with 0 ≤ d≤ 2m, $q(n) = \left\lfloor \frac{m^2+d}{m} \right\rfloor = m + \left\lfloor \frac{d}{m} \right\rfloor$

Which is non-decreasing.

95. Let a, b, c be positive real numbers.
  (a) Prove that $4(a^3+b^3) \geq (a+b)^3$
  (b) Prove that $9(a^3+b^3+c^3) \geq (a+b+c)^3$

Sol.: Both parts follow from the Power Mean inequality: for r > 1 and $x_1, \dots, x_n$ positive,

$$\left(\frac{{x_1}^r + \cdots + {x_n}^r}{n}\right)^{1/r} \geq \frac{x_1 + \cdots + x_n}{n},$$

Which in turn follows from Jensen's inequality applied to convex function x^r.

96. Find all Solutions in non-negative integers x, y, z of the equation.

$$\mathbf{2^x + 3^y = z^2}$$

Sol.: If y = 0, then $2^x = z^2 - 1 = (z+1)(z-1)$, so z +1 and z -1 are powers of 2. The only powers of 2 which differ by 2 are 4 and 2, so (x, y, z) = (3, 0, 3).

If y > 0, then 2^x is a quadratic residue modulo 3, hence x is even. Now we have $3^y = z^2 - 2^x = \left(z + 2^{\frac{x}{2}}\right)(z - 2^{\frac{x}{2}})$. The factors are powers of 3, say z+$2^{x/2} = 3^m$ and z $-2^{x/2} = 3^n$, but then $3^m - 3^n = 2^{\frac{x}{2}+1}$. Since the right side is not divisible by 3, we must have n = 0 and $3^m - 1 = 2^{\frac{x}{2}+1}$.

If x = 0, we have m = 1, yielding (x, y, z) = (0, 1, 2). Otherwise, $3^m - 1$ is divisible by 4, so m is even and $2^{\frac{x}{2}+1} = \left(3^{\frac{m}{2}}+1\right)\left(3^{\frac{m}{2}}-1\right)$. The two factors on the right are powers of 2 differing by 2, so they are 2 and 4, giving x = 4 and (x, y, z) = (4, 2, 5).

97. The sides a, b, c and u, v, w of two triangles ABC and UVW are related by the equations.

$$\boldsymbol{u(v+w-u) = a^2},$$

$$\boldsymbol{v(w+u-v) = b^2},$$

$$\boldsymbol{w(u+v-w) = c^2}.$$

Prove that ABC is acute, and express the angles U, V, W in terms of A, B, C.

Sol.: Note that $a^2 + b^2 - c^2 = w^2 - u^2 - v^2 + 2uv = (w+u-v)(w-u+v) > 0$ by the triangle inequality, so $\cos C > 0$. By this reasoning, all of the angles of triangle ABC are acute. Moreover,

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

$$= \sqrt{\frac{(w+u-v)(w-u+v)}{4uv}}$$

$$= \sqrt{\frac{w^2-u^2-v^2+2uv}{4uv}} = \frac{1}{\sqrt{2}}\sqrt{1-\cos U}$$

From which we deduce U = $1 - 2cos^2 A = \cos(\pi - 2A)$.

Therefore U = π -2A, and similarly V = π- 2B, W = π- 2C.

98. Two circles S_1 and S_2 touch each other externally at K; they also touch a circle S internally at A_1 and A_2, respectively. Let P be one point of intersection of S with the

common tangent to S_1 *and* S_2 at K. The line PA_1 *meets* S_1 *again at* B_1, and PA_2 meets S_2 again at B_2. Prove that B_1B_2 is a common tangent to S_1 *and* S_2.

Sol.: It suffices to show that $\angle B_2B_1O_1 = \angle B_1B_2O_2 = \frac{\pi}{2}$, where O_1 *and* O_2 are the centers of S_1 *and* S_2, respectively. By power – of-a-point. $PA_1.PB_1 = P\,K^2 = PA_2.PB_2$, so triangles PA_1A_2 *and* PB_2B_1 are similar. Therefore $\angle PB_1B_2 = \angle PA_2A_1 = \frac{1}{2}\angle POA_1$, where O is the center of S.

Now note that the homothety at A_1 carrying S_1 to S takes O_1 *to* O *and* B_1 *to* P, so $\angle POA_1 = \angle B_1O_1A_1$. From this we deduce $\angle PB_1B_2 = \angle B_1O_1N$, where N is the midpoint of A_1B_1. Finally, $\angle B_2B_1O_1 = \pi - \angle PB_1B_2 - \angle O_1B_1N = \frac{\pi}{2}$, as desired.

99. Find all solutions in positive real numbers a, b, c, d to the following system of equations:

a + b+ c + d = 12

abcd = 27 +ab +ac +ad +bc +bd +cd.

Sol.: The first equation implies abcd = ≤ 81 by the arithmetic geometric mean inequality, with equality holding for a = b = c = d = 3. Again by AM-GM,

abcd ≥ 27 +6 $(abcd)^{1/2}$

However, $x^2 - 6x - 27 \geq 0$ *for* $x \leq -3$ *or* $x \geq 9$, so $(abcd)^{1/2} \geq 9$, hence abcd ≥ 81. We conclude abcd = 81, and hence a = b= c= d= 3.

100. Prove that the average of numbers n $\sin n°\ (n = 2, 4, 6, \ldots, 180)$ *is* cot 1°.

Solution: All arguments of trigonometric functions will be in degrees. We need to prove

$$2\sin 2 + 4\sin 4 + \cdots + 178\sin 178 = 90\cot 1 \qquad (2)$$

Which is equivalent to

$$2\sin 2\sin 1 + 2(2\sin 4\,.\sin 1) + \ \ldots + 89\,(2\sin 178\,.\sin 1) = 90\cos 1. \qquad (3)$$

Using the identity $2\sin a\,.\sin b = \cos(a-b) - \cos(a+b)$, we find

$2\sin 2\,.\sin 1 + 2(2\sin 4\,.\sin 1) + \cdots + 89$

$(2\sin 178\,.\sin 1)$

$= (\cos 1 - \cos 3) + 2(\cos 3 - \cos 5) + \cdots + 89(\cos 177 - \cos 179)$

$= \cos 1 + \cos 3 + \cos 5 + \cdots + \cos 175$

$\cos 177 - 89\cos 179$

$$= \cos 1 + (\cos 3 + \cos 177) + \cdots + (\cos 89 + \cos 91) - 89\cos 179$$

$= \cos 1 + 89\cos 1 = 90\cos 1,$

So (1) is true.

Note: An alternate solution involves complex numbers. One expresses sin n as

$\frac{\left(e^{\frac{\pi i n}{180}} - e^{\frac{-\pi i n}{180}}\right)}{(2i)}$ *and uses the fact that*

$x + 2x^2 + \cdots + nx^n = (x + \cdots + x^n) + (x^2 + \cdots + x^n) + \cdots + x^n$

$= \frac{1}{x-1}[(x^{n+1} - x) + (x^{n+1} - x^2) + \cdots + (x^{n-1} - x^n)]$

$= \frac{nx^{n+1}}{x-1} - \frac{x^{n+1} - x}{(x-1)^2}.$

101. For any nonempty set S of real numbers, let σ(S) denote the sum of the

elements of S. Given a set A of n positive integers, consider the collection of all distinct sums σ(S) as S ranges over the nonempty subsets of A. Prove that this collection of sums can be partitioned into n classes so that in each classes, the ratio of the largest sum of the smallest sum does not exceed 2.

Sol.: Let A = $\{a_1, a_2, \dots, a_n\}$ where $a_1 < a_2 < \cdots < a_n$. For i = 1, 2, …, n let $s_i = a_1 + a_2 + \cdots + a_i$ $and\ take\ s_0 = 0$. All the sums is question are less than or equal to s_n, and if σ is one of them, we have

$$s_{i-1} < \sigma < s_i \qquad (1)$$

For an appropriate i. Divide the sums into n classes by letting C_i denote the class of sums satisfying (1). We claim that these classes have the desired property. To establish this, it suffices to show that (1) implies.

$$\frac{1}{2}s_i < \sigma < s_i \qquad (2)$$

Suppose (1) holds. The inequality $a_1 + a_2 + \cdots + a_{i-1} = s_{i-1} < \sigma$ shows that the sum σ contains at least one addend a_k $with\ k \geq i.$ Then since then $a_k \geq a_i$, we have

$s_i - \sigma < s_i - s_{i-1} = a_i \leq a_k \leq \sigma$, which together with $\sigma \leq s_i$ implies (2).

Note: The result does not hold if 2 is replaced by any smaller constant c. To see this, choose n such that $c < 2 - 2^{-(n-1)}$ and consider the set $\{1, \dots, 2^{n-1}\}$. If this set is divided into n subsets, two of 1,……, 2^{n-1}, 1 + …….+ 2^{n-1} must lie in the subset, and their ratio is at least (1+…..+ $2^{n-1})/(2^{n-1}) = 2 - 2^{(n-1)}$ > c.

102. Let ABC be a triangle. Prove that there is a line l (in the plane of triangle ABC) such that the intersection of the interior of triangle ABC and the interior of its reflection A'B'C' in l has area more than $\frac{2}{3}$ the area of triangle ABC.

Solution :

In all of the solutions, a, b, c denote the lengths of the sides BC, CA, AB, respectively, and we assume without loss of generality that a ≤ b ≤ c.

Choose l to be the angle bisector of ∠A. Let P be the intersection of l with BC. Since AC ≤ AB, the intersection of triangles ABC and A'B'C' is the disjoint union of two congruent triangles. APC and APC'. Considering BC as a base, triangles APC and ABC have equal altitudes, so their areas are in the same are in the same ratio as their bases:

$$\frac{Area\ (APC)}{Area\ (ABC)} = \frac{PC}{BC}.$$

Since AP is the angle bisector of ∠A, we have $\frac{BP}{PC} = \frac{c}{b}$, so

$$\frac{PC}{BC} = \frac{PC}{BP + PC} = \frac{1}{\frac{c}{b} + 1}$$

Thus it suffices to prove

$$\frac{2}{\frac{c}{b} + 1} > \frac{2}{3}. \qquad (1)$$

$But\ 2b \geq a + b > c$ by the triangle inequality, so $\frac{c}{b} < 2$ and thus (1) holds.

103. An n –term sequence $\{x_1, x_2, \dots, x_n\}$ in which each term is either 0 or 1 is called a binary sequence of length n. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n

that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n.

Sol.: We refer to the binary sequences counted by (a_n) and (b_n) as "type A" and "type B", respectively. For each binary sequence $(x_1, x_2, \dots, x_n)$ there is a corresponding binary sequence $(y_0, y_1, \dots, y_n)$ obtained by setting $y_0 = 0\ and\ y_1 = x_1 + x_2 + \cdots + x_i$ mod 2, i = 1, 2, …, n. (2)

(Addition mod 2 is defined as follows: 0 +0 = 1 +1 = 0 and 0 + 1 = 1 +0 = 1.) Then

$$x_i = y_i + y_{i-1}\ mod\ 2, i = 1, 2, \dots, n,$$

And it is easily seen that (1) provides a one-to – one correspondence between the set of all binary sequences of length n and the set of binary sequences of length n +1 in which the first term is 0. Moreover, the binary sequence $(x_1, x_2, \dots, x_n)$ has three consecutive terms equal 0, 1, 0 in that order if and only if the corresponding sequence $(y_0, y_1, \dots, y_n)$ has four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order, so the first is of type A if and only if the second is of type B. The set of Type B sequences of length n +1 in which the first term is 0 is exactly half the total number of such sequences, as can be seen by means of the mapping in which 0's and 1's are interchanged.

104. Triangle ABC has the following property: there is an interior point P such that ∠PAB = 10°, $\angle PBA = 20°, \angle PCA = 30°, and\ \angle PCA = 40°$. Prove that triangle ABC is isosceles.

First Solution :

All angles will be in degrees. Let x = ∠PCB. Then ∠PBC = 80 –x. By the Law of Sines,

$$1 = \frac{PA\ PB\ PC}{PB\ PC\ PA} = \frac{\sin\angle PBA \sin\angle PCB \sin\angle PAC}{\sin\angle PAB \sin\angle PBC \sin\angle PCA}$$

$$= \frac{\sin 20 \sin x \sin 40}{\sin 10 \sin(80-x) \sin 30} = \frac{4 \sin x \sin 40 cos\ 10}{\sin(80-x)}.$$

The identity $2\sin a . \cos b = \sin(a-b) + \sin(a+b)$now yields

$$1 = \frac{2\sin x(\sin 30 + \sin 50)}{\sin(80-x)} = \frac{\sin x(1+2\cos 40)}{\sin(80-x)},$$

So,

$$2\sin x \cos 40 = \sin(80-x) - \sin x = 2\sin(40-x)\cos 40.$$

This gives x = 40 –x and thus x = 20. It follows that ∠ACB = 50 = ∠BAC, so triangle ABC is isosceles.

105. Solve the system of equations:

$$\sqrt{3x}\left(1 + \frac{1}{x+y}\right) = 2$$
$$\sqrt{7y}\left(1 - \frac{1}{x+y}\right) = 4\sqrt{2}$$

Sol.: Let u =$\sqrt{x}, y = \sqrt{y}$, so the system becomes

$$u + \frac{u}{u^2+v^2} = \frac{2}{\sqrt{3}}$$

$$v - \frac{v}{u^2+v^2} = \frac{4\sqrt{2}}{\sqrt{7}}.$$

Now let z = u +vi; the system then reduces to the single equation

$$z + \frac{1}{z} = 2\left(\frac{1}{\sqrt{3}} + \frac{2\sqrt{2}}{\sqrt{7}} i\right).$$

Let t denote the quantity inside the parentheses; then

$$z = t \pm \sqrt{t^2 - 1}$$

$= \frac{1}{\sqrt{3}} + \frac{2\sqrt{2}}{\sqrt{7}} i \pm \left(\frac{2}{\sqrt{21}} + \sqrt{21}\right)$

From which we deduce

$$u = \left(\frac{1}{\sqrt{3}} \pm \frac{2}{\sqrt{21}}\right)^2, v = \left(\frac{2\sqrt{2}}{\sqrt{7}} \pm \sqrt{2}\right)^2.$$

106. Let ABCD be a tetrahedron with AB = AC = AD and circumcenter O. Let G be the centroid of triangle ACD, let E be the midpoint of BG, and let F be the midpoint of AE. Prove that OF is perpendicular to BG if and only if OD is perpendicular to AC.

Sol.: We identify points with their vectors originating from the circumcenter, so that A. B = A. C = A.D and

$$A^2 = B^2 = C^2 = D^2.$$
$$Now\ (O - F).(B - G)$$
$$= \frac{1}{2}(A + E).(B - G)$$
$$= \frac{1}{4}[(2A + B + G).(B - G)]$$
$$= \frac{1}{36}[18A.B - 6A.(A + C + D) + 9B^2 - (A + C + D)^2]$$
$$= \frac{1}{36}[2A.D - 2C.D]$$

Therefore OF ⊥ BF if and only if OD ⊥ AC.

107. Determine, as a function of n, the number of permutations of the set {1, 2, …, n} such that no three of 1, 2, 3, 4 appear consecutively.

Sol.: There are n! permutations in all. Of those, we exclude (n -2)! Permutations for each arrangement of 1, 2, 3, 4 into an ordered triple and one remaining element, or 24(n -2)! in all. However, we have twice excluded each of the 24(n-3)! Permutations in which all four of 1, 2, 3, 4 occur in a block. Thus the number of permutations of the desired from is n! -24 (n - 2)! + 24(n -3)!

108. Determine all function f: ℕ ⟶ℕ satisfying (for all n ∈ℕ)

$$\boldsymbol{f(n) + f(n+1) = f(n+2)f(n+3) - 1996}.$$

Sol.: From the given equation, we deduce

$$f(n) - f(n+2) = f(n+3)[f(n+2) - f(n+4)]$$

If f(1) > f(3), then by induction, f(2m -1) > f(2m +1) for all m > 0, giving an infinite decreasing sequence f(1), f(3), …. Of positive integers, a contradiction. Hence f(1) ≤ f(3), and similarly f(n) ≤ f(n +2) for all n.

Now note that

$$0 = 1996 + f(n) + f(n+1) - f(n+2)f(n+3)$$
$$\le 1996 + f(n+2) + f(n+3) - f(n+2)f(n+3)$$
$$= 1997 - [f(n+2) - 1][f(n+3) - 1].$$

In particular, either f(n +2) = 1 or f(n +3) ≤ 1997, and vice versa. The numbers f(2m+1) – f(2m-1) are either all zero or all positive, and similarly for the numbers f(2m+ 2) –f(2m). If they are both positive, eventually f(n +2) and f(n+3) both exceed 1997, a contradiction.

We now split into three cases. If f(2m) and f(2m +1) are both constant, we have [f(2m) - 1][f(2m+1)-1] = 1997 and so either f(2m) = 1 and f(2m +1) = 1997 or vice versa. If f(2m +1) is constant but f(2m) is not, then $f(2m + 1) = 1$

for all m and $f(2m+2) = f(2m) + 1997, so\ f(2m) = 1997(m-1) + f(2)$.

Similarly, if f(2m) is not constant, then f(2m)= 1 and f(2m +1) = 1997m + f(1).

109. Consider triangles ABC where BC = 1 and ∠BAC has a fixed measure $\boldsymbol{\alpha > \frac{\pi}{3}}$. Determine which such triangle minimizes the distance between the incenter and centroid of ABC, and compute this distance in terms of $\boldsymbol{\alpha}$.

Sol.: If we fix B and C and force A to lie above the line BC, then A is constrained to an arc. The centroid of ABC is constrained to the image of that arc under a $\frac{1}{3}$ homothety at the midpoint of BC. On the other hand, the incenter subtends an angle of $\frac{(\pi/\alpha)}{2}$ at BC, so it is also constrained to lie on an arc, but its arc passes through B and C. Since the top of the incenter arc lies above the top of the centroid arc, the arcs cannot intersect (or else their circles would intersect four times). Moreover, if we dilate the centroid arc about the midpoint of BC so that its image is tangent to the incenter arc at its highest point, the image lies between the incenter arc and BC. In other words, the distance from the incenter to the centroid is always at least the corresponding distance for ABC isosceles. Hence we simply compute the distance for ABC isosceles. Hence we simply compute the distance in that case. The incenter makes an isosceles triangle of vertex angle $\frac{(\pi/\alpha)}{2}$, so its altitude is $\frac{\frac{1}{2\cot(\pi-\alpha)}}{4}$.

Meanwhile, the distance of the centroid to BC is $\frac{1}{3}$ that of A to BC, or $\frac{1}{6\cot(\alpha/2)}$. The desired distance is thus

$\frac{1}{2}\cot\frac{\pi+\alpha}{4} - \frac{1}{6}\cot\frac{\alpha}{2}$.

110. Let a, b, c, d be four nonnegative real numbers satisfying the condition

$$\boldsymbol{2(ab+ac+ad+bc+bd+cd) + abc+abd+acd + bcd = 16}$$

Prove that

$$\boldsymbol{a+b+c+d \geq \frac{2}{3}(ab+ac+ad + bc+bd+cd)}$$

And determine when equality occurs.

Sol.: For i = 1, 2, 3, define s_i as the average of the products of the i-element subsets of {a, b, c, d}. Then we must show

$$3s_2 + s_3 = 4 \Rightarrow s_1 \geq s_2.$$

It suffices to prove the (unconstrained) homogeneous inequality

$$3{s_2}^2{s_1}^2 + s_3{s_1}^3 \geq 4{s_2}^3,$$

As then $3s_2 + s_3 = 4$ will imply

$$(s_1 - s_2)^3 + 3({s_1}^3 - {s_2}^3) \geq 0.$$

We now recall two basic inequalities about symmetric means of nonnegative real numbers. The first is Schur's inequality:

$$3{s_1}^3 + s_3 \geq 4s_1s_2.$$

While the second,

$${s_1}^2 \geq s_2$$

Is a case of Maclaurin's inequality ${s_i}^{i+1} \geq {s_{i+1}}^i$.These combine to prove the claim:

$$3{s_2}^2{s_1}^2 + s_3{s_1}^3 \geq 3{s_2}^2{s_1}^2 + \frac{{s_2}^2s_3}{s_1} \geq 4{s_2}^3.$$

Finally, for those who have only seen Schur's inequality in three variables, note that in general any inequality involving $s_1, \dots, s_k$ which holds for n ≥ k variables also holds for n +1 variables, by replacing the variables $x_1, \dots, x_{n+1}$ by the roots of the derivative of the polynomial $(x - x_1) \dots (x - x_{n-1})$.

111. Let ABCD be a quadrilateral with AB = BC = CD = DA. Let MN and PQ be two segments perpendicular to the diagonal BD and such that the distance between them is d >$\frac{BD}{2}$, with M ∈ AD, N ∈ DC, P ∈AB, and Q ∈ BC. Show that the perimeter of the hexagon AMNCQP does not depend on the positions of MN and PQ so long as the distance between them remains constant.

Solution: The lengths of AM, MN, NC are all linear in the distance between the segments MN and AC; if this distance is h, extrapolating from the extremes MN = AC and M = N = D gives that

$$AM + MN + NC = AC + \frac{2AB - AC}{BD/2}$$

In particular, if the segments MN and PQ maintain constant total distance from AC, as they do if their distance remains constant, the total perimeter of the hexagon is constant.

112. Let m and n be positive integers such that n ≤ m. Prove that

$$\boldsymbol{2^n n! \leq \frac{(m+n)!}{(m-n)!} \leq (m^2+m)^n}.$$

Sol.: The quantity in the middle is $(m+n)(m+n-1)\dots(m-n+1)$. If we pair off terms of the form $(m+x) and (m+1-x)$, we get products which do not exceed $m(m+1)$, since the function $f(x) = (m+x)(m+1-x)$ is a concave parabola with maximum at $x = \frac{1}{2}$. From this the right inequality follows. For the left, we need only show $(m+x)(m+1-x) \geq 2x$ for x ≤n; this rearranges to $(m-x)(m+1+x) \geq 0$, which holds because $m \geq n \geq x$.

113. Let $\boldsymbol{P_1, P_2, P_3, P_4}$ be four points on a circle, and let $\boldsymbol{I_1}$ be the incenter of the triangle $\boldsymbol{P_2P_3P_4}, \boldsymbol{I_2}$ be the incenter of the triangle $\boldsymbol{P_1P_3P_4}, \boldsymbol{I_3}$ be the incenter of the triangle $\boldsymbol{P_1P_2P_4}$ and $\boldsymbol{I_4}$ be the incenter of the triangle $P_1P_2P_3$. Prove that I_1, I_2, I_3 and I_4 are the vertices of a rectangle.

Sol.: Without loss of generally, assume P_1, P_2, P_3, P_4 occur on the circle in the order. Let $M_{12}, M_{23}, M_{34}, M_{41}$ be the midpoints of arcs $P_1P_2, P_2P_3, P_3P_4, P_4P_1$, respectively.

Then the line P_3M_1 is the angle bisector of $\angle P_2P_3P_1$ and so passes through I_4. Moreover, the triangle M_{12}, P_2I_4 is isosceles because $\angle I_4M_{12}P_2 = \angle P_1P_2P_3$

$$= \pi - 2\angle P_1P_2I_4 - 2\angle M_{12}P_2P_1$$

$$= \pi - 2\angle M_{12}P_2I_4$$

Hence the circle centered at M passing through P_1 and P_2also passes through I_4, and likewise through I_3.

From this we determine that the angle bisector of $\angle P_3M_{12}P_4$ is the perpendicular bisector of I_3I_4. On the other hand, this angle bisector passes through M_{34}, so it is simply the line $M_{12}M_{34}$; by symmetry, it is also the perpendicular bisector of I_1I_2. We conclude that $I_1I_2I_3I_4$ is a parallelogram.

To show that $I_1I_2I_3I_4$ is actually a rectangle, it now suffices to show that $M_{12}M_{34} \perp M_{23}M_{41}$. To see this, simply note that the angle between these lines is half the sum of measure of the arcs $M_{12}M_{23}$ and $M_{34}M_{41}$, but these arcs clearly comprise half of the circle.

114. The national Marriage Council wishes to invite n couples to from 17 discussion groups under the following conditions:

(a) All members of the group must be of the same sex, i.e. they are either all male or all female.

(b) The difference in the size of any two groups is either 0 or 1.

(c) All groups have at least one number.

(d) Each person must belong to one and only one group.

Find all values of n, n ≤ 1996, for which this is possible. Justify your answer.

Sol.: Clearly n ≤ 9 since each of 17 groups must contain at least one member. Suppose there are k groups of men and 17 –k groups of women; without loss of generality, we assume k ≤ 8. If m is the minimum number of members in a group, then the number of men of women is at most $k(m+1)$, while the number of women is at least $(k+1)m$. As there are the same number as men as women, we have $k(m+1) \geq (k+1)m, so\ m \leq k \leq 8$, and the maximum number of couples is $k(k+1) \leq 72$. In fact, any number of couples between 9 and 72 can be distributed: divide the men as evenly as possible into 8 groups, and divide the women as evenly as possible into 9 groups. Thus 9 ≤ n ≤ 72 is the set of acceptable numbers of couples.

115. Let a, b and c be the lengths of the sides of a triangle. Prove that

$$\sqrt{\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c}}+\sqrt{\boldsymbol{b}+\boldsymbol{c}-\boldsymbol{a}}+\sqrt{\boldsymbol{c}+\boldsymbol{a}-\boldsymbol{b}} \leq \sqrt{\boldsymbol{a}}+\sqrt{\boldsymbol{b}}+\sqrt{\boldsymbol{c}}$$

Sol.: By the triangle inequality, $b+c-a\ and\ c+a-b\ abc$ positive. For any positive x, y, we have

$$2(x+y) \geq x+y+2\sqrt{xy} = \left(\sqrt{x}+\sqrt{y}\right)^2$$

By the AM-GM inequality, with equality for x = y. Substituting $x=a+b-c, y=b+c-a$ we get

$$\sqrt{a+b-c}+\sqrt{b+c-a} \leq 2\sqrt{a},$$

Which added to the two analogous inequalities yields the desired result. Inequality holds for $a+b-c=b+c-a=c+a-b, i.e. a=b=c$.

116. Let k ≥1 be an integer. Show that there are exactly 3^{k-1} positive integers n with the following properties:

(a) The decimal representation of n consists of exactly k digits.

(b) All digits of k are odd.

(c) The number n is divisible by 5.

(d) The number m = $\frac{n}{5}$ has k odd (decimal) digits.

Sol.: The multiplication in each place must produce an even number of carries, since these will be added to 5 in the next place and an odd digit must result. Hence all of the digits of m must be 1, 5 or 9 and the first digit must be1, since m and n have the same number of decimal digits. Hence there are 3^{k-1} choices for m and hence for n.

117. A convex hexagon ABCDEF satisfies the following conditions:

(a) Opposite sides are parallel (i.e. AB ∥DE, BC ∥EF, CD ∥FA).

(b) The distances between opposite sides are equal (i.e. d(AB, DE)= d(BC, EF)= d(CD, FA), where d(g, h) denotes the distance between lines g and h).

(c) The angles ∠FAB and ∠CDE are right.

Show that diagonals BE and CF intersects at an angle of $45°$.

Sol.: The conditions imply that A and D are opposite vertices of a square APDQ such that B, C, E, F lie on AP, PD, DQ, QA, respectively and that all six sides of the hexagon are tangent to the inscribed circle of the square. The diagonals BE and CF meet at the center O of the square. Let T, U, V be the feet of perpendiculars from O to AB, BC, CD; then ∠TOB = ∠BOU by reflection across OB, and similarly ∠UOC = ∠COV. Therefore $\frac{\pi}{2} = 2\angle BOC$, proving the claim.

118. The polynomials $P_n(x)$ are defined by $P_0(x) = 0, P_1(x) = x$ and $P_n(x) = xP_{n-1}(x) + (1-x)P_{n-2}(x)\ \ n \geq 2$. For every natural number n ≥ 1, find all real numbers x satisfying the equation $P_n(x) = 0$.

Sol.: One shows by induction that

$$P_n(x) = \frac{x}{x-2}[(x-1)^n - 1]$$

Hence $P_n(x) = 0$ if and only if x = 0 or $x = 1 + e^{2\pi ik/n}$ for some k ∈ {1, …, n -1}.

119. The real numbers x, y, z, t satisfy the equalities $x + y + z + t = 0$ and $x^2 + y^2 + z^2 + t^2 = 1$. Prove that $-1 \leq xy + yz + zt + tx \leq 0$.

Sol.: The inner expression is $(x+z)(y+t) = -(x+z)^2$, so the second inequality is obvious. As for the

first, note that

$1 = (x^2 + z^2) + (y^2 + t^2) \geq \frac{1}{2}[(x+z)^2 + (y+t)^2] \geq [(x+z)(y+t)]$

By two applications of the power mean inequality.

120. A convex polyhedron P and a sphere S are situated in space such that S intercepts on each edge AB of P a segment XY with AX = XY = YB = $\frac{1}{3}AB$. Prove that there exists a sphere T tangent to all edges of P.

Sol.: Let AB and BC be two edges of the polyhedron, so that the sphere meets AB in a segment XY with AX = XY = YB and meets BC in a segment ZW with BZ = ZW = WC. In the plane ABC the points X, Y, Z, W lie on the cross-section of the sphere, which is a circle. Therefore BY. BX = BZ. BW by power –of-a point; this clearly implies AB = BC, and so the center of S is equidistant from AB and BC. We conclude that any two edges of P are equidistant from S and so there is a sphere concentric with S tangent to all edges.

121. Natural numbers k, n are given such that 1 < k < n. Solve the system of n equations.

$$x_i^3(x_i^2 + \cdots + x_{i+k-1}^2) = x_{i-1}^2 \quad 1 \leq i \leq n$$

in n real unknowns $x_1, \ldots, x_n$. (Note: $x_0 = x_n, x_1 = x_{n+1}, etc.$)

Sol.: The only solution is $x_1 = \cdots x_n = k^{-1/3}$. Let L and M be the smallest and largest of the x_i, respectively. If M = x_i, then

$$kM^3L^2 \leq x_i^3(x_i^2 + \cdots + x_{i+k-1}^2) = x_{i-1}^2 \leq M^2$$

And so M ≤ $\frac{1}{(kL^2)}$. Similarly, if L = x_j, then

$$kL^3M^3 \geq x_1^3(x_i^2 + \cdots + x_{i-k+1}^2) = x_{i-1}^2 \geq L^2$$

and so $L \geq \frac{1}{(kM^2)}$. Putting this together, we get

$L \geq \frac{1}{kM^2} \geq kL^4$

And so $L \geq k^{-1/3}; similarly, M \geq \ k^{-1/3}$. Obviously L ≤ M, so we have L = M = $k^{-1/3}\ and\ x_1 = \cdots = x_n = \ k^{-1/3}$.

122. Shows that there do not exist nonnegative integers k and m such that k! +48 = 48 $(\boldsymbol{k}+\mathbf{1})^{\boldsymbol{m}}$.

Sol.: Suppose such k, m exist. We must have 48 │k!, so k ≥ 6; one checks that k = 6 does not yield a solution, so k ≥ 7. In that case k! is divisible by 32 and by 9, so that $\frac{(k!+48)}{48}$ is relatively prime to 6, as then is k +1.

If k +1 is not prime, it has a prime divisor greater than 3, but this prime divides k! and not k! +48. Hence k +1 is prime, and by Wilson's theorem k! +1 is a multiple of k +1. Since k! +48 is as well, we find k +1 = 47, and we need only check that $\frac{46!}{48+1}$ is not a power of 47. We check that $\frac{46!}{48+1} = 29(mod\ 53)$ (by cancelling as many terms as possible in 46! Before multiplying), but that 47 has order 13 modulo 53 and that none of its powers is congruent to 29 modulo 53.

123. We are given a collection of rectangular bricks, no one of which is cube. The edge lengths are integers.

For every triple of positive integers (a, b, c), not all equal, there is sufficient supply of $\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c}$ bricks. Suppose that the bricks completely tile a $\mathbf{10} \times \mathbf{10} \times \mathbf{10}$ box.

(a) Assume that at least 100 bricks have been used. Prove that there exist at least two parallel bricks, that is, if AB is an edge of one of the bricks, A'B' is an edge of the other and AB ||A'B' then AB = A'B'.

(b) Prove the same statement with 100 replaced by a smaller number. The smaller the number, the better the solution.

Sol.: We prove the claim with 97 bricks. For each integer up to 16, we tabulate the number of nonparallel bricks that volume (disallowing cubical bricks and bricks with a dimension greater than 10) and their total volume:

Volume	2 3 4 5 6 7 8 9 10 12 14 15 16
Number	3 3 6 3 9 3 9 6 9 15 6 6 12
Total	6 9 24 15 54 21 72 54 90 180 74 90 192

Assuming no two bricks are parallel, the 90 smallest bricks have total volume 891. The 7 other bricks each have volume at least 18, giving a total volume of at least 1017, a contradiction.

We have not determined the optimal constant (one can improve the above bound to 96 easily), but we note that an arrangement with 73 nonparallel bricks is possible.

124. Let O and G be the circumcenter and centroid, respectively, of triangle ABC, If R is the circumradius and r the inradius of ABC, show that

$$\boldsymbol{OG} \leq \sqrt{\boldsymbol{R}(\boldsymbol{R}-\mathbf{2}\boldsymbol{r})}.$$

Sol.: Using vectors with original at O, we note that $OG^2 = \frac{1}{9}(A+B+C)^2 = \frac{1}{3}R^2 + \frac{2}{9}R^2(\cos 2A + \cos 2B + \cos 2C)$.

Hence $R^2 - OG^2 = \frac{(a^2+b^2+c^2)}{9}$. On the other hand, by the standard area formula K = $rs = \frac{abc}{4R}$, we have $2rR = \frac{abc}{(a+b+c)}$. We now note that

$$(a^2 + b^2 + c^2)(a + b + c) \geq 9abc$$

By two applications of the AM-GM inequality, so $2rR \leq R^2 - OG^2$, proving the claim.

125. Let ABCDE be a convex pentagon, and let M, N, P, Q, R be the midpoints of sides AB, BC, CD, DE, EA, respectively. If the segments AP, BQ, CR, DM have a common point, show that this point also lies on EN.

Sol.: Let T be the common point, which we take as the origin of a vector system. Then $A \times P = 0$, or equivalently $A \times (C + D) = 0$, which we may write $A \times C = D \times A$.

Similarly, we have $B \times D = E \times B, C \times E = A \times C, D \times A = B \times D$. Putting these equalities together gives $E \times B = C \times E, or\ E \times (B + C) = 0$, which means the line EN also passes through the origin T.

126. Show that there exists a subset A of the set {1, 2, …, 1996} having the following properties:

(a) 1, $\mathbf{2^{1996} - 1} \in A$;
(b) Every element of A, except 1, is the sum of two (not necessarily distinct) elements of A;
(c) A contains at most 2012 elements.

Sol.: We state the problem a bit differently: we want to write down at most 2012 numbers, starting with 1 and ending with $2^{1996} - 1$, such that every number written is the sum of two numbers previously written. If $2^n - 1$ has been written, then $2^n(2^{n-1})$ can be obtained by n doublings, and $2^n - 1$ can be obtained in one more step.

Hence we can obtain $2^2 - 1, 2^4 - 1, \dots, 2^{256} - 1\ in\ (1 + 1) + (2 + 1) + \dots + (128 + 1) = 263$ steps. In 243 steps, we turn $2^{256} - 1\ into\ 2^{499} - 2^{243}$. Now notice that the numbers $2^{243} - 2^{115}, 2^{115} - 2^{51}, 2^{51} - 2^{19}, 2^{19} - 2^3, 2^3 - 2^1, 2^1 - 1$ have all be written down; in 6 steps, we now obtain $2^{499} - 1$. We make this into $2^{998} - 1$ in 500 steps, and make $2^{1996} - 1$ in 999 steps. Adding 1 for the initial 1, we count

$$1 + 263 + 243 + 6 + 500 + 999 = 2012$$

Numbers written down, as desired.

127. Let $\mathbb{Z}^+$ denote the set of nonzero integers. Show that an integer p > 3 is prime if an only if for any a, b ∈ $\mathbb{Z}^+$, exactly one of the numbers

$$\boldsymbol{N_1 = a + b - 6ab + \frac{p-1}{6}},$$
$$\boldsymbol{N_2 = a + b + 6ab + \frac{p+1}{6}}$$

belongs to $\mathbb{Z}^+$.

Sol.: If $N_1 = 0$, then $p = (6a - 1)(6b - 1)$ is composite; similarly, $N_2 = 0\ implies\ p = -(6a + 1)(6b + 1)$ is composite. Conversely, suppose that p is composite. If p ≡ 0, 2, 3 or 4 (mod 6), then $N_1\ and\ N_2$ are not integers.

Otherwise, all divisors of p are congruent to ± 1 (mod 6). So there exist natural numbers c, d such that

$$p = (6c + 1)(6d + 1) or (6c - 1)(6d - 1) or (6c + 1)(6d - 1).$$

In the first case, N_2 is not an integer and $N_1 = 0$ for $a = -c, b = -d$.

In the second case, N_2 is not an integer and $N_1 = 0$ for $a = c, b = d$.

In the third case, N_1 is not an integer and N_2= 0 for $a = c, b = -d$.

128. Let M be a nonempty set and *a binary operation on M. That is, to each pair (a, b) ∈$\boldsymbol{M} \times \boldsymbol{M}$ one assigns an element a * b. Suppose further that for any a, b ∈M, (a* b)*b = a and a*(a* b) = b.

(a) Show that a *b = b *a for all a, b ∈M.

(b) For which finite sets M does such a binary operation exist?

Sol.:

(a) First note that [a* (a* b)] *(a* b) = a by the first rule. By the second rule, we may rewrite the left side as b* (a *b), so b* (a *b) = a and so b *a = b*[b* (a* b)]. b* a = b*[b*(a* b)]. By the second rule this equals a *b, so a* b = b* a.

(b) Such sets exist for all finite sets M. Identify M with {1, …, n} and define a* b = c ⟺a + b+ c = 0 (mod n). It is immediate that the axioms are satisfied.

129. Determine whether there exist a function f: ℤ⟶ℤ such that for each k = 0, 1, …, 1996 and for each m ∈ℤ the equation $\boldsymbol{f(x) + bx = m}$ has at least one solution x ∈ℤ.

Sol.: Each integer y can written uniquely as 1997m +k with m ∈ℤ and k ∈ {0, …, 1996}. Define the function f by $f(y) = m - ky$; then $f(x) + kx = m$ has the solution $x = 1997m + k$, so the condition satisfied.

130. Two sets of intervals A, B on a line are given. The set A contains $\boldsymbol{2m-1}$ intervals, every two of which have a common interior point. Moreover, each interval in A contains at least disjoint intervals of B. Show that there exists an interval in B which belongs to at least m intervals from A.

Sol.: Let $\alpha_1 = [a_i, b_i](i = 1, \dots, 2m-1)$ be the intervals, indexed so that $a_1 \le a_2 \le \cdots \le a_{2m-1}$. Choose k ∈$\{m, \dots, 2m-1\}$ to minimize b_k. By assumption, the interval α_k contains two disjoint intervals from B, say $\beta_1 = [c_1, d_1]$ *and* $\beta_2 = [c_2, d_2]$. Without loss of generality, assume

$$a_k \le c_1 < d_1 < c_2 < d_2 \le b_k.$$

If $d_1 \le b_i$ for i = 1, 2, …, m, then $\beta_1 \subset \alpha_1$ for i = 1, 2, …, m, so β_1 satisfies the desired property. Otherwise, $d_1 > b_x$ for some s ∈ {1, 2, …, m}. By assumption, $c_2 > d_1 > b_8$. Since no two of the α are disjoint, we have $b_8 \ge a_i$ for all i, so $c_2 > a_i$. On the other hand, by the choice of $k, b_k < b_1$ for i = m, …, $2m_1$. Therefore $a_i < c_2 < d_2 \le b_k \le b_i$ for each $i \in \{m, m+1, \dots, 2m-1\}, and\ so\ \beta_2$ has the desired property.

131. The points E and D lie in the interior of sides AC and BC respectively, of a triangle ABC. Let F be the intersection of the lines AD and BE. Show that the area of triangles ABC and ABF satisfies.

$$\boldsymbol{\frac{S_{ABC}}{S_{ABF}} = \frac{|AC|}{|BE|} + \frac{|BC|}{|BD|} - 1}$$

Sol.: Let the line parallel to BC through F meet AB at K and AC at N, Let the line parallel to CA through F meet BC at ME and AB at P; let the line parallel to AB through F meet BC at L and CA at O. Let v_C *and* v_F be the distances of C and F, respectively, to the line AB. Then

$$\frac{S_{ABC}}{S_{ABF}} = \frac{v_C}{v_F} = \frac{BC}{FK} = \frac{BL + LM + MC}{FK}.$$

Under the homothety through B carrying F to E, the segment PM maps to AC. Thus

$$\frac{LM}{FK} = \frac{FM}{FP} = \frac{EC}{AC} = \frac{AC}{AE} - 1.$$

And similarly

$$\frac{CM}{FK} = \frac{NF}{FK} = \frac{CD}{BD} = \frac{BC}{BD} - 1.$$

The required assertion follows by putting this all together and nothing BL = FK.

132. Let n be a natural number. A cube of side length n can be divided into 1996 cubes whose side lengths are also natural numbers. Determine the smallest possible value of n.

Sol.: Since 1996 >12^3, we must have $n \geq 13$, and we now show n = 13 suffices, Inside a cube of edge 13, we place one cube of edge 5, one cube length 4, and 2 of length 2, and fill the remainder with cubes of edge 1. The number of cubes used is

$13^3 - (5^3 - 1) - (4^3 - 1) - 2(2^3 - 1) = 2197 - 124 - 63 - 2(7) = 1996$, as desired.

133. Let M be the midpoint of the median AD of triangle ABC. The line BM intersects side AC at the point N. Show that AB is tangent to the circumcircle of NB if and only if the following equality holds:

$$\boldsymbol{\frac{BM}{BN} = \frac{BC^2}{BN^2}}.$$

Sol.: First note that (by the Law of Sines in triangles ABM and AMN)

$$\frac{BM}{BN} = \frac{\sin\angle MAB \sin\angle MNA}{\sin\angle ABM \sin\angle NAM}.$$

Then note that (by the Law of Sines in triangle ABD and ADC)

$$\frac{\sin\angle MAB}{\sin\angle NAM} = \frac{BD}{DC}\frac{\sin\angle ABD}{\sin\angle DCA}.$$

By the law of Sines in triangle BNC,

$$\frac{BC^2}{BN^2} = \frac{sin^2\angle BNC}{sin^2\angle BCN};$$

Therefore $\frac{BM}{MN} = \frac{BC^2}{BN^2}$ if and only if

$$\frac{\sin\angle ABD}{\sin\angle ABM} = \frac{\sin\angle BNC}{\sin\angle BCN},$$

Which if we put

$\alpha = \angle ABM, \beta = \angle BCN, \theta = \angle NBC$ becomes

$$\sin(\alpha + \theta)\sin\beta = \sin(\beta + \theta)\sin\alpha.$$

Rewriting each side as a difference of cosines and cancelling, this becomes

$$\cos(\alpha + \theta - \beta) = \cos(\beta - \alpha + \theta).$$

Both angles in this equation are between $-\pi$ and π, so the angles are either equal or negatives of each other. The latter implies θ = 0, which is untrue, so we deduce $\alpha = \beta$, and so $\frac{BM}{MN} = \frac{BC^2}{BN^2}$ if and only if ∠ABM = ∠BCN, that is, if AB is tangent to the circumcircle of BNC.

134. Three counters A, B, C are placed at the corners of an equilateral triangle of side n. The triangle is divided into triangles of side length 1. Initially all lines of the figure are painted blue. The counters move along the lines, painting their paths red, according to the following rules:

(i) First A moves, then B, then C, then A, and so on in succession. On each turn, each counter moves the full length

of a side of one of the short triangles.

(ii) No counter may retrace a segment already painted red, through it can stop on a red vertex, even if another counter is already there.

Show that for all integers n >0 it is possible to paint all of the segments red in this fashion.

Sol.: The cases n = 1, 2, are trivial; we use them as the base cases for an inductive proof. We describe the moves for A, understanding that the moves for B and C are the same moves rotated by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, respectively. To fix directions, imagine the triangle is oriented with one side parallel to the horizontal and the third vertex above it, and suppose A starts at the bottom left. We first move A right for n –1 steps. We then alternate moving it up to the left and down to the left for a total of 2n –5 steps.

We then trace a path through the inner triangle of side n –2 using the induction hypothesis, ending at another corner. Finally, we follow the unused edges from that corner, ending three steps later.

135. Fifty numbers are chosen from the set {1, …, 99}, no two which sum to 99 or 100. Prove that the chosen numbers must be 50, 51, …, 99.

Sol.: In the sequence

99, 1, 98, 2, 97, 3,… 51, 49, 50, any two adjacent numbers sum to 99 or 100, so both cannot occur. Grouping the numbers into 49 pairs plus one extra, we see at most 50 numbers can occur, and 50 must be one of them. Since we must step at least two terms along the list to make the next choice, the numbers must indeed be 50, 51, …, 99. Clearly we maximize the number of chosen numbers by taking them two apart, and the list has odd length, so taking 99, 98 …, 50 is the only draw a graph with {1, …, 99} as vertices, where two numbers are adjacent if they sum to 99 or 100.

136. Let M be the intersection of the diagonals of the trapezoid ABCD. A point P such that ∠APM = ∠DPM is chosen on the base BC. Prove that the distance from C to the line AP is equal to the distance from B to the line DP.

Sol.: Since M lies on the internal angle bisector of angle ∠APD, it lies at the same distance from the lines AP and DP. The ratio of this distance to the distance from C to AP is $\frac{AM}{AC}$, while the ratio of this distance to the distance from B to DP is $\frac{BM}{MD}. But \frac{AM}{MC} = \frac{BM}{MD}$ by similar triangles, so the latter two distances are indeed the same.

137. In a group of several people, some are acquainted with each other and some are not. Every evening, one person invites all of his acquaintances to a party and introduces them to each other. Suppose that after each person has arranged at least one party, some two people are still unacquainted. Prove that they will not be introduced at the next party.

Sol.: We claim that two people unacquainted after each person has held at least one party lie in different connected components of the original (and final) graph of acquaintance. If two people are connected by a path of length n, they will be connected by a path of length n -1 after one person along the path (including either of the two people at the ends) holds a party, by a path of length n -2 after two of them hold a party, and so on. After each person holds a party, the two people on the ends will be acquainted.

138. There are n parking spaces along a one –way road down which n drivers are travelling. Each driver goes to his favorite parking space and parks there if it is free; otherwise, he parks at the nearest free place down the road. If there is no free space after his favorite, he drives away. How many lists $\boldsymbol{a_1}, \dots, \boldsymbol{a_n}$of favorite parking spaces are there which permit all of the drivers to park?

Sol.: There are $(n+1)^{n-1}$ such lists. To each list of preferences $(a_1, \dots, a_n)$ which allows all drivers to park, associate the list $(b_2, \dots, b_n)$, where b_i is the difference mod n+ 1 between the numbers of the space driver i wants and the space the previous driver took. Clearly any two lists give rise to different sequences of b_i.

We now argue that any list of b_i comes from a list of preferences. Imagine that the n parking spaces are arranged in a circle with an extra phantom space put in at the end. Put the first driver in any space, then for i = 2, …, n, put driver i in the first available space after the space b_i away from the space taken by driver i-1; this gives a list of preferences if and only if the one space not taken at the end is the phantom space. However, by shifting the position of the first driver, we can always ensure that the phantom space is the space not taken. Thus the sequences of b_i are equal in number to the lists of preferences, so there are $(n+1)^{n-1}$ of each.

139. Find all positive integers n such that $\boldsymbol{3^{n-1} + 5^{n-1}}$ *divides* $\boldsymbol{3^n + 5^n}$.

Sol.: This only occurs for n = 1. Let $s_n = 3^n + 5^n$ and note that

$$s_n = (3+5)s_{n-1} - 3.5.\, s_{n-2}$$

So s_{n-1} must also divide $3.5.\, s_{n-2}$. If n >1, then s_{n-1} is coprime to 3 and 5, so s_{n-1} must divide s_{n-2}, which is impossible since $s_{n-1} > s_{n-2}$.

140. Let M be the midpoint of side BC of triangle ABC, and let $\boldsymbol{r_1}$ *and* $\boldsymbol{r_2}$ be the radii of the incircles of triangles ABM and ACM. Prove that $\boldsymbol{r_1} < 2\boldsymbol{r_2}$.

Sol.: Recall that the area of a triangle equals its in radius times half its perimeter. Since ABM and ACM have equal area, we have

$$\frac{r_1}{r_2} = \frac{AC + AM + CM}{AB + AM + BM}$$

And it suffices to show AC + AM + CM <2AB + 2AM +2BM;

Since BM = CM, this simplifies to AC < 2AB + AM +CM.

In fact, by the triangle inequality, AC < AM +CM, so we are done.

141. Several positive integers are written on a blackboard. One can erase any two distinct integers and write their greatest common divisor and least common multiple instead. Prove that eventually the numbers will stop changing.

Sol.: If a, b are erased and c < d are written instead, we have c ≤ min (a, b) and d ≥ max (a, b); moreover, ab = cd. From this we may conclude $a + b \le c + d$ by writing $ab + a^2 = cd + a^2 \le ac + ad$ (the latter since $(d - a)(c - a) \le 0$) and dividing both sides by a. Thus the sum of the numbers never decreases, and it is obviously bounded (e.g. by n times the product of the numbers, where n is the number of numbers on the board); hence it eventually stops changing, at which time the numbers never change.

142. No three diagonals of a convex 1996 – gon meet in a point. Prove that the number of triangles lying in the interior of the 1996-gon and having sides on its diagonal is divisible by 11.

Sol.: There is exactly one such triangle for each choice of six vertices of the 1996-gon: if A, B, C, D, E, F are the six vertices in order, the corresponding triangle is formed by the lines AD, BE, CF. Hence the number of triangle is $\binom{1996}{6}$; since 1991 is a multiple of 11, so is the number of triangles.

143. Prove that for every polynomial $x^2 + px + q$ with integer coefficients, there exists a polynomial $2x^2 + rx + s$ with integer coefficients such that the sets of values of the two polynomials on the integers are disjoint.

Sol.: If p is odd, then $x^2 + px + q$ has the same parity as q for all integers x, and it suffices to choose r even and s of the opposite parity as q. If p = 2m is even, then $x^2 + px + q = (x + m)^2 + (q - m)^2$ which is congruent to $q - m^2 \; or \; q - m^2 + 1$ modulo 4. Now it suffices to choose r even and s congruent to $q - m^2 + 2$ modulo 4.

144. Sergey found 11 different solutions to the equation $f\left(19x - \frac{96}{x}\right) = 0$. Prove that if he had tried harder, he could have found at least one more solution.

Sol.: The equations $19x - \frac{96}{x} = t$ can be rewritten $19x^2 - tx - 96 = 0; since\; t^2 + 19.96 > 0$, it always has two real roots. Therefore the number of zeroes of f (if finite) is an even integer, so Sergey can find at least one more zero.

145. Find all quadruples of polynomials $P_1(x), P_2(x), P_3(x), P_4(x)$ with real coefficients such that for each quadruple of integers x, y, z, t such that $xy - zt = 1$, one has

$$P_1(x)P_2(y) - P_3(z)P_4(t0 = 1.$$

Sol.: If $P_1(1) = 0$, then $P_3(z)P_4(t) = -1$ for each pair of integers z, t, and so $P_3 \; and \; P_4$ are constant functions; moreover, $P_1(x)P_2(y) = 0$, so one of $P_1 \; and \; P_2$ is identically zero. Ignoring such cases, which are easily enumerated, we assume $P_i(1) \neq 0$ for all i.

We first note that $P_1(x)P_2(1) = P_1(1)P_2(x)$ for all nonzero integers x, so that $P_1 \; and \; P_2$ are equal up to a scalar factor; similarly, $P_3 \; and \; P_4$ are equal up to a scalar factor. Now note that $P_1(x)P_2(ay) = P_1(ax)P_2(y)$ for all nonzero a, x, y, so that the difference between the two sides is identically zero as a polynomial in a. In particular, that means no term in $P_1(x)P_2(y)$ has unequal exponent in x and y, and the same is true of $P_1(x)P_1(y)$ on the other hand, if $P_1(x)$ has term of more than one degree, then $P_1(x)P_1(y)$ contains a term with different degrees in x and y. Hence $P_1(x) = cx^k$ for some integer k and some constant c, and similarly $P_2(x) = dx^k, P_3(x) = ex^m, P_4(x) = fx^m$.

Thus we must determine when $cdx^ky^k - efz^mt^m = 1$ whenever $xy - zt = 1$ in integers. Clearly k = m since otherwise one of the two terms on the left dominates the other, and cd =1 by setting x = y = 1 and z = t = 0, and similarly ef = 1. Now note that $(xy)^k - (zt)^k = 1$ can only happen in general for k = 1, since for k >1, there are no consecutive perfect k-th powers. We conclude $P_1(x) = cx, P_2(x) =$

$\frac{x}{c}, P_3(x) = ex, P_4(x) = \frac{x}{e}$ for some nonzero real numbers c, e.

146. Two players play the following game on a 100×100 board. The first player marks a free square, then the second player puts a 1×2 domino down covering two free squares, one of which is marked. This continuous until one player is unable to move. The first player wins if the entire board is covered, otherwise the second player wins. Which player has a winning strategy?

Sol.: The first player has a winning strategy. Let us say a position is stable if every square below or to the right of a free square is free. Then we claim the first player can always ensure that on his turn, either the position is stable or there is a free square with exactly one free neighbor (or both).

Let us label the square in the i-th row and j-th column as (i, j), with (1, 1) in the top left. We call a free square a corner if is not below or to the right of another free square. Let $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ be the corners from top to bottom.

First notice that if (a, b) is a corner such that both $(a+1, b-1) and\ (a-1, b+1)$ are nonfree (or off the board), then the first player may mark (a, b), and however the second player moves, the result will be a stable position. More generally, if $(a, b), (a+1, b-1), \ldots, (a+k, b-k)$ are corners and $(a-1, b+1) and\ (a+k+1, b-k-1)$ are both nonfree or off the board, the first player can be sure to return to a stable position.

To show this, first note that we cannot have both a = 1 and b –k = 1, or else the number of non-free squares would be odd, which is impossible. Without loss of generality, assume that b –k ≠ 1 is not the final corner. The first player now marks (a, b). If the second player covers (a, b) and (a, b+1), the position is again stable. Otherwise, the first player marks $(a+1, b-1)$ and the second player is forced to cover it and $(a+2, b-1)$. Then the first player marks $(a+2, b-2)$ and the second player is forced to cover it and $(a+3, b-2)$, and so on. After $(a+k, b-k)$ is marked, the result is a stable position.

(Note that the assumption b –k ≠ 1 ensures that the moves described do not cross the edge of the board.) To finish the proof, we need to show that such a chain of corners must exist. Write the labels $(a_1, b_1), \ldots, (a_k, b_k)$ in a row, and join two adjacent labels by a segment if they are of the form $(a, b), (a+1, b-1)$. If two adjacent labels $(a, b), (a+i, b-j)$ are not joined by a segment, then either i = 1 or j = 1 but not both. If i = 1, draw an arrow between the labels pointing towards $(a+i, b-j)$; otherwise draw the arrow the other way. Also draw arrows pointing to $(a_1, b_1) and\ (a_k, b_k)$. There is now one more chain of corners (joined by segments) than arrows, so some chain has two arrows pointing to it. That chain satisfies the condition above, so the first player can use it to create another stable position. Consequently, the first player can ensure victory.

147. Let BD be the bisector of angle B in triangle ABC. The circumcircle of triangle BDC meets AB at E, while the circumcircle of triangle ABD meets BC at F. Prove that AE = CF.

Sol.: By power-of-a-point. AE .AB = AD. AC and CF. CB = CD. CA, so $\frac{AE}{CF} = \left(\frac{AD}{CD}\right)\left(\frac{BC}{AB}\right)$. However,

$\frac{AB}{CB} = \frac{AD}{CD}$ by the angle bisector theorem, so AE = CF.

148. A $\mathbf{10 \times 10}$ table consists of positive integers such that for every five rows and five columns, the sum of the numbers at their intersections is even. Prove that all of the integers in the table are even.

Sol.: We denote the first five entries in a row as the "head" of that row. We first show that the sum of each head is even. We are given that the sum of any five heads is even; by subtracting two such sums overlapping in four heads, we deduce that the sum of any two heads is even. Now subtracting two such relations from a sum of five heads, we determine that the sum of any head is even.

By a similar argument, the sum of any five entries in a row is even.

By the same argument as above, we deduce that each entry is even.

149. Prove that there are no positive integers a and b such that for each pair p, q of distinct primes greater than 1000, the number $\boldsymbol{ap + bq}$ is also prime.

Sol.: Suppose a, b are so chosen, and let m be a prime greater than a +b, by Dirichet's theorem, there exist infinitely many primes in any nonzero residue class modulo m; in particular, there exists a pair p, q such that p ≡ b (mod m), q ≡ -a (mod m), giving $ap + bq$ is divisible by m, a contradiction.

150. There are 2000 towns in a country, each pair of which is linked by a road. The Ministry of Reconstruction proposed all of the possible assignments of one way traffic to each road. The ministry of Transportation rejected each assignment that did not allow travel from any town to any other town. Prove that more of half of the assignments remained.

Sol.: We will prove the same statement for n ≥ 6 towns. First suppose n = 6. In this case there are 2^{15} assignments, and an assignment is rejected only if either one town has road to all of the others in the same direction, or if there are two sets of three towns, such that within each town the roads point in a circle, but all of the roads from one set to the other point in the same direction. There are 5.2^{11} had assignments of the first kind and 20.8 of the second kind, so the fraction of good assignments is at least $\frac{5}{8}$.

For n ≥ 6, we claim that the fraction of good assignments is at least

$$\frac{5}{8}\prod_{i=6}^{n-i}\left(1-\frac{1}{2^{i-1}}\right).$$

We show this by induction on a good assignment or r h -1 vertices can be extended to a good assignment on vertices simply by avoiding having all edges from the last vertex pointing in the same direction, which occurs in 2 cases out of 2^{n-1}.

Now it suffices to show that the above expression is more than $\frac{1}{2}$.

In fact,

$$\prod_{i=5}^{\infty}\left(1-\frac{1}{2}\right)^{-1} \leq 1+\sum_{i=5}^{\infty}\frac{i-4}{2^i}$$

$$= 1+\frac{1}{2^5}\sum_{i=0}^{\infty}\frac{i+1}{2^i} = 1+\frac{1}{2^5}\sum_{i=0}^{\infty}\sum_{k=i}^{\infty}\frac{1}{2^i}$$

$$= 1 + \frac{1}{2^5}\sum_{i=0}^{\infty}\frac{1}{2^i - 1} = 1 + \frac{4}{2^5} = \frac{9}{8}$$

Thus the fraction of good assignments is at least $\left(\frac{5}{8}\right)\left(\frac{8}{9}\right) = \frac{5}{9} > \frac{1}{2}$.

151. Find all real numbers satisfying $\boldsymbol{6^x + 2^{2x} + 24^x - 36^x - 16^x = 1}$.

Ans:- Rewrite the given relation as:

$6^x + 4^x - 36^x + 24^x - 16^x$= 1

Let $6^x = a, 4^x = b,$ we have

a+ b -$a^2 + ab - b^2$ = 1

$\Rightarrow a^2 - ab + b^2 - a - b + 1$=0

$\Rightarrow 2a^2 - 2ab + 2b^2 - 2a - 2b + 2$=0

$\Rightarrow (a^2 - 2ab + b^2)$+ $(a^2 - 2a + 1)$+ $(b^2 - 2b + 1)$=0

$\Rightarrow (a - b)^2 + (a - 1)^2 + (b - 1)^2$=0

∴ a= 1 and b= 1 when a= b.

$\Rightarrow 4^x = 1 \; and \; 6^x = 1$, giving x= 0 only.

152. Two boxes contain between them 65 balls of several different sizes. Each ball is white, black, red, or yellow. If you take any five balls of the same colour, at least two of them will always be of the same size (radius). Prove that there are at least three balls which lie in the same box, have the same colour and are of the same size.

Sol: We will make repeated use of pigon–hole- principle (PHP). As there are 65 balls and 2 boxes , one of these boxes must contain at least $\left[\frac{65}{2}\right]$+1 = 33 balls.

Consider that box, now we have four colours (white, black, red, yellow) and hence there must be at least $\left(\frac{33}{4}\right)$+1 = 9 balls of the same colour.

There can be at most 4 different sizes available for these 9 balls of the same colour, For if there were 5 (or more)different sizes, then collection of 5 balls, all of different sizes, would not satisfy the given property.

Thus of these 9 balls there must be at least 3 balls of the same size.

153. Find all continuous function f : (0, ∞)⟶(0, ∞) ∋ f (1)= 1 and

$$\boldsymbol{\frac{1}{2}\int_0^x (f(t))^2 dt = \frac{1}{x}\left(\int_0^x f(t) dt\right)^2}$$

Ans:- Define , F (x) = $\int_0^x f(t)dt$ and G (x)= $\int_0^x (f(t))^2 dt$

Since f: (0, ∞) ⟶ (0, ∞)

we have F (x)> 0 ∀ $x > 0$

Also, $\frac{1}{2}G(x) = \frac{1}{x}\{F(x)\}^2$, from the given condition on differentiation, we have

$\frac{1}{2}G'(x)$= $\frac{1}{x}.2F(x).F'(x) - \frac{1}{x^2}(F(x))^2$

This means that $\frac{1}{2}(F(x))^2$=$\frac{2}{x}F(x)\,F'(x) - \frac{1}{x^2(F(x))^2}$

or, $\frac{1}{2}\left(\frac{xF'(x)}{F(x)}\right)^2 = 2\frac{xF'(x)}{F(x)} - 1$

Solving this equation as a quadratic in $\frac{xF'(x)}{F(x)}$we have

$\frac{xF'(x)}{F(x)} = 2 \pm 2 = k$(say)

On integration, we obtain $\int \frac{dF(x)}{F(x)} = k \int \frac{dx}{x}$

$\Rightarrow$ ln $F(x)$= klnx + ln$\lambda \Rightarrow F(x)= \lambda x^k$

$\Rightarrow$ f (x)=λk$x^{k-1} \Rightarrow$ f (1) = 1

$\Rightarrow \lambda$k=1

$\therefore$ f(x)= $x^{k-1} = x^{1+\sqrt{2}}/x^{1-\sqrt{2}}$

154. Let x $\geq$ 1, $f(x) = \frac{\sqrt{[x]}+\sqrt{\{x\}}}{\sqrt{x}}$, where [.] denotes G.I.F. and { } denotes fractional part. Determine the smallest number k ∋ f(x)$\leq k$ for each x $\geq$ 1

Ans:- Let x = a+ b where a= [x], b= {x}

f(x)= $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a+b}}$

(f$(x))^2 = \frac{a+b+2\sqrt{ab}}{a+b} = 1 + \frac{2\sqrt{ab}}{a+b}$

Using AM$\geq GM, \leq 1 + 1 \Rightarrow f(x) \leq \sqrt{2}$.

155. Solve the equation $(\sqrt{2} + \sqrt{2})^x + (\sqrt{2} - \sqrt{2})^x = 2^x$

Ans:- 1+ $\frac{\sqrt{2}}{2} = 1 + cos\frac{\pi}{4} = 2\cos^2\frac{\pi}{8}$

$(\frac{2+\sqrt{2}}{4})^{x/2} + (\frac{2-\sqrt{2}}{4})^{x/2}$

= $(cos\frac{\pi}{8})^x + (sin\frac{\pi}{8})^x$

$\Rightarrow$ x= 2

156. Maximize x+ y subject to the condition that $2x^2 + 3y^2 \leq 1$.

Ans:- $\frac{x^2}{1/2} + \frac{y^2}{1/3} \leq 1$

Let z= x+ y

Now, 4x + 6y$\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{2x}{3y}$

At the touching point. -$\frac{2x}{3y}$ = -1

$\Rightarrow$ 2x= 3y and $2x^2 + 3y^2$=1 $\Rightarrow 2(\frac{3y}{2})^2 + (3y^2) = 1$

$\Rightarrow 15y^2 = 2 \Rightarrow$ y= $\pm\sqrt{\frac{2}{15}}$

$\therefore$ x= $\frac{3}{2}\left(\pm\sqrt{\frac{2}{15}}\right) = \pm\sqrt{\frac{3}{10}}$ $\therefore$ Max (z)= $\sqrt{\frac{3}{10}} + \sqrt{\frac{2}{15}} = \frac{5}{\sqrt{30}}$.

157. For any positive a, b prove that $(a+\frac{1}{a})^2 + (b+\frac{1}{b})^2 \geq 8$.

Ans: AM $\geq$ GM

$(a+\frac{1}{a})^2 + (a+\frac{1}{a})^2 \geq 2\sqrt{(a+\frac{1}{a})^2 + (b+\frac{1}{b})^2}$

$\geq$ 2(ab+$\frac{1}{ab} + \frac{a}{b} + \frac{b}{a}$)

$\geq$ 2(2+2) [$\because$ ab + $\frac{1}{ab} \geq$ 2]

158. Find the following limit: $\lim\limits_{x\to\infty}(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}})$

Ans:- Let $u_n = \frac{n}{\sqrt{n^2+n}}$

$\therefore \lim\limits_{x\to\infty} u_n = \lim\limits_{x\to\infty}\frac{n}{\sqrt{n^2+n}} = \lim\limits_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}} = 1$.

By Cauchy's first theorem:-
$\lim\limits_{x\to\infty}\left(\frac{u_1+\cdots+u_n}{n}\right) = 1.$

So, $\lim_{x\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right)=1$.

159. For any real number x and for any positive integer n show that

$[x]+[x+\frac{1}{n}]+\left[x+\frac{2}{n}\right]+\cdots+\left[x+\frac{n-1}{n}\right]=[nx]$

Ans:- Let x= [x]+y, where $0 \le y < 1$,

Let p be an integer such that P-1 $\le ny < P$

Now, x+$\frac{k}{n}=[x]+y+\frac{k}{n}$

Also, $\frac{P+k-1}{n}<y+\frac{k}{n}<\frac{P+k}{n}$

So, long as $\frac{P-1+k}{n}<1$, i.e. , k < n-(P-1)

So, $y+\frac{k}{n}<1$ and consequently

[x+$\frac{k}{n}$] = [x]for k= 0, 1, 2, …, n-P.

But [x+$\frac{k}{n}$] = [x]+1 for k= n-P+1, …..n-1.

∴[x]+ [x+$\frac{1}{n}$]+…+[x+$\frac{n-1}{n}$]

$= \underbrace{([x]+[x]+\cdots+[x])}+\underbrace{(([x]+1)+([x]+1)+\cdots+([x+1]))}$

= n[x]+(P-1)……………………(1)

Also, [nx]=[n[x]+ny]= n[x]+(P-1)

Since P-1 ≤ ny < P……(2)

From equation (1) & (2),

[x]+[x+$\frac{1}{n}$]+…+[x+$\frac{n-1}{n}$] = [nx].

160. Prove that for n > 1, $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}<2-\frac{1}{n}$

Ans:- P(1)= $1+\frac{1}{2^2}=\frac{5}{4}<2-\frac{1}{2}=\frac{3}{2}=\frac{6}{4}$.

The statement is true for n= 2.

Let , the statement is true for n= m.

∴P(m)= $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{m^2}<2-\frac{1}{m}$.

Now, we need to show that the statement is also true for n=m+1.

P(m+1)= $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{m^2}<2-\frac{1}{(m+1)^2}$

$<2-\frac{1}{m}+\frac{1}{m(m+1)}$. $[\because \frac{1}{(m+1)^2}<\frac{1}{m(m+1)}\forall m>1]$

< 2-$\frac{1}{m+1}$.

∴ The statement is true for n= m+1

So, for all n ∊ ℕ the statement is true.

Hence proved.

161. In a ΔPQR, ∠R = $\frac{\pi}{2}$. If $\tan\frac{P}{2}$ and $\tan\frac{Q}{2}$ are the roots of equation ax^2+bx+c= 0 (a≠0), then show that a + b = c.

Ans:- (a) $\tan\frac{p}{2}+\tan\frac{Q}{2}=-\frac{b}{a}$, $\tan\frac{P}{2}\tan\frac{Q}{2}=\frac{c}{2}$ $\quad\therefore \frac{P}{2}+\frac{Q}{2}=\frac{\pi}{4}$

$$\therefore \frac{\tan\frac{p}{2}+\tan\frac{Q}{2}}{1-\tan\frac{p}{2}\tan\frac{Q}{2}}=\tan\frac{\pi}{4}=1$$

$$\Rightarrow \frac{-\frac{b}{a}}{1-\frac{c}{a}}=1 \Rightarrow b=c-a \Rightarrow a+b=c$$

162. If A and B are real orthogonal matrices of the same order and |B|+|A|= 0.

Prove that |A+ B|= 0

Ans:- |A|+|B|=0

⇒ |A|= $-$|B|

|A|.|B|=-1 [∵ |B|=|B^{-1}|as they are orthogonal]

Let, C = A ($A^T + B^T$)B

⇒ |C|= |A$A^T B + AB^T B| = |B + A|$ ………..(i)

And |C|= |A||$A^T + B^T||B| = -|A^T + B^T|$

⇒ - $|(A+B)^T| = -|A+B|$…………..(ii)

$|A+B| = -|A+B|$

⇒ 2 |A +B|=0

⇒ |A+ B|=0

163. Determine whether there is a one –to – one function f: ℝ⟶ ℝ such that

$\mathbf{f(x^2)\text{-}[f(x)]^2 \geq \frac{1}{4} \forall\, x}$

Ans:- Take x= 0, then f(0) – (f(0))^2 $\geq \frac{1}{4}$

⇒ (f(0))^2 $+ (\frac{1}{2})^2 - 2.\frac{1}{2}.$f(0) ≤ 0

⇒ (f(0) $- \frac{1}{2})^2 \leq 0$

⇒ f(0) $- \frac{1}{2}$ = 0 ⇒ f(0) $= \frac{1}{2}$

Also, taking x=1 we have f(1) - $\frac{1}{2}$ = 0 ∴ f(0) = f(1)= $\frac{1}{2}$

∴ This is not one -to –one function.

164. If 0 < u < 1 and $u_{n+1} = 1 - \sqrt{1-u_n}$ ∀ $n > 1$,

Prove that (i) $\{u_n\}$ converges to zero

(ii) $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{2}$

Ans:- (i) 0 < $u_1 < 1$

⇒ 0 < $\sqrt{1-u_1} < 1$

⇒ 0 < 1 -$\sqrt{1-u_1} < 1$

i.e. 0 < $u_2 < 1$

Similarly, 0 < $u_3 < 1$ ….. and so on.

Let 0 < $u_n < 1$, then 0 < 1- $\sqrt{1-u_n} < 1$, i.e. 0 < u_{n+1} < 1

Thus $\{u_n\}$ is bounded.

Again, $u_{n+1} - u_n$= 1-$\sqrt{1-u_n}$ -u_n

=(1- u_n)- $\sqrt{1-u_n}$

= $(\sqrt{1-u_n})^2 - \sqrt{1-u_n}$

= $\sqrt{1-u_n}\ (\sqrt{1-u_n} - 1)$

< 0 as 0 < $\sqrt{1-u_n} < 1$

∴ $u_{n+1} < u_n$ as 0 < $\sqrt{1-u_n}$<1

∴ $\{u_n\}$ is monotonically decreasing. ∴ $\{u_n\}$ converges two zero.

(ii) Let $\lim\limits_{n\to\infty} u_n = l$, then $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n}$

∴ $\lim\limits_{n\to\infty} \frac{1-\sqrt{1-l}}{l} = \lim\limits_{n\to\infty} \frac{l}{l(1+\sqrt{1-l})} = \frac{1}{1+\sqrt{1-0}} = \frac{1}{2}$; Since u_n converges to zero.

165. Let g: ℝ⟶ ℝ be a continuous function ∋ g(x) = g $(\frac{x-1}{2})$ ∀ x.

Show that g must be a constant function.

Ans:- g(x) = g $(\frac{x-1}{2})$

⇒ g $(\frac{x-1}{2})$ = g $(\frac{\frac{x-1}{2}-1}{2})$ = g $(\frac{x-3}{4})$

Again putting x= $\frac{x-1}{2}$

g $(\frac{x-1}{2})$= g $(\frac{x-7}{2})$ and so on

Generally we have, g$(\frac{x-1}{2})$= g $(\frac{x-(2^n-1)}{2^n})$

∴ g (x)= g $(\frac{x}{2^n} - 1 + \frac{1}{2^n})$

∴ $\lim_{n\to\infty} g\ (x) = g\ (-1)$

⇒ g (x)= g (-1)= constant ∀ x.

166. Find the greatest and least value of the function f (x)= $x^3 - 3x^2 + 2x + 1$ in [2, 3].

Ans:- f (x)= $x^3 - 3x^2 + 2x + 1$

f′(x)= $3x^2 - 6x + 2$; f′ (2) = 2 > 0; f′ (3) =11 > f′ (2)

∴ f(x) is an increasing function

Note that f″(x) = 6x -6 > 0 ∀ x ∊ [2, 3]

∴ f(x) is concave.

Thus the function has min. value at x= 2 and max. Value at x= 3

∴ minimum value= f(2)=1 ∴ maximum value = f(3) = 7

167. Let F (x) = $\sum_{k=0}^{n} a_k x^k$, where a_k satisfy $\sum_{k=0}^{n} \frac{a_k}{k+1} = 0$ so that there exists a real root of f (x) = 0 in the interval (0, 1)

Ans:- F (x) = $\int_0^x f(t)dt =$ $\int_0^x (\sum_{k=0}^{n} a_k t^k)dt$ = $\sum_{k=0}^{n} a_k \cdot \frac{x^{k+1}}{k+1}$

Clearly F (x) satisfies the conditions of Rolle's Theorem as F (0) = 0 and

F (1) =$\sum_{k=o}^{n} \frac{a_k}{k+1} = 0$

Hence ∃ a 'c' ∊ (0, 1) ∋ F′(c) =0 ⇒ F (c) =0

168. Show that 1+ $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots . + \frac{1}{n}$ can never be an integer value.

Ans:- We are to show:- 1+ $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots . + \frac{1}{n}$ = $\frac{an\ odd\ number}{an\ even\ number}$ $\forall\ n > 1$

Let, P (n): 1+ $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots . + \frac{1}{n}$= $\frac{an\ odd\ number}{an\ even\ number}$ $\forall\ n > 1$

When n=2, LHS= 1+ $\frac{1}{2} = \frac{3}{2} = \frac{an\ odd\ number}{an\ even\ number}$

∴ P (2) is true. Let P (m)be true

⇒ 1+ $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots . + \frac{1}{m}$ = $\frac{an\ odd\ number}{an\ even\ number}$ = $\frac{k}{r}(say)$

Now, P (m+1) = $\frac{k}{r} + \frac{1}{m+1}$

m is odd or even,

But in case, it can be shown that

P (m)= $\frac{an\ odd\ number}{an\ even\ number}$

∴P (n) is true for all n ∊ℕ

169. Let f: ℝ → ℝ be differentiable and assume there is no x $\boldsymbol{in}$ ℝ ∋ f(x) = f'(x) = 0. Show that S = {x| 0≤ $\boldsymbol{x \leq 1}$; $\boldsymbol{f(x) = 0}$} is finite.

Ans:- Consider $f^{-1}(\{0\})$. Since {0} is closed and f is continuous $f^{-1}(\{0\})$ is closed. Therefore, S=[0,1]∩ $f^{-1}(\{0\})$ is closed and bounded subset of ℝ. Hence, S is complete.

Assume S is infinite.

Then there is a limit point $x \in S$;

i.e. there is a sequence $\{x_n\}$ of distinct points in S which converges to x.

Also, as all points are in S, $f(x_n) = f(x) = 0 \; \forall \; n \in N$.

We now show that f'(x) =0.

Since.$|x_n - x|$→0 ,

so

f'(x) = $\lim_{n\to\infty} \frac{f(x+(x_n-x))-f(x)}{x_n-x}$

= $\lim_{n\to\infty} \frac{f(x_n)-f(x)}{x_n-x}$

= 0

The last equality holds since f(x) = f(x_n) =0 holds $\forall \; n \in N$.

170. The four digit number aabb is a square. Find the number.

Ans:- aabb = n^2

Then n^2 = 1100a + 11b

= 11(100a + b)

= 11(99a + a + b)

Since, n^2 is divisible by11^2, we see that 11 | (a+b)

i.e. a+b=11. Since n^2 is a square , bcan't be 0,1,2,3,5,7 or 8 . Checking the remaining we see that 7744= 88^2

171. Find the maximum value of $\boldsymbol{cos\alpha_1 . cos\alpha_2 . cos\alpha_n}$, under the restrictions

$0 < \alpha_1, \alpha_2, \ldots \alpha_n \leq 2 \;\; cot\alpha_1 . cot\alpha_2 . cot\alpha_n$ = 1.

Ans.

Given, $(\cot\alpha_1).(\cot\alpha_2) \ldots. (\cot\alpha_n) = 1$

$\Rightarrow \cos\alpha_1 . \cos\alpha_2 \ldots. \cos\alpha_n = \sin\alpha_1 . \sin\alpha_2 \ldots. \sin\alpha_1$ ……………(1)

Now, $(\cos\alpha_1 . \cos\alpha_2 \ldots. \cos\alpha_n)^2 = (\cos\alpha_1 . \cos\alpha_2 \ldots. \cos\alpha_n)(\cos\alpha_1 . \cos\alpha_2 \ldots. \cos\alpha_n)$

$= (\cos\alpha_1 . \cos\alpha_2 \ldots. \cos\alpha_n)(\sin\alpha_1 . \sin\alpha_2 \ldots. \sin\alpha_n)$ [from (1)]

$= \frac{1}{2^n} \sin 2\alpha_1 . \sin 2\alpha_2 \ldots. \sin 2\alpha_n \leq \frac{1}{2^n}$

$\therefore (\cos\alpha_1 . \cos\alpha_2 \ldots. \cos\alpha_n)^2 \leq \frac{1}{2^n}$

$\therefore \cos\alpha_1 . \cos\alpha_2 \ldots. \cos\alpha_n \leq \sqrt{\frac{1}{2^n}} \leq \frac{1}{2^{\frac{n}{2}}}$

$[\because \cos\alpha_i \geq 0]$

172. Let f(x, y) = 0 is a circle such that f(0, $\boldsymbol{\lambda}$) = 0 and f($\boldsymbol{\lambda}$,0) = 0 have equal roots and f(1,1)= - 2 then the radius of the circle is

(a) 4 **(b) 8**
(c) 2 **(d) 1**

Ans. (c)

Let f(x, y) = $x^2 + y^2 + 2gx + 2fy + c = 0$ *be the required circle*

f(0, λ) = $\lambda^2 + 2f\lambda + c = 0$ ………….(1)

f(λ, 0) = $\lambda^2 + 2g\lambda + c$= 0…………….(2)

∵ (1) and (2) have equal roots.

∴ D= 0

$\Rightarrow f^2 = g^2 = c$

∴ f(x, y) = $x^2 + y^2 + 2gx \pm 2fy + g^2 = 0$

f(1, 1) = - 2 $\Rightarrow g^2 + 2g \pm 2g + 4 = 0$

when f = - g, $g^2 = -4$ (not possible)

∴ f = g and g = f = -2 and c = 4

∴ Radius of circle is 2.

173. Let s=$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{10000}$ and I=$\int_0^{1000} \sqrt{x}\, dx$. Show that I $\leq s \leq I + 100$.

Ans:- I=$\int_0^{1000} \sqrt{x}\, dx = \int_0^1 \sqrt{x}\, dx + \int_1^2 \sqrt{x}\, dx + \cdots + \int_{9999}^{10000} \sqrt{x}\, dx$

$\rightarrow \int_0^1 0.dx + \int_1^2 \sqrt{1}$ dx +….+$\int_{9999}^{10000} \sqrt{9999}\, dx \leq I \leq \int_0^1 1.dx + \int_1^2 \sqrt{2}\, dx + \cdots + \int_{9999}^{10000} \sqrt{10000}$ dx

$\rightarrow \sqrt{1} + \sqrt{2} + \ldots + \sqrt{9999} \leq I \leq \sqrt{1} + \sqrt{2} + \ldots + \sqrt{10000}$

$\rightarrow$I $\leq S$

Also, S $\leq I + \sqrt{10000}$

∴ I $\leq S \leq I + 100$

174. Prove that

$$1 < \frac{1}{1001} + \frac{1}{1002} + \cdots + \frac{1}{3001} < \frac{4}{3}$$

Ans:- consider 2001 numbers $\frac{1}{k}, 1001 \leq k \leq 3001$

Using AM- HM inequality, we get

$(\sum_{k=1001}^{3001} k)(\sum_{k=1001}^{3001} \frac{1}{k} \geq (2001)^2$

But $\sum_{k=1001}^{3001} k = (2001)^2$

Hence we get the inequality $\sum_{k=1001}^{3001} \frac{1}{k} > 1$

On the other hand grouping 500 terms at a time, we also have

S= $\sum_{k=n+1}^{3n+1} \frac{1}{k} < \frac{500}{1000} + \frac{500}{1500} + \frac{500}{2000} + \frac{500}{2500} + \frac{1}{3001} < \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{3001} = \frac{3851}{3000} < \frac{4}{3}$

[Remarks:- if S =$\sum_{k=1001}^{3001} \frac{1}{k}$, there are (2n+ 1) terms in the sum and the middle term is $\frac{1}{2n+1}$; then $\frac{29}{27} < S < \frac{7}{6}$].

175. Let A be a set containing n elements. If the number of elements in the set,

B = {(x, y, z) : x ∈A, y ∈A, z ∈A and x, y, z, are not all distinct} is equal to 280,

then find the value of n?

Ans; According to question $n^3 - \binom{n}{3}.3! = 280$

∴ $n^3 - n(n-1)(n-2) = 280$

$\Rightarrow$ n($n^2 - n^2 + 3n - 2$)= 280

$\Rightarrow$ n(3n- 2)= 280= 10(3.10- 2)

TOPIC WISE SOLVED PROBLEMS

INDUCTION

(Objective Type)

1. The sum of n terms of the series $1.3^2 + 2.5^2 + 3.7^2 + \cdots \infty$ is

(a) $4n^3 + 4n^2 + n$ (b) $\frac{n}{6}(n+1)(6n^2 + 14n + 7)$ (c) $\frac{n(n+1)}{6}$ (d) none.

Sol. : Let $p(n): 1.3^2 + 2.5^2 + 3.7^2 + \cdots + (n)(2n+1)^2$

$\therefore P(n): \sum n\,(2n+1)^2 \Rightarrow P(n): \sum n\,(4n^2 + 4n + 1)$

$\Rightarrow P(n): \sum n\,(4n^2 + 4n^2 + n) \Rightarrow P(n): 4\sum n^3 + 4\sum n^2 + \sum n \;\therefore P(n) = \frac{1}{6}n(n+1)(6n^2 + 14n + 7).$

2. $10^n + 3.4^{n+2} + K$ K is divisible by 9 for all n ∈N. Then the least +ve integral value of k is-

(a) 5, (b) 3, (c) 1, (d) none.

Sol. Let $P(n): 10^n + 3.4^{n+2} + k = 9\lambda\ (where\ \lambda \in I)$

$\Rightarrow P(1): 10 + 3.4^3 + k = 9\ \lambda_1$
$\Rightarrow P(1): 202 + k = 9\lambda_1 \ldots\ldots\ldots(1) Also, P(2): 100 + 3.4^4 + k = 9\lambda_2$

$\Rightarrow P(2): 868 + k = 9\lambda_2 \ldots\ldots\ldots\ldots(2)$

From (1) and (2), then minimum value of k is 5 such that P(n) is divisible by 9.

3. $x^n - 1$ is divisible by $k - k$. Then the least +ve integral value of k is

(a) 1, (b) 2, (c) 3, (d) none

Sol.: Let $P(n): x^n - 1 = \lambda(x-k)$

$Now, P(1): x - 1 = \lambda_1(x-k)\ Also, P(2): x^2 - 1 = \lambda_2(x-k)$
$\Rightarrow P(2): (x-1)(x+1) = \lambda_2(x-k)$

∴ Least value of k which the proposition P(n) is true is k = 1.

4. $If\ 1 + 5 + 12 + 22 + 35 + \cdots$ to n terms $= \frac{n^2(n+1)}{2}$, nth term of series is

(a) $\frac{(2n-1)}{2}$, (b) $\frac{n(3n-1)}{2}$, (c) $\frac{2n(2n+1)}{3}$, (d) none.

Sol.: Let $P(n): 1 + 5 + 12 + 22 + 35 + \cdots$ (n terms)

$= \frac{n^2(n+1)}{2}\ nth\ term\ of\ LHS = P_{(n)} - P_{(n-1)}$

$\Rightarrow P_{(n)} - P_{(n-1)} = \frac{n^2(n+1)}{2} - \frac{(n-1)^2 n}{2}$

$\Rightarrow P_{(n)} - P_{(n-1)} = \frac{n}{2}\{n^2 + n - n^2 + 2n - 1\}$

$\therefore T_n = P_{(n)} - P_{(n-1)} = \frac{n}{2}(3n-1)$

5. $1^3 + 2^3 + 3^3 + \cdots + 100^3 = k^2$, $then\ k =$

(a) 1010, (b) 5050, (c) 10501, (d) none

Sol.: $1^3 + 2^3 + 3^3 + \cdots + 100^3 = k^2 \Rightarrow \sum_{n=1}^{100} n^3 = k^2$

$$\Rightarrow \left\{\frac{100(100+1)}{2}\right\}^2 = k^2 \Rightarrow k = \frac{100.101}{2} = 5050$$

Note that k here will not be negative as k is sum of cubes of +ve integers.

6. Sum till n terms of the series $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}=$

(a) $\frac{4n^2+3}{5}$, (b) $\frac{n(n+1)^3}{6}$, (c) $\frac{n(2n^2+9n+13)}{24}$, (d) none

Sol.: Let $P_{(n)} = \frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\cdots(n\ terms)$

$\Rightarrow P_{(n)} : \sum\frac{1^3+2^3+\cdots+n^3}{1+3+5\ldots\ldots.(n\ terms)} \Rightarrow P_{(n)}: \sum\left\{\frac{\sum n^3}{n^2}\right\}$

$\Rightarrow P_{(n)}: \sum\left\{\frac{1}{4}\frac{n^2(n+1)^2}{n^2}\right\} \Rightarrow P_{(n)}: \frac{1}{4}\sum(n^2+2n+1)$

$\Rightarrow P_{(n)} : \frac{1}{4}\{\sum n^2+2\sum n+\sum(1)\}$

$\Rightarrow P_{(n)}: \frac{1}{4}\left\{\frac{n(n+1)}{2}+\frac{1}{3}n(n+1)(2n+1)+n\right\}$

$\Rightarrow P_{(n)}: \frac{1}{24}n\,\{3(n+1)+2(n+1)(2n+1)+6\}$

$\therefore P_{(n)}: \frac{1}{24}n(2n^2+9n+13)$

7. $\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots$ up to(n+1) terms is equal to

(a) $\frac{2n}{n+1}$, (b) $\frac{n}{n+1}$, (c) $\frac{2n}{n+2}$ (d) none

Sol.: Let $P_{(n)}: \frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots(n+1)terms$

$$t_n = \frac{1}{1+2+3+\cdots+n} = \frac{2}{n(n+1)} \Rightarrow t_n = 2\left[\frac{1}{n}-\frac{1}{n+1}\right]$$

$$\Rightarrow P_{(n)}: \left\{\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right\}$$

$$\therefore P_{(n)} = 2\left(1-\frac{1}{n+1}\right) = \frac{2n}{n+1}$$

8. For all n∈ N, $\int_0^\pi \frac{\sin 2nx}{\sin x}\boldsymbol{dx} =$

(a) $-\pi$, (b) 0, (c) $\frac{\pi}{2}$ (d) none

Sol.: Let $P_{(n)} = \int_0^\pi \frac{\sin 2nx}{\sin x}dx$

STEP I : $\Rightarrow P_{(1)} = \int_0^\pi \frac{\sin 2nx}{\sin x}dx = 2\int_0^\pi \cos x = 0$

$For\ n = 1$

STEP II: $\Rightarrow$ For n = n_1, Let us assume that the proposition is true for n = m

$\therefore P_{(m)} = \int_0^\pi \frac{\sin 2kx}{\sin x}dx = 0 \ldots\ldots\ldots..(1)$

STEP III: Let us prove the proposition for n = m +1. If the proposition is true for n = m+1, then it was also true for n = m. Thus the proposition will be true for all n∈ N.

$$P_{(m+1)} - P_{(m)} = \int_0^\pi \frac{\sin[2(k+1)x]-\sin 2kx}{\sin x}dx$$

$$P_{(m+1)} + P_{(m)} = \int_0^\pi \frac{2\cos[(2k+1)x]-\sin x}{\sin x}\,dx$$

$$\Rightarrow P_{(m+1)} - P_{(m)} = \left[\frac{\sin(2k+1)\,x}{2k+1}\right]_0^\pi$$

$$\Rightarrow P_{(m+1)} - P_{(m)} = \frac{2}{2k+1}(0-0) = 0$$

$\therefore P_{(m+1)} = P_{(m)}$

Since the proposition is true for n = m+1, thus it was also true for n = m. Hence the proposition is true ∀ n ∈ N.

9. $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\cdots$ **to n terms**

(a) $\frac{1}{5n-1}$, **(b)** $\frac{1}{5n+1}$, **(c)** $\frac{1}{n+4}$, **(d) none**

Sol.: Let $P_{(n)}: \frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\cdots n\ terms$

$$\Rightarrow P_{(n)}: \frac{1}{3}\left(1-\frac{1}{4}\right)+\frac{1}{3}\left(\frac{1}{4}-\frac{1}{7}\right)+\frac{1}{3}\left(\frac{1}{7}-\frac{1}{10}\right)+\cdots(n\ terms)$$

$$\Rightarrow P_{(n)}: \frac{1}{3}\left\{1-\frac{1}{4}+\frac{1}{4}-\frac{1}{7}+\cdots+\frac{1}{(3n-2)}-\frac{1}{3n+1}\right\}$$

$$\Rightarrow P_{(n)}: \frac{1}{3}\left\{1-\frac{1}{3n+1}\right\} \therefore\ P_{(n)}: \frac{n}{3n+1}.$$

10. For every positive integer $n, \frac{n^7}{7}+\frac{n^5}{5}+\frac{2n^3}{3}-\frac{n}{105}$

(a) an integer (b) a rational number (c) an odd integer (d) none

Sol.: Let $P_{(n)}: \frac{n^7}{7}+\frac{n^5}{5}+\frac{2n^3}{3}-\frac{n}{105}$

$$\Rightarrow P_{(1)}: \frac{1}{7}+\frac{1}{5}+\frac{2}{3}-\frac{1}{105} \Rightarrow P_{(1)} = \frac{15+21+70-1}{105}$$

$$= 1 \Longrightarrow\ P_{(2)}: 8\left(\frac{16}{7}+\frac{4}{5}+\frac{2}{3}\right)-\frac{2}{105} = 15$$

∴ By induction $P_{(n)}$ is an integer ∀ n ∈ N.

(SUBJECTIVE TYPE)

1) Given $a_{n+1} = 3a_n - 2a_{n-1}\ and\ a_0 = 2, a_1 = 3$**, show that** $a_n =\ 2^n + 1$ **∀ n∈ N**

Sol.: Step I. For n = 1 $\quad a_1 =\ 2^1+1 = 3$

$n = 2$

$a_{1+1} = 3a_1 - 2a_0 = 3_{(3)} - 2_{(2)} = 5$

Therefore, the result is true for $a_2 =\ 2^2 + 1\quad n = 1$ and 2

Step II: Assume that the result is true for n = k.

$a_k =\ 2^k + 1$

Step III: for n = k +1

$a_{k-1} = 3a_k - 2a_{k-1} \qquad = 3\left(2^k+1\right) - 2\left(2^{k-1}+1\right) = 3.2^k + 3 - 2^k - 2 = 2.2^k + 1 =\ 2^{k+1} + 1$

(By the induction assumption)

This shows that the result is true for n = k +1. Hence by the principle of mathematical induction. The result is true for all n ∈ N.

2) Prove that $\frac{d^xy}{dx^n} =\ (-1)^{n-1}.(n-1)!\sin\left\{n\frac{\pi}{2}-y\right\}\ sin^n\left\{\frac{\pi}{2}+y\right\}\forall\ n\ \in N, where\ y =\ \tan^{-1}x.$

Sol.: Let $P_{(n)} = \frac{d^xy}{dx^n} =\ (-1)^{n-1}.(n-1)!\sin\left\{n\left(\frac{\pi}{2}-y\right)\right\}\ sin^n\left\{\frac{\pi}{2}+y\right\}$ ……………(1)

Step I: For n = 1

LHS of (1) $\frac{dy}{dx} = \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} + \frac{1}{1+tan^2y}$

$$(\because\ y =\ \tan^{-1}x) = \frac{1}{sec^2y}$$

$$= cos^2y\ \ldots\ldots\ldots\ldots\ldots(2)$$

$$= \sin\left(\frac{\pi}{2}-y\right)\sin\left(\frac{\pi}{2}-y\right)$$

$$= \ (-1)^{1-1}(1-1)!$$

$$\sin\left(\frac{\pi}{2}-y\right)\sin\left(\frac{\pi}{2}+y\right) = R.H.S\ of\ (1)$$

Therefore, P(1) is true.

Step II: Assume it is true for n = k, then

$P_{(k)}: \frac{d^k y}{dx^k} = (-1)^{k-1}(k-1)! \sin\left\{k\frac{\pi}{2} - y\right\} sin^k\left\{\frac{\pi}{2}+y\right\}$

Step III: For n = k+1.

$P(k+1): \frac{d^{k+1}y}{dx^{k+1}} = (-1)^k k! \sin\left\{k + 1\left(\frac{\pi}{2} - y\right)\right\} sin^{k+1}\left\{\frac{\pi}{2}+y\right\}$

L. H. S.

$\frac{d^{k+1}y}{dx^{k+1}} = \frac{d}{dx}\left[\frac{d^k y}{dx^k}\right] = \frac{d}{dx}\left\{(-1)^{k-1}(k-1)! \sin\left\{k\frac{\pi}{2} - y\right\} sin^k\left\{\frac{\pi}{2}+y\right\}\right\}$

(by assumption step)

$$= (-1)^{k-1}(k-1)!\frac{d}{dy}\left[\sin\left\{k\left(\frac{\pi}{2} - y\right)\right\} sin^k\left(\frac{\pi}{2}+y\right)\right]$$

$$\frac{dy}{dx} = (-1)^{k-1}(k-1)!$$

$\left[\sin\left\{k\left(\frac{\pi}{2}-y\right)\right\}.\frac{d}{dy} sin^k\left(\frac{\pi}{2}+y\right) + sin^k\left(\frac{\pi}{2}+y\right)\frac{d}{dy}\sin\left\{k\left(\frac{\pi}{2}-y\right)\right\}\right]\frac{dy}{dx}$

$= (-1)^{k-1}(k-1)!\left[\sin\left\{k\left(\frac{\pi}{2} - y\right)\right\} k\ sin^{k-1}\left(\frac{\pi}{2}+y\right)\cos\left(\frac{\pi}{2}+y\right) - sin^k\left(\frac{\pi}{2}+y\right)\cos\left\{k\left(\frac{\pi}{2} - y\right)\right\}\right].cos^2 y\ from\ (2)$

$= (-1)^{k-1}(k-1)!\ sin^{k-1}\left(\frac{\pi}{2} + y\right)\left[\sin\left\{k\left(\frac{\pi}{2}-y\right)\right\}\cos\left(\frac{\pi}{2}+y\right) - \sin\left(\frac{\pi}{2}+y\right)\cos\left\{k\left(\frac{\pi}{2}-y\right)\right\}\right] sin^2\left(\frac{\pi}{2}+y\right)$

$= (-1)^{k-1}k!\ sin^{k+1}\left(\frac{\pi}{2}+y\right)\sin\left\{\frac{k\pi}{2} - ky - \frac{\pi}{2} - y\right\}$

$= (-1)^{k-1}k!\ sin^{k+1}\left(\frac{\pi}{2}+y\right)\sin\left\{(k-1)\frac{\pi}{2} - (k+1)y\right\}$

$= (-1)^{k-1}k!\ sin^{k+1}\left(\frac{\pi}{2}+y\right)\times - \sin\left\{\pi + (k-1)\frac{\pi}{2} - (k+1)y\right\}$

$= (-1)^{k-1}k!\ sin^{k+1}\left(\frac{\pi}{2}+y\right)\sin\left\{(k+1)\frac{\pi}{2} - (k+1)y\right\}$

$= (-1)^{k-1}k!\ sin^{k+1}\left(\frac{\pi}{2}+y\right)\sin\left\{(k+1)\left(\frac{\pi}{2} - y\right)\right\}$

$= R.$ H. S.

This shows that the result is true for n = k+1. Hence by the principle of mathematical induction the result is true for all n ∈ N.

3) Given that $a_r, b_r,$ and c_r are (+) ve real numbers for r = 1, 2, 3, ..., n and that
$a_r^2 = b_r^2 + c_r^2, r = 1, 2, \ldots, n$
$A_n = a_1 + a_2 + \cdots + a_n$; $B_n = b_1 + b_2 + \cdots + b_n$;
$C_n = c_1 + c_2 + \cdots + c_n$
Prove that $A^2 n \geq B^2 n + C^2 n$

Sol.: Let $P_{(n)}: A^2 n \geq B^2 n + C^2 n$

Step I: For n = 1, $A_1 = a_1, B_1 = b_1, C_1 = c_1$

$Since, a_1^2 = b_1^2 + c_1^2\ therefore, A_1^2 \geq B_1^2 + C_1^2$
So the result is true for n = 1

Step II: Assume it is true for n = k, then ${A_k}^2 \geq {B_k}^2 + {C_k}^2$

Step III: For n = k+1

Now $A_{k+1} = A_k + a_{k+1}, B_{k+1} = B_k + b_{k+1}, C_{k+1} = C_k + c_{k+1}$
Let $B_k = r\cos\theta$ and $C_k = r\sin\theta$

$$\therefore B^2k + C^2k = r^2 \tan^{-1}\left(\frac{C_k}{B_k}\right) = \theta$$

$\because B^2k + C^2k \leq A^2k$ (by assumption step)
$\therefore r^2 \leq A^2k \ i.e. r \leq A_k$
$Further, since\ {a_{k+1}}^2 \geq {b_{k+1}}^2 + {c_{k+1}}^2 (for\ r = k+1)$

$$1 = \left(\frac{b_{k+1}}{a_{k+1}}\right)^2 + \left(\frac{c_{k+1}}{a_{k+1}}\right)^2$$

$$Put\ \frac{b_{k+1}}{a_{k+1}} = \cos\alpha, \frac{c_{k+1}}{a_{k+1}} = \sin\alpha$$

$So\ that\ B_{k+1} = B_k + b_{k+1} = r\cos\theta + a_{k+1}\cos\alpha$ and $C_{k+1} = C_k + c_{k+1} = r\sin\theta + a_{k+1}\sin\alpha\ then$
${B_{k+1}}^2 + {C_{k+1}}^2 = r\cos\theta + a_{k+1}\cos(\theta - \alpha) \leq r^2 + {a_{k+1}}^2 + 2ra_{k+1}$ $(\because \cos(\theta-\alpha) \leq 1) = (r + a_{k+1})^2$
$= {A_{k+1}}^2$ $(\because r \leq A_k)$
$Thus\ {B_{k+1}}^2 + {C_{k+1}}^2 < {A_{k+1}}^2$

This shows that the result is true for n = k+1. Hence, by the principle of mathematical induction the result is true for all n ∊ N.

4) Prove that $\boldsymbol{\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{(-1)^n . n!}{x^{n+1}}\left(\log x - 1 - \frac{1}{2} \ldots . . \frac{1}{n}\right) \forall\, n \in N\ and\ x = 0}$.

Sol.: **Step I:** For n = 1 L. H. S. $\frac{d}{dx}\left(\frac{\log x}{x}\right) = \frac{1-\log x}{x^2}$

$= \frac{(-1)^1 1}{x^{1+1}}(\log x - 1)$ R. H. S. which is true for n = 1

Step II: Assume it is true for n = k.

$i.e. \frac{d^k}{dx^k}\left(\frac{\log x}{x}\right) = \frac{(-1)^k K!}{x^{k+1}}\left(\log x - 1 - \frac{1}{2} \ldots . \frac{1}{k}\right)$

Step III: For n = k +1

We have $\frac{d^{k+1}}{dx^{k+1}}\left(\frac{\log x}{x}\right) = \frac{d}{dx}\left[\frac{d^k}{dx^k}\left(\frac{\log x}{x}\right)\right]$

$$= \frac{d}{dx}\left[\frac{(-1)^k K!}{x^{k+1}}\left(\log x - 1 - \frac{1}{2} \ldots . . \frac{1}{k}\right)\right]$$

(by assumption)

$= (-1)^k\ K!\left[\frac{1}{x^{k+1}}\left(\frac{1}{x}\right) + \left(\log x - 1 - \frac{1}{2} \ldots . . \frac{1}{k}\right) - \left(\frac{(k+1)}{x^{k+2}}\right)\right]$

$= \frac{(-1)^k K!}{x^{k+2}}\left[1 - (k+1)\left[\log x - 1 - \frac{1}{2} \ldots . - \frac{1}{k}\right]\right]$

$= \frac{(-1)^{k+1} K!}{x^{k+2}}\left[(k+1)\left(\log x - 1 - \frac{1}{2} - \cdots - \frac{1}{k}\right) - 1\right]$

$= \frac{(-1)^{k+1}(k+1)!}{x^{k+2}}\left[\log x - 1 - \frac{1}{2} - \cdots - \frac{1}{k} - \frac{1}{k+1}\right]$

This shows that the result is true for n = k+1. Hence by the mathematical induction, the result is true for all n ∊ N.

5) Let $\boldsymbol{f . R \to R}$ such that $\boldsymbol{f(x+y) = f(x). f(y) \forall\ x, y\ \in R}$, to show that $\boldsymbol{f(n) = k^n}$, where k is real number and n is non-negative integer. Hence or otherwise show that $\boldsymbol{f(x) = k^x}$ for all rational x.

Sol.: $\because f(x+y) = f(x)f(y) \ldots \ldots \ldots \ldots \ldots (1)$

$Putting\ x = 0, y = 0\ we\ get\ f(0) = 1, let\ k = f_{(1)}$

We will show that $f_{(n)} = k^n\ for\ all\ n \geq 0$

Step I: For $n = 0, f(0) = k^0 1$, so the result is true for n = 0.

Step II: Assume it is true for n = p, when p is positive integer $f_{(p)} = k^p$

Step III: For n = p + 1

$f_{(p+1)} = f_{(p)}f_{(1)} = k^p.k = k^{p+1}$ (by assumption step)

This show that the result is true for all non-negative integers.

Again if p is negative integer $i.e. p = -m$. Where m is positive integer. Then we can write

$f_{(0)} = f_{(p+m)} \Longrightarrow f_{(p)}f_{(m)} = 1\ (by\ assumption\ step)$

$\Longrightarrow f_{(p)}k^m = 1$

$\Longrightarrow f_{(p)} = k^{-m} \Longrightarrow f_{(p)} = k^p$ it is true for negative integer

Finally let $n = \frac{p}{q}$ when p, q> 0 and then

$f_{(p)} = f_{(nq)} = f(n + n + \cdots q\ times)$

$= f_{(n)}f_{(n)}f_{(n)} \ldots\ldots q\ factors = \left[f_{(n)}\right]^2$

$\therefore f_{(p)} = \left[f_{(n)}\right]^q\ (by\ assumption\ step)$

$\Longrightarrow f_{(n)} = k^{\frac{p}{q}} = k^n.$

Hence the result is true for all rational number.

6) **Show that for all n ∈ N,**

$$\sqrt{\boldsymbol{a} + \sqrt{\boldsymbol{a} + \sqrt{\boldsymbol{a} + \cdots + \sqrt{\boldsymbol{a}}}}} \leq \frac{\mathbf{1}+\sqrt{\mathbf{4}\boldsymbol{a}+\mathbf{1}}}{\mathbf{2}}$$

where 'a' is fixed positive number and n radical signs are taken on L.H.S.

Sol.: Let $P_{(n)} = \sqrt{a + \sqrt{a + \sqrt{a + \cdots + \sqrt{a}}}} < \frac{1+\sqrt{(4a+1)}}{2}$

Step I: For n = 0, then $\sqrt{a} < \frac{1+\sqrt{(4a+1)}}{2}$

$\Longrightarrow 2\sqrt{a} < 1 + \sqrt{(4a+1)} \Longrightarrow 4a < 1 + 4a + 1 + 2\sqrt{(4a+1)}$
$\Longrightarrow 2\sqrt{(4a+1)} + 2 > 0$ which is true
$Therefore, P_{(1)}\ is\ true$
Step II: Assume it is true for n = k, then

$P_{(k)}$: $\sqrt{a + \sqrt{a + \sqrt{a + \cdots + \sqrt{a}}}} < \frac{1+\sqrt{(4a+1)}}{2}$
K-radical signs
Step III: For n = k+1
$P_{(k+1)}$: $\sqrt{a + \sqrt{a + \sqrt{a + \cdots + \sqrt{a}}}} < \frac{1+\sqrt{(4a+1)}}{2}$
$(k+1)radical\ signs$
For assumption step
$\sqrt{a + \sqrt{a + \sqrt{a + \cdots + \sqrt{a}}}} < \frac{1+\sqrt{(4a+1)}}{2}$
k-radical signs
$\Longrightarrow$

$$\sqrt{a + \sqrt{a + \sqrt{a + \cdots + \sqrt{a}}}}$$
$$< a + \frac{1 + \sqrt{(4a+1)}}{2}$$

$$\Rightarrow \sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\cdots+\sqrt{a}}}}}$$

$$< \sqrt{a+\frac{1+\sqrt{(4a+1)}}{2}}$$

$$= \sqrt{\frac{2a+1+\sqrt{(4a+1)}}{2}}$$

$(k+1)$ radical signs

$$= \sqrt{\frac{4a+2+2\sqrt{(4a+1)}}{4}}$$

$$= \sqrt{\frac{\left(\sqrt{4a+2}\right)^2+1+2\sqrt{(4a+1)}}{4}}$$

$$= \sqrt{\left(\frac{1+\sqrt{(4a+1)}}{2}\right)^2} = \frac{1+\sqrt{(4a+1)}}{2}$$

$$= \sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\cdots+\sqrt{a}}}}}$$

$$< \frac{1+\sqrt{(4a+1)}}{2}$$

$(k+1)$ radical signs

Which is true for n = k +1
Hence by the principle of mathematical induction the result is true for all n ∈ N.

7) Show that for all $\boldsymbol{n \geq 1}$, $\boldsymbol{(XY)_n = \sum_{r=0}^{n} n_{C_r}\ X_{n-r}\ Y_r}$ where X and Y are functions of x and $\boldsymbol{x_n}$ denotes the nth derivation of X with respect to x.

Sol.: **Step I:** For n = 1,

$(XY)_1 = XY_1 + YX_1 = X_0Y_1 + Y_0X_1 = \sum_{R=0}^{1} 1_{C_r} X_{1-r}Y_r$ which is true for all n = 1

Step II: Assume it is true for n = k, i.e. $(XY)_k = \sum_{r=0}^{k} k_{C_r} X_{k-r}Y_r$

Step III: For n = k+1, we have $(XY)_{k+1} = \left[k_{C_0}X_{k-0}Y_0 + k_{C_1}X_{k-1}Y_1 + k_{C_2}X_{k-2}Y_2 + \cdots + k_{C_k}X_0Y_k\right]_1$

$= k_{C_0}(X_kY_1 + Y_0X_{k+1}) + k_{C_1}(X_{k-1}Y_2 + Y_1X_k) + k_{C_2}(X_{k-2}Y_3 + Y_2X_{k-1}) + \cdots + k_{C_k}(X_0Y_{k+1} + Y_1X_k)$

$= k_{C_0}X_{k+1}Y_0 + \left(k_{C_0} + k_{C_1}\right)X_kY_1 + \left(k_{C_1} + k_{C_2}\right)X_{k-1}Y_2 + \left(k_{C_2} + k_{C_3}\right)X_{k-1}Y_3 + \cdots + \cdots + k_{C_k}X_0Y_{k+1}$

$$= k+1_{C_0}X_{k+1}Y_0 + k+1_{C_1}X_kY_1 + k+1_{C_2}X_{k-1}Y_2 + \cdots + k+1_{C_{k+1}}X_0Y_{k+1}$$

$= \sum_{r=0}^{k+1} k+1_{C_r} X_{k+1-r}Y_r$.

This shows that the result is true for n = k+1. Hence by the principle of mathematical induction, the result is true for all n ∈ N.

8) Suppose the natural number are divided into groups (1); (2, 3); (4, 5, 6); (7, 8, 9, 10); ……… and that every second group is deleted. Prove that the sum of the terms of the first k groups, which remain after deletion, is always $\boldsymbol{k^4}$.

Sol.: The remaining groups are (1); (4, 5, 6); (11, 12, 13, 14, 15); ……….. **step I.** Sum of the terms in first group = 1 = 1^4 and sum of the terms in the terms in the 1st two groups = 1+(4+5+6)= 16= 2^4. The result is true for k = 1 and k = 2 **step II.** Assume that the sum of the terms of the first k groups k^4

Step III. Now considering the $(k+1)th$ group

1st term in the 1st group = 1

1st term in the 2nd group = (1+2)+1= 4

1st term in the 3rd group = (1+2+3+4)+1= 11

……………………………………………

……………………………………………

1st term in the (k+1) th group = (1+2+3+…+2k)+1

=k(1+2k)+1 = ($2k^2$+k+1)

The number of terms in the (k+1)th group = (2k +1)

∴ The sum of the terms in the (k+1) the group

$$= \frac{(2k+1)}{2}[2\,(2k^2+k+1)+2k]$$

$$= (2k+1)(2k^2+2k+1)$$

$$= 4k^3+6k^2+4k+1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1)$$

Sum of the terms in the first (k+1) groups = (sum of first k group) + (sum of terms in the (k+1)th group) = $k^4+4k^3+6k^2+4k+1$ by assumption step and (1) = $(k+1)^4$.

This show that the result is true for n = k+1. Hence, by the principal of mathematical induction, the result is true for all n ∈N.

9) Show that $\int_0^{\frac{\pi}{2}} \cos^n x \cos nx\, dx = \frac{\pi}{2^{n+1}}$

Sol.: Let $I_n = \int_0^{\frac{\pi}{2}} cos^n x \cos nx\, dx = \frac{\pi}{2^{n+1}}$

Step I: For n = 1, $I_1 = \int_0^{\frac{\pi}{2}} \cos x \cos x\, dx$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}(1+\cos 2x)\,dx$$

$$= \frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{2^2}$$

Therefore, the result is true for n -1

Step II: For n = k+1

$$I_{k+1} = \int_0^{\frac{\pi}{2}} cos^{k+1}x \cos(k+1)x\,dx$$

$$= \int_0^{\frac{\pi}{2}} cos^k x\,(\cos(k+1)x\;\;dx)dx$$

$$I_{k+1} - I_k = \int_0^{\frac{\pi}{2}} cos^k x\,\{\cos(k+1)x\cos x - \cos kx\}dx$$

$$= \int_0^{\frac{\pi}{2}} cos^k x\,\{\cos(k+1)x\cos x - \cos(k+1)x\cos x - \sin(k+1)x\sin x\}dx$$

$$\{\because \cos kx = \cos[(k+1)x - x] = \cos(k+1)\,x\cos x + \sin(k+1)x\sin x\}$$

$$= \sin(k+1)\,x.\,cos^k x(-\sin x)dx$$

Integrating by parts

$$I_{k+1} - I_k = \left[\sin(k+1)\,x\,\frac{cos^{k+1}x}{(k+1)}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}(k+1)\cos(k+1)x\,\frac{cos^{k+1}}{(k+1)}\,dx = 0 - I_{k+1}$$

$$I_{k+1} = \frac{1}{2}I_k = \frac{1}{2}.\frac{\pi}{2^{k+1}}.I_{k+1} = \frac{\pi}{2^{k+2}}$$

This shows that the result is true for n = k+1. Hence, by the principal of mathematical induction, the result is true for all n ∈ N.

10) Show that $\sum_{k=0}^{n} k^2\, n_{C_k} = n(n+1)2^{n-2}$ $for\ n \geq 1$.

Sol.: Let $P_{(n)} = \sum_{k=0}^{n} k^2\, n_{C_k}$

$$= n(n+1)2^{n-2}\ for\ n \geq 1.$$

Step I: For n = 1

$$P_{(1)} = \sum_{k=0}^{1} k^2\, 1_{C_k} = 0 + 1\ 1_{C_1} = 1$$
$$= 1(1+1)\ 2^{1-2}$$

$which\ is\ true\ for\ n = 1$

Step II: Assume it is true for n –m

$$i.e. P_{(m)} = \sum_{k=0}^{m} K^2\, m_{C_k} = m(m+1)2^{m-2}$$

Step III: For n = m+1

$$P_{(m+1)} = \sum_{k=0}^{m+1} k^2\, m + 1_{C_k} = 0 + \sum_{k=1}^{m+1} K^2$$

$$m + 1_{C_k} = \sum_{k=1}^{m} k^2\, m + 1_{C_k} + (m+1)^2 m + 1_{C_{m+1}}$$
$$= \sum_{k=1}^{m} k^2$$

$$\left(m_{C_k} + m_{C_{k-1}}\right) + (m+1)^2$$
$$= \sum_{k=1}^{m} k^2\, m_{C_k} + \left[\sum_{k=1}^{m} k^2\, m_{C_{k-1}} + (m+1)^2\right]$$
$$= P_{(m)} + \sum_{k=1}^{m+1} k^2\ m_{C_{k-1}}$$

$$= P_{(m)} + \sum_{k=0}^{m} (k+1)^2\, m_{C_k} (Replace\ k\ by\ k + 1)$$

$$= P_{(m)} + \sum_{k=0}^{m} (k^2 + 2k + 1)\, m_{C_k}$$
$$= P_{(m)} + \sum_{k=0}^{m} k^2\, m_{C_k} + 2\sum_{k=0}^{m} k\, m_{C_k} + \sum_{k=0}^{m} m_{C_k}$$

$$= 2P_{(m)} + 2_m 2^{m-1} + 2^m (from\ binomial\ theorem)$$

$$= 2m(m+1)2^{m-2} + 2_m . 2^{m-1} + 2. 2^{m-1}$$

$$= (m+1)(m+2)2^{m-1}$$

This show that the result is true for n = m+1. Hence, by the principal of mathematical induction, the result is true for all n ∊ N.

11) Show that the sequence $\{a_n\}$, where $a_n = \frac{1.3.5....(2n-1)}{2.4.6.....2n}\sqrt{(2n+1)}$ is a monotonic decreasing sequence.

Sol.: $\therefore a_n = \frac{1.3.5......(2n-1)}{2.4.6.....2n}\sqrt{(2n+1)}$

Step I. For n= 1, 2 $a_1 = \frac{\sqrt{3}}{2}, a_2 = \frac{1.3}{2.4}\sqrt{(4+1)} = \frac{3\sqrt{5}}{8}$

$Here\ a_1 > a_2\ (\therefore 2 > 1)$

It is monotonic decreasing function which is true for n = 1, 2

Step II. Assume it is true for n = k, then $a_{k-1} > a_k (\because k > k-1)$

Step III. For n = k+1

a_{k+1}
$= \frac{1.3.5\ldots\ldots(2k-1)(2k+1)}{2.4.6\ldots\ldots2k(2k+2)} . \sqrt{(2k+3)}$
$= \frac{a_k}{\sqrt{(2k+1)}} . \frac{(2k+1)-\sqrt{(2k+3)}}{(2k+2)}$

$\Rightarrow a_{k+1}$
$= \frac{a_k\sqrt{(2k+1)(2k+3)}}{(2k+2)} \ldots\ldots\ldots\ldots\ldots(1)$

$Assume\ \frac{\sqrt{(2k+1)(2k+3)}}{(2k+2)} < 1 \ldots\ldots\ldots.(2)$

$then\ (2k+1)(2k+3) < (2k+2)^2$

$\Rightarrow 4k^2+8k+3 < 4k^2+8k+4 \Rightarrow 0 < 1$ which is true.

From (1) and (2), we get $a_{k+1} < a_k (\because k+1 > k)$

This show that the result is true for n = k+1. Hence, by the principal of mathematical induction, the result is true for all n ∈ N.

12) If 'a' fixed real number ≥ 2, then show that $\frac{1+a+a^2+\cdots+a^n}{a+a^2+\cdots+a^{n-1}} \geq \frac{n+1}{n-1}, n \in N, n > 1$.

Sol.: let $P_{(n)}: \frac{1+a+a^2+\cdots+a^n}{a+a^2+\cdots+a^{n-1}} \geq \frac{n+1}{n-1}$

Step I. For n = 2 $P_{(2)}: \frac{1+a+a^2}{a} \geq \frac{3}{1}: 1+a+a^2 \geq 3a$

$: a^2-2a+1 \geq 0 \quad : (a-1)^2 \geq 0$
$which\ is\ true\ for\ n=2$

Step II. Assume it is true for n = k, then

$$P_{(k)}: \frac{1+a+a^2+\cdots+a^k}{a+a^2+\cdots+a^{k-1}} \geq \frac{k+1}{k-1}$$

Step III. For n = k+1, we have to show that

$P_{(k+1)}: \frac{1+a+a^2+\cdots+a^k+a^{k+1}}{a+a^2+\cdots+a^{k-1}+a^k} \geq \frac{k+2}{k}$
$or\ 1+a+a^2+\cdots+a^k+a^{k+1} \geq \frac{k+2}{k} \left(a+a^2+\cdots+a^k\right) \ldots\ldots\ldots\ldots\ldots.(1)$
$By\ assumption\ step, we\ get$
$1+a+a^2+\cdots+a^k \geq \left(\frac{k+1}{k-1}\right)\left(a+a^2+\cdots+a^{k-1}\right)$
$Adding\ a^{k+1}$ to both sides, we get
$1+a+a^2+\cdots+a^k+a^{k+1} \geq \left(\frac{k+1}{k-1}\right)\left(a+a^2+\cdots+a^{k-1}\right) + a^{k+1} \ldots\ldots\ldots(2)$
$assume\ \left(\frac{k+1}{k-1}\right)\left(a+a^2+\cdots+a^{k-1}\right) + a^{k+1} \geq \left(\frac{k+2}{k}\right)\left(a+a^2+\cdots+a^k\right) \ldots\ldots\ldots\ldots\ldots.(3)$
$then \left(\frac{k+1}{k-1} - \frac{(k+2)}{k}\right)\left(a+a^2+\cdots+a^{k-1}\right) + a^{k+1} \geq \left(\frac{k+2}{k}\right)a^k$
$\Rightarrow \frac{2}{k(k-1)}\left(a+a^2+\cdots+a^{k-1}\right) - \frac{(k+2)}{k}a^2 \geq 0$
$\Rightarrow 2\left(a+a^2+\cdots+a^{k-1}\right) + k(k-1)a^{k+1} - \frac{k+2}{k}a^k \geq 0$
$\Rightarrow 2\left(a+a^2+\cdots+a^{k-1}\right) + a^k\{(k-1)ka - (k+2)\} \geq 0$
$\Rightarrow 2\left(a+a^2+\cdots+a^{k-1}\right) + a^k(k-1)(k(a-1) - 2) \geq 0$

Which is true

$$\begin{cases} \because a \geq 2 \; and \; k \geq 2 \\ \therefore k-1 \geq 1, k(a-1)-2 \geq 0 \end{cases}$$

From (2) and (3), we get

$$1 + a + a^2 + \cdots + a^k + a^{k+1} \geq \left(\frac{k+2}{k}\right)(a + a^2 + \cdots + a^k)$$

$$\therefore \frac{1 + a + a^2 + \cdots + a^k + a^{k+1}}{a + a^2 + \cdots + a^k} \geq \frac{k+2}{k} \ldots\ldots\ldots (4)$$

Hence, the statement (4) is true for n = k+1 and by the principle of mathematical induction, it is true for all natural numbers.

13) Let $a_1, \ldots, a_n$ be positive integers s.t. $a_1 \leq \cdots \leq a_n$.

Prove that $\frac{1}{a_1} + \cdots + \frac{1}{a_n} = 1 \Rightarrow a_n < 2^n$

Sol.: Suppose $a_n \geq 2^n$! By backward induction, we prove that $a_k \geq 2^k! \, for \; k = 1, \ldots, n$

Suppose that the assumption is proved for $k = n, n-1, \ldots\ldots, m+1$. Then,

$$\frac{1}{a_m} \leq \sqrt[m]{\frac{1}{a_1 \ldots\ldots a_m}} \leq m\sqrt{1 - \frac{1}{a_1} - \cdots - \frac{1}{a_m}}$$

$$= m\sqrt{1 - \frac{1}{a_{m+1}} + \cdots + \frac{1}{a_n}} \leq m\sqrt{\frac{1}{a_1 \ldots a_m}}$$

$$\leq m\sqrt{\sum_{i=m+1}^{n} \frac{1}{2^i!}} \leq \frac{1}{2m!}$$

It remains to be observed that $\frac{1}{2^{1!}} + \frac{1}{2^{2!}} + \cdots + \frac{1}{2^{k!}} < 1$.

14) The positive integers $x_1, x_2, \ldots, x_n \; and \; y_1, y_2, \ldots, y_n$ are given. The sums $x_1 + x_2 + \cdots + x_n \; and \; y_1 + y_2 + \cdots + y_n$ are equal and less than mn. Prove that one may cross out some of the terms in the equality $x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n$. So that one, again gets an equality

Sol.: The conditions of the problem imply that $S = x_1 + \cdots + x_m = y_1 + \cdots + y_n$ is at least 2 (since $m \leq s, n \leq s, s < mn$). If $m = n = 2, 2 \leq S \leq 3$, the assertion is easy to check. We prove in the general case by induction on $m + n = k$, if $k \geq 4. Let \; x_1 > y_1$be the largest numbers among $x_i \; and \; y_i$ respectively ($1 \leq i \leq m, 1 \leq j \leq n$). The case $x_i = y_i$ is obvious. To apply the induction hypothesis to the equality $(x_1 - y_1) + x_2 + \cdots + x_m = y_2 + \cdots + y_n \; with \; k - 1 = m + n - 1$ on both sides, it is sufficient to check the inequality $S' = y_2 + \cdots + y_n < m(n-1). Since, y_1 > \frac{s}{n}, we \; have \; S' < S - \frac{s}{n} = mn\frac{(n-1)}{n} = m(n-1)$.

NUMBER THEORY

(Objective Type)

1) If the unit digit in $459 \times 46 \times 28 \times 484$ is 2, then the digit in place of * is

(a) 3; (b) 5; (c) 7; (d) none

Sol.: $(9 \times 6 \times 4) = 216$. In order to obtain 2 at the unit place he must multiply 216 by 2 or 7.

$\therefore$ of the given numbers, we have 7.

2) **If the unit's digit in $(\mathbf{3127})^{\mathbf{173}}$ is**
   (a) 1; (b) 3; (c) 7; (d) none

Sol.: Unit digit is $(3127)^{173}$= unit digit in 7^{173}. Now, 7^4 gives unit ……digit 1.

$\therefore 173 = (7^4)^{43} \times 7^1$. Thus 7^{173} gives unit digit 7.

3) $\mathbf{4^{61} + 4^{62} + 4^{63} + 4^{64}}$ **is divisible by**
   (a) 3; (b) 10; (c) 11; (d) none

Sol.: $4^{61}(1 + 4 + 4^2 + 4^3) = 4^{61} \times 85 = 4^{60} \times 340$ which is clearly divisible by 10.

4) **A number when divided successively by 4 and 5 leaves remainders 1 and 4 respectively. When it is successively divided by 5 and 4, then the respectively remainder will be**
   (a) 2, 3; (b) 3, 2; (c) 4, 1; (d) none

Sol.:

$\therefore$ y = $(5 \times 1 + 4) = 9$

$\therefore$x = (4x+1)=4 × 9+1= 37

Now, 37 when divided successively by 5 and 4.

$\therefore$ Respectively remainders (2, 3)

5) **When the sum $\mathbf{1^5 + 2^5 + 3^5 + \cdots + 99^5 + 100^5}$ is divided by 4, then remainder is**
   (a) 1; (b) 2; (c) 3; (d)none

Sol.: We see that the unit of 5th power of any no. is the unit digit of the same no………… So, the sum of the unit digit of 5th powers of numbers from 1 to 100 = $(0 + 1 + 2 + \cdots + 9) \times 10 = 450$

The remaining parts of the given sum will be divisible by 4. Now, if we divide 450 by 4, then the quotient is 112 and the remainder is 2, which is the required remainder.

(6) **The last two digits in $\mathbf{19^{1991}}$ is**
   (a) 17; (b)18; (c) 19; (d) none

Sol.:

Because $19^{10} = 1\ (and\ 100), 19^{1991} = (19^{10})^{199}.19 = 1.19\ (mod\ 100)$. The last two digit number is 19.

(7) **The sum $\mathbf{1^{1999} + 2^{1999} + \cdots + 2000^{19}}$ is multiple of**
   (a) 1998; (b) 1999; (c) 2000; (d) none

Sol.:

Because $\sum_{1001}^{2000} i^{1999} = \sum_{1}^{1000}(j + 1000)^{1999} = \sum_{1}^{1000}(j - 1001)^{1999} = \sum_{1}^{1000}(-k)^{1999}$

$= \sum_{1}^{1000}(i)^{1999}\ (mod\ 2001)$, it follows that

$$\sum_{1}^{2000} i^{1999} = \sum_{1}^{1000} i^{1999} + \sum_{1}^{2000} i^{1999}$$
$$= \sum_{1}^{1000} i + \sum_{1}^{1000} (-i)^{1999}$$
$$= 0\ (mod\ 2001)$$

(8) **January 1, 2000, falls on a Saturday, the day of the week will January 1, 2020 is**
   (a) Sunday; (b) Wednesday; (c) Friday; (d) none

Sol.: Because there are 20 years in the range 2000-2019 of which five are leap years, January 1, 2020, falls on day 20+5≡ 4 (mod 7) i.e. Wednesday.

(9) **The number Zeros at the end, if 100! Is fully expanded and written out is**
   (a) 23; (b) 24; (c) 25; (d) none

Sol.: Highest power of 2 in 100! is 97. Similarly power of 5 in 100! is 24.

Each pair of 5 and 2 will give rise to a 10 or a zero at the end. Hence the number of zero in 100! is equal to 24.

(10) If n is positive odd integer., then $n^3 - n$ is divisible by
(a) 15; (b) 20; (c) 24; (d) none

Sol.: $n^3 - n = (n-1)n(n+1)$= product of three consecutive positive intergers and is divisible by 3!= 6.

Also $(n-1) and\ (n+1)$ are consecutive even integers and their product is divisible by 4.

Hence $n^3 - n$ is divisible by $6 \times 4 = 24$ if n is positive odd.

(11) Sum of all the divisors of 360 excluding 1 and itself is
(a) 1170; (b) 924; (c) 809; (d) 723

Sol.: $N = 360 = 2^3 \times 3^2 \times 5^1 = {P_1}^{\alpha_1}.{P_2}^{\alpha_2}.{P_3}^{\alpha_3}$

The sum of all the distinct positive integral divisor of 360

$$= \left(\frac{{P_1}^{\alpha_1+1} - 1}{P_1 - 1}\right)\left(\frac{{P_2}^{\alpha_2+1} - 1}{P_2 - 1}\right)\left(\frac{{P_3}^{\alpha_3+1} - 1}{P_3 - 1}\right)$$

$$= \frac{2^4 - 1}{2 - 1}.\frac{3^3 - 1}{3 - 1}.\frac{5^2 - 1}{5 - 1} = \frac{15}{1}.\frac{26}{2}.\frac{24}{4} = 1170$$

This includes 1 and the number 360. Thus sum of all the divisors of 360 excluding 1 and itself is $1170 - 361 = 809$.

(12) The sum of all cubes of three consecutive integers is divisible by
(a) 9; (b) 12; (c) 18; (d) none

Sol.: Try by trial method

$$\left.\begin{aligned} 1^3 + 2^3 + 3^3 &= 36 \\ 2^3 + 3^3 + 4^3 &= 36 \\ 3^3 + 4^3 + 5^3 &= 36 \end{aligned}\right\} \text{All these numbers are}$$

divisible by 9.

(13) $2^{2^n} + 1$ where n is a positive integer greater than one ends in
(a) 5; (b) 7; (c) 9; (d) none

Sol.: For $2^{2^n} + 1\ for\ n > 1, 2^{2^n} + 1 = 2^4 + 1 = 17\ for\ n = 2$

$2^{2^n} + 1 = 2^8 + 1 = 257\ for\ n = 3\ , 2^{2^n} + 1 = 2^{16} + 1 = 56537\ for\ n = 4$ and so, on we observe that it ends with 7.

(14) When $2^n - 1$ is a prime, then the sum of the reciprocals of all the divisors of the number $2^n(2^n - 1)$ is
(a) $\frac{1}{2^n}$; (b)$\frac{1}{2^{n+1}}$; (c) $\frac{1}{2}$; (d) none

Sol.: Sum of all reciprocals of all the divisors of the number $2^n(2^n - 1)$. Sum of all divisors including

$$= \frac{1\ and\ itself\ 2^{n-1}(2^n - 1)}{2^{n-1}.(2^n - 1)}$$

$$= \frac{2^n(2^n - 1)}{2^{n-1}.(2^n - 1)} = 2$$

(Subjective Type)

1) Prove that $\frac{13}{2^{70}} + 3^{70}$

Sol.: Observe that $2^{70} + 3^{70} = 4^{35} + 9^{35}$, and that 35 is odd. Now $a^n + b^n$ is divisible by $a + b$ when n is odd.

From this is follows that $4^{35} + 9^{35}$ is divisible by 13.

2) Find the number of positive integer n less than 1991 for which $\frac{6}{n^2} + 3n + 2$.

Sol. Note that $n^2 + 3n + 2 = (n+1)(n+2)$ and that $6 = 2 \times 3$. So if 6 is to be a divisor of $n^2 + 3n + 2$, then either (a) 6 is divisor of $n + 1$; or (b) 6 is a divisor of $n + 2$; or (c) 3 is a divisor of $n + 1$, and 2 is a divisor of $n + 2$; or (d) 2 is a divisor of $n + 1$ and 3 is a divisor of $n + 2$.

Possibility (a) holds for n = 5, 11, 17, ………… , 1991, or 332 values in all. Possibility (b) holds for n = 4, 10, 16, ….. , 1990 another 332 values. Possibility (c) holds for n = 2, 8, 14, …… , 1998, another 332 values, and possibility (d) holds for n = 1, 7, 13, ….. , 1987, yet another 332 values. So there are $4 \times 332 = 1328$ values of n between 1 and 1991 for which $n^2 + 3n + 2$ is divisible by 6.

3) Prove that the positive integers that have an odd number of divisors are the squares.

Sol.: We know this by exhibiting a pairing between the divisors of an integer. Let n be a given positive integer and let d be any divisors of n. Then n/d is an integer, and it is also a divisors of n because $n = (d \times n/d)$. If n is not a square, then each divisors acquires one and precisely one mate The divisors now get grouped into pairs, and this tell us that the number of divisor is even (for it is twice the number of pairs.)

4) Given that a, b, c are positive integers with no common factor and such that $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$, then prove that (a+ b) is a square.

Sol.: This is clearly to be analyzed along the same links as problem 7, but it is an trickier one. From the relation $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$. We obtain $c(a+b) = ab \text{ or } ab - c(a+b) = 0$.

Adding c^2 to both sides and factorizing we get $(a-c)(b-c) = c^2$.

So $a - c \text{ and } b - c$ are a pair of complementary factors of c^2. Suppose that $a - c \text{ and } b - c$ share a prime factor P.

Then p^2 is a divisor of c^2, so p is a divisor of c. This means that P is a divisors of a and b as well which cannot happen as a and b are coprime.

Since the product of the coprime numbers $a - c \text{ and } b - c$ is a square, each of them is a squares. Let $a - c = u^2, b - c = \frac{c^2}{u^2}$, where u is a divisors of c. This leads to $a + b = c + u^2 + c + \frac{c^2}{u^2} = u^2 + 2c + \frac{c^2}{u^2} = \left(u + \frac{c}{u}\right)^2$, and so a+ b is a square.

5) Find the number of pairs (x, y) of integers for which $2xy - 5x + y = 55$.

Sol.: We need to find all pairs (x, y) of integers such that $2xy - 5x + y = 55$. Write the equation as $y(2x+1) = 5x + 55, or\ y = \frac{5x+55}{2x+1}$. From this equation it follows that $2x + 1$ is a divisor of $5x + 55$. Since 5x +55= $2(2x+1) + (x+53)$, it further follows that $2x + 1$ is a divisor of x +53 and therefore also of 2(x+53) = 2x +106. Since 2x +106= $(2x+1) + 105$, this means that 2x +1 is a divisors of 105.

Next, since $105 = 3 \times 5 \times 7$, the divisors of 105 are $\pm1, \pm3, \pm5, \pm7, \pm15, \pm35, and \pm 105$. Since 2x +1 may assume any of these values, the possible values of x are $0, -1, 1, -2, 2, -3, 3, -4, 7, -8, 10, -11, 17, -18, 52\ and -53$. The values of y are readily found from the relation $y = \frac{(5x+55)}{(2x+1)}$ and we obtain the following pairs (x, y), that solve the given equation

$(-53, 2), (-18, 1), (-11, 0), (-8, -1), (-4, -5),$

$(-3, -8), (-2, -15), (-1, -50), (0, 55),$

$$(1,20),(2,13),$$

$$(3,10),(7,6),(10,5),(17,4)\,and\;(52,3)$$

or sixteen pairs in all.

6) N is a 50 digit number (in base -10). All digits except the 26th (from the left) are 1. Given that N is divisible by 13, find its 26th digit.

Sol.: Let x be the 26th digit of N. In base −10.

$$N = \underbrace{11111111\ldots\ldots1111}_{25\ ones}\,x\ \underbrace{11111111\ldots\ldots1111}_{24\ ones}$$

We now apply the 'alternating 3-digit sum' testing divisibility by 13. Recalling that the sum has to be computed starting from the right. We computer the All sum as $A + \overline{11}x - B + 11$, where

$$A = \underbrace{111 - 111 + \cdots - 111}_{8\ blocks}\ B$$
$$= \underbrace{111 - 111 + \cdots + 111}_{7\ blocks}$$

Clearly A = 0 and B = 111. So, the All sum is $\overline{11}x - 100\ or\ \overline{1}x$, which obviously is divisible by 13 only for x = 3.

Hence, the 26th digit of N is 3.

7) Find integers a, b such that $x^2 - x - 1$ is a divisor of the polynomial $ax^{17} + bx^{16} + 1$

Sol.: This is a difficult problem, and its solution should be studied with great care. A few new ideas are used in the analysis, which were not discussed in the earlier chapters.

Consider the equation $x^2 - x - 1 = 0$. It has two solutions of which one is the number ϕ (the 'golden ratio') given by $\phi = \frac{\sqrt{5}+1}{2} =$ 1.6180339 ………… It can be shown (by arguments similar to those used earlier, in showing that $\sqrt{2}$ is not rational), that ϕ is not rational; that is, it does not equal the ratio of two non-zero integers. This has the following implication. If c and d are integers such that cϕ +d = 0, then c and are both 0. For if cϕ +d = 0 and c ≠ 0 then we have $\phi = -\frac{d}{c}$, a rationed number. Since this can't be, we must have c = 0; but this forces d to be 0 too.

By definition, substituting 0 for x in $x^2 - x - 1$ yields 0, Since $ax^{17} + bx^{16} + 1$ is a multiple of $x^2 - x + 1$, substituting ϕ for x in $ax^{17} + bx^{16} + 1$ must yield 0; that is $a\,\phi^{17} + b\phi^{16} + 1 = 0$. Since the result of substituting ϕ in $x^2 - x - 1$yields 0, we must have $\phi^2 = \phi + 1$. This relation allows us to express all powers of ϕ in the form cϕ +d where c and d are integers. For example, $\phi^3 = \phi + \phi^2 = \phi \times (\phi + 1) = \phi^2 + \phi = 2\phi + 1$, the relation, $\phi = \phi +$ I being used repeatedly. Similarly, $\phi^4 = \phi \times \phi^3 = \phi \times (2\phi + 1) = 2\phi^2 + \phi = 3\phi + 2$. In general, if we have expressed ϕ^{n-1} in the form cϕ+d, then, we have $\phi^n = \phi \times \phi^{n-1} = \phi \times (c\phi + d) = c\phi^2 + d\phi = c(\phi + 1) + d\phi = (c + d)\phi + c$

Since we have already expressed ϕ^4 in the stated form, We may now do the same for any higher power of ϕ . We display below some of the results of these computations,

$$\phi^5 = 50 + 3, \quad \phi^6 = 8\phi + 5,$$

$$\phi^7 = 1.3\phi + 8, \qquad \phi^8 = 21\phi + 13, \ldots \ldots$$

$$\ldots\ldots\ldots\ldots \qquad \phi^{15} = 610\phi + 377,$$

$$\phi^{16} = 987\phi + 610, \quad \phi^{17} = 1597\phi + 987,$$

As already noted that if c and d are integers, such that cϕ +d = 0, then we have c = 0 and d = 0. Since a and b are integers, so are

$1597_a + 987_b$ *and* $987_a + 610_b + 1$ *and thus we deduce that*

$$1597_a + 987_b = 0, 987_a + 610_b = -1$$

This is a pair of simultaneously equations in a and b, and it is easily solved (by the usual elimination technique) we obtain a = 987, b = -1597. We thus obtain the required answer and we see that there is just one pair of integers (a, b) such that $x^2 - x - 1$ is a divisors of $ax^{17} + bx^{16} + 1$; namely (a, b) = (987, -1597).

8) Find all prime numbers P such that the number $P^2 + 11$ has exactly 6 divisors.

Sol.: We first note that the choice p =2 does not work as $2^2 + 11 = 15 = 3 \times 5$ has 4 and not 6 divisors. So P must be an odd prime, implying that $P^2 + 11$ is even and therefore contains the prime 2 as a factor.

We know now use the formula for d(n), the divisor function. Since $6 = 3 \times 2$, there are precisely two categories of number with 6 divisors, those of the kind q^5 (with q prime) and those of the kind q^2r(with q, r unequal primes). So, if $P^2 + 11$ has 6 divisors, then $P^2 + 11 = q^5$ *or* q^2r where q, r are primes, q ≠ r. The 1st case is quickly ruled out, our earlier observation tells us that q = 2, but this does not work, as there is no prime P with $P^2 + 11 = 2^5$.

Now we shall consider $P^2 + 11 = q^2r$. Observe that P = 3 works; for $3^2 + 11 = 20 = 2^2 \times 5$, which has 6 divisors. We need to only consider the case when P>3. Since P is prime, it is indivisible by 3, so p ≡±1, which means that $p^2 \equiv 1 \ (mod\ 3)$ and therefore that $P^2 + 11 \equiv 0 \ (mod\ 3)$ so, 3 is a divisor of $P^2 + 11$. This means that q, r are 2, 3 in some order. However, neither possibility works; for neither $2^2 3 = 12$ *nor* $3^2 2 = 12$ is the form $P^2 + 11$ for any time P.

9) Find the positive integer n for which the following holds if its divisors are listed in increasing order as $d_1, d_2, d_3, \ldots,$ *with* $d_1 = 1$ *then* $n = d_{13} + d_{14} + d_{15}$ *and* $(d_5 + 1)^3 = d_{15} + 1$

Sol.: An impossible problem? Not quite; Let $a = \frac{n}{d_{13}}, b = \frac{n}{d_{14}}$ *and* $c = \frac{n}{d_{15}}$. Then since $d_{13} + d_{14} + d_{15} = n$ *and* $d_{13} < d_{14} < d_{15}$, we deduce that $a > b > c$, *and* $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$.

In this equation we must have $c > 1$. (*If* $c = 1$, *the left side would exceed*1.) If $c \geq 3$, then a>b>3, and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$; the above equation could never hold good so we must have c>3, which implies that c = 2 and therefore that a>b>2 and $\frac{1}{a} + \frac{1}{b} = \frac{1}{2}$ Next, suppose that $b \geq 4$. Then a >4 and $\frac{1}{a} + \frac{1}{b} < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$; the equation $\frac{1}{a} + \frac{1}{b} = \frac{1}{2}$ could never hold good so, b<4, which means that b= 3. Since $\frac{1}{a} + \frac{1}{b} = \frac{1}{2}$. We get a = 6. So the values of a, b, c are now all known using these values we see that $d_{13} = \frac{n}{6}, d_{14} = \frac{n}{3}$ *and* $d_{15} = \frac{n}{2}$. Three deductions now follows:

(a) n is divisible by 6, so $\frac{2}{n}, \frac{3}{n}$;

(b) as n cannot have a proper divisor greater than $\frac{n}{2}$ (this is true for any n), there can be only divisor after d_{15}, namely n itself; implying that n has exactly 16 divisors, with $d_{16} = n$;

(c) $\frac{n}{2} + 1$ is a cube (its cube root is $d_5 + 1$)

Next, suppose that 5 is a divisor of n. Then, depending upon whether n is divisible by 4 or not, the first five divisors of n are either1, 2, 3, 4, 5 or 1, 2, 3, 5, 6. So either $d_5 = 5$ or $d_5 = 6$. The first possibility leads to $\frac{n}{2} + 1 = (5+1)^3 = 216$ or $n = 430$. while the second leasts to $\frac{n}{2} + 1 = (6+1)^3 = 343$ or $n = 684$. But 430 is not divisible by 6 (whearas, it is already know that $\frac{6}{n}$) and 684 is not divisible by 5. This contradictory state of affairs tells us that n does not have 5 as a prime factor. Since 16 has following five factorization; 16 $=8 \times 2 = 4 \times 4 = 4 \times 2 \times 2 = 2 \times 2 \times 2 \times 2$, there are five classes of numbers with 16 divisors; those of the kinds $p^{15}, p^7 \times q, p^3 \times q^3, p^3 \times q \times r$ and $p \times q \times r \times s$ where p, q, r, s are distinct primes, each greater than 5. We shall now consider each of these cases in turn.

We already know that n is divisible by the primes 2 and 3, so the possibility n = p^{15} is ruled out.

If $n = p^7 \times q$. then we must have $n = 2^7 \times 3 = 384$ or $n = 3^7 \times 2 = 4374$; $while\ if\ n = p^3 \times q^3, then\ we\ must\ have\ n = 2^3 \times 3^3 = 216$. In one of these three cases is $\frac{n}{2} + 1$ a cube. So, this possibility too is ruled out.

If $p \times q \times r \times s$, then, as earlier, 2, 3 ∊ {p, q, r, s}. So, n is of the form 6pq where p, q are distinct primes, say with $5 < pq$. The 16 divisors of n are in this case 1, 2, 3, 6, $p(= d_5), \dots, pq(= d_{13}), 2pq(d_{14}, 3pq = d_{15}), 6pq$. The relation $\frac{n}{2} + 1 = (d_5 + 1)^3$ reduces to $3pq + 1 = (p + 1)^3$, $which\ yeilds$ $3p = p^2 + 3p + 3$. Since 3 is a divisor of each of the quantities 3q, 3p and 3, we must have $\frac{3}{p^2}$, which is absurd as p is a prime number greater than 5. So this possibilities is also ruled out.

If $n = p^3 \times q \times r$, $then$ $2, 3 \in \{p, q, r\}$, so n is one of the following forms: $24p, 54p, 6p^3$, where p is a prime number greater than 5. Each of these cases must now be considered. In each case, the first four divisors of n are 1, 2, 3, 6. We quickly ascertain that p = 7 does not yield a solution because none of the numbers $12 \times 7 + 1 = 85, 27 \times 7 + 1 = 190, 3 \times 7^3 + 1 = 1030$ is a cube. Therefore, p> 7, which in facts means that $p \geq 11$. If n = 24p then the next divisor after 6 is $d_5 = 8$, yielding $\frac{n}{2} + 1 = (8+1)^3 = 729$, or $n =$ 1456, but this cannot be, as 24 is not even a divisor of 1456.

If n = 54p then the next divisor after 6 is $d_5 9$, $yeilding$ $\frac{n}{2} + 1 = (9+1)^3 = 1000$, or $n = 1998 = 54 \times 37$. Therefore p = 37. Does this fit the given condition the 16 divisors of 1998 are 1, 2, 3, 6, 9, 18, 27, 37, 54, 74, 111, 222, 333, 666, 999 and 1998, therefore $d_5 = 9, d_{13} = 333, d_{14} = 999$. The conditions do indeed hold good!

Finally, of $n = 6p^3$, $with$ $p \geq 11$, $then$ $d_5 = p$. So we obtain the equation $3p^3 + 1(p+1)^3$, or $3p^3 = p^3 + 3p^2 + 3p$. This implies that $\frac{3}{p}$, an absurdity. So there is no solution to be found here.

Thus there is just one number which fits the given conditions: $n = 1998 = 2 \times 3^3 \times 37$.

10) Let n be the positive integer with at least 4 divisors and let its divisors is be $d_1, d_2, d_3, d_4, \dots$, where $d_1 < d_2 < d_3 < d_4 < \cdots$, $with$ $d_1 = 1$. find all possible values of n it is known that n = $d_1{}^2 + d_2{}^2 + d_3{}^2 + d_4{}^2$

Sol.: Obviously $d_1 = 1$. If n were odd, then all its divisors would be old and 1+$d_2{}^2 + d_3{}^2 + d_4{}^2$ would be a sum of four old numbers, therefore even; a contradiction. So n is even, which means that $d_2 = 2$. Let d_3 and d_4 be denoted by a and b, for the case of writing; then $n = 5 + a^2 + b^2, with \frac{a}{n}, \frac{b}{n},\ 2 < a < b$.

Suppose that a is even. Then $a\ = 2^2 = 4$ (the only possibility) so $n = 21 + b^2$ and since b must be old, it is the least odd prime divisor of n. Since $\frac{2b}{n}$ it follows that $\frac{2b}{21} + b^2, so\,\frac{b}{21}$ and therefore b = 3 or 7, which means that n = 30 or 70. It may be checked however that neither 30 nor 70 fits given conditions.

So a is odd, and therefore is even, can be 4?

Then it can only be the case that a = 3 and $n = 5 + 3^2 + 4^2 = 30$. But 30 does not fit the conditions so b > 4, and since b is first even divisors of after 2, it must happen that b = 2a. So, $n =\ 5 + a^2 + (2a)^2\ or\ n = 5 + (1 + a^2)$. Since $\frac{a}{n}$ we must have $\frac{a}{5} + (1 + a^2)$. Since a and 1+a^2 have no factors in common, we conclude that $\frac{a}{5}$ ∴ a = 5, forcing $n = 5 \times 26 =$ 130. And this does fit the given conditions: The first four divisors of 130 are 1, 2, 5, 10 and indeed 130 = $1^2 + 2^2 + 5^2 + 10^2$. So there is precisely one number which satisfies the given conditions.

11) Let p > 2 be a prime suppose, the sum $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}$ be denoted by $\frac{Ap}{Bp}$ where Ap and Bp are co-prime positive integers. For example, when p = 5, we get the sum $\frac{25}{12}, so\ A_5 = 25, B_5 = 12\ and\ when\ p =$ 11 we get the sum $\frac{7381}{2520}, so\ A_{11} =$ $7381, B_{11} =$ $2520\ observe\ that \frac{5}{A_5}\ and\ \frac{11}{A_{11}}$. Show that $\frac{P}{A_p}$ is always hold.

Sol.: Since $P - 1$ is an even number, we pay pair the numbers 1, 2, 3, …, P -1 thus; $\{1, p-1\}, \{2, p-2\} \ldots \ldots,$ the sum of the numbers in each pair being P. Now observe that $\frac{1}{1} + \frac{1}{p-1} = \frac{p}{p-1}, \frac{1}{2} + \frac{1}{p-2} = \frac{p}{2(p-2)}$, and more generally, for any i($1 < i \le p - 1$)

$\frac{1}{i} + \frac{1}{p-i} = \frac{p}{i(p-i)}$ so the sum $\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{(p-1)}$ may be written in the from $\frac{p}{p-1} + \frac{p}{2(p-2)} +$ $\frac{p}{3(p-3)} \ldots \ldots, with\,\frac{(p-1)}{2}$ fractions in all, each with a numerator p. The 1cm of the denominators of these fraction is not divisible by p, as the numbers $p - 1, 2(p-2), 3(p -$ $3)$, …., are not divisible by p(each is a product of number less than p, therefore not divisible by p). So the p in the numerator remains-it does not get cancelled away with anything in the denominator. It follows that the numerator of sum contains a factor of p. In other words, A_p is divisible by p. In fact, for primes p greater than 3, A_p is divisible not just by p but by p^2; but this is lot harder to prove and we shall not attempt it here. The reader should refer to the text by Hardy and wright for a proof.

12) If the sum $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{99} + \frac{1}{100}$ be computed and written as $\frac{A}{B}$. Where A and B are positive integer with no common factors. Show that neither A nor B is divisible by 5.

Sol.: We start by grouping the denominators 1, 2, …, 100 into different, subsets depending upon their divisibility by 5. Specifically for k = 0, 1 and 2, let S_k be the set of numbers n between 1 and 100 (both inclusive) for which power (5, n)= k. That is, so has all the non-multiple of 5, S_1 has the multiples of 5 which are not multiplies of 25, and S_2 has the multiplies of 25:

$$S_0 = \{1, 2, 3, 4, 5, 6, 7, 8, \ldots, 97, 98, 99\}$$
$$S_1 = \{5, 10, 15, 20, 30, \ldots, 85, 90, 95\}$$

$$S_2 = \{25, 50, 75, 100\}$$

Next, let A_k for k = 0, 1, 2 be defined thus:

$$A_0 = 100!\left(1! + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{99}\right).$$

$$A_1 = 100!\left(\frac{1}{5} + \frac{1}{10} + \frac{1}{15} + \frac{1}{20} + \frac{1}{30} + \cdots + \frac{1}{95}\right).$$

$$A_2 = 100!\left(\frac{1}{25} + \frac{1}{50} + \frac{1}{75} + \frac{1}{100}\right)$$

(The bracketed expressions on the right are, respectively the sums of the reciprocals of the of the numbers in s_0, s_1 and s_2) obviously, A_0, A_1, A_2 are integers (multiplication by 100! "clears the function") and $\frac{A}{B} = \frac{A_0+A_1+A_2}{100!}$.

Now observe that $A_2 = \frac{100!}{25}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) - 2p$ implying that power (5, A_2)= power (5, 100!) since power (5, 100!) = $\left[\frac{100}{5}\right] + \left[\frac{100}{25}\right] =$ $20 + 4 = 24$, we deduce that power (5, A_2) = 24.

Next, let us consider A_1. We shall make repeated use of the following readily-verified identity

$$\frac{1}{5n+1} + \frac{1}{5n+2} + \frac{1}{5n+3} + \frac{1}{5n+4} = \frac{50(2n+1)(5n^2+5n+1)}{(5n+1)(5n+2)(5n+3)(5n+4)}$$

Observe that the denominator on the right is indivisible by 5, as is the quantity $5n^2 + 5n + 1$. So, if the sum on the left is written as $\frac{a}{b}$ where a, b are coprime, then b is indivisible by 5, and power (5, a) is at least 2(by virtue of the factor of 50). Now rewrite the expression for A_1 as

$$\frac{100!}{5}\left[\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \cdots + \left(\frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19}\right)\right]$$

The dots represent two more bracketed expressions (there are four such expression in all). Each bracketed expression on the right yields a fraction with a numerator which is a multiple of 25. So it follows that, power (5, A_1) $\geq 24 - 1 + 2 = 25$. When the same argument is applied to the defining expression for A_0, it yields the following sum:

$$100!\left[\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{96} + \frac{1}{97} + \frac{1}{98} + \frac{1}{99}\right)\right]$$

And as each bracketed expression on the right yields a fraction with denominator invisible by 5 and a numerator which is a multiple of 25, we have power $(5, A_0) \geq 24 + 2 = 26$. Summarizing our finding, we have power $(5, A_0) \geq 26$,

$Power\ (5, A_1) =\geq 25$,

$Power\ (5, A_2) \geq 24$

From which it follows that power $(5, A_0 + A_1 + A_2) = 24$.

Finally using the fact that power (5, 100!) = 24, we deduce that in the fraction $\frac{A}{B} = \frac{A_0+A_1+A_2}{100!}$ the numerator and denominator are divisible by the same power of 5. This means that when common factors are cancelled away, 5 is not present as a factor in either the numerator or denominator; that is both A and B are indivisible by 5.

13) Show that if the positive integer n is such that 2n +1 and 3n +1 are both squares, then $\frac{40}{n}$.

Sol.: Let $2x + 1 = x^2$ *and* $3x + 1 = y^2$ where x and y are n for which these equations holds are shown in the following table:

x 1 9 89 881

y 1 11 109 1079 ....

n 0 40 3960 388080.........

This table is full of tantalizing patterns!

We need to so that n is a multiple of 40. Since $n = \frac{(x^2-1)}{2}$, this is the same as showing that $x^2 - 1$ is a multiple of 80. So , it suffices to show the following if x, y are integers such that $3x^2 - 2y^2 = 1, then\ x^2 - 1\ \ (mod\ 80)$.

To show this, it is enough to show that $x^2 \equiv 1(mod\ 5) and\ x^2 \equiv 1\ (mod\ 16); for\ if\ \frac{5}{x^2} - 1\ and\ also\ \frac{16}{x^2} - 1\ then\ certainly \frac{80}{x^2} - 1$.

First we note that x must be odd; for $3x^2 = 1 + 2y^2$, an odd number. Next, note that since $(\pm 1) \equiv 1(mod\ 5) and (\pm 2) \equiv 4$ mod, all squares are congruent to 0, 1 or 4 (mod 5); and since $(\pm 3)^2 \equiv (\pm 5)^2 \equiv 9$ (mod 16) all odd suppose that $x^2 = 0$ (mod 5); then $3x^2 - 1 \equiv 1$,

$\therefore 2y^2 \equiv -1, so\ y^2 \equiv 2\ (mod\ 5)$; but this is not possible. The possibility $x^2 \equiv 4$ (mod 5) is ruled out similarly it leads to $y^2 \equiv 3$ (mod 5). Therefore, $x^2 \equiv 1$ (mod 5). Next, suppose that $x^2 \equiv 9$ (mod 16); then $3x^2 - 1 \equiv 10$ (mod 16) therefore $2y^2 \equiv 10$, leading to $y^2 \equiv 5$ (mod 16) or $y^2 \equiv 13$ (mod 16). But both these are impossible! All odd squares are congruent to 1 or 9 (mod 16)

Therefore, $x^2 \equiv 1$ (mod 16)

Since $x^2 \equiv 1$ (mod 5) and $x^2 \equiv 1$ (mod 16), it follows that $x^2 \equiv 1$ (mod 80), and therefore that $\frac{40}{n}$.

14) Let T be the set of all triplets (a, b, c) of integers such that $1 \leq a \leq b \leq c \leq 6$. For each triplet (a, b, c) in T, take the number a$\times b \times c$ and odd all these nmubers corresponding to all the triplets in T. Show that this sum is divisible by 7.

Sol.: If (a, b, c) is a valid triplet then $(7 - c, 7 - b, 7 - a)$ is also a valid triple as $1 \leq (7 - c) \leq (7 - b) \leq (7 - a) \leq 6$. Note $(7 - b) \neq b$ etc.

Let S $= \sum_{1 \leq a \leq b \leq c \leq 6}(a, b, c)$ then by the above

$$S = \sum_{1 \leq a \leq b \leq c \leq 6} (7-6)(7-b)(7-c)$$

$$\therefore 2S = \sum_{1 \leq a \leq b \leq c \leq 6} [(a, b, c) + (7-a)(7-b)(7-c)]$$

In the R. H. S. every term is divisible by 7. i.e. $\frac{7}{2s}$ and hence, $\frac{7}{s}$.

15) A sequences of numbers $a_n, n - 1, 2, ...$ is defined as follows: $a_1 = \frac{1}{2}$ $and\ for\ each\ n \geq 2, a_n = \left(\frac{2n-3}{2n}\right) a_{n-1}\ show\ that\ \sum_{k=1}^{n} a_k <$ $1\ for\ all\ n \geq 1$.

Sol.: Given: $a_1 = \frac{1}{2}\ for\ n \geq 2$

$$so, a_n = \frac{2k-3}{2k} a_{k-1}\ for\ k \geq 2.$$

$$or\ 2k_{a_k} = (2k-3)a_{k-1}$$
$$\Longrightarrow 2k_{a_k} - (2k-3)a_{k-1} = 0$$

$$\Longrightarrow 2k_{a_k} - 2(k-1)a_{k-1} - a_{k-1} = 0$$

$$\Longrightarrow 2k_{a_k} - 2(k-1)a_{k-1} = -a_{k-1} \ldots\ldots\ldots\ldots..(1)$$

Now adding up Eq. (1) from k = 2 to k = (n +1), we have

$$\left.\begin{aligned} 4a_2 - 2a_1 &= -a_1 \\ 6a_3 - 2a_2 &= -a_2 \\ 8a_4 - 6a_3 &= -a_3 \\ &\vdots \\ &\vdots \end{aligned}\right\} \ldots\ldots\ldots\ldots(2)$$

$$2na_n - 2(n-1)a_{n-1} = -a_{n-1}, 2(n+1)a_{n+1} - 2na_n$$

$$= -a_n \; summing, Eq.(2), we\ get, 2(n+1)a_{n+1} - 2a_1$$

$$= -\sum_{k=1}^{n} a_k \Rightarrow \sum_{k=1}^{n} a_k = 2a_1 - 2(n+1)a_{n+1} = 1 - 2(n+1)a_{n+1}$$

$$a_1 = \frac{1}{2}, a_n = \left(1 - \frac{3}{2n}\right)a_{n-1}$$

$$\Rightarrow a_2 = \left(1 - \frac{3}{4}\right)\frac{1}{2}, = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

$$\Rightarrow a_3 = \left(1 - \frac{3}{6}\right)\frac{1}{8}$$

$= \frac{1}{2} \times \frac{1}{8} = \frac{1}{16}$ a_n: $\left(1 - \frac{3}{2n}\right)a_{n-1}$ is positive as $\left(1 - \frac{3}{2n}\right) for\ all\ n \geq 2$ is positive and $a_1, a_2, a_3, \ldots\ldots\ldots$ are all positive. Since each $a_i\ is\ a\ product\ of\ \left(1 - \frac{3}{2_i}\right)a_{i-1}\ and\ a_i > 0$ implies that $a_2 > 0 \ldots\ldots\ a_{i-1} > 0$ and hence,

$$\sum_{k=1}^{n} a_k = 1 - 2(n+1)a_{n+1} < 1. [\because 2(n+1)a_{n+1} > 0$$

16) Prove that $n = \frac{1}{8}\left\{\left(17 + 12\sqrt{2}\right)^m + \left(17 - 12\sqrt{2}\right)^m\right\} + 6$ is an integer for all n ∈ N and hence, show that both $(n-1)(2n-1)$ are perfect squares for all n ∈ N.

Sol.: As is problem 73, the terms containing $\sqrt{2}$ vanishes in the expansion of $\left(17 + 12\sqrt{2}\right)^m + \left(17 - 12\sqrt{2}\right)^m$ and integral terms are all multiplies of 8 and hence, n is an integers, (prove it) $n - 1 = \frac{1}{8} \times \left[\left(17 + 12\sqrt{2}\right)^m + \left(17 - 12\sqrt{2}\right)^m + 6 - 8\right] = \frac{1}{8} \times \left[\left(17 + 12\sqrt{2}\right)^m + \left(17 - 12\sqrt{2}\right)^m - 2\right]$ comparing the above expansion from the result of problem 71. We get,

$$17 + 12\sqrt{2} = (3 + 2\sqrt{2})^2, 17 - 12\sqrt{2} = (3 - 2\sqrt{2})^2\ again\ both\ \left(17 + 12\sqrt{2}\right)\left(17 - 12\sqrt{2}\right) and\ \left(3 + 2\sqrt{2}\right)\left(3 - 2\sqrt{2}\right) are\ equal\ to\ 1.$$

$$So, \quad \frac{1}{8} \times \left[\left(17 + 12\sqrt{2}\right)^m + \left(17 - 12\sqrt{2}\right)^m - 2\right]$$

$$= \frac{1}{8} \times \left\{\left[\left(3 + 2\sqrt{2}\right)^{m-2}\right] + \left[(3 - 2\sqrt{2})^{m-2}\right] - 2 \times \left(3 + 2\sqrt{2}\right)\left(3 - 2\sqrt{2}\right)\right\}$$

$$= \frac{1}{8} \times \left[\frac{\left(3 + 2\sqrt{2}\right)^{m^2} + \left(3 - 2\sqrt{2}\right)^{m^2}}{2\sqrt{2}}\right]$$

$$and\ 2n - 1 = \frac{1}{4} \times \left[\left(17 + 12\sqrt{2}\right)^m + \left(17 - 12\sqrt{2}\right)^m + 6 - 4\right]$$

$$= \frac{1}{4} \times \left[\left(17 + 12\sqrt{2}\right)^m + \left(17 - 12\sqrt{2}\right)^m + 2\right]$$

$= \left[\frac{(3+2\sqrt{2})^m+(3-2\sqrt{2})^m}{2}\right]$ and hence the result show that $(3+2\sqrt{2})^m - (3-2\sqrt{2})^m$ and $\frac{(3+2\sqrt{2})^m+(3-2\sqrt{2})^m}{2}$ are and so $\frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{2\sqrt{2}}$ is also an integer and hence, their sum is also an integer. Thus,

$\frac{1}{32}\left[(17+12\sqrt{2})^m + (17-12\sqrt{2})^m - 2\right]$ is a square integer. To show that Exp. (1) can be written as $\frac{1}{2}m(m+1)$ consider the Exp. (2) $\frac{1}{32} \times \left[(17+12\sqrt{2})^m + (17-12\sqrt{2})^m - 2\right]$

$$= \left\{\frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}\right\}^2 \left\{\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}\right\}^2$$

$$= \frac{1}{2}\left[\frac{\left\{(1+\sqrt{2})^n - (1-\sqrt{2})^n\right\}^2}{4}\right].$$

$$\left[\frac{\left\{(1+\sqrt{2})^n + (1-\sqrt{2})^n\right\}^2}{4}\right]$$

For all n, we shall show that

$\frac{\left\{(1+\sqrt{2})^n-(1-\sqrt{2})^n\right\}^2}{4}$ $\frac{\left\{(1+\sqrt{2})^n+(1-\sqrt{2})^n\right\}^2}{4}$ are consecutive integers clearly, for n = 1, we get

$$\frac{\left\{(1+\sqrt{2})^n - (1-\sqrt{2})^n\right\}^2}{4} = \frac{8}{4}$$

$$= 2 \text{ and } \frac{\left\{(1+\sqrt{2})^n + (1-\sqrt{2})^n\right\}^2}{4}$$

$$= \frac{4}{4} = 1 \text{ and hence for } n = 1$$

$$\frac{\left\{(1+\sqrt{2})^n + (1-\sqrt{2})^n\right\}^2}{4} \text{ and}$$

$$\frac{\left\{(1+\sqrt{2})^n - (1-\sqrt{2})^n\right\}^2}{4}$$

Are consecutive integers. For any n,

$$\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2} \text{ and } \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}$$

Are integers (prove) and hence

$$\left\{\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2}\right\}^2$$

$$= \left\{\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{4}\right\}^2$$

$$\text{and } \left\{\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2}\right\}^2$$

$$= \left\{\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{4}\right\}^2$$

are integers. Now

$$= \left\{\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{4}\right\}^2$$

$$= \frac{(1+\sqrt{2})^{2n} + (1-\sqrt{2})^{2n} - 2}{4}$$

$$= \frac{(3+2\sqrt{2})^n + (3-2\sqrt{2})^n - 2}{4} \dots\dots\dots..(3)$$

$$\text{and similarly, } \left\{\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{4}\right\}^2$$

$$= \frac{(3+2\sqrt{2})^n + (3-2\sqrt{2})^n - 2}{4} \dots\dots\dots.(4)$$

∴ From Exp (3) and (4), we find that

$$\frac{\left\{(1+\sqrt{2})^n + (1-\sqrt{2})^n\right\}}{4} \text{ and}$$

$$\frac{\left\{\left(1+\sqrt{2}\right)^n-\left(1-\sqrt{2}\right)^n\right\}}{4}$$

$are\ integers\ of\ the\ form\frac{2k-2}{4}\ and\frac{2k+2}{4}\ or\frac{1}{2}(k-1) and\frac{1}{2}(k+1)$ and hence, they differ by $\frac{1}{2}(k+1)-\frac{1}{2}(k-1)=1$.

$$So\ \frac{1}{32}\times\left\{\left(17+12\sqrt{2}\right)^n-\left(17-12\sqrt{2}\right)^n-2\right\}$$

$$=\frac{1}{2}\times\left\{\frac{\left(1+\sqrt{2}\right)^n-\left(1-\sqrt{2}\right)^n}{4}\right\}^2\times\left\{\frac{\left(1+\sqrt{2}\right)^n+\left(1-\sqrt{2}\right)^n}{4}\right\}^2$$

$=\frac{1}{2}\times\frac{(k-1)}{2}\times\frac{(k+1)}{2}\ or=\frac{1}{2}(m-1)m\ or\ equivalently\frac{m(m+1)}{2}$ and hence, the result.

Note: This $\frac{1}{32}\left[\left(17+12\sqrt{2}\right)^n+\left(17-12\sqrt{2}\right)^n-2\right]$ gives you an infinite family of square and triangular numbers.

17) Show that for $f(m)=\frac{1}{8}\left\{\left(3+2\sqrt{2}\right)^{2m+1}+\left(3-2\sqrt{2}\right)^{2m+1}-6\right\}$ $both\ f_{(m)}+1\ and\ 2f_{(m)}+1$ are perfect squares for all n∈ N by showing that $f(m)$ is an integer.

Sol.: First let us show that the expression

$$f(m)=\frac{1}{8}\left[\left(3+2\sqrt{2}\right)^{2m+1}+\left(3-2\sqrt{2}\right)^{2m+1}-6\right]$$

$$for\ m=1,\frac{1}{8}\times\left[\left(3+2\sqrt{2}\right)^{2m+1}+\left(3-2\sqrt{2}\right)^{2m+1}-6\right]$$

$$=\frac{1}{8}\times\left[2\times 3_{C_0}\times 3^3+2\times 3_{C_2}\times 3^1\times\left(2\sqrt{2}\right)^2-6\right]$$

$=\frac{1}{8}\times[54+144-6]=\frac{1}{8}\times[192]=$ 24 and hence, is an integer. For any m>1 let us prove that the expression,

$f(m+1)=\frac{1}{8}\left[\left(3+2\sqrt{2}\right)^{2m+1}+\left(3-2\sqrt{2}\right)^{2m+1}-6\right]$ is an integer. Expanding and cancelling the terms, we get

$$f(m+1)=\frac{1}{8}\times\left[\left(3+2\sqrt{2}\right)^{2m+1}+\left(3-2\sqrt{2}\right)^{2m+1}-6\right]$$

$$=\frac{1}{4}\times\left[3^{2m+1}+2m+1_{C_2}.3^{2m-1}\left(2\sqrt{2}\right)^2+2m+1_{C_4}.3^{2m-3}.\left(2\sqrt{2}\right)^4+\cdots+2m+1_{C_2}.3\left(2\sqrt{2}\right)^{2m-1}-3\right]$$

$$=\frac{1}{4}\times\left[2m+1_{C_2}.3^{2m+1}.\left(2\sqrt{2}\right)^2+2m+1_{C_4}.3^{2m-3}.\left(2\sqrt{2}\right)^4+\cdots+2m+1_{C_{2m}}.3\left(2\sqrt{2}\right)^{2m}+3^{2m+1}-3\right]$$

All the terms in the above expression except $3^{2m+1}-3$ are multiplies of 4, as the even power of $\left(2\sqrt{2}\right)$ is a multiple of 4. $3^{2m+1}-3=3[9^m-1]$ is also multiple of 4.

Now, $f(m)+1=\frac{1}{8}\times\left[\left(3+2\sqrt{2}\right)^{2m+1}+\left(3-2\sqrt{2}\right)^{2m+1}-6\right]+1$

$= \frac{1}{8} \times \left[\left(3+2\sqrt{2}\right)^{2m+1} + \left(3-2\sqrt{2}\right)^{2m+1} - 6 + 8\right]$

$= \frac{1}{8} \times \left[\left(3+2\sqrt{2}\right)^{2m+1} + \left(3-2\sqrt{2}\right)^{2m+1} + 2\right]$

$Now, 3 + 2\sqrt{2} = \left(1+\sqrt{2}\right)^2, 3 - 2\sqrt{2} = \left(1-\sqrt{2}\right)^2$

$So, \frac{1}{8} \times \left[\left(3+2\sqrt{2}\right)^{2m+1} + \left(3-2\sqrt{2}\right)^{2m+1} + 2\right]$

$= \frac{1}{8} \times \left[\left\{\left(1+\sqrt{2}\right)^2\right\}^{2m+1} + \left\{\left(1-\sqrt{2}\right)^2\right\}^{2m+1} + 2\right]$

$= \frac{1}{8} \times \left[\left\{\left(1+\sqrt{2}\right)^{2m+1}\right\}^2 + \left\{\left(1-\sqrt{2}\right)^{2m+1}\right\}^2 + 2\right]$

$= \frac{1}{8} \times \left[\left\{\left(1+\sqrt{2}\right)^{2m+1}\right\}^2 + \left\{\left(1-\sqrt{2}\right)^{2m+1}\right\}^2 - 2(-1)\right]$

$= \frac{1}{8} \times \left[\left\{\left(1+\sqrt{2}\right)^{2m+1}\right\}^2 + \left\{\left(1-\sqrt{2}\right)^{2m+1}\right\}^2 - 2 \times \left(1+\sqrt{2}\right)^{2m+1}\left(1-\sqrt{2}\right)^{2m+1}\right]$

$Since \left(1+\sqrt{2}\right)^{2m+1}\left(1-\sqrt{2}\right)^{2m+1}$

$= \left[\left(1+\sqrt{2}\right)\left(1-\sqrt{2}\right)\right]^{2m+1}$

$= (-1)^{2m+1} = -1$

So the given expression is equal to

$$\left\{\frac{\left(1+\sqrt{2}\right)^{2m+1} - \left(1-\sqrt{2}\right)^{2m+1}}{2\sqrt{2}}\right\}^2$$

$Note\ that\ \frac{\left(1+\sqrt{2}\right)^{2m+1} - \left(1-\sqrt{2}\right)^{2m+1}}{2\sqrt{2}}$ is an integer, as all the left over terms contain $2\sqrt{2}$ as a factor in the numerator.

$Now, 2f(m) + 1 = \frac{1}{4} \times \left[\left(3+2\sqrt{2}\right)^{2m+1} + \left(3-2\sqrt{2}\right)^{2m+1} - 6\right] + 1$

$= \frac{1}{4} \times \left[\left(3+2\sqrt{2}\right)^{2m+1} + \left(3-2\sqrt{2}\right)^{2m+1} - 2\right]$ Since n is shown. Now, (2n +1) can be written as

$= \frac{1}{4} \times \left[\left\{\left(1+\sqrt{2}\right)^{2m+1}\right\}^2 + \left\{\left(1-\sqrt{2}\right)^{2m+1}\right\}^2 - 2\right]$

$= \frac{1}{4} \times \left[\left\{\left(1+\sqrt{2}\right)^{2m+1}\right\}^2 + \left\{\left(1-\sqrt{2}\right)^{2m+1}\right\}^2 + 2 \times \left\{\left(1+\sqrt{2}\right)\left(1-\sqrt{2}\right)\right\}^{2m+1}\right]$

$$= \left\{\frac{\left(1+\sqrt{2}\right)^{2m+1} + \left(1-\sqrt{2}\right)^{2m+1}}{2}\right\}^2$$

By a similar reasoning, the expression

$\frac{\left(1+\sqrt{2}\right)^{2m+1} + \left(1-\sqrt{2}\right)^{2m+1}}{2}$ is an integer.

Hence, the result.

18) Suppose f is a function on the positive integers, which takes integers (i.e. f: N→Z) with the following properties
(a) $f_{(2)} = 2$, (b) $f_{(m,n)} = f_{(m)} \cdot f_{(n)}$, (c) $f_{(m)} > f_{(n)}$ if m > n. Find f(1983).

Sol.: $f_{(2)} = 2, f_{(4)} = f_{(2.2)} = f_{(2)} . f_{(2)} = 2.2 = 4,$

$f_{(8)} = f_{(2.4)} = f_{(2)}.f_{(4)} = 2.4 = 8$. Thus we infer $f_{(24)} = 2^n$ that let us use M. I for proving $f_{(2^1)} = 2$ by hypothesis………..(1)

$Assume\ f(2^n) = 2^n \ldots\ldots\ldots\ldots..(2)$

$f(2^{n+1}) = f(2.2^n) = f_{(2)}.f_{(2^n)} = 2.2^n \ldots\ldots\ldots\ldots\ldots(3)$

By hypothesis and Eq. (1) and (2), we need to find f(n) for all n.

Let us see that happens for $f_{(1)}, f_{(3)}\ at\ first\ f_{(1)} < f_{(2)}$.

$Now, f_{(2)} = f_{(1\times2)} = f_{(1)} \times f_{(2)} \Longrightarrow f_{(1)} = 1$

$similarly\ f_{(2)} < f_{(3)} < f_{(4)}, 2 < f_{(3)} < 4$

But the only integer lying between 2 and 4 is 3. Thus $f_{(3)} = 3$. So, again we guess that $f_{(n)} = n$, for all n. Let us prove by using the strong principle of mathematical induction.

Let $f_{(n)} = n$ for all n < a, fixed m ∊ N.

Now, we should prove that $f_{(m)} = m$. If m is an even integer, then $f_{(m)} = 2k, and\ k < m$.

So, $f_{(m)} = f_{(2k)} = f_{(2)} \times f_{(k)} = 2 \times k = 2k = m$. So all even m, $f_{(m)} = m$. If m is an odd integer, let m = 2k +1, and $f_{(2k)} < f_{(2k+1)} < f_{(2k+2)}, 2k < f_{(2k+1)} < f_{(2k+2)}$

(Because the function $f_{(n)} = n$ is true for all even integer n). But only integer lying between 2k and $2k + 2\ is\ 2k + 1$,(since the range of f is integer)

Thus, $f_{(k+1)} = 2k + 1, i.e. f_{(m)} = m$, in the case of odd m also. Thus, $f_{(n)} = n$, for all n∊ N.

$\therefore f_{(1983)} = 1983$.

19) Let a sequence $\boldsymbol{x_1, x_2 + x_3, \ldots \ldots}$ of complex numbers de defined by $\boldsymbol{x_1 = 0, x_{n+1} = x_{n-1}{}^2\ for\ n} > 1$ where $\boldsymbol{i^2 = -1}$. Find the distance of $\boldsymbol{x_{2000}\ from\ x_{1997}}$ in the complex plane.

Sol.: Let a sequence $x_1 = 0, x_2 = 0^2 - i$,

$$x_3 = (-i)^2 - i = -1 - i = -(1 + i),$$

$$x_4 = [-(1 + i)]^2 - i = 2i - i = i,$$

$$x_5 = (i)^2 - i = -1 - i = x_3,$$

$$x_6 = (-1 - i)^2$$

$-i = i = x_4, x_6 = x_4\ and\ hence\ x_7 = x_5\ and\ so\ on\ x_{2n} = i\ for\ n \geq 1, x_{2n+1} = -1 - i\ x_{2000} = i = (0, 1)$ in the complex plane, $x_{1997} = (-1, -i) = (-1, -1)$ in the complex plane.

So the distance between $x_{2000}\ and\ x_{1997}$ is $\sqrt{1^2 + 2^2} = \sqrt{5}$.

20) Show that $\boldsymbol{F(P_1{}^{x_1} \times P_2{}^{x_2}) = F(P_1{}^{x_1}) \times F(P_2{}^{x_2})}$

Sol.: Any divisors of $P_1{}^{\alpha_1}$is $P_1{}^{r_1}, where\ 0 \leq r \leq \alpha_1$

$F(P_1{}^{\alpha_1}) = \sum_{r=0}^{\alpha_1} T_3\,(P_1{}^r) = \sum_{r=0}^{\alpha_1} (r + 1)^3 =$ sum of the cubes of the first $\alpha_1 + 1$ natural numbers,

$$= \left[\frac{(\alpha_1 + 1)(\alpha_1 + 2)}{2}\right]^2$$

$$similarly, F(P_2{}^{\alpha_2}) = \left[\frac{(\alpha_1 + 1)(\alpha_2 + 2)}{2}\right]^2$$

$$F(P_1{}^{\alpha_1}.P_2{}^{\alpha_2}) = \sum_{\substack{0 \leq r \leq \alpha_1 \\ 0 \leq r \leq \alpha_2}} T_3(P_1{}^r.P_2{}^s)$$

$$= \sum_{r=0}^{\alpha_1}\sum_{s=0}^{\alpha_2}(r+1)^3\,(s+1)^3$$

$$= \sum_{r=0}^{\alpha_1}(r+1)^3\left(\sum_{s=0}^{\alpha_2}(s+1)^3\right)$$

$$= \sum_{r=0}^{\alpha_1}(r+1)^3\left[\frac{(\alpha_2+1)(\alpha_2+2)}{2}\right]^2$$

$$F({P_2}^{\alpha_2}).\sum_{r=0}^{\alpha_1}(r+1)^3$$

$$= F({P_2}^{\alpha_2})\left[\frac{(\alpha_1+1)(\alpha_2+2)}{2}\right]^3$$

$= F({P_2}^{\alpha_2})\,F({P_1}^{\alpha_2})$. Hence proved.

21) If n_1 *and* n_2 are two numbers, such that the sum of all the divisors of n_1 other than n_1 is equal to sum of all the divisors of n_2 other than n_2, then the pair (n_1, n_2) is called an anticable number pair. Given $a = 3.2^n - 1$, primes numbers, then shows that $(2^n ab, 2^n c)$ is an anticable pair.

Sol.: If N = ${P_1}^{\alpha_1}, {P_2}^{\alpha_2}, \dots, {P_n}^{\alpha_n}$, then sum of the divisors N is given by the formula.

$$\sum d(N) = \frac{{P_1}^{\alpha_1+1}-1}{P_1-1}\times\frac{{P_2}^{\alpha_2+1}-1}{P_2-1}\times\dots\dots\frac{{P_n}^{\alpha_n+1}-1}{P_n-1}$$

So the sum of the divisors of $2^n\, a.b = (2^{n+1}-1)\times\frac{a^2-1}{a-1}\frac{b^{2-1}}{b-1} = (2^{n+1}-1)(a+1)(b+1)$

$= (2^{n+1}-1)(9.(2^{2n-1})$

$But, 2^n\,ab = \ 2^n[9.2^{2n-1} - 9.2^{n-1} + 1](on\ simplification)$

The sum of the divisors of $2^n\ ab$ other than $2^n\ a.b$ is

$9.2^{2n-1}(2^{+1}-1) - 2^n(9.2^{2n-1} - 9.2^{n-1} + 1)$

$= 9.2^{3n} - 9.2^{2n-1} - 9.2^{3n-1} + 9.2^{2n-1} - 2^n = 9.2^{3n-1} - 2^n$

$= 2^n\ (9.2^{2n-1} - 1) = \ 2^n.c.$

Thus the sum of the divisors of $2^n.ab$ other than itself is $2^n.c$.

Now, sum of the divisors of $2^n c$ other than itself is $\frac{2^{n+1}-1}{2-1}\times\frac{c^2-1}{c+1} - 2^n.c = (2^{n+1}-1)(c+1) - 2^n.c$

$$\begin{aligned} &= (2^{n+1}-1)9.2^{2n-1} - 2^n(9.2^{2n-1}-1)\\ &= 9.2^{3n} - 9.2^{2n-1} + 2\\ &= \ 2^n[9.2^{2n-1} - 9.2^{n-1} + 1]\\ &= \ 2^n\ ab \end{aligned}$$

i.e. the sum of the divisors of $2^n c$ other than $2^n c$ equal to $2^n\ ab$.

22) If n = P_1, P_2, P_3 and P_1, P_2 and P_3 are distinct prime numbers. If $\sum_{\frac{d}{n}} d = 3N\ or\ Oc(N) = \ 3N$, then show that $\sum_{i=1}^{N}\frac{1}{di} = 3$.

Sol.: The divisors of N are $1, P_1, P_2, P_3, P_1P_2, P_1P_3, P_2P_3, P_1P_2P_3$. It is given that

$$1 + P_1 + P_2 + P_3 + P_1P_2 + P_1P_3 + P_2P_3 + P_1P_2P_3 = 3N$$

$$Now, \sum_{i=1}^{N}\frac{1}{di} = \frac{1}{1} + \frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} + \frac{1}{P_1P_2} + \frac{1}{P_1P_3} + \frac{1}{P_2P_3} + \frac{1}{P_1P_2P_3}$$

$$= \frac{P_1P_2P_3 + P_2P_3 + P_1P_3 + P_1P_2 + P_3 + P_2 + P_1 + 1}{P_1P_2P_3}$$

But the numerator is the sum of the divisors of N.

$$i.e. \sum_{\frac{d}{n}} d = 3N = 3P_1P_2P_3 \text{ and hence},$$

$$\sum_{i=1}^{N} \frac{1}{di} = \frac{3P_1P_2P_3}{P_1P_2P_3} = 3$$

23) Determine with proof all the arithmetic program with integer terms with the property that for each positive integer n, the sum of the first n terms is a perfect square.

Sol.: When n = 1, the first term itself is a perfect square

Let it be k^2. The sum to n terms of the Ap is

$$S_n = \frac{n}{2}[2a + (n-1)d], where\ a = p^2$$

Since S_n is a perfect square for every n, the nth term $2a + (n-1)d > 0$, for every n and hence d>0.

If n is an odd prime, say P, then

$$S_p = \frac{p}{2}[2a + (n-1)d]$$

Since S_p is a perfect square $P^1[2a + (p-1)d]$ i.e. $\frac{P}{[(2a-d)+pd]}$

$$But \frac{P}{pd}, so \frac{P}{(2a-d)}.$$

$This\ is\ possible\ for\ all\ prime\ p, if\ and\ only\ if\ 2a - d = 0 or\ 2a = d$ i.e. d= $2k^2$

So, the required A.p is $k^2, 3k^2, 5k^2, \dots, (2n-1)k^2$ where k is any natural number.

24) All two digit numbers from 10 to 99 are written consecutively, that is N= 101112 ..... 99.
Show that $\frac{3^2}{N}$. From which other two digit number you should start so that N is divisible by (i) 3 and (ii) 3^2.

Sol.: N is divisible by 9, if the digit sum is divisible by 9. The digit sum of N:

The number of 1^s occurring in the digits from 10 to 19 = 11 and from 20 to 99 = 8

So, that of ones is 11+8= 19. Similarly,

No of $2^s, 3^s, \dots .9$ are all equal to 19.

So, sum of all digits = $19(1 + 2 + 3 + \cdots + 9) = \frac{19\times9\times10}{2} = 19 \times 5 \times 9 = 855$ and hence, 1011……. 99 is divisible by 9.

When the numbers start from 12, the sum of the digits becomes $855 - 3 = 852$ (since 10, 11 account for the digital sum 3) and, hence is divisible by 3

(a) For divisibility by 3, it could start from 13, 15, 16, 18, 19, 21, 22, 24, 25 ……….
(b) For divisibility by $3^2 = 9$ the numbers may start from any of 18, 19, 27, 28, 36, 37 ……….

25) When the numbers from 1 to n are written in decimal notation, it is found that the total number of digit in writing all these in 1998, find n.

Sol.: To write the first nine single digit number from 1 to 9 both inclusion the no. of digit used = 9

To write the two digits number from 10 to 99, no of digit used = $(99 - 9) \times 2 = 180$.

So, the number of digit used to write numbers from 1 to 99 is 189.

Total number of digit used in writing up to n is 1998.

The total no. of all three digits numbers = $(999 \times 99) \times 3 = 2700 > 1998$. So, n should be less than 999.

No. of digits used to write the three digit numbers up to N is $1998 - 189 = 1809$.

In each 3 digit number, we use three digits.

So, the number of three digits number in N = $\frac{1809}{3} = 603$

So, therefore N = $703 - 1 = 702$

Since up to 702, there are 603, three digit numbers 90 two digit numbers and 9 one digit numbers.

26) Find all integers values of a such that quadratic expressions $(\boldsymbol{x}+\boldsymbol{a})(\boldsymbol{x}+\mathbf{1991})+\mathbf{1}$ can be factored as $(\boldsymbol{x}+\boldsymbol{b})(\boldsymbol{x}+\boldsymbol{c})$ where b and c are integers.

Sol.: $(x+a)(x+1991) += (x+b)(x+c)$

$$\Rightarrow 1991 + a = b + c \; and \; 1991a + 1 = bc$$

$$\therefore (b-c)^2 = (b+c)^2 - 4bc = (1991+9)^2 - 4(1991a+1)$$

$$= \underbrace{(1991+a)^2 - 4 \times 1991a} - 4 = (1991-a)^2 - 4$$

$$or \; (1991-a)^2 - (b-c)^2 = 4$$

If the difference between two perfect square is 4, then one of them is 4 and the other is zero. Therefore, $1991 - a = \pm 2, (b-c)^2 = 0$.

$$\Rightarrow a = 1991 + 2 = 1993 \; and \; b - c = 0 \; or \; a = 1991 - 2 = 1989 \; and \; b = c. But \; b + c = 2b = 1991 + a = 1991 + 1993 \; or \; 1991 + 1989 \;\Rightarrow b = c = 1992 \; or \; 1990.$$

So, the only 2 values of a are 1993 and 1989

27) Find the last two digit in $(\mathbf{56789})^{\mathbf{41}}$

Sol.: $56789 \equiv 89 \pmod{100} = -11 \; (mod \; 100)$

$$\therefore (56789)^{49} \equiv (-11)^{41} \; (mod \; 100) \equiv (-11)^{40} \times (-11)(mod \; 100) \equiv (11)^{40} \times (-11)(mod \; 100)$$

$11^2 \equiv 21 \; (mod \; 100), 11^4 \equiv 21 \times 4 \; (mod \; 100), 11^6 \equiv 21 \times 41 = 61(mod \; 100), 11^{10} \equiv 41 \times 61 \equiv 01(mod \; 100), 11^4 \equiv (01)^{40} \equiv 1(mod \; 100), (-11)^{41} \equiv 11^{40} \times (-11)(mod \; 100) \equiv \; 1 \times (-11)(mod \; 100) \equiv -11 \equiv 89 \; (mod \; 100).$

That is the last two digits of $(56789)^{41}$ are 8 and 9 in that order.

28) Prove that $[\boldsymbol{x}] + [\mathbf{2}\boldsymbol{x}] + [\mathbf{4}\boldsymbol{x}] + [\mathbf{8}\boldsymbol{x}] + [\mathbf{16}\boldsymbol{x}] + [\mathbf{32}\boldsymbol{x}] = \mathbf{12345}$ has no solution.

Sol.: $12345 \leq x + 2x + 4x + 8x + 16x + 32x = 63x$

$\therefore x \geq \frac{12345}{63} = 195\frac{20}{21}$, when x = 196, the L H S of the given equation becomes 12348.

$\therefore 195\frac{20}{21} \leq x < 196$. Consider x in the interval$\left(195\frac{31}{32}, 196\right)$. The L H S expression of the given equation= $195 + 0 + 390 + 1 + 780 + 3 + 1560 + 7 + 3120 + 15 + 6240 + 31 = 12342 < 12345$.

When $x < 195\frac{31}{32}$, the LHS is less than 12342.

$\therefore$ For no value of x. The given equality will be satisfied.

29) Consider the following multiplication in decimal notation $\mathbf{999 \times abc = def\,132}$. Determine the digits a, b, c, d, e, f

Sol.: Since $999 \times abc = \mathrm{def}\,132$,

$therefore\ (1000 - 1) \times abc = \mathrm{def}\,132.$

$i.e. abc\ 000 = \mathrm{def}\,132 + abc$

This implies that c = 8, b = 6, a = 8, so that $abc = 868$

$Now, 86800 - 868 = \mathrm{def}\,132\,.\,i.e. 867132 = \mathrm{def}\,132\,, so\ that\ \mathrm{def}\,867.$

This digits a, b, c, d, e, f are → 8, 6, 8, 8, 6, 7 respectively.

30) Given with justification, a natural number n for which $\mathbf{3^9 + 3^{12} + 3^{15} + 3^n}$ is a perfect cube (of an integer).

Sol.: $3^9 + 3^{12} + 3^{15} + 3^n = 3^9(1 + 3^3 + 3^6 + 3^{n-9}) = (3^3)^3\{1 + 3.3^2 + (3^2)^s + 3^{n-9} - 3.(3^2)^2\}$

$= (3^3)^3(1 + 3^2)^3, provided\ 3^{n-9} - 3^5 = 0 = (270)^3$, Provided $3^{n-9}=3^5$

i.e., provided n = 14.

31) Two prime's numbers $\boldsymbol{P_1, P_2}$ $\boldsymbol{with}$ $(\boldsymbol{P_1 < P_2})$ are called twin primes if they differ by 2. (e.g. 17, 19 or 41, 43). Prove that if $\boldsymbol{P_1, P_2}$ are twin primes with $\boldsymbol{P_1}$ bigger than 3, then $\boldsymbol{P_1 + P_2}$ is always divisible by 12.

Sol.: Since P_1, P_2 are twin primes with $P_1 < P_2$ and $P_1 < 3$, therefore $P_1, P_1 + 1, P_2$ are three consecutive integers P_1, P_2 are both odd and neither of them is divisible by 3. Therefore P_1 is of the form $6k - 1$ and P_2 is of the form $6k + 1$. Therefore $P_1 + P_2 = 12k$. i.e. $P_1 + P_2$ is a multiple of 12.

32) Determine with proof all the arithmetic progressions with integer terms with the property that for each positive integer n, the sum of the first n terms is a perfect is a perfect square.

Sol.: Let a be the first term and d the common difference.

The sum of n terms, S_n say, is given by

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

Since $S_1 (= a)$ must be a perfect square, therefore a must be a perfect square, say k^2, where k is an integer. Also since S_n is a perfect square for every n, therefore $2a + (n-1)d > 0$ for every n. Consequently d must be a integer.

Let us consider the case when n is an odd prime, say p .

Then $S_p = \frac{P}{2}\{2a + (p-1)d\}$

Since S_p must be a perfect square, and $\frac{p}{sp}$,

therefore $\frac{P^2}{S_P}, so\ that\ P|\{(2a-d) + pd\}\ i.e. P|\{2a-d\}$.

This is possible for all primes P, if and only if $2a - d = 0$

i.e., if and only if d $= 2k^2$. Therefore the A.P. is $k^2, 3k^2, 5k^2, 7k^2, \dots\dots..$

33) How many zeros are there at the end in the product of the numbers 1, 2, 3, …., 1994?

Sol.: We are required to find the highest power of 10 contained in the product 1994!

If P be the highest power of 5 contained in 1994! And q be the highest power of 2 contained in 1994! Then highest power of 10 contained in 1994! $= \min\{p, q\}$

Since $2 < 5$, therefore the highest, power of 5 contained in 1994! Is less than the highest power of 2 contained in 1994! i.e. p< q therefore $\min\{p, q\} = p$. To find p we proceed as follows: let [x] denote the greatest integer not exceeding x. Then the highest power of 5 contained in 1994! Is equal to $\sum_{k=1}^{\infty}\left[\frac{1994}{5^k}\right]$ observe that the above expression is not an infinite series because $\left[\frac{1994}{5^k}\right] = 0$,

Whenever $5^k > 1994$.

Therefore, P $= \left[\frac{1994}{5}\right] + \left[\frac{1994}{5^2}\right] + \left[\frac{1994}{5^3}\right] + \left[\frac{1994}{5^4}\right] + \left[\frac{1994}{5^5}\right] + \cdots = 398 + 79 + 15 + 3 + 0 \dots . = 495$

Hence the highest power of 10 contained in 1994! = 495.

34) If a, b, x, y are integer greater than 1, such that a and b have no common factor except 1 and $x^a = y^b$, show that $x = n^b, y = n^a$ for some integer n greater than 1.

Sol.: Since $x^a = y^b$, therefore if a prime, say p, divides x, then it must divide y^b, and consequently it must divide y as well. Similarly, if a prime, say q, divide y, then it must divide x as well. Thus we find that exactly the same primes must occur in the prime factorization of x and y.

Let $x = P_1^{e_1}\ P_2^{e_2}\ \ P_3^{\alpha_3} \dots ....$

$y = P_1^{d_1}\ P_2^{d_2}\ \ P_3^{d_3}$

$Then\ x^a = y^b \Longrightarrow (P_1^{e_1}\ P_2^{e_2}\ \dots)^a = \left(P_1^{d_1}\ P_2^{d_2} \dots .\right)^b$

$\Longrightarrow\ c_1 a = d_1 b, c_2 a = d_2 b, \dots \dots ..$

Since a prime to b, therefore it follows that a divides $d_1\ and\ b\ divides\ c_1\ i.e \dots .,$

$c_1 = u_1 b$, which gives $d_1 = u_1 a$, similarly we have

$c_2 = u_2 b, d_2 = u_2 a, \dots,$ for some positive integers u_1, u_2 …. etc.

$Let\ n = P_1^{u_1}\ P_2^{u_2}\ \ P_3^{u_3} \dots \dots . then\ n^b = x, x^a = y.$

35) Let $m_1, m_2, m_3, \dots, m_n$ be a re-arrangement of numbers 1, 2, 3, …, n. Let that n is odd. Prove that the product $(m_1 - 1)(m_2 - 2) \dots (m_n - n)$ is an even integer.

Sol.: Since n is odd, we have n = 2m +1 for some positive integer m. Out of the integer 1, 2, …, n, there are m+ 1 odd ones namely 1, 3, 5, ……, (2m+1) and m even ones, namely 2, 4, 6, ...2m. Consider the pairs $(m_1, 1), (m_2, 2), \ldots, (m_n, n)$.

Since there are m +1 odd integers among $m_1, m_2, \ldots, m_n$ and only m even integers among 1, 2, ..., n therefore at least one of the odd $m_i's$ must be paired with an odd i, consequently, for some positive integer i, $m_i - i$ must be even, and therefore the product $(m_1 - 1), (m_2 - 2), \ldots, (m_n - n)$ must be even.

36) Determine, with proof, all the positive integers n for which (i) n is not the square of any integer and (ii) $[\sqrt{n}]^3$ *divides* n^2.

Sol.: Suppose $[\sqrt{n}] = t$. Since n is not the square of any integer, therefore $\sqrt{n}$ must lie strictly between t and t +1 i.e. $t < \sqrt{n} < t + 1. So\ that\ t^2 < n < (t+1)^2$.

By hypothesis (ii), $t^3\ divides\ n^2$. This implies that t^2 must divide n^2, and consequently t must divide n. Now $t^2 + t\ and\ t^2 + 2t$ are the only positive integers lying between $t^2\ and\ (t+1)^2$ which are multiples of t. Therefore we must have either $n = t^2 + t\ or\ n = t^2 + 2t$.

Case I: $n = t^2 + 1, t^3|n^3 \Rightarrow t^3|(t^2+t)^2 \Rightarrow \frac{t}{(t+1)^2}$

$$\Rightarrow \frac{t}{1} \Rightarrow t = 1 \Rightarrow n - 2.$$

Case II: $n = t^2 + 2t, t^3|n^2 \Rightarrow t^3|(t^3 + 2t)^2 \Rightarrow \frac{t}{(t+2)^2}$

$$\Rightarrow \frac{t}{4} \Rightarrow t = 1, 2\ or\ 4 \Rightarrow n = 3, 8\ or\ 24.$$

Thus the possible values of n are 2, 3, 8, 24. By actual verification we find all these values of n satisfy the given conditions.

Thus n = 2, 3, 8, 24

37) Determine the largest 3-digit prime factor of the integer $2000_{C_{1000}}$.

Sol.: $2000_{C_{1000}} = \frac{2000.1999....1001}{1.2.3....1000}$

Every three digit prime is a factor of 1, 2, 3, ..., 1000. Also $2000_{C_{1000}}$ is an integer. So every three digit prime occurs in the prime factors of the denominator at least once. The greatest three digit prime factor of $2000_{C_{1000}}$ is the one which occurs once in the denominator and at least twice in the numerator.

Therefore it must be less than $\frac{1}{3} \times 2000$. i.e. less than 666 and as close to it as possible. (Because then only will it occur twice in the numerator and once in the denominator). Checking the numbers 666, 665, ... for primality we find that 661 is the first prime in this sequences, which is the desired answer.

38) Prove that any number N written in base 7 will be even or odd according as the sum of its digits is even or odd.

Sol.: Let us first observe that for P = 2, $\left(\frac{2^{p-1}-1}{p}\right)$ is not an integer. Therefore 2 cannot satisfy the given condition. Next, let p be a prime of the form 4k +1. Suppose $\left(\frac{2^{p-1}-1}{p}\right) = m^2$, for some odd integer m. It is obvious that m cannot be even. Then $2^{4k} - 1 = (4k+1)m^2$.

Since every perfect square leaves a remainder 1 when divided by 4, therefore R. H. S. will leave a remainder 1 when divided by 4. But the L. H. S will leave a remainder 3 when divided by 3. This is not possible. Therefore p cannot be of the form 4k +1. Let now p be of the form 4k +3. First consider the case p = 3 (i.e. k = 0)in this case $\left(\frac{2^{p-1}-1}{p}\right) = \left(\frac{2^{3-1}-1}{3}\right) = 1$

Which is a perfect square. Therefore p = 3 is one of the primes t that we are looking for.

Let Now $P = 4k + 3, with\ k > 0, 2^{p-1} - 1 = 2^{4k+2} - 1(2^{2k+1} - 1)(2^{2k+1} + 1)$.

Since $2^{2k+1} - 1\ and\ 2^{2k+1} + 1$ are relatively prime, therefore if their product is of from Pm^2, one of them must be Pu^2 and the other must be v^2, where u and v are relatively prime, Since $2^{2k+1} - 1$ is of the form $4_s + 3\ and\ v^2$ must be of the from $4_s + 1$, therefore it follows that $2^{2k+1} - 1$ cannot be of the form v^2. Therefore we must have $2^{2k+1} - 1 = Pu^2, 2^{2k+1} + 1 = v^2$.

Now, $2^{2k+1} + 1 = v^2 \Longrightarrow 2^{2k+1} = (v-1)(v+1)$, so that $v - 1\ and\ v + 1$ must be both powers of 2. Suppose $v + 1 = 2^a, v - 1 = 2^b. So\ that\ 2^a - 2^b = 2.2^{a+b} = 2^{2k+1}$

$Now, 2^a - 2^b = 2 \Longrightarrow 2^b(2^{a-b} - 1) = 2 \Longrightarrow 2^b = 1$

$2^{a-b} - 1 = 1 \Longrightarrow b = 1, a = 2\ \therefore v - 1 = 2, v + 1 = 4,$

$v^2 = 9\ 2^{2k+1} + 1 = 9 \Longrightarrow k + 1 \Longrightarrow P = 7$

Therefore the only possibility for P is 7. Since $\left(\frac{2^{7-1}-1}{7}\right) = 9$ which is a perfect square, therefore 7 is another prime that we are looking for.

Thus the only primes satisfying the given condition are 3 & 7.

39) Each of the positive integers $a_1, \dots, a_n$ is less than 1951. The least common multiple of any two of these is greater than 1951. Show that $\frac{1}{a_1} + \cdots + \frac{1}{a_n} < 2$.

Sol.: The numbers of integers from 1 to m, which are multiple of b is $\left[\frac{m}{b}\right]$. From the assumption, we know that none of the integers 1, …, 1951 is simultaneously divisible by two of the numbers $a_1, \dots, a_n$. Hence the number of integers 1, …., 1951 divisible by one of $a_1, \dots, a_n\ is\ \left[\frac{1951}{a_1}\right] + \cdots + \left[\frac{1951}{a_n}\right]$. This number does not exceed 1951. Hence $\frac{1951}{a_1} - 1 + \cdots + \frac{1951}{a_n} - 1 < 1951 \frac{1951}{a_1} + \cdots + \frac{1951}{a_n} < n + 1951 < 2.1951$.

$\frac{1}{a_1} + \cdots + \frac{1}{a_n} < 2$. This problem was used at the MMO 1951. It is due to paul Erdos. The 2 can be replaced by $\frac{6}{5}$, but even this is not the best possible bound.

40) If the positive integers x, y satisfy $2x^2 + x = 3y^2 + y$, then show that $x - y, 2x + 2y + 1, 3x + 3y + 1$ are perfect square.

Sol.: (a) From $2x^2 + x = 3y^2 + y, we\ get\ x^2 = x - y + 3x^2 - 3y^2 = (x - y)(3x + 3y + 1), y^2 = x - y + 2x^2 - 2y^2 = (x - y)(2x + 2y + 1)$. Since $3(x + y) + 1\ and\ 2(x + y) + 1$ are prime to each other and $x - y = \gcd(x^2, y^2) = \gcd(x, y)^2, the\ integers\ 3x + 3y + 1 = b^2\ and\ 2x + 2y + 1 = a^2$ must also be squares. This proves (a) (b) with $x = d.b, y = d.a, \gcd(a, b) = 1, we\ get\ d^2 = x - y$ From (a) we get $3a^2 - 2b^2 = 1\ and\ d^2 = db - da \Longrightarrow d = b - a, x = (b - a)b, y = (b - a)a$. The solutions of $3a^2 - 2b^2 = 1$ can be obtained from.

$\left(\sqrt{3}+\sqrt{2}\right)^{2n+1} = a_n + \sqrt{3} + b_n\sqrt{2}$ by powering or, simpler, by recurrence . From $a_{n+1} + \sqrt{3} + b_{n+1}\sqrt{2} = \left(a_n\sqrt{3} + b_n\sqrt{2}\right)(5 + 2\sqrt{6})$ we get $a_{n+1} = 5a_n + 4b_n, b_{n+1} = 6a_n + 5b_n, a_1 = 1, b_1 = 1$. The next solutions $a_2 = 9, b_2 = 11\ yeilds\ x_2 = 22, y_2 = 18$.

41) Several different positive integers lie strictly between two successive square. Prove that their pair wise products are also different.

Sol.: Let $n^2 < a < b < c < d < (n+1)^2, ab = bc$. Then $d - a < 2n$ our aim is to produce a contradiction to (1).

From $ad = bc$, we conclude that

$a[(a+d) - (b+c)] = (a-b)(a-c) > 0$. Hence

$a + d > b + c. Now\ (a+d)^2 - (d-a)^2 = 4ad = 4bc < (b+c)^2\ .We\ conclude\ that\ (d-a)^2 > (a+d)^2 - (b+c)^2 = (a+d+b+c)(a+d-b+c)$. Each term of the first factor on the R. H. S. is larger than n^2, and the second is $n^2 \geq 1$.

Thus we have $d - a > 2n$, which contradicts (1).

42) Let a, b, c , d be integers with a > b >c >d >0. Suppose that

$ac + bd = (b + d + a - c)(b + d - a + c)$. Prove that $ab + cd$ is not prime.

Sol.: Three different arguments are presented hence. The first is the most elementary, using only number theory and counting arguments, and a detailed proof is given. The second arguments uses technical topic, so only a sketch of the proof is provided. The third argument, which only a sketch, is a lovely hybrid of algebra, number theory and combinatory. It was discovered by a Bulgarian contestant who received a special prize for his creativity.

For any p-element subset A of {1, 2, ……….2p} denote the sum of the elements of A by Γ(A) of the $\binom{2p}{p}.such\ \ subsets\ , L = \{1, 2, \dots, p\}\ and\ R = \{p+1, p+2, \dots ,2p\} satisfy$ Γ (L) ≡Γ(R)≡ 0 (mod P). For A≠ LR . we have A ∩ L ≠ ϕ ≠ A ∩ R.

Portion the $\binom{2p}{p} - 2p$ elements subsets other than L and R into group of size p as follows.

For any set E of integers, define x ⨁E = $\{x + e(mod\ p): e\ \ \in E\}$, where the sums are positive and no greater than P. Let A be any p-element subset other than P. Let A be any p-element subset other than L or R. Define $A_L = A \cap L\ and\ A_R = A \cap R$. (Note that both of these sets are non empty). Then the group of P subsets in which A lies is $A_L \cup A_R, (1 \oplus A_L) \cup A_R, (2 \oplus A_L) \cup A_R, (P - 1 \oplus A_L) \cup A_R$

(In more sophisticated language, we are partitioning the subsets into equivalence classed where two subset A and A’ are in the same class if and only if A’∩R= A∩ R and A’ ∩L is a cyclic permutation of A ∩ L within L.) This method of grouping subsets has the following properties.

- Each group contain P distinct subsets, and each subsets has a different element sum modulo P. To see this, assume that (x ⨁A_L) ∪$A_R = (y \oplus A_L) \cup A_R$
  This implies that $x \oplus A_L = y \oplus A_L$. Let A_L have n elements. This we have $a(x \oplus A_L) = a(y \oplus A_L)$. But this implies that $nx - ny \equiv 0$ (mod P),

which forces x = y because P is prime and 0<n<p.

- Every subset other than L or R lies in exactly one group. Assume that two groups shared the same subset E. In other words, assume that $E = (x \oplus A_L) \cup A_R and\ E(y \oplus B_L)B_R$.

  Since both x $\oplus A_L$ and y$\oplus B_L$ are subsets of L, we have $A_R = E \cap R = B_R$. Thus x $\oplus A_L = y \oplus B_L$ or $B_L = (x+y) \oplus A_L$. This shows that the collection of subsets (y$\oplus B_L) \cup B_R, y = 0,1,2, \dots, P-1$ will be the same as the collection (x $\oplus A_L)\cup A_R, x = 0,1,2, \dots, P-1$. It follows that exactly one subsets A in each group satisfies σ(A) ≡ 0 (mod P), and the total number of such sub sets is
  $\frac{1}{P}\left[\binom{2p}{2} - 2\right] + 2$
  In fact, we have proved a little bit more than for any r ≠0 (mod P), there are exactly $\frac{1}{p}\left[\binom{2p}{2} - 2\right]$
  Subsets with element sum congruent to r modulo P.

43) Let p, q, n be positive integers with $p + q < n$. Let $(x_0, x_1, \dots, x_n)$ be an (n+1) tuple of integers satisfying the following conditions.

(a) $x_0 = x_n = 0$, (b) For each i with $1 \le i \le n$, either $x_i - x_{i-1} = P\ or\ x_i - x_{i-1} = -q$. Show that there exist indicates i > j with (i, j) ≠ (0, n) such that $x_i = x_j$.

Sol.: Let d be the greatest common factor of p and q; then the problem with p, q x_i replaced by $\frac{p}{d}, \frac{q}{d}, \frac{x}{d}$ is equivalent to the original problem. Hence without loss of generality, we may assume p and q are relatively prime.

Let r = p+ q observe that $x_{i+1} \equiv x_r + p(mod\ r)$. Since $-q \equiv p(mod\ r)$. By induction, $x_{i+k} \equiv x_i + p_k(mod\ r)$. In particular, $x_n \equiv x_0 + n_p\ (mod\ p+q)$. Since $x_n = x_0 = 0$, we deduce that r divides up. However, by assumption p is relatively prime to 9 and hence also to r, and so n = mr for some integer m. Since n > r is assumed, we have m > 1.

Let $s_i = x_{i+p} - x_i$ for I = 0, …, $(m-1)r$. By the previous observation, s_i is a multiple of r, Moreover $s_{i+1} - s_i = (x_{i+r+1} - x_{i+r}) - (x_{i+1} - x_i) \le p — q = r\ and\ similarly\ s_{i+1} - s_i > -r$.

It suffices to show that $s_i = 0$ for some I, for then we can take our pair to be $(i, i+r)$.If $s_0 = 0$, we are done so assume that $s_0 > 0$ (the argument for $s_0 > 0$ is similar). Let s_i be the first non-negative term among $s_0, \dots, s_{(m-1)r} = x_n - x_0 = 0$. Then $s_i - 1$ is negative and s_i is non negative, but both are multipliers of r and they differ by at most r. This can only occur if $s_{i-1} = -r\ and\ s_i = 0$.

44) For any positive integer n, let d(n) denote the number of positive divisors of n (including 1 and n itself). Determine all positive integers k such that $\frac{d(n)^2}{d(n)} = k$ for some n.

Sol.: Let n = $P_1^{a_1} \dots \dots \dots . P_r^{a_r}$

$Then\ d_{(x)} = (a_1+1)(a_2+1) \dots . . (a_r + 1), and$

d$(n^2) = (2a_1+1)(2a_2+1) \dots \dots \dots (2a_r+1)$.
So the a_i must be chosen so that

$(2a_1+1)(2a_2+1) \dots \dots \dots . . (2\alpha_r+1) = K(a_1+1)(a_2+1) \dots (a_r+1)$

Since all $(2a_i + 1)$ are odd, this clearly mplies that k must odd. We know that conversely, given any odd k, can find a_i.

We use a form of induction on k. First, it is true for k = 1 (take n = 1). Second, we show that if it is true for k, then it is true for $2^m k - 1$. That is sufficient, since any odd number has the form $2^m k - 1$ for some smaller odd number k. Take $a_i = 2^i[(2^m - 1)k - 1] for\ i = 0, 1, \dots, m - 1$.

$Then\ 2a_i + 1 = 2^{i+1}(2^m - 1)K - (2^{i+1} - 1) and$

$$a_i + 1 = 2^i(2^m - 1)K - (2^i - 1).$$

So the product of the $(2a_1 + 1)'S$ divided by the product of the $(a^i + 1)'S$ is $2^m(2^m - 1)K - (2^m - 1) divided\ by\ (2^m - 1)K, or \frac{(2^m k - 1)}{k}$.

Thus if we take these $a_i s$ together with those giving k, we get $2^m k - 1$ which completes the induction.

45) Find all pairs (n, p) of positive integers, such that :P is prime; $\boldsymbol{n \le 2p}$; $\boldsymbol{and\ (p - 1)^n + 1}$ is divisible by $\boldsymbol{n^{p-1}}$

Sol.: Evidently (1, P) is a solution for every prime p. Assume n >1 and take q to be the smallest prime divisor of n. we first show that q = p. Let x be the smallest positive integer for which $(p - 1)^x = 1\ (mod\ q)$. Certainly y exists and indeed y < q.

Since $(p - 1)^{q-1} = 1\ (mod\ q)$. We know that $(p - 1)^n = -1(mod\ q)$ so x exists also.

Writing $n = sy + r, with\ 0 \le r < y$, we conclude that $(p - 1)^r = -1(mod\ q)$ and hence $x \le r < y$ (r cannot be zero sine 1 is not -1(mod q))

Now write $n = hx + K\ with\ 0 \le k \le x$.

$Then - 1 = (p - 1)^n = (-1)^n(p - 1)^k(mod\ q)$. h cannot be even. Because then $(p - 1)^k = -1(mod\ q)$, contradicting the minimality of x. so h is odd and hence $(p - 1)^k = 1\ (mod\ q)$ with $0 \le k < x < y$. This contradicts the minimality of y unless k = 0, so n = hx.

But x <q, so x = 1, so $(p - 1) = -1$ (mod q) p and q are primes so q = p as claimed.

So p is the smallest prime divisors of n. We are also given that $x < 2p$. so either p = n, or p = 2, n = 4. The latter dues not work, so we have shown that n = p Evidently n = p = 2 and n = p = 3 work. Assume now that p > 3 we show that there are no solutions of this type.

Expand $(p - 1)^p + 1$ by the binomial theorem, to get (since $(-1)^p = -1$):

$$1 + -1 + p^2 - \frac{1}{2}p(p - 1)p^2 \frac{p(p - 1)p - 2}{6p^3}.$$

$The\ terms\ of\ the\ from\ p^i\ with\ i \ge 3$ are obviously divisible by p^3.

Since the binomial co-efficient by are integral. Hence the sum is $P^2 + (a\ multiple\ of\ P^3)$. So the sum is not divisible by P^3. But for $p > 3, P^{p-1}$ is divisible by p^3. So it cannot divide $(p - 1)^p + 1$, and there are no more solutions.

46) Determine whether or not there exists a positive integer n such that n is divisible by exactly 2000 different prime numbers, and $\boldsymbol{2^n + 1}$ is divisible by n.

Sol.: Note that for a odd b, we have

$2^{ab}+1=(2^4+1)(2^a(b-1)-2^a(b-2)+\cdots+1)$ and so 2^a+1 is a factor of $2^{ab}+1$. It is sufficient therefore to find m such that (1) m has only a few distinct prime factors (2) 2^m+1 has a large number of distinct prime factors (3) m divides 2^m+1 (but not m), so that km has exactly 2000 factors then km still divides 2^m+1 and hence $2^{km}+1$.

The simplest case is where m has only one distinct prime factor P, in other words it is a power of P. But if P is a prime, then p divides 2^p-2, so the only p for which p divides. 2^p+1 is 3. So the questions are (1) whether $a_n=2^m+1$ is divisible by $m=3^4$ $and\ (2) a_n\ has\ a\ large\ number$

$of\ distinct\ prime\ factor\ a_{n+1}=a_n(2^m-2^m+1), where\ m=\ 3^n$ but $2^m=(a_{n-1}), So\ a_{n+1}=\ a_n({a_n}^2-3a_n+3). Now\ a_1=9$, so an induction show that 3^{n+1} divides a_n, which answer (1) affirmatively.

Also, since a_n is a factor of a_{n+1}, any prime dividing $a_n\ also\ divides\ a_{n+1}$

$Put\ a_n=\ 3^{n+1}\ bn$. Then $b_{n+1}b_n\left(3^{2n+1}\ {b_n}^2-3^{n+2}b_n+1\right)$

$Now\ \left(3^{2n+1}\ {b_n}^2-3^{n+2}b_{n+1}\right)>1$, So, it must have some prime factor p > 1.

But P be 3 or divide $b_n\ since\ (3^{2n+1}\ {b_n}^2-3^{n+2}b_{n+1})$ is a multiple of $3b_n\ plus\ 1. So\ b_{n+1}$ has at least one prime factor p >3 which does not divide $b_n. so\ b_{n+1}$ has at least h distinct prime factors greater than 3, which answers (2) affirmatively. But that is all we need we can take m in the first paragraph above to be 3^{2000}.

(1) m has only one distinct prime factor.
(2) $2^{m+1}=\ 3^{2001}b_{2000}\ has\ at\ least$ 1999 distinct prime factors other than 3.
(3) m divides 2^m+1. Take k to be a product of 1999 distinct prime factors dividing b_{2000}. Then $n=\ k_m$ is the required number with exactly 2000 distinct prime factors which divides 2^n+1.

47) Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many negative terms suppose that for each positive integer n, the numbers $a_1, a_2, \ldots, a_n$ leave n different remainder on division by n. Prove that each integer occurs exactly once in the sequence.

Sol.: Let $A_n=\{a_1,\ldots,a_n\}$. Elements of A_n are distinct because they are distinct modulo n. Observe that, for $a_i, a_j\ \in A_n, k\ :=\left|a_i-a_j\right|<n$, because, otherwise, $a_1\ a\ \in a_k\ and\ A_i\equiv a_j\ mod\ k$. Therefore $\max A_n-\min A_n<n$.

But A_n consists of n distinct integers. Therefore, for $m_n=\min A_n, A_n=\{m_n, m_{n+1},\ldots,m_n+n-1\}$

There are infinitely many negative and positive numbers in the sequence, therefore all integers have to appear in our sequence. This finishes the proof.

48) Determine all pairs (x, y) of integers such that $1+2^x+2^{2x+1}=\ y^2$.

Sol.: If (x, y) is a solution then obviously $x\geq 0$ and $(x,-y)$ is a solution too. For x = 0, we get the two solutions (0, 2) and $(0,-2)$

Now let (x, y) be a solution with $x>0$, without loss of generality confine attention to y > 0. The equation rewritten as $2^x(1+2^{x+1})=(y-1)(y+1)$. Shows that the factors $y-1\ and\ y+1$ are even exactly one of them divisible by 9. Hence $x\geq 3$ and one of these factors is divisible by 2^{x-1} but not by 2^x. So $y = 2^{x+1}\, m^{+e}, m\ odd, e = \pm 1$. Plugging this into the original equation we obtain.

$2^x(1 + 2^{x+1}) = (2^{x-1}m + e)^2 - 1 = 2^{2x-2}m^2 + 2^x$ me or equivalently, $1 + 2^{x+1} = 2^{x-2}\, m^2 + me$

Therefore, $1 - em = 2^{x-2}(m^2 - 8) \dots\dots\dots\dots (ii) for\ e = 1$ this yields $m^2 - 8 \leq 0$ i.e. m = 1, which fails to satisfy (ii) For e = -1 equation (ii) gives us.

$1 + m = 2^{x-2}(m^2 - 8) \geq 2\,(m^2 - 8) implying$

$2m^2 - m - 17 \leq 0\ hence\ m \leq 3$; on the other hand m cannot be 1 by (ii). Because m is odd . these values indeed satisfy the given equation. Recall that then $y = -23$ is also good. Thus we have the complete list of solutions (x, y): $(0, 2), (0, -2), (4, 23), (4, -23)$

49) Let n be a positive integer and $\boldsymbol{P_1, P_2, \dots, P_n}$ be n prime numbers all larger than 5. Such that $\boldsymbol{P_1}^2 + \boldsymbol{P_2}^2 + \cdots + \boldsymbol{P_n}^2$ is divisible by 6. Prove that 6 divides n.

Sol.: Through possible remainders when divided by 6 are 0, 1, 2, 3, 4, 5, p_i being prime $p_i{}^2$ will have to leave only remainder 1 or 5. i.e. $p_i{}^2$ is of the from $6m^r \pm 1$: so should be pi, Hence $p_i{}^2$ is of the form $6m^r + 1$: $so\ p_i{}^2 + \cdots + p_n{}^2$ is divisible by 6 only when n is divisible by 6.

50) Find all pairs (m, n) of (+) integers such that $\boldsymbol{\frac{m^2}{2mn^2 - n^3 + 1}}$ $\boldsymbol{is\ (+)\ ve\ integer}$.

Sol.: The denominator is $2mn^2 - n^3 + 1 = n^2(2m - n) + 1, so\ 2m \geq n > 0. If\ n = 1,$ then m must be even, in other words, we have the solution (m, n) = (2k, 1). So assume n > 1. Put $n = \frac{m^2}{2mn^2 - n^3 + 1}$.

Then we have a quadratic equation for m, namely $m^2 - 2hn^2m + (n^3 - 1)h = 0$. This has solutions $hn^2 \pm N$, where N is the positive square root of $n^2x^4 - hx^3 + h$. Since n >1, $n \geq 1$, n is certainly real. But the sum and product of the roots are both positive, so both roots must be positive. The sum is an integer, so, if one root is a positive integer, than so is the other.

The larger root $hn^2 + N$ is greater than hn^2, so the smaller root $< \frac{h(n^3-1)}{hn^2} < n$. But note that if $2m - n > 0$, then since n > 0, we must have the denominator $(2m - n)n^2 + 1$ smallest than the numerator and hence $m - n$. So for the smallest root we cannot have $2m - n > 0$. But $2m - n = 0$ for the smaller root. Hence $hn^2 - N = \frac{n}{2}$.

Now $N^2 = \left(hn^2 - \frac{h}{2}\right)^2 = h^2n^4 - hn^3 + h, so\ h = \frac{n^2}{4}$ Thus n must be even, put n = 2k and get the solutions $(m, n) = (k, 2k) and\ (8k^4 - k, 2k)$. We have shown that any solutions must be of one of the three forms given, out it is trivial to check that they are all indeed solutions.

QUADRATIC EQUATIONS AND EXPRESSIONS

(Objective Type)

1) If $\boldsymbol{0 < \alpha < \beta < \gamma < \frac{\pi}{2}}$, then equation $\boldsymbol{\frac{1}{x - \sin\alpha} + \frac{1}{x - \sin\beta} + \frac{1}{x - \sin\gamma} = 0}$ has

(a) Imaginary root, (b) real and equal roots, (c) real and unequal root, (d) rational roots.

Sol.: Since, $0 < \alpha < \beta < \gamma < \frac{\pi}{2} (given)$

$\Rightarrow \sin\alpha < \sin\beta < \sin\gamma$

Now the given equation is

$$(\chi - \sin\beta)(\chi - \sin\gamma) + (\chi - \sin\alpha)(\chi - \sin\gamma) + (\chi - \sin\alpha)(\chi - \sin\beta) = 0$$

$$Let\ f(x) = (\chi - \sin\beta)(\chi - \sin\gamma) + (\chi - \sin\alpha)(\chi - \sin\gamma) + (\chi - \sin\alpha)(\chi - \sin\beta) = 0$$

$\Rightarrow f(\sin\alpha) = (\sin\alpha - \sin\beta)(\sin\alpha - \sin\gamma) > 0$

$\Rightarrow f(\sin\beta) = (\sin\beta - \sin\alpha)(\sin\beta - \sin\gamma) < 0$

$\Rightarrow f(\sin\gamma) = (\sin\gamma - \sin\alpha)(\sin\gamma - \sin\beta) > 0$

Hence equation f(x) = 0 has once root between $\sin\alpha\ \ and \sin\beta$ and other between $\sin\beta\ \ and \sin\gamma$.

2) If $\boldsymbol{y = \frac{1}{\cot x \tan 3x}, then}$

(a) $\boldsymbol{y < \frac{1}{3}\ or\ y > 3}$; (b) $\boldsymbol{\frac{1}{3} \le y \le 3}$;

(c) $\boldsymbol{y \le \frac{1}{3}\ or\ y \ge 3}$; (iv) $\boldsymbol{\frac{1}{3} \le y \le 1}$

Sol.: $y = \frac{\tan x}{\tan 3x} = \frac{1-3tan^2x}{3-tan^2x} \Rightarrow y = \frac{1-3t^2}{3-t^2}$

$\therefore (y-3)t^2 + 1 - 3y = 0\ \ since\ t\ is\ real\ \Rightarrow \Delta \ge 0$

$\therefore 0 - 4(y-3)(1-3y) \ge 0 \Rightarrow (y-3)(1-3y) \le 0$

$\Rightarrow y \le \frac{1}{3} or\ y \ge 3\ \Rightarrow y < \frac{1}{3} or\ y > 3.$

$\left(Here\ y\ \ne \frac{1}{3}, 3\right)$

Case I: For $y = \frac{1}{3} \Rightarrow \frac{1-3t^2}{3-t^2} = \frac{1}{3} \Rightarrow 3 - 9t^2 = 3 - t^2$

$\Rightarrow t = 0$

Case II: For $= 3 \Rightarrow \frac{1-3t^2}{3-t^2} = 3, \Rightarrow 1 - 3t^2 = 9 - 3t^2$

{not possible}

3) If $\boldsymbol{\frac{\sum_{r=0}^{k-1} x^{2r}}{\sum_{r=0}^{k-1} x^r}}$ is a polynomial in x for two values p and q of k, then roots of equation, $\boldsymbol{x^2 + px + q = 0}$ cannot be

(a) Real; (b) positive (c) rational; (d) irrational

Sol.: $\frac{\sum_{r=0}^{k-1} x^{2r}}{\sum_{r=0}^{k-1} x^r}$ is a polynomial in x. i.e.

$\{1 + x^2 + x^4 + \cdots + x^{2(k-1)}\}$ is divisible by $\{1 + x + x^2 + \cdots + x^{k-1}\}$

$\Rightarrow \frac{\left(\frac{1-x^{2k}}{1-x^2}\right)}{\left(\frac{1-x^k}{1-x}\right)} = \frac{1+x^k}{1+x} (x \ne 1)$ is a polynomial in x if and only if: $(1 + x^k)$ is divisible by $(1 + x) if\ x = -1$

But it is not possible, because the equation is not defined for $x = -1$.

Hence, there are no real values of x for which the equation is defined.

Thus, it can be easily said that the roots of the equation cannot be Rational.

4) If one root of the equation $\boldsymbol{ax^2 + bx + c = 0}$ is reciproed of the other root of the equation $\boldsymbol{a_1x^2 + b_1x + c_1 = 0, then}$

(a) $\boldsymbol{(aa_1 - cc_1)^2 = (bc_1 - b_1a)(b_1c - a_1b)}$

(b) $(\boldsymbol{ab_1 - a_1b})^2 = (\boldsymbol{bc_1 - b_1c})(\boldsymbol{ca_1 - c_1a})$
(c) $(\boldsymbol{bc_1 - b_1c})^2 = (\boldsymbol{ca_1 - a_1c})(\boldsymbol{ab_1 - a_1b})$
(d) None

Sol.: Let α be a root of the equation $ax^2 + bx + c = 0$

$Then,\frac{1}{\alpha}$ is a root of $a_1x^2 + b_1x + c_1 = 0$

$\Longrightarrow a\alpha^2 + b\alpha + c = 0 \ldots\ldots\ldots(1) \& \frac{a_1}{\alpha^2} + \frac{b_1}{\alpha} + c_2 = 0$

$\Longrightarrow c_1\alpha^2 - b_1\alpha + a_1 = 0 \;\ldots\ldots\ldots.(2)$

Since (1) & (2) have one root in common,

$\Longrightarrow \frac{\alpha^2}{ba_1 - b_1c} = \frac{\alpha}{cc_1 - aa_1} = \frac{1}{ab_1 - c_1b}$

$\Longrightarrow \alpha^2 = \frac{ba_1 - b_1c}{ab_1 - c_1b}, \alpha = \frac{cc_1 - aa_1}{ab_1 - c_1b}, Now, \alpha^2 = (\alpha)^2$

$\Longrightarrow (ba_1 - b_1c)(ab_1 - c_1b) = (cc_1 - aa_1)^2$

5) If α and $\boldsymbol{\beta}$ are the roots of the equation $\boldsymbol{x^2 - ax + b = 0}$ and $\boldsymbol{v_n = \alpha^n + \beta^n}$ then which of the following is true?

(a) $\boldsymbol{v_{n+1} = av_n + bv_{n-1}}$; **(b)** $\boldsymbol{v_{n+1} = bv_n + av_{n-1}}$; **(c)** $\boldsymbol{v_{n+1} = av_n - bv_{n-1}}$; **(d)** $\boldsymbol{v_{n+1} = bv_n - av_{n-1}}$;

Sol.: $\alpha + \beta \in x^2 - ax + b = 0 \; \alpha^2 + a\alpha + b = 0 \ldots\ldots\ldots\ldots.(1)$

$or \; \beta^2 + a\beta + b = 0 \ldots\ldots\ldots\ldots\ldots(2)$

$Multiplying \; (1) by \; \alpha^{n-1} \; and \; (2) by \; \beta^{n-1}$

$\Longrightarrow \alpha^{n-1} - a\alpha^n + b\alpha^{n-1} = 0 \ldots\ldots\ldots\ldots.(3)$

$\Longrightarrow \beta^{n-1} - \alpha\beta^n + b\beta^{n-1} = 0 \ldots\ldots\ldots(4),$

Adding (3) & (4), we have

$(\alpha^{n+1} + \beta^{n+1}) - a(\alpha^n + \beta^n) + b(\alpha^{n-1} + \beta^{n-1}) = 0$

$\Longrightarrow v_{n+1} - av_n + bv_{n-1} = 0$

$\therefore v_{n+1} - av_n - bv_{n-1}$

6) Let a, b, c be non-zero real numbers, such that

$$\int_0^1 (\boldsymbol{1 + cos^8x})(\boldsymbol{ax^2 + bx + c})\boldsymbol{dx} = \int_0^2 (\boldsymbol{1 + cos^8x})(\boldsymbol{ax^2 + bx + c})\boldsymbol{dx}$$

Then the quadratic equation $\boldsymbol{ax^2 + bx + c = 0}$ has

(a) No root in (0, 2); (b) at least one root in (0, 2); (c) two root in (0, 2); (d) two imaginary roots

Sol.: Consider the function

$$\phi(x) = \int_0^x (1 + cos^8x)(ax^2 + bx + c)dx \Longrightarrow f(1) = f(2) \ldots\ldots.(1)$$

Obviously, (a), ϕ(x) is continuous on [1, 2] and (b) differentiable on (1, 2), (c) Also, $\phi_{(1)} = \phi_{(2)}$ (given)

Therefore, by Rolle's theorem there exists at least point k ∈ (1, 2) such that ϕ'(k) = 0

Now, ϕ(x) = $(1 + cos^8x)(ax^2 + bx + c)$

$\therefore \phi'(k) = 0$

$\Longrightarrow (1 + cos^8x)(ak^2 + bk + c) = 0$

$\Rightarrow ak^2 + bk + c = 0 \{\because (1 + cos^8 x) \neq 0\}$

$\Rightarrow k \text{ is a root of } ax^2 + bx + c = 0 \text{ in } (1, 2)$

7) If $2a + 3b + 6c = 0$ $(a, b, c \in R)$ then the quadratic equation $ax^2 + bx + c = 0$ has
   (a) At least one in [0, 1]; (b) at least one root in [2, 3]; (c) at least one root in [4, 5]; (d) none

Sol.: (A) Let $f(x) = \frac{a}{3}x^3 + \frac{b}{2}x^2 + cx$. We have $f(0) = 0 \text{ and } f(1) = \frac{a}{3} + \frac{b}{2} + c = \frac{2a+3b+6c}{6} = 0$

$(\because 2a + 3b + 6c = 0)$.

Thus, 0 and 1 are two roots of f(x) = 0. So, $f'(x) = 0 \text{ i.e.} ax^2 + bx + c = 0$ has at least one real root between 0 and 1.

8) If α, β be the roots of the equation $6x^2 - 6x + 1 = 0, then \frac{1}{2}(a + bx + cx^2 + dx^3) + \frac{1}{2}(a + b\beta + c\beta^2 + d\beta^2) =$
   (a) $\frac{d}{1} + \frac{c}{2} + \frac{b}{3} + \frac{a}{12}$; (b) $12a + 6b + 4c + 9d$; (c) $\frac{a}{1} + \frac{b}{2} + \frac{c}{3} + \frac{d}{4}$; (d) none

Sol.: Here $\alpha + \beta = 1, \alpha\beta = \frac{1}{6}$. Thus the given expression $\frac{1}{2}\{(a + b) + b(\alpha + \beta) + c(\alpha^2 + \beta^2) + d(\alpha^3 + \beta^3)\} = \frac{a}{1} + \frac{b}{2} + \frac{c}{3} + \frac{d}{4}$ after calculation of values.

9) If a is a positive integer, the number of values of a satisfying

$$\int_0^{\frac{\pi}{2}} \left\{a^2 \left(\frac{\cos 3x}{4} + \frac{3}{4}\cos x\right) + a \sin x - 20 \cos x\right\} dx \leq -\frac{a^2}{3},$$

   (a) Only one; (b) two; (c) three; (d) none

Sol.: Given $\int_0^{\frac{\pi}{2}} \left\{a^2 \left(\frac{\cos 3x}{4} + \frac{3}{4}\cos x\right) + a \sin x - 20 \cos x\right\} dx \leq \frac{a^2}{3} \Rightarrow \left\{a^2 \left(\frac{\sin 3x}{4} + \frac{3}{4}\sin x\right) - a \cos x - 20 \sin x\right\} \Big|_0^{\frac{\pi}{2}} \leq -\frac{a^2}{3} \Rightarrow a^2 \left(-\frac{1}{12} + \frac{3}{4}\right) - 0 - 20 + a \leq \frac{a^2}{3}$

$\Rightarrow \frac{2}{3}a^2 + a - 20 + \frac{a^2}{3} \leq 0 \Rightarrow a^2 + a - 20 \leq 0$

$\Rightarrow (a + 5)(a - 4) \leq 0$

Thus, a = 1, 2, 3, 4 $\{\because a \in I\}$

10) If $(x - c)$ is a factor of order m of the polynomial f(x) of degree n $(1 < m < n)$, then x = c is a root of the polynomial
   (a) $f^m(x)$; (b) $f^{m-1}(x)$; (c) $f^n(x)$; (d) none

Sol.: since $(x - c)$ is a factor of order m of the polynomial $f(x)$

$\therefore f(x) = (x - c)^m \phi(x)$

Where ϕ(x) is a polynomial of degree $(n - m)$

$\Rightarrow f(x), f'(x), \dots, f^{(m-1)}(x)$ are all zero for x = c but $f^m(x) \neq 0 \text{ at } x = c$.

11) The value of a for which one root of $(a^2 - 5a + 3)x^2 + x(3a - 1)a + 2 = 0$ is twice as large as the other is
   (a) $\frac{1}{3}$; (b) $-\frac{1}{3}$; (c) $\frac{2}{3}$, (d) none

Sol.: Let α and 2α be the two roots of the given equation. So,

$\alpha + 2\alpha = -\frac{3a-1}{a^2-5a+3} \text{ and}$

$2\alpha^2 = \frac{2}{a^2-5a+3}, \Rightarrow \alpha = -\frac{3a-1}{3(a^2-5a+3)} \text{ and}$

$$\alpha^2 = \frac{1}{a^2 - 5a + 3}, \Rightarrow \frac{1}{a^2 - 5a + 3} = \frac{(3a-1)^2}{9(x^2 - 5a + 3)}$$

$$[\because \alpha^2 = (\alpha)^2], \Rightarrow 9(a^2 - 5a + 3) = (3a-1)^2$$

$$(\because a^2 - 5a + 3 \neq 0), \Rightarrow -39 + 26 = 0, \Rightarrow a = \frac{2}{3}.$$

12) Equation $\frac{a^2}{x-\alpha} + \frac{b^2}{x-\beta} + \frac{c^2}{x-\nu} = m - n^2 x$ $(a, b, c, m, n \in R)$ has necessarily

(a) All the roots real; (b) all the roots imaginary; (c) two real and two imaginary (d) none

Sol.: Let $P + L_q$ be a root of given equation

$$\therefore \frac{a^2}{p-\alpha+l_q} + \frac{b^2}{p-\beta+l_q} + \frac{c^2}{p-\gamma+l_q}$$

$$\Rightarrow \frac{a^2\{p-\alpha+l_q\}}{(p-\alpha)^2+q^2} + \frac{b^2\{(p-\beta)-l_q\}}{(p-\beta)^2+q^2} + \frac{c^2\{(p-\gamma)-l_q\}}{(p-\gamma)^2+q^2} = m - n^2 p - ln^2 q$$

According to law of equality of complex numbers, we have

$$q\left[\frac{a^2}{(p-\alpha)^2+q^2} + \frac{b^2}{(p-\beta)^2+q^2} + \frac{c^2}{(p-\gamma)^2+q^2} + n^2\right] = 0 \therefore q = 0$$

$Hence\ p + lq = p$ is a real number.

13) If $x = (\beta - \gamma)(\alpha - \delta), y = (\gamma - \alpha)(\beta - \delta)$ $z = (\alpha - \delta)(\gamma - \delta)$ then the value of $x^3 + y^3 + z^3 - 3xyz$ is

(a) 0; (b) $\alpha^6 + \beta^6 + \gamma^6 + \delta^6$; (c) $\alpha^6\beta^6\gamma^6\delta^6$; (d) none

Sol.: Since on solving, we have $x + y + z = 0$

$\therefore x^3 + y^3 + z^3 = 3xyz$ (by definition)

14) The range of values of a for which all the roots of the equation

$$(a-1)(1 + x + x^2) = (a+1)(1 + x^2 + x^4)$$ **are imaginary is**

(a) $[-\infty, -2]$; (b) $[2, \infty]$; (c) $-2 < a < 2$; (d) none

Sol.:
$(1 + x + x^2)[(a-1)(1 - x + x^2) - (a+1)(1 - x + x^3)]$

$= 0$ $(1 + x + x^2 = 0\ has\ imaginary\ roots)$

$\Rightarrow -2(1 + x^2) + 2ax = 0$, must have imaginary roots.

$\Rightarrow x^3 - ax + 1$, must have imaginary roots

$\Rightarrow a^2 - 4 < 0 \Rightarrow -2 < a < 2.$

15) If a, b , c are non-zero, unequal rational number then the roots the equation $abc^2x^2 + (3a^2 + b^2)cx - 6a^2 - ab + 2b^2 = 0$ are

(a) Rational; (b) imaginary; (c) irrational; (d) none

Sol.:

$\Delta = c^2(3a^2 + b^2)^2 - 4abc^2(-6a^2 - ab + 2b^2)$

$\Rightarrow \Delta = c^2(3a^2 - b^2 + 4ab)^2$

16) If the equation $x^2 - 3xk + 2e^{2\log k} - 1 = 0$ has real roots such that the product of roots is 7, then the value of k is

(a) ± 1; (b) ± 2; (c) ± 3; (d) none

Sol.: Since, $e^{2\log k}\, e^{\log k^2} = k^2$

$\therefore$ The given equation is $x^2 - 3kx + (2k^2 - 1) = 0$

Now, product of roots = 7 (given)

$\Rightarrow 2k^2 - 1 = 7 \;\therefore k = \pm 2$

17) Product of real root of $t^2x^2 + |x| + 9 = 0$,

(a) Is always positive; (b) is always negative; (c) does not exist; (d) none.

Sol.: Since the equation $t^2x^2 + |x| + 9 = 0$ is always positive for all x ∈ R.

$\therefore$ The equation does not possess real root.

18) If a, b, c are real and $x^3 - 3b^2x + 2c^3$ is divisible by $(x-a) and\ (x-b)$, then

(a) $a = -b = -c$; (b) $a = 2b = 2c$; (c) $a = b = c$; $or\ a = -2b = -2c$; (d) none

Sol.: Since, $f(x) = x^3 3b^2x + 2c^3$ is divisible by $x - a\ and\ x - b \;\therefore f(a) = 0$

$\Rightarrow a^3 - 3b^2a + 2c^3 = 0 \ldots\ldots\ldots..(1) and\ f(b) = 0$

$\Rightarrow b^3 - 3b^3 + 2c^3 = 0 \ldots\ldots(2) and \;\Rightarrow -2b^3 + 2c^3 = 0$

$\therefore b = c. Putting, b = c\ in\ (1), we\ get.$

$a^3 - 3ab^2 + 2b^3 = 0 \Rightarrow (a-b)(x^2 + ab - 2b^2) = 0$

$\Rightarrow a - b\ or\ a^2 + ab + 2b^2. Thus, a = b = c$

$or\ a^2 + ab = 2b^2 \Rightarrow a = b = c\ and$

$(a + 2b)(a - b) = 0 \Rightarrow a = b = c\ and\ a = -2b$

$\therefore a = b = c\ and\ a = -2b = -2c$

19) Both the root of the equation $(x-b)(x-c) + (x-c)(x-a) + (x-a)(x+b) = 0$ are always:

(a) Positive; (b) negative; (c) real; (d) none

Sol.: The given equation can be written as $3x^2 - 2x(a+b+c) + ab + bc + ca = 0.$

$Now, \Delta\ 4(a+b+c)^2 - 12(ab+bc+ca)$

$\Rightarrow \Delta = 2\,[(b-c)^2 + (c-a)^2 + (a-b)^2] \Rightarrow \Delta \geq 0$. Hence the roots are real.

20) If x denotes the set of real number p for which $x^2 = p(x+p)$ has a its roots greater than P, then x is equal to

(a) $\left(-2, -\frac{1}{2}\right)$, (b) $\left(-\frac{1}{2}, \frac{1}{4}\right)$, (c) null set, (d) $(-\infty, 0)$

Sol.: Since the roots are greater than p, i.e. p lies outside both the roots, such that $af(p) > 0$

$\Rightarrow 1(p^2 - 2p^2) > 0 \;\Rightarrow p^2 > 0$ which is impossible.

$Hence, x = \{p | p \in \phi\}$

21) If a and b are rational and α, β be the roots of $x^2 + 2ax + b = 0$, then the equation with rational coefficients are one of whose roots in $\alpha + \beta + \sqrt{\alpha^2 + \beta^2}$ is

(a) $x^2 + 4ax + 2b = 0$; (b) $x^2 + 4ax - 2b = 0$; (c) $x^2 - 4ax + 2b = 0$; (d) $x^2 - 4ax - 2b = 0$

Sol.: Since, $\alpha, \beta \in x^2 + 2ax + b = 0 (given)$

$\therefore \alpha + \beta = -2a\ and\ \alpha\beta = b$

$Let\ y = \alpha + \beta + \sqrt{\alpha^2 + \beta^2}$

$\Rightarrow (y+2a)^2 = \alpha^2+\beta^2 =$
$(\alpha^2+\beta^2)^2 - 2\alpha\beta = 4a^2 - 2b$

$\Rightarrow y^2 + 4ay + 2b = 0.$

The required equation is $x^2 + 4ay + 2b = 0$

22) Let $f(x) = x^3 + x^2 + 100x + 7\sin x$, then the equation $\frac{1}{y-f_{(1)}} + \frac{2}{y-f_{(2)}} + \frac{3}{y-f_{(3)}} = 0$ has

(a) No real root; (b) one real root; (c) two real roots; (d) more than two real roots.

Sol.: $f(x) = x^3 + x^2 + 100x + 7\sin x$

$\therefore f'(x) = 3x^2 + 2x + 100 + 7\cos x$

$\Rightarrow f'(x) = 3x^2 + 2x + 93 + 7(1+\cos x) > 0$

$\therefore$ f(x) is an increasing function.

$\Rightarrow f_{(1)} < f_{(2)} < f_{(3)}\ let\ f_{(1)} = a, f_{(2)} = b, f_{(3)} = c, then\ a < b < c \ldots\ldots\ldots (1)$

Now given equation is $\frac{1}{y-a} + \frac{2}{y-b} + \frac{3}{y-c} = 0$

$\Rightarrow (y-b)(y-c) + 2(y-a)(y-c) + 3(y-a)(y-b) = 0$

Let

$g(y) = (y-b)(y-c) + 2(y-a)(y-c) + 3(y-a)(y-b)$

$\Rightarrow g(a) = (a-b)(a-c) > 0$

$\Rightarrow g(b) = 2(b-a)(b-c) < 0$

$\Rightarrow g(c) = 3(c-a)(c-b) > 0$

$\therefore$ given equation $g(y) = 0$ has one real root between a and b and other between b and c.

23) If $\sin\alpha\ and \cos\alpha$ are roots of the equation $px^2 + qx + r = 0, then$

(a) $p^2 - q^2 + 2pr = 0$; (b) $(p+r)^2 = q^2 - r^2$; (c) $p^2 + q^2 - 2pr = 0$; (d) $(p-r)^2 = q^2 + r^2$

Sol.: $\sin\alpha, \cos\alpha\ px^2 + qx + r = 0$

$\Rightarrow \sin\alpha + \cos\alpha = -\frac{q}{p}$ and since $\alpha\cos\alpha = \frac{r}{p}$

$\Rightarrow (\sin\alpha + \cos\alpha)^2 = \frac{q^2}{p^2}$

$\Rightarrow 1 + 2\sin\alpha\cos\alpha = \frac{q^2}{p^2} \Rightarrow 1 + \frac{2r}{p} = \frac{q^2}{p^2}$

$\therefore P^2 - q^2 + 2pr = 0$

24) The roots of the equation $8x^2 - 10x + 3 = 0$ are α and $\beta^2 > \frac{1}{2}$, then the equation whose roots are $(\alpha + i\beta)^{100}\ and\ (\alpha - i\beta)^{100}$ is

(a) $x^2 - x + 1 = 0$; (b) $x^2 + x + 1 = 0$; (c) $x^2 - x - 1 = 0$; (d) none

Sol.: Since $f(x) = (2x-1)(4x-3) \therefore x = \frac{1}{2}, \frac{3}{4}$

$Now, \alpha = \frac{1}{2} and\ \beta^2 = \frac{3}{4}\left\{\therefore\ \beta^2 > \frac{1}{2}(given)\right\}$

$\alpha + i\beta = \frac{1}{2} + \frac{\sqrt{3}}{2} = re^{i\theta} = e^{\frac{l\pi}{3}} \Rightarrow r = 1\ and\ \Theta = \frac{\pi}{3}$

$Also, \alpha - i\beta = e - \frac{i\pi}{3} (conjugate)$

$$\therefore (\alpha+i\beta)^{100} = e^{\frac{l100\pi}{3}} = e^{i.3.3\pi}.e^{\frac{l\pi}{3}}$$
$$= -e^{\frac{l\pi}{3}}$$

$$Also, (\alpha-i\beta)^{100}$$
$$= -e^{-\frac{l\pi}{3}}(conjugate)$$

$$Sum = -\left(e^{\frac{l\pi}{3}} + e^{\frac{l\pi}{3}}\right) = -2\cos\frac{\pi}{3}$$
$$= -1$$

Product = 1

$\therefore$ Required equation is $x^2+x+1=0$

(SUBJECTIVE TYPE)

1) If the root of $f_{(x)} = ax^3+bx^2+cx+d=0$ are $\alpha_1, \alpha_2, \alpha_3$ and the root of ϕ(y) = $ay^3 + \frac{f''(y)}{2!}y^2 + \frac{f'(y)}{1!}z + f_{(y)} = 0$ are $\beta_1, \beta_2, \beta_3$, then show that $\alpha_1-\beta_1 = \alpha_2-\beta_2 = \alpha_3-\beta_3$.

Sol.: Consider $f_{(x)} = ax^3+bx^2+cx+d \ldots\ldots\ldots\ldots(1)$

$$\phi(y) = ay^3 + \frac{f''(y)}{2!}y^2 + \frac{f'(y)}{1!}y + f_{(y)} = 0 \ldots\ldots\ldots.(2)$$

$Now\ f_{(y)} = ay^3+by^2+cy+d, f'_{(y)} = 3ay^2+2by+c$

$$f''_{(y)} = 6ay+2b$$

Substituting above values in equation (2), we get

$\phi_{(y)} = ay^3 + \frac{(6ay+2b)}{2!}y^2 + (3ay^2+2by+c)y + ay^3+by^2+cy+d$

$= ay^3 + (3ay+b)y^2 + (3ay^2+2by+c) + ay^3+by^2+cy+d$ collecting all a, b, & c terms

$= a[y^3+y^3+3y^2y+3yy^2] + b[y^2+y^2+2yy] + c(y+y) + d$

$= a(y+y)^3 + b(y+y)^2 + c(y+y)d \ldots\ldots(3)$

$\Longrightarrow \phi_{(y)} = f_{(y+y)}$ key point. Let $y+y=x \Longrightarrow y = x-y$

Hence which means the roots of (1) are decreased by quality y.

$\therefore$ If $\alpha_1, \alpha_2, \alpha_3$ a are the roots of (1), and $\beta_1, \beta_2, \beta_3$, are the roots of (2), then $\alpha_1 - y = \beta_1;\ \alpha_2 - y = \beta_2;\ \alpha_3 - y = \beta_3$

$\Longrightarrow \alpha_1-\beta_1 = y;\ \ \alpha_2-\beta_2 = y;\ \ \alpha_3-\beta_3 = y$

$\Longrightarrow\ \alpha_1-\beta_1 = \alpha_2-\beta_2 = \alpha_3-\beta_3\ \ Proved.$

2) If α, β, β, y, y, α are the roots of $a_ix^2 + b_ix + c_i = 1, 2, 3$; then show that $(\alpha+\beta+\gamma) + (\alpha\beta+\beta y+y\alpha) + \alpha\beta\gamma = \left\{\prod_{i=0}^{3}\frac{a_i-b_i+c_i}{a_i}\right\}^{\frac{1}{2}} - 1$.

Sol.: Since α, β are roots of $a_ix^2+b_ix+c_i = 0$

$$\therefore \alpha+\beta = \frac{-b_1}{a_1}\ \&\ \alpha\beta = \frac{c_1}{a_1}$$

$$1+\alpha+\beta+\alpha\beta = 1 - \frac{-b_1}{a_1} + \frac{c_1}{a_1}$$
$$= \left(\frac{a_1-b_1+c_1}{a_1}\right)$$

$$\Longrightarrow (1+\alpha)(1+\beta) = \left(\frac{a_1-b_1+c_1}{a_1}\right)\ldots\ldots\ldots..(1)$$

$similarly\ (1+\beta)(1+\gamma)$
$= \left(\frac{a_2 - b_2 + c_2}{a_2}\right) \dots\dots\dots\dots (2) and$

$(1+\gamma)(1+\alpha) = \left(\frac{a_3 - b_3 + c_3}{a_3}\right) \dots\dots\dots . (3)$

Multiplying equation (1),(2) and (3). We get,
$(1+\alpha)^2(1+\beta)^2(1+\gamma)^2 = \prod_{i=1}^{3}\left(\frac{a_i - b_i + c_i}{a_i}\right) \Rightarrow 1 + \alpha + \beta + \gamma + \alpha\beta + \beta\gamma + \gamma\alpha + \alpha\beta\gamma = \left(\prod_{i=1}^{3}\left(\frac{a_i - b_i + c_i}{a_i}\right)\right)^{\frac{1}{2}}$

$$\Rightarrow (\alpha+\beta+\gamma) + (\alpha\beta + \beta\gamma + \gamma\alpha) + \alpha\beta\gamma = \left(\prod_{i=1}^{3}\left(\frac{a_i - b_i + c_i}{a_i}\right)\right)^{\frac{1}{2}} - 1.$$

3) If a, b, c, d are the roots of the equation $x^4 + p_1x^3 + p_2x^2 + p_4 = 0$, $then\ show\ that$ $(1+a^2)(1+b^2)(1+c^2)(1+d^2) = (1 - p_2 + p_4) + (p_3 - p_1)^2$

Sol.: As a, b, c, d are the roots of the given equation

$$\therefore x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = (x-a)(x-b)(x-c)(x-d) \dots\dots\dots\dots .. (1)$$

Equation (1) is an identity, so we put x = i on both sides

$$= i^4 + p_1i^3 + p_2i^2 + p_3i + p_4 = (i-a)(i-b)(i-c)(i-d)$$

$$\Rightarrow (1 - p_2 + p_4) + i(p_3 - p_1) = (i-a)(i-b)(i-c)(i-d) \dots\dots\dots (2)$$

Again putting $x = -i$ on equation (1), we get

$$(1 - p_2 + p_4) - i(p_3 - p_1) = (-i-a)(-i-b)(-i-c)(-i-d) \dots\dots\dots .. (3)$$

Multiplying equation (2) and (3), we get

$$\{(1 - p_2 + p_4) + i(p_3 - p_1)\}\{(1 - p_2 + p_4) - i(p_3 - p_1)\}$$

$$= (i-a)(i-b)(i-c)(i-d)(-i-a)(-i-b)(-i-c)(-i-d)$$

$$\Rightarrow (1 - p_2 + p_4)^2 + (p_3 - p_1)^3 = (i + a^2)(i + b^2)(i + c^2)(1 + d^2).$$

4) If $\beta + cos^2\alpha, \beta + sin^2\alpha$ $are\ the\ roots\ of$ $x^2 + 2bx + c = 0$ and $y + cos^4\alpha, y + sin^4\alpha$ are the roots of $x^2 + 2bx + c = 0$, the prove that $B^2 - B^2 = c - C$.

Sol.: If α, β are the roots of equation $ax^2 + bx + c = 0 then\ (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = \frac{b^2}{a^2} - \frac{4c}{a} = \frac{b^2 - 4ac}{a^2}$ $since$

$(\beta + cos^2\alpha), (\beta + sin^2\alpha)$ are roots of equation $x^2 + 2bx + c = 0$

$$\therefore \{(\beta + cos^2\alpha), -(\beta + sin^2\alpha)\}^2 = \frac{4b^2 - 4c}{l^2}$$

$$\Rightarrow cos^2 2\alpha = 4(b^2 - c) \dots\dots\dots . (1)$$

$$Also, [(\gamma + cos^4\alpha) - (\gamma + sin^4\alpha)]^2 = \frac{4B^2 - 4c}{l^2}$$

$$\Rightarrow (cos^4\alpha - sin^4\alpha)^2 = 4(B^2 - C) \dots\dots . (2)$$

$$\Rightarrow (cos^2\alpha + sin^2\alpha)(cos^2\alpha - sin^2\alpha) = 4(B^2 - C).$$

From equation (1) and (2), we get
$4(b^2 - c) = 4(B^2 - C)$

$$\Rightarrow b^2 - B^2 = c - C \quad Proved$$

5) Show that for any real numbers $a_3, a_4, \dots, a_{85}$, the roots of the equation $a_{85}x^{85} + a_{84}x^{84} + \cdots + a_3x^3 + 3x^2 + 2x + 1 = 0$ are real.

Sol.: Let

$$P(x) = a_{85}x^{85} + \cdots + a_3x^3 + 3x^2 + 2x + 1 = 0 \dots\dots\dots(1)$$

$since\ P_{(x)} = 1,$ then 0 is not a root of (1).

$Let\ \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{85}$ be the complex root of (1)

$Then\ the\ \beta_i \left(let \frac{1}{\alpha_i}\right)$ are the complex roots of the polynomial

$Q_{(y)} = y^{85} + 2y^{85} + 3y^{85} + a_3y^{82} + \cdots + a_{85}.$

$It\ follows\ that\ \sum_{i=1}^{85} \beta_i = -2\ and\ \sum_{i<j}^{85} \beta_i\beta_j = 3$

$Then\ \sum_{i=1}^{85} \beta_i^{\,2} = \left(\sum_{i=1}^{85} \beta_i\right)^2 - 2\sum_{i<j} S_iS_j = -2 < 0$

Thus, the $\beta_i{}'s$ are not real and then the $\alpha_i{}'s$ are not all real.

Conclusion: The equation $a_{85}x^{85} + a_{84}x^{84} + \cdots + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ are not real root, if $a_0 \neq 0\ and\ a_1{}^2 < 2a_0a_2$.

6) Suppose that $a_1 > a_2 > a_3 > a_4 > a_5 > a_6$ and $p = a_1 + a_2 + a_3 + a_4 + a_5 + a_6, q = a_1a_3 + a_3a_5 + a_5a_1 + a_2a_4 + a_4a_6 + a_6a_2\ and\ r = a_1a_3a_5 + a_2a_4a_6$. Then show that all the roots of the equation $2x^3 - px^2 + qx - r = 0$ are real

Sol.: let $f_{(x)} = 2x^3 - px^2 + qx - r = 2x^3 - (a_1 + a_2 + \cdots + a_6)x^2 + (a_1a_3 + a_3a_5 + a_5a_1 + a_2a_4 + a_4a_6 + a_6a_2)x - (a_1a_3a_5 + a_2a_4a_6)$

$From\ (a_1a_3a_5 + a_2a_4a_6) we\ can\ write\ f_{(x)}\ in\ form\ of$

$f_{(x)} = p(x - a_1)(x - a_3)(x - a_5) + q(x - a_2)(x - a_4)(x - a_6)$

$\Longrightarrow f_{(x)} = (x - a_1)(x - a_3)(x - a_5) + (x - a_2)(x - a_4)(x - a_6) \dots\dots(1)$

$f_{(x)} > 0\ \forall\ x > a_1, f_{(x)} < 0\ \forall a_2 > x > a_1$

$f_{(x)} > 0\ \forall\ a_4 > x > a_6, f_{(x)} < 0\ \ \forall\ a_6 > x$

So from properties of continuous function we say that equation $f_{(x)} = 0$ have three real roots.

7) Suppose the root of the equation $x^n - a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$ are real

Show that if α is a real root, then

$$\boldsymbol{a_1 - \left[\frac{(a_1{}^2 - 2na_2)}{(n-1)}\right]^{\frac{1}{2}} \leq n\alpha + a_1 + \left[\frac{(a_1{}^2 - 2na_2)}{(n-1)}\right]^{\frac{1}{2}}}$$

Sol.: Let $f_{(x)} = x^n - a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0\ \ \dots\dots(1)$

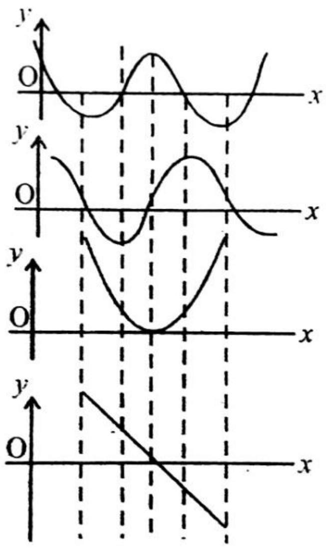
After differentiating $(n-2)$ times, we get

$[n(n-1)(n-2)\dots\dots\dots.3]x^2 - [(n-1)(n-2)\dots.2]$

$a_1x + [(n-2)\dots1]a_n = 0$

$\Longrightarrow n(n-1)x^2 - 2(n-1)$

$a_1x + 2a_2 = 0$



$\Longrightarrow$ roots of this equation is

$$x=\frac{a_1(n-1)\pm\sqrt{(n-1)^2{a_1}^2-2n(n-1)a_2}}{n(n-1)}$$

If α is a root of equation (1)

$$\Longrightarrow \frac{1}{n}\left[a_1-(n-1)\sqrt{\frac{\left({a_1}^2-\frac{2na_2}{(n-1)}\right)}{(n-1)^2}}\right]\le\alpha$$

$$\le\frac{1}{n}\left[a_1+(n-1)\sqrt{\frac{\left({a_1}^2-\frac{2na_2}{(n-1)}\right)}{(n-1)^2}}\right]$$

$$\Longrightarrow \alpha_1-\sqrt{\frac{{a_1}^2-2na_2}{(n-1)}}\le n\alpha$$

$$\le \alpha_1+\sqrt{\frac{{a_1}^2-2na_2}{(n-1)}}\ \ proved.$$

8) If all the coefficient of the equation $\boldsymbol{f_{(x)}=x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_n=f_{(x)}=0}$ Be whole number and if (0) and $\boldsymbol{f_{(1)}}$ be each odd integers, then prove that the equation can not have integral root.

Sol.: $f_{(0)}=\ p_n=odd, f_{(1)}=1+p_1+p_2+\cdots+p_n=odd$

Case I: If x = 2m, then $f_{(x)}=\ \underbrace{x^n}_{odd}+p_1x^{n-1}+\cdots+\underbrace{p_n}_{odd}=odd$ ∴ x = 2m cannot be a root of the odd equation $f_{(x)}=0$

Case II: $x=2m+1, then, f_{(x)}=\ \underbrace{x^n}_{odd}+p_1x^{n-1}+\cdots+\underbrace{p_n}_{odd}$

$$=\ x^n+p_1\,(even+1)+p_2(even+1)+\cdots+p_{n-1}(even+1)+p_n$$

$$=x^n+p_n+(p_1\,even+p_2\,even+\cdots+p_{n-1}\,even)+(p_1+p_2+\cdots+p_{n-1})$$

$$=\underbrace{x^n}_{odd}+\underbrace{p_1\,even+\cdots+p_{n-1}\,even}_{even}+\underbrace{p_1+\cdots+p_{n-1}}_{even}$$

$\Longrightarrow x=(2m+1)$ cannot be a root of the equation $f_{(x)}=0$

Therefore equation cannot have an integral root.

9) If the equations $\boldsymbol{ax^2+2bx+c=0\ and\ x^2+2p^2x+1=0}$, has one root commons. If a, b, c are in arithmetic progression and $\boldsymbol{p^2\neq1}$, then find the second root of second equation.

Sol.: Let α is the common root of both equations

$$\therefore a\alpha^2+2b\alpha+c=0\ \ldots\ldots\ldots..(1)$$

$$\alpha^2+2p^2\alpha+1=0\ \ldots\ldots\ldots\ldots.(2)$$

Now by cross multiplication, we get

$$\therefore \frac{\alpha^2}{2b-2p^2c} = \frac{\alpha}{c-a} = \frac{1}{2ap^2-2b} \ldots\ldots\ldots(3)$$

$$\Rightarrow \alpha^2 = \frac{2b-2p^2c}{2ap^2-2b} = \frac{(c-a)^2}{(2ap^2-2b)^2}$$

$$\therefore (2b-2p^2c)(2ap^2-2b)^2 = (c-a)^2(2ap^2-2b)$$

$$\Rightarrow 4acp^4 - 4b(a+c)p^2 + 4b^2 + (c-a)^2 = 0$$

$$\Rightarrow p^2 = \frac{4b(a+c) \pm \sqrt{16b^2(a+c)^2 - 16ac\{4b^2+(c-a)^2\}}}{8ac}$$

Let d is the common difference of A.P.
$a, b, c \Rightarrow a = b-d, c = b+d,$

$$so, p^2 = \frac{4a(2b) \pm \sqrt{16b^2(2b)^2 - 16(b-d)(b+d)(4b^2+4d^2)}}{8(b-d)(b+d)}$$

$$= \frac{b^2 \pm \sqrt{b^4-(b^4-d^4)}}{b^2-d^2} = \frac{b^2 \pm d^2}{b^2-d^2}$$

$$so, p^2 = \frac{b^2+d^2}{b^2-d^2}\left(p^2 \neq \frac{b^2-d^2}{b^2-d^2}\ as\ p^2 \neq 1\right)$$

$$From\ equation\ (1)\alpha = \frac{c-a}{2ap^2-2b}$$

$$\Rightarrow \alpha = \frac{2d}{2(b-d)\frac{(b^2+d^2)}{b^2-d^2} - 2h} = -\left(\frac{b+d}{b-d}\right) = -\frac{c}{a}$$

Let second root of equation (2) be β

$$\therefore \alpha\beta = 1 \Rightarrow \beta = \frac{1}{\alpha} = \frac{1}{-c/a} = -\frac{a}{c}$$

10) If α, β are the roots of the equation $ax^2 + 2bx + c = 0$ and $\alpha^4\ \beta^4$ are the roots of the equation $lx^2 + mx + n = 0$, then prove that the roots of the equation $a^2lx^2 - 4aclx + 2c^2l + a^2m = 0$ are always real and opposite in sign (α, β are real and different).

Sol.: We have $\alpha + \beta = -\frac{b}{a}$; $\alpha\beta\frac{c}{a}$ & $\alpha^4 + \beta^4 = -\frac{m}{1}$;

$$\alpha^4\beta^4 = \frac{n}{l}, Now\ \alpha^4+\beta^4 = -\frac{m}{l} = [(\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2]$$

$$\Rightarrow -\frac{m}{l} = [\{(\alpha+\beta)^2 2\alpha\beta\}^2 - 2\alpha^2\beta^2] = \left\{\frac{b^2}{a^2} - \frac{2c}{a}\right\} - \frac{2c^2}{a^2}$$

$$\Rightarrow \left(\frac{b^2}{a^2}\right)^2 - 4\left(\frac{c}{a}\right)\left(\frac{b^2}{a^2}\right) + 2\left(\frac{c^2}{a^2}\right) + \frac{m}{l} = 0$$

It shows $\left(\frac{b^2}{a^2}\right)$ is a root of the equation $x^2 - 4\left(\frac{c}{a}\right)x + 2\left(\frac{c^2}{a^2}\right) + \frac{m}{l} = 0 \ldots\ldots\ldots\ldots(1)$

$$i.e., a^2lx^2 - 4aclx + 2c^2l + ma^2 = 0$$

It show one of the equation (i) is $\frac{b^2}{a^2}$ (positive)

Now, $y + \frac{b^2}{a^2} = \frac{4c}{a} \Rightarrow y = -\left(\frac{b^2-4ac}{a^2}\right) < 0$

Hence roots are real and opposite in sign.

11) If $\tan x - \tan y = a$ and $x + y = 2b$, prove that $\tan z$ and $\tan y$ are the roots of the equation $x^2 - (1-a)\tan 2b\,.\,x + a = 0$

Sol.: Let $\tan x$, $\tan y$ are the roots of the equation $x^2 - (1-a)\tan 2b\,.\,x + a = 0$

$\tan x + \tan y = (1 - a)\tan 2b$ $and \tan x . \tan y = a \dots\dots\dots\dots (1)$

$and \tan x - \tan y = a\ (it\ is\ given). Given\ x + y = 2b \dots\dots\dots\dots..(2)$

Taking tan on both sides of equation (2)

$$\frac{\tan x + \tan y}{1 - \tan x \tan y} = \tan 2b \Rightarrow \frac{\tan x + \tan y}{1 - ay} = \tan 2b$$

So, equation (1) is true

∴ Our assumption that $\tan x$, $\tan y$ are the roots of the equation $x^2 - (1 - a).\tan 2b\,.x + a = 0, is\ true.$

12) Let $p_n(x)$ be the polynomial, $p_n(x) = 1 + 2x + 3x^2 + \cdots + (n+1)x^n$, show that $p_n(x)$ has no real root if n is even and exactly one real root if n is odd and root lies between -1 and 0.

Sol.: $p_n(x) = 1 + 2x + 3x^2 + \cdots + (n+1)^{x^n}$ (where x >0; $p_n(x)$>0) so, $p_n(x)$ have no positive real root.

$$p_n(x) = 1 + 2x + 3x^2 + \cdots + (n+1)x^n,$$

$$xp_n(x) = x + 2x^2 + \cdots + nx^4 + (n+1)x^{n+1}$$

$$\Rightarrow (1-x)p_n(x) = 1 + x + x^2 + x^3 + \cdots + x^4 - (n+1)x^{n+1}$$

$$= \frac{1(1 - x^{n+1})}{1 - x} - (n+1)x^{n+1}$$

$\Rightarrow p_n(x) = \frac{1-(n+2)x^{n+1}+(n+1)x^{n+2}}{(1-x)^2}$ for negative values of x, $p_n(x)$will vanish when ever

$$f_{(x)} = 1 - (n+2)x^{n+1} + (n+1)x^{n+2}$$

$$f_{(-x)} = 1 - (n+2)(-x)^{n+1} + (n+1)(-x)^{n+2}.$$

If n is even, there is no change of sign in this expression and so there is no negative real root. If n is odd, there is one charge of sign. So there can be one negative real root.

In this case $f_{(-1)} = 1 - (n+2) - (n+1) = -(n+1) = -(2n+2) < 0\& f_{(0)} > 0.$

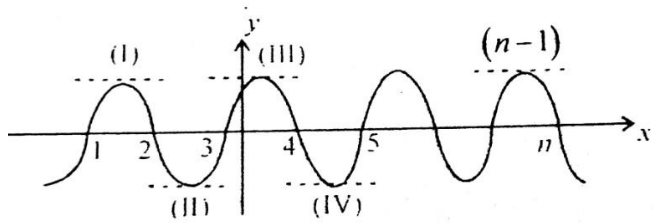
So we can say that when n is odd, the real root lies between 0 and -1.

13) Show that the roots of the equation $x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$ can not be real if $(n-1){a_1}^2 - 2na_2 < 0$.

Sol.: If $f_{(x)} = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0 \dots\dots\dots\dots (1)$ has n roots, then $f'_{(x)} = 0\ has\ (n-1) roots. f'_{(x)}$ has $(n-2)$ roots.

So differential equation (1) $(n-2)$ times we get,

$n(n-1)(n-2)\dots.3x^2 + (n-1)(n-2)\dots.2a_1x + (n-2)(n-3)\dots.1.a_2 = 0$ has two roots.



If equation (2) not has two real roots, then equation (1) not has n (all) real roots.

Equation (2) not has two real root if $B^2 - 4Ac$ (discriminant of (2) <0)

$$\Rightarrow [(n-1)(n-2)\ldots..2]^2.a_1{}^2 - 4.n(n-1)(n-2)\ldots\ldots\ldots..3.(n-2)(n-3)\ldots..2.1.a_2 < 0.$$

$\Rightarrow (n-1).a_1{}^2 - 2na_2 < 0$ Proved.

14) a, b, c, d are four distinct real numbers and they are in A. P. If $2(a-b)+x(b-c)^2+(c-a)^3 = 2(a-1)+(b-d)^2+(c-d)^3$ then prove that $x \geq 16$ or $x \leq -8$.

Sol.: Since a, b, c, d are in A.P.

$\therefore (b-a) = c-b = d-c =$ Let d (common difference)

$\therefore d = a+3D \Rightarrow a-d = -3D$ and $d = b+2D$

$\Rightarrow b-d = -2D$ $c = a+2D \Rightarrow c-a = 2D.$

$\therefore$ *Given equation*, $2(a-b)+x(b-c)^2+(c-a)^3$

$= 2(a-d)+(b-d)^2+(c-d)^3$ *becomes*

$-2D + xD^2 + (2D)^3 = -6D + 4D^2 - D^3$

$\Rightarrow 9D^2 + (x-4)D + 4 = 0$ *since D is real*

$\Rightarrow (x-4)^2 - 4.9.4 \geq 0 \Rightarrow x^2 - 8x - 128 \geq 0$

$\Rightarrow (x-16)(x+8) \geq 0 \therefore x \geq 16$ *or* $x \leq -8$ *proved.*

15) Find the value of 'a' for which the equation $(x^2+x+2)^2 - (a-3)(x^2+x+2)(x^2+x+1) + (a-4)(x^2+x+1)^2 = 0$, has at least one real root.

Sol.: The given equation can be written as

$$\left(\frac{x^2+x+2}{x^2+x+1}\right) - (a-3)\left(\frac{x^2+x+2}{x^2+x+2}\right) + (a+4) = 0 \ldots\ldots\ldots(1)$$

$$Let\,\frac{x^2+x+2}{x^2+x+2} = t \Rightarrow t = 1 + \frac{1}{x^2+x+1}$$

$$Since\ (x^2+x+1) = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} \Rightarrow x^2+x+1 \geq \frac{3}{4}$$

$\Rightarrow t \in \left(1, \frac{7}{3}\right)$ now equation (1) becomes;

$t^2 - (a-3)t + (a-4) = 0 \ldots\ldots\ldots.(2)$

At least one root of this equation must lie in $\left(1, \frac{7}{3}\right)$ from equation (2), $t = a-4, 2.$

For one root lie in $\left(1, \frac{7}{3}\right)$, *we has* $1 < a \leq \frac{7}{3} \Rightarrow 5 < a \leq \frac{19}{3}.$

16) Let $f_{(x)}$ be a polynomial leaving remainder, A_1, when divided by $(x-a_1)$. The remainder A_2 when divided by $(x-a_2)$………. And finally A_m, *if divide by* $(x-a_m)$. Find the remainder left by the polynomial, when divided by $(x-a_1)(x-a_2)\ldots..(x-a_m)$

Sol.: $f_{(a_1)} = A_1, f_{(a_2)} = A_2, \ldots\ldots\ldots.., f_{(a_m)} = A_m$

$f_{(x)} = (x-a_1)(x-a_2)\ldots.(x-a_m)Q(x) + R(x) \ldots\ldots\ldots.(y)$

Here $R_{(x)}$ will be a polynomial of degree $(m-1)$

$$R_{(a_1)} = A_1, R_{(a_2)} = A_2, \dots, R_{(am)} = A_m$$

$$R_{(x)} = p_{m-1} - x^{m-1} + p_{m-2} - x^{m-2} + \cdots + p_0 \; and \; ,$$

$$R_{(x)} = B_1(x-a_2)(x-a_3)\dots(x-a_m) + B_2(x-a_1)(x-a_3)(x-a_m) + B_3(x-a_1)(x-a_2)(x-a_4)\dots + \cdots + B_m(x-a_1)(x-a_2)\dots(x-a_{m-1}) \dots\dots.(1)$$

Putting x = $a_1$on both sides, we get

$$R_{(a_1)} = \frac{A_1}{(a_1-a_2)(a_1-a_3)\dots(a_2-a_m)} = B_1$$

… … … … … ….

… … … … … … …

$$Similarly\; B_2 = \frac{A_2}{(a_2-a_1)(a_2-a_3)\dots.(a_2-a_m)}$$

… … … … … … … …

… … … … … … … ..

$$B_m = \frac{A_m}{(a_m-a_1)(a_m-a_2)\dots..(a_m-a_{m-1})}$$

$$\therefore R_{(x)} = \frac{A_1.(x-a_2)(x-a_3)\dots(x-a_m)}{(a_1-a_2)\dots.(a_1-a_m)} + \frac{A_2.(x-a_1)(x-a_3)\dots..(x-a_m)}{(a_2-a_1)(a_2-a_3)\dots\dots(a_2-a_m)} + \cdots$$

$$\Rightarrow R_{(x)} = \sum_{i=1}^{m} A_i \left(\prod_{\substack{k=2 \\ k\neq 1}}^{m} \left(\frac{x-a_k}{a_i-a_k} \right) \right)$$

17) Given that α, γ are the roots of the equation $Ax^2 - 4x + 1 = 0$ and β, δ the roots of the equation $Bx^2 - 6x + 1 = 0$ find values of A and B such that α, β, γ and δ are in H.P.

Sol.: Given equation are $Ax^2 - 4x + 1 = 0 \dots\dots\dots.(1)$

$$Bx^2 - 6x + 1 = 0 \dots\dots\dots\dots..(2)$$

$$\alpha + \gamma = \frac{4}{A} \dots\dots\dots(i) \quad \beta + \delta = \frac{6}{B} \dots..(ii) \quad \alpha\gamma = \frac{1}{A} \dots\dots\dots..(iii) \quad \beta\delta = \frac{1}{B} \dots\dots\dots(iv)$$

Given $\alpha, \beta, \gamma, \delta$ are in H. p.

$$\therefore \beta = \frac{2\alpha\gamma}{\alpha+\gamma} = \frac{1}{2} \, and \, \gamma = \frac{2\beta\delta}{\beta+\delta} = \frac{1}{3}$$

Since β is a root of equation (2)

$$\therefore B\beta^2 - 6\beta + 1 = 0 \Rightarrow B \times \frac{1}{4} - 6.\frac{1}{2} + 1 = 0 \;\Rightarrow B = 8$$

Since γ is a root of equation (1) $\quad \therefore A\gamma^2 - 4\gamma + 1 = 0, \Rightarrow \frac{4}{9} - \frac{4}{3} + 1 = 0 \Rightarrow A = 3$

18) If α, β are the roots of the equations $x^2 - px + q = 0$ then find the quadratic equation whose roots are

(a) $(\alpha^2 - \beta^2)(\alpha^3 - \beta^3) and\; \alpha^3\beta^2 + \alpha^2\beta^2$

(b) $(\alpha^2 + \beta^2)(\alpha^3 + \beta^3) and\; \alpha^5\beta^3 + \alpha^3\beta^5 - 2\alpha^4\beta^4$

Sol.:

(a) α, β are roots of $x^2 - px + q = 0$

$\therefore\ \alpha+\beta=p\ \&\ \alpha\beta=q\ \ldots\ldots\ldots..(1)$

$Now\ (\alpha^2-\beta^2)(\alpha^3-\beta^3)=\{(\alpha+\beta)(\alpha-\beta)\}\{(\alpha-\beta)\big((\alpha^2-\beta^2)(\alpha^2+\alpha\beta+\beta^2)\}$

$=(\alpha-\beta)^2(\alpha+\beta)(\alpha^2+\beta^2+\alpha\beta)$

$=\{(\alpha+\beta)^2-4\alpha\beta\}(\alpha+\beta)^2\{(\alpha+\beta)^2-\alpha\beta^2\}$

$=(p^2-49)p(p^2-q)=\ p^5-5p^3q+4pq^4$

$\alpha^3\beta^3+\alpha^2\beta^2=\ \alpha^2\beta^3(\alpha+\beta)=(q)^2.p=pq^2$

$\therefore$ Quadratic equation whose roots are

$(\alpha^2-\beta^2)(\alpha^3-\beta^3)\ \&\ \alpha^3\beta^2+\alpha^2\beta^3$ is

$$x^2-[\{(\alpha^2-\beta^2)(\alpha^3-\beta^3)\}+\{\alpha^3\beta^2+\alpha^2\beta^3\}]x+(\alpha^2-\beta^2)(\alpha^3-\beta^3)\{\alpha^3\beta^2+\alpha^2\beta^3\}=0$$

$\Longrightarrow x^2-\ [(p^5-5p^3q+4pq^4)+pq^2]x-(p^5-5p^3q+4pq^4)$

$pq^2=0\Longrightarrow x^2-\ \ (p^5-5p^3q+4pq^4+pq^2)x-(p^6q^2-5p^4q^3+4p^2q^6)=0$

(b) $(\alpha^2+\beta^2)(\alpha^3+\beta^3)=\{(\alpha+\beta)^2-2\alpha\beta\}\{(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)\}$

$=\{(\alpha+\beta)^2-2\alpha\beta\}\{(\alpha+\beta)\{(\alpha+\beta)^2-2\alpha\beta-\alpha\beta\}\}$

$=(p^2-2q)(p^3-3pq)=p(p^2-2q)(p^2-3q)$

$\alpha^5\beta^3+\alpha^3\beta^5-2\alpha^4\beta^4=\alpha^3\beta^3(\alpha^2+\beta^2-2\alpha\beta)$

$=\alpha^3\beta^3\{(\alpha+\beta)^2-4\alpha\beta\}=q^3(\pi^2-49)$

$\therefore$ Quadratic equation whose roots are $(\alpha^2+\beta^2)(\alpha^3+\beta^3)\ \&\ \alpha^5\beta^2+\alpha^3\beta^5-2\alpha^5\beta^4\ is$

$x^2-[\{(\alpha^2+\beta^2)(\alpha^3+\beta^3)\}+\{\alpha^5\beta^3+\alpha^3\beta^5-2\alpha^4\beta^4\}]$

$x+\{(\alpha^2+\beta^2)(\alpha^3+\beta^3).\alpha^5\beta^3+\alpha^3\beta^5-2\alpha^4\beta^4\}=0$

$\Longrightarrow x^2—[p(p^2-2q)(p^2+3q)+q^3(p^2-4pq)]$

$x+p(p^2-2q)(p^2+3q)+q^3(p^2-49)=0$

19) Let p(x) be a polynomial of degree n>1 with integer coefficients and let k be a positive integer. Consider the polynomial $\boldsymbol{Q(x)=p\left(p.\left(p\left(p_{(x)}\right)\right)\right)}$, where p occurs k times. Prove that there are at most n integer t such that Q(t)= t.

Sol.: The claim is obvious of every integer fixed point of Q is a fixed point of p itself. For the sequel assume that this is not the case. Take any integer x_0 such that $\mathrm{Q}(x_0)=x_0, P(x_0)\neq x_0$ and define inductively $x_{i+1}=p_{(x_0)}\ for\ i=0,1,2,\ldots\ldots\ldots,$ then $x_k=\ x_0$. it is evident that $p_{(u)}-p_{(v)}$ is divisible $u-v$ for distinct integer u, v ……..(i)

[Indeed, if $p(x)=\ \sum a_ix_i$ then each $q_i(u_i-v_i)$ is divisible by $u-v$].

Therefore each term in the claim of (non-zero) differences.

$x_0-x_1,x_1-x_2,\ldots.,x_{k-1}-x_k,x_k-x_{k+1}\ldots\ldots.(ii)$ is a divisor of the next one; and since $x_k-x_{k+1}=\ x_0-x_1;$ all these difference have equal absolute values. For $x_m=\min(x_1,\ldots.,x_k)$ this means that $x_{m-1}-x_m=\ -(x_m-x_{m+1})$. Thus $x_{m+1}(\neq x_m)$ It follows that consecutive difference in the sequence (ii) have opposite signs. Consequently $x_0,x_1,x_2\ldots\ldots$ is an alternating sequence of two distinct values. In other words, every integer fixed point of Q is a fixed point of the polynomial $P\big(p_{(x)}\big)$. Out task is to prove that there are at most n such points.

Let a be one of them so that b= p(a) ≠ a (we have assumed that such an a exists); then a = $p_{(b)}$. Take any other integer fixed point α of $P(p_{(x)})$ *and let* $p_{(\alpha)} = \beta$, *so that* $p_{(\beta)} = \alpha$; the numbers α and β need not be distinct (α can be a fixed point of p), but each of α, β is different from each of a, b. Applying property (1) to the four pairs of integers (α, a) (β, b), (a, b), (β, a) we get that the members $\alpha - a$ *and* $\beta - b$ divide each other, and also $\alpha - \beta$ *and* $\beta - a$ divide each other.

Consequently, $\alpha - b \pm (\beta - a), \alpha a = \pm(\beta - b) \ldots\ldots\ldots\ldots\ldots (iii)$

Suppose we have a plus in both in stances : $\alpha - b = \beta - a$ *and* $\alpha - a = \beta - b$.

Subtraction yields $a - b = b - a$, a contradiction, as a ≠ b therefore at least one equality in (iii) holds with a minus sign. For each of them this means that α+β = a +b; equivalently a+ b –$\alpha = p_{(\alpha)} = 0$.

Denote a +b by c. We have shown that every integer fixed point of Q other that a and b is a root of the polynomial. $F_{(x)} = c - x - p_{(x)}$. This is of course true for a and b as well. And since p has degree n >1, the polynomial F has the same degree. S, it cannot have more than n roots. Hence the result.

20) If α areal root of $x^5 - x^3 + x - 2 = 0$, then show that $[\alpha^6] = 3$ (for any real number we denote by [x] the greatest integer not exceeding x)

Sol.: If α is solution of $x^5 - x^3 + x - 2 = 0$, then we have $\alpha(\alpha^4 - \alpha^2 + 2) = 2$. Now,

$\alpha^4 - \alpha^2 + 1 = \left(\alpha^2 - \frac{1}{2}\right)^2 + \frac{3}{4} > 0$. So, we must have α> 0. Also $0 \le x \le 1 \Longrightarrow x^5 - x^3 - x + 2$

$x^3(x^2 - 1) + x - 1 - 1 = -[1 + (1 - x) + x^3(1 - x)^2]$

$\therefore \alpha < 1.$ *Again* α is a solution of the given equation implies that $\alpha^5 - \alpha^3 + \alpha - 2 = (\alpha - 1)(\alpha^4 + \alpha^3 + 1) - 1 = 0$

If $\alpha \ge 2$, *then* $(\alpha - 1)(\alpha^4 + \alpha^3 + 1) - 1 \ge 24$

$\therefore \alpha < 2$ *we have* $1 < \alpha < 2, \alpha^6 = \alpha^4 - \alpha^2 + 2\alpha$.

We want to prove that $[\alpha^6] = 3$ *or* $3 \le \alpha^6 < 4$

or $\alpha^4 - \alpha^2 + 2\alpha - 3 \ge 0$ *and* $\alpha^4 - \alpha^2 + 2\alpha - 4 < 0$

But $\alpha^5 - \alpha^3 = 2 - \alpha$.

So that want to prove that $2\alpha^2 - 5\alpha + 2 < 0$

Now, $2\alpha^2 - 5\alpha + 2 = (a + 2)(2\alpha - 1) < 0$ *for*

$\frac{1}{2} < \alpha < 2$. We have already seen that $1 < \alpha < 2$.

Hence $\alpha^6 < 4$.

Again $\alpha^4 - \alpha^2 + 2\alpha - 3 \ge 0$ is equivalent to

$\alpha^5 - \alpha^3 + 2\alpha^2 - 3\alpha \ge 0$ is equivalent to

$2\alpha^3 - 4\alpha + 2 = 2(\alpha - 1)^2 \ge 0$

$\therefore$ *we proved* $3 \le \alpha^6$

21) Let α and β be the roots of the equation $x^2 - mx - 1 = 0$ where m is an odd integer. Let $\lambda n = \alpha^n + \beta^n$ *for* $n \ge 0$. *Show that for* $n \ge 0$

(a) λ_n **is an integer**
(b) $\mathbf{gcd}(\boldsymbol{\lambda n}, \boldsymbol{\lambda_{n+1}}) = \mathbf{1}$

Sol.: α and β are the root of the equation $x^2 - mx - 1$ $we\ have\ \alpha^2 + m\alpha - 1 = 0 \dots\dots\dots(1) \beta^2 + m\beta - 1 = 0 \dots\dots\dots(2)$

$Multiply\ Eq.(1) by\ \alpha^{n-2}, \alpha^n + m\alpha^{n-1} - \alpha^{n-2} = 0 \dots\dots\dots(3)$

Multiply Eq. (2) by $\beta^{n-2}, \beta^n + m\beta^{n-1} - \beta^{n-2} = 0 \dots\dots\dots\dots(4)$

Adding Equations. (3) and (4), we get

$\alpha^n + \beta^n = -m(\alpha^{n-1} + \beta^{n-1}) + (\alpha^{n-2} + \beta^{n-2})$ Which gives a recurrence relation for $n \geq 2$ $i.e.$,

$$\lambda n = -m\,\lambda_{n-1} + \lambda_{n-2}\ for\ n \geq 2 \dots\dots\dots(A)$$

(a) $\lambda_0 = 1 + 1 = 2, \lambda_1 = \alpha + \beta = -m.$ Thus $\lambda_0\ and\ \lambda_1$ are integers. By induction, it follows from (A) that λ_n is an integer for each $n \geq 0$.
(b) We again use (A) to prove by production that $\gcd(\lambda n, \lambda_{n+1}) = 1$. This is clearly true for $\mathrm{n} = 0$ as $\gcd(-2, -m) = 1$ Let $\gcd(\lambda_{n-2}, \lambda_{n-1}) = 1, n \geq 2$.

It were to happen that $\gcd(\lambda_{n-1}, \lambda_n) > 1$

Take a prime p, that divides both $\lambda_{n-1}\ and\ \lambda_n$. Then from A, we find that p divides λ_{n-2} also.

Thus ϕ is a factor of $\gcd(\lambda_{n-2}, \lambda_{n-1})$ a contradiction, so $\gcd(\lambda_{n-2}, \lambda_n)$ is equal to 1.

Hence we have $\gcd(\lambda, \lambda_{n+1}) = 1\ \forall\ n \geq 0$.

22) If p, q, r be positive real numbers, but not all equal such that two of the equations. $\boldsymbol{px^2 + 2qx + r = 0, qx^2 + 2rx + p = 0, rx^2 + 2px + q = 0}$ **have a common root say $\boldsymbol{\alpha}$. Show that**

(a) $\boldsymbol{\alpha}$ **is real and negative ,**
(b) **The third equation has non – real roots.**

Sol.: Consider the discriminantes of three equations.

$$px^2 + 2qr + r = 0 \dots\dots\dots(1)$$

$$qx^2 + 2rx + p = 0 \dots\dots\dots..(2)$$

$$rx^2 + 2px + q = 0 \dots\dots..(3)$$

Let us denote by D_1, D_2, D_3 respectively.

So, we have

$D_1 = 4(q^2 - rp), D_2 = r(r^2 - pq), D_3 = 4(p^2 - q^r)$

Adding we get

$D_1 + D_2 + D_3 = 4(p^2 + q^2 + r^2 - pq - qr - rp)$

$= 2\{(p-q)^2 + (q-r)^2 + (r-p)^2\} > 0$

∴, q, r are not all equal.

Hence at least one of $D_1 . D_2 . D_3$ must be positive we may assume $D_1 > 0$.

Let us suppose $D_2 < 0$ and $D_3 < 0$. In this case both the equation (2) and (3) have only non –real roots but equation (1) has only real roots. Hence, the common roots α must be between (2) and (3).

But then the conjugate $\bar{\alpha}$ of α is the other root of both (2) and (3).

Hence it follows that (2) and (3) have same set of roots

$\Rightarrow \frac{q}{r} = \frac{r}{p} = \frac{p}{q}$. This p= q= r contradicting the given condition. Hence D_2 and D_3 cannot be negative we may assume $D_2 \geq 0$.

So we have $q^2 - rp > 0, r^2 - pq \geq 0$. which give $q^2r^2 > p^2q^r$

$\therefore$p, q, r are all positive

Hence, we get $qr > p^2. or\ D_3 > 0$

We conclude that the common root must be between eqs. (1) and (2),. Thus $p\alpha^2 + 2q\alpha + r = 0, q\alpha^2 + 2r\alpha + p = 0$. Eliminating α^2, we get

$2(q^2 - pr)\alpha = p^2 - qr \ \therefore\ q^2 - pr > 0, p^2 - qr < 0$.

So we conclude $\alpha < 0$.

$\Rightarrow$ (B) as only non-real roots.

23) If a, b, c are three positive real numbers such that $a + b + c = 1$ and Let

$$\boldsymbol{\lambda = \min\{a^3 + a^2bc, -b^3 + ab^2c, c^3 + abc^2\}}$$

Show that the roots of the equations $x^2 + x + 4\lambda = 0$ are real.

Sol.: Suppose the equation $x^2 + x + 4\lambda = 0$ has no real roots. Then $1 - 16\lambda < 0 \Rightarrow 1 - 16(a^3 + a^2bc) < 0$,

$1 - 16(b^3 + ab^2c) < 0\ and\ 1 - 16(c^3 + abc^2) < 0$.

$Now\ 1 - 16(a^3 + a^2bc) < 0 \Rightarrow 1 - 16a^2(a + bc) < 0$

$\Rightarrow 1 - 16a^2(1 - b - c + bc) < 0$

$\Rightarrow 1 - 16a^2(1 - b)(1 - c) < 0 \Rightarrow \frac{1}{16} < a^2(1 - b)(1 - c)$

$Similarly\ \frac{1}{16} < b^2(1 - c)(1 - a) and \frac{1}{16} < c^2(1 - a)(1 - b)$ Multiplying these we get, $a^2b^2c^2(1 - a)^2(1 - b)^2(1 - c)^2 > \frac{1}{16^3}$

$$However, 0 < a < 1 \Rightarrow a(1 - a) \leq \frac{1}{4}$$

$Hence, a^2b^2c^2(1 - a)^2(1 - b)^2(1 - c)^2 = \big(a(1 - a)\big)^2\big(b(1 - b)\big)^2\big(c(1 - c)\big)^2 < \frac{1}{16^3}$ a contradiction.

So, we conclude that the given equation has real roots.

24) Let $f_{(x)}, g_{(x)}\ and\ h_{(x)}$ be three polynomials such that $\frac{f_{(x)}}{h_{(x)}}, \frac{g_{(x)}}{h_{(x)}}, (f_{(x)}, g_{(x)}) = 1$, show that $f_{(x)}, \frac{g_{(x)}}{h_{(x)}}$

Sol.: Since $\frac{f_{(x)}}{h_{(x)}}$ is a polynomial $p_{(x)}$ such that $h_{(x)} = g_{(x)}.p_{(x)}$ … … … … (1)

Again $\frac{g_{(x)}}{h_{(x)}}$ is a polynomial $q_{(x)}$ such that $h_{(x)} = q_{(x)}.g_{(x)}$ … … … (2)

Further $\because (f_{(x)}, g_{(x)}) = 1$.

$\therefore \exists$ polynomial $a_{(x)}\ and\ b_{(x)}$

Such that $f_{(x)}a_{(x)} + g_{(x)}b_{(x)} = 1$ … … . (3)

Multiplying by $h_{(x)}$ on both sides, we get

$f_{(x)}h_{(x)}a_{(x)} + g_{(x)}h_{(x)}b_{(x)} = h_{(x)}$ using (2) and (1), we get

$\Rightarrow f_{(x)}[g_{(x)}q_{(x)}]a_{(x)} + g_{(x)}[f_{(x)}p_{(x)}]b_{(x)} = h_{(x)} \Rightarrow f_{(x)}g_{(x)}[q_{(x)}a_{(x)} + p_{(x)}b_{(x)}] = h_{(x)} \Rightarrow f_{(x)} \frac{g_{(x)}}{h_{(x)}}$

25) If m, n integers ≥ 0 *and* $f_{(x)}, g_{(x)}$ are polynomial such that $(x-a)^m f_{(x)} = (x-\alpha)^n g_{(x)}$ *with* $f_{(x)} \neq 0, g_{(x)} \neq 0$. Show that $m = n$ *and* $f_{(x)} = g_{(x)}$.

Sol.: Given $(x-a)^m f_{(x)} = (x-\alpha)^n g_{(x)}, f_{(\alpha)} \neq 0, g_{(\alpha)} \neq 0$. We want to prove that m=n and $f_{(x)} = g_{(x)}$. If possible let m≠n, without loss of any generality. Let m>n.

$\therefore n-m$ is a +ve integer, so that

$$(x-\alpha)^m f(x) = (x-\alpha)^{n-m} g(x)$$

$$i.e. \frac{(x-\alpha)}{f_{(x)}} \Longrightarrow \alpha \text{ is a root of } f_{(x)} = 0$$

$\Longrightarrow f_{(\alpha)} = 0$ which is contrary to the given hypothesis.

$\therefore$ Our supposition is wrong, Hence m = n and $(x-a)^m f_{(x)} = (x-\alpha)^n g_{(x)} \Longrightarrow f_{(x)} = g_{(x)}$

26) The root of the equation $x^3 - ax^2 + bx - c = 0$ are α, β, γ from the equation whose roots are $\alpha+\beta, \beta+\gamma, \gamma+\alpha$. Also express $\frac{1}{\alpha+\beta} + \frac{1}{\beta+\gamma} + \frac{1}{\gamma+\alpha}$ in terms of a, b, c.

Sol.: Roots of the equation $x^3 - ax^2 + bx - c = 0 \ldots\ldots\ldots\ldots (1)$ are α, β, γ, if $\gamma = \alpha + \beta = (\alpha+\beta+\gamma) - \gamma = a - \gamma$

$[\because \alpha+\beta+\gamma = a]$

$\therefore y = a - x$ *or* $x = a - y$. Putting this value of x in (1) we have $(a-y)^3 - a(a-y)^2 + b(a-y) - c = 0$

or, $y^3 - 2ay^2 + (a^2+b)y + (c-ab) = 0 \ldots\ldots\ldots. (2)$

Which is required equation. Its roots are $\alpha+\beta, \beta+\gamma, \gamma+\alpha$. Changing into $\frac{1}{y}$ and Multiplying by y^3, we get $(c-ab)y^3 + (a^2+b)y^2 - 2ay + 1 = 0 \ldots\ldots. (3)$.

Roots of this equation are the reciprocals of the roots of (2), $\therefore$ Roots of (3) are

$$\frac{1}{\alpha+\beta}, \frac{1}{\beta+\gamma}, \frac{1}{\gamma+\alpha}$$

Now, $\frac{1}{\alpha+\beta} + \frac{1}{\beta+\gamma} + \frac{1}{\gamma+\alpha}$= sum of roots of (3)

$$= \frac{-a^2+b}{c-ab} = \frac{a^2+b}{ab-c}$$

27) If α, β, γ are the roots of the cubic equation $x^3 + 3x + 2 = 0$ from an equation whose roots are $(\beta-\gamma)^2, (\gamma-\alpha)^2, (\alpha-\beta)^2$ and hence show that $x^3 + 3x + 2 = 0$ has imaginary roots.

Sol.: $x^3 + 3x + 2 = 0 \ldots\ldots\ldots\ldots. (1)$

$\therefore$ It roots are α, β, γ

$\therefore \alpha+\beta+\gamma = 0, \alpha\beta+\beta\gamma+\gamma\alpha = 3, \alpha\beta\gamma = -2.$

Let γ be a root of the transformed equation

$$\therefore \gamma = (\beta-\gamma)^2 = (\beta+\gamma)^2 - 4\beta\gamma = (-\alpha)^2 - \frac{4\alpha\beta\gamma}{\alpha}$$

$$[\because \alpha+\beta+\gamma = 0] = \alpha^2 + \frac{8}{\alpha} [\because \alpha\beta\gamma = -2]$$

$$\textit{Replacing } \alpha \textit{ by } x, \therefore y = x^2 + \frac{8}{x} \textit{ or } x^3 - xy + 8 = 0 \ldots\ldots. (2)$$

Subtracting (2) from (1) $(3+y)x - 6 = 0$

$\therefore x = \frac{6}{3+y}$. Putting this value of x in (1), we get

$$\left(\frac{6}{3+y}\right)^3 + 3.\frac{6}{3+y} + 2 = 0$$

$216 + 18(3+y)^2 + 2(3+y)^2 = 0.\ y^3 + 18y^2 + 82y + 216 = 0$. Which is the required equation product of all its roots $= -216$

$\therefore (\alpha-\beta)^2\ (\beta-\gamma)^2\ (\gamma-\alpha)^2 = -216$

R H S being – ve, one of the factors.

$\therefore$ that will make all the three roots imaginary which is not possible. Every odd integer equation with real coefficients has at least one real root on the L H S $(\alpha-\beta)^2\ is -ve$.

$\therefore\ \alpha-\beta$ is purely imaginary.

$\therefore\ \alpha$ and β are conjugate complex roots. Hence two roots of (1) are imaginary.

28) The roots x_1, x_2, x_3 of the equation $x^3 + ax + a = 0$, where a is a non zero real, satisfy $\frac{x_1{}^2}{x_2} + \frac{x_2{}^2}{x_3} + \frac{x_3{}^2}{x_1} = -8$. Find x_1, x_2, x_3

Sol.: We are given $x_1{}^3x_3 + x_2{}^3x_1 + x_3{}^3x_2 = 8x_1x_2x_3$

$x_1 + x_2 + x_3 = 0;\ x_1x_2 + x_2x_3 + x_3x_1 = a;$

$$x_1x_2x_3 = -a\ and\ for\ i = 1, 2, 3.$$

$$x_i{}^3 + ax_i + a = 0, x_1{}^3 + ax_1 + a = 0$$

$$Now, x_2{}^3 + ax_2 + a = 0, x_3{}^3 + ax_3 + a = 0.$$

$$\Longrightarrow (x_1{}^3x_3 + x_2{}^3x_1 + x_3{}^3x_2) + a(x_1x_3 + x_1x_3 + x_2x_1 + x_3x_2) + a(x_3 + x_1 + x_2) = 0.$$

$$i.e., 8a + a^2 = 0 \Longrightarrow a = -8$$

So, given equation is $x^3 - 8x - 8 = 0, one\ root\ is -2$, other roots are given by $x^2 - 2x - 4 = 0\ i.e.\ x = 1 \pm \sqrt{5}$

$So, \{x_1, x_2, x_3\} = \{-2, 1-\sqrt{5}, 1+\sqrt{5}\}$

29) If a, b, c ∈ R, a ≠ 0, then solve the system of equation: $ax_1{}^2 + bx_1 + c = x_2$; $a\,x_2{}^2 + bx_2 + c = x_3$; … … $ax_{n-1}{}^2 + bx_{n-1} + c = x_n\ and\ ax_n{}^2 + bx_n + c = x_1\ is\ n\ unknowns\ x_1, x_2, x_n\ then$

(i) $(b-1)^2 < 4ac$;

(ii) $(b-1)^2 = 4ac$;

(iii) $(b-1)^2 > 4ac$

Sol.: Given system of equation can be written as

$$ax_1{}^2 + (b-1)x_1 + c = x_2 - x_1 = f_{(x_1)}\ say$$

$$a\,x_2{}^2 + (b-1)x_2 + c = x_3 - x_2 = f_{(x_2)}\ say$$

$$ax_{n-1}{}^2 + (b-1)x_{n-1} + c = x_n - x_{n-1} = f_{(x_{(n-1)})}\ say$$

$$ax_n{}^2 + (b-1)x_n + c = x_1 - x_n = f_{(x_{(n)})}\ say$$

$$f_{(x_1)} + f_{(x_2)} + \cdots + f_{(x_n)} = 0 \ldots\ldots (1)$$

Case 1: When $(b-1)^2 < 4ac$. Each roots of $ax_1{}^2 + (b-1)x_1 + c = 0$ are imaginary. If a > 0, then

$$f_{(x_1)} + f_{(x_2)} + \cdots + f_{(x_n)} > 0. If\ a < 0, then$$

$$f_{(x_1)} + f_{(x_2)} + \cdots + f_{(x_n)} \neq 0$$

$\therefore$No solution.

Case 2: When $(b-1)^2 = 4ac$. In case 1 and 2 all of

$f_{(x_1)}, f_{(x_2)}, \ldots, f_{(x_n)} \geq 0, f_{(x_1)}, f_{(x_2)}, \ldots, f_{(x_n)} \leq 0$

From equation (1),

$$f_{(x_1)} + f_{(x_2)} + \cdots + f_{(x_n)} = 0,$$

$$f_{(x_1)} = f_{(x_2)} = \cdots = f_{(x_n)} = 0$$

But $f_{(x_i)} = 0 \Longrightarrow ax_i^2 + (b-1)x_i + c = 0$

$$x_i = \frac{-(b-1) \pm 0}{2a}$$
$$= \frac{1-b}{2a} [\because \ (b-1)^2 = 4ac]$$

$$Hence\ x_1 = x_2 = \cdots = x_n = \frac{1-b}{2a}$$

Case 3: When $(b-1)^2 > 4ac$. Roots of $ax_i^2 + (b-1)x_i + c = 0$, are real and unequal. Let α and β be roots.

If $a < 0\ \forall x_1 \in [\alpha, \beta]\ a\, x_1^2 + (b-1)x_1 + c \geq 0$

$i.e.\, f(x_1) \geq 0$

$Similarly\ for\ all\ x_i \in [\alpha, \beta],$

$(i = 1, 2, 3 \ldots n) i.e.\, f_{(x_1)} \geq 0$

$But\ f_{(x_1)} + f_{(x_2)} + \cdots + f_{(x_n)} = 0,$

$$f_{(x_1)} = f_{(x_2)} = \cdots = f_{(x_n)} = 0$$

$\therefore x_1 = x_2 = \cdots = x_n$

$then\ each\ ax_i^2 + (b-1)x_i + c = 0$

$$So, x_1 = x_2 = \cdots = x_n = -(b-1) \pm \frac{\sqrt{(b-1)^2 - 4ac}}{2a}$$

$$Also, \forall\ x_1 \notin (\alpha, \beta)(i = 1, 2, 3, \ldots n) i.e.\, f(x_i) \leq 0$$

$$but\ f_{(x_1)} + f_{(x_2)} + \cdots + f_{(x_n)} = 0,$$

$$so\ f_{(x_1)} = f_{(x_2)} = \cdots = f_{(x_n)} = 0$$

$\therefore x_1 = x_2 = \cdots = x_n\ then\ each\ ax_i^2 + (b-1)x_i + c = 0$

$$so\ x_1 = x_2 = \cdots = x_n = \frac{(1-b) \pm \sqrt{(b-1)^2 - 4ac}}{2a}$$

$when\ a > 0\ we\ get$

$$x_1 = x_2 = \cdots = x_n = \frac{(1-b) \pm \sqrt{(b-1)^2 - 4ac}}{2a}$$

30) Let $P_{(x)} = 0$ be a fifth degree polynomial equation with integer coefficients that has at least one integer root. If $P_{(2)} = 13\ and\ P_{(10)} = 5$. Compute a value of x that must satisfy $P_{(x)} = 0$.

Sol.: Let $P_{(x)} \equiv (x-2)\, q_{(x)} + P_{(2)}, q_{(x)}$ would have integer coefficients.

Let r be an integer such that $P_{(r)} = 0$

Then $P_{(r)} = (r-2).\, q_{(r)} + 13 = 0, So\ r - \frac{2}{13}$

$Thus, r-2$ can only equal $\pm\ or \pm 13$. Leading to r = 3, 1, 15, or −11.

$Let\ P_{(x)} \equiv (x = 10), F_{(x)} + P_{(10)}$. Leads to $r - \frac{10}{5}$.

So r can only be 11, 9, 15 or 5. Thus, r = 15.

31) If x_1, x_2, x_3 are the roots of $x^3 - x^2 + 4 = 0$, from the equation whose roots are $x_1 + x_2^2 + x_3^2;\ x_2 + x_3^2 + x_1^2;\ x_3 + x_1^2 + x_2^2$

Sol.: x_1, x_2, x_3 are the roots of equation

$$x^3 - x^2 + 4 = 0 \ldots\ldots\ldots\ldots (1)$$

$\therefore x_1 + x_2 + x_3 = 1;\ x_1x_2 + x_2x_3 + x_3x_1 = 0;$

If the transformed equation is in terms of y, then

$$y = x_1 + {x_2}^2 + {x_3}^2 = x_1 + (x_2 + x_3)^2 - 2x_2x_3$$

$$= x_1 + (1 - x_1)^2 - \frac{2x_1x_2x_3}{x_1}$$

$$= x_1 + (1 - x_1)^2 + \frac{8}{x_1}$$

$$\therefore y = x + (1 - x)^2 + \frac{8}{x}$$

$$= x^2 - x + 1 + \frac{8}{x}$$

$$or\ x^3 - x^2 + x - xy + 8 = 0 \ldots\ldots(2)$$

Subtracting (2) from (1), we get $xy - x - 4 = 0 \Longrightarrow x = \frac{4}{y-1}$

Putting this value of x in (1), we get

$$\frac{64}{(y-1)^3} - \frac{16}{(y-1)^2} + 4 = 0$$

$$\Longrightarrow (y-1)^3 - 4(y-1) + 16 = 0$$

32) If α, β, γ be the roots of the equation $x^3 + 3x + 2 = 0$, find the equation whose roots are $(\alpha - \beta)(\alpha - \gamma), (\beta - \gamma)(\beta - \alpha), (\gamma - a)(\gamma - \beta)$.

Hence show that the above equation has two imaginary roots.

Sol.: Let $z = (\alpha - \beta)(\alpha - \gamma) = \alpha^2 - \alpha\beta - \alpha\gamma + \beta\gamma$

$= \alpha^2 - \sum\alpha\beta + \frac{2\alpha\beta\gamma}{\alpha}$ $or\ \alpha z = \alpha^3 - 3\alpha + 2(-2) \ldots\ldots\ldots.(1)$

$[\because\ \sum\alpha\beta = 3;\ \alpha\beta\gamma = -2]also\ \alpha^3 + 3\alpha + 2 = 0, we\ get$

$\alpha^2 - 3\alpha = -6\alpha - 2$ on putting this value in(1)

$$\alpha z = -6\alpha - 6\ or\ \alpha(z + 6) = -6, \alpha = -\frac{6}{z+6}$$

But α is a root of $x^3 + 3x + 2 = 0$

$$\therefore \left[-\frac{6}{z+6}\right]^3 + 3\left[-\frac{6}{z+6}\right] + 2 = 0$$

$$(z+6)^3 - 9(z+6)^2 - 108 = 0, z^3 + 9z^2 - 216 = 0$$

$let\ z_1, z_2, z_3$ be the roots of above equation then;

$z_1, z_2, z_3 = (\alpha - \beta)(\alpha - \gamma)\ (\beta - \gamma)(\beta - \alpha)\ (\gamma - a)(\gamma - \beta) = 216$

$or - (\alpha - \beta)^2(\beta - \gamma)^2(\gamma - a)^2 = 216,$

$(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - a)^2 = 216$

Hence, one of the factors in R H S must be $-ve$ say $(\alpha - \beta)^2 is - ve\ i.e. \alpha - \beta$ =pure imaginary showing that α and β are conjugate complex. Hence, the given equation has two imaginary roots.

33) If α, β, γ be the roots of $x^3 - x^2 + 4 = 0$, find that the equation whose roots are $\alpha - \alpha^2 + \left(\sum\alpha^2\right),\ \beta - \beta^2 + \left(\sum\alpha^2\right), \gamma - \gamma^2 + \left(\sum\alpha^2\right)$

Sol.: $\sum\alpha = 1, \sum\alpha\beta = 0\ \ \therefore \sum\alpha^2 = \sum\alpha^2 - 2\sum\alpha\beta = 1 \ldots..(1)$

$Also\ \alpha^3 - \alpha^2 + 4 = 0$ [$\therefore\alpha$ is a root of the given equation]

$$\therefore \alpha^2 - \alpha = -\frac{4}{\alpha} or\ \alpha - \alpha^2 = \frac{4}{\alpha} \ldots\ldots(2)$$

Let γ be a root of the new equation

$\therefore \gamma = \alpha - \alpha^2 + (\alpha^2 + \beta^2 + \gamma) = \frac{4}{\alpha} + 1[by\ (1) and\ (2)]$

$$\therefore \gamma - 1 = \frac{4}{\alpha} or\ \alpha = \frac{4}{\gamma - 1}$$

Since α is a root of $x^3 - x^2 + 4 = 0$, we get

$$\frac{64}{(\gamma-1)^3} - \frac{16}{(\gamma-1)^2} + 4 = 0, (\gamma-1)^3 - 4(\gamma-1) + 16 = 0$$

$$or,\ y^3 - 3y^2 - y + 19 = 0$$

34) A polynomial $f_{(x)}$ with rational coefficients leaves remainder 15. When divided by $x-3$ and remainder $2x+1$, when divided by $(x-1)^2$. Find the remainder when $f_{(x)}$ is divided by $(x-3)(x-1)^2$.

Sol.: Let quotient be $q_{(x)}$ and remainder be $r_{(x)}$ when $f_{(x)}$ *is divided by* $(x-3)(x-1)^2$.

Now as divisor is a polynomial of degree 3 the remainder must be polynomial of degree at most 2. i.e., it must be of the form $ax^2 + bx + c$; a, b, c are some rational numbers.

$$ax^2 + bx + c = a[(x-1)+1]^2 + b[(x-1)+1] + c$$

$$= a(x-1)^2 + (2a+b)(x-1) + a + b + c$$

By division algorithm,

$$f_{(x)} \equiv q_{(x)}(x-3)(x-1)^2 + a(x-1)^2 + (2a+b)(x-1) + a + b + c \ldots\ldots(1)$$

Now according to given condition $f_{(x)}$ leaves a remainder 15 when divided by $x-3$

$$\therefore f_{(3)} = 15$$

Now putting x = 3 in (1), we have

$$(2a+b)(x-1) + (a+b+c) = 2x + 1$$

Putting x = 1, we get $a + bc = 3 \ldots\ldots\ldots\ldots(3)$

Putting x = 0, throughout, we get $-a + c = 1 \ldots\ldots\ldots(4)$

From (2), (3) and (4), we get $a = 2, b = -2, c = 3$

Remainder = $ax^2 + bx + c = 2x^2 - 2x + 3.$

35) If $p_{(x)} = x^2 + ax + b$ be a quadratic polynomial in which a, b are inters. Given any integer n, show that there is an integer M such that $p_{(n)}p_{(n+1)} = p_{(M)}$

Sol.: Let the zero's of $p_{(x)}$ be α, β so that $p_{(x)} = (x-\alpha)(x-\beta). Then\ p_{(n)} = (n-\alpha)(n-\beta.)$

$p_{(n+1)} = (n+1-\alpha)(n+1-\beta)$ we have to show that $p_{(n)}p_{(n+1)}$ can be written as $(t-\alpha)(t-\beta)$ for some integer t (which will depend upon)

$p_{(n)}p_{(n+1)} = (n-\alpha)(n-\beta.)(n+1-\alpha)(n+1-\beta)$

$$= \{(n-\alpha)(n+1-\beta)\}\{(n-\beta)(n+1-\alpha)\}$$

$$= \{n(n+1) - n(\alpha+\beta) - \alpha + \alpha\beta\} \times \{n(n+1) - n(\alpha+\beta) - \beta + \alpha\beta\}$$

$$= \{n(n+1) + n\alpha + b - \alpha\}\{\{n(n+1)\} + na + b - \beta\}$$

$$= (t-\alpha)(t-\beta); t = n(n+1) + an + b = p_{(t)}$$

$Thus, p_{(n)}p_{(n+1)}$ can be written as $p_{(M)}$ for $p_{(n)}M = n(n+1) + an + b$

36) If n is an odd integers no divisible by 3, show that $xy(x+y)(x^2+xy+y^2)$ is a factor of $(x+y)^n - x^n - y^n$

Sol.: We have

$xy(x+y)(x^2+y^2+xy) = xy(x+y)(x - wy)(x - w^2y)$

[w, w^2 are non real cube roots of unity]

It is enough to show that $(x+y)^n - x^n - y^n$ vanishes for x = 0; y = 0. Now, $x = -y; x = wy \; and \; x = \omega^2 y$

The polynomial obviously vanishes for x = wy

$(wy+y)^n - (wy)^n - y^n = y^n[(w+1)^n - w^n - 1]$

$= y^n[(-w^2)^n - w^n - 1] = -y^n[(w+1)^n - w^n - 1]$

$= y^n[(-w^2)^n - w^n - 1] = -y^n[w^{2n} + w^n + 1]$

(∵ n is odd)

Let $n = 3p+2, then \; w^n = w^{3p+1}, w^{2n} = w^{6p+2} = w^2$

∴ Above expression = $-y^n[w^2+w+1] = 0$

If n =3p+2, then $\omega^n = \omega^2, \omega^{2n} = \omega$ and the above expression is zero. We can similarly prove that the given polynomial vanishes for $x = \omega^2 y$. If n is an odd positive integer not divisible by 3, then $(x+y)^n - x^n - y^2$ is divisible by $xy(x^2+y^2+xy)$.

37) Find out at what n the polynomial $1 + x^2 + x^4 + \cdots + x^{2n-2}$ is divisible by the polynomial $1 + x + x^2 + x^{n-1}$

Sol.: 1+$x^2 + x^4 + \cdots + x^{2n-2} = \frac{x^{2n-1}}{x^2-1}$,

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}$$

It is required to find out at what n.

$\frac{\left[\frac{x^{2n}-1}{x^2-1}\right]}{\left[\frac{x^n-1}{x-1}\right]}$ will be a polynomial in x,

$we\; find, \frac{\left[\frac{x^{2n}-1}{x^2-1}\right]}{\left[\frac{x^n-1}{x-1}\right]} = \frac{x^n+1}{x+1}$

For $x^n + 1$ to be divisible by x +1, it is necessary and sufficient that $(-1)^n + 1 = 0$ i.e. n is odd.

Thus, $1 + x + x^n + \cdots + x^{2n-2}$ is divisible by $1 + x + x^2 + \cdots + x^{n-1}$, if n is odd.

38) Find out at what values of p and q where $x^4 + 1$ is divisible by $p^2 + px + q$?

Sol.: Let us suppose

$$x^4 + 1 = (x^2+px+q)(x^2+p'x+q') = x^4(p+p')x^3(q + q'pp')x^2.(pq'+qp')x + qq'$$

For determining p, q, p' and q' we have four equations.

p +p' = 0, $pp' + q + q' = 0, pq' + qp' = 0, qq' = 1$

From(1) and (3), we find $p' = -p(q'-q) = 0$ assume

Case 1: $p = 0, p' = 0, q + q' = 0, qq' = 1, q^2 = -1$

$q = \pm i; q' = \pm i.$

The corresponding factorization has the from $x^4 + 1 = (x^2+i)(x^2-i)$

Case 2: $q' = q, q^2 = 1, q = \pm 1$, suppose first $q' = q = 1 \; then \; pp' = -2, p + p' = 0, p^2 = 2, p = \pm\sqrt{2}, p' = \pm\sqrt{2}$

The corresponding factorization is

$x^4 + 1 = \left(x^2 - \sqrt{2x} + 1\right)(x^2 - \sqrt{2x} + 1)$

Assume then,

$q = q' = 1, p + p' = 0, pp' = 2; p = \pm\sqrt{2i}, p' = \sqrt{2i}$

Factorization will be

$x^4 + 1 = \left(x^2 + \sqrt{2x_i} - 1\right)(x^2 - \sqrt{2x_1} - 1$

39) Show that if $a_1, a_2, \dots, a_n$ are all distinct, then the polynomial $(x - a_1)^2(x - a_2)^2 \dots . (x - a_n)^2 + 1$ can never be written as the product of two polynomials with integer coefficients.

Sol.: Suppose that there exists polynomial $f_{(x)} . g_{(x)}$ with integer coefficients such that

$f_{(x)} . g_{(x)} = (x - a_1)^2(x - a_2)^2 \dots . (x - a_n)^2 + 1 \dots \dots . (1)$

$\therefore$ R H S is always +ve.

$\therefore$ $f_{(x)}$ can never vanish.

So its sign never changes.

Similarly $g_{(x)}$ can never vanish and its sign never changes.

$\therefore f_{(x)}, g_{(x)}$ are always +ve, so $f_{(x)}$ *and* $g_{(x)}$ are both always +ve.

Substituting $x = a_1, a_2, \dots, a_n$ in (1) we get

$f_{(a_1)}\ g_{(a_1)} = 1, f_{(a_2)}\ g_{(a_2)} = 1, \dots . ., f_{(a_n)}\ g_{(a_n)} = 1$

$\because f_{(a_1)}, \dots \dots, f_{(a_n)}$ are all +ve integers.

It follows that

$f_{(a_1)} = f_{(a_2)} = \cdots = f_{(a_n)} = 1$

Similarly $g_{(a_1)} = g_{(a_2)} = \cdots = g_{(a_n)} = 1$

$\because f_{(x)} - 1, g_{(x)} - 1$ vanish when $x = a_1, a_2, \dots, a_n$

$\therefore f_{(x)} - 1 = p_{(x)}(x - a_1)(x - a_2) \dots . (x - a_n)$

By factor theorem,

$g_{(x)} - 1 = q_{(x)}(x - a_1)(x - a_2) \dots . (x - a_n)$

$p_{(x)}, q_{(x)}$ are polynomial with integer coefficients.

$\because f_{(x)} . g_{(x)}$ *is a degree of* $2n, p_{(x)} . q_{(x)}$ must be both constants. Suppose $p_{(x)} = a,\ q_{(x)} = b$

Then $f_{(x)} = a(x - a_1)(x - a_2) \dots . (x - a_n) + 1$

$g_{(x)} = b(x - a_1)(x - a_2) \dots . (x - a_n) + 1$

(substituting these conditions these conditions imply $a^2 = -1, b^2 = -1$)

$\therefore$ There is a contradiction and given polynomial cannot be expressed as the product of two polynomials with integer coefficients.

40) If $\alpha^{13} = 1$, *and* $\alpha \neq 1$, find an equation, whose roots are $\alpha + \alpha^3 + \alpha^4 + \alpha^{-4} + \alpha^{-3} + \alpha^{-1}$ *and* $\alpha^2 + \alpha^5 + \alpha^6 + \alpha^{-6} + \alpha^{-5} + \alpha^{-2}$

Sol.: Let A $= \alpha + \alpha^3 + \alpha^4 + \alpha^{-4} + \alpha^{-3} + \alpha^{-1}$

$= \alpha + \alpha^3 + \alpha^4 + \alpha^9 + \alpha^{10} + \alpha^{12} (\because \alpha^{13} = 1)$

$B = \alpha^2 + \alpha^5 + \alpha^6 + \alpha^{-6} + \alpha^{-5} + \alpha^{-2} = \alpha^2 + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^8 + \alpha^{11}$

$A + B = \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^8 + \alpha^9 + \alpha^{10} + \alpha^{11} + \alpha^{12} = (1 + \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^{12}) - 1 = \frac{(\alpha^{13} - 1)}{(\alpha - 1)} - 1 = 1$

$A \times B = (\alpha + \alpha^3 + \alpha^4 + \alpha^9 + \alpha^{10} + \alpha^{12}) \times (\alpha^2 + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^8 + \alpha^{11})$

$$= 3(\alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^{12}) \leq 3(-1) = -3$$

Required equation is $x^2 + x - 3 = 0$

41) Show that $f_{(x)} = x^{100} - x^{500} + x^{100} + x + 1 = 0$ has no rational roots.

Sol.: If there a rational root. Let it be $\frac{p}{q}$, where $(p, q) = 1, q \neq 0$. Then q should divide the coefficient of the leading term and p should divide the constant term.

Thus, $\frac{q}{1} \Longrightarrow q = \pm 1$ and $\frac{p}{1} = p = \pm 1$

$Thus, \frac{p}{q} = \pm 1. If\ the\ root \frac{p}{q} = 1, then$

$$f_{(1)} = 1 - 1 + 1 + 1 + 1 = 3 \neq 0.$$

So 1 is not a root.

If $\frac{p}{q} = -1, then\ f_{(-1)} = 1 \neq 0$

Hence, -1 is not a root. Thus, there exists no rational roots for given polynomial.

42) If $p_{(x)}$ be a real polynomial function $p_{(x)} = ax^3 + bx^2 + cx + d$. Show that if $|p_{(x)}| \leq 1$ for all x such that $|x| \leq 1$, $then$ $|a| + |b| + |c| + |d| \leq 7$.

Sol.: Considering the polynomials $\pm p_{(\pm x)}$. We may assume without loss of generality that a, b $>$ 0

Case 1: If c, d $>$ 0, then $p_{(1)} = a + b + c + d \leq 1 < 7$

Case 2: If $d \leq 0$ and $c \leq 0, then$ $|a| + |b| + |c| + |d|$

$$= a + b + c + d = (a + b + c + d) - 2d = p_{(1)} - 2p_{(0)} \leq 1 + 2 = 3 < 7$$

Case 3: $d \geq 0, c > 0$ $then$ $|a| + |b| + |c| + |d|$

$$= a + b + c + d$$

$$= \frac{4}{3}p_{(1)} - \frac{1}{3}p_{(-1)} - \frac{8}{3}p_{\left(\frac{1}{2}\right)} + \frac{8}{3}$$

$$p_{\left(-\frac{1}{2}\right)} \leq \frac{4}{3} + \frac{1}{3} + \frac{8}{3} + \frac{8}{3} = \frac{21}{3} = 7$$

Case 4: If d$<$ 0, c$<$ 0 then |a|+|b|+|c|+|d| = $a + b - c - d$

$$= \frac{5}{3}p_{(1)} - 4p\left(\frac{1}{2}\right) + \frac{4}{3}p\left(-\frac{1}{2}\right)$$
$$\leq \frac{5}{3} + 4 + \frac{4}{3} = \frac{21}{3} = 7$$

43) If all the coefficients in the equation $f_{(n)} = x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0$ be whole numbers and if $f_{(0)}$ amd $f_{(1)}$ be each odd integers. Show that the equation cannot have a common measurable root.

Sol.: Given $x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = f_{(x)} = 0 \ldots\ldots\ldots\ldots..(1)$

Cannot have fractional root as all coefficients are integers and coefficients of x^n is 1.

It cannot have even roots as of $f_{(0)}$ $i.e., p^n$ is odd.

Hence $f_{(2m)}$ will be odd

∵ all the terms expect the last term are even. In the given equation it cannot have odd roots, for it x is odd. Then x^n = an odd number = an even number $+f_{(x)}$= an odd number, as so it cannot vanishes. Thus equation (1) cannot have common surable roots.

44) Find all polynomials $\boldsymbol{f_{(x)}}$ with real coefficients which satisfy the equality $\boldsymbol{f_{(a-b)} + f_{(b-c)} + f_{(c-a)} = f(a+bc)\forall}$ real numbers a, b, c such that $\boldsymbol{ab + bc + ca = 0}$

Sol.: Let $f_{(x)}$ be a polynomial which satisfy the equation if a = b = 0

Then $ab + bc + ca = 0\ for\ each\ c \in R\ so\ we\ get\ f_{(0-0)} + f_{(0-c)} + f_{(c-0)} = 27(0 + 0 + c), \forall\ c \in R$

$\Rightarrow f_{(0)} + f_{(-c)} + f_{(c)} = 27_{(c)}, \forall\ C\ \in R$

$\Rightarrow f_{(0)} + f_{(-c)} = f_{(c)}\ \forall\ C\ \in R\ Let\ c = 0\ we\ get\ f_{(0)} = 0$

$so\ that\ f_{(c)} = f_{(c)}\ \forall\ C \in R.$

Hence, f is even which must be of the from

$f_{(x)} = a_n x^{2n} + a_{n-1}x^{2n-2} + \cdots + a_1 x^2\ with\ a_1, a_2, a_3, \dots \dots, a_n \in R$

For any real number u and v the triplet (a, b, c) will satisfy $ab + bc + ca$.

$Let\ a = uv, b = (1-u)v, c = (u^2 - u)v$

$\therefore\ ab + bc + ca\ will\ becme\ (a + b)c + ab$

$= v + (u^2 - u)v + uv(1 - u)v = v^2 + (u^2 - u) + v^2(u - u^2) = v^2 + (u^2 - u) + v^2(u^2 - u) = 0$

∴The given equation results

$f[(2u - 1)v] + f[(1 - u^2)v] + f[(u^2 - 2u)v]$

$= 27(v^2 - u + 1)u\forall\ u, v \in R \dots \dots \dots . . (1)$

$[\because\ (a - b) = (2u - 1)v, (b - c) = (1 - u^2)v, (c - a) = (u^2 - 2u)v]$

Let us fix u regard this is as a polynomial with variable v.

$f_{(2u-1)}v - (2u - 1)^{2n}v^{2n} + (2u - 1)^{2n-2}v^{2n-2} \dots \dots$

$f_{(1-u^2)}v = (1 - u^2)^{2n}v^{2n} + (1 - u^2)^{2n-2}\ v^{2n-2} \dots \dots$

$f_{(u^2-2u)}v = (u^2 - 2u)^{2n} + v^{2n} + \cdots$

$f_{(u^2-u+1)}v = (u^2 - u + 1)^{2n}\ v^{2n} + \cdots$

How equality the leading coefficients of both sides of (1), we get $(2u - 1)^{2n} + (1 - u^2)^{2n} + (u^2 - 2u)^{2n} = 2(u^2 - u + 1)^{2n}\ \forall\ u\ \in R$

$Let\ u = -2, then\ (-5)^{2n} + (-3)^{2n} + (8)^{2n} = 2(7)^{2n}$

$5^n + 3^{2n} + 8^{2n} = 2(7)^{2n}.$

Now above result is true only for x = 1 and x = 2

$\therefore f_{(x)}$ is either $\alpha n^2\ for\ n = 1. or\ \beta x^2\ for\ n = 2$

Hence $f_{(x)}$ can be written as a linear combination i.e. $f_{(x)} = \alpha x^2 + \beta x^4$

Hence, $f_{(x)} = \alpha x^2 + \beta x^4$ is a polynomial which satisfies the given equation for $\alpha, \beta \in$ R

45) If $\boldsymbol{\alpha + \beta + \gamma = 0}$, then show that $\boldsymbol{\alpha^{n+3} + \beta^{n+3} + \gamma^{n+3} = \alpha\beta\gamma(\alpha^n + \beta^n + \gamma^n) + \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2)(\alpha^{n+1} + \beta^{n+1} + \gamma^{n+1})}$

Sol.: $\alpha + \beta + \gamma = 0$ i.e., Let α, β, γ denote the roots of equation $x^3 + qy + r = 0\ i.e.\ x = \alpha\ or\ \beta\ or\ \gamma$

Multiplying this equation by x^n. We have

$$x^{n+3} + qx^{n+1} + rx^n = 0$$

Substituting in succession x = $\alpha, \beta\ \gamma$

We have $\alpha^{n+3} + q\alpha^{n+1} + r\alpha^n = 0 \ldots\ldots\ldots\ldots (1)$,

$\beta^{n+3} + q\beta^{n+1} + r\beta^n = 0 \ldots\ldots\ldots\ldots (2)$,

$\gamma^{n+3} + q\gamma^{n+1} + r\gamma^n = 0 \ldots\ldots\ldots\ldots (3)$

Adding (1), (2) and (3)

$$\alpha^{n+3} + \beta^{n+3} + \gamma^{n+3} + q(\alpha^{n+1} + \beta^{n+1} + \gamma^{n+1}) + r(\alpha^n + \beta^n + \gamma^n) = 0 \ldots\ldots (4)$$

But by $x^3 + qx + r = 0$, we have $\alpha\beta + \beta\gamma + \gamma\alpha = q, \alpha\beta\gamma = -r$.

$i.e., \quad \sum \alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{1}{2}(2\alpha\beta + 2\beta\gamma + 2\gamma\alpha)$

$$= \frac{1}{2}[(\alpha+\beta+\gamma)^2 - (\alpha^2+\beta^2+\gamma^2)]$$

$$= -\frac{1}{2}(\alpha^2+\beta^2+\gamma^2)[\because \alpha + \beta + \gamma = 0]$$

$\therefore (4) gives\ \alpha^{n+3} + \beta^{n+3} + \gamma^{n+3} = -q(\alpha^{n+1} + \beta^{n+1} + \gamma^{n+1}) - r(\alpha^n + \beta^n + \gamma^n)$

$$= \frac{1}{2}(\alpha^2+\beta^2+\gamma^2)(\alpha^{n+1} + \beta^{n+1} + \gamma^{n+1}) + \alpha\beta\gamma(\alpha^2+\beta^2+\gamma^2$$

MISCELLANEOUS EQUATIONS AND INEQUATIONS

(OBJECTIVE TYPE)

1) **The solution $\left|\frac{x}{x-1}\right| + |x| = \left|\frac{x^2}{x-1}\right|$ is**
(a) $x \geq 0$; (b) $x > 0$; (c) $x \in (1, \infty)$; (d) none

Sol.: Let $\frac{x}{(x-1)}\ and\ b = x \quad \therefore a + b = \frac{x^2}{(x-1)}$

The given equation becomes $|a| + |b| = |a + b|$.

But his equality holds if $ab \geq 0$

$\therefore \frac{x^2}{(x-1)} \geq 0$ critical points are 0, 1

$\therefore x \in \{0\} \cup (1, \infty)$

2) **The number of solutions of the equations $1! + 2! + 3! + \cdots + (x-1)! + x! = k^2\ and\ k \in I$ are**
(a) 2; (b) 3; (c) 4; (d) none

Sol.: The given equation is 1! +2! +3! +...+$(x-1)!\,x! = k^2$

We can readily check that for x < 4 the given equation has the only solutions $= 1, k = \pm 1\ and\ x = 3, k \pm 3$.

Now let us prove that there are no solutions for $x \geq 4$.

The expressions $\left.\begin{matrix} 1! + 2! + 3! + 4! = 33 \\ 1! + 2! + 3! \ldots + 5! = 153 \\ 1! + 2! + \cdots + 6! = 873 \\ 1! + 2! + \cdots + 7! = 5913 \end{matrix}\right\}$

End with the digit 3.

Now for $x \geq 4$ the last digit of the sum 1! +2! +...+x! is equal to 3 and therefore this sum can not be equal to a square of a whole number k (because a square of a whole number cannot end with 3)

3) **The solution set of $\left(\frac{3}{5}\right)^x = x - x^2 - 9$ is**
(a) ϕ; (b) all real; (c) all x ∈ N; (d) none

Sol.: We have $x - x^2 - 9 = -(x^3 - x + 9) = -\left\{\left(x - \frac{1}{2}\right)^2 + \frac{35}{4}\right\} < 0 \forall\ x \in R$

$But\ \left(\frac{3}{5}\right)^x > 0$ for each x ∈ R. Thus $\left(\frac{3}{5}\right)^x = x - x^2 - 9$ has no solution.

4) **If $0 \le p \le \pi$, then the quadratic equation**

$$(\cos p - 1)x^2 + (\cos p)x + \sin p = 0,$$

(a) **Real roots; (b) imaginary roots; (c) nothing can be said, (d) none.**

Sol.: The discriminant D of the quadratic equation (1) is given by D= $cos^2 p - 4(\cos p - 1)\sin p$

$$= cos^2 p - 4\cos p \sin p + 4\sin p = (\cos p - 2\sin p)^2 + 4\sin p - 4sin^2 p = (\cos p - 2\sin p)^2 + 4\sin p\,(1 - \sin p)$$

$As\ 0 \le p \le \pi, \sin p \le 0. Also\ 1 - \sin p \ge 0\ \forall\ p\ \in R.$

Therefore, $D \ge 0$. Hence (1) has real roots.

5) **The range of values of a for which all the roots of the equation**
$(a-1)(1+x+x^2)^2 = (a+1)(1+x^2+x^4)$ **are imaginary is**
(a) $(2,\infty)$; **(b)** $(-\infty,-2]$; **(c)** $-2 < a < 2$; **(d) none**

Sol.: $(1+x+x^2)[(a-1)(1+x+x^2) - (a+1)(1-x+x^2)] = 0$

$(1+x+x^2) = 0$ has imaginary roots

$\Rightarrow -2(1+x^2) + 2ax = 0,$ must have imaginary roots

$\Rightarrow x^2 - ax + 1 = 0$, must have imaginary root.

$\Rightarrow a^2 - 4 < 0, \Rightarrow -2 < a < 2.$

6) **Let f(x) = $(1+b^2)x^2 + 2bx + 1$ *and* let m(b) be the minimum value of f(x). As b varies, the range of m(b) is**
(a) [0, 1]; (b) $\left[0,\frac{1}{2}\right]$; (c) $\left[\frac{1}{2},1\right]$; (d) none

Sol.: $f(x) = (1+b^2)x^2 + 2bx + 1 \dots\dots\dots\dots (1)$

$$f'_{(x)} = 2(1+b^2)x + 2b = 0 \Rightarrow x = -\frac{b}{1+b^2}.$$

$$f''_{(x)} = 2(1+b^2) > 0$$

$\therefore f_{(x)}$ has min. value at

$x = -\frac{b}{1+b^2}$ min. value of $f_{(x)}$ $i.e.,$

$$m_{(b)} = \frac{b^2}{1+b^2} - \frac{2b^2}{1+b^2} + 1$$

$$or\ m_{(b)} = 1 - \frac{b^2}{1+b^2} = \frac{1}{1+b^2}$$

$clearly, 0 < m_{(b)} \le 1. [\because b^2 \ge 0\ max. value\ of\ m_{(b)} = 1]$

7) **If $y = 2[x] + 3 = 3[x-2] + 5$, *then* $[y+y]$ *is* $[x]$ denotes the integral part of x**
(a) 10; (b) 11; (c) 12; (d)none

Sol.: $\because\ y = 2[x] + 3 = 3[x-2] + 5 \dots\dots\dots. (1)$

$or,\ 2[x] + 3 = 3[x-2] + 5 \Rightarrow 2[x] + 3 = 3\{[x] - 2\} + 5$

$\Rightarrow 2[x]3 = 3[x] - 6 + 5 \Rightarrow [x] = 4\ from\ (1)$

$y = 2,4 + 3, y = 11$

$\therefore 4 \le x < 5$

$\Rightarrow 4 + y \leq x + y < 5 + y \Rightarrow 15 \leq x + y < 16$

$(\because y = 11)$

$\therefore (x + y) = 15$

8) The solution set of the following equation is $\begin{cases} \mathbf{4\log_2{}^2 x + 1 = 2\log_2 y} \\ \mathbf{\log_2 x^2 \geq \log_2 y} \end{cases}$

(a) $(\sqrt{2}, 2)$; **(b) (2, 2); (c)** $(\sqrt{2}, 1)$; **(d) none**

Sol.: The system of equation is $4\log_2{}^2 x + 1 = 2\log_2 y \ldots\ldots\ldots\ldots.(1)$

$\log_2 x^2 \geq \log_2 y \;\; 0 \ldots\ldots\ldots\ldots(2)$

Substituting $\log_2 y$ from (1)in (2)

$2\log_2 x \geq \left(\frac{1}{2}\right)\left(4 \log_2{}^2 x + 1\right)$

$\Rightarrow 4\log_2 x \geq 4(\log_2 x)^2 + 1 \Rightarrow (2\log_2 x - 1)^2 \leq 0$

$Hence\ 2\log_2 x - 1 = 0, \therefore x = \sqrt{2}\ from\ (1) y = 2$

Hence solutions set of system is $\left(\sqrt{2}, 2\right)$

9) Values of a for which exactly one root of $\mathbf{5x^2 + (a + 1)n + a = 0}$ **lies in the interval** $\mathbf{1} < x < 3$ **is**

(a) $\boldsymbol{a} > 0$; **(b)** $\boldsymbol{a} > 2$; **(c)** $\mathbf{-12} < a < -3$; **(d) none**

Sol.: $f_{(1)} = 5 + a + 1 + a = 6 + 2a,$

$f_{(3)} = 45 + 3(a + 1) + a = 4a = 48$

$\therefore f_{(1)}.f_{(3)} < 0, (6 + 2a)(4a + 48) < 0,$

$(a + 3)(a + 12) < 0$

$\therefore -12 < a < -3$

10) The number of real roots of $\mathbf{(6 - x)^4 + (8 - x)^4 = 16}$ **is**

(a) 0; (b) 2; (c) 4; (d) none

Sol.: Consider $y = \frac{[(6-x)+(8-x)]}{2}$

Let $y = f(-x)$. Then the given equation becomes

$(y + 1)^4 + (y - 1)^4 = 16 \Rightarrow y^4 + 6y^2 - 7 = 0$

$\Rightarrow (y^2 - 1)(y^2 + 7) = 0 \Rightarrow y^2 - 1 = 0$

$\because y^2 + 7 \neq 0$

$\Rightarrow y = \pm 1 \Rightarrow 7 - x = \pm 1 \Rightarrow x = 6, 8$

11) All solutions of equations $\mathbf{x^2 + y^2 - 8x - 8y = 20}\ \boldsymbol{and}\ \mathbf{xy + 4x + 4y = 40}$ **satisfy the following equations (s).**

(a) $\mathbf{x + y = 10}$; **(b)** $\mathbf{|x + y| = 0}$; **(c)** $\mathbf{|x - y| = 10}$; **(d) none**

Sol.: Given, $x^2 + y^2 - 8x - 8y = 20 \ldots\ldots\ldots..(1)$

$xy + 4x + 4y = 40 \ldots..(2)$

$(1) + 2.(2) \Rightarrow (x + y)^2 = 100 \Rightarrow x + y = \pm 10 \quad \therefore (x + y) = 10.$

12) If $\mathbf{5\{x\} = x + [x] - \{x\} = \frac{1}{2}}$, $\boldsymbol{where}\ \{x\}\boldsymbol{and}\ [x]$ **are fractional and integral part of x then the number of solutions f the equation is**

(a) 1 ; (b) 2; (c) 3; (d) none

Sol.: $5\{x\} = x + [x] \ldots\ldots\ldots..(1)$

$[x] - \{x\} = \frac{1}{2} \ldots\ldots\ldots.(2)$

$\therefore x = [x] + \{x\} \ldots\ldots\ldots(3)$ from (1) & (3) we get

$[x] + 2\{x\} \ldots\ldots\ldots(4)$. Solving (2) & (4) we get [x] = 1, {x} $\frac{1}{2}$ $\quad \therefore from\ (3)\ x = \frac{3}{2}$.

13) Let F(x) be a function defined by $\boldsymbol{F(x) = x - [x], R}$ where [x] is the greatest integer less than or equal to x. Then the number of solutions of $\mathbf{F(x) + F\left(\frac{1}{x}\right) = 1}$ is

(a) 0; (b) 1; (c) 2; (d) none

Sol.: $F(x) = x - [x]\ \ 0 \neq x \in R \quad \therefore p_{(x)} + F_{\left(\frac{1}{x}\right)} = 1$

$$\Rightarrow x - [x] + \frac{1}{x} - \left[\frac{1}{x}\right] = 1$$

$$\Rightarrow \left(\frac{x+1}{x}\right) - \left([x] + \left[\frac{1}{x}\right]\right) = 1 \ldots\ldots(1)$$

$$\Rightarrow x + \frac{1}{x} = [x] + \left[\frac{1}{x}\right] \neq 1$$

$\therefore$ R H S is an integer. Hence LHS is also integer

$Let\ [x] + \left[\frac{1}{x}\right] + 1 = A$ (Integer)

The equation (1) becomes $x + \frac{1}{x} = A$

$$\Rightarrow x^2 - Ax + 1 = 0 \ \therefore x = \frac{A \pm \sqrt{A^2 - 4}}{2}$$

$For\ real\ x, A^2 - 4 \geq 0$

$\therefore A \geq 2\ \&\ A \leq -2, A = 2\ \&\ A = -2$ does not satisfy of (1)

$\therefore A > 2\ \&\ A < -2\ \&\ A \in I$.

Then equation (1) has infinite many solutions.

14) Number of solutions of $\mathbf{3^{|x|} = \big|2 - |x|\big|}$ is

(a) 0; (b) 2; (c) 4; (d) none

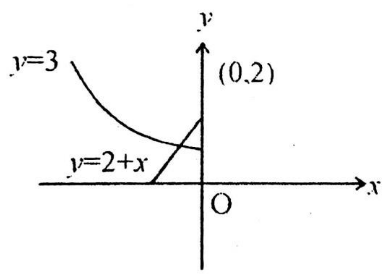
Sol.: Given equation is $\left(\frac{1}{3}\right)^x = -2 - x,$

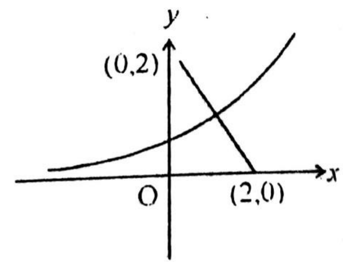
$$-\infty < x \leq -2 + x, -2 \leq x \leq 0$$

$$3^x = 2 - x, 0 \leq x \leq 2 = x - 2,$$

$$2 \leq x < \infty\ at\ x = 2,\ 3^x - x + 2 = 9$$

(as $3^x - x + 2$ is an increasing function for x>2)





For $x = -2, \left(\frac{1}{3}\right)^x + 2 + x = 9\ for\ x < -2, \left(\frac{1}{3}\right)^x + 2 + x = 9\ \left(as\ \left(\frac{1}{3}\right)^x + 2 + x\ is\ decreasing\right)$

Hence given equation has only two solution – 2 and 2.

15) The system of equation $\mathbf{|x - 1| + 3y = 4, x - |y - 1| = 2}$ has

(a) 1; (b) 2; (c) 3; (d) none

Sol.: The given equations are $|x - 1| + 3y = 4$

$$\Rightarrow \begin{cases} x + 3y = 5 \quad x \geq 1 \;\ldots..(1) \\ -x + 3y = 3, x < 1 \ldots\ldots\ldots(2) \end{cases} \text{and } x - |y-1| = 2$$

$$\Rightarrow \begin{cases} x - y = 1 \; y \geq 1 \ldots\ldots\ldots\ldots.(3) \\ x + y = 3, x < 1 \ldots\ldots.(4) \end{cases}$$

Solving (1) & (3) we get x = 2, y = 1

Solving (1) & (4) we get x = 2, y = 1 no solution

($\because x \geq 1, y < 1$)

Solving (2) & (3) we get x = 3, y = 2 no solving

($\because x \geq 1, y \geq 1$)

Solving (2) & (4) we get $x = \frac{5}{2}, y = \frac{3}{2}$

No solving (∵x<1, y<1)

Here solution is x = 2, y = 1 (a unique solution)

16) The number of integral roots of the equation

$$\sqrt{(\boldsymbol{x}+\mathbf{3}) - \mathbf{4}\sqrt{\boldsymbol{x}-\mathbf{1}}} + \sqrt{(\boldsymbol{x}+\mathbf{8}) - \mathbf{6}\sqrt{\boldsymbol{x}-\mathbf{1}}} = \mathbf{1} \textbf{ is}$$

(a) 1; (b) 2; (c) 3; (d) none

Sol.: Taking $\sqrt{x-1} = t (t \geq 0)$ the equation reduces to $\sqrt{t^2 + 4 - 4t} + \sqrt{t^2 - 6t + 9} = 1$

$\Rightarrow |t - 2| + |t - 3| = 1$

It is necessary for t to satisfy $2 \leq t \leq 3$.

$\therefore 2 \leq -\sqrt{x-1} \leq 3 \Rightarrow 4 \leq (x-1) \leq 9 \Rightarrow 5 \leq x \leq 10$

17) If $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ be the roots of $\boldsymbol{f}_{(x)} = \mathbf{0}. \boldsymbol{where}\ \boldsymbol{f}_{(x)} = \boldsymbol{x}^3 + \boldsymbol{x}^2 - \mathbf{5}\boldsymbol{x} - \mathbf{1} = \mathbf{0}, \boldsymbol{then}\ [\boldsymbol{\alpha}] + [\boldsymbol{\beta}] + [\boldsymbol{\gamma}]$, where |.| denotes, the greatest integer is equal to
(a) 1; (b) –2; (c) –c ; (d) none

Sol.: Let $f_{(x)} = x^3 + x^2 - 5x - 1$

$\therefore f'_{(x)} = 3x^2 + 2x - 5$

Now the sign scheme for $3x^2 + 2x - 5$ is

Also $f_{(\infty)} = -\infty < 0; f\infty = \infty > 0;\ f_{(1)} = -4$

$$f\left(-\frac{5}{3}\right) = \frac{148}{27}$$

$\therefore f(-3) = -27 + 9 + 15 - 1$

$= -4 < 0 \;\; \therefore f_{(-2)} = -8 + 4 + 10 - 1 > 0;$

$f_{(-1)} = 4 > 0, f_{(0)} = -1 < 0;\ f_{(2)} = -1 > 0$

$\therefore -3 < \alpha < -2;\ -1 < \beta < 0; 1 < \gamma < 2$

$\therefore [\alpha] + [\beta] + [\gamma] = -3 - 1 + 1 = -3.$

18) If S be the solution of the equation $(\boldsymbol{x})^2 + [\boldsymbol{x}]^2 = (\boldsymbol{x}-\mathbf{1})^2[\boldsymbol{x}+\mathbf{1}]$, where (x) = least integer, [x] = greatest integer, R= real numbers, Z = integer, N = natural numbers, then
(a) $\boldsymbol{S} = \boldsymbol{R}$; (b) $\boldsymbol{S} = \boldsymbol{R} = -\boldsymbol{Z}$; (c) $\boldsymbol{S} = \boldsymbol{R} = -\boldsymbol{N}$; (d) none

Sol.: Here (x)= least integer $\geq x \text{ and } [x] =$ greatest integer $\leq x$, so $(x) - [x] = 1$, if x is not integer and [x] = (x) if x ∊z.

Now, $(x-1) = (x) - 1, [x+1] = [x] + 1, so,$

$(x)^2 + [x]^2 = (x-1)^2\ (x+1)^2,$

$\Rightarrow (x)^2 + [x]^2 = (x)^2 - 2(x) + 1 + [x]^2 + 2[x] + 1$

$\Rightarrow [x] - (x) + 1 = 0, \Rightarrow\ -1 + 1 = 0\ if\ x \notin z, and\ 0 + 1 \neq 0\ if\ x \in z,$

Hence the solution set S= R –Z.

19) The number of triplets (x, y, z) satisfying the equation $x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2z^2x^2 = 24$ is (where x, y, z are integer)
(a) 0; (b) 1; (c) 2; (d) none

Sol.: Since 24 is even number $x^4 + y^4 + z^4$ has to be even

Two cases arise:

(i) All of x, y and z are even which is not possible since in that case 16 divides each term of (and hence the whole of) the left hand side while 16 does not divide 24.
(ii) Two of x, y and z are odd and one of them is even say, x is even.

We have that: $x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2z^2x^2 = x^4 - 2x^2(y^2 + z^2) + (z^2 - y^2)^2$

$= x^4 - 2x^2(y^2 + z^2) + (z - y)^2(z + y)^2$

Here again 16 divides each term since y and z are odd.

Thus in either case the equation has no solutions in integers.

(SUBJECTIVE TYPE)

1) Solve $z + ay + a^2x + a^3 = 0; z + by + b^2x + b^3 = 0; z + cy + c^2x + c^3 = 0.$

Sol.: The given equation show that the polynomial $\alpha^3 + x\alpha^3 + y\alpha + z$ vanishes at three different values of α namely at α= a, α= b and at α = c (assuming that a, b, c are not equal to one another)

Set up different $\alpha^3 + x\alpha^2 + y\alpha + z - (\alpha - a)(\alpha - b)(\alpha - c)$

This differences also becomes zero at α equal to a, b, c. Expanding this expression in powers of α, we get $(x + a + b + c)\alpha^2 + (y - ab - ac - bc)\alpha + z + abc$ this second degree thrinomial vanishes at three different values at α and therefore it equals zero identically and consequently, all its coefficients are equal to zero. i.e. x +a +b +c = 0; $y = ab - ac - bc = 0; z + abc = 0$

$Hence, x = -(a + b + c), y = ab + ac + bc,$

$z = -abc$ is solution of our system.

2) Solve: $x_1 + x_2 = a_1$; $x_2 + x_3 = a_2$; $x_3 + x_4 = a_3$; …. $x_{n-1} + x_n = a_{n-1}$ $x_n + x_1 = a_n$

Sol.: we have $x_2 = a_1 - x_1, x_3 = a_2 - x_2 = a_2 - a_1 + x_1$ $x_4 = a_3 - x_3 = a_3 - a_2 + a_1 = x_1$ … ..

$$x_n = a_{n-1} - a_{n-2} + \cdots \pm a_2 \pm a_1 \pm x_1$$

It should be noted that in the last quality the upper signs will occur when n is odd and let the lower signs when n is even.

Consider the two cases separately.

1. Let n be odd, then $x_n = a_{n-1} - a_{n-2} + \cdots + a_2 - a_1 + x_1$ on other hand $x_n + x_1 = a_n$ from these two equalities we get $x_1 = \frac{a_n - a_{n+1} + a_{n-2} \ldots . - a_2 + a_1}{2}$

$$x_2 = \frac{a_1 - a_n + a_{n-1} \ldots . - a_3 + a_2}{2}$$
$$x_3 = \frac{a_2 - a_1 + a_n \ldots . - a_4 + a_3}{2}$$

2. Let now n be even, then $x_n = a_{n-1} - a_{n-2} + \cdots - a_2 + a_1 - x_1$ on the other hand $x_n = a_n - x_1$ consequently for the given system of equations to be compatible the following equality must be satisfied.
   $a_{n-1} - a_{n-2} + \cdots - a_2 + a_1 = a_n$
   $i.e. a_n + a_{n-2} + \cdots + a_2 = a_{n-1} + a_{n-3} + \cdots + a_1$
   (The sum of coefficients with even subscript must equal the sum of coefficients with odd subscript) It is apparent that in this case the system will be indeterminate. i.e. will allow an infinite number of solutions namely.
   $x_1 = \lambda;\ x_2 = a_1 - \lambda;\ x_3 = a_2 - a_1 + \lambda;\ x_4 = a_3 - a_2 + a_1 - \lambda;\ x_n = a_{n-1} - a_{n-2} + \cdots + a_3 - a_2 + a_1 - \lambda$
   Where λ is an arbitrary quantity

3) Solve: $\boldsymbol{x\sin a + y\sin 2a + z\sin 3a = \sin 4a\quad x\sin b + y\sin 2b + z\sin 3b = \sin 4b\,,\, x\sin c + y\sin 2c + z\sin 3c = \sin 4c}$

Sol.: We have $\sin 2a = 2\sin a\cos a\,, \sin 3a = \sin a\,(4\,cos^2 a - 1)\ \sin 4a = 4\sin a\,(2\,cos^3 a - \cos a)$.

The first equation of our system is rewritten in following way. $x + 2y\cos a + z(4cos^2 a - 1) = 4$

$(2\,cos^3 a - \cos a)$ The remaining two are similar. Expand this equation in powers of $\cos a\,, we\ have$

$8\,cos^3 a - 4z\,cos^2 a - (2y + 4)\cos a + z - x = 0$

Putting $\cos a = t$ and dividing both members by 8, we get $t^3 - \frac{z}{2}t^2 - \frac{y+2}{4}t + \frac{z-x}{8} = 0$

Our system of equations is equivalent to the statement that the equation has three roots:

$t = \cos a\,; t = \cos b\,; and\ t = \cos c\,, which\ follows$:

$$\frac{z}{2} = \cos a + \cos b + \cos c$$

$$\frac{y+2}{4} = -(\cos a\cos b + \cos a\cos c + \cos b\cos c)$$

$$\frac{x-z}{4} = \cos a\cos b\cos c$$

∴The solution of our system will be

$x = 2(\cos a + \cos b + \cos c) + 8(\cos a + \cos b + \cos c)$

$y = -2 - 4\,(\cos a\cos b + \cos a\cos c + \cos b\cos c)$

$z = 2(\cos a + \cos b + \cos c)$

4) Solve : x+ y+ z= 14 ; $\boldsymbol{x^2 + y^2 + z^2 = 91}$; $\boldsymbol{y^2 = zx}$

Sol.: We have $x + y + z = 14 \ldots\ldots..(1)$

$x^2 + y^2 + z^2 = 91 \ldots\ldots..(2)$

$y^2 = zx \ldots\ldots.(3)$

Squaring the 1st we get $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = 196$ Putting the values of $x^2 + y^2 + z^2$ from (2)

And of zx from (3), we get $91 + 2xy + 2yz + 2y^2 = 196.2xy + 2yz + 2y^2 = 105, 2x(x + y + z) = 105$

$$2y(14) = 105\ or\ v = \frac{105}{28} = \frac{15}{4}$$

$$Hence, x+z = 14 - \frac{15}{4}\ from\ (1) or\ x + z = \frac{41}{4}$$

$$Also, zx = \left(\frac{15}{4}\right)^2\ or\ zx = \frac{225}{116}\ from\ (2)$$

Hence, x and z are the roots of the equation

$$t^2-(x+z)t+(xz) = 0\ or\ t^2 - \frac{41}{4}t + \frac{225}{16} = 0$$

$$or\ 16t^2 - 164t + 225 = 0$$

$$\Rightarrow t = \frac{164 \pm \sqrt{(164)^2 - 4 \times 16 \times 225}}{32}$$

$$\Rightarrow t = \frac{41 \pm \sqrt{41^2 - 900}}{8}$$

$$= \frac{41 \pm \sqrt{(41+30)(41-30)}}{8} = \frac{41 \pm \sqrt{71 \times 11}}{8}0 = \frac{41 \pm \sqrt{781}}{8}$$

$$Hence, x = \frac{41 \pm \sqrt{781}}{8}, y = \pm\frac{15}{4}, z = \frac{41 \pm \sqrt{781}}{8}$$

5) Solve: x+ y +z = $\boldsymbol{ab}$; $\boldsymbol{x^{-1} + y^{-1} + z^{-1} = a^{-1}b; xyz = a^3}$

Sol.: $x+y+z = ab, \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{b}{a}, xyz = a^3$

By (2) and (3), we have $xy + yz + zx = a^2b$

Now by (1), (3) and (4), it is clear that x, y, z are roots of $t^3 - abt^2 + a^2bt - a^3 = 0$ we see that the above equations vanishes for t = a, i.e. $(t-a)$ is a factor of (5).

So by remainder theorem (5) is

$$t^2(t-a) + at(t-a) + a^2(t-a) - abt(t-a) = 0$$

$$or\ (t-a)(t^2 - at + a^2 - abt) = 0\ or\ (t-a)$$

$$\{t^2 + t(a-ab) + a^2\} = 0\ i.e., either\ t = a.$$

$$or,\ t = \frac{-(a-ab) \pm \sqrt{[(a-ab)^2 - 4a^2]}}{2}$$

$$i.e., x, y, z\ are\ a, \frac{1}{2}a\left[b - 1 + \sqrt{b^2 - 2b - 3}\right]$$

$$\frac{1}{2}a\left[b - 1 - \sqrt{b^2 - 2b - 3}\right] respectively.$$

6) Examine x, y, z from the equations $\boldsymbol{(x+y-z)(x-y+z) = ayz; (y+z-x)(y-z+x) = bzx; (z+x-y)(z-x+y) = cxy}$

Sol.: Given equation are $(x+y-z)(x-y+z) = ayz \dots\dots\dots\dots(1)$

$$(y+z-x)(y-z+x) = bzx \dots\dots\dots\dots\dots(2)$$

$$(z+x-y)(z-x+y) = cxy \dots\dots\dots\dots(3)$$

Multiplying (1), (2), (3) we get

$$(x+y-z)^2(x-y+z)^2(y+z-x)^2 = abcx^2y^2z^2$$

$$or,\quad (-x^3 - y^3 - z^3 + y^2z + yz^2 + z^2x + zx^2 + x^2y + xy^2 - 2xyz)^2 = abcx^2y^2z^2 \dots\dots\dots(4)$$

Or, dividing both sides of equation (4) by $x^2y^2z^2$

$$or, \quad abc = \left(-\frac{x^2}{yz} - \frac{y^2}{zx} - \frac{z^2}{xy} + \frac{y}{x} + \frac{z}{x} + \frac{x}{y} + \frac{x}{z} + \frac{y}{z} - 2\right) \ldots\ldots\ldots\ldots\ldots.(5)$$

By equation (1) may be written as

$$a = \frac{x^2 - y^2 - z^2 + 2yz}{yz} = \frac{x^2}{yz} - \frac{y}{z} - \frac{z}{y} + 2$$

$$or\ a - 2 = \frac{x^2}{yz} - \frac{y}{z} - \frac{z}{y}$$

$$Similarly, b - 2 = \frac{y^2}{zx} - \frac{z}{x} - \frac{x}{z} and\ c - 2 = \frac{z^2}{xy} - \frac{x}{y} - \frac{y}{x}$$

Now from equation (5)

$abc = \ [(2-a) + (2-b) + (2-c) - 2]^2$

$abc = \ (4 - a - b - c)^2$

7) Eliminate x, y, z from the equations $\boldsymbol{ax^2 + by^2 + cz^2 = ax + by + cz = yz + zx + xy = 0}$

Sol.: Given equation are $ax^2 + by^2 + cz^2 = 0 \ldots\ldots\ldots..(1)$

$ax + by + cz = 0 \ldots\ldots\ldots\ldots\ldots(2)$

$yz + zx + xy = 0 \ldots\ldots\ldots\ldots..(3)$

Multiplying (2) by (x+ y+ z), we have

$(ax + by + cz)(x + y + z) = 0\ or, ax^2 + by^2 + cz^2 + xy$

$(a + b)yz(b + c) + zx(c + a) = \ 0$

$But\ ax^2 + by^2 + cz^2 = 0$

$\therefore xy(a + b) + yz(b + c) + zx(c + a) = 0$

$Also\ xy + yz + zx = 0$

$$Hence \frac{xy}{b-a} = \frac{yz}{c-b} = \frac{zx}{a-c} = \frac{1}{k}(say)$$

Dividing each ratio by xyz,

$$\frac{1}{z(b-a)} = \frac{1}{x(c-b)} = \frac{1}{y(a-c)} = \frac{1}{k}$$

$$\therefore x = \frac{k}{(c-b)}, y = \frac{k}{(a-c)}, z = \frac{k}{(b-a)}$$

Substituting these values in (2)

$$a\frac{k}{c-b} + b\frac{k}{a-c} + \ c\frac{k}{b-a} = 0$$

$a(b-a)(a-c) + b(c-b)(b-a) + c(c-b)(a-c) = 0$

$or\ a^3 + b^3 + c^3 - (a+b)(b+c)(c+a) + 5abc = 0$

$or\ a^3 + b^3 + c^3 - 3(a+b)(b+c)(c+a) - 4(a+b)(b+c)(c+a) + 5abc = 0$

$or\ (a+b+c)^3 - 4(a+b)(b+c)(c+a) + 5abc = 0$

8) Solve: $\boldsymbol{(12x-1)(6x-1)(4x-1)(3x-1) = 5}$

Sol.: We can write the equation in the from

$\left(x - \frac{1}{12}\right)\left(x - \frac{1}{6}\right)\left(x - \frac{1}{4}\right)\left(x - \frac{1}{3}\right) = \frac{5}{1.2.6.} \ldots\ldots\ldots\ldots(1)$

$\therefore \frac{1}{12} < \frac{1}{6} < \frac{1}{4} < \frac{1}{3} and \frac{1}{6} - \frac{1}{12} = \frac{1}{3} - \frac{1}{4}$

We can introduce a new variable

$y = \frac{1}{4}\left[\left(x - \frac{1}{12}\right) + \left(x - \frac{1}{6}\right) + \left(x - \frac{1}{4}\right) + \left(x - \frac{1}{3}\right)\right] = x - \frac{5}{24}$

$Substitute\ x = y + \frac{5}{24}\ in(1)$, we get

$$\left(y + \frac{3}{24}\right)\left(y + \frac{1}{24}\right)\left(y - \frac{1}{24}\right)\left(y - \frac{3}{24}\right) = \frac{5}{12.6.4.3}$$

$$\left(y^2 - \left(\frac{1}{24}\right)^2\right)\left(y^2 - \left(\frac{3}{24}\right)^2\right) = \frac{5}{12.6.4.3}$$

$$So, y^2 = \frac{49}{24^2}\ i.e.\ y_1 = \frac{7}{24}\ and\ y^2 = -\frac{7}{24},$$

$$corresponding\ roots\ are -\frac{1}{12}\ and\ \frac{1}{2}.$$

Note: An equation of the from $(x - a)(x - b)(x - c)(x - d) = Ax^2$ where $ab = cd$ can be reduced to a collection of two quadratic equations by a change of variable $y = x + \frac{ab}{x}$.

9) Solve the equation $\boldsymbol{x^3 - [x] = 3\ where\ [x] denotes}$ the greatest integer.

Sol.: $\because x = [x] + f, 0 \le f < 1$. And given equation is $x^3 - [x] = 3 \implies x^3 - (x - f) = 3 \implies x^3 - x = 3 - f,$

Hence it follows that $2 < x^3 - x \le 3\ further\ for\ x \ge 2.$

We have $x^3 - x = x(x^2 - 1) \ge 2(4 - 1) = 6 > 3$

$for\ x < -1\ we\ have\ x^3 - x = 0 < 2;$

For $-1 < x \le 0\ we\ have\ x^3 - x \le -x < 1\ and$

$for\ 0 < x \le 1\ we\ have\ x^3 - x < x < x^3 \le 1$

Therefore x, must be $1 < x < 2$.

10) Solve:

$$\boldsymbol{\sqrt{x + 2\sqrt{x + 2\sqrt{x + \cdots + 2\sqrt{x + 2\sqrt{3x}}}}} = x}$$

Sol.: The given equation

$\sqrt{x + 2\sqrt{x + 2\sqrt{x + \cdots + 2\sqrt{x + 2\sqrt{x + 2}}}}} = x \ldots..(1)$

On replacing the last letter x on the L H S of equation (1) by the value of x, expressed by (1)we obtain

$$x = \sqrt{x + 2\sqrt{x + 2\sqrt{x + \cdots + 2\sqrt{x + 2x}}}}$$

$(2n\ radical\ signs)$

Further, let us replace the last letter x by the same expression, again and again yields

$$\therefore x = \sqrt{x + 2\sqrt{x + 2\sqrt{x + \cdots + 2\sqrt{x + 2x}}}}$$

$(3x\ radical\ signs)$

$$\sqrt{x + 2\sqrt{x + 2\sqrt{x + \cdots + 2\sqrt{x + 2x}}}} = \cdots$$

$we\ can\ write = x$

$$= \sqrt{x + 2\sqrt{x + 2\sqrt{x + \cdots}}}$$

$$= \lim_{N\to\infty} \sqrt{x + 2\sqrt{x + 2\sqrt{x + \cdots + 2\sqrt{x + 2x}}}}$$

$(N\ radical\ signs)$

It follows that

$$x = \sqrt{x + 2\sqrt{x + 2\sqrt{x + \cdots}}} =$$
$$\sqrt{x + 2\left(\sqrt{x + 2\sqrt{x + \cdots}}\right)} = \sqrt{x + 2x}$$

Hence $x^2 = x + 2x \Longrightarrow x^2 - 3x = 0$

Therefore x = 0, 3.

11) Solve: $1 - \frac{x}{1!} + \frac{x(x-1)}{2!} - \frac{x(x-1)(x-2)}{3!} + \cdots + (-1)^n . \frac{x(x-1)(x-2)-(x-x+1)}{n!} = 0$

Sol.: The given equation is

$$1 - \frac{x}{1!} + \frac{x(x-1)}{2!} - \frac{x(x-1)(x-2)}{3!} + \cdots + (-1)^n . \frac{x(x-1)(x-2)-(x-x+1)}{n!} = 0$$

$$Put\ x = n\ \in N \therefore\ 1 - \frac{n}{1!} + \frac{n(n-1)}{2!} - \frac{n(n-1)(n-2)}{3!} + \cdots + (-1)^n .$$

$$\therefore n_{C_0} \pm n_{C_1} + n_{C_2} - n_{C_3} + \cdots + (-1)^n n_{C_n} = 0$$

$$\frac{n(n-1)(n-2)}{n!} = 0$$

$\therefore\ (1-1)^n = 0\ or\ 0^n = 0$ it is true for all natural numbers

Hence solution x ∈ N

12) Find the number of roots in the equations $\sin x = \log x$

Sol.: $\sin x = \log x\ \ then\ x \le 10$ (because if otherwise the L H S ≤ 1 and RHS >1). Since $2.2\pi > 10$, the interval of the axis on the from x = 0 to x = 10 contains only one wave of the since curve $y = \sin x$ and a part of the next wave (see the figure). The graph of the function y = $\log x$ obviously intersects the first wave of the since curve at one point.

Further, $2\pi + \frac{\pi}{2} < 10, for\ the\ point\ x = \frac{5\pi}{2}$ we have $\sin x = 1 > \log x\,, the\ graph\ of\ y = \log x$ also intersects the first half of the second positive half wave of the since curve; further since at the point x = 10. We have $\log x = 1 > \sin x$, the graph of $y = \log x$ must intersect the second half wave as well. We see that the total number of the roots of the equation $\sin x = \log x$ is equal to three.

13) Solve: $4^{|x^2-8x+12|-\log 4^7} = 7^{2y-1}$ $and\ |y-3| - 3|y| - 2(y+1)^2 \ge 1$

Sol.: The equation of the system is equivalent to the equation $4^{|x^2-8x+12|} = 7^{2y}$

Its both sides are positive, and therefore it is equivalent to the equation $|x^2 - 8x + 12| = (2y)\log_4 7$

$Since\ \log_4 7 > 0\ and\ x^2 - 8x + 12 = 0$ for any x, it follows that y≥ 0. Therefore we should solve the inequality of given system only for two cases, $0 \le y \le 3\ and\ y < 3. For\ 0 \le y \le 3$ the equality of given system assumes the from $3 - y - 3y - 2y^2 - 4y - 2 - 1 \ge 0\ i.e. y^2 + 4y \ge 0, whence\ we\ find - 4 \le y \le 0.$

$Nothing\ that\ 0 \le y \le 3$, we find that y = 0 for y > 3 the inequality of the given system assumes the from $y-3-3y-2y^2-4y-2-1 \geq 0$ $i.e. -2y^2-6y-6>=0.$

This inequality has no solutions. Substituting the value y = 0 into the equation of given system, we obtain

$$\Rightarrow \begin{cases} y=0 \\ x^2-8x+12=0 \end{cases} \Rightarrow \begin{cases} y=0 \\ (x-2)(x-6)=0 \end{cases}$$

$$\Rightarrow [\begin{cases} x=2, \\ y=0, \end{cases} \begin{cases} x=6 \\ y=0 \end{cases}$$

Thus two pairs of numbers (2, 0) and (6, 0) are the solutions of given system.

14) Find a, where the equation $a^3+a^2|a+x|+|a^2x+1|=1$ has no less than four different integers solutions.

Sol.: Given equation is $a^3+a^2|a+x|+|a^2x+1|=1 \ldots\ldots\ldots..(1)$

We can write equation (1) as $|a^2x+1|+|a^3+a^2x|=(a^2x+1)-(a^3+a^2x)$

It follows from the properties of modulus that the inequality $|A|+|B|=A-B$ holds true if $A \geq 0$ and $B \leq 0$, equation (1) is equivalent to the system

$$\begin{cases} a^2x+1 \geq 0 \\ a^3+a^2x \leq 0 \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

The values a = 0 satisfies the hypothesis since in this case system (2) and consequently equation (1) have all x ∈ R as their solutions. Let a ≠ 0. Then system (2) is equivalent to

$$\begin{cases} x \geq -a^{-2} \\ x \leq -a \end{cases} \ldots\ldots\ldots.(3)$$

Thus we have to find all values of a for which system (3) has no less than four different integer solutions.

Let us compare the numbers $-a$ and $-\frac{1}{a^2}$, we find their difference, $-\frac{1}{a^2}-(-a)=--\frac{1}{a^2}+a=\frac{a^3-1}{a^2}$

$=\frac{(a-1)a^2+a+1}{a^2}$ $since$ $a^2+a+1>0$ for any a, it follows that a^2+a+1 $does\ not\ efect$

$the\ sign\ of\ the\ difference\ of\ the\ numbers$

$being\ compared$. In accordance with the method of intervals, we have $-a^{-2}<-a$ if $a<1, a\neq 0, -a^{-2}=-a=-1$ if $a=1, -a^{-2}<-a$ if $a>1$. Consequently; (a) if a > 1, then system (3) has no solutions. (b) If a = 1 then (3) ⟹ x = -1, there is a unique solution and the condition of the problem are not satisfied, (c) if $0<a<1$, then $-1<-a<0$, $and\ therefore\ the\ interval[-a^{-2},-a]$ contains no less than four integers provided that the inequality $-a^{-2} \leq -4$ holds true.

Let us solve the system $\begin{cases} 0<a<1 \\ -\frac{1}{a^2} \leq -4 \end{cases} \Rightarrow$ $\begin{cases} 0<a<1 \\ 1-4a^2 \geq 0 \end{cases}$

$$\Rightarrow \begin{cases} 0<a<1 \\ \left(\frac{1}{2}-a\right)\left(\frac{1}{2}+a\right) \geq 0 \end{cases} \Rightarrow \begin{cases} 0<a<1 \\ a \leq \frac{1}{2} \end{cases}$$

$$\Rightarrow \{0<a \leq \frac{1}{2}$$

Thus if $0<a \leq \frac{1}{2}$, then the given equation has no less than four different integer solutions.(d) if $-1<a<0$, $then$ $0<-a<1$ and the interval.

$[-a^{-2},-a]$ contains at least four integers. Provided that the inequality $-a^{-2} \leq -3$ holds true Let us solve the system $\begin{cases}-1 < a < 0\\ -a^{-2} \le -3\end{cases} = \begin{cases}-1 < a < 0\\ -1 \le -3a^2\end{cases} \Rightarrow \begin{cases}-1 < a < 0\\ 3a^2 - 1 \le 0\end{cases}$

$$\Rightarrow \begin{cases} -1 < a < 0 \\ \left(a - \frac{1}{\sqrt{3}}\right)\left(a + \frac{1}{\sqrt{3}}\right) \le 0 \end{cases}$$

$$\Rightarrow \begin{cases} -1 < a < 0 \\ -\frac{1}{\sqrt{3}} \le a \le \frac{1}{\sqrt{3}} \end{cases}$$

$$\Rightarrow \frac{\sqrt{3}}{3} < a < 0$$

$if\ \frac{-\sqrt{3}}{3} \le a < 0$, then the equation has no less than four integer solutions. (e) if a = -1, then the interval [-1, 1] contains only three integers i.e. conditions of the problem are not satisfied. (f) if a <-1, then $-1 < -a^{-2} < 0$, and for the interval $[-a^{-2}, -a]$ to contain no less then four, it is necessary that the inequality $-a \ge 3$ hold true, i.e. the inequality $a \le -3$ be valid. Thus, for $a \le -3$, the given equation has no less than four integer solutions.

Combining all the results, we get the set of required values of the number a namely the interval $(-\infty, -3) and\ the\ interval\ \left[-\frac{\sqrt{3}}{2}, \frac{1}{2}\right]$.

15) Solve: $\log_{(2-x)}(2-y) > 0$ $and\ \log_{(4-y)}(2x-2) > 0$

Sol.: If the numbers x and y satisfy this systems them they also satisfy the conditions. $2 - x > 0.2 - x \ne 1, 2x - 2 > 0.4 - y > 0, 4 - y \ne 1.2 - y > 0$. i.e. the system of inequality $1 < x < 2, y < 2$. On this domain for the bases of the logarithms of the initial system we have $0 < 2 - x < 1, 4 - y > 2$.

Thus the original system is equivalent to the system

$$\begin{cases} 1 < x < 2 \\ y < 2 \\ 0 < 2 - y < 1 \\ 2x - 2 > 1 \end{cases} \Rightarrow \begin{cases} \frac{3}{2} < x < 2 \\ 1 < y < 2 \end{cases}$$

Consequently, the set of all solutions of the original system is the set of pairs (x, y) where x belongs to the interval $\left(\frac{3}{2}, 2\right)$ and y belongs to the interval (1, 2).

16) Find out whether the system of equations $x + y = 0\ x^2 + y^2 = 0\ and\ \sin(x + y) = 0, x^2 + y^2 = b$ are equivalent for (i) b = 2 and (ii) b = 5.

Sol.: It is clear that both for b = 2 and for b = 5 the second system is a consequence of the first. Since the equation $\sin(x + y) = 0$ is a consequence of the equation $x + y = 0$

The first system has solutions $(1, -1), (-1, 1) when\ b = 2\ and\ \left[\sqrt{\frac{5}{2}}, -\sqrt{\frac{5}{2}}\right], \left[\sqrt{\frac{5}{2}}, \sqrt{\frac{5}{2}}\right]\ when\ b = 5$

Let us find the set of solutions of the second system.

From its first equation we have $x + y = n\pi (n \in I) and,$ consequently, it is equivalent to the collection of system

$$\begin{cases} x + y = n\pi \\ x^2 + y^2 = b, n \in I \end{cases} \dots\dots\dots\dots (1)$$

Consequently, the collection of system (1) is equivalent to the collection of systems

$$\begin{cases} x + y = n\pi \\ xy = \frac{1}{2}n^2\pi^2 - \frac{1}{2b}, \quad n \in I \end{cases} \dots\dots\dots\dots\dots (2)$$

To find the set of solutions of (2), x and y are the roots of the quadratic equation.

$t^2 + n\pi t + \frac{1}{2}(n^2\pi^2 - b) = 0, n \in I.$
Discriminant $D \geq 0$ $i.e., n^2\pi^2 - 2(n^2\pi^2 - b) \geq 0$ i.e. when $x^2 \leq \frac{2b}{\pi^2}$.

It follows that for b = 2 the collection of system (2) has a solutions only for n = 0 and for b = 5 it has a solution for n = -1, n = 0, n = 1. Thus, for b = 2 these systems are equivalent.

Comparing the sets of solutions for b = 5, we find that the initial system are not equivalent.

17) Solve the equation $(\mathbf{144})^{|x|} - \mathbf{2}(\mathbf{12})^{|x|} + \boldsymbol{a} = \mathbf{0}$ for every value of the parameter a.

Sol.: The given equation can be written as $(12)^{2|x|} - 2(12)^{|x|} + a = 0$. Let us write $y = (12)^{2|x|}$.

Then the above equations becomes $y^2 - 2y + a = 0$

$$\Rightarrow y = \frac{2 \pm \sqrt{4-4a}}{2} \Rightarrow y = 1 \pm \sqrt{1-a} \ldots\ldots(1)$$

The equation (1) is valid if $1 - a > 0$ $i.e.\, a < 1$.

No solution is possible if $a > 1. If\ a = 1, then\ y = 1.$

$$\Rightarrow 12^{|x|} = 1 \Rightarrow |x| = 0 \Rightarrow x = 0$$

$let\ us\ consider\ the\ case\ when\ a < 1. From\ (1) we\ have\ 12^{|x|} = 1 + \sqrt{1-a} \ldots\ldots(2) and\ 12^{|x|} = 1 - \sqrt{1-a} \ldots\ldots\ldots(3)$

But the equation (3) is unacceptable, since $12^{|x|} > 1$.

Hence, we have $12^{|x|} = 1 + \sqrt{1-a}$

$\Rightarrow |x| = \log_{12}\left(1 + \sqrt{1-a}\right) \Rightarrow x = \pm\log_{12}\left(1 + \sqrt{1-a}\right)$

Whenever a < 1.

18) Solve: $\boldsymbol{x^2}\ \mathbf{2}^{\boldsymbol{x+1}} + \mathbf{2}^{|\boldsymbol{x-3}|\boldsymbol{+2}} = \boldsymbol{x^2}.\mathbf{2}^{|\boldsymbol{x-3}|\boldsymbol{+4}} + \mathbf{2}^{\boldsymbol{x-1}}$

Sol.: We consider two cases according as $x \geq 3\ or\ x < 3$. As we know, if $x \geq 3, then\ [x - 3] = x - 3$

Case I : $Let\ x \geq 3. Then\ |x-3| = (x-3).$

Hence the given equations becomes

$x^2.2^{x+1} + 2^{x-3+2} = x^2.2^{x-3+4} + 2^{x-1}$

$\Rightarrow x^2.2^{x+1} + 2^{x-1} = x^2.2^{x+1} + 2^{x-1}$ which is satisfied for every x.

Hence the given equations is satisfied for every $x \geq 3$

Case II : $Let\ x < 3, then\ |x-3| = -(x-3).$

Hence the given equations becomes,

$x^2.2^{x+1} + 2^{-(x-3)+2} = x^2.2^{-(x-3)+4} + 2^{x-1}$

$\Rightarrow x^2.2^{x+1} + 2^{5-x} = x^2.2^{7-x} + 2^{x-1}$

$\Rightarrow x^2.2^{x+1} - 2^{x+1} = x^2.2^{7-x} + 2^{5-x}$

$\Rightarrow x^2.2^{x-1} - 2^{x-1} = x^2.2^2.2^{5-x} - 2^{5-x}$

$\Rightarrow\ 2^{x-1}(4x^2 - 1) = 2^{5-x}(4x^2 - 1)$

$\Rightarrow 2^{x-1}(4x^2-1) - 2^{5-x}(4x^2-1) = 0$

$\Rightarrow (4x^2-1)(2^{x-1} - 2^{5-x}) = 0 \Rightarrow 4x^2 - 1 = 0 \ldots\ldots\ldots(1)$

$and\ 2^{x-1} - 2^{5-x} = 0 \ldots\ldots\ldots\ldots(2)$

$(1) \Rightarrow 4x^2 = 1 \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = 1\frac{1}{2}$

$(2) \Rightarrow 2^{x-1} = 2^{5-x} \Rightarrow x - 1 = 5 - x \Rightarrow 2x = 6 \Rightarrow x = 3$

But x is < 3.

Hence the $x = 3$ does not give the solution of the equations. Hence the solutions of the original solutions are $x \geq 3$. (from case 1) and $x = \pm\frac{1}{2}$(from case 2)

19) Solve the inequality

$$\log_{\frac{25-x^2}{16}}\left(\frac{14-2x-x^2}{14}\right) > 1$$

Sol.: As we know, if the log function $\log_a x$ is meaning ful then its base a should be > 0 and $a \neq 1$.Also $x > 0$ therefore we shall discuss two cases according as the $0 < \frac{25-x^2}{16} < 1$ and $\frac{25-x^2}{16} > 1$.

Case 1 : Let $\frac{25-x^2}{16} > 1$.

This $\Rightarrow 25 - x^2 > 16 \Rightarrow x^2 < 9, \Rightarrow (x^2 - 9) < 0$

$\Rightarrow (x+3)(x-3) < 0 \Rightarrow 25 - x^2 > 16 \Rightarrow -3 < x < 3 \ldots \ldots .(1)$

In this case, the given inequality is equivalent to

$$\frac{24-2x-x^2}{14} > \frac{25-x^2}{16}, \Rightarrow \frac{24-2x-x^2}{7} > \frac{25-x^2}{8}$$

$\Rightarrow 192 - 16x - 8x^2 > 175 - 7x^2 \Rightarrow x^2 + 16x - 17 < 0$

$\Rightarrow (x+17)(x-1) < 0 \Rightarrow -17 < x < 1 \ldots \ldots \ldots (2)$

$\therefore$ Taking (1) and (2) together we find that $-3 < x < 1$ ………….(3)

Case 2 : Let $0 < \frac{25-x^2}{16} < 1$. In this case, the original inequality is equivalent to the double inequality.

$$0 < \frac{24-2x-x^2}{14} < \frac{25-x^2}{16}$$

Thus in this case, we have to solve the following system of double inequalities:

(i) $0 < \frac{25-x^2}{16} < 1$

(ii) $0 < \frac{24-2x-x^2}{14} < \frac{25-x^2}{16}$

The first inequality is reduced to $0 < 25 - x^2$ and $25 - x^2 < 16$ $i.e.$ $9 < x^2 < 25$. But $9 < x^2 \Rightarrow x^2 - 9 > 0 \Rightarrow (x-3)(x+3) > 0$

$\Rightarrow x < -3$ or $x > 3$ and $x^2 < 25 \Rightarrow x^2 - 25 < 0 \Rightarrow (x-5)(x+5) < 0$

$\Rightarrow -5 < x < 5$.

Hence taking together $9 < x^2 < 25 \Rightarrow -5 < x < -3$ and $3 < x < 5 \ldots \ldots \ldots ..(4)$

The second double inequality is equivalent to the system of inequalities (iii) $24 - 2x - x^2 > 0 \Rightarrow x^2 + 2x - 24 < 0$ and $(iv) x^2 + 16x - 17 > 0$

(iii) $\Rightarrow (x+6)(x-4) < 0 \Rightarrow -6 < x < 4$

(iv) $(x+17)(x-1) > 0 \Rightarrow x < -17$ or $x > 1$.

Thus (iii)+ (iv) $\Rightarrow 1 < x < 4$ ………….(5)
Finally (4) +(5) i.e. , $(3 < x < 5$ and $1 < x < 4)$ $together$ $\Rightarrow 3 < x < 4$.

Hence combining the two cases we have the solution of the original inequality which consists of two intervals : $-3 < x < 1$ and $3 < x < 4$.

20) Solve : $\left(\frac{1}{2}\right)^{\sqrt{x^2-2x^3+1}} < \left(\frac{1}{2}\right)^{1-x}$

Sol.: Since the base of the exponential inequality (i.e. $\frac{1}{2}$) is less than 1, hence the original inequality is equivalent to the inequality. $\sqrt{x^6-2x^3+1} > 1-x$

$Since\ \sqrt{x^6-2x^3+1} = \sqrt{(x^3-1)^2}|x^3-1|,$

$\therefore$ the above inequality can be written as $|x^3-1| > 1-x \ldots\ldots\ldots\ldots\ldots.(1)$

$If\ 1-x<0\ i.e.x>1$, then the inequality (1) is automatically satisfied since the left member is non negative.

$\therefore$ The solution of the inequality (1) is the set of all x >1 we now consider $x \le 1$. In this case $x^3 \le 1. i.e. x^3-1 \le 0\ and\ so\ |x^3-1| = (x^3-1)$ and then we can write the inequality (1) as $-(x^3-1) > 1-x \Rightarrow -x^3+1 > 1-x \Rightarrow x^3-x>0, \Rightarrow x(x^2-1)>0 \Rightarrow x(x-1)(x+1)<0$ Solving this inequality by the method of intervals we find that it is true for $x<-1$ and for x located in the interval $0<x<1 \ldots\ldots(3)$ Hence combining (2) and (3), we concluded that the original inequality is valid for $x<1, 0<x<1$ and also $x<1$.

INEQUALITIES

(OBJECTIVE TYPE)

1) **If $n^4 < 10^n$ for a fixed positive integer $n \ge 2, then$**
   (a) $(n+1) < 10^{n+1}$; (b) $(n+1)^n \ge 10^{n+1}$; (c) $n^4+1 < 10^{n+1}$; (d) none

Sol.: we have $\left(\frac{n+1}{n}\right)^4 = \left(1+\frac{1}{n}\right)^4 \le \left(1+\frac{1}{2}\right)^4 = \left(\frac{3}{2}\right)^4 < 10$

$[\because n \ge 2] \Rightarrow (n+1)^4 \le n^4.10 < 10^n.10 = 10^{n+1}$

2) **If a, b, c are the sides of a triangle, then $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ is also the sides of the triangle is,**
   (a) Always false; (b) always true; (c) Sometimes; (d) none

Sol.: Assume that $a \ge b \ge c$. We must have $b+c>a$.

Also, note that $b+c \le c+a \le a+b$

$$\Rightarrow \frac{1}{b+c} \ge \frac{1}{c+a} \ge \frac{1}{a+b}$$

To show that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are sides of a triangle, it is sufficient to show that $\frac{1}{c+a} = \frac{1}{a+b} > \frac{1}{b+c}$

$As\ a \ge b \ge c, we\ get\ 2a \ge a+b, and\ 2a \ge a+c$

$$\Rightarrow \frac{1}{2a} \le \frac{1}{a+b}, \frac{1}{2a} \le \frac{1}{a+c}$$

$$\Rightarrow \frac{1}{a+b} + \frac{1}{a+c} \ge \frac{1}{2a} + \frac{1}{2a} = \frac{1}{d} > \frac{1}{b+c}$$

$[\because a<b<c] \therefore$ it represents a triangle.

3) **The product of three positive reals is 1 and their sum is greater then sum of their reciprocals. Exactly one of them is greater than**
   (a) -1; (b) 0; (c) 1; (d) none

Sol.: Let three positive reals be a, b and $\frac{1}{ab}$, We are given $a+b+\frac{1}{ab}>\frac{1}{a}+\frac{1}{b}+ab$ ………(1)

$$Now\ (a-1)(b-1)\left(\frac{1}{ab}-1\right) = 1+\left(a+b+\frac{1}{ab}\right) - \left(ab+\frac{1}{a}+\frac{1}{b}\right)-1 = \left(a+b+\frac{1}{ab}\right) - \left(ab+\frac{1}{a}+\frac{1}{b}\right) > 0$$

Using (1)$\Longrightarrow$ either all these $a-1, b-1$ $and \frac{1}{ab}-1$ are positive or exactly one of them is positive.

But $a>1, b>1$ $and \frac{1}{ab}>1$. Thus exactly one of $a, b, \frac{1}{ab}$ exceed 1.

4) **If $\boldsymbol{a+b+c=6}$, *then* $\boldsymbol{\sqrt{4a+1}+\sqrt{4b+1}+\sqrt{4c+1}}$ is**
(a) ≤ 9; (b) >9; (c) <9; (d) none

Sol.: By the Cauchy Schwarz inequality,

$\left(\sqrt{4a+1}+\sqrt{4b+1}+\sqrt{4c+1}\right)^2 \le (1+1+1)$

$(4a+1+4b+1+4c+1) = 3[4(a+b+c)+3]$

$=(3)(27) \Longrightarrow \sqrt{4a+1}+\sqrt{4b+1}+\sqrt{4c+1} \le 9$

5) **If a, b, c ∈ R, then $\boldsymbol{\sqrt{a^2+b^2+c^2-bc-ca-ab}} \ge$**
(a) $\frac{\sqrt{3}}{4}$max$\{|b-c|,|c-a|,|a-b|\}$;
(b) max$\{|b-c|,|c-a|,|a-b|\}$
(c) $\frac{\sqrt{3}}{2}$ $\{$max$|b-c|,|c-a|,|a-b|\}$
(d) *none*

Sol.: We have, $a^2+b^2+c^2-bc-ca-ab=\frac{1}{2}$

$$[(b^2+c^2-2bc)+(c^2+a^2-2ca)+(a^2+b^2-2ab)=\frac{1}{2}[(b-c)^2+(c-a)^2+(a-b)^2]\ge 0$$

$$Also, a^2+b^2+c^2-bc-ca-ab-\frac{3}{4}(b-c)^2$$

$$=\frac{1}{4}[4a^2+4b^2+4c^2-4bc-4ca-4ab-3(b^2+c^2-2bc)] = \frac{1}{4}[4a^2+b^2+c^2+2bc-4a(c+b)]$$

$=\frac{1}{4}[4a^2+(b+c)^2-4a(b+c)] = \frac{1}{4}[2a-(b+c)]^2 \ge 0$

$$Similarly, a^2+b^2+c^2-bc-ca-ab \ge \frac{\sqrt{3}}{2}|c-a| and\ a^2+b^2+c^2-bc-ca-ab \ge \frac{\sqrt{3}}{2}|a-b|$$

$$\Longrightarrow\ a^2+b^2+c^2-ab-bc-ca \ge \frac{\sqrt{3}}{2}\max\{|b-c|,|c-a|,|a-b|\}$$

6) **If x >0, $\boldsymbol{\lambda>0}$ and $\boldsymbol{\lambda x+\frac{1}{x}-1}$ is always non-negative, then the least value of $\boldsymbol{\lambda}$ is:**
(a) $\frac{1}{4}$; (b) $\frac{1}{2}$; (c) $\frac{1}{3}$; (d) none

Sol.: $\lambda x+\frac{1}{x}-1\ge 0 \Longrightarrow \lambda x^2-x+1\ge 0$

$$\Longrightarrow (-1)^2-4.\lambda.1\le 0 \Longrightarrow \lambda \ge \frac{1}{4}$$

$\therefore$ least value of λ is $\frac{1}{4}$.

7) If x ∈ R and y = $\frac{x^2}{(1+x^4)}$ then

(a) $0 \le y \le \frac{1}{2}$; (b) $0 \le y \le 1$; (c) $0 \le y \le 2$; (d) none

Sol.: $y + yx^4 = x^2, where\ y \ge 0, yx^4 - x^2 + y = 0$

$x^2 = \frac{1 \pm \sqrt{1-4y^2}}{2}$, for this exists. $1 - 4y^2 = 0$

$$\Rightarrow -\frac{1}{2} \le y \le \frac{1}{2}\ but\ y = 0 \Rightarrow 0 \le y \le \frac{1}{2}.$$

1

8) If $x_1, x_2, \dots, x_n$ are any real numbers and n is position integer, then

(a) $\sum_{i=1}^{n} x_i^2 \ge n(\sum x_i)^2$; (b) $n \sum x_i^2 < (\sum_1^n x_i)^2$; (c) $n \sum_{i=1}^{n} x_i^2 \ge (\sum_1^n x_i)^2$

(d) none

Sol.: Here, $\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} \ge \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^2$

$$\Rightarrow n \sum_{i=1}^{n} x_i^2 \ge \left(\sum_{1}^{n} x_i\right)^2$$

9) If x, y, z are positive real number, such that $x + y + z = 2$, then

(a) $(2 - x)(2 - y)(2 - z) \le 8xyz$;

(b) $(2 - x)(2 - y) \times (2 - z) \le 8xyz$

(c) $(2 - x)(2 - y)(2 - z) \ge \frac{1}{2}$;

(d) *none*

Sol.: $x + y + z = 2 \ \therefore (2 - x)(2 - y)(2 - z) = (y + z)(z + x)(x + y)$

$$\Rightarrow (y + z) \ge 2\sqrt{yz}, (x + y) \ge 2\sqrt{xz},$$

$$(x + y) \ge 2\sqrt{xy}.$$

$$\therefore (y + z)(z + x)(x + y) \ge 8xyz$$

$$Also, \left(\frac{x^{-1} + y^{-1} + z^{-1}}{3}\right) \ge \left(\frac{x + y + z}{3}\right)^{-1}$$

$$\Rightarrow x^{-1} + y^{-1} + z^{-1} \ge 3.\left(\frac{2}{3}\right)^{-1}$$

$$\Rightarrow x^{-1} + y^{-1} + z^{-1} \ge \frac{9}{2}$$

10) If the product of n positive numbers is n^n, them their sum is

(a) & n^2; (b) $= n + \frac{1}{n}$; (c) $> n$; (d) none

Sol.: Let $a_1, a_2, \dots \dots \dots, a_n$ be n positive integers such that $a_1 a_2, \dots, a_n = n^n$. Since $A.M \ge G.M$

$$\therefore \frac{a_1 + a_2 + \cdots + a_n}{n} \ge (a_1 a_2, \dots a_n)^{\frac{1}{n}}$$

$$\Rightarrow \frac{a_1 + a_2 + \cdots + a_n}{n} \ge n$$

$$\Rightarrow a_1 + a_2 + \cdots + a_n \ge n^2$$

11) For positive real number a, b, c such that a +b +c = p which one holds?

(a) $\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} \ge p$;

(b) $(P - a)(P - b)(P - c) \le 8abc$

(c) $(P - a)(P - b)(P - c) \ge \frac{8}{27}p^3$;

(d) *none*

Sol.: Using $A.M \ge G.M$ one can show

$$(b + c)(c + a)(a + b) \ge 8abc$$

$$\Rightarrow (p - a)(p - b)(p - c) \ge 8abc \Rightarrow (b) holds$$

$$Also, \frac{(p-a)+(p-b)+(p-c)}{3} \geq [(p-a)(p-b)(p-c)]^{\frac{1}{3}}$$

$$\Rightarrow \frac{3p-(a+b+c)}{3} \geq [(p-a)(p-b)(p-c)]^{\frac{1}{3}}$$

$$\Rightarrow \frac{2p}{3} \geq [(p-a)(p-b)(p-c)]^{\frac{1}{3}}$$

$$\Rightarrow (p-a)(p-b)(p-c) \leq \frac{8p^3}{27} \Rightarrow (c) does\ not\ holds$$

$$Again, since \frac{1}{2}\left(\frac{bc}{a}+\frac{ca}{b}\right) \geq \sqrt{\left(\frac{bc}{a}.\frac{ca}{b}\right)} etc$$

∴Adding the inequalities, we get

$$\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c} \geq a+b+c=p$$

$\Rightarrow$ (a) does not holds.

12) If $0 < \alpha < \beta < \frac{\pi}{2}$ then
(a) $\beta \tan\alpha < \alpha \tan\beta$; (b) $\beta \tan\alpha > \alpha \tan\beta$; (c) $\beta \tan\alpha < \alpha \sin\beta$; (d) none

Sol.: We know, $f_{(x)} = \frac{\sin x}{x}$ is decreasing

$$f_{(\beta)} < f_{(\alpha)}: 0 < \alpha < \beta < \frac{\pi}{2}. \frac{\sin\beta}{\beta} < \frac{\sin\alpha}{\alpha}\ or\ \alpha\sin\beta < \beta\sin\alpha. Also, f_{(x)} = \frac{\tan x}{x}\ is\ increasing$$

$\therefore\ f_{(\beta)} > f_{(\alpha)}\ as\ \alpha < \beta.$

$$\Rightarrow \frac{\tan\beta}{\beta} > \frac{\tan\alpha}{\alpha}\ or\ \alpha\tan\beta > \beta\tan\alpha.$$

13) If x, y, z are real, distinct and $u = x^2 + 4y^2 + 9z^2 - 6yz - 3zx - 2xy$, *then* u is
(a) Zero; (b) non-negative; (c) non-positive; (d) none

Sol.: $u = \frac{1}{2}\{2x^2 + 8y^2 + 18z^2 - 12yz - 6zx - 4xy\} = \frac{1}{2}\{(x-2y)^2 + (2y-3z)^2 + (3z-x)^2\}$

$\Rightarrow u \geq 0$

14) The minimum value of p = $bcx + cay + abz$ when $xyz = abc$, *is*
(a) abc; (b) $4abc$; (c) 5 abc; (d)none

Sol.: $A.M \geq G.M \Rightarrow \frac{bcx+cay+abz}{3} \geq (a^2b^2c^2, xyz)^{\frac{1}{3}}$

$bcx + cay + abz \geq 3xyz\ or$

$bcx + acy + acy + abz \geq 3abc$

15) If a, b , c, d are positive real numbers such that a +b+ c+ d = 2, then $M = (a+b)(c+d)$ satisfies the relation
(a) $0 \leq M \leq 1$; (b) $1 \leq M \leq 2$ (c) $2 \leq M \leq 3$; (d) none

Sol.: (a) Using $A.M \geq G.M, then \frac{(a+b)+(c+d)}{2} \geq \{(a+b)(c+d)\}^{\frac{1}{2}}$,

$$\Rightarrow \frac{2}{2} \geq M^{\frac{1}{2}}, \Rightarrow M \leq 1.$$

As a, b, c, d > 0. So, M = $(a+b) \times (c+d) > 0$

$i.e.,\ 0 \leq M \leq 1.$

(SUBJECTIVE TYPE)

1) **If** $-1 \le a_1 \le a_2 \le \cdots \le a_n \le 1$, **prove that**

$$\sum_{i=1}^{n-1} \sqrt{1 - a_i a_{i+1} - \sqrt{(1-a_i^2)(1-a_{i+1}^2)}} \le \frac{\pi\sqrt{2}}{2}$$

Sol.: It is natural to make the trigonometric substitution $a_i = \cos x_i \ for\ some\ x_i \in 0, \pi, i = 1, 2, \dots, n$. Note that the monotonicity of the cosine function combined with the given inequalities show that the $x_i's$ from a decreasing sequence. The expression on the left becomes

$$\sum_{i=1}^{n-1} \sqrt{1 - \cos x_i \cos x_{i+1} - \sin x_i \sin x_{i+1}}$$

$$= \sum_{i=1}^{n-1} \sqrt{1 - \cos(x_{i+1} - x_i)}$$

$$= \sqrt{2} \sum_{i=1}^{n-1} \sin \frac{x_{i+1} - x_i}{2}$$

Here we used a subtraction and a double-angle formula. The sine function is concave down on $[0, \pi]$; hence we can Jensen's inequality to obtain

$$\frac{1}{n-1} \sum_{i=1}^{n-1} \sin \frac{x_{i+1} - x_i}{2} \le \sin\left(\frac{1}{n-1} \sum_{i=1}^{n-1} \frac{x_{i+1} - x_i}{2}\right)$$

Hence,

$$\sqrt{2} \sum_{i=1}^{n-1} \sin \frac{x_{i+1} - x_i}{2} \le (n-1)\sqrt{2} \sin \frac{x_n - x_1}{2(n-1)} \le \sqrt{2}(n-1)$$

$\sin \frac{\pi}{2(n-1)}. Since\ x_n - x_1 \in (0, \pi)$.

Using the fact that $\sin x < x \ for\ x > 0\ yeilds$

$$\sqrt{2}(n-1) \frac{\sin \pi}{\left(2(n-1)\right)} \le \frac{\sqrt{2}\pi}{2}$$

2) **Let** $x_0 = 0$ ***and*** $x_1, x_2, \dots, x_n > 0$ **with** $\sum_{k=1}^{n} x_k = 1$

Prove that

$$\sum_{k=1}^{n} \frac{x_k}{\sqrt{1 + x_0 + \cdots + x_{k-1}}\sqrt{x_k + \cdots + x_n}} < \frac{\pi}{2}$$

Sol.: Since $x_i's$ are positive and add up to 1, we can make the substitution $x_0 + x_1 + \cdots + x_k = \sin a_k$, with $a_0 < a_1 < \cdots < a_n = \frac{\pi}{2}, k = 0, 1, \dots, n$. The inequality becomes $\sum_{k=1}^{n} \frac{\sin a_k - \sin a_{k-1}}{\sqrt{1+\sin a_{k-1}}\ \sqrt{1 - \sin a_{k-1}}} < \frac{\pi}{2}$,

Which can be written as

$$\sum_{k=1}^{n} \frac{2 \sin \frac{a_k - a_{k-1}}{2} \cos \frac{a_k + a_k - 1}{2}}{\cos a_{k-1}}$$

For $0 < x < \frac{\pi}{2}$, $\cos x$ is a decreasing function and $\sin x < x$. Hence the left side of the inequality is strictly less than

$\sum_{k=1}^{n} \frac{2 \frac{a_k - a_{k-1}}{2} \cos a_{k-1}}{\cos a_{k-1}} = \sum_{k=1}^{n} (a_k - a_{k-1}) = \frac{\pi}{2}$ and problem is solved.

3) **If a, b, c be the edge of a right parallelepiped and d its diagonal. Show that $a^2b^2 + b^2c^2 + c^2a^2 \geq abcd\sqrt{3}$.**

Sol.: Since in a right parallelepiped the diagonal is given by the formula $d = \sqrt{a^2 + b^2 + c^2}$, the inequality is equivalent to $(a^2b^2 + b^2c^2 + c^2a^2)^2 \geq 3a^2b^2c^2$

$(a^2 + b^2 + c^2)$ After regrouping term this becomes $\frac{c^4}{2}(a^2 - b^2) + \frac{a^4}{2}(b^2 + c^2)^2 + \frac{b^4}{2}(c^2 - a^2) \geq 0$

Note that the equality holds if and only if = a = b = c i.e. the parallelepiped is a cube.

4) **If $a_1, a_2, \dots, a_n$ are real numbers, show that**

$$\sum_{i=1}^{n}\sum_{j=1}^{n} ij\cos(a_i - a_j) \geq 0$$

Sol.: By using the addition formula for the cosine we obtain

$$\sum_{i=1}^{n}\sum_{j=1}^{n} ij\cos(a_i - a_j)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\big(ij\cos a_i\cos a_j + ij\sin a_i + \sin a_j\big)$$
$$= \sum_{i=1}^{n} i\cos a_i \sum_{j=1}^{n} j\cos a_j + \sum_{i=1}^{n} i\sin a_i \sum_{j=1}^{n} j\sin a_j$$
$$= \left(\sum_{i=1}^{n} i\cos a_i\right)^2 + \left(\sum_{i=1}^{n} i\sin a_i\right)^2 \geq 0$$

5) **The non-negative numbers a, b, c, A, B, C and k satisfy a +A = b +B = c+ C = k. Prove that $aB + bC + cA \leq k^2$.**

Sol.: The inequality is equivalent to $a(k-b) + b(k-c) + c(k-a) \leq k^2$. If we view the left side as a function in a, it is linear. The conditions from the statement imply that interval of definition is [0, k]. It follows that in order to maximize the left and side we need to choose a ∈ {0, k}. Repeating the same argument for b and c, it follows that the maximum of the left hand side is attained for some (a, b, c)∈ $\{0, k\}^3$. Checking the eight possible situations, we obtain that this maximum is k^2, and we are done.

6) **Let $0 \leq x_k \leq 1$ for all k = 1, 2, ….., n. Show that $x_1 + x_2 + \cdots + x_n - x_1x_2 \dots\dots\dots . x_n \leq n - 1$**

Sol.: Let us fix $x_2, x_3 \dots\dots x_n$ and then consider the function f: [0, 1] →R. f(x)= $x_1 + x_2 + \cdots + x_n - xx_2 \dots\dots\dots . x_n$. The function is linear in x, hence attains its maximum the left side of the inequality one must choose x_1 *to* be 0 or 1, and by symmetry, the same is true for the other variables of occurs, if all x_i are equal to 1, then we have equality. If at least one of them is 0, then their product is also zero, and the sum of the other $n - 1$ terms is at most $n - 1$. Which proves the inequality.

7) **Find the maximum value of the sum $S_n = a_1(1 - a_2) + a_2 + (1 - a_3) + \cdots + a_n(1 - a_1)$, where $\frac{1}{2} < a_j \leq 1$ *for every* $i = 1, 2, \dots, n$.**

Sol.: The expression is linear in each of the variables. So, as in the solutions to the previous problems, the maximum is attained for $a_k = \frac{1}{2} or\ 1, k = 1,2, \dots\dots, n. If\ a_k = \frac{1}{2}\ for\ all\ k, then\ S_n = \frac{n}{4}$, Let us show that the value of S_n cannot exceed this number. If exactly m of the $a_k's$ are equal to 1, then m terms of the sum are zero. Also, at most m trams are equal to $\frac{1}{2}$, *namely* those of the from $a_k(1 - a_{k+1}) with\ a_k = 1\ and\ a_{k+1} = \frac{1}{2}$. Each of the remaining terms has both factors equal to $\frac{1}{2}$, and hence is equal to $\frac{1}{4}$. Thus the value of the sum is at most $m.0 + \frac{m}{2} + \frac{(n-2m)}{4} = \frac{n}{4}$. Which shows that the maximum is $\frac{n}{4}$.

8) **If $n \geq 2\ and\ 0 < x_{i\leq 1}\ for\ all\ i = 1, 2, \dots$, n show that $(x_1 + x_2 + \dots + x_n) - (x_1x_2 + x_2x_3 + \dots + x_nx_1) \leq \left[\frac{n}{2}\right]$ and determine when there is equality [.] denote the greatest integer function.**

Sol.: Denote the left side of the inequality by $S(x_1, x_2 \dots x_n)$ This expression is linear in each of the variables x_i As before, it follows that it is enough to prove the inequality when the $x_i's$ are equal to 0 or 1. If exactly k of the $x_i's$ are equal to 0, and the others are equal to 1, then $S(x_1, x_2, \dots, x_n) \leq n - k$, and since the sum $x_1x_2 + x_2x_3 + \dots + x_nx_1$ is at least $n - 2k$,$S(x_1, x_2, \dots, x_n)$ is less than or equal to $n - k - (n - 2k) = k$. So the maximum of S is less than or equal to $\min(k, n - k)$. *Which* is at most $\left[\frac{n}{2}\right]$. If follows that for n even. Equality holds when $(x_1, x_2, \dots,)$ =(1, 0, 1, 0, ……, 1, 0) or (0, 1, 0, 1, ……0, 1). For n odd. Equality holds when all pairs $(x_i, x_{i+1}), i = 1, 2, \dots, n$. consist of a zero and a one, except for one pair which consist of two ones (with the convention $x_{n+1} = x_i$).

9) **Prove that for numbers a, b, c in the interval [0, 1] $\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1 - a)(1 - b)(1 - c) \leq 1$**

Sol.: For any non-negative numbers α, β, the function $x \to \frac{\alpha}{x+\beta}$ is convex for $x \geq 0$. Viewed as a function in any of the three variables, the given expression is a sum of two convex functions and two linear functions so it is convex. Thus when two of the variables are fixed, the maximum is attained when the third is at one of the end points of the interval, so the values of the expression are always less than the largest value obtained by choosing a, b, c ∈[0, 1]. An easy check of the eight possible cases shows that the value of the expression cannot exceed 1.

10) **If a, b, c, d, e ∈[p, q] with p > 0, prove that (1+b+c+d+e) $\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\right) \leq 25 + 6\left(\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right)^2$**

Sol.: If we fix four of the numbers and regard the fifth as a variable x, then the left side becomes a function of the form $\alpha x + \frac{\beta}{x+y}$, with α, β, y positive and x ranging over the interval [p, q]. This function is convex on the interval [p, q] being the sun of a linear and a convex function, so it attains its maximum at one (or possibly both) of the end points of the interval of definition. As the value of the expression, it is enough to let a, b, c, d, e take the values p and q.

If n of the numbers are equal tom p, and $5-n$ are equal to q, then the left side is equal to

$$x^2+(5-n)^2+n(5-n)\left(\frac{p}{q}+\frac{q}{p}\right) = 25 + n(5-n)\left(\frac{\sqrt{p}}{q}-\frac{\sqrt{q}}{p}\right)^2$$

The maximal value of $n(5-n)$ is attained when n=2 or 3 in which case $n(5-n)$=6, and the inequality is proved.

11) Prove that if $1 \le x_n \le 2, k=1,2,\ldots\ldots,n$ then $\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n}\frac{1}{x_k}\right)^2 \le n^3$

Sol.: Using the AM- GM inequality we can write

$$\sqrt[3]{\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n}\frac{1}{x_k}\right)^2} \le \frac{1}{3}\left(\sum_{k=1}^{n} x_k + \sum_{k=1}^{n}\frac{1}{x_k}+\sum_{k=1}^{n}\frac{1}{x_k}\right)$$

$$\sum_{k=1}^{n}\frac{x_k+\frac{1}{x_k}+\frac{1}{x_k}}{3}.$$

The function $x+\frac{2}{x}$ is convex on the interval [1, 2], so it attains its maximum at one of the end points of the interval. Also, the value of the function at each of the end point is equal to 3. This shows that

$$\sum_{k=1}^{n}\frac{x_k+\frac{1}{x_k}+\frac{1}{x_k}}{3} \le n$$

And the inequality is proved.

Let us point out that the same idea can be used to prove the more general from of this inequality.

12) Prove that

$$\frac{\mathbf{1}}{\sqrt{\mathbf{1}}+\sqrt{\mathbf{3}}}+\frac{\mathbf{1}}{\sqrt{\mathbf{5}}+\sqrt{\mathbf{7}}}+\cdots+\frac{\mathbf{1}}{\sqrt{\mathbf{9999}}+\sqrt{\mathbf{9999}}} > 24$$

Sol.: There are some terms missing to make this sum telescope. However, since the left hand side is greater than $\frac{1}{\sqrt{3}+\sqrt{5}}+\frac{1}{\sqrt{7}+\sqrt{9}}+\cdots+\frac{1}{\sqrt{9999}+\sqrt{10001}}$ the inequality will show from $\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{5}}+\frac{1}{\sqrt{5}+\sqrt{7}}+\cdots+\frac{1}{\sqrt{9999}+\sqrt{10001}} > 48$ Now we are able to telescope. Rationalize the denominators and obtain the equivalent inequality.

$$\frac{\sqrt{3}-\sqrt{1}}{2}+\frac{\sqrt{5}-\sqrt{3}}{2}+\frac{\sqrt{7}-\sqrt{5}}{2}+\cdots+\frac{\sqrt{10001}+\sqrt{9999}}{2} > 48.$$

The left side is equal to $\frac{(\sqrt{10001}-1)}{2}$, and an easy check shows that this is larger than 48.

13) If $a_k=\frac{k}{(k-1)^{4/3}+k^{4/3}+(k+1)^{4/3}}$. Show that $a_1+a_2+\cdots+a_{9999} < 50.$

Sol.: The idea is first to decreases the denominator of a_n, replacing $k^{4/3}\ by\ (k-1)^{2/3}\ (k+1)^{2/3}$ and then to rationalize it. We have

$$a_n < \frac{k}{(k-1)^{\frac{4}{3}}+(k-1)^{\frac{2}{3}}(k+1)^{\frac{2}{3}}+(k+1)^{\frac{4}{3}}} = \frac{k((k+1)^{\frac{2}{3}}(k-1)^{\frac{2}{3}})}{(k+1)^2\,(k-1)^2} = \frac{1}{4}((k+1)^{\frac{2}{3}}(k-1)^{\frac{2}{3}})$$

It follows that $\sum_{n=1}^{999} a_n < \frac{1}{4}\sum_{n=1}^{999} a_n\left((k+1)^{\frac{2}{3}}(k-1)^{\frac{2}{3}}\right)$

$$= \frac{1}{4}\left(1000^{\frac{2}{3}} + 999^{\frac{2}{3}} - 1^{\frac{2}{3}} - 0^{\frac{2}{3}}\right)$$

$$< \frac{1}{4}(100 + 100 - 1)$$

$$< 50.$$

14) Show that $\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n}} < 2.$

Sol.: It is natural to transform the terms of the sum as

$$\frac{1}{\sqrt{n}(n+1)} = \frac{1/n}{n(n+1)} = \frac{\sqrt{n}}{n} - \frac{\sqrt{n}}{n+1}.$$

This allow us to rewrite the sum as $1 + \sum_{n=2}^{\infty} \frac{\sqrt{n}-\sqrt{n-1}}{n}$.

The sum does not telescope, but it is bounded from above by

$$1 + \sum_{n=2}^{\infty} \frac{\sqrt{n} - \sqrt{n-1}}{\sqrt{n}\sqrt{n-1}}$$

$$= 1 + \sum_{n=2}^{\infty}\left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}\right)$$

With telescope to 2. This proves the inequality.

15) For each positive integer n. Show that

$$\left(\mathbf{1} + \frac{\mathbf{1}}{\boldsymbol{n}}\right)^{\boldsymbol{n}} < \left(\mathbf{1} + \frac{\mathbf{1}}{\boldsymbol{n} + \mathbf{1}}\right)^{\boldsymbol{n}+\mathbf{1}}$$

Sol.: This is an important inequality that can be proved in a number of ways. Here we will give a proof based on comparing corresponding terms in the binomial expansions of each side,

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k$$

$$= \sum_{k=0}^{n} \frac{n(n-1)(n-2)\dots(n-k+1)}{n.n.n\dots\dots\dots n} \frac{1}{k!} \sum_{k=0}^{n} \frac{1}{k!}$$

$$\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots.\left(1 - \frac{k-1}{n}\right).$$

In a similar manner,

$$\left(1 + \frac{1}{n+1}\right)^{n+1} = \sum_{k=0}^{n+1} \frac{1}{k!}\left(1 - \frac{1}{n+1}\right).$$

$$(1 - \frac{2}{n+1})\dots\left(1 - \frac{k-1}{n+1}\right)$$

$$= \left(\frac{1}{n+1}\right)^{n+1}$$

$$+ \sum_{k=0}^{n}\left(1 - \frac{1}{n+1}\right)$$

$$\left(1 - \frac{2}{n+1}\right)\dots.\left(1 - \frac{k-1}{n+1}\right).$$

The inequality is now obvious. Since comparing the coefficients of $\frac{1}{k}$: in these expressions. We see that for each k, k = 0, 1, 2, ….., n.

$$\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\dots\dots\left(1 - \frac{k-1}{n}\right)$$

$$< \left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\dots\dots\dots\left(1 - \frac{k-1}{n+1}\right)$$

It is worth nothing that

$\left(1 + \frac{1}{n}\right)^2 = \sum_{k=0}^{n} \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\dots.\left(1 - \frac{k-1}{n}\right) < \sum_{k=0}^{n} \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\dots\dots\left(1 - \frac{k-1}{n}\right) < \sum_{k=0}^{n} \frac{1}{k!} = 1 + \sum_{k=0}^{n} \frac{1}{k!} < 1 + \sum_{k=0}^{n} \frac{1}{2^{k-1}} = 1 + \sum_{k=0}^{n-1} \frac{1}{2^k} < 1 + \sum_{k=0}^{\infty} \frac{1}{2^k} = 3.$

Thus the sequence $\left(1 + \frac{1}{n}\right)^n$ is increasing and bounded above by 3. (It can be shown that the sequence converges to the number e)

The next result is important theoretically and is very useful.

16) Show that $n\left\{(n+1)^{\frac{1}{n}}-1\right\} < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} < n-(n-1)n^{-1/(n-1)}$

Sol.: Let $S_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$. The left most inequality is equivalent to proving.

$$\frac{n+S_n}{n} > (n+1)^{1/n}$$

Which has vaguely the look of an arithmetic mean geometric mean inequality. We can make the idea work in the following way:

$$\frac{n+S_n}{n} = \frac{n+\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)}{n}$$

$$= \frac{(1+1)+\left(1+\frac{1}{2}\right)+\cdots+\left(1+\frac{1}{n}\right)}{n}$$

$$= \frac{2+\frac{3}{2}+\frac{4}{3}+\cdots+\frac{(n+1)}{n}}{n}$$

$$\left(2.\frac{3}{2}.\frac{4}{3}\ldots\ldots..\frac{n+1}{n}\right)^{1/n}.$$

For the right most inequality. We need to show that $\frac{n-S_n}{n-1} > n^{-1/(n-1)}$. Again, using the arithmetic mean geometric mean inequality. We have

$$\frac{n-S_n}{n-1} = \frac{n-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)}{n-1}$$

$$= \frac{(1-1)+\left(1-\frac{1}{2}\right)+\cdots+\left(1-\frac{1}{n}\right)}{n-1}$$

$$= \frac{\frac{1}{2}+\frac{2}{3}+\cdots+\frac{(n-1)}{n}}{n-1}$$

$$> \left(\frac{1}{2}.\frac{1}{3}.\frac{3}{4}\ldots..\frac{n-1}{n}\right)^{\frac{1}{(n-1)}}$$

$$= \frac{1}{n}^{1/(n-1)} = n^{-1/(n-1)}$$

17) Let $a_1, a_2, \ldots, a_n$ are real (n > 1) and $A + \sum_{i=1}^{n} a_i^2 < \frac{1}{n-1}\left(\sum_{i=1}^{n} a_i\right)^2$ *Prove that* $A < 2a_ia_j$ *for* $1 \le i \le j \le n$.

Sol.: By the Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^{n} a_i\right)^2 = [(a_1+a_2)-a_3+\cdots+a_n]^2$$
$$\le (1+\cdots 1)((a_1+a_2)^2 + a_3{}^2 \ldots.+a_n{}^2)$$
$$= (n-1)\left[\sum_{i=1}^{n} a^2 + 2a_1a_2\right]$$

This, together with the given inequality, implies that

$$A < -\left(\sum_{i=1}^{n} a_i{}^2\right) + \frac{1}{n-1}\left(\sum_{i=1}^{n} a\right)^2$$
$$< -\left(\sum_{i=1}^{n} a_i{}^2\right) + \frac{1}{n+1}\left[(n-1)\left[\left(\sum_{i=1}^{n} a^2 - 2a_1a_2\right)\right]\right]$$
$$= 2a_1a_2.$$

In a similar manner, $A < 2a_1a_2$ for $1 \le i \le j \le n$.

18) If positive numbers p, q, r such that $2p = q+r, q \neq r$ show that $\frac{p^{q-r}}{q^2r^2} < 1$.

Sol.: Suppose that q and r are positive integers, and consider the q number $\frac{1}{q}\ldots\ldots\frac{1}{q}$ and the r. By the arthmatic mean geometric mean equality.

$$\frac{1}{q^q}\cdot\frac{1}{r^r}^{1/(q+r)} < \frac{q^{(1/q)}+r^{(1/r)}}{q+r} = \frac{1}{p},$$

$which\ is\ equiva$lent to the desired inequality of course, this method breaks down if either q or r is not an integer, so how shall we proceed? One idea is to rewrite the inequality in the following manner:

$$p^{q+r} < q^{q_r r}, \left(\frac{q+r}{2}\right)^{q+r} < q^q r^r, \left(\frac{1}{2}\right)^{q+r} < \left(\frac{q}{q+r}\right)^q$$

$$\left(\frac{r}{q+r}\right)^r \cdot \frac{1}{2} < \left(\frac{q}{q+r}\right)^{q/(q+r)} \left(\frac{r}{q+r}\right)^{r/(q+r)}.$$

$$Set\ x = \frac{q}{(q+r)}\ and\ y = \frac{r}{(q+r)}.$$

$observe\ that\ x+y=1\ and\ 0<x,y<i.$ Then the problem is equivalent to proving that

$$F_{(x)} \equiv x^x(1-x)^{1-x} > \frac{1}{2}, 0<x<1, x \neq \frac{1}{2}.$$

By introducing the function in this way, we are able to use the methods of analysis. The idea is to find the minimum value of F on (0, 1). To simplify the differentiation, we will consider the function $G_{(x)} = \log F_{(x)}$. To find the critical points, we differentiate:

$$G'_{(x)} = \frac{d}{dx}[x\log x + (1-x)\log(1-x)] = (\log x + 1) - 1 - \log(1-x) = \log\frac{x}{1-x}.$$

$We\ see\ that\ G'_{(x)} = 0\ if\ and\ only\ if\ x = \frac{1}{2}.$ Furthermore, $G'_{(x)} < 0$ on the interval $\left(0, \frac{1}{2}\right)$ $and\ G'_{(x)} > 0$ on the interval $\left(\frac{1}{2}, 1\right)$. Therefore $G_{(x)}$ lakes its minimum value on $(0,1) at\ x = \frac{1}{2}$. Thus, the minimum value of $F_{(x)}$ on (0, 1) is $F_{\left(\frac{1}{2}\right)} = \left(\frac{1}{2}\right)^{1/2} = \frac{1}{2}$. It follows that $F_{(x)} > \frac{1}{2}$ for all x in (0, 1) $x \neq \frac{1}{2}$ and the proof is complete.

19) If $\boldsymbol{0 \leq x_i < \pi, i = 1, \dots, n}$ and set $\boldsymbol{x = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)}$. *Show that* $\boldsymbol{\prod_{i=1}^{n}\left(\frac{\sin x_i}{x}\right) \leq \left(\frac{\sin x}{x}\right)^n}$

Sol.: The problem is equivalent to proving that $\sum_{i=1}^{n} \log\frac{\sin x_i}{x_i} \leq n\log\frac{\sin x}{x}$ consider the function $f_{(t)} = \log\frac{\sin t}{t}$.

It is a straight forward matter to show that f is concave $\left({f_{(t)}}^n < 0\right)$ on the interval $(0, \pi)$

Therefore,

$$f\left(\frac{x_1+x_2}{2}\right) \geq \frac{f_{(x_1)} + f_{(x_2)}}{2}.$$

In a manner completely analogous to the proof it follows that

$$f\left(\frac{x_1+\cdots+x_n}{2}\right) \geq \frac{f_{(x_1)}+\cdots+f_{(x_n)}}{2}$$

Direct substitution into this inequality completes the proof.

$$\log\left(\frac{\sin x}{x}\right) \geq \frac{1}{n}\left(\log\frac{\sin x_1}{x_1} + \cdots + \log\frac{\sin x_n}{x_n}\right)$$

20) If a, b, c, d are non-negative numbers such that $\boldsymbol{a \leq 1, a+b \leq 5, a+b+c \leq 14, a+b+c+d \leq 30}$. Prove that $\boldsymbol{\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d} \leq 10}$.

Sol.: We will prove a more general statement.

If $a_1, a_2, \dots, a_n$ are positive, $0 \leq b_1 \leq b_2 \leq \cdots \leq b_n$ and for all $k \leq n$, $a_1 + a_2 + \cdots + a_k \leq b_1 + b_2 + \cdots + b_k$, then $\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n} \geq \sqrt{b_1} + \sqrt{b_2} + \cdots + \sqrt{b_n}$. The special case of the original problem is obtained for n = 4, by setting $b_k = k^2, k = 1, 2, 3, 4$. Let us prove the above result, we have $\frac{a_1}{\sqrt{b_1}} + \frac{a_2}{\sqrt{b_2}} + \cdots + \frac{a_n}{\sqrt{b_n}}$

$$= a_1\left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right)(a_1 + a_2)\left(\frac{1}{\sqrt{b_2}} - \frac{1}{\sqrt{b_3}}\right) + (a_1 + a_2 + a_3)\left(\frac{1}{\sqrt{b_3}} - \frac{1}{\sqrt{b_4}}\right) + \cdots + (a_1 + a_2 + \cdots + a_n)\frac{1}{\sqrt{b_n}}$$

The differences in the parenthesis are all positive. Using the hypothesis we obtain that this expression is less than or equal to

$$b_1\left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) - (b_1 + b_2)\left(\frac{1}{\sqrt{b_2}} - \frac{1}{\sqrt{b_3}}\right) + \cdots + (b_1 + b_2 + \cdots + b_n)\frac{1}{\sqrt{b_n}}$$

$$= \sqrt{b_1} + \sqrt{b_2} + \cdots + \sqrt{b_n}.$$

Therefore,

$$\frac{a_1}{\sqrt{b_1}} + \frac{a_2}{\sqrt{b_2}} + \cdots + \frac{a_n}{\sqrt{b_n}} \leq \sqrt{b_1} + \sqrt{b_2} + \cdots + \sqrt{b_n}$$

Using this result and the Cauchy-Schwarz inequality, we obtain. $\left(\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n}\right)^2$

$$= \left(4\sqrt{b_1}.\sqrt{\frac{a_1}{b_1}} + 4\sqrt{b_2}.\sqrt{\frac{a_2}{b_2}} + \cdots + 4\sqrt{b_n}.\sqrt{\frac{a_n}{b_n}}\right)^2 \leq \left(\sqrt{b_1} + \sqrt{b_2} + \cdots + \sqrt{b_n}\right)$$

$$\frac{a_1}{\sqrt{b_1}} + \frac{a_2}{\sqrt{b_2}} + \cdots + \frac{a_n}{\sqrt{b_n}} \leq \left(\sqrt{b_1} + \sqrt{b_2} + \cdots + \sqrt{b_n}\right)^2$$

This gives $\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n} \leq \sqrt{b_1} + \sqrt{b_2} + \cdots + \sqrt{b_n}$.

21) Suppose $\boldsymbol{a_1, a_2, \dots, a_n}$ be non-negative numbers such that $\boldsymbol{a_1 a_2 - a_k \leq \frac{1}{(2k)!}}$ for all k. Show that $\boldsymbol{a_1 + a_2 + \cdots a_n \geq \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}}$.

Sol.: We have

$$a_1 + a_2 + \cdots + a_n = \left(1 - \frac{1}{2}\right)(1.2a_1) + \left(\frac{1}{3} - \frac{1}{4}\right)(3.4a_2) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right)\left((2n-1).2na_n\right)$$

$$= \left(1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4}\right)(1.2a_1) + \left(\frac{1}{3} - \frac{1}{4} - \frac{1}{5} + \frac{1}{6}\right)$$

$$(1.2a_1 + 3.4a_2) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right)(1.2a_1 + 3.4a_2) + \cdots + (2n-1).2na_n$$

Using the AM-GM inequality and the hypothesis we obtain.

22) The numbers $\boldsymbol{a_1} \geq \boldsymbol{a_2} \ldots \geq \boldsymbol{a_n} > 0$ and $\boldsymbol{b_1} \geq \boldsymbol{b_2} \geq \cdots \geq \boldsymbol{b_n} > 0.$ $Satisfy$ $\boldsymbol{a_1} \geq \boldsymbol{b_1}, \boldsymbol{a_1} + \boldsymbol{a_2} \geq \boldsymbol{b_1} + \boldsymbol{b_2}, \boldsymbol{a_1} + \boldsymbol{a_2} + \cdots + \boldsymbol{a_n} \geq \boldsymbol{b_1} + \boldsymbol{b_2} + \cdots + \boldsymbol{b_n}$. Prove that

$\boldsymbol{a_1}^{\boldsymbol{k}} + \boldsymbol{a_2}^{\boldsymbol{k}} + \cdots + \boldsymbol{a_n}^{\boldsymbol{k}} \geq \boldsymbol{b_1}^{\boldsymbol{k}} + \boldsymbol{b_2}^{\boldsymbol{k}} + \cdots + \boldsymbol{b_n}^{\boldsymbol{k}}$ for every positive integer k.

Sol.: We can write

$$a_i^k - b_i^k (a_i - b_i)\left(a_i^{k-1} + a_i^{k-2} + \cdots + a_i b_i^{k-2} + b_i^{k-1}\right)$$

To simplify computations, set $c_i = a_i = b_i$ and $d_i = a_2^{k-1} + a_i^{k-2} b_i + \cdots + a_i b_i^{k-2} + b_i^{k-1}$.

The hypothesis implies $c_1 + c_2 + \cdots + c_j \geq 0$ for all j and $d_i > d_{i+1} > 0$ the latter since a_i and b_i are decreasing positive sequences.

Hence $a_1^k - b_1^k + a_2^k - b_2^k + \cdots + a_n^k - b_n^k = c_1 d_1 + c_2 d_2 + \cdots + c_n d_n = (d_1 - d_2)c_1 + (d_2 - d_3)(c_1 + c_2) + \cdots + d_n(c_1 + c_2 + \cdots + c_n) \geq 0$ the inequality is proved.

23) If $\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_n}$ and$\boldsymbol{y_1} \geq \boldsymbol{y_2} \geq \cdots \geq \boldsymbol{y_n}$ be two sequence of positive numbers such that $\boldsymbol{x_1} \geq \boldsymbol{y}$.

$\boldsymbol{x_1 x_2} \geq \boldsymbol{y_1 y_2} \ldots\ldots\ldots\ldots \boldsymbol{x_1 x_2} \ldots \boldsymbol{x_n} \geq \boldsymbol{y_1 y_2} \ldots\ldots \boldsymbol{y_n}$.

***Show that* $\boldsymbol{x_1} + \boldsymbol{x_2} + \cdots + \boldsymbol{x_n} \geq \boldsymbol{y_1} + \boldsymbol{y_2} + \cdots + \boldsymbol{y_n}$.**

Sol.: We want to reduce the inequalities involving products to inequality involving sums. For this we use the A.M. GM inequality we have

$$\frac{x_1}{y_1} + \frac{x_2}{y_2} + \cdots + \frac{x_k}{y_k} \geq K \sqrt[k]{\frac{x_1}{y_1} \frac{x_2}{y_2} \ldots\ldots.. \frac{x_k}{y_k}} \geq K,$$

$where\ the\ last\ inequality\ foilows\ from\ the\ hypothesis.$

Returning to the original inequality we have

$$x_1 + x_2 + \cdots + x_n = \frac{x_1}{y_1} y_1 + \frac{x_2}{y_2} y_n = \frac{x_1}{y_1}$$

$$(y_1 - y_2) + \left(\frac{x_1}{y_1} + \frac{x_2}{y_2}\right)(y_2 - y_3) + \cdots + \left(\frac{x_1}{y_1} + \frac{x_2}{y_2} + \cdots + \frac{x_n}{y_n}\right) In.$$

By using the inequality deduced at the beginning of the solution for the first factor in each term, we obtain that this expression is greater than or equal to

$$1 - (y_1 - y_2) + 2(y_2 - y_3) + \cdots + x y_n$$

$= y_1 + y_2 + \cdots + y_n$ and we and done.

24) Let $\{\boldsymbol{a_n}\}$ be a sequence of positive numbers such that for all n, $\sum_{k=1}^{n} \boldsymbol{a_k} \geq \sqrt{\boldsymbol{n}}$. *Show that* $\sum_{k=1}^{n} \boldsymbol{a^2}_{\boldsymbol{k}} \geq \frac{1}{4}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$ for all n.

Sol.: We start by proving another inequality, namely that if $a_1, a_2, \ldots., a_n$ are positive and

$b_1 \geq b_2 \geq \cdots \ldots\ldots b_n \geq 0$ and if for all $K \geq n, a_1 + a_2 + \cdots + a_k \geq b_1 \geq b_2 + \cdots + b_k$, than $a_1^2 + a_2^2 + \cdots + a_n^2 \geq b_1^2 + b_2^2 + \cdots + b_n^2$.

This inequality is the same as the one in problem 2 in the particular case where the exponent is 2, but with a weaker hypothesis using the Able summation formula, we can write

$a_1b_1 + a_2b_2 + \cdots + a_nb_n = a_1(b_1 - b_2) + (a_1 + a_2)(b_2 - b_3) + (a_1 + a_2 + a_3)(b_3 - b_4) + \cdots + (a_1 + a_2 + \cdots + a_n)\, b_n$ this inequalities in the statement show that this is greater than or equal to $b_1(b_1 - b_2) + (b_1 + b_2)(b_2 - b_3) + \cdots + (b_1 + b_2 + \cdots + b_n)b_n = {b_1}^2 + {b_2}^2 + \cdots + {b_n}^n$ combining this with the canchy Schwarz inequality we obtain

$({a_1}^2 + {a_2}^2 + \cdots + {a_n}^2)({b_1}^2 + {b_2}^2 + \cdots + {b_n}^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \geq ({b_1}^2 + {b_2}^2 + \cdots + {b_n}^2)^2$ and the proof is complete.

Returning to our problem, note first that $\sqrt{n} - \sqrt{n-1} > \frac{1}{2\sqrt{n}}$ Indeed, multiplying by the rational conjugate of the left side, this becomes

$$n(n-1) > \frac{\left(\sqrt{n} - \sqrt{n-1}\right)}{\left(2\sqrt{n}\right)}.$$

After eliminating the denominators and cancelling out terms, this becomes $\sqrt{n} - \sqrt{n-1}$.

The conclusion of the problem now follow from the inequality proved in the beginning by choosing $b_n = \sqrt{n} - \sqrt{n-1}$.

25) Suppose $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ be positive real numbers such that

(i) $\boldsymbol{x_1y_1 < x_2y_2 < \cdots < x_ny_n}$

(ii) $\boldsymbol{x_1 + x_2 + \cdots + x_k \geq y_1 + y_2 + \cdots + y_k \text{ where } 1 \leq k \leq n.}$

Sol.:

(a) Let $S_k = (x_1 - y_1) + (x_2 - y_2) + \cdots + (x_k - y_k) and$

$Z_k = \frac{1}{x_ky_k}$ Then we have $S_k \geq 0\ and\ z_k - z_{k+1} > 0\ for\ any\ K = 1, 2, \ldots, n-1.$
It follows that

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} - \frac{1}{y_1} - \frac{1}{y_2} \ldots \frac{1}{y_n}$$

$$= \left(\frac{1}{x_1} - \frac{1}{y_1}\right) + \left(\frac{1}{x_2} - \frac{1}{y_2}\right) + \cdots + \left(\frac{1}{x_n} - \frac{1}{y_n}\right)$$

$$= \frac{y_1 - x_1}{x_1y_1} + \frac{y_2 - x_2}{x_2y_2} + \cdots + \frac{y_n - x_n}{x_ny_n}$$

$$= S_1Z_1 - (S_2 - Z_2)Z_2 \ldots\ldots - (S_n - S_{n-1})Z_n = -S_1(Z_1 - Z_2) - S_2(Z_2 - Z_3) \ldots\ldots S_{n-1}(Z_{n-1} - Z_n)$$

$-S_nZ_n \leq 0$ with equality if and only if $S_k = 0, K = 1, 2, \ldots, n$ that is, when $x_k = y_k, K = 1, 2, \ldots, x.$

(b) We can assume without loss of generality that $a_1 < a_2 < \cdots < a_n$. From the hypothesis it follows that if or any partition of the set $\{a_1, a_2, \ldots, a_n\}$ into two subset since we can perform such a partition in 2^k ways it follows that $a_1 + a_2 + \cdots + a_k \geq 2^k$. we now apply (a) to the numbers $a_1 + a_2 + \cdots + a_n\ and\ 1.2.2^2 \ldots 2^{n-1}$.
(whose sum is $2^n - 1$) . It follows that $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \leq \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}$

26) If $\boldsymbol{0 = a_0 < a_1 < \cdots < a_n\ and\ a_{i+1} - a_i \leq 1\ for\ 0 \leq i \leq n-1}$ then show that $\boldsymbol{(\sum_{i=0}^{n} a_i)^2 \geq \sum_{i=0}^{n} {a_i}^3}$

Sol.: Try to prove that

$$\left(\sum_{i=0}^{n} a_i\right)^2 - \sum_{i=0}^{n} {a_i}^3 = 2\sum_{i=0}^{n}\sum_{j=0}^{n} a_i \,\frac{a_j + a_j - 1}{2}$$

$\left[1-\left(a_j-a_{j-1}\right)\right]$ we have equality if $a_j - a_{j-1} = 1$ for j = 1, …, n.

This gives the well known $(\sum_{i=0}^{n} i)^2 - \sum_{i=0}^{n} i^3$

27) Prove that, for any positive numbers $x_1, x_2, \dots, x_k (k \geq 4)$ $\frac{x_1}{x_k+x_2} + \frac{x_2}{x_1+x_3} + \cdots + \frac{x_1}{x_{k-1}+x_1} \geq 2$. Can you replace 2 by a greater number?

Sol.: Denote the L.H. S of the inequality by L_k. For K = 4, we have

$$L_4 \frac{x_1}{x_4+x_2} + \frac{x_2}{x_1+x_3} + \frac{x_3}{x_2+x_4} + \frac{x_4}{x_1+x_3} = \frac{x_1+x_3}{x_2+x_4} + \frac{x_2+x_4}{x_1+x_3} \geq 2$$

Now suppose that the proposed inequality is true for some $k \geq 4$. i.e., that $L_k \geq 2$. Consider k+1 arbitary positive numbers $x_1, x_2, \dots., x_k, x_{k+1}$. Since L_{k+1}, is symmetric with respect to these number without loss of generality , we may assume that $x_i \geq x_{k+1}\ for\ i = 1, \dots., k. Thus,$

$$L_{k+1} = \frac{x_1}{x_{k+1}+x_1} + \cdots + \frac{x_k}{x_{k-1}+x_{k-1}} + \frac{x_{k+1}}{x_k+x_1} > L_k \geq 2.$$

Now we prove 2 cannot be replaced by a larger number. Consider the case k = 2m, where m is a positive integer >1 set $x_1 = x_{2m} = 1, x_2 = x_{2m-1}, x_3 = x_{2m-2} = t^2, \dots., x_m = x_{m+1} = t^{m-1}$

Where t is an arbitrary positive number. Then L_k simplifies to $L_k = 2$.

$$1 + \frac{(m-2)t}{1+t^2}. Hence, lim \to \infty\ L_k = 2.$$

We can proceed similarly in the case $k = 2m + 1$.

28) If x, y, z be positive reals with $xy + yz + zx = 1$ prove that $\frac{2x(1-x^2)}{(1+x^2)^2} + \frac{2y(1-y^2)}{(1+y^2)^2} + \frac{2z(1-z^2)}{(1+z^2)^2} \leq$

$$\frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2}$$

Sol.: This reminds of the formula $\sin\alpha = 2\tan\left(\frac{\alpha}{2}\right)\left[1 + tan^2\ \left(\frac{\alpha}{2}\right)\right] and \cos\alpha = \frac{\left[1-tan^2\left(\frac{\alpha}{2}\right)\right]}{\left[1+tan^2\left(\frac{\alpha}{2}\right)\right]}$.

$$So\ let\ us\ set\ x = \tan\left(\frac{\alpha}{2}\right), y = \tan\left(\frac{\beta}{2}\right), z = \tan\left(\frac{\gamma}{2}\right)$$

The inequality now becomes, $\cos\alpha\sin\alpha + \cos\beta\sin\beta + \cos\gamma\sin\gamma \leq \frac{(\sin\alpha+\sin\beta+\sin\gamma)}{2}$

$\sin 2\alpha + \sin 2\beta + \sin 2\gamma \leq \sin\alpha + \sin\beta + \sin\gamma$ (1) until now we ignored $xy + yz + zx = 1$. It is satisfied if $\alpha+\beta+\gamma = \pi$. Indeed $z = \tan\left(\frac{\pi}{2} - \frac{\alpha}{2} - \frac{\beta}{2}\right) = \cot\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \frac{(1-xy)}{(x+y)}, and\ xy + yz + zx = xy + (x+y)z = xy + 1 - xy = 1.$

We may assume that in (1) we are dealing with the angles α, β, γ of a triangle. By the sine law, for the RHS we have

$$\sin\alpha + \sin\beta + \sin\gamma = \frac{a+b+c}{2R} = \frac{2S}{2R} = \frac{S_r}{R_r} = \frac{A}{r^R}$$

Denote the distance of the circumcentre M. from a, b, c by x, y, z. Then, for the LHS we get

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma$$
$$= 2(\sin\alpha\cos\alpha + \sin\beta\cos\beta + \sin\gamma\cos\gamma)$$
$$= \frac{a\cos\alpha + b\cos\beta + c\cos\gamma}{R}\ but$$

$$a\cos\alpha + b\cos\beta + c\cos\gamma = a.\frac{x}{R} + b.\frac{y}{R} + c.\frac{z}{R}$$
$$= \frac{2.4}{R}$$

$$Hence, \frac{\sin\alpha + \sin\beta + \sin\gamma}{\sin 2\alpha + \sin 2\beta + \sin 2\gamma} = \frac{R}{2_r} \geq 1.$$

29) Prove that for real numbers $x_1 \geq x_2 \geq \cdots \geq x_n > 0$.

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \leq \frac{x_2}{x_1} + \frac{x_3}{x_2} + \cdots + \frac{x_n}{x_{n-1}} + \frac{x_1}{x_n}.$$

Sol.: Transfer all terms to the left side and look at all terms with an x_n: $f_{(x)} = \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} - \frac{x_n}{x_{n-1}} - \frac{x_1}{x_n}$. Let us find the minimum of this function on the interval $[x_{n-1}, \infty]$. the derivation of $f_{(x_n)}$ on this interval is positive, and hence the minimum is attained at $x_n = x_{n-1}$. Inserting $x_n = x_{n-1}$ into the inequality, we get same inequality, but for variables $x_1\ to\ x_{n-1}$. We finish the proof by induction.

30) Let n > 2 and $x_1, x_2, \ldots, x_n$ be non negative reals. Prove that

$$(x_1 x_2, \ldots, x_n)^{\frac{1}{n}} + \frac{1}{n}\sum_{i<j}|x_i - x_j| \geq \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

Sol.: We may assume that $x_1 \geq x_2 \geq \cdots \geq x_n$. Then all the points $x_1, \ldots, x_n$ lie on the segment $[x_n, x_1]$. Hence $|x_i - x_j| \leq |x_n - x_1|$. In addition, $|x_1 - x_k| + |x_k - x_n| = x_1 - x_n$ for k = 2, …., n -1. Together with $|x_1 - x_n|$we get the estimate $\sum_{i>j}|x_i - x_j| \geq (n-1)(x_1 - x_n)$

$Since\ (x_1 \ldots x_n)^{\frac{1}{n}} \geq x_n$, it is sufficient to prove that $x_n + \frac{1}{n}(n-1)\left(x_1 - x_n \geq \frac{x_1 + \cdots + x_n}{n}\right)$

$or,\ x_n + (n-1)x_i \geq x_1 + \cdots + x_n$, which is valid. The proof of this weak inequality was so simple since. We could get by with huge over estimations.

SET THEORY

(OBJECTIVE TYPE)

1) **Which of the following has only one subset?**
   (a) { }; (b) {4}; (c) {0}; (d) none

   Sol.: Subsets of { }i.e. ϕ is ϕ. Subsets of {4} are ϕ, {4} subsets of {4, 5} are ϕ {4}, {5}, {4, 5}. Subsets of {0} are ϕ, {0}

   ∴ cannot answer is (a).

2) **If A = $\{x \in c;\ x^2 = 1\}$ *and* B = $\{x \in c;\ x^4 = 1\}$, then A Δ B =**
   (a) $\{-1, 1\}$; (b) $\{-i, i\}$; (c) $\{-1, 1, i, -i\}$ (d) none

Sol.: $x^2 = 1 \implies x = -1, 1$ $\therefore A = \{-1, 1\}$

$x^4 = 1 \implies x^2 = -1, 1 \Rightarrow x = -i, i, -1, 1$

$\therefore B = \{-i, i, -1, 1\}$

$\therefore A \,\Delta B = (A - B) \cup (B - A)$
$= \phi \cup \{-i, i\} = \{-i, i\}$

3) **If $\boldsymbol{n(A) = 3, n(B) = y, then\ n(A \times A \times B) =}$**
(a) 36; (b) 102; (c) 108; (d) none

Sol.: $n(A \times A \times B) = n(A) \times n(A) \times n(B) = 3 \times 3 \times 4 = 36$

4) **If u = R and let A = $\{\boldsymbol{x} \in \boldsymbol{R}: \boldsymbol{0} < x < 2\}, \boldsymbol{B} = \{\boldsymbol{x} \in \boldsymbol{R}: \boldsymbol{1} < x \leq 3\}$ which of the following is false?**
(a) $\boldsymbol{A \cap B} = \{\boldsymbol{x} \in \boldsymbol{R}: \boldsymbol{1} < x < 2\}$
(b) $\boldsymbol{A' = \{x \in R: x \leq 0\ or\ x \leq 2\}}$
(c) $\boldsymbol{A \cup B = \{x \in R: 0 \leq x \leq 3\}}$;
(d) *none*

Sol.: $A' = R - A = \{x \in R: x \leq 0\ or\ x \geq 2\}$

$B' = R - B = \{x \in R: x \leq 1\ or\ x > 3\}$

$A \ \cup B = \{x \in R: x \in A\ and\ x \ \in B\} = \{x \in R: 0 < x \leq 3\}$

5) **The set (A ∪ B ∩C) ∩$\boldsymbol{(A \cup B' \cup C')'nc'}$=**
(a) A ∩ C; (b) B’ ∩ C’; (c) B ∩ C’ (d) none

Sol.: (A ∪ B ∪C) ∩$(A \cup B' \cap C')' \cap C'$

$= (A \cup B \cup C) \cap (A' \cup B \cup C) \cap C'$

$= (\phi \cup B \cup C) \cap C' = (B \cup C) \cap C' = (B \cap C') \cup (C \cap C')$

$= (B \cap C') \cup \phi = B \cap C'$

6) **If $\boldsymbol{A = \left\{(x, y): y = \frac{1}{x}, 0 \neq x \in R\right\} and\ B = \{(x, y): y = -x, x \in R\}, then}$**
(a) A ∩B = $\boldsymbol{\phi}$ (b) A ∩B = A; (c) A∩ B = B; (d) none

Sol.: Here A and B can be shown as;

⇒A ∩ B = ϕ

7) **Let x be the universal set for sets A and B. If $\boldsymbol{n(A) = 200, n(B) = 300\ and\ n(A \cap B) = 100}$, then $\boldsymbol{n(A' \cap B') = 300}$ provides n(x)=**
(a) 500; (b) 600; (c) 700; (d) none

Sol.: We have $n(A \cap B) = n(A) + n(B) - n(A \cap B)$

$\therefore n(A \cup B) = 200 + 300 - 100 = 400$

$Also, n(A' \cup B') = n(A \cup B') = n(x) - n(A \cup B)$

$\therefore 300 = n(x) - 400\ or\ n(x) = 700.$

8) **If A ∪ B = A ∪ C and A ∩B = A ∩ C, then**
(a) B = C only when A ⊆ C
(b) B = C;
(c) B= C only when A ⊆ B
(d) Done

Sol.: Let x ∈ B ⟹ x ∈ A ∪ B ⟹ x ∈ A ∪ C

Case I: x ∈ A, x ∈ A ∩ B or x ∈ A ∩ C or x ∈ C, B⊆C

Case II: x ∈ C, x∈ B⟹ x ∈ C or B ⊆ C

Similarly C⊆ B

∴ B = C

9) If A $= \{\boldsymbol{\theta}: 2\, \boldsymbol{cos^2\theta} + \sin\boldsymbol{\theta} \le 2\}$ ***and***

$$\boldsymbol{B} = \left\{\boldsymbol{\theta}: \frac{\boldsymbol{\pi}}{2} \le \boldsymbol{\theta} \le \frac{3\boldsymbol{\pi}}{2}\right\}\ \boldsymbol{then}\ \boldsymbol{A} \cap \boldsymbol{B} =$$

(a) $\left\{\boldsymbol{\theta}: \boldsymbol{\pi} \le \boldsymbol{\theta} \le \frac{3\pi}{2}\right\}$;

(b) $\left\{\boldsymbol{\theta}: \frac{\pi}{2} \le \boldsymbol{\theta} \le \frac{5\pi}{6}\ \boldsymbol{or}\ \boldsymbol{\pi} \le\ \boldsymbol{\theta} \le \frac{3\pi}{2}\right\}$;

(c) $\left\{\boldsymbol{\theta}: \boldsymbol{\pi} \le \boldsymbol{\theta} \le \frac{5\pi}{6}\right\}$

(d) None

Sol.: Let $2\, cos^2\theta + \sin\theta \le 2\ and \frac{\pi}{2} \le \theta < \frac{3\pi}{2}$

$\Rightarrow 2 - 2\, sin^2\theta + \sin\theta \le 2 \Rightarrow 2\, sin^2\theta - \sin\theta \ge 0$

$$\Rightarrow \sin\theta\,(2\sin\theta - 1) \ge 0 \Rightarrow \frac{\pi}{2} \le \theta \le \frac{5\pi}{6}\ or\ \pi \le \theta \le \frac{3\pi}{6}$$

$$\therefore A \cap B = \left\{\theta: \frac{\pi}{2} \le \theta \le \frac{5\pi}{2}\ or\ \theta \le \pi \le \frac{3\pi}{2}\right\}$$

10) If A and B be two sets such that $\boldsymbol{n(A \times B) = 6}$.
Let three elements of $\boldsymbol{A} \times \boldsymbol{B}\ \boldsymbol{are}\ \boldsymbol{(3, 2)(7, 5)(8, 5)}$then

(a) A= {3, 7, 8}; (b) B = {2, 5}; (c) C = {3, 5}; (d) none

Sol.: Since (3, 2), (7, 5), (8, 5)∈ A × B, we have 3, 7, 8∈ A and 2, 5 ∈B.

Also $n(A \times B) = 6 = 3 \times 2$

A = {3, 7, 8} and B = {2, 5}

Since the graphs of $xy = 4$ and y = x, x > 0 intersect ay one point, we have $A \cap B = \phi$ and A ∩B is a single set.

11) A set contains n elements, then its power set

(a) n element; (b) $\boldsymbol{2^n}$ elements; (c) $\boldsymbol{n^n}$ elements (d) none

Sol.: As power set is set of all subsets, and we know number of subsets of a set containing n element is 2^n.

∴Power set contains $2^n\ elements$.

12) If A and B are sets, then $\boldsymbol{A} \cap \left(\frac{\boldsymbol{B}}{\boldsymbol{A}}\right)$is

(a) $\boldsymbol{\phi}$; (b) A; (c) B; (d) none

Sol.: Let x ∈ $A \cap \left(\frac{B}{A}\right) \Rightarrow x\ \in A\ and\ \left(x \in \frac{B}{A}\right)$

$\Rightarrow x\ \in A\ and\ (x \in B\ and\ x\ \notin A) \Rightarrow x \in \phi$

$$\therefore A\ \cap \left(\frac{B}{A}\right) \subset \phi\ \ldots\ldots\ldots\ldots..(1)$$

$$Since\ \phi \subset A \cap \left(\frac{B}{A}\right) \ldots\ldots\ldots\ldots(2)$$

$$\therefore From\ (1)\ and\ (2)\ A\ \cap \left(\frac{B}{A}\right) = \phi$$

13) Let R be set of points inside a rectangle of sides a and b (a, b>1) with two sides along the positive direction of x-axis and y-axis and C be the set of points inside a unit circle central at origin, then

(a) $\boldsymbol{R} = \{(\boldsymbol{x}, \boldsymbol{y}): \boldsymbol{0} \le \boldsymbol{x} \le \boldsymbol{a}, \boldsymbol{0} \le \boldsymbol{y} \le \boldsymbol{b}\}$;

(b) $\boldsymbol{R} = \{(\boldsymbol{x}, \boldsymbol{y}): \boldsymbol{0} < x < a, 0 < y < b\}$;

(c) $\boldsymbol{R} \cup \boldsymbol{C} = \boldsymbol{R}$;

(d) None

Sol.: Since, R denotes the set of points inside the rectangle of sides a and b for both a and b >1, then $R\{(x, y): 0 < x <$

$a, 0 < y < b\}$. Also C is the set of points inside the unit circle, centred at origin, such that $S = \{(x, y): x^2 + y^2 < 1\}$

14) Which of the following is not correct?
(a) $A \subseteq A^c$ if and only if A = ϕ;
(b) $A = B$ is equivalent to A ∪ C= B ∩ C and A ∩ C= B ∩ C;
(a) $A^c \subseteq A$ if and only if A = x, where x is a universal set.
(b) *none*

Sol.: A^c satisfies (A) and (B) by definition (D) also follows trivially.

Assuming A to be any set other than the empty set also $B = A \ and \ C = \phi, we \ have \ A \cup B = A - A \cup C \ But \ B \neq C, so$ (c) is incorrect.

15) Let S is the set of points inside the square. T is the set of points inside the triangle and C is the set of the points inside the circle. If the triangle and circle intersect each other and are contained in the square , then
(a) S∩ T ∩ C ≠ ϕ; (b) S ∪T ∪C =C; (c) S ∪T =S ∪C; (d) none

Sol.: Since, T ∩ C≠ ϕ and S∩ T ∩ C= T ∩ C so option (c) is true

Also T ⊂ S and C ⊂ S, So S ∪ T ∪ C = S

Also, S∪ T = S = S∪ C.

16) The set $(A \cap B^c)^c \cup (B \cap C) =$
(a) $A^c \cup B$; (b) $A^c \cup B \cup C$; (c) $A^c \cup B^c$; (d) none

Sol.: Let S = $(A \cap B^c)^c \cup (B \cap C)$

$\Rightarrow S = (A^c \cup B) \cup (B \cap C) \ (De \ Morgan's Law)$

$\Rightarrow S = A^c \cup \left(B \cup (B \cap C)\right) \therefore S = A^c \cup B$

17) Of the numbers of 3 teams in a college 21 are in the cricket team, 26 are in hockey team and 29 are in the football team. Among team, 14 play football and cricket. Eight play all the three games. The total number of members in the three teams is
(a) 43; (b) 49; (c) 64; (d) none.

Sol.: (a) Let C, H, F denote the sets members who are on the cricket, hokey and football team respectively.

∴ n(C) = 21, n(H) = 26, n(F) = 29, n(C ∩ B) = 14, n(C∩ F) = 15, n(F ∩ C) = 12 and n(C ∩ H ∩ F)= 8.

∴n(C ∪ H ∪ F) = n(C) + n(H)+ n(F) –n(C ∩ H) –n (H∩ F) –n (F ∩) +n(C∩ H ∩ F)

$= (21 + 26 + 29) - (14 + 15 + 12) + 9 = 43.$

(SUBJECTIVE TYPE)

1) If the collection of all these three elements subsets drawn from the set {1, 2, 3, …., 300}. Find the number of these subsets for which the sum of the three elements is a multiple of 3.

Sol.: For $0 \leq j \leq 2, let \ A_j$ denote the set of all integers between 1 and 300 which leave remainder j when divided by 3. Then $|A_j| = 100 \ for \ 0 \leq j \leq 2$. If a, b, c is a 3-element subset of the given set

$s' = 1, 2, \ldots, 300$ then 3 divides a+ b+ c if and only if

(i) All a, b, c are inA_0 or in $A_1 \ or \ in \ A_2$
(ii) One of the a, b, c is in A_0, another in A_1, and the third one in A_2.

The number of 3 element subsets of $A_1, 0 \leq j \leq 2$ is $\binom{100}{3}$. For each choice of a in A_0. b in A_1 and c in A_2.

We get a 3-element subset such that 3 divides a+ b+ c.

Thus the total number of 3-element subsets {a, b, c} such 3 divides a +b +c is equal to $3\binom{100}{3} + 100^3 = 1495100$

2) **How many 3-element subset of the set {1, 2, 3, ..., 20} are there such that the product of the three number in the subset is divided by 4?**

Sol.: We cannot the 3-element subset {a, b, c} such that 4 does not divide abc. This is possible if and only if either all the three are odd numbersor any two of them are odd and the other is an even number not divisible by 4. There are 10 odd numbers in the set {1, 2, 3, ...20} and 5 even numbers not divided by 4. Thus the numbers of 3-element subset {a, b, c} such that 4 does not divide abc is equal to $\binom{10}{3} + 5\binom{10}{3} = 345$. The number of 3 element subset is

$\binom{20}{3} = 1140$. Thus the number of 3 element subsets such that the product of these element is divisible by 4 is equal to $1140 - 345 = 795$.

3) **Suppose $A_1, A_2, \dots, A_6$ are six sets each with 4 elements and $B_1, B_2, \dots, B_n$ are n sets each two elements such that $A_1 \cup A_2 \cup \dots \cup A_6 = B_1 \cup B_2 \cup \dots \cup B_n = S\ (say)$. Given that each element of S belongs to exactly 4 of the $A_i{}'s$ and exactly 3 of the $B_j{}'s$ then find the value of n.**

Sol.: Since each A_i contains 4 elements, totally we get 24 elements of which some may be repeated. But each element is repeated 4 times as each element belongs to exactly 4 of the $A_i's$. Hence there are $\frac{24}{4} = 6\ distinct\ element\ in\ S$.

Since S $= B_1 \cup B_2 \cup \dots \cup B_n$ and each B_i consists of each element appears exactly 3 times. Thus the number of distinct elements in S is also equal to $\frac{2n}{3}$.

Therefore $\frac{2n}{3} = 6$hich gives n $= 9$.

4) **If A = {1, 2, 3, ..., 100} and B is a subset of A having 48 elements. Show that B has two distinct elements x and y whose sum is divisible by 11.**

Sol.: For each n, $0 \leq n \leq 10, let\ A_n\ denote\ the\ set\ of\ integer\ between\ 1\ to\ 100$ which leaves remainder n after division by 11. Then A_1 consists of 10 elements and A_n for n ≠1 consists of 9 elements each. If (a, b) is any two element subset of (1, 2, 3, ..., 100) the 11 divides a +b if and only if either both a and b are in A_0 or else a is A_k and b is in A_{11-k} for some $k, 1 \leq k \leq 10$.

Consider any set B with 48 elements. If B contains two element from the set A_0, then we are done. Similarly if B contains an element from A_k and another from $A_{11-k}, 1 \leq k \leq 10$ then again, their sum is divisible by 11. Thus B can contain one element from A_0, 10 from A_1 and 9 from the sets A_k for some 4 values of k (≠10), say k_1, k_2, k_3, k_4 no two of which add up to 11.

But these account only for 47 elements. Hence there must be an element which is either in A_{10} or in $A_{11-k}j$ $1 \leq j \leq 4$. Thus we can always find an element a in A_k *and b in* A_{11-k}. Here a, b are in B and 11 divides a +b.

5) **If A⊂ {1, 2, 3, …100}, |A| =50 such that no two numbers from A have their sum as 100 show that A contains a square.**

Sol.: If 100∈ A then we are done , so assume A ⊂ {1, 2, 3, …, 100} consider the two element subsets {1, 99}, {2, 98}, {3, 97}, …., {49, 51} along with the singleton set (50). These fifty sets are disjoint, and their un ion is the set {1, 2, 3, …., 99} and the sun of the two numbers in each of the two element set is 100. The hypotheses implies that A can contain at most one elements it has to contain exactly one element from each of the fifty sets. Since (36, 64) is one of the pairs and both 36 and 64 are squares we are done.

6) **Find the number of un ordered pairs (A, B) (i.e. the pair (A, B) and (b, A) are considered to the same) of subsets of an n-element set x which satisfy the condition (a) A ≠ b; (ii) A∪ B = x.**

Sol.: Suppose A has r elements, $0 \leq r \leq n$. Such an A can be choosen in $\binom{n}{r}$ ways. For each such A, the set B must necessarily have the remaining (n –r) elements and possible some elements of A. Thus there are $\sum_{r=0}^{n}\binom{n}{r}2^r = (1+2)^n = 3^n$ ways of choosing two sets A and B satisfying the given condition. Among these choices, only in one case A = B (=x), and in all other cases A ≠ B, since the order does not manner, we essentially have $\frac{(3^n-1)}{2}$ pairs.

7) **Let x be a set containing n elements. Find the number of all ordered triplets (A, B, C) of subsets of x such that A is a subset of B is a proper subset of C.**

Sol.: Let x be an n-element set and let B be a subset of x containing r elements. Thus there are $\binom{n}{r}$ choices for B, Hence there are 2^1 choices for A $2^{n-1} - 1$ choices for C. Thus we obtain the total number of triplets (A, B, C) such that A ⊂ B ⊂ C, but B ≠ C as $\sum_{r=0}^{n} 2^r \binom{n}{r}(2^{n-r} - 1)$ which simiplifies to $4^n - 3^n$.

Aliter: Let us denote by 0 or 1 the absence or presence of element of x in the sets A, B, C, for any fixed element of x, there are only four choices, namely, 000, 011, 111. Hence there are 3^n triplets (A, B, B). The number of triples (A, B, C) with A ⊂ B ⊂ C but B ≠ C is therefore $4^n - 3^n$

8) **Show that the number of 3-element subsets (a, b, c) of the set {1, 2, 3, …, 63} with a+ b+ c<95 is less than the number of those with a +b +c > 95.**

Sol.: Suppose that (a, b, c) is a subset of {1, 2, 3, …, 63} with a +b +c < 95. Then $(64-a, 64-b, 64-c)$ is a subset of {1, 2, 3, …, 95} with $(64-a, 64-b, 64-c)$ $= 192-(a+b+c) > 192 - 95 = 97$. Conversely, if (a, b, c) is a subset of {1, 2, 3, …, 63} with a +b +c>97. Then $(64-a, 64-b, 64-c)$ is such that $(64-a)+(64-b)+(64+c) = 192 - (a+b+c) < 95$. Thus there is one-one correspondence between 3-element subsets (a, b, c) with a +b +c <95 and those such that a +b +c > 97.

Hence the number of subsets with a +b +c < 95 is equal to that with a +b +c >97. Thus the set of 3-element subsets (a, b, c) with a+ b +c > 95 will contain those with a +b +c => 97 and a few more.

9) For which positive integral values of n can the set {1, 2, 3, …, 4n} be split into n disjoint 4-element subset {a, b, c, d} such that in each of these sets $\boldsymbol{a = \frac{(b+c+d)}{3}}$.

Sol.: Suppose {a, b, c, d} is a group in which $a = \frac{(b+c+d)}{3}$. Then a + b+ c +d = 4a. Hence if such an n-exists, then 4 divides 1+ 2 +…+ 4n. However this sum is $2n(4n+1)$.

Thus a necessary condition for existence of such a set is that n be even.

We show that this condition is also sufficient i.e. if n = 2k for some k, then it is possible to partition {1, 2, 3, …., 8k} into groups of 4 elements {a, b, c, d} such that a = $\frac{(b+c+d)}{3}$. To this end, divide {1, 2, 3, …, 8k} into groups of 8 integers such that each group contains 8 consecutive integers. If $\{a+1, a+2, a+3, \dots, a+8\}$ is one such set, we can divide this set into two 4 integers each as follows: $\{a+4, a+1, a+3, .0 \dots, a+8\}, \{a+5, a+2, a+6, \dots, a+7\}$.

The desired partition is obtained since $a+4 = \frac{a+1+a+3+a+8}{3}$,

$$a+5 = \frac{a+2+a+6+a+7}{3}$$

10) Find the number of ways to choose an ordered pair (a, b) of numbers from the set {1, 2, …., 10} such that $\boldsymbol{|a-b| \le 5}$.

Sol.: Let $A_1 = [(a,b)a, b \in \{1, 2, 3, \dots., 10\}, |a-b| = \{i\}, i = 0, 1, 2, 3, 4, 5. A_0 = \left\{\frac{(i,i)}{i} = 1, 2, 3, \dots, 10\right\} and$

$$|A_0| = 10, A_1 = \left\{\frac{(i, i+1)}{i} = 1, 2, 3, \dots, 9\right\} \cup \left\{\frac{(i, i-1)1}{i} = 1, 2, 3, \dots, 10\right\} and\ |A_1| = 9 + 9 = 18.$$

$$A_2 = \left\{\frac{(i, i+2)}{i} = 1, 2, 3, \dots, 8\right\} \cup \left\{\frac{(i, i-2)}{i} = 3, 4, \dots., 10\right\}\ and\ |A_2| = 8 + 8 = 16$$

$$A_3 = \left\{\frac{(i, i+3)}{i} = 1, 2, \dots, 7\right\} \cup \left\{\frac{(i, i-3)}{i} = 4, 5, \dots, 10\right\} and\ |A_3| = 7 + 7 = 14$$

$$A_4 = \left\{\frac{(i, i+4)}{i} = 1, 2, 3, \dots, 6\right\} \cup \left\{\frac{(i, i-4)}{i} = 5, 6, \dots, 10\right\} amd\ |A_4| = 6 + 6 = 12.$$

$$A_5 = \left\{\frac{(i, i+5)}{i} = 1, 2, \dots, 5\right\} \cup \left\{\frac{(i, i-5)}{i} = 6, 7, \dots, 10\right\} and\ |A_5| = = 5 + 5 = 10$$

∴the required set of pairs $(a, b) = \bigcup_{i=0}^{5} A_i$ and the number of such pairs, (which are disjoint)

$$= \left|\bigcup_{i=0}^{5} A_1\right| = \sum_{i=0}^{5} |A_i| = 10 + 18 + 16 + 14 + 12 + 10 = 80.$$

11) Identify the set S by the following information:

(i) S ∩ {3, 5, 8, 11} = {5, 8}
(ii) S ∪ {4, 5, 11, 13} = {4, 5, 7, 8, 11, 13}
(iii) {8, 13} ⊂ S
(iv) S ⊂ {5, 7, 8, 9, 11, 13}

Also show that no three of the condition suffices to identify S uniquely.

Sol.: From(i), 5, 8 ∈ S ……….(1) From (ii), 7, 8 ∈ S …… (2)From (iii), 8, 13∈ S ……….(3)

Therefore from eqns. (1) (2) and (3), we find that 5, 7, 8, 13 ∈ S. S ⊂ {5, 7, 8, 9, 11, 13} ……….(4) (given)

If at all S contain any others element other than those given in (4), it may be 9 or 11 or both.

But 9 ∉ S [∵ 9 ∈ S ∪{4, 5, 11, 13}= {4, 5, 7, 8, 11, 13}]

Again 11 ∉ S, for 11 ∉ S ∩{3, 5, 8, 11}= {5, 8}

∴ S = {5, 7, 8, 13}

If condition (i) is not given, then S is not unique as S may be {7, 8, 13} or {5, 7, 8, 13} or {5, 7, 8, 11, 13}. Similarly deleting any other data leads to more than one solution to S (verify)

12) Let x ⊂ {1, 2, 3, …, 99} and n(x) =10. Show that it is possible to choose two distinct non empty proper subsets y, z of x such that $\sum\left(\frac{y}{y} \in \boldsymbol{y}\right) = \sum\left(\frac{z}{z} \in \boldsymbol{z}\right)$.

Sol.: Since n(x) = 10, the number of non-empty, proper subsets of x is $2^{10} - 2 = 1022$.

The sum of the elements of the proper subsets of x can possibly range from 1 to $\sum_{i=1}^{9}(90 + i)$. That is 1 to (91+ 92+ … + 99) i.e. 1 to 855.

That is the 1022 subsets can have sums from 1 to 855. By pigeon-hole principle, at least two distinct subsets B and C will have the same sum.

(∵ there are 855 different sums, and so if we have more than 855 subsets then at least two of then have the same sum.) If B and C are not disjoint, then let

$X = B - (B\ \cap\ C) and\ Y = C - (B\ \cap C)$.

Clearly, X and Y are disjoint and non-empty and have the same sum of their elements.

Define S(A) = sum of the elements of A. We have B and C not necessarily disjoint such that S(B) = S(C).

Now, S(X)= S(B) –S(B∩ C), S(Y) = S(C) –S (B∩ C) but S(B) = S(C).

Hence, S(X)= S(Y)

Also X $\neq \phi$. For if x is empty, then B ⊂C which implies S(B)< S(C) (a contradiction). Thus x and y non empty and S(X)= S(Y).

13) A, B, C are the set of all the positive divisors of 10^{60}, 20^{50} and 30^{40} respectively. Find $n(A \cup B \cup C)$.

Sol.: Let n(A) = number of positive divisors of $10^{60} = 20^{60} \times 5^{60}$ is 61^2 n(B) = number of positive divisors of $20^5 = 2^{100} \times 5^{50}$ is 101×51 and n(C) = number of positive divisors of $30^{40} = 2^{40} \times 3^{40} \times 5^{40} = 41^3$.

The set of common factors of A and B will be of the form $2^m.5^n$ $where$ $0 \leq m \leq 60$ and $0 \leq n \leq 50. So, n(A \cap B) = 61 \times 51$.

Similarly, since the common factors of B and C and A are C are also of the from $2^m \times 5^n$, and in the former case $0 \leq m \leq 40, 0 \leq n \leq 40$, and in the latter case $0 \leq m \leq 40, 0 \leq n \leq 40$

$\therefore n(B \cap C) = 41^2$ also $n(A \cap C) = 41^2$ and $n(A \cap B \cap C) is$ $also$ 41^2.

$$\therefore n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

$61^2 + 101 \times 51 + 41^3 - 61 \times 51 - 41^2 - 41^2 + 41^2$

$= 61(61 - 51) + 41^2(41 - 1) + 101 \times 51$

$= 610 + 1681 \times 40 + 101 \times 51 = 73001.$

14) A student an vacation for a d days observed that

(i) It rained 7 times morning or afternoon.

(ii) When it rained in the afternoon, it was clear in the morning.

(iii) There were five clear afternoon and

(iv) There were 6 clear mornings. Find the value of d.

Sol.: Let the set of days in rained in the morning be M, and the set of days it rained in the afternoon be A_r. Then, clearly the set of days when there were clear morning is M'_r and the set of days when there were clear afternoon is A_r'

By condition (b), we get $M_r \cap A_r = \phi$, by (d), we get $M_r' = 6$ by $(c), we$ get $A_r' = 5, and$ by $(a), we$ get $M_r \cup A_r = 7, M_r$ and A_r are disjoint sets, and $n(M_r) = d - 6, n(A_r) = d - 5$

∴ Applying the principle of inclusion and exclusion we get

$n(M_r \cup A_r) = n(M_r) + n(A_r) - n(M_r \cap A_r)$

$\Rightarrow 7 = (d - 6) + (d - 5) - 0 \Rightarrow d = 18, \Rightarrow d = 9.$

15) It is proposed to partition the set of positive integers into two disjoint subsets A and B. Subject to the following conditions: (i) 1 is in A; (ii) No two distinct numbers of A have a sum of the from$2^k + 2(k = 0, 1, 2, \ldots.)$; (ii) No two distinct members of 3 have a sum of the form $2^k + 2(k = 0, 1, 2, \ldots.)$ Show that this

partitioning can be carried out in a unique manner and determine the subsets to which 1987, 1988, 1989, 1997, 1998 belong.

Sol.: Since it is given that 1 ∈ A, 2 ∉ A For if 2 ∈ A then $2^0 + 2 = 3$ is generated by 2 members of A violating the condition for the partitioning.

∴ 2 ∈B similarly, 3∉ A as $3 = 4 = 2^1 + 2 \therefore 3 \in B$.

But 4 ∉B. For if 4∈ B, then $2^2 + 2 = 4 + 2 = 6$ is generated by two members of B.

∴ The partitioning for the first few positive integers is

A = {1, 4, 7, 8, 12, 13, 15, 16, 20, 23, ….}

B = {2, 3, 5, 6, 9, 10, 11, 14, 17, 18, 19, 21, 22, ….}

Suppose 1, 2, …, n-1 (for $n \geq 3$) have already been assigned to A ∩ B in such a way that no two distinct members of A or B have a sum = $2^1 + 2(l = 0, 1, 2, \ldots \ldots)$ Now, we need to assign n to A or B.

Let k be a positive integer such that $2^{k-1} + 2 \leq n < 2^k + 2$. The assign 'n' to the complement of the set to which $2^k + 2 - n$ belongs. But for this, we need to check that $2^k + 2 - n$ has already been assigned. Now as $n \geq 2^{k-1} + 2 < 2^k + 1$. $2n > 2^k + 2 \therefore n > 2^k + 2 - n$.

Since all numbers below n have been assumed to be assigned to either A or , $2^k + 2 - n$ has already been assigned and hence n is also assigned uniquely. For example, consider k =1, $3 = 2^0 + 2 \leq n < 2^1 + 2 = 4. Consider\ n = 3, 4 - n = 1$ Now 1 ∈ A (given)

∴ 3 ∈ B consider k = 2.

∴ $2^{2-1} + 2 \leq n < 2^2 + 2 = 6, 4 \leq n < 6$.

When n = 4, as $6 - n = 2 \in B, we\ assign$ 4 to A. when n = 5, as $6 - 5 = 1 \in A$, we assign 5 to B. Since the set to when n gets assigned is uniquely determined by the set to which $2^k + 2 - n$ belongs, the partitioning is unique. Looking at the pattern of the partitioning of the initial set of positive integers, we conjecture the following:

(1) n ∈ A if $\frac{4}{n}$.
(2) n ∈B if $\frac{2}{n} but \frac{4}{n}$
(3) If n = $2^r.k + 1(r \geq 1, k\ odd, then\ n \in A$ if k is of the form $4m - 1$).

Proof the conjecture: we note that 1, 4∈ A and 2, 3 ∈B. $2^{k-1} + 2 \leq n < 2^k + 2$ and all numbers less than n have been assigned to A or B and satisfy the above conjectures, then if $\frac{4}{n}, as\ 2^k + 2 - n$ is divisible by 2 but not 4, $2^k + 2 - n \in B$. Hence n ∈ A. Similarly, if 2 divides n but not 4, then $2^k + 2 - n$ is divisible by 4 and hence, is in A

∴ n ∈ B. If n = $2^r.k + 1$

Where r >1, k is odd and k = $4m - 1, then\ 2^k + 2 - n = 2^k - 2^r.k + 1 = 2^r(2^{k-r} - k) + 1$, where clearly $2^{k-r} - k$ is odd and equals 1 (mod 4)

∴ $2^k + 2 - n \in B$

Hence, n ∈ A similarly, it can be shown that if n = $2^r.k + 1$ where k ≡1 (mod 4), then n ∈ B. Thus, the conjecture is proved.

Now, 1988 is divisible by 4.

∴ 1998 ∈ A.

$1987 = 2^1.993 + 1$ where 993= 1 (mod 4) $\therefore$ 1988 $\in B$

$1989 = 2^2.497 + 1$ where 497 = 1 (mod 4) $\therefore$ 1989 $\in B$.

$$\frac{2}{1998}\ but \frac{4}{1998}\quad \therefore 1998\ \in B.$$

$1997 =\ 2^2.449 + 1$ where 499= 3(mod 4)

$\therefore$ 1997 ∊ A

16) If A denote the subsets of the set {1, 11, 21, 31, ….541, 551} having the property that no two elements of A odd up to 552. Show that A cannot have more than 28 elements.

Sol.: Observe that S consists of 56 numbers in A.P. Hence first term is 1 and common difference is 10. The sum of every pair of numbers equidistant from the beginning and the end is 552. Also, the sum of no two others elements can be 552.

We divide S into 28 pairs: (1, 551), (11, 541), (21, 531) ………….. (271, 282), if A consists of at the most 28 elements, then it is possible to choose these elements in such a way that at most one element from a pair is in A. however if A contains 29 (or more) elements, then by the pigeon hole principle, A must contain both the elements of at least one pair, and therefore A contain two elements whose sum is 552.

Since A has the property that no two element of A add up to 552, therefore A cannot have more than 28 elements.

17) Show that in any set of 20 distinct integers chosen from the set {1, 4, 7, …, 100} there will always be two distinct integers whose sum is 104.

Sol.: There are 34 integers in A.P. 1, 4, 7, …, 100. Let us denote the set {1, 4, 7, …., 100} by s. Let us group them into 17 pairs.

(4, 100), (7, 97), (10, 94), …, (49, 55) and (1, 52). The sum of the integers in each of the first sixteen pairs is 104. The last pair consists of the two integers which cannot be paired with any other integer in the given A.P. So as to have the sum 104. It is obvious that the sum of two integers from the given A.P. can be 104 if and only if two integers both belong to some one of the first sixteen pairs written above.

Let us try to construct a subset of S which is as big as possible, and has the property that no two numbers of the set add up to 104. Such a set can have at the most 18 members, namely the two integers 1 and 100, and exactly one out of each of the remaining16 pairs.

The moment we odd one more member of S to it, it will have both the one of the sixteen pairs (4, 100) …. (49, 55) i.e. it will have two distinct integers whose sum is 104.

Therefore in any set of 20 (in fact 19!) distinct integers chosen forms, there will always be two distinct integers whose sum is 104.

18) Find all possible sets of consecutive positive integers such that the sum of the numbers in the set is 795 (e. g. the sets {30}, {9, 10, 11}, {4, 5, 6, 7, 8} and {6, 7, 8, 9} are the sets of consecutive positive integers with sum 30.)

Sol.: Suppose $n+1, n+2, \ldots, n+k$ is a set of k consecutive integers whose sum is 795 i.e. $(n+1)+(n+2)+\cdots+(n+k) = 795 \implies$ $\left\{kn + \frac{1}{2}k(k+1)\right\} = 795 \ldots\ldots..(1)$

We are required to find all solutions of (1) in integers.

Case I: If k is doubly even, say = 4m, then $4mn + 2n(4m + 1) = 795$.

Here L HS is even and RHS is odd, and consequently no solution is possible.

Case II: If k is singly even, say $= 4m + 2$, the (1) becomes $(4m + 2)n + (2m + 1)(4m + 3) = 795$.

So that 2m +1 must divide 795, i.e. 2m +1 = 1, 3, 5, 15, 53, 159, 265, 795. Also then $2n = \frac{795}{2m+1} - (4m + 3) > 0$ i.e. 2m+1 cannot have values other than 1, 3, 5, 15.

For these values, we have k = 4m +2 = 2, 6, 10, 30 and the corresponding values of n are $\frac{\left[795-\frac{k(k+1)}{2}\right]}{k}$ i.e. 396, 129, 74, 11.

Thus the sets are {397, 398}, {130, 131, …, 135}{75, 76, …, 84,}{12, …, 13, 41} …. (A)

Case III: If k is odd, say 2m +1, then (1) becomes $(2m + 1)n + (m + 1)(2m + 1) = 795$.

So that 795 must be divided by 2m +1. The possible values of 2m +1 are 1, 3, 5, 15, 53, 159, 265, 795 Further more $n = \frac{795}{(2m+1)} - (m + 1) > 0$, which gives n = 794, 263, 157, 45 respectively, when 2m +1 = 1, 3, 5, 15

The other values do not give positive values of n and therefore must be rejected. The corresponding sets are {795}, {264, 265, 266}, {158, 159, 160, 161, 162}, {46, 47, …., 60} …… (B) All the possible sets are those given in (A) and (B).

19) If the 7 element set A= {a, b, …, g}, find a collection T of 3- element subsets of A such that each pair of element from A occurs exactly in one of the subsets of T.

Sol.: If the 3-element subsets in the class are pairwise disjoint, then one of the subsets has number of elements and so at least one pair has one element in common but not two or more. If just one pair of subsets has common elements, they have to have two elements in common. So this is also not possible. Thus the problem reduces to finding a class of all 3-element subsets with precisely one element common between any two of the subsets. Clearly, {(a, b, d), (b, c, e), (c, d, f), (d, e, g), (e, f, a), (a, c, g), (b, f, g)}, in one such class. Any permutation of A will give another class.

20) Let S in the set {1, 2, 3, …, 10^6}. Show that for any subset A of S with 101 elements. We can find 100 distinct elements x_i of S, Such that the sets $x_i + A$ are all pair wise disjoint.

{Note that $x_i + A$ is the set $\left\{a + \frac{x_i}{a}\right\}$ *is in A*]

Sol.:

Having found $x_1, x_2, \dots, x_k$ there are k. 101 - 100 for –bidden values for x_{k+i} of the form $x_1 + a_m - a_n$ with m and n unequal and another k forbidden values with m= n.

Since 99.101.100+99=$10^6 - 1$.

We can successively choose 100 distinct x_i.

COMBINATORICS

(OBJECTIVE TYPE)

1) If S = 1.1!+2.2!+3.3!+...+n.n! then $\frac{1}{n!}(S+1)$ is

(a) Not integer; (b) integer; (c) undefined; (d) none

Sol.: We have S $= \sum_{k=1}^{n} k(k!) = \sum_{k=1}^{n}\{(k+1)-1\}(k!)$

$$= \sum_{k=1}^{n}\{(k+1)!-k!\} = (n+1)!-1 \Rightarrow S+1$$
$$= (n+1)!$$

Thus, $\frac{S+1}{n!} \in$ integer. Hence (b) is correct answer.

2) If 1! +2! +3!+...+n! cannot be the square of a natural number except for n =

(a) 1, 3; (b) 2, 3; (c) 3, 3; (d) none

Sol.: For n = 1, we have $S_1 = 1! = 1$, which is a perfect square.

For n = 2, we have $S_2 = 1! + 2 = 1 + 2 + 3$, which is not a perfect square.

For n = 3, We have $S_3 = 1! + 2! + 3! = 1 + 2 + 6 = 9$, *which* is a perfect square.

For n = 4, we have $S_4 = 1! + 2! + 3! + 4! = 1 + 2 + 6, \ldots, +24 = 33$ which is not a perfect square.

For $n \geq 5$, *we find that the digits* at units place in n! is 0and $S_4 = 1! + 2! + 3! + 4!$ has 3 as the digit ay units place. Therefore for $n \geq 5$, S_n has 3 at units place. Therefore S_n is not a perfect square for $n \geq 5$.

Hence, $S_n = 1! + 2! + 3! + \cdots + n!$ is not a perfect square of a natural number except for n = 1, 3. Hence (a) is correct answer.

3) The value of

$$\left(7_{C_0} + 7_{C_1}\right) + \left(7_{C_1} + 7_{C_2}\right) + \cdots + \left(7_{C_6} + 7_7\right) \textit{is}$$

(a) $2^8 - 2$; (b) $2^8 - 3$; (c) 2^8; (d) none

Sol.: $\left(7_{C_0} + 7_{C_1}\right) + \left(7_{C_1} + 7_{C_2}\right) + \cdots + \left(7_{C_6} + 7_7\right) = 8_{C_1} + 8_{C_2} + \cdots + 8_{C_7} = 8_{C_0} + 8_{C_1} + 8_{C_2} + \cdots + 8_{C_7} + 8_{C_8} - \left(8_{C_0} + 8_{C_8}\right) = 2^8 - 1(1+1) = 2^8 - 2$

Hence (a) is the correct answer.

4) The value of n for which

$$n - 1_{C_4} - n - 1_{C_3} - \frac{5}{4}, n - 2_{P_2} < 0.$$

where n ∈ N is

(a) $(-\infty, 1) \cup (3, 10)$; (b) (0, 2) ∪(3, 10); (c) {1, 2, 3}; (d) none

Sol.: we have

$$n - 1_{C_4} - n - 1_{C_3} - \frac{5}{4}.n - 2_{P_2} < 0$$

$$\Rightarrow \frac{(n-1)(n-2)(n-3)(n-4)}{4!} - \frac{(n-1)(n-2)(n-3)}{3!} - \frac{5}{4}(n-2)(n-3) < 0$$

$$\Rightarrow \frac{(n-2)(n-3)}{24}\{(n-1)(n-4) - 4(n-1) - 30\} < 0$$

$$\Rightarrow (n-2)(n-3)(x^2 - 9x - 22) < 0$$

$$\Rightarrow (n-2)(n-3)(n-11)(n+2) < 0$$

$$\Rightarrow (n-2)(n-3)(n-11) < 0$$

$[\because n+2>0\ for\ n\ \in N]$

$\Rightarrow n\ \in(-\infty,2)\cup(3,11)\Rightarrow n\ \in(0,2)\cup(3,11)$

$\Rightarrow n=1,4,5,6,7,8,9,10$

But $n-1_{C_4}\ and\ n-1_{P_2}$ both are meaningful for $n\geq 5$.

Hence, n = 5, 6, 7, 8, 9, 10.

5) (n!)! is divisible by
(a) $(\boldsymbol{n}!)^{\boldsymbol{n}!}$; (b) $(\boldsymbol{n}!)^{\boldsymbol{n}!-\boldsymbol{1}}$; (c) $(\boldsymbol{n}!)^{(\boldsymbol{n}-\boldsymbol{1})!}$ (d) none

Sol.: Clearly. (n!) is the product of natural numbers from 1 to n!

$$\therefore (n!)!=\{1\times 2\times 3\ldots xn\}\times\{(n+1)(n+2)\ldots(2n)\}\times$$

$$\vdots \quad \vdots \quad \vdots$$

$$\{(2n+1)(2n+2)\ldots.(3n)\}\times\{(n!-n+1)(n!-n+2)(n!-n+3)\ldots.n!\}\ldots\ldots\ldots..(1)$$

We observe that: Last term of the first bracket on RHS of (1) in n

Last term of second bracket RHS of (1)is 2n.

Last term of third bracket on RHS of (1) is 3.n and so on

Last term of the last bracket on RHS of (1) is (n−1)! n

It is clear from this that there are $(n-1)!$ brackets on the RHS of (1) and each bracket there is product of n consecutive natural numbers. From (1) we have,

$$(n!)!=\prod_{r=1}^{(n-1)!}[(r-1)n+1\{(r-1)n+3\}-\{(r-1)n+n\}]$$

We know that the product of n consecutive natural number is divisible by n!

So, let $\{(r-1)n+1\}\{(r-1)n+2\}\ldots\{(r-1)n+n\}=n!\ I_r, r=1,2,\ldots,(n-1)!$

$$\therefore (n!)!=\prod_{r=1}^{(n-1)!}n!\,I_r=\ (n!)^{(n-1)!}$$

$$\therefore (n!)!=\prod_{r=1}^{(n-1)!}n!\,l_r=\ (n!)^{(n-1)!}.\prod_{r=1}^{(n-1)!}l_r$$
$$=\ (n!)^{(n-1)!}\times a\ natural\ No.$$

Thus (n!)! is divisible by $(n!)^{(n-1)!}$

6) Number of positive integer n <17, for which n!(n+ 1)!+ (n+ 2)! Is an integral multiple of 49 is
(a) 5; (b) 6; (c) 7; (d) none

Sol.: Here n! + (n +1)!+(n+2)! = n! (1+(n+1)+(n+2) (n+1)) = n! $(n+2)^2\Rightarrow$ either 7 divides (n+2) or 49 divides n!

i.e., n = 5, 12, 14, 15, 16 (as n< 17). Thus the number of solution if five

7) Number of ordered triplets (x, y, z) such that x, y, z are primes and $\boldsymbol{x^y+1=z}$ is
(a) 0, (b) 1, (c) 2, (d) none

Sol.: Here, $x^y + 1 = z$, where x, y, z are prime.

Thus, y cannot be odd, as if y is prime

$\Rightarrow x^y + 1$ is divisible by (x+1). Now, z must be odd

$\Rightarrow$ x must be even (as, $x^y = z - 1$).

Thus only even i.e. prime is x = 2

$\Rightarrow$ x = 2, y = 2, z = 5

So, there is only one such triplet (2, 2, 5)

Hence (b) is correct answer.

8) If $n_1 = x_0x_2x_3$ and $n_2 = y_1y_2y_3$ be two 3-digit numbers, then the pairs n_1 and n_2 can be formed, so that n_1 can be subtracted from n_2 without borrowing is
(a) 45.55; (b) $55.(45)^2$; (c) $55^2.45^2$; (d) none

Sol.: Here, $n_1 = x_1x_2x_3$ and $n_2 = y_1y_2y_3$

$\Rightarrow n_1$ can be subtracted from n_2 without borrowing if $y_i \geq x_i$ for $i = 1, 2, 3$.

$\therefore$ Let $x_1 = r \Rightarrow \begin{cases} r = 0, 1, 2, \ldots, 9 & for\ x_2\ or\ x_3 \\ r = 1, 2, 3, \ldots, 9 & for\ x_1 \end{cases}$

$\therefore y_i = r, r+1, \ldots, 9$. Thus for y_1, y_2 and y_3 we $have$ $(10 - r)$ choices, each
$\Rightarrow$ Total number of ways for choosing y_i and x_i

$= \{\sum_{r=1}^{9}(10 - r)\}\{\sum_{r=0}^{9}(10 - r)\}.\{\sum_{r=0}^{9}(10 - r)\} = 45.55.55 = 45.55^2$

9) $\sum_{0\leq i}\sum_{\leq j} 10_{C_j} . j_{C_i} =$
(a) $3^{10} - 1$; (b) $2^{10} - 1$; (c) $3^{10} - 2^{10}$; (d) none

Sol.: $\sum_{0\leq i}\sum_{\leq j} 10_{C_j} . j_{C_i} = 10_{C_1}(1_{C_0} + 1_{C_1}) + 10_{C_2}(2_{C_0} + 2_{C_1} + 2_{C_2})10_{C_3}(3_{C_0} + 3_{C_1} + 3_{C_2} + 3_{C_3}) + \cdots + 10_{C_{10}}(10_{C_0} + 10_{C_1} + 10_{C_2} + \cdots + 10_{C_{10}})$

$= 10_{C_1}.2 + 10_{C_2}.2^2 + 10_{C_3}.2^3 + \cdots + 10_{C_{10}}.2^{10}$

$= (1+2)^{10} - 1 = 3^{10} - 1$

10) If f(n) denotes the number of different ways the position integer 'n' can be expressed as the sum of $1's$ and $2's$. For example $f_{(4)} = 5$, since 4 = 1+ 1+ 1 +1, 1 +1 +2, 1+ 2+ 1, 2 +1 +1, 2 +2 note that order of $1's$ and $2's$ is important then $f(f_{(6)})$=
(a) $f_{(6)}$; (b) $f_{(10)}$; (c) $f_{(13)}$; (d) none

Sol.: As: $f_{(4)} = 5$ given

$\therefore f_{(6)}$ can be written using $1's$ and $2's$ as

Number of 1's	Number of 2'S	No. of arrangement
0	3	$\frac{3!}{3!} = 1$
2	2	$\frac{4!}{2!\,2!} = 6$
4	1	$\frac{5!}{4!} = 5$
6	0	$\frac{6!}{6!} = 1$
		Total = 13

$\therefore f_{(6)} = 13$

$\therefore f\left(f_{(6)}\right) = f_{(13)}$

11) The number of ways of choosing triplets (x, y, z) such that $z \geq$ $\mathbf{max}\{\boldsymbol{x}, \boldsymbol{y}\}$ $\boldsymbol{and}\ \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \{\mathbf{1}, \mathbf{2}, \dots, \boldsymbol{n}, \boldsymbol{n}+\mathbf{1}\}$ is
(a) $\boldsymbol{n}+\mathbf{1}_{\boldsymbol{C}_3}+\boldsymbol{n}+\mathbf{2}_{\boldsymbol{C}_3}$; (b) $\boldsymbol{n}+\mathbf{1}_{\boldsymbol{C}_2}+\boldsymbol{n}+\mathbf{2}_{\boldsymbol{C}_3}$; (c) n (n+1); (d) none

Sol.: When z = n +1, we can choose x, y from {1, 2, …, n}.

Thus when z = n +1, x, y can be chosen n^2. When z = n, x, y can be chosen in$(n-1)^2$ ways and so on.

Thus, there are $n^2+(n-1)^2+\cdots+1^2 = \frac{1}{6}n(n+1)(2n+1)$ ways of choosing the triplets.

Alternatively, triplets with x = y < z, x < y <z, y < z < x can be chosen in $n+1_{C_2}, n+1_{C_3}, n+1_{C_3}$ *ways*

$\therefore$ There are $n+1_{C_2}+2\left(n+1_{C_3}\right) =$ $n+2_{C_3}+n+1_{C_3}$.

12) The number of ordered pairs (m, n), m, n ∈ {1, 2, …., 100} such that $\mathbf{7^m}+\mathbf{7^n}$ is divisible by 5 is
(a) 1000; (b) 12000; (c) 3000, (d) none

Sol.: Note that $7^r (r \in N)$ ends in 7, 9, 3, or 1 (corresponding to r = 1, 2, 3 and 4 respectively). Thus $7^m +$ 7^n *cannot end in* 5 *for any values of* $m.n \in N$. In other words, for 7^m+7^n *to* be divisible by 5, it should end in 0.

For 7^m+7^n to end in 0, the forms of m and n should be as follows:

	m	n
1	4r	4s+2
2	4r +1	4s +3
3	4r +2	4s
4	4r +3	4s +1

Thus for a given value of m there are just 25 values of n for which 7^m+7^n ends in 0. For instance, if m = 4r, then = 2, 6, 10, 98.

$\therefore$ There are 100 × 25= 2500 ordered pairs (m, n) for which 7^m+7^n is divisible by 5.

13) A 7-digit number is divisible by 9 is to be formed by using 7 out of numbers {1, 2, 3, 4, 5, 6, 7, 8, 9}. The number of ways in which this can be done is
(a) 4. 7! ; (b) 3, 7!; (c) 2. 7!; (d) none

Sol.: Sum of 7 digits = a multiple of 9. We know, sum of numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 is 45. So, two left number should also have sum as 9.

The pairs to be left are (1, 8)(2, 7), (3, 6)(4, 5) which each pair left number of 7 –digit number is 7! So, with all 4 pairs = 4 × 7!

14) The number of ways af arranging m members out of 1, 2, 3, …, n so that maximum is (n –2) and minimum is 2 (repetitions of number is allowed) such that maximum and minimum both occur exactly once ($\boldsymbol{n}>5, m>3$) is
(a) $(\boldsymbol{n}-\mathbf{1}_{\boldsymbol{C}_{m-2}})^{\mathbf{2}}$; (b) $\boldsymbol{m}(\boldsymbol{m}-\mathbf{1})(\boldsymbol{n}-\mathbf{5})^{\boldsymbol{m}-\mathbf{2}}$ (c) $\boldsymbol{n}_{\boldsymbol{C}_2}.\boldsymbol{n}_{\boldsymbol{C}_m}$; (d) none

Sol.: First we take one number as 2 and one as $(n-2)$ and put them in m $(m-1)$ ways.. Now remaining $(m-2)$ numbers can be any one from, 3, 4, …., (n-4), (n-3),

Which we can do in $(n-5)^{(m-2)}$.

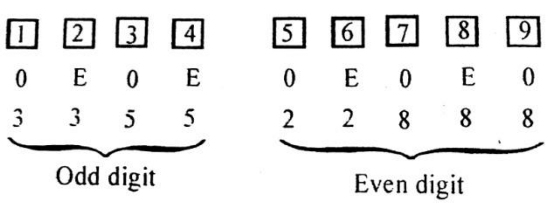
$\therefore$ Total number of ways $= m(m-1)(n-5)^{(m-2)}$.

15) The number of rational numbers lying in the interval (2002, 2003) all whose digits other the decimal point are non- zero and are in decreasing order is
(a) $2^{10}-1$; (b) 2^9-1 ; (c) $2^{10}-2$; (d) none

Sol.: A rational number of the desired category is of the form 2002. $x_1, x_2, \dots, x_k$ where $1 \le k \le 9$ and $9 \ge x_1 > x_2 > \cdots > x_k \ge 1$. We can choose k digits out in 9_{C_k} ways and arrange them is decreasing order in just one way. Thus, the desired number of rational number is $9_{C_1} + 9_{C_2} + \cdots + 9_{C_9} = 2^9 - 1$.

16) How many different 9 digit numbers can be formed from the number 22, 33, 55, 888 by rearranging its digits so that the odd digits occupy even positions?
(a) 16; (b) 32; (c) 64; (d) none

Sol.:



Out of 4 odd digits, 4 even places can be occupied in n_{C_4} ways.

$\therefore$ Total number of ways $= \left(\frac{4_{C_4}\,4!}{2!2!}\right)\left(\frac{5_{C_5}\,5!}{3!2!}\right)$

17) The sum of all possible numbers greater than 10^4 formed by using the digits from {1, 3, 5, 7, 9} is
(a) 6, 66, 66, 600; (b) 6, 66, 600; (c) 6, 66, 660; (d) none

Sol.:

If 1 were at units place (i.e. 5)then the remaining first 4 places (1- 4)can be filled in 4! Ways.

$\therefore$ sum of all 4! (1)= 24 similarly for 3, 5, 7 and 9 filled in units place the rest four places can be filled in 4! Ways in each case

$\therefore$ If sum of all digits in units place is S.

$\Longrightarrow$ S = 4! (1+3+5+7+9)$\Longrightarrow$ S = 4! × 25 = 600.

Similarly, sum of all digits in ten places, hundred's place, thousands place, ten thousands place and hundred thousands place in 600 in all cases.

18) The number of ways of arranging letters AAAAABBBCCCDEEF in a row if the letters C are Separated from one another
(a) 95135040; (b) 95135039; (c) 95135041; (d) none

Sol.: $\underbrace{A\ A\ A\ A\ A}_{5}\ \underbrace{B\ B\ B}_{3}\ \underbrace{C\ C\ C}_{3}\ \underbrace{E\ E}_{2}\ \underbrace{F}_{1}$

Number of ways of arranging AAAABBDEEF are $\frac{12!}{5!3!2!}$

Now, there are 13 places in between or on the sides of 12 characters, and since we want to separate all the c's this can be done in placing these c's in these 13 places This can be done in 13_{C_3} ways.

$\therefore$ Total number of ways $= \frac{12!}{5!3!2!} \times 13_{C_3}$

$\therefore$ Total number of ways $= 95135040$.

19) The number of rectangles in the following fig is
(a) 5×5; (b) $5^2 \times 3^3$; (c) $5_{C_2} \times 5_{C_2}$; (d) none

Sol.: Since, there are 5 horizontal lines and 5 vertical lines, and each choice of a pair of horizontal lines and a pair of vertical lines gives us a rectangle. Hence the no. of rectangles $= 5_{C_2} \times 5_{C_2}$

20) From a company of 15 soldiers any 4 are placed on guard, each both to catch for 4 hour. For what length of time (in hour) can different batches be selected?
(a) 5460; (b) 5410; (c) 54090; (d) none

Sol.: The number of ways in which 4 soldiers can be selected out of 15 are the number of ways batches can be formed.

Now 4 soldiers can be selected out of 15 in 15_{C_4} ways $= 1365$. Again, if can batch has two watch for 4 hours, then 1365 batches will watch for $1365 \times 4 = 5460$.

21) A parallelogram is cut by two sets of m lines parallel to its sides. The number of parallelograms thus formed is
(a) m_{C_2}; (b) $(m_{C_2})^2$; (c) $(m + 2_{C_2})^2$; (d) none

Sol.: Each set is having $(m + 2)$ parallel lines and each parallelogram is formed by choosing two straight lines from the first set and two straight lines from the second set.

Two straight line from the first set can be chosen in $m + 2_{C_2}$ ways and two strainght lines from the second set can be chosen in $m + 2_{C_2}$ ways.

Hence, the total number of parallelograms formed $= m + 2_{C_2} . m + 2_{C_2} = \left(m + 2_{C_2}\right)^2$

22) If n dice from an even number is 189, then n =
(a) 3; (b) 4; (c) 8; (d) none

Sol.: Number of all possible outcomes, on all the n-sides $\equiv 6^n$. Now, for any one dice odd number on it can occur in 3_{C_1} ways.

So, for all the dice, an odd number can occur in 3^n ways

$$\begin{bmatrix} No.\ of\ ways\ in\ which\ at\ least\ one\ of\ the\ dice \\ shows\ an\ even\ number \end{bmatrix}$$

$$[No.\ of\ all\ possible\ outcomes\ on\ n$$

$$.\ dices][every\ dice\ shown\ o\ odd\ number]$$

$\Longrightarrow$ Required ways $= 6^n - 3^n = 189\ (given)$.

$$By\ Hit\ and\ Trial\ the\ equality\ is\ true\ for\ n = 3.$$

23) If a, b, c, d, e are primes, the number of divisions of ac^2de is
(a) 73; (b) 72; (c) 71; (d) none

Sol.: Let N $= ab^2c^2de$

Where a, b, c, d and e are prime out of one factor a, we can have either one or none.

∴Number of possible divisors of a = (1+1) = 2, for b, there are two factors, we can have either one, two or none

∴ Number of possible divisors of b = (2+1) = 3

Similarly, the number of possible factors of c, d and e is (2+1), (1+1) and (1+1) respectively.

Hence, the number of all possible divisors of $ab^2c^2de\ are\ (1+1)(2+1)(2+1)(1+1)(1+1)$.

Now, there exists only one possible worst case in which the factors is $a^0b^0c^0d^0e^0 = 1$ which a factor of every number. ∴ Number of ways = 72 -1 = 71.

24) If 3dices are thrown together, then the number of ways in which the sum of the number of ways in which the sum of the numbers appearing on the dice is n, $9 \leq n \leq 14$ is
(a) $-n^2+21n-83$; (b)$(-n^2-21n-83$; (c) $-n^2+21n+83$ (d) none

Sol.: If $9 \leq n \leq 14, then\ 6 \leq k-3 \leq 11$. Thus the coeff. of $x^{n-3}\ in\ (1-x^6)^3\ (1-x)^3 = coeff.of\ x^{n-3}$ in

$\left(3_{C_0} - 3_{C_{1}x^6} + 3_{C_{2}x^{12}} - 3_{C_{3}x^{18}}\right) \times (1-x)^{-3} = 3_{C_0}$

$Coeff.of\ x^{n-3}\ in\ (1-x)^{-3} - 3_{C_1}\ coeff.of\ x^{n-9}\ in\ (1-x)^{-3} = 3_{C_0} \times n-3+3-1_{C_{3-1}} - 3_{C_1} \times n-9+3-1_{C_{3-1}}$

$= n-1_{C_2} - 3 \times n-7_{C_2} = 21n - n^2 - 83 = -n^2+21n-83$

25) The number of ways of choosing 10 balls from infinite white, red , blue and green balls is
(a) 286; (b) 295; (c) 312; (d) none.

Sol.: Required ways = $\{coefficient\ of\ x^{10}\ in\ (1+x+x^2+\cdots)^4\}$

$\Rightarrow$ coefficient of $x^{10}\ in\ \left(\frac{1}{1-x}\right)^4 \Rightarrow$ coefficient of x^{10} in $(1-x)^{-4} \Rightarrow$ $coefficient\ of\ x^{10}$ in

$$\left(1+4x+\frac{5.4}{2!}x^2+\frac{4.5.6}{3!}x^3+7_{C_{4}x^4}+8_{C_{5}x^5} + 9_{C_{6}x^6}+\cdots+13_{C_{10}x^{10}}\right)$$

∴ Required ways= $13_{C_{10}} = \frac{13.12.11}{3.2.1} = 286$

26) In how many ways can 6 coins be chosen from 20 one rupee coins, 10 fifty paise coins, 7 twenty paise coins?
(a) 37_{P_6}; (b) 37_{C_6}; (c) $37_{P_{10}}$; (d) none

Sol.: Since, the distribution equation is x + y + z = 6, where x, y and z represents one rupee, fifty paise and twenty paise coins respectively.

∴Number of ways of choosing r things out of n things = $n+r-1_{C_r}$.

Where everything occur any number of time.

∴Required ways = $3+6-1_{C_6} = 8_{C_2} =$ 28.

27) The number of non-negative solution of $x_1+x_2+x_3+\cdots+x_n \leq n$.(where n is possible integer) is
(a) $2n_{C_{n-1}}$; (b) $2n_{C_{n-1}} - 1$; (c) $2n_{P_n} - 1$; (d) none

Sol.: In general, we know that, for the distribution equation $x_1 + x_2 + x_3 + \cdots + x_n = n$. The number of ways in which n things can be distributed among r in such a ways end can receive none, one or more or all of n items are $n + r - 1_{C_{r-1}}$.

$\therefore$ for the distribution equation

$x_1 + x_2 + x_3 + \cdots + x_n \leq n$. Let required ways = w

$\Rightarrow W = \{No. of\ ways\ of\ distributing\ 1\ item\}\{No. of\ ways\ of\ distributing\ 2\ item\}$

$\{No. of\ ways\ of\ distributing\ in\ item\}$

$\Rightarrow W = 1 + n - 1_{C_{n-1}} + 2 + n - 1_{C_{n-1}} + \cdots + n + n - 1_{C_{n-1}}$

$\Rightarrow W = n_{C_{n-1}} + n + 1_{C_{n-1}} + \cdots + 2n - 1_{C_{n-1}}$

$\Rightarrow W = \left(n_{C_{n-1}} + n_{C_n}\right) + n + 1_{C_{n-1}} + \cdots + 2n - 1_{C_{n-1}}$

$\Rightarrow W = \left\{\left(n + 1_{C_n} + n - 1_{C_{n-1}}\right) + \cdots + 2n - 1_{C_{n-1}}\right\} - n_{C_n}$

… … … … … … … … … … … … … … … … … … … …

… …

$\Rightarrow W = \left(2n_{C_n} + 2n - 1_{C_{n-1}}\right) - n_{C_n} \Rightarrow W = 2n_{C_n} - n_{C_n}$

$\therefore W = 2n_{C_{n-1}} - 1.$

28) For $2 \leq r \leq n, \binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} =$
(a) $\binom{n+2}{r}$; (b) $\binom{n}{r}$; (c) $\binom{n+1}{r+1}$; (d) none

Sol.: Let n be the number of newspaper which are read

$\Rightarrow$60n = (300)(5)

$\therefore$ n = 25

29) In how many ways the letters of the word PERSON can be placed in the squares of the adjoining fig. So that no row remains empty?
(a) 81; (b) 18720; (c) 18721; (d) none

Sol.: In PERSON total letters = 6 which are to be filled in 8 squares.

6 number of ways of choosing 6 letters to fill in 8 squares= $8_{C_6} - 2 = 28 - 2 = 26$

Required ways = $26 \times 6! = 18720$.

30) The number of ways of arranging 5 players to through the cricket ball so that the youngest way no thrown first is
(a) 97; (b) 98; (c) 99; (d) none.

Sol.: Keeping the youngest player aside, one of four players can throw the cricket ball at first place in 4_{C_1} ways.

Now the three players (not able to throw) the ball in first placed and 1 youngest player i.e. 4 can arrange themselves in 4! Ways to thrown the ball.

$\therefore$ Required ways = $4_{C_1}\ 4! = 96$.

31) The total number of ways in which a bigger can be given at least one rupee from four 25 paise coins three 50 paise coins and 2 one rupee coin is
(a) 55; (b) 54; (c) 53; (d) none

Sol.: (b)

32) In how many ways can 4 prizes be distributed in a class of 20 students when each student is eligible for all prizes?
(a) 1600; (b) 16000; (c) 160000; (d) none

Sol.: The first prize can be given in 20 ways.

The next prize can be given in 20 ways

The next prize can be given in 20 ways.

The last prize can be given in 20 ways.

Total number of ways all the four prizes can be given is $20 \times 20 \times 20 \times 20 = 160000$.

33) The number of ways in which 4 particular persons A, B, C, D and 6 move persons can stand in a queue. So that A always stand before B. B stand C and C before D is
(a) 10! 4!; (b) 10! -4!; (c) $\frac{10!}{4!}$; (d) none

Sol.: Total number of arrangements of 10 persons when there is no restriction = 10!

Number of ways in which A, B, C, D can be arranged among themselves = 4!

∴ Number of arrangements of 10 persons when A, B, C, D occurs in a particular order = $\frac{10!}{4!}$

34) A father with 8 children taken 3 at a time to Nicco Park, as often as he can without talking the same children together more than once. How often will be father go?
(a) 56; (b) 106 ; (c) 206; (d) none.

Sol.: [The number of times he can select 3 children out of 8]= [The number of visits he (the father) can make]

∴ The number of ways of selecting 3 children out of 8 = $8_{C_3} = 56$.

35) In a steamer there are stalls for 12 animals and there are horse cows and calves (not less than 12 each) ready to be shipped. In how many ways can the ship load be made?
(a) 3^{12}; (b) $3^{12} - 12$; (c) $3^{12} + 1$; (d) none

Sol.: First stall can be filled in 3 ways, 2nd stall in 3 ways and so on. Similarly, 12th stall in 3 ways.

∴ Number of ways of loading steamer is $3 \times 3 \times 3 \ldots\ldots \times 3(12\ times) = 3^{12}$.

(SUBJECTIVE TYPE)

1) Evaluate: $\sum_{r=0}^{\infty} \frac{n_{C_r}}{2n-1_{C_r}}$

Sol.: A general method of finding the sum of a series $\sum_{r=0}^{\infty} u_r$ is to express $u_r\ and\ v_r - v_{r+1}$ so that $v_n \to 0\ as\ n \to \infty, we\ get.$

$$\operatorname*{Lt}_{n\to\infty} \sum_{r=0}^{\infty} u_r = \operatorname*{Lt}_{n\to\infty} \sum_{r=0}^{\infty} (v_r - v_{r+1})$$

$$= \operatorname*{Lt}_{n\to\infty} (v_0 - v_{r+1}) = v_0.$$

In the present case, it is easily verified that

$$\frac{n_{C_r}}{2n_{C_r}} - \frac{n_{C_{r+1}}}{2n_{C_{r+1}}} = \frac{\llcorner n}{\llcorner r} \frac{\llcorner r}{\llcorner n-r} \frac{\llcorner 2n-r}{\llcorner 2n} - \frac{\llcorner n}{\llcorner r+1} \frac{\llcorner r+1}{\llcorner n-r-1}$$

$$\frac{\llcorner 2n-r-1}{\llcorner 2n}$$

$$= \frac{\llcorner n}{\llcorner 2n}\left[\frac{\llcorner 2n-r}{\llcorner n-r} \frac{\llcorner 2n-r-1}{\llcorner n-r-1}\right]$$

$$= \frac{\llcorner n}{\llcorner 2n} \frac{\llcorner 2n-r-1}{\llcorner n-r}$$

$[(2n-r)-(n-r)]$

$$= \frac{1}{2} \frac{\llcorner n}{\llcorner 2n-1} \frac{\llcorner 2n-r-1}{\llcorner n-r}$$

$$= \frac{1}{2} \frac{n_{C_r}}{2n-1_{C_r}} \sum_{r=0}^{\infty} \frac{n_{C_r}}{2n-1_{C_r}}$$

$$= 2 \sum_{r=0}^{\infty} \left[\frac{n_{C_r}}{2n_{C_r}} - \frac{n_{C_{r+1}}}{2n_{C_{r+1}}} \right]$$

$$= 2 \frac{n_{C_0}}{2n_{C_0}} = 2$$

2) **Given that the number C is greater than 1, show that one of the two number $\sqrt{c+1} - \sqrt{C}, \sqrt{C} - \sqrt{C-1}$ is always greater than the number.**

Sol.: In fact $\sqrt{c+1} - \sqrt{c}, \sqrt{c} - \sqrt{c-1}$

To show this we have to show that $\sqrt{c+1} + \sqrt{c-1} \leq 2\sqrt{c}$

$or\ 2c + 2\sqrt{c^2 - 1} \leq 4c\ or \sqrt{c^2 - 1} \leq c$

$which\ is\ true.$

Alternatively, consider the parabola $y = \sqrt{x}$. and $\sqrt{x}$ is a concave function.

3) **Show that the product of 2n consecutive negative integers is divisible by (2n)!**

Sol.: Let r be a natural number, Then $-r, -r-1, -r-2, \dots, -r-(2n-1)$ are an consecutive negative integers.

Let P be their product. Then, $P = (-r)(-r-1)(-r-2) \dots . \big(-r - (2n-1)\big)$

$$= (-1)^{2n}\, r(r+1)(r+2) \dots . (r + 2n - 1)$$

$$= r(r+1)(r+2) \dots . (r + 2n - 1)$$

$$= \frac{(r+1)!\ r(r+1)(r+2) \dots . (r+2n-1)}{(r-1)!}$$

$$= \frac{(r+2n-1)!}{(r-1)!}$$

$$= \left\{ \frac{(2n+r-1)!}{(2n)!\,(r-1)!} \right\} (2n)!$$

$$= (2n)!\, 2n + r - 1_{C_{2n}}$$

$$= (2n)!$$

(A natural number). Hence, P is divisible by (2n)!

4) **How many 3-digit numbers are of the from abc, with a, c < b and a ≠ 0?**

Sol.: Since the digit at hundred's place cannot be zero

Therefore, we must have $a \geq 1$.

But it is given that a, c < b. Therefore $b \geq 2$.

⟹b = 2, 3, 4, 5, …., 9

Let b = r, where r = 2, 3, ….., 9. Then a can take $(r-1)$ values 1, 2, …., r-1. Thus for each value of r, abc can take r(r−1) values. But, r can take value from 2 to 9. Therefore, by the fundamental principle of addition. Required number of numbers

$$= \sum_{r=2}^{9} r(r-1) = \sum_{r=2}^{9} r^2 - \sum_{r=2}^{9} r$$

$$= \sum_{r=1}^{9} r^2 - \sum_{r=1}^{9} r$$

$$= \frac{9(9+1)(9 \times 2 + 1)}{6} - 9(9+1).$$

5) **There are two sets of parallel lines, their equations $x \cos \alpha + y \sin \alpha = P; P = 1, 2, \dots, m\ and$ $y \cos \alpha - x \sin \alpha = q; q = 1, 2, \dots, n(n > m)$ where α is a given const. Show that the lines from $\frac{1}{6} m(m-1)(3n - m - 1)$ squares.**

Sol.: The equation $x\cos\alpha + y\sin\alpha = p; p = 1, 2, 3, \dots, m$ represents m parallel lines such that the distance between two consecutive lines is one unit. Similarly the equation $y\cos\alpha - x\sin\alpha = q; q = 1, 2, \dots, n$ represents n parallel lines such that the distance between any two consecutive lines is one unit.

We observe that the slope of each line of first set is $m_1 = -\cot\alpha$ and the slope of each line of second set is $m_2 = \tan\alpha$.

Clearly, $m_1 m_2 = -1$

Therefore every line of first set is perpendicular to every line of second set.

We observe that four lines consisting of two lines of the first set and two lines of the second set will form a square, if the distance between two parallel lines of first set is same as the distance between two parallel lines of second set.

Since m < n therefore the length of the side of the largest square formed by the two sets of lines is $(m-1)$ units and the length of the side of the smallest square is 1 unit. Clearly, two lines at a unit distance from the set of m parallel lines can be chosen in $(m-1)$ ways, namely (1, 2) (2, 3) (3, 4)……., (m-1, m) and two lines at a unit distance from the set of n parallel lines can be chosen in $(m-1)$ ways, namely (1, 2)(2, 3), …., (n-1, n). Therefore number of squares whose sides are of length 1 unit (m-1)(n-1).

Similarly two lines at a distance of 2 units from the set of m parallel lines can be chosen in $(m-2)$ ways, namely (1, 3) (2, 4), …, (m-2, m) and two lines at a distance of 2 units from the set of n parallel lines can be chosen in (n - 2) ways, namely (1, 3)(2, 4), ….., (n -2, n)

Therefore, number of squares whose sides are of length 2 units = $(m-2)(n-2)$ containing in this, manner, we find that the number of squares whose sides are of length 2 units = $\big(m-(m-1)\big)\big(n-(m-1)\big)$

Hence, Total number of squares

$= (m-1)(n-1) + (m-2)(n-2) + \cdots + \big(m-(m-1)\big)\big(n-(m-1)\big)$

$$= \sum_{r=1}^{m-1} (m-r)(n-r)$$

$$= \sum_{r=1}^{m-1} \{mn - r(m+n) + r^2\}$$

$$= mn(m-1) - (m+n)$$

$$\sum_{r=1}^{m-1} r + \sum_{r=1}^{m-1} r^2$$

$$= mn(m-1) - (m+n)\frac{m(m-1)}{2} + \frac{(m-1)m(2m-1)}{6}$$

$$= \frac{m(m-1)}{6}\{6n - 3(m+n) + (2m-1)\}$$

$= \frac{m(m-1)}{6}\{6n - 3m - 3n + 2m + 1\} = \frac{m(m-1)(3n-m-1)}{6}$.

6) **There are n straight lines in a plane such that $\boldsymbol{n_1}$ are parallel in different direction, $\boldsymbol{n_2}$ are parallel in different direction and so on, $\boldsymbol{n_k\ are}$ parallel in another direction such that $\boldsymbol{n_1 + n_2 + \cdots n_2 = n}$. Also no three of the given lines meet a point. Show that the total number of inter section is $\boldsymbol{\frac{1}{2}\left(n^2 - \sum_{r=1}^{k} {n_r}^2\right)}$.**

Sol.: If no two of n given lines are parallel and no three of them meet at a point, then the total number of points of intersection is n_{C_2}. But it is given that there are k sets of $n_1, n_2, n_3, \dots, n_k$ parallel lines such that no line in one set is parallel to a lines in any other set. Also lines of one set do not intersect with each other.

Therefore, lines of one set do not provide any points of intersection. Hence, total number of points of intersection

$$= n_{C_2} - \left(n_{1_{C_2}} + n_{2_{C_2}} + \cdots + n_{k_{C_2}} +\right)$$

$$= \frac{n(n-1)}{2} - \left\{\frac{n_1(n_1-1)}{2} + \frac{n_2(n_2-1)}{2} + \cdots + \frac{n_k(n_k-1)}{2}\right\}$$

$$= \frac{n(n-1)}{2} - \frac{1}{2}\{(n_1{}^2 + n_2{}^2 + \cdots + n_k{}^2) - (n_1 + n_2 + \cdots + n_k)\}$$

$$= \frac{n(n-1)}{2} - \frac{1}{2}\{(n_1{}^2 + n_2{}^2 + \cdots + n_k{}^2) - n\}$$

$$= \frac{n_2}{2} - \frac{1}{2}\,(n_1{}^2 + n_2{}^2 + \cdots + n_k{}^2)$$

$$= \frac{1}{2}\left[n^2 - \sum_{r=1}^{k} n_r{}^2\right]$$

7) **There are 15 seats in as row numbered as 1 to 15. In how many ways can 4 persons sit in such a way that seat number 6 is always occupied and no two person sit in adjacent seats.**

Sol.: Since seat number '6' is always occupied and no two persons can occupy adjacent seats. Therefore , at most two persons can sit on the left side of sixth seat.

Thus, we have the following cases for the selection of seats.

Case I: When two seats are selected on the right side sixth seat and one seat on its right side;

Since no two adjacent seats are selected, so we can select either 1st and 3rd or 2nd and 4th or 1st and 4th seats. So, there are 3 ways to select 2 seat on the left side of sixth seat one the right side of sixth seat there are 9 seats. Therefore, one seat (excluding 7th seat) on the right side of sixth seat can be chosen in 8 ways.

Case II: When two seats are selected on the right side of sixth seat and one seat on its left side.

In this case, one seats are selected on the right side of sixth seat and one seat on its left side.

In this case, one seat on the left side of sixth seat can be chosen in 4 ways (any one of the first four seats) and the number of ways of selecting two seats on the right side of sixth seat is same the number of non-negative integral solutions of the equation $x_1 + x_2 + x_3 = 7$, where $x_1 \geq 1, x_2 \geq 1$ and $x_3 \geq 0$.

Here, x_1 is the number of vacant seats between sixth seat and the first seat selected on the right side of sixth seat, x_2 is the number of vacant seats between sixth seat and the first seat selected on the right side of sixth seat, x_2 is the number of vacant seats between first and second seat selected on the right side of sixth seat and x_3 is the number of vacant seats on the right side of the second selected seat.

Let $y_1 = x_1 - 1, y_2 = x_2 - 1$ and $y_3 = x_3$. Then $x_1 + x_2 + x_3 = 7$

$$\Rightarrow y_1 + y_2 + y_3 = 5, \text{ where } y_1, y_2, y_3 \geq 0$$

Total number of integral solutions of this equation is $5 + 3 - 1_{C_{3-1}} = 7_{C_2} = 21$.

Thus, the number of ways in which two seats can be chosen on the right side of sixth seat = 21. Hence total number of selection of seats in this case $4 \times 21 = 84$.

Case III: When all the three persons sit on the right side of sixth seat:

Let x_1 be the number of vacant seats between 6th seat and first seat selected on the right of 6th seat, x_2 be the number of vacant seats between first selected seat and the second selected seat, x_3 be the number of seats between second and third selected seat and x_1 be the numbers of vacant seats on the right side of fourth selected seat. Then, the number of ways of selecting 3 seats on the right side of sixth seat is equal to the number of the integral solution of the equation $x_1 + x_2 + x_{32} + x_4 = 6$, where $x_1 \geq 1, x_2 \geq 1, x_3 \geq 1, x_4 \geq 0$. Let $z_1 = x_1 - 1, z_2 = x_2 - 1, z_3 = x_3 - 1$ and $z_4 = x_4$. Then, we have $z_1 + z_2 + z_3 + z_4 = 3, where\ z_i \geq 0; i = 1, 2, 3, 4$

Total number of solution of this equation is

$3 + 4 - 1_{C_{4-1}} = 6_{C_3} = 20$

Thus the number of ways of selecting 3 seats on the right side of 6th seat = 20.

Hence total number of ways of selection of 4 seats 24 +84+ 20= 128.

But, corresponding to each way of selection of 4 seats there are 4! Arrangements of 4 persons. Hence total number of seating arrangement =128× 4! = 3072.

8) In the given figure, you have the road plane of a city. A man standing at x wants to reach the house at y by the shortest path. What is the number of different paths that he can take?

Sol.: As the man wants to travel by one many possible shortest paths, he will never turn up the trim down words. So a travel by one of the shortest path is to take a horizontal pieces and 4 vertical pieces of roads. As he cannot take a right turn, he will use only one of the five horizontal pieces in the same vertical column. Similarly same horizontal row.

∴A shortest path is an arrangement of eight things

$L_1, L_2, L_3, L_4, U_1, U_2, U_3, U_4$. So that the order of L_5 and U_5 do not change.

(∵ clearly L_2 cannot be taken without talking L_1, L_2 can not taken without taking U_1 etc.)

Hence, the number of shortest path = the number of arrangements of $L_1, L_2, L_3, L_4, U_1, U_2, U_3, U_4$ where the order of L_5 as well as the order of U_5 do not change = the number of arrangements treating L_5 $identical\ and$ U_5 as identical $= \frac{8!}{4!4!} = 70$.

9) Find the number of permutations $(\boldsymbol{P_1}, \boldsymbol{P_2}, \boldsymbol{P_3}, \boldsymbol{P_4}, \boldsymbol{P_5})\boldsymbol{of}$ $\mathbf{1, 2, 3, 4, 5, 6}$ such that for any k, $\mathbf{1} \leq \boldsymbol{k} \leq \mathbf{5}$, $(\boldsymbol{P_1}, \boldsymbol{P_2}, \ldots, \boldsymbol{P_k})$ does not from a permutation of 1, 2, …., k.

So.: Let T_n is the required number of permutations. If k is the least positive integer such that $(P_1, P_2, \ldots, P_k)$ is a permutations of 1, 2, 3, …, k.

Now are desire of count the number of permutation for k = n. Now,

$$\sum_{k=1}^{n} T_k\,.\,(n-k)! = n!$$

$\Longrightarrow T_n = n! - T_1.(n-1)! - T_2.(n-2)! - \cdots - T_{n-1}.1!$

$Clearly, T_1 = 1;\ T_2 = 2! - T_1.1! = 1, T_3 = 3; T_4 = 13;\ T_5 = 71\ and\ T_6 = 461.$

10) Find the best and the greatest value of $\sum_{j=1}^{n}\sum_{i=1}^{n}|x_i - x_j|$; $\boldsymbol{where\ 0} < x < 1\ \forall\ 1 \le i \le n; 1 \le i \le j \le n.$

Sol.: For last value: ∵It is possible to have $x_i = x_j;\ i = j = 1, 2, 3, \dots, x$

$$\therefore S_{min} = 0$$

For greatest value: without loss of generality, we can assume that $0 \le x_1 \le x_2 \le \cdots \le\ x_n \le 1$ (supposing the equality sign for 0 and 1 also).

Then $\mathbb{S} = (x_2 - x_1) + [(x_3 - x_1) + (x_3 - x_2)] + [(x_4 - x_1)] + [(x_4 - x_2) + (x_4 - x_3)] + \cdots + [(x_4 - x_1) + (x_4 - x_2) + \cdots + x_x - (x_x - 1)]$

Case I: If n = 2m,

$\mathbb{S}= \sum_{K=1}^{2m}(2k - 2m - 1)\,x_k$

If k= 1, 2, …, m; then coefficients are negative.

∴To maximize S', We chose $x_1 =\ x_2 = \cdots = x_n = 0$

If $k = m+1, m+2, \dots, 2m$ then coefficients are positive.

∴ To maximize $S' \sim$, we choose $x_{m+1} = x_{m+2} = \cdots \sim=\ x_{2m} = 1$

$$\therefore S_{max} = 1 + 3 + \cdots + (2m-1) =\ m^2 = \frac{n^2}{4}$$

Case II: If $n = 2m+1\ \ S =\ \sum_{K=1}^{(2m+1)}(2k - 2m - 1)\,x_k$

Now we choose $x_1 = x_2 = \cdots = x_m = 0, x_{m+1}$ can take any$x_{n+2} = \cdots = x_{2m+1} = 1$

$$\therefore S_{m+x} = 2 + 4 + \cdots + 2m = m(m+1) = \frac{(n^2-1)}{4}$$

Combining, $S'_{max} = \left[\frac{n^2}{4}\right]$.

By actually, $x_i\ \neq 0$ & $x_i = 1\ \forall\ i = 1, 2, \dots \dots \dots \dots, n$

∴S'_{max} is not possible.

Even if you show S= $\sum_{k=1}^{2m}(2k - n - 1)\,x_k$

and mentions than to get S_{max} we have to put $x_i, S = 0$ and some x_i,

$S = 1\ which\ is\ not\ possible$. He will get full credit.

11) If $(\boldsymbol{1+x})^{\boldsymbol{n}}\sum_{\boldsymbol{r=0}}^{\boldsymbol{n}}\boldsymbol{n_{C_r}}\,.\,\boldsymbol{x^r}$= then, show that

	Number of times occurring with negative sign	Number of times with positive sign	Total
x_1 x_2 ⋮ ⋮ x_n	$(n-1)$ $(n-2)$ ⋮ ⋮ $(n-k)$	0 1 ⋮ ⋮ (k- 1)	$-(n-1)x_1$ $-(n-3)x_2$ ⋮ ⋮ $(2k-n-1)x_k$

$$\left\{\sum_{r=1}^{\infty}(-1)^{r-1}.c_{2r-2}\right\}^2+\left\{\sum_{r=1}^{\infty}(-1)^{r-1}.c_{2r-1}\right\}^2=\sum_{r=0}^{n}c_r$$

Sol.: Let

$$\left\{\sum_{r=1}^{\infty}(-1)^{r-1}.c_{2r-2}\right\}^2=(c_0-c_2+c_4\ldots.)^2\ldots\ldots\ldots.(i)and$$

$$\left\{\sum_{r=1}^{\infty}(-1)^{r-1}.c_{2r-1}\right\}^2=(c_1-c_3+c_5\ldots.)^2\ldots\ldots(ii)$$

$Given: (1+x)^n=c_0+c_1x+c_2x^2+\cdots+c_nx^4\ldots\ldots.(iii)$

$Put\ x=i\ in\ (iii): (1+i)^n=c_0+c_1i-c_2-c_3i+c_4\ldots\ldots\ldots=(c_0-c_2+c_4\ldots.)+i(c_1-c_3+c_5\ldots.)\ldots\ldots\ldots\ldots(iv)$

$Put\ x=i\ in\ (iii): (1-i)^n=c_0-c_1i-c_2+ic_3+c_4\ldots\ldots\ldots=(c_0-c_2+c_4\ldots.)+i(c_1-c_3+c_5\ldots.)\ldots..(v)$

$(iv)\times(v)=\{(1+i)(1-i)\}^n=(c_0-c_2+c_4\ldots.)^2+(c_1-c_3+c_5\ldots.)^2$

$\Rightarrow 2^n=(c_0-c_2+c_4\ldots.)^2+(c_1-c_3+c_5\ldots.)^2,$

$\Rightarrow \sum_{r=0}^{n}c_r=(c_0-c_2+c_4\ldots.)^2+(c_1-c_3+c_5\ldots.)^2$

12) Prove that $\sum_{k=0}^{9}x^k\ divides\ \sum_{k=0}^{9}x^{kkk}$

Sol.: Let

$$B=\sum_{k=0}^{9}x^{kkk}=x^0+x^{1111}+x^{2222}+\cdots+x^{9999}\ and\ A=\sum_{k=0}^{9}x^k=x^0+x^1+x^2+\cdots+x^9$$

$$Now, B-A=\sum_{k=0}^{9}\left(x^{kkk}-x^k\right)=\sum_{k=0}^{9}x^k\ \{(x^{10})^{kkk}-1\}=\{(x^{10})^{kkk}-1\}\sum_{k=0}^{9}x^k=M\sum_{k=0}^{9}x^k,$$

$\Rightarrow B=(M+1)\sum_{k=0}^{9}x^k, i.e. \sum_{k=0}^{9}x^{kkk}$ is divisible by $\sum_{k=0}^{9}x^k$.

13) After several operation of differentiation and multiplying by (x+ 1) performed in an arbitrary order the polynomial x^8+x^7 is changed to ax +b. Prove that the difference between the in tegers a and b is always divisible by 49.

Sol.: Let $f(x)=x^m, then\ f^n(x)=m(m-1)(m-2)\ldots.(m+1-n)x^{m-n} where\ f^n(x) is\ nth$ derivative of f(x).

$$\therefore\ f^n(x)=\frac{m!}{(m-n)!}x^{m-n}$$

$Let\ g(x)=x^8+x^7, then\ g^n(x)=\frac{8!}{(8-n)!}x^{8-n}+\frac{7!}{(7-n)!}x^{7-n}$ Multiplying both sides by (1+ x), then

$$(1+x)g^n(x) = \frac{8!}{(8-n)!}x^{8-n} + \frac{7!}{(7-n)!}x^{7-n} + \frac{8!}{(8-n)!}x^{9-n}$$

After $(8-n)^{nth}$ differentiation of (i) we get $\frac{7!}{(7-n)!}x^{8-n} \ldots\ldots(i)$ if into the from ax +b.

Let h(x) = (x+1) $g^n(x)$.

$$Now\ h^{8-n}(x) = \frac{8!}{(8-n)!}.\frac{(8-n)!}{0!} + \frac{7!}{(7-n)!}.\frac{(9-n)!}{7!} + \frac{8!}{(8-n)!}.\frac{(9-n)!}{7!}x = 8!(9-n)x + 7!(8-n) + 8.7!$$

$= 8!(9-n)x + 7!(16-n)comparing\ a = 8!(9-n)x$

$= 7!(72-8n)!\,b = 7!(16-n)$

$\therefore a-b = 7!(72-8n-16+n) = 7!(56-7n) = 7\times 7!(8-n) = 49.6!(8-n)$

$\therefore a-b$ is divisible by 49.

14) Let n be an odd integer greater than 1 and $\boldsymbol{k_1, k_2, \ldots, k_n}$ be given integers. For each of the n! Permutations $\boldsymbol{a = (a_1, a_2, \ldots, a_n) of\ 1, 2, \ldots\ldots, n}$. Let $\boldsymbol{S_{(a)} \sum_{i=1}^{n} k_i\ a_i}$. Show that there are two permutations b and c, b ≠ c such that n! is a divisors of $\boldsymbol{S_{(b)} - S_{(c)}}$.

Sol.: Let $\sum S_{(a)}$ be the sum of $S_{(a)}$ over all n! permutation a = $(a_1, a_2, \ldots, a_n)$. We compute $\sum S_{(a)}$ mod n! two ways one of which depends on the desired conclusion being false, and reach a contradiction when n is odd.

First way $I_n\ \sum S_{(a)}$, k_1 is multiplied by each i ∈ {1, …, n} a total of (n -1)! Times. Once for each permutation of {1, …, n} in which $a_1 = i$. Thus the coefficient of $k_1\ in\ \sum S_{(a)}\ is\ (n-1)!\,(1+2+\cdots+n) = \frac{(n+1)!}{2}$.

The same is true for all k, so $\sum S_{(a)} = \frac{(n+1)!}{2}\sum_{i=1}^{n} k_i \ldots\ldots\ldots(1)$

Second way, if n! is not a divisors of $S_{(b)} - S_{(c)}$ for any b ≠ c, then each $S_{(a)}$ must have a different remainder mod n! Since there are n! permutations, these remainders must be precisely the numbers 0, 1, 2, …, n! -1. Thus

$\sum S_{(a)} = \frac{(n!-1)n!}{2}\ \ mod\ n! \ldots\ldots\ldots\ldots(2)$

Combining (1) and (2), we get

$$\frac{(n+1)!}{2}\sum_{i=1}^{n} k_i \equiv \frac{(n!-1)n!}{2}\ \ mod\ n! \ldots\ldots\ldots\ldots(3)$$

Now, for n odd, the left side of (3) is congruent to 0 modulo n!, while for n >1 the right side is not congruent to 0 (n! -1 is odd) For n> 1 and odd, we have a contradiction.

15) If $\boldsymbol{x_1, x_2, \ldots, x_n}$ be real numbers satisfying the conditions: $\boldsymbol{|x_1 + x_2 + \cdots + x_n| = 1\ and\ |x_i| \le \frac{n+1}{2}\ or\ i = 1, 2, \ldots, n}$. show that there exist a permutations $\boldsymbol{y_1, y_2, \ldots\ldots, y_n\ of\ x_1, x_2, \ldots, x_n}$ such that $\boldsymbol{|y_1 + 2y_2 + \cdots + ny_n| \le \frac{n+1}{2}}$.

Sol.: For any permutation $\pi = (y_1, y_2, \ldots, y_n)$ of $(x_1, x_2, \ldots, x_n)$ Let S(π)= $y_1 + 2y_2 + 3y_3 + \cdots + ny_n$ Let π_0 be the identity permutation, $\pi_0 = (x_1, x_2, \ldots, x_n)$ and let π be the reverse permutation, $\pi =$

$(x_n, x_{n-1}, \dots, x_1), if\ |S(\pi_0)| \le \frac{(n+1)}{2}\ or\ |S(\pi)| \le \frac{(n+1)}{2}$, then we are done. Thus we assume $|S(\pi_0)| > (n+1) and\ |S(\pi)| > \frac{(n+1)}{2}$

Note that $S(\pi_0) + S(\pi) = (x_1 + 2x_2 + \cdots + nx_n) + (x_n + 2x_{n-1} + \cdots + nx_1) = (n+1)(x_1 + x_2 + \cdots + x_n) and\ hence\ that\ |S(\pi_0) + S(\pi)| = n + 1$. Since each of $S(\pi_0) and\ S(\pi)\ exceeds\ \frac{(n+1)}{2}$ is absolute value, they must have opposite signs. Thus, one of $S(\pi_0) and S(\pi)$ is greater than $\frac{(n+1)}{2}$, and the other is less than $-\frac{(n+1)}{2}$. Now, starting from π_0, we can obtain any permutation by successive, transpositions of neighboring elements. In particular, there exists a chain $\pi_0, \pi_1, \dots, \pi_m$ of permutations π_{x+1} is obtained from π_i by interchanging two of its neighboring terms.

This means that if $\pi_i = y_1, y_2, \dots\dots, y_n \quad and\ \pi_{i+1} = (z_1, z_2, \dots, z_n)\ then\ there\ is\ an\ index\ k,\ 1 \le k \le n-1$, such that $z_k = y_{k+1}, z_{k+1} = y_k\ and\ z_i = y_i, j \ne k, j \ne k+1$.

Because the numbers x_i do not exceed $\frac{(n+1)}{2}$ in absolute value, we have $|S(\pi_{i+1}) - S(\pi_i)| = |kx_k + (k+1)z_{k+1} - ky_k - (k+1)y_{k+1}| = |y_k - y_{k+1}| \le |y_k| + |y_{k+1}| \le n+1$. It follows that the difference between any two consecutive numbers in the sequence $S(\pi_0), S(\pi_1), \dots, S(\pi_m)$ is at most n +1 in absolute value. Recall that the numbers $S(\pi_0)\ and\ S(\pi_m) = S(\tilde{\pi})$ regarded as points on the real line, lie outside of and on opposite sides of the interval $\left[-\frac{(n+1)}{2}, \frac{(n+1)}{2}\right]$.

Because this interval has length n +1, it follows that at least. One of the numbers $S(\pi_i)$ must lie in this interval. For this particular π_i we have $|S(\pi_i)| \le \frac{(n+1)}{2}$.

16) Find the number of non-degenerate triangle whose vertices lie in set of points (s, t) in the plane such that $0 \le S \le 4, 0 \le t \le 4$, S and t are integers.

Sol.: There are 25 points in the given set, we can choose 3 out of them in $\binom{25}{3}$ ways. Let us count the number of ways in which the 3 points chosen will lie on a line. L : The given set S contains 5 horizontal lines 5 points each.

We can choose 3 points from any of them in $\binom{5}{3}$ ways.

Hence the number of ways in which L can be a horizontal line is $5.\binom{5}{3} = 50$. Similarly the number of ways in which L can be a vertical line is 50.

As shown in fig.(ii) S contains 5 lines of slope 1; one line contain 5 points, 2 lines contain 4 points each and 2 lines contain 3 points each. So the number of ways in which L can be line of slope 1 is $\binom{5}{3} + 2\binom{4}{3} + 2\binom{3}{3} = 20$.

Similarly, the number of ways in which L can be a line of slope $-1\ is\ 20$.

As shown in fig (ii) there are 3 lines of slope $\frac{1}{2}$ each containing 3 points; and there are 3 lines of slope 2, each containing 3 points. So the number of ways in which L can have slope $\frac{1}{2}$ or 2 is 6 $\binom{3}{3}$= 6 similarly L can have slope $-\frac{1}{2}\ or -2\ in$ 6 ways.

Since no other line can contain more than two points of S, the number of ways in which the 3 points chosen will lie is 50+50+20+20+6+6= 152

The required number of triangle is therefore $\binom{25}{3} - 152 = 2148$.

17) For non-negative integers n, r the binomial coefficient $\binom{n}{r}$ denotes the number of combinations of n objects chosen r at a time, with the convention that $\binom{n}{0} = \mathbf{1}\ \boldsymbol{and}\ \binom{n}{r} = \mathbf{0}\ \boldsymbol{if}\ \boldsymbol{n} < r$.

Prove that $\sum_{d=1}^{\infty}\binom{n-r+1}{d}\binom{r-1}{d-1} = \binom{n}{r}$ for all integers n, r with $\mathbf{1} \leq \boldsymbol{r} \leq \boldsymbol{n}$.

Sol.: We use a combinatorial argument to establish the obviously equivalent identity

$$\sum_{d=1}^{\infty}\binom{n-r+1}{d}\binom{r-1}{d-1} = \binom{n}{r} \ldots\ldots(*) where\ k = \min\{r, n-r+1\}.$$

It clearly suffices to demonstrate that the left hand side of (i) counts the number of ways of selecting r objects from n distinct objects (without replacements). Let $S_2 = r - 1$. For each fixed d = 1, 2, …, k any selection of d objects from $S_1\left(\frac{S}{S_2}\right)$ together with any selection of $r - d$ objects from S_2 would yield a selection of r objects from S. The total number of such electrons is $\binom{n-r+1}{d}\binom{r-1}{d-1}$. Conversely each selection of r objects from S clearly much arise in this manner. Summing over d = 1, 2, ….(*) follows.

18) If S be the set of natural numbers whose digits are chosen from {1, 2, 3, 4} such that (i) when no digits are repeated, find n(s) and the sum of all numbers in S and (ii) when $\boldsymbol{S_1}$ is the set of up to 4-digits numbers where digits are repeated. Find $|\boldsymbol{S_1}|$ and also find the sum of all the numbers in $\boldsymbol{S_1}$.

Sol.: (i) S consists of single digit numbers, two digits numbers three digits numbers and four digit numbers.

No. of single digit number = 4, No. of two digit number = $4 \times 3 = 12$ (since repeatation is not allowed, there are four choices for ten's place and three choices for unit's place)

No. of three digit number = $4 \times 3 \times 2 = 24$

No. of four digit number $= 4 \times 3 \times 2 \times 1 = 24$

∴ n(s) = 4 +12+ 24+ 24 = 64.

Now for the sum of these 64 numbers, sum of all the single digit number is 1 +2 +3 +4= 10. (since there are exactly 4 digits 1, 2, 3, 4 and their numbers are 1, 2, 3 and 4).

Now, to find the sum of all the two digit numbers. No of two digit number is 12.

The digit used in units place are 1, 2, 3 and 4. In the 12 numbers, each of 1, 2, 3 and 4 occurs thrice in unit digit $\left(\frac{12}{4} = 3\right)$.

Again in ten's place, each of these digits occurs thrice also so, sum of these 12 numbers = $30 \times (1+2+3+4) + 3 \times (1+2+3+4) = 300 + 30 = 330$. No. of the digit numbers is 24. So, the number of times each of 1, 2, 3, 4 occurs in each of unit's ten's and hundred's place is $\frac{24}{4} = 6$.

So, sum of all these three digit number is $100 \times 6(1+2+3+4) + 10 \times 6(1+2+3+4) + 1 \times 6(1+2+3+4) = 6,000 + 600 + 60 = 6660$.

Similarly for the four digit numbers, the sum is computed as $100 \times 6(1+2+3+4) + 100 \times 6(1+2+3+4) + 10 \times 6(1+2+3+4) + 1 \times 6(1+2+3+4) = 60{,}000 + 6{,}000 + 600 + 60 = 66,660$.

[Since there are 24 digit numbers, each of 1, 2, 3, 4 occurs in each of the four digits in $\frac{24}{4} =$ 6 times]

So, the sum of all the single digit, two digit, three digit and four digit number = 10+330+6660+66660= 73, 660.

(i) There are just four single digits numbers 1, 2, 3, 4.
(ii) There are $4 \times 4 = 16$ two digits numbers, as digit can be repeated.
(iii) There are $4 \times 4 \times 4 = 64$ three digit numbers.
(iv) There are $4 \times 4 \times 4 \times 4 = 256$ four digit numbers.

So, that total number of numbers up to 4 digit numbers that could be formed using the digits 1, 2, 3 and 4 is $4 + 16 + 64 + 256 = 340$.

Sum of the 4 single digit numbers = 1+2+3+4= 10. To find the sum of 16, two digit number each of 1, 2, 3, 4 occur in each of units and ten's place $\frac{16}{4} = 4\ times$. So, the sum of all these 16 numbers is $= 10 \times 4(1 + 2 + 3 + 4) + 4(1 + 2 + 3 + 4) = 400 + 40 = 440$.

Similarly, the sum of all the 64 three digit numbers $100 \times \frac{64}{4} \times (1 + 2 + 3 + 4) + 10 \times \frac{64}{4} \times (1 + 2 + 3 + 4) + 1 \times \frac{64}{4} \times (1 + 2 + 3 + 4) = 16{,}000 + 1{,}600 + 160 = 17{,}760$.

Again the sum of al the 256 four digit numbers $= 1000 \times \frac{256}{4} \times (1 + 2 + 3 + 4) = 6{,}40{,}000 + 64{,}000 + 6{,}400 + 640 = 7{,}11{,}040$.

Therefore, sum of all the number is 10 + 440 +17,760+ 7,11,040= 7,29,250.

19) Find the number of 6 digit natural numbers where each digit appears at least twice.

Sol.: We consider number like 222222 or 233200 but not 212222. Since the digit 1 occurs only once.

The set of all such 6 digits can be divided into following classes.

S_1 = the set of all 6 digit numbers where a single digit is repeated 6 times.

$n(S_1)$= 9.

Since '0' cannot be a significant number when all its digits are zero.

$Let\ S_2$ be the set of all six digit numbers, made up of three distinct digits.

Here we should have two cases : $S_2(a)$ one with the exclusion of zero as a digit and other $S_2(b)$ with the inclusion of zero as a digit.

$S_2(a)$ The numbers of ways, three digit could be chosen from 1, 2, …, 9 is 9_{C_3}. Each of these three digits occurs twice. So, the number of six digit number in this case is

$$9_{C_3} \times \frac{6!}{2! \times 2! \times 2!} = \frac{9 \times 8 \times 7}{1 \times 2 \times 3} \times \frac{720}{8}$$
$$= 9 \times 8 \times 7 \times 15 = 7560.$$

$S_2(b)$ the three digits used include one zero, implying we have to choose the other two digits from the 9 non zero digits.

This could be done in $9_{C_2} = \frac{9 \times 8}{1.2} = 36$. Since zero can not be in the leading digit. So let us fix one of the fixed non-zero number in the extreme left. Then the other five digits are made up of 2 zeros, 2 fixed non zero number and the another non-zero number, one of which is put in the extreme left.

In this case the number of six digit numbers that could be formed is $\frac{5}{2!\times2!\times2!}\times 2$ (since from either of the pairs of fixed non-zero numbers, one can occupy the extreme digit) $= 60$.

So, the total number in this case $= 36 \times 60 = 2160$.

$\therefore$ n$(S_2) = n(S_2a) + (S_2b) = 7560 + 2160 = 9720$

Now, Let S_3 be the set of six digit numbers, whose digits are made up to of two distinct, digits each of which occurs thrice. Here again, there are two cases: $S_{3(a)}$ excluding the digit zero and $S_{3(b)}$ including the digit zero.

$S_{3(a)}$ is the set of six digit numbers, each of whose digits are made up of two non-zero digits each occurring thrice

$\therefore$n$\left[S_{3(a)}\right] = \ 9_{C_2} \times \frac{6!}{3!\times3!} = 36 \times 20 = 720$

$S_{3(b)}$ consists of 6 digits numbers whose digits are made up of three zeros and one of non-zero digit, occurring thrice. If you fix one of the nine non-zero digit, use that digit in the extreme left.

This digit should be used thrice. So in the remaining 5 digits, this fixed non zero digit is used twice and the digit zero occurs thrice.

So, the number of 6 digit numbers formed in these cases is $9 \times \frac{5!}{3!\times2!} = 90 \ \therefore n(S_3) = nS_3(a) + nS_3(b) = 720 + 90 = 810$.

Now let us take S_4, the case where the six digit number consists of exactly two digits, one of which occurs twice and the other four times.

Here, again, there are two cases; $S_4(a)$ excluding zero and $S_4(b)$ including zero.

If a and b are the two non-zero numbers a used twice and b four times, then we get $\frac{6!}{2!\times4!}$ and when a used four times, b twice, we again get $\frac{6!}{2!\times4!}$. So, when 2 of the nine non-zero digits are used to from the six digit number in this case, the total numbers got is $9_{C_2} \times 2 \times \frac{6!}{4!\times2!} = 36 \times 5 \times 6 = 1080$.

Thus n$[S_4(a)] = 1080$.

For counting the numbers in $S_4(b)$.

In this case we may use 4 zeros and a non-zero number twice or 2 zeros and a non-zero number for times. In the former case, assuming the one of the fixed non-zero digit occupying the extreme left, we get the other five digits consisting of 4 zeros and one non-zero number.

This result in $9 \times \frac{5!}{4!\times1!} = 45$ six digit numbers. When we use the fixed non-zero digit 4 times and use zero twice, then we get $9 \times \frac{5!}{3!\times2!} = 90$ six digit numbers, as fixed number occupies the extreme left and for the remaining three times it occupies 3 of the remaining digit, other digits being occupied by the two zeros. So, n$(S_4) = n\ [S_4(a)] + n[S_4(b)] = 1080 + 45 + 90 = 1215$.

Hence, the total number of six digit numbers satisfying the given condition $= n(S_1) + n(S_2) + n(S_3) + n(S_4)$= 9+720+810+1215 = 2754

20) If x = {1, 2, 3, … 4}, where n ∈N, show that the number of r combinations of x which contain no consecutive integer is given by $\binom{n-r+1}{r}$, $\boldsymbol{where}\ \mathbf{0} \le \boldsymbol{r} \le \boldsymbol{n} - \boldsymbol{r} + \mathbf{1}$.

Sol.: From the hypothesis $r \le n - r + 1, we\ get\ 2r \le n + 1$. Each such r combinations can be represented by a binary sequences $b_1, b_2, b_3 \dots \dots . b_n$ where $b_i = 1, if\ i\ is$ a number of the r combinations and 0, otherwise with no consecutive $b_i's$ =1 (the above r combinations contain no consecutive integers). The number of 1s in the sequence is r. Now, this amounts to counting such binary sequences. Now, look at the arrangements of the following boxes;

And the balls in them

1	2	3	4	5	6	7
00	000	00	0000	0	0	000

Here, the balls stand for the binary digits zero, and the boundaries on the left and right of each box can be taken as the binary digit one. In this display of boxes and balls as interpreted gives previously how we want the binary numbers. here there are 7 boxes, and 6 left/right boundary for the boxes.. So, this is an illustration of 6 combinations of non-consecutive numbers.

The reason for zeros in the front and at the end is that we can have leading zeros and trailing zeroes in the binary sequence $b_1, b_2, \dots . b_n$

Now clearly finding the r combination amounts to distribution of (n-r) balls into $(r+1)$ distinct boxes $[(n-r) balls = (n-r)]$ zeros as these are r ones, in the n number sequence]. Such that the 2nd, 3rd-rth boxes are non-empty.(The first and the last boxes may or may not be empty in the illustration 1st and the 7th may have zeros or may not have balls as we have already had 6 combinations!). Put $(r-1)$ balls one in each of 2nd, 3rd, …., rth boxes. (So, that no two 1's occurs consecutively).

Now we have $(n-r)-(r-1)$ balls to be distributed in r +1 distinct boxes.

21) If $S = \{1, 2, 3, \dots., (n+1)\} where\ n \geq 2$ and let $t - \left\{\frac{(x,y,z)}{x,y,z} \in S, x < z, y < z\right\}$. By counting the numbers of T in two different ways, show that $\sum_{k=1}^{n} k^2 = \binom{n+1}{2} + 2\binom{n+1}{3}$.

Sol.: T can be written as $T = T_1 \cup T_2, T_1 = \left\{\frac{(x,x,z)}{x,z}, \in S, x < z\right\} and\ T_2 = \left\{\frac{(x,y,z)}{x,y,z} \in S, x + y < z\right\}$

The number of elements in T_1 is the same as choosing two elements from the set S, where n(S)= (n +1). i.e. $n(T_1) = \binom{n+1}{2}$ (as every subset of two elements the larger elements will be z and the smaller will be x and y.)

In T_2. we have $2\binom{n+1}{3}$ elements, other choosing 3 elements from the set S, two of the smaller elements will be x and y and they may be either taken as (x, y, z) or as (y, x, z) or in other words, every three element subset of S, say {a, b, c} the greatest is z, and the other two can be placed in two different ways in the first two positions,

∴n(T) $(or\ |T|) = \binom{n+1}{2} + 2\binom{n+1}{3}T,$ can also be considered as $\bigcup_{i=2}^{n+1} S_i$,

where $S_i = \left\{\frac{(x,y,i)}{x} y < i, x, y \in S\right\}$.

All these sets are pair wise disjoint as for different i, we get different ordered triplets (x, y, i).

Now in S_i, the first two components of (x, y, i) namely (x, y) can be any element from the set 1, 2, 3, …., (i-1) equal or district.

∴ The number of ways of selecting (x, y), x, y ∈{1, 2, 3, ….., (i-1)} is $(i-1)^2$

Thus, n$(S_i) for\ each\ i\ is (i-1)^2, i \geq 2$.

For example, $n(S_2) = 1, n(S_3) = 2^2 = 4$ and so on

Now, $n(T) = n\left(\bigcup_{i=2}^{n+1} S_i\right) = \sum_{i=2}^{n+1} n(S_i)$
(because all $S_i's$ are pair –wise disjoint)

$$= \sum_{i=2}^{n+1} (i-1)^2$$
$$= \sum_{i=1}^{n} i^2 \ and\ hence, \binom{n+1}{2} + 2\binom{n+1}{3}$$
$$= \sum_{k=1}^{n} k^2$$

22) Show that the number of ways in which 3 numbers in A.P. can be selected from 1, 2, 3, …., n is $\frac{1}{4}(n-1)^2$ or $\frac{1}{4}n(n-2)$ according as n is odd or even.

Sol.:

Let us assume that n is odd, so $n = 2m - 1$.

Now, we will count the set of all triplets of numbers which are in A.P.

Observe the following sequence of triplets in A.P. with common difference 1.

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ \vdots & \vdots & \vdots \end{matrix}$$

$$or, \begin{matrix} n-2 & n-1 & n \\ 2m-3 & 2m-2 & 2m-1 \end{matrix}$$

AP is common difference 2.

$$\begin{matrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \\ \vdots & \vdots & \vdots \\ 2m-5 & 2m-3 & 2m-1 \end{matrix}$$

Thus there are $2m - 5$ A.P.'s here with common difference 2.

Now let us consider an A.P. with common difference $m - 1$, then (1, m, 2m-1) will be the only Ap with this common difference m -1. Thus the greatest value for the common difference of the Ap's in equation is d $= m - 1$

Now, let us taken all the AP's with common difference d,

$$\begin{matrix} 1 & 1+d & 1+2d \\ 2 & 2+d & 2+2d \\ \vdots & \vdots & \vdots \\ 2m-1-2d & 2m-1-2d & 2m-1 \end{matrix}$$

Therefore, there are exactly $(2m - 1 - 2d)$ triplets in Ap with common difference d, but d varies from 1 to m -1.

So, the total number of triplets in AP in this case is

$$\sum_{d=1}^{m-1} (2m-1-2d)$$
$$= (m-1)(2m-1) - 2\frac{(m-1)m}{2}$$

$$\therefore \sum_{d=1}^{m-1} 2m-1-2d$$
$$= (m-1)(2m-1) - 2\sum_{d=1}^{m-1} d$$
$$= (m-1)(2m-1) - 2\frac{(m-1)m}{2}$$
$$= 2m^2 - 3m + 1 - m^2 + m$$
$$= m^2 - 2m + 1 = (m-1)^2$$

$$But\ n = 2m - 1 \Rightarrow m = \frac{n+1}{2} \Rightarrow m - 1 = \frac{n-1}{2}\ and$$

Hence, when n is odd, the total number of A.P.'s is $\left(\frac{n-1}{2}\right)^2 = \frac{1}{4}(n-1)^2$

For the case where n is even, assume = 2m. In this case also we can show that the AP with the biggest value of the common difference is again $m - 1. for\ 1, m, 2m - 1$, it will form an Ap with common difference $m - 1\ and\ 1, m, 2m - 1$ all belong to the given set of natural numbers up to n. If m is the common difference then $1, 1 + m, 1 + 2m$, will be triplet in A.P., but 2m +1 does not belong to the given set.

However there are two AP's with common differences $m - 1\ as\ (2, m + 1, 2m)$ will from the Ap, with all the three numbers belonging to the set, whereas there is just one AP with biggest possible common difference (m -1) in the case of n, an odd number 1.

Now, consider the Aps with common differences d, $(1,1 + d, 1 + 2d), (2, 2 + d, 2 + 2d), (3, 3 + d, 3 + 2d), \ldots, (2m - 2d, 2m - d, 2m)$

So for each d, there are $2m - 2d$ Ap's d varying from 1 to $m - 1$.

So, the total number of AP's in this case where n = 2m is

$$\sum_{d=1}^{m-1}(2m - 2d) = 2\sum_{d=1}^{m-1}(m - d)$$
$$= 2\left[m(m-1) - \frac{1}{2}(m-1)m\right]$$
$$= 2 \times \frac{1}{2}m(m-1)$$
$$= m(m-1)$$

$$but\ m = \frac{1}{2}n\ and\ m - 1 = \frac{1}{2}(n-2),$$

$$we\ have\ the\ total\ number\ of\ AP's\ in\ this\ case = \frac{1}{2}n \times \frac{1}{2}(n-2) = \frac{1}{4}n(n-2).$$

23) There are two boys, each containing m balls. A person has to select an equal number of balls from both bags. Find the number of ways in which he can select at least one ball from each bags.

Sol.: He may choose one ball or two balls ….. or m balls from each bags.

Choosing one ball from one of the bags can be done in $m_{C_1} ways$. Then, choosing one ball from the other bag also can be done in m_{C_1} ways.

Thus, there are $m_{C_1} \times m_{C_1}$ ways of choosing one ball from each bag. Similarly if r balls, $1 \leq r \leq m$ are chosen from each of the two bags, the number of ways of doing this is $(m_{C_r}).(m_{C_r}) = (m_{C_r})^2$

Thus , the total number of ways of choosing at least one ball from both the bag is $\sum_{r=1}^{m}(m_{C_r})^2 = \sum_{r=0}^{m}(m_{C_r})^2 + (m_{C_0})^2 = 2n_{C_{n-1}} = \frac{2n!}{n!n!} - 1\ as\ m_{C_0} = 1$

$$\left[\sum_{r=0}^{m}(m_{C_r})^2 = 2m_{C_m}\right]$$

24) If $A_i = 1, 2, \dots, 2$ be the vertices of a 21 sided regular polygon inscribed in a circle with centre 0. Triangles are formed by joining the vertices of the 21 sided polygon. How many of them are acute angled triangles? How many of them are obtuse angles triangles? How many of them are equilateral. How many of them are isosceles?

Sol.: Since this is a regular polygon with odd number of vertices, no two of the vertices are placed diagonally opposite, so there are no right angled triangles. Hence number of right angled triangles is zero. Let A be the number of acute angled triangles. To from a triangle we need to choose 3 vertices out of the 21 vertices which can be done in C (21, 3) = $\frac{21\times20\times19}{6} = 1330$ ways.

Since the triangles are either acute or obtuse get A+0 = 133D

To find A, the number of acute angled triangles The 3 vertices of a triangle (say $A_i\ A_j\ A_k, 1 \le i \le j < k \le 21$). Divide the 21 spaces between the vertices into say x, y, z such that x + y +z = 21. We will count now the acute angled triangle with A_i as one of the vertices. For $A_i\ A_j\ A_k$ to be acute angled, $j \le 11$ and the distance between $A_j\ and\ A_k$ is less than 10, here $x = j - 1, y = k - jand\ z = 22 - k$ (as we want the distance from $A_k\ to\ A_1$). The problem can be modeled as distributing 21 identical balls' into 3 boxes with each box getting at least one ball. This can be done in $(21 - 3 + 2)_{C_2} = \frac{20\times19}{2} = 190$. But these, note that $j - 1, k - j, and\ 22 - k\ all\ must\ be \le 10$. Now, we need to find the number of distribution of these balls in 3 boxes where at least one box gets more than 10. Note that only one box can get more than 10 as we have only 21 balls. Also, any one of the 3 boxes can get more than 10 balls. To find the number of ways where box 1 gets more than 10. i.e. at least 11 balls' in box 1. 1 each in boxes 2 and 3. We are left with 8 balls now. Now, number of ways of distributing 8 balls unconditionally in 3 boxes in $8 + 2_{C_2} = \frac{10\times9}{2} = 45$ ways. Thus number of acute angled triangles with A_1 as vertex $= 190 - 3 \times 45$ (3 times, as each box could get at least 11 balls) = 55. Now, for each vertex, we get 55 such triangles. But a triangle A_i, A_j, A_k will be counted in A_i vertex, $A_j\ vertex\ and\ A_k\ vertex. i.e. thrice$ so, each triangle will be counted thrice. Thus , the total number of acute angled triangles.

$$A = 55 \times \frac{21}{3} = 385, 0 = 1330 - 385 = 955$$

A triangle A_i, A_j, A_k is equilateral if A_i, A_j, A_k are equally spaced out of $A_1, \dots, A_{21}$. We have only 7 such triples $A_1A_8A_{15}, A_2A_9A_{16}, \dots\dots, A_7A_{14}A_{21}$.

Therefore, there are only 7 equilateral triangles.

Consider the diameter A_1OB where B is the point. Where A_1O meets the circle. If we have an isosceles triangle A_1 as its vertex then A_1B is the altitude and the base is bisected by A_1B. This means that the other 2 vertices $A_j\ and\ A_k, are\ equally\ spaced\ from\ B$.

We have 10 such pairs, so we have 10 isosceles triangle with A_1 as vertex of which one is equilateral.

Because proper isosceles triangles with A_1 as vertex (non equilateral) are 9. With each vertex $A_i, i = 1, 2, \dots, 21$ we have such isosceles triangles.

So, total number of isosceles but non-equilateral triangles are $9 \times 21 = 189$. But the 7 equilateral triangles are also to be considered as isosceles.

∴Total number of isosceles triangle are 196. Note this problem can be generalized to a polygon having n vertices. Find the number of acute, obtuse, right, isosceles and equilateral triangles.

25) Show that for any set of 10 points chosen within a square whose sides are of length 3 units, there are two points in the set whose distance most $\sqrt{2}$.

Sol.: Divide the square into 9 unit squares as given in the figure. Out of the 10 points distributed in the big square, at least one of the small squares must have at least two points by the pigeon hole principle (p.p.). These two points being in a unit square, are at the most $\sqrt{2}$ unit distance a part as $\sqrt{2}$ is the length of the diagonal of the unit square.

26) Show that given a regular hexagon of side 2cm. and 25 points inside it, there are at least two points among them which are at most 1 cm. distance a part.

Sol.: If ABCDE is the regular hexagon of side 2cm and P, Q, R, S, T and u are respectively the midpoints of AB, BC, CD, DE, EF, and FA respectively, then by joining the opposite vertices, and joining PR, RT, TP, UQ, QS, and SU. We get in all 24 equilateral triangles of side 1 cm.

We have 25 points so, these 25 points inside the hexagon ABCDEF, at least 2 points lie inside any one triangle whose sides are 1 cm long. So, at least two points among them will be 1cm apart.

27) Find the number of integer solutions to the equation $\boldsymbol{x_1 + x_2 + x_3 = 28\ where\ 3 \leq x_1 \leq 9, 0 \leq\ x_2 \leq p\ \ and\ 7 \leq\ x_3 \leq 17}$

Sol.: considered three numbered boxes whose contents are denoted as x_1, x_2, x_3 respectively. The problem now reduces to distributing 28 balls in the three boxes such that the first box has at least 3 and not more than 9 balls, the second box has at most 8 balls and the third box has at least 7 and at most, 17 balls. In first put 3 balls in the first box, and 7 balls in the third box. So, now the problem reduces to finding the number of distribution of 18 balls in 3 boxes such that the first has at most (9- 3) = 6, the second at most 8 and the third at most (17-7)=10. The number of ways of distributing 18 balls in 3 boxes with no condition is $\binom{18+3-1}{3-1} = \binom{20}{2} = 190$.

[The number of ways of distributing r identical objects in n distinct boxes is $\binom{n+r-1}{r}\binom{n+r-1}{n-1}$ where 'n' stands for the numbers of boxes and r for balls.]

Let a_1 be the distributions where the second box gets at least 7; d_2 the distributions where the third gets at least 9; and d_3 the distributions where the third gets at lest 11.

$$|d_1| = \begin{vmatrix} 18-7+3-1 \\ 3-1 \end{vmatrix} = \binom{13}{2}$$

$$= \frac{13 \times 12}{1.2} = 78, |d_2| = \begin{vmatrix} 18-9+3-1 \\ 3-1 \end{vmatrix}$$

$$= \binom{11}{2} = \frac{11 \times 10}{1.2} = 55,$$

$$|d_3| = \begin{vmatrix} 18-11+3-1 \\ 3-1 \end{vmatrix} = \binom{9}{2} = \frac{9 \times 8}{1.2}$$

$$= 36,$$

$$\therefore d_1 \cap x d_2 = \begin{matrix} 18-7-9+3-1 \\ 3-1 \end{matrix} = \begin{matrix} 4 \\ 2 \end{matrix}$$

$$= 6,$$

$$|d_2 \cap d_3| = \binom{18-9-11+3-1}{3-1}$$
$$= \binom{0}{2} = 0,$$

$$|d_3 \cap d_1| = \frac{18-11-7+3-1}{3-1} = \frac{2}{2} = 1$$

Therefore, $|d_1 \cap d_2 \cap d_3| = 0$ *and* $|d_1 \cup d_2 \cup d_3| = 78 + 55 + 36 - 6 - 0 - 1 + 0 = 162$.

So the required number of solutions $= 190 - 162 = 28$.

Note that the number of ways the first box gets at most 6, the second utmost 8 and the third utmost 10= total number of ways of getting 18 balls distributed in 3 boxes –(the numbers of ways of getting at least 7 in the first box, at least 9 in the second box and at least 1 in the third box) and $n(A \cup B \cup C) = n(A' \cap B' \cap C')$.

28) If repetition of digits is not allowed in any number (in base 10) show that among three four digit numbers two have a common digit occurring in them. Also, show that in base 7 system any two four digit numbers without repetition of digit will have a common number occurring in their digits.

Sol.: In base 10, we have ten digits 0, 1, 2, 3, 4,5 ,6, 7 8 and 9. Thus, for 3 digit numbers without repetition of digits, we have to use in all 12 digits but in base 10 we have just 10 digits. Thus, at least at least any two of the three 4 digit numbers have a common number occurring in their digits by pigeon hole principle. Again for base 7 system, we have seven digits 0, 1, 2, 3, 4, 5, 6. For two four digit numbers without repetition we have to use eight digits and again by pigeonhole principle they have at least one common number in their digits.

29) In base $2k$, $k \geq 1$ number system, any 3non-zero k-digit numbers are written without repetition of digits. Show that two of them have a common digit among them. In base $2k+1$, $k \geq 1$ among any 3k +1 digit non-zero numbers, there is a common numbers occurring in any two digits.

Sol.: **case (i):** in case k = 1, we have the digits 0, 1 and the k-digit non-zero number (s) is 1 only. Thus, all the three numbers in this case are trivially the some 1.

For k >1. There 'k' digit (non-zero) numbers will have altogether 3k digits and the total number of digits in base 2k system is 2k. Since repetition of digit is not allowed and 3k>2k implies that among the digits of at least two of the numbers, there is at least one digit common among them (by pigeon-hole principle)

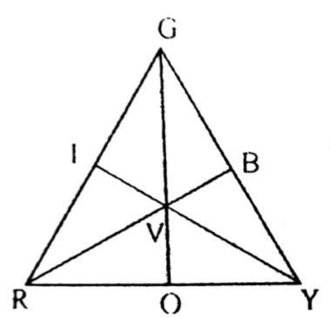
Case(ii) in the case of k = 1,2k+1=3, the three digits in base 2k+1=3 systems are 0,1 and 2k+1=1+1=2 and the digits non-zero numbers here are 10, 20, 12, 21. So, we can pick up 10,20 and 12 or 10,20,20,21,……..in each of the cases there is a common digit among two of them. (in fact, any two numbers will have to a common digit 1) in general case, 3 (k+1) digit numbers will have 3k+3 digits in all. But it is a base (2k+1) system.

The numbers are written without repetition of digits since 3k+ 3 > 2k+ 1. In fact, any two k+1 digit numbers could also have the same property as 2k+2 >2k+1, again by the pigeon-hole principle at least two of the numbers, will have at least one common number in their digits.

30) There are certain number of all balls and they are painted with the following conditions:

(i) Every two colours appear on exactly one ball.
(ii) Every two balls have exactly one colour in common.
(iii) There are four colours such that any three of them appear on one ball.
(iv) Each ball has three colours. Find the number of balls and colour used.

Sol.: Let us represent each of the balls by a line segment with three points to show the 3 colours. Thus, Roy is a ball with three colours red, orange and yellow. We have to have three more balls such that on yellow. So, next drawn lines through R, O, Y to meet at a common point G standing for green colour. But the balls with colours RG, OG AND YG must have a third colour in them say indigo (i), violet (v) and Blue (B). Thus we have 7 balls and 7 colours in all. 7 colours R, O, Y, G, I, V, B and 7 balls. 1. ROY, 2. RIG, 3. RVB, 4. Ova, 5. YBa, 6. YVI, 7. IBO



Clearly any pair of the above 7 balls have exactly one colour in common (satisfying condition 2). Each of the balls contribute 3 pairs of colours. In all, we have 21 pairs of columns in all the 7 balls. Now 7 colours lead to $\frac{7\times 6}{2} = 21$ pairs of colours and each pair of colours is found in exactly one ball satisfying condition 1. Each ball has 3 colours (condition 4 satisfied). Now, consider the four colour GRYV. No. three of these colours are found on a ball. (condition 3 is satisfied). Thus, the total number of colour is 7 and the total number of balls is also 7.

31) A mathematical conjection consisted of a part I and part II with a combined total of 28 problems. Each contestant solved 7 problems altogether. For each pair of problems there were exactly two contestants who solved both of them. Show that there was a constants who in part I solved either no problem or at least 4 problems.

Sol.: We will find the total number of contestants. Since for each pair of problems there were exactly two contestants let us assume that an arbitrary problem P_1 was solved by r contestants. Each of these r constestants solved 6 more problems, solving 6r more problems in all counting multiplicants. Since every problem, other than P_1, was paired with P_1 and was solved by exactly two constants, each of the remaining 27 problems (i.e. other than P_1) is counted twice among the problems solved by the r contestants. i.e. $6\text{r} = 2 \times 27 \ or\ r = 9$. Therefore an arbitrary problem P_1 is solved by 9 contestants.

Hence, in all we have $\frac{9\times 28}{7} = 36$ contestants, as each contestant solves 7 problems.

From the rest of the proof, let us assume the countrary that is every constant solved either 1, 2, or 3 problems in part 1.

Let us assume that there are n problems in part 1 and let x, y, z be the number of contestants who solved 1, 2, and 3 problems in part 1.

Since every one of the contestants solve either 1, 2, or 3 problems in part 1,

we get $x + y + z = 36 \ldots\ldots\ldots (1)$

$x + 2y + 3z = 9n \ldots\ldots\ldots. (2)$ (since each problem was solved by 9 contestants).

Since every contestant among y solves of a pair of problems in part I and every contestants among z solves 3 pairs of problems was solved by exactly two contestants, we get the following equations

$$y + 3z = 2.n_{C_2} = 2.\frac{n(n-1)}{2} = n(n-1) \ldots\ldots\ldots\ldots (3)$$

From eq.(1). Eq. (2) and Eq. (3), we get

$z = n^2 - 10n + 36$ *and* $y = -2n^2 + 29n - 108 = -2\left(n - \frac{29}{4}\right)^2 - \frac{23}{8} < 0.$

As y < 0 is not an acceptable result, our assumption is wrong. Hence, there is at least one contestants who solved either no problem from part 1 or solved at least 4 problems from part 1.

32) Find a recurrence relation for the number a_n of ten nary sequence of length n that contain 2 consecutive digits that are the same. What are the initial conditions? Find a_6.

Sol.: Clearly, no ternary sequence of length 1 can contain 2 consecutive identical digits and so $a_1 = 0$. Next the only ternary 2sequence of the required type are 00, 11, 22 and so $a_2 = 3$. *Let* $n \geq 3$. Every n sequence of the required from satisfies exactly one of the following conditions:

(i) It first 2 digits are unequal.

(ii) It first 2 digits are identical.

Let (i) hold. Then the sequence starts with one of 1, 02, 12, 20, 21. First suppose that it starts with 01. Now the condition that the sequence contain "2 consecutive identical digits" is symmetric w. r. t. all 3 digits 0, 1, 2. Hence these are

a_n *equal a number, namely* $m = \frac{1}{3}a_{n-1}$, of sequence of length n -1 and starting with 0, 1, or 2. So by appending 0 as first digit to each $(n-1)$ sequences starting with 1, we get m sequence of length n which start with 01. Similarly, there are m sequences of length n starting with 02, 01, 12, 20 or 21.

Thus there are $6m = 2a_{n-1}$ sequence in this case. Let (ii) hold. Then the sequences starts with 00 or 11 or 22 and its remaining $n-2$ digits can form any $(n-2)$ ternary sequences Hence rhere are 3^{n-2} n sequence starting with 00; and the same holds for 11 and 22.

Thus there are $3 \times 3^{n-2} = 3^{n-1}$ sequences in this case.

Required recurrence relation is $a_n = 2a_{n-1} + 3^{n-1}$ with initial conditions$a_1 = 0, a_2 = 3$.

Hence $a_3 = 15, a_4 = 57, a_5 = 195, a_6 = 633$.

33) For every real number x_1 construct the sequence $x_1, x_2, \ldots..$ by setting $x_{n+1} = x_n\left(x_n + \frac{1}{n}\right)$ for each$n \geq 1$. Show that there exists exactly one value of x_1 for which $0 < x_n < x_{n+1} < 1$ for every n.

Sol.: Let $P_1(x) = x, P_{n+1}(n) = P_n(n)\left[P_n(x) + \frac{1}{2}\right]$ $for\ n = 1, 2, \ldots.$

(i) from this recursive definition, we see inductively that (i) P_n is an polynomial of degree 2^{n-1}

(ii) P_n has positive coefficients is therefore an increasing convex function for $x \geq 0$.
(iii) $P_n(0) = 0, P_n(1) \geq 1$.
(iv) $P_n(x_1) = x_n$.

Since the condition $x_{n+1} > x_n$ is equivalent to $x_n > 1 - \frac{1}{n}$

We can reformulate the problem as follows show that there is unique positive real number t such that $1 - \frac{1}{n} < P_n(t) < 1$ for every n.

Since P_n is continuous and increases from 0 to a value of ≥ 1 for $0 \leq x \leq 1$, there is unique values a_n and b_n such that $a_n < b_n, P_n(a_n) = 1 - \frac{1}{n}, P_n(b_n) = 1$ ………(2)

By definition (1)

$$P_{n+1}(a_n) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n} + \frac{1}{n}\right) = 1 - \frac{1}{n}$$

$$P_{n+1}(a_{n-1}) = 1 - \frac{1}{n+1}. We\ see\ that\ a_n < a_{n+1} \ldots\ldots\ldots (3)$$

$$Also\ since\ P_{n+1}(b_n) = 1 + \frac{1}{n}\ and\ P_{n+1}(b_{n+1}) = 1$$

$b_n > b_{n+1}$. Since $P_n$is convex, the graph of $P_n(x)$ lies below, the chord $y = \frac{1}{b_n} x\ for\ 0 \leq x \leq b_n$

$In\ particular\ P_n(a_n) = 1 - \frac{1}{n} \leq \frac{a_n}{b_n}$ from this and the fact than $b_n \leq 1$.we find that $b_n - \frac{b_n}{n} \leq a_n, b_n - a_n \leq \frac{b_n}{n} \leq \frac{1}{n}$ for all n.

Thus we have2 in finite bounded sequences$\{a_n\}, \{b_n\}$ the first is increasing the second decreasing $a_n > b_n$ and the different between their nth numbers approaches 0 as n on creases. We conclude that there is a unique common value to that they approach $a_n < t < b_n\ \forall\ n$.

Number of uniquely satisfies $1 - \frac{1}{n} < P_n(t) < 1 \forall\ n$.

34) Find the number of isosceles triangle with integer sides, if no sides a exceeds 1994.

Sol.: Let 2 equal sides of an isosceles Δ be P units each and let remaining sides be q units.

Case I : P > q. q can take values 1, 2, 3, ….., P -1 (if $P - 1 > 0$) condition for p, q, q be a sides of a Δ is automatically satisfied here, for each positive integer P > 1, we can have P -1 isosceles Δ is

$$\sum_{p=2}^{1994} (P-1) = 1 + 2 + 3 + \cdots + 1993$$
$$= \frac{(1993 \times 1994)}{2}$$
$$= 1998721.$$

case II: p < q in order that p, q, may be sides of Δ we must have 2p > q. i.e. $p < q < 2p$.

If p is even say 2m, then q can take value 1, 2, …., m -1 if p is odd say $2m - 1$ then q can take values 1, 2, …., m -1 = $\left(\frac{p-1}{2}\right)$. Numbers of possible isosceles Δ is $\frac{1-1}{2} + \frac{3-1}{2} + \cdots + \frac{1993-1}{2} + 1 + 2 + 3 + \cdots$ for q = 1994, p +q >p is true. Also, we must have $\frac{q}{2} < p < q$. If q is even there are $q - \frac{q-1}{2} - \frac{q-2}{2} =$ possible values for p. If q is odd, there $(q-1) - \frac{q-1}{2} - \frac{q-1}{2} =$ possible value for p.

There are in all isosceles $\sum_{q\ even} \frac{q-2}{2} + \sum_{q\ odd} \frac{q-1}{2}\ \Delta s\ \ 1 \leq q < 1994$

$1 \leq q \leq 1994 \quad 1 \leq q \leq 1994$

$$i.e. (1 + 2 + \cdots + 996) + (1 + 2 + \cdots + 996)\Delta s = 2.\frac{996.997}{2} \;\Delta s = 993012\;\Delta s$$

Total number of isosceles $\Delta s = 1998721+993012 = 2991733$.

35) Define a hook to be a figure made up to 6 unit sequences as shown in the diagram or any of the figures obtained by applying rotations and reflections to this, figure. Find all $m \times n$ rectangles that can be covered with hooks so that

(i) The rectangle is covered without gaps and without overlaps.

(ii) No part of hook covers are outside the rectangle.

Sol.: Consider a covering of an $m \times n$ rectangle satisfying the conditions.

For any hook A there is a unique hook B which covered the inside square of A with one of its end most squares. On the other hand the inside square of B must be covered by an end most square of A. Thus in a tasting all hooks are matched into pairs.

There are only 2 possible way to place B so that it does not overlap with A and no gaps occur.

In one of the base A and B from 3×4 rectangle and the other case their union has an rectangle shape with side lengths 3, 2, 1, 3, 2, 1, 2

So an $m \times n$ rectangles can be covered with hooks and only if it can be covered with the 12 square titles as discussed above.

Suppose that such a tiling exists then mn is divisible by 12. We now show that one of m and n is divisible by 4. Suppose on the contrary that this is not the case then m and n are both even because mn is divisible by 4. Imagine that the rectangle divided into unit squares with the rows and columns formed labeled 1, ……, m and 1, ….., n write 1 in the square (i, j) if exactly one of i and j is divisible by 4. Since the number of square in each row and column is even the sum of all numbers, written is even.

Now 3×4 rectangle always covers number with sum 3 or 7 other 12 square shape always covers number with sum 5 or 7.

Consequently, the total number of 12 square shape is even. But the mn is divisible by 24 and hence by 8 country to the assumption that m and n are not divisible by 4. Also, neither m nor n can be 1, 2, 5

If a tiling is possible when one of m and n is divisible be 4, one is divisible by 4 and m $\notin$ {1, 2, 5}

Conversely, if these conditions are satisfied the tilling is possible (using only 3×4 rectangle at that)

This is immediate if 3 divides m and 4 divides n. Let m be divisible by 12 and n $\notin$ {1, 2, 5} then n can be represented as the sum of several 3's and 4's.

Hence the rectangle can be partitioned into $m \times 3$ and $m \times 4$ rectangle which are easy to cover only with 3×4 tiles again.

FUNCTIONAL EQUATIONS

1) If f be a function satisfying $f(x+y) = f_{(x)} + f_{(y)} \forall\; x, y \in R \text{ and } f_{(1)} = k$, then $f_{(x)}$ where n ∈N is

(a) nk; (b) n^k; (c) k^n; (d) none

Sol.: Since $f(x+y) = f_{(x)} + f_{(y)}. x = 1, y = 1$,

$f(1+1) = f_{(1)} + f_{(1)} \Rightarrow f_{(1)} = 27_{(1)}$

$x = 2\ y = 1\ f(2+1) = f_{(2)} + f_{(1)} \Rightarrow f_{(3)} = 2f_{(1)} + f_{(1)} = 3f_{(1)}$

$x = 2\ \ y = 2\ f(2+2) = f_{(2)} + f_{(2)} \Rightarrow f_{(4)} = 4f_{(1)}$

$In\ general, we\ have\ f_{(n)} = nf_{(1)}$

$\therefore f_{(n)} = nk\ for\ k = f_{(1)}$

2) **The function $f_{(x)} = \sin\left(\frac{\pi x}{n!}\right) - \cos\left\{\left(\frac{\pi x}{(n+1)!}\right)\right\}$ *is***
   (a) Not periodic; (b) period (2n!); (c) period (n+1); (d) none

Sol.: $f_{(x)} = \sin\left(\frac{\pi x}{n!}\right) - \cos\left\{\left(\frac{\pi x}{(n+1)!}\right)\right\} = t_1 - t_2\ period\ of$

$$t_1 = \frac{2\pi}{\left(\frac{\pi}{n!}\right)} = 2(n!) = A\ (say)$$

Period of $t_2 = 2\big((n+1)!\big) = B\ (say)$

Now, LCM of A and B is $2\big((n+1)!\big)$. Hence the function $f_{(x)}$ is periodic with period $2((n+1)!)$.

3) **If $f: [-4, 0] \to R$ is defined by $e^x + \sin x$, its even extension to $[-4, 4]$ is given by**
   (a) $-e^{-|x|} - \sin|x|$; (b) $e^{-|x|} - \sin|x|$; (c) $e^{-|x|} + \sin|x|$; (d) none

Sol.: To make $f_{(x)}$ an even function, in the interval $[-4, 4] f_{(x)}$ can be re-defined as under

$f_{(x)} = \{f_{(x)} - 4 \le 0 \le 0. f(-x) 0 < x \le 4\}$

Hence, even extension of the function from [0, 4] is $f(-x) = e^{-x} + \sin(-x) \Rightarrow f(-x) = e^{-x} - \sin x$

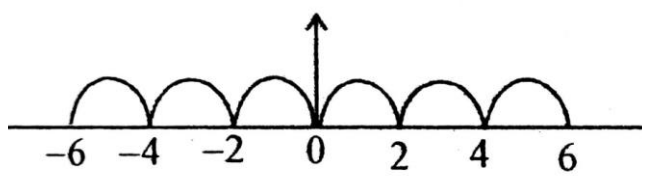
$\therefore f(x) = e^{-|x|} - \sin|x|$

4) **If $f_{(x)} = x(2-x), 0 \le x \le 2$ and the definition of f is extended over the set $R - [0, 2]$ *by* $f(x+1) = f(x)$, *then f is***
   (a) Period 1 ; (b) non-period ; (c) period 2 ; (d) none

Sol.: In $R - [0, 2], we\ have f_{(x+2)} = f_{((x+1)+1)}$

$\Rightarrow f_{(x+2)} = f_{(x+1)} = f_{(x)}\ \{given\}$

Graphically



5) **The value of the b and c for which the identity $f_{(x+1)} - f_{(x)} = 8x + 3$ is satisfied, where $f_{(x)} = bx^2 + cx + d$, *are***
   (a) 2, 1; (b) 4, -1; (c) 2, -2; (d) none

Sol.: Since, $f_{(x)} = b\,x^2 + cx + d.$

$Now, f_{(x+1)} - f_{(x)} = 8x + 3$

$\Rightarrow b(x+1)^2 + c(x+1) + d - bx^2 - cx - d = 8x + 3$

$\Rightarrow bx^2 + 2bx + b + cx + c - bx^2 - cx = 8x + 3$

$\Rightarrow 2bx + (b+c) = 8x + 3$

Comparing respective coefficients, we have b = 4, and c = -1.

6) **If $f_{(x)} = \cos(\log x)$, then**

$$f_{(x^2)}f_{(y^2)} - \frac{1}{2}\left[f\left(\frac{x^2}{y^2}\right) + f(x^2y^2)\right]$$

(a) -2; (b) -1; (c) $\frac{1}{2}$; (d) none

Sol.: $f\left(\frac{x^2}{y^2}\right) + f(x^2y^2) = \cos\left(\log\frac{x^2}{y^2}\right) + \cos\{\log(x^2y^2)\} = \cos(p-q) + \cos(p+q)$,

$where\ p = \log x^2$.

$q = \log y^2 = 2\cos p\,.\cos q$.

$\therefore Reqd.value = \cos p\,.\cos q - \frac{1}{2}(\cos p\,.\cos q) = 0$

7) **If $f_{(x_1)} - f_{(x_2)} = f\left(\frac{x_1-x_2}{1-x_1x_2}\right)$ $for\ x_1, x_2 \in (-1,1), then\ f_{(x)} =$**

(a) $\log\frac{1-x}{1+x}$; (b) $\tan^{-1}\frac{1-x}{1+x}$; (c) $\cot^{-1}\frac{1-x}{1+x}$; (d) none

Sol.: For (a) is correct is because LHS = $\log\frac{1-x_1}{1+x_1} - \log\left(\frac{1-x_2}{1+x_2}\right) = \log\frac{(1-x_1)(1+x_2)}{(1+x_1)(1-x_2)}$ and

$RHS = \log\frac{\left\{1-\frac{(x_1-x_2)}{1-x_1x_2}\right\}}{\left\{1+\frac{(x_1-x_2)}{1-x_1x_2}\right\}} = \log\frac{(1-x_1)(1+x_2)}{(1+x_1)(1-x_2)}$.

8) **If $2\,f_{(x)} - 3f\left(\frac{1}{x}\right) = x^2$, x is not equal to zero, then $f_{(2)} =$**

(a) -1; (b) $-\frac{7}{4}$; (c) 0; (d)none

Sol.: $2f_{(2)} - 3f\left(\frac{1}{2}\right) = 4 \ldots\ldots\ldots.(i) and\ 2f\left(\frac{1}{2}\right) - 3f(2) = \frac{1}{4}\ldots\ldots\ldots\ldots(ii)$

Thus 2(ii) +3(iii), $\Longrightarrow -5f(2) = 8 + \frac{3}{4}, \Longrightarrow f_{(2)} = -\frac{7}{4}$

9) **If $f_{(x)}$ be defined for all x $>$ 0 and be continuous,. Let $f_{(x)}$ satisfy $f\left(\frac{x}{y}\right) = f_{(x)} - f_{(y)}$ for all x, y and $f_{(e)} = 1\ then\ f_{(x)} =$**

(a) Bounded; (b) $xf_{(x)} \to 1\ as\ x \to 0$; (c) $\log x$; (d) none

Sol.: If we have $f_{(x)} = \log x \ldots\ldots\ldots.(i)$, then the conditions $f\left(\frac{x}{y}\right) = f_{(x)} - f_{(y)}\ and\ f_{(e)} = 1$ are satisfied (i). $f_{(x)}$ is not bounded as $f_{(x)}$ is increasing function.

10) **If $f_{(x)} = \frac{2^x - 2^{-x}}{2}, then\ f_{(x+y)}f_{(x+y)} =$**

(a) $\frac{1}{2}\{f_{(2x)} + f_{(2y)}\}$; (b) $\frac{1}{2}\{f_{(2x)} - f_{(2y)}\}$; (c) $\frac{1}{2}\{f_{(2y)} - f_{(2x)}\}$; (d) none

Sol.: $f(x+y) - f(x-y) = \left\{\frac{2^{x+y}+2^{-(x+y)}}{2}\right\} \times \left\{\frac{2^{x-y}+2^{-(x-y)}}{2}\right\} = \frac{1}{4}(2^{2x} + 2^{-2x} + 2^{2y} + 2^{-2y})$

$$= \frac{1}{2}\{f_{(2x)} + f_{(2y)}\}$$

11) **If f is even function defined on the interval $(-5, 5)$ then the real values of x satisfying the equation $f_{(x)} = f\left(\frac{x+1}{x-1}\right)$ are**

(a) $\frac{-1\pm\sqrt{5}}{2}$; (b) $\frac{-2\pm\sqrt{5}}{2}$; (c) 0; (d) none

Sol.: $f_{(x)} = f\left(\frac{x+1}{x+2}\right), \Rightarrow x = \frac{x+1}{x+2}, \Rightarrow x^2 + 2x = x + 1, \Rightarrow x = \frac{-1\pm\sqrt{5}}{2}$. Both lie in $(-5, 5)$.

For an even function: $f_{(x)} = f_{(-x)}, \Rightarrow -x = \frac{x+1}{x+2}$;

$\Rightarrow x^2 - 2x = x + 1; \Rightarrow x^2 + 3x + 1 = 0,$

$\Rightarrow x = \frac{-3\pm\sqrt{5}}{2}$ which lies in $(-5, 5)$.

12) If $f_{(x)} = xe^{x(1-x)}$, then $f_{(x)}$ is

(a) Increasing on $\left[-\frac{1}{2}, 1\right]$; (b) decreasing on $\left[-\frac{1}{2}, 1\right]$; (c) increasing on R; (d) none

Sol.: $f'_{(x)} = e^{x(1-x)}.(1 + x - 2x^2) = -e^{x(1-x)}.(x - 1)(2x + 1)$

$and\ e^{x(1-x)} > 0 \forall\ x, f_{(x)}\ is\ decreasing. \Rightarrow f'_{(x)} < 0,$

$\Rightarrow (x - 1)(2x + 1) < 0, \Rightarrow -\frac{1}{2} \leq x \leq 1.$

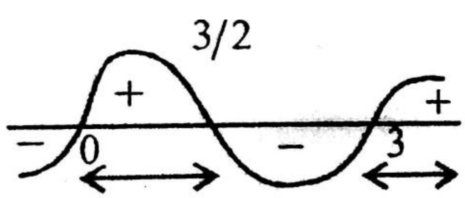
13) If $f_{(x)} = \{x(x-3)\}^2$ increase for the values of x lying the interval

(a) $1 < x < 3$; (b) $0 < x < \infty$; (c) $-\infty < x < 0$; (d) none

Sol.: $\frac{dy}{dx} = 2x(x - 3)(2x - 3)$ and for increasing,

$\frac{dy}{dx} > 0, \Rightarrow 0 < x < \frac{3}{2}, 3 < x < \infty,$

$\Rightarrow In\ \ Particular\ 0 < x < \frac{3}{2}.$



(SUBJECTIVE TYPE)

1) Find all subjection functions f: No → No with the property that for all $n \geq 0, f_{(n)} \geq n + (-1)^n$

Sol.: If we let $g: No \ \to No, g_{(n)} = x + (-1)^n$, then g satisfies the equation. Moreover, g is bijective. We will show that for any solution f we must have f = g.

In fact , we will prove a more general property, namely that if f and g are two functions defined on the non-negative integers such that $f_{(n)} \geq g_{(n)}$ for all n, and f is subjective and bijective, then f = g. The proof is based on the wall ordering of the set of positive integers, namely on the fact that any set of positive integers has a smallest element.

Assume f ≠ g, and let no be such that $f_{(n0)} > g_{(n0)}$ If we let M= $g_{(n0)}$, then the set A = $\{k, g_{(k)} \leq n\}$ has exactly M +1 elements, since g is bijective. On the other hand, since, $f \geq g$ and no does not belong to A, the set B = $\{k, f_{(k)} \leq M\}$ is included in A but has at least one less element, namely no. Hence the values of f do not exhaust all numbers less than M +1. Which contradicts the subjectivity of f. Therefore, $f_{(n)} = g_{(n)} = n + (-1)^n$ is the only solution.

2) Find all function f: N →N with the property that $f\{f_{(m)} + f_{(n)}\} = m + n$ for all m and n.

Sol.: The solution is done by manipulating the equation and plugging in particular values for the variables.

Thinking of $f_{(m)}$ and $f_{(n)}$ as positive integers. We have $f\big(f\big(f_{(m)}+f_{(n)}\big)+f_{(k)}\big) = f_{(m)} + f_{(n)} + k$ on the other hand by using the given relation for f

$f_{(m)} + f_{(n)} + f_{(k)}$ $we\ obtain$

$f\big(m+n+f_{(k)}\big) = f\big(f\ \ f_{(m)}+f_{(n)} + f_{(k)}\big) = f_{(m)} + f_{(n)} + k$

For m = n = 0 this reduces to $f\big(f_{(k)}\big) = 2f_{(0)} + k$.

Also for k = 0 and m, n arbitrary, we have

$f\big(m+n+f_{(0)}\big) = f_{(m)} + f_{(n)}$.

$Hence\ f\left(f\big(m+n+f_{(0)}\big)\right)$.

The left hand side of this equality is equal to $2f_{(0)} + m + n + f_{(0)}$ and the right hand side is equal to m +n. It follows that $f_{(0)} = 0$ and for all m, n, $f_{(m+n)} = f_{(m)} + f_{(n)}$, that is, f is additive. Choose m = 1 and use induction to show that $f_{(n)} = f_{(1)}{}^{n}$. From $f\big(f_{(m)}+f_{(n)}\big) = m+n$ one obtains $f_{(1)^2}(m+n) = m+n$, for all m, n. This can happen only if $f_{(1)} = 1$, so the only solution to the functional equation is the identity function f: N→ N, $f_{(n)} = n$

3) Find all pairs of functions f, g: N → N satisfying $\boldsymbol{f_{(n)} + f\{n + g_{(n)}\} = f_{(n+1)}}$

Sol.: One possibility is that f is identically equal to 0 and g is arbitrary. Another possibility is that g is identically equal to zero and $f_{(n)} = 2^n\, f_{(0)}$.

Let us find the remaining pairs of functions. Note that the identically implies $f_{(n+1)} \geq f_{(n)}$ for all n; hence f is increasing. If for a certain n, $g_{(n)} \geq 1$, then $f_{(n+1)} \leq f_{(n+g(n))}$; hence $f_{(n)} = 0$. A backwards induction shows that, $f_{(n-1)} = f_{(n-2)} = \cdots = f_{(0)} = 0$

Hence in order for f not be identically zero, there must exist m such that $g_{(k)} = 0$ for all k$\geq m$.

Assume m minimal, that is, $g_{(m-1)} \neq 0$, then on the one hand, $f_{(k)} = 0\ for\ k \geq m-1$, and on the other hand $f_{(k)} = 2^{k-m} f_{(m)}\ for\ k > m, so\ for\ k \geq m$ the function is strictly increasing. This together with $f_{(n)} + f_{(n+g(n))} = f_{(n+1)}$ implies that $n + g(n)$ cannot exceed m; hence $g(n) > m-n$. Thus all other solutions $(f.g) satisfy\ f_{(0)} - f_{(1)} = \cdots = f_{(m-1)} = 0$,

$f_{(k)} = 2^{k-m} a\ for\ k \leq m$ and a arbitrary, and $g(k) \leq m-k, for\ k \leq m, g_{(k)} = 0\ for\ k \geq m$.

4) Let F: N→ be such that $\boldsymbol{f_{(n+1)} > \{f_{(n)}\}}$ for all n ∊ N. Show that $\boldsymbol{f_{(n)} = n\ \forall\ n\ \in N}$.

Sol.: This problem might look easy to people familiar with the axiomatic description of the set of positive integers. The solution uses again the property that every set of natural numbers has a smallest element.

Let us look at the set $\{f\big(f_{(1)}\big), f_{(2)}, f\big(f_{(2)}\big), f_{(3)}, f\big(f_{(3)}\big), f_{(n)}, f\big(f_{(n)}\big), \ldots.\}$

Note that these are, exactly the numbers that appear in the inequality $f\big(f_{(n)}\big) < f_{(n+1)}$. This set has a smallest element, which cannot be of the from $f_{(n+1)}$ because then it, would be larger than $f(f_{(n)})$. Thus it is of the form $f(f_{(n)})$. The same argument shows that for this n, $f_{(n)} = 1. If\ n\ itself\ were\ greater\ than\ 1, we\ would\ get\ 1 =$

$f_{(n)} > f\left(f_{(n-1)}\right)$, which is impossible. Hence $f_{(1)} = 1$ *and* $f_{(n)} > 1$ *for* $n > 1$.

Considering the restriction$f: \{n \geq 2\} \rightarrow \{n \geq 2\}$, the same argument applies maintains mutadis to show that $f_{(2)} = 2$ *and* $f_{(n)} > 2$ *for* $n > 2$. By induction one shows that $f_{(k)} = k$, *and* $f_{(n)} k$ *for* $n > k$ thus the unique solution to the problem is identify function.

5) **Find all functions f: N →N with the property that for all n ∊ N, $\frac{1}{f_{(1)}f_{(2)}} + \frac{1}{f_{(2)}f_{(3)}} + \cdots + \frac{1}{f_{(n)}f_{(n+1)}} = \frac{f\{f_{(n)}\}}{f_{(n+1)}}$**

Sol.: The equality from the statement reminds us of the well-known identify $\frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$, which shows that the function f: $N \rightarrow N$, $f_{(n)} = n$ is a solutions.

Let us prove that this is only function with the required property.

The ratio $f\frac{(f_{(n)})}{f_{(n+1)}}$ remainds us of the previous problem. In fact, we will reduce the present problem to the previous one.

Plugging in n = 1 into the given relation yields

$$f\left(f_{(1)}\right)f_{(1)} = 1; \;\; lhence \; f_{(1)} = 1.$$

Replacing the given equality for into the one for n+1 we obtain

$$\frac{f(f_{(1)})}{f_{(n+1)}} + \frac{1}{f_{(n+1)}\, f_{(n+2)}} = \frac{f(f_{(n+1)})}{f_{(n+2)}}$$

This is equivalent to $f\left(f_{(n)}\right)f_{(n+2)} + 1 = f\left(f_{(n+1)}\right)f_{(n+1)}$ *Note that* $f_{(n+1)} = 1$ *implies that* $f\left(f_{(n+1)}\right) = 1$;

hence $f\left(f_{(n)}\right) f_{(n+2)} = 0$, which is impossible. Therefore $f_{(n)}$ *for* $n > 1$. we use induction to show that $f\left(f_{(n)}\right) < f_{(n+1)}$. The inequality is true for n =1, since $f_{(2)} > 1 = f\left(f_{(1)}\right)$. *Also if* $f_{(n+1)} > f(f_{(n)})$, then $f_{(n+1)} \geq f(f_{(n)})$+1.

Hence $f\left(f_{(n)}\right)f_{(n+2)} + 1 \geq f\left(f_{(n+1)}\right)f\left(f_{(n)}\right) + f\left(f_{(n+1)}\right)$. Since $n + 1 > 1$, *we have* $f_{(n+1)} > 1$, *thus* $f\left(f_{(n+1)}\right) > 1$, which implies that $f_{(n+2)} > f\left(f_{(n+1)}\right)$.

Therefore the function satisfies $f_{(n+1)} > f(f_{(n)})$ for all N. In view of problem 6, the only function with this property is the identify function, and we done.

6) **Find all function f : No → No satisfying following two conditions: (ii) For any m , n ∊N_0, $\boldsymbol{with\ m \geq n, f_{(m^2)} \geq f_{(n^2)}}$ (i) for any m, n ∊N_0, $2f\left(m^2 + n^2\right) = \left\{f_{(m)}\right\}^2 + \left\{f_{(n)}\right\}^2$;**

Sol.: Substituting successively m = 0 and n = 0 in (a) and subtracting the two relations yields $f_{(m)^2} - f_{(n)^2} = 2\left(f_{(m^2)}\right)\left(f_{(n^2)}\right)$ which together with (b) implies that f is increasing i.e. if $m \geq n$, then $f_{(m)} \geq f_{(n)}$. Plugging m = n = 0 into (b) yields $f_{(0)} = 0$ *or* 1.

Case I : $f_{(0)} = 1$, *then* $2f_{(m^2)} = f_{(m^2)} + 1$, so $f_{(1)} = 1$ *plugging* $m = n = 1$ in (a) we get $f_{(2)} = 1$. *Also* $f_{(2^n)} = \frac{1}{2}(f(2^{2n-1})^2 + 1)$. This implies that $f_{(2^k)} = 1$ for all non-negative integers k. By the monotonicity of f, we conclude that $f_{(n)} = 1$for all non-negative integers n.

Case II : $f_{(0)} = 0. Then\ 2f_{(m^2)} = f_{(m^2)}\ or \frac{f_{(m^2)}}{2} = \left(\frac{f_{(m)}}{2}\right)^2\ since\ f_{(2)} = f_{(1)^2}$, we obtain

$$\frac{f(2^{2n})}{2} = \left(\frac{f(2^{2n-1})^2}{2}\right) = \left(\frac{f(2^{2n-1})}{2}\right)^2$$
$$= \cdots = \left(\frac{f_{(2)}}{2}\right)^2$$
$$= \frac{f(1)^{2n+1}}{2^{2n}}.$$

On the other hand, (a) implies that $f(1) = f_{(1)^2}$. So either $f_{(1)} = 0\ or\ f_{(1)} = 2$

If $f_{(1)} = 0$, the above chain of equalities implies that $f_{(2^{2n})} = 0\ for\ n \geq 0$. Monotonicity implies that f is identically equal to zero.

If $f_{(1)} = 2, then\ f_{(2^{2n})} = 2.\,2^{2n}. Since \frac{f_{(m^2)}}{2} = \left(\frac{f_{(m)}}{2}\right)^2$ $f_{(m)}$ is always even. We have$f_{(m+1)}{}^2 = 2f(m+1)^2 \geq 2f(m^2+1) = f_{(m)^2} + f_{(1)^2} > f_{(m)^2}$. which implies that $f_{(m+1)} > f_{(m)}$.

Consequently, $f_{(m+1)} - f_{(m)} - 2 \geq 0$

$But\ \sum_{m=0}^{2^{2n-1}}\left(f_{(m+1)} - f_{(m)} - 2\right) = f_{(2^{2n})} - f_{(0)} - 2.\,2^{2n} = 0\ varying\ n\ we\ conclude\ that\ f_{(m+1)} = f_{(m)} + 2$ for all $m \geq 0$. Thus $f_{(n)} = 2n\ for\ all\ n\ \in N_0$.

$In\ comclusion, f_{(n)}$ identically equal to zero. $f_{(n)}$ identically equal to 1, or $f_{(n)} = 2n$ for all n, are the only possible solutions.

7) Let P be a given odd prime. Find all functions f: z → z satisfying the following conditions (i) If m = n (mod p) for m, n ∈ z, then $\boldsymbol{f_{(m)} = f_{(n)}}$; (ii) $\boldsymbol{f_{(mn)} = f_{(m)} f_{(n)}}$ for all m, n ∈z.

Sol.: Show first that $f_{(0)} = 0\ or\ f_{(n)} = 1$ for all n ∈ z consider non constant solution of the given equation. Show that $f_{(kp)} = 0$ for all integers k. Using Fermat's little theorem, prove that $f_{(m)} = f_{(m)}{}^p$ for each integer m. This $f_{(m)} = 0\ or\ f_{(m)} = \pm1$. Choose m = a, a primitive root, with respect to p, Then $f_{(a)} \neq 0$. Consider the cases $f_{(a)} = 1\ and\ f_{(a)} = -1$ separately.

$$f_{(n)} \equiv 0, f_{(n)} \equiv 1, f_{(n)} = \begin{cases} 0 & if\ \frac{p}{n} \\ 1 & if\ p \times n \end{cases}$$

$$f_{(n)} = \begin{cases} 0 & if\ p/n \\ 1 & if\ p \times n\ , n\ is\ a\ square \\ 1 & if\ p \times n,\quad n\ is\ not\ square \end{cases}$$

The last function is precisely Legendre's symbol.

8) Find all the function f : z → z which satisfies the equation $\boldsymbol{f(a^3 + b^3 + c^3) = \{f_{(a)^3} + f_{(b)^3} + f_{(c)}\}^3}$

Sol.: Show that $f_{(0)} = 0$ and hence $f_{(x)} = -f_{(x)}$ for all x∈ z. Prove that $f_{(1)} = -1, 0\ or\ 1$ and hence $f_{(2)} = 2f_{(1)}\ f_{(3)} = 3f_{(1)}. For\ x > 3$ prove that x^3 is a sum of five cubes has absolute value smaller than x, using the identity. $(2k+1)^3 = (2k-1)^3 + (k+4)^3 + (4-k)^3 + (-5)^3 + (-1)^3$ using this representation, prove that $f_{(x)} = xf_{(1)}$.

$$f_{(x)} = -x, f_{(x)} = 0\ or\ f_{(x)} = x$$

9) For what integers k, there exists a functions f : N →z which satisfies, (i)

f(1995) = 1996 and (ii) $f_{(xy)} = f_{(x)} + f_{(y)} + kf\{\gcd(x,y)\}$ for all x, y ∈N?

Sol.: Using (b), get an expression for $f(x^2)$ and hence for $g(x^4)$. Using $x^4 = x.x^3, x^3 = x.x^2$, get another expression for $g(x^4)$. Show that k = 0 or -1. Using prime decomposition, define if f suitably for these value of k.

10) The set of all positive integers is the union of two disjoint subsets : $N\{f_{(1)}, f_{(2)}, \ldots, f_{(n)}, \ldots\} \cup \{g_{(1)}, g_{(2)}, \ldots, g_{(n)}, \ldots\}$ $where\ f_{(1)} < f_{(2)} < \cdots < f_{(n)} \ldots, g_{(1)} < g_{(2)} < \cdots < g_{(n)}$ $and\ g_{(n)} = f\{f_{(n)}\} + 1; for\ all\ n \geq 1. find\ f_{(240)}$.

Sol.: Show that $f_{(1)} = 1\ and\ g_{(1)} = 2. Suppose\ f_{(n)} = k$ for some n. Show that the disjoint sets $\{f_{(1)}, f_{(2)}, \ldots, f_{(k)}\} and \{g_{(1)}, g_{(2)}, \ldots, g_{(n)}\}$, together exhaust all the number from 1 to $g_{(n)}$. Conclude that $g_{(n)} = k + n$. Prove that $f_{(k)} = k + n - 1$. Show also that no two consecutive integers lie in the set $\{g_{(m)} : m \in N\}$. Conclude that $f_{(k+1)} = k + n$ use these three implication to get $f_{(240)} = 388$.

11) If f: w → N be a strictly increasing function such $f_{(2)} = 2\ and\ f_{(mn)} = f_{(m)} f_{(n)}$ for every relatively prime pair of natural numbers m and m. Show that $f_{(n)} = n\ for\ every$ positive integer n.

Sol.: One can easily see that $f_{(n)} = n$ satisfies the given property. Let us show this is the only function. The proof is based on factorizations of positive integers.

We start by computing the value of $f_{(3)}$. Since the function is increasing, $f_{(3)} f_{(5)} = f_{(15)} < f_{(18)} = f_{(2)} f_{(9)}\ hence\ f_{(3)} f_{(5)} < 2f_{(9)}\ and\ f_{(9)} < f_{(10)} = f_{(2)} f_{(5)} = 2f_{(5)}$. Combining the two inequalities we get $f_{(3)} f_{(5)} < 4f_{(5)}\ hence\ f_{(3)} < 4$. We also have that $f_{(3)} > f_{(2)} = 2$; thus $f_{(3)}$ can be equal only to 3.

Since 2 and 3 are relatively prime, we deduce that $f_{(6)} = 6$, and from monotinicity it follows that $f_{(4)} = 4\ and\ f_{(5)} = 5$. We will prove by induction that $f_{(n)} = n$ for all n ∈ N. For n = 1, 2, 3, 4, 5, 6 the property is true, as shown above. Let n > 6 and assume that $f_{(k)} = k < n$. Let us show that, $f_{(n)} = n$. Consider $2^r(2m+1)$ to be the smallest even integer greater than or equal to n that is not a power of 2. This number is equal either to n, $n + 1, n + 2\ or\ n + 3$, and since n >6, both $2^r\ and\ 2m + 1$ are strictly less than n.

Hence $f(2^r(2m+1)) = f(2^r) f(2m+1) = 2^r(2m+1)$ by the induction hypothesis. From monotonicity and the fact that are exactly $2^r(2m+1)$ values that the function can take in the interval $[1, 2^r(2m+1)]$ it follows that $f_{(k)} = k\ for\ k \leq 2^r(2m+1)$. In particular, $f_{(n)} = n$, and the proof is finished.

12) Find a bijective function f: No →No such that for all m, n; $f(3mn + m + n) = 4f_{(m)} f_{(n)} + f_{(m)} + f_{(n)}$

Sol.: The solutions, as in the case of the previous problem, uses of the factorization of positive integers. Suppose that a function f having the required property has been found. We use f to be defined a function.

$$g: 3N_0 + 1 \to 4N_0 + 1 \; by \; g(x) = 4f\left(\frac{x-1}{3}\right) + 1$$

This is certainly well defined and one can check immediately that g is a bijection from $3N_0 \to 1$ onto $4N_0 \to 1$. with the inverse function given by $g^{-}(y) = 3f^{-1}\left(\frac{y-1}{4}\right) + 1. For\ m, n \in N_0$ by using the definition of f and g, we obtain $g(3m+1)(3n+1) = g(3(3mn+m+n)+1) = 4f(3mn+m+n)+1$

$= 4(4f_{(m)}f_{(n)} + f_{(m)}+f_{(n)})$ +1= $(4f_{(m)}+1)+1$

$= (4f_{(m)}+1)(4f_{(n)}+1) = g(3m+1)g(3n+1)$.

This g is multiplicative, in the case sense that $g(xy) = g_{(x)}g_{(y)}\ for\ all\ x, y \in 3N_0 + 1$.

Conversely, given any multiplicative bijection from $3N_0 + 1\ onto\ 4N_0 + 1$, we can construct a function f having the required property by letting $f_{(x)} = (g(3x+1))$.

It remains only to exhibit such a bijection. Let $P_1\ and\ P_2$ denotes the sets of primes of the for 3n +1 and 3n +2, respectively and let $Q_1\ and\ Q_2$ denote the sets of primes of the form 4n +1 and 4n +3 respectively. Since each of these sets is infinite, there exists a bijection h from $P_1 \cup P_2\ to\ Q_1 \cup Q_2\ that\ maps\ P_1$ bijectively onto $Q_1\ and\ Q_2$. Define g as following $g_{(1)} = 1$, $and\ for\ n > 1, n \in 3\,N_0 + 1$, let the prime factorization of n be n = $\prod p_i$(with possible repititions among the $p_i's$), then define $g(n) = \prod h(p_i)$.

Note that g is well –defined, because if $n \in 3\,N_0 + 1$, then there must be an even number of P_2 type primes that divide n. Each of these primes gets mapped by h to a prime in Q_2, and since there are an even number of such primes, their product lies in $n \in 4\,N_0 + 1$. The multiplicavity of g follows easily.

13) Find whether there exists a function f: N → N such that $f\{f_{(n)}\} = n^2 - 19n + 99$ for all positive integer n.

Sol.: Such a function does exist. Let $P_{(n)} = n^2 - 19n + 99\ and\ note\ that\ P_{(n)} = P(19-n) and\ that\ p_{(n)} \geq 9\ for\ all\ n \in N$. we first set $f_{(9)} = f_{(10)} = 9\ and\ f_{(8)} = f_{(11)} = 11$. (one could alternatively set $f_{(9)} = f_{(10)} = 11\ and\ f_{(8)} = f_{(11)} = 9$)

Write ${P_{(n)}}^{(k)}$ for the kth composite of P. That is ${P_{(n)}}^{(0)} = n\ and\ {P_{(n)}}^{(k+1)} = {p_{(n)}}^{(k)}. For\ n \geq 12, let\ g_{(n)}$ be the smallest integer k such that n is not in the image of $P^{(k)}$. Such a k exists because a side from 9 and 11, every integer in the image of ${P_{(n)}}^{(k)}\ for\ k > 0$ is greater than or equal to ${P_{(12)}}^{(k)}$, and an easy induction shows that ${P_{(m)}}^{(k)} > n + k\ for\ n \geq 12$.

Let 12= $S_1 \leq S_2 \leq \cdots$ be the integers greater than or equal to 12, not in the image of P, in creasing order. Then for every integer $n \geq 12$, there exists a unique integer h(n) such that $n = P^{(g_{(n)})}S_{(n_{(n)})}$

$For\ n \geq 12, set$

$$f_{(n)} = \begin{cases} P(g^{(n)})(S\,h(n)+1)h(n)\ odd \\ P(g^{(n)+1})\ (S\,h(n)-1)\ h(n)\ even \end{cases}$$

$For\ n \leq 7, put\ f_{(n)} = f_{(19-n)}$. To show that $f(f_{(n)}) = P_{(n)}$

We need only consider $n \geq 12$, and we may examine two cases. If h(n) is odd, then $g(f_{(n)}) = g_{(n)}\ and\ h(f_{(n)}) = h_{(n)} +$

$1\ is\ even, so\ f\left(f_{(n)}\right) = f\left(P^{(g_{(n)})}(S\ h_{(n)} + 1)\right) = P^{(g_{(n)}+1)}\ S\ h_{(n)} = P_{(n)}$

Similarly, if h(n) is even, then $g\left(f_{(n)}\right) = g_{(n+1)}\ and\ h\left(f_{(n)}\right) = h_{(n)} - 1 is\ odd,$

$so\ \ f\left(f_{(n)}\right) = f P^{(g_{(n)}+1)}\left(s\ h_{(n)} - 1\right) = P^{(g_n+1)}(\ s\ h_{(n)}) = P_{(n)}$

14) Find all functions f : R→ R which obey the equation $\boldsymbol{f\{(x-y)^2\} = \{f_{(x)}\}^2 - 2xf_{(y)} + y^2}$.

Sol.: One can easily guess that $f_{(x)} = x$ is a solution of this functional equation. Are there any other solutions which are not obvious but hidden in the equation? Indeed there is one more solution, $f_{(x)} = x + 1$ which is not apparent from the equation. We see that for the function, $f_{(x)} = x + 1\ we\ have$

$f_{((x-y)^2)} = (x-y)^2 + 1 = x^2 - 2xy + y^2 + 1, and$

$f_{(x)^2} - 2xf_{(y)} + y^2 = (x+1)^2 - 2x(y+1) + y^2 = x^2 - 2xy + y^2 + 1.$

How do we compute these two and other if any? Put y = 0 in (1) to obtain $f(x^2) = f(x)^2 - 2xf_{(0)}\ and\ put\ x = 0$ to get, $f_{(y^2)} + f_{(0)^2} + y^2$.

$Taking\ y = 0\ in\ (3) we\ see\ that\ f_{(0)^2} = f_{(0)}\ giving\ f_{(0)} = 0\ or\ f_{(0)} = 1. Taking\ x = y$ in (1), we obtain $f_{(0)} = f_{(x)^2} - 2xf_{(x)} + x^2 = (f_{(x)} - x)^2$

$If\ f_{(0)} = 0,$ then the above relation shows that $f_{(x)} = x$ for all x ∈R. If $f_{(0)} = 1, then\ f_{(x)} - x = \pm i\ and\ hence\ f_{(x)} = x \pm 1,$ which sign should we choose here? It may also happen that $f_{(x)} = x + 1$ for some real number x and $f_{(y)} = y - 1$ for some other real number y. We have to resolve this before concluding anything. $Suppose$

$$f_{(x_0)} = x_0 - 1$$

$$for\ some\ real\ number\ x_0. Then\ using\ (3)$$

$and\ (2), we\ get\ 1 + x_{0^2} = f(x_{0^2}) = f(x_0)^2 - 2x_0 = (x_0 - 1)^2 - 2x_0 = x_0{}^2 - 4x_0 + 1. This\ faces\ x_0 = 0.$ but the we obtain $1 = f_{(0)} = f_{(x_0)} = x_0 - 1 = -1$, which is absurd, we concluded that $f_{(x)} = x + 1$ for real number x.

If follows that $f_{(x)} = x$ and $f_{(x)} = x + 1$ and the only solutions of the given functional equation.

15) If f : R →R be a function such that (i) $\boldsymbol{f_{(x+y)} = f_{(x)} + f_{(y)} \forall\ x, y \in R\ and}$ (ii) $\boldsymbol{f\left(\frac{1}{x}\right) = \frac{f_{(x)}}{x^2} \forall\ x\ \neq 0, show\ that\ f_{(x)} = cx \forall\ x\ \in R}$ where c is constant.

Sol.: It is easy to check that (a) gives $f_{(0)} = 0\ andf\ f_{(-x)} = -f_{(x)}$ for all real x, we know, for x ≠ 0 and x ≠ 1, the identity $\frac{1}{x-1} - \frac{1}{x} = \frac{1}{x(x-1)}$

This in conjunction with the property (a) gives

$$f\left(\frac{1}{x-1}\right) - f\left(\frac{1}{x}\right) = f\left(\frac{1}{x(x-1)}\right)$$

Now an application of (b) yields

$$\frac{f(x-1)}{(x-1)^2} - \frac{f(x)}{x^2} = \frac{f(x(x-1))}{x^2(x-1)^2}$$

$This\ simplifies\ to$

$$x^2f(x-1)-(x-1)^2f(x)=f(x^2-x)$$

If we use (a) and $f(-y) = -f_{(y)}$ *here, we obtain*

$$f(x^2)+x^2f(1)=2xf_{(x)}.$$

Replacing x by $x+\left(\frac{1}{x}\right)$ and simplifying, we obtain $f_{(x)}=\left(\frac{f_{(2)}+2f_{(1)}}{4}\right)x$, valid for all $x \neq 0$ and $x \neq 1$.

Putting x = 2 in this relation, we see that $f_{(2)}=2f_{(1)}$ Thus we obtain $f_{(x)}= f_{(1)^x}$, for all $x \neq 0$ and $x \neq 1$.

This remains valid for x = 0 and x = 1 as may be seen by inspection.

The above problems reveal the fact that using simple manipulations, we can solve some functional equations on R. We have not exactly effectively used any structure of R to arrive at the solution, next few problems tell us how to use the known structure(s) of real numbers to solve equations.

16) Let f : R →R is a function such that $f\left(\frac{x+y}{x-y}\right)=\frac{f_{(x)}+f_{(y)}}{f_{(x)}-f_{(y)}}$, $\boldsymbol{for\ all\ x\ \neq y}$. Show that $f_{(x)}=x, f\ x\ \in R$

Sol.: we use similar techniques here as in the earlier problem but in a more subtle way. We start with the observation that such a function is one−one and hence cannot be constant on any interval. Otherwise the right hand side is not defined since the denominator reduces to zero.

Taking y = 0 in (1). We obtain $f(1)=\frac{f_{(x)}+f_{(0)}}{f_{(x)}-f_{(0)}}$

This can be solved for $f_{(x)}$ to get

$$f_{(x)}\left(f_{(1)}-1\right)= f_{(0)}\left(f_{(1)}+1\right).$$

$If\ f_{(1)}\neq 1, then\ we\ get\ f_{(x)}=\frac{f_{(0)}\left(f_{(1)}+1\right)}{f_{(1)}-1}$, showing that f is constant function. Since we have ruled out constant function. We conclude that $f_{(1)}=1$ and hence $f_{(0)}=0$. Now replacing y by $x-2$ in (i), we obtain $f_{(x-1)}=\frac{f_{(x)}+f_{(x-2)}}{f_{(x)}-f_{(x-2)}}\ldots\ldots.(2)$

If we replace x by $x-1$ and y by 1 in (1), we get

$$f\left(\frac{x}{x-2}\right)=\frac{f_{(x+1)}+1}{f_{(x-1)}-1},\ldots\ldots.(3)$$

Where we have used $f_{(1)}=1$. If we use the value of $f_{(x-1)}\ from\ (2) in\ (3) and\ simplify, we\ get\ f\left(\frac{x}{x-2}\right)=\frac{f_{(x)}}{f_{(x-2)}}\ldots\ldots\ldots..(4)$

A comparison of (3) and (4) shows that

$$f_{(x)}=\ f_{(x-2)}\left\{\frac{f_{(x-1)}+1}{f_{(x-1)}-1}\right\}\ldots\ldots\ldots..(5)$$ putting x = 3 in (3), we get

$$f_{(3)}=\frac{f_{(2)}+1}{f_{(2)}-1}.$$

$Similarly\ the\ substitution\ x=4\ in$ (4) leads to $f_{(4)}=\ f_{(2)^2}$. Taking x = 5 in (5).

We also obtain $f_{(5)}=\ f_{(3)}=\left\{\frac{f_{(4)}+1}{f_{(4)}-1}\right\}=\left\{\frac{f_{(2)}+1}{f_{(2)}-1}\right\}\left\{\frac{f_{(2)^2+1}}{f_{(2)^2-1}}\right\}=\frac{f_{(2)^2+1}}{\left(f_{(2)^2-1}\right)^2}$.

However we can also express $f_{(5)}$ in a different way using (1). $f_{(5)}=f\left(\frac{3+2}{3-2}\right)=\frac{f_{(3)}+f_{(2)}}{f_{(3)}-f_{(2)}}$ using the expression for $f_{(3)}, which$ we have obtained earlier, we get $f_{(5)}=\frac{f(2)^2+1}{1+2f_{(2)}-f(2)^2}$

Comparing two expressions for $f_{(5)}$, we see that $(f_{(2)} - 1)^2 = 1 + 2f_{(2)} - f_{(2)^2}$.

The quadratic equation for $f_{(2)}$ simplifies to $f_{(2)^2} = 2f_{(2)}$.

We conclude that $f_{(2)} = 0 \; or \; f_{(2)} = 2. \; since \; f \; is \; one - one \; and \; f_{(0)} = 0,$ we cannot have $f_{(2)} = 0$.

The only possibility is $f_{(2)} = 2$.

This is the most difficult and important step is getting a solution of our problem. The set follows familiar track. We compute $f_{(3)} = 3, f_{(4)} = 4 \; and \; f_{(5)} = 5$. Suppose $f_{(k)} = k$ for all natural numbers $k <-= n$, where n is a natural number.

Then (5) shows that $f_{(n+1)} = \; f_{(n-1)} \left\{\frac{f_{(n)}+1}{f_{(n)}-1}\right\}$

$Since \; f_{(n-1)} = n - 1 \; and \; f_{(n)} = n$, we obtain $f_{(n+1)} = n + 1$. We conclude that $f_{(n)} = n$ for all natural numbers n. Replacing y by xz in (1), we get

$$f\left(\frac{x + xz}{x - xz}\right) = \; f\left(\frac{1 + z}{1 - z}\right) = \frac{1 + f_{(z)}}{1 - f_{(z)}},$$

$where \; we \; have \; used$ (1) again, comparing these two expressions and solving for $f_{(xz)}$, we obtain $f_{(zx)} = \; f_{(z)}f_{(x)}. A \; priori$ this is valid for x ≠ 0 and z ≠ 1. But since $f_{(0)} = 0 \; and \; f_{(1)} = 1$, we see that this multiplicative property is valid for all x, z in R. Taking $y = -x$ in (1), we see that $f_{(0)} = \frac{f_{(x)}+f_{(y)}}{f_{(x)}-f_{(y)}}, given \; us \; f_{(-x)} = \; -f_{(x)}$. This f is also an odd function. Since $f_{(n)} = n$ for all natural numbers n, now it follows that $f_{(k)} = k$ for all integers k. This with multiplicativity (6) implies that $f_{(r)} = r$ for all rational number r. Since (6)implies that $f_{(x^2)} = f(x)^2$. It follows that maps non-negative reals to non-negative reals, since f is one-one and $f_{(0)} = 0$. We conclude that $f_{(x)} > 0$. Whenever x > 0. Suppose x > y we consider different cases: (a) suppose x > y≥ 0. Here we obtain

$\frac{f_{(x)}+f_{(y)}}{f_{(x)}-f_{(y)}} = f\left(\frac{x+y}{x-y}\right) > \; 0,$ showing that $f_{(x)} > f_{(y)}$. (b) Suppose y< 0<x. In this case $f_{(y)} < 0 \; and \; f_{(x)} > 0$ show that $f_{(y)} < f_{(x)}$.

$(c) consider$ the case y < x < 0. Then $0 < -x < -y$ and by (a), we conclude that $f_{(-x)} < f_{(-y)}$.

Using the fact that f is an odd function, this reduces to $f_{(y)} < f_{(x)}$. It follows that f is a strictly increasing function on R. Since $f_{(r)} = r$ for all rational number r, we obtain $f_{(x)} = x$ for real number x.

17) Find all f : R →R such that $f(f_{(x)} + y) = f(x^2 - y) + 4(x)y \; \forall \; x, y \; \in R$.

Sol.: It is easy to check that $f_{(x)} \equiv 0 \; and \; f_{(x)} = \; x^2$ are solutions of this problem. We show that these are the only solutions of the problem.

Suppose $f_{(a)} \neq a^2$ for some a. Replacing by y in (1) by $\frac{(x^2-f_{(x)})}{2}, we \; get \; f_{(x)}(x^2 - f_{(x)}) = \; 0.$

$Since \; f_{(a)} \neq a^2, it \; follows \; that \; f_{(a)} = 0$. This also shows that a ≠ 0, for then $a^2 = 0 = \; f_{(a)}$ contradicting the choice of a. We further observe that $f_{(x)} = 0 \; or \; f_{(x)} = \; x^2$ for any x. In any case $f_{(0)} = 0. Taking \; x = 0$ in (1), we get $f_{(y)} = \; f_{(-y)}$.

Putting x = a and replacing y by –y, we also see that $f_{(a^2+y)} = f_{(-y)} = \; f_{(y)}$

Thus f is periodic with period a^2. This implies that $f(f_{(x)}) = f(f_{(x)} + a^2) = f_{(x^2-a^2)} + 4f_{(x)}a^2$. Putting y = 0 in (1) we get another expression $f(f_{(x)}) = f_{(x^2)}$

Invoking the periodicity of f. we note that $f_{(x)}a^2 = 0$

However, we have observed that a ≠0 by our choice of a. It follows that if $f_{(x)} \neq x^2$, then we must have $f_{(x)} \equiv 0$. This completes our claim and determines all the solutions of the problem.

18) Find all f: $(-\mathbf{1}, \infty) \to (-\mathbf{1}, \infty)$ such that the function

(i) $\boldsymbol{f\{x + f_{(y)} + xf_{(y)}\} = y + f_{(x)} + yf_{(x)}, for\ all\ x, y \in (-1, \infty)}$

(ii) $\frac{f_{(x)}}{x}$ **is strictly increasing on each of the intervals (-1, 0) and (0, ∞)**

Sol.: Let f: $(-1, \infty) \to (-1, \infty)$ be a function of the desired type since $\frac{f_{(x)}}{x}$ is strictly increasing on the interval (-1, 0) the equation $f_{(x)} = x$ can have at most one solution in (0, ∞). Moreover x = 0 may be a solution in $f_{(x)} = x$. Thus the equation $f_{(x)} = x$ can have at most three solution in $(-1, \infty)$. In other words, there are at most three fixed points of $f_{(x)}$ in the domain $(-1, \infty)$.

Suppose u ∈ (-1, 0) is a fixed point of $f_{(x)}$. Thus we have $f_{(u)} = u$. Taking x = y = u in (a). we see that $f(2u + u^2) = 2u + u^2$. This shows that $2u + u^2$ is also a fixed point $f_{(x)}$. We claim that $2u + u^2$ is also a fixed point is in the interval (-1, 0). In fact $2u + u^2 = u(2 + u) < 0$. Since u <0 and 2+u>1>0 because u>-1. On the other hand $2u + u^2 > -1$ because $2u + u^2 + 1 = (u + 1)^2 > 0$.

Since there can be at most one fixed point of $f_{(x)}$ in (-1, 0). We conclude that

$2u + u^2 = u. This\ forces\ u(u + 1) = 0$, we contradicting the assumption that u ∈(-1, 0). It follows that there is no fixed point of $f_{(x)}$ in (-1, 0). Similar analysis shows that $f_{(x)}$ has no fixed in (0, ∞) as well. Thus 0 is the only possible fixed point of $f_{(x)}$ if at all it has any. However taking x = y in (i), we see that $f(x + f_{(x)} + xf_{(x)}) = x + f_{(x)} + xf_{(x)}\ for\ all\ x \in (-1, \infty)$. Thus each $x + f_{(x)} + xf_{(x)}, x \in (-1, \infty)$ is a fixed point of f. We conclude that $x + f_{(x)} + xf_{(x)} = 0$ for all x∈ (-1, ∞) we see that

$$x + f_{(y)} + xf_{(y)} = x - \frac{y}{1 + y} - \frac{xy}{1 + y} = \frac{x - y}{1 + y}.$$

$$Thus\ we\ obtain\ f(x + f_{(y)} + xf_{(y)}) = f\left(\frac{x - y}{1 + y}\right) = \frac{y - x}{1 + y}.$$

It follows that $f_{(x)} = -\left(\frac{x}{1+x}\right)$ indeed satisfies (a) we can easily check that is also satisfied (b).

The fixed points, can also be used in proving non-existence of solutions to some functional equations. The following problem illustrates this point.

19) Find all functions f: R →R such that $\boldsymbol{f\{x - f_{(y)}\} = f\{f_{(y)}\} + xf_{(y)} + f_{(x)} - 1}$ **holds for all x, y ∈R.**

Sol.: We easily see that $f_{(x)} = 1 - \frac{x^2}{2}$ satisfies the equation (1), We show that this is the only function which obey the relation (1). Let S denote the range of f. Put c = $f_{(0)}$. Taking x = y= 0 in (1), we obtain

$f(-c) = f_{(c)} + c - 1$

This shows that c ≠ 0. Taking x = $f_{(y)}$ in (1), we also get c = $f_{(x)} + x^2 + f_{(x)} - 1$.

This gives $f_{(x)} = \frac{c+1}{2} - x^2 \ldots\ldots\ldots..(2)$ whenever x = $f_{(y)}$

This determines f on s, the range of f. Taking y = 0 in (1), we get $f(x-c) = f_{(c)} + cx + f_{(x)} - 1$

This can be written in the form $f(x-c) - f_{(x)} = cx + f_{(c)} - 1$ consider the set $\{cx + f_{(c)} - 1; x \in R\}$. Since c ≠ 0, it follows that this set is R itself. Thus we conclude that $\{f(x-c) - f_{(x)}; x \in R\} = R$. We use this to determine f on R. Fix any x ∊ R. we find $y_1, y_2 \in S\ such\ that\ x = y_1 - y_2. Let\ y_2 = f_{(x_2)}. Then f_{(x)} = f\left(y_1 - f_{(x_2)}\right) = f\left(f_{(x_2)}\right) + y_2\, f_{(x_2)} + f_{(y_1)} - 1 = f_{(y_2)} + y_1 y_2 + f_{(y_1)} - 1.$

But we know f on s: from (2) we see that $f_{(y_2)} = \frac{c+1}{2} - \frac{{y_2}^2}{2},\ f_{(y_1)} = \frac{c+1}{2} - \frac{{y_1}^2}{2}$. Putting these values, we obtain $f_{(x)} = C - \frac{(y_2-y_1)^2}{2} = C - \frac{x^2}{2}$.

Comparing these expressions, we conclude that c = 1.

Thus we obtain $f_{(x)} = 1 - \frac{x^2}{2}$, for all x ∊ R

Alternate Solution: As in the first solution, we take c = $f_{(0)}$. Putting x = $f_{(y)}$ in (1), we can solve for $f\left(f_{(y)}\right)$: $f\left(f_{(y)}\right) = \frac{c+1-f(y)^2}{2} \ldots\ldots..(3)$

$Introducing\ g(x) = f_{(x)} + \frac{x^2}{2}$ it is easy to compute $g\left(x - f_{(y)}\right) = g(x) + \frac{c-1}{2}$

Note that the given equation has no constant solutions. Thus we may find $y_0\ such\ that\ f_{(y_0)} \neq 0. Taking\ x = \frac{1}{f_{(y_0)}}\ and\ y = y_0\ in\ (1), we\ obtain\ f\left(x - f_{(y_0)}\right) = f\left(f_{(y_0)}\right) + f_{(x)}. Setting\ x - f_{(y_0)} = a, and\ f_{(y_0)} = b, we\ obtain\ f_{(a)} = f_{(b)} + f_{(x)}$

Thus (4) gives $g(x) + \frac{c-1}{2} - g\left(x - f_{(a)}\right) = g\left(x - f_{(b)} - f_{(x)}\right) = g\left(x - f_{(b)}\right) + \frac{c-1}{2} = g_{(x)} + c - 1$.

It follows that c = 1 and now (4) shows that $g\left(x - f_{(y)}\right) = g_{(x)}$, for all reals x, y. Thus we obtain that every element in the range of f is a period for g. However putting $f_{(0)} = c = 1\ in$ (3), we obtain $f_{(1)} = f\left(f_{(0)}\right) = \frac{1}{2}$. Also taking y = 0 in (1), we see that $f_{(x-1)} = x + f_{(x)} - \frac{1}{2}$. We have proved that $\frac{1}{2}, f_{(x)}\ and\ x + f_{(x)} - \frac{1}{2}$ are periods of g. Since a linear combination of several periods is again a period, x it self is period for g.

Since this is true for every real number x, we conclude that g is constant function. However $g_{(0)} = f_{(0)} = 1\ and\ we\ get\ g_{(x)} \equiv 1$. The definition of g shows that $f_{(x)} = 1 - \frac{x^2}{2}$.

Some of the functional equations may require a single or a combination of several ideas in their solutions. This is illustrated in the solution of the following few problems.

20) Find all functions f : $[1, \infty) \rightarrow [1, \infty)$ which satisfy,

(i) $\boldsymbol{f_{(x)} \leq 2(1+x) \forall x \in [1, \infty)}$;

(ii) $\boldsymbol{x f_{(x+1)} = \{f_{(x)}\}^2 - 1 \forall\, x \in [1, \infty)}$.

Sol.: It is easy to verify that $f_{(x)} = x + 1$ satisfies both (a) and (b). We show that is the only solutions.

We have $f_{(x)^2} = xf(x+1)+1 \le x\big(2(x+1)\big)+1 = 1+4x+2x^2 < 2(1+2x+x^2) = 2(1+x)^2. It\ follows\ that\ f_{(x)} < \sqrt{2}(1+x)$ using this fresh bound, we obtain

$$\begin{aligned} f_{(x)^2} = xf(x+1)+1 &< \sqrt{2}x(2+x)+1 \\ &= \sqrt{2}x^2 + 2\sqrt{2}x + 1 \\ &< \sqrt{2}(x^2+2x+1) \\ &= \sqrt{2}(x+1)^2 \end{aligned}$$

Thus we obtain another bound; $f_{(x)} < 2^{\frac{1}{4}}(x+1)$.

Continuing by induction, we arrive at $f_{(x)} < 2^{\frac{1}{2^k}}(1+x), for\ all\ k\ \in N, and\ x\ \in [1,\infty)$.

It follows that $f_{(x)} \le 1+x\ for\ all\ x\ \in [1,\infty)$

$Suppose\ f_{(x_0)} < 1+x_0\ for\ some\ x_0 \in [1,\infty)$. Let $f_{(x_0)} = 1+x_0 - \in\ where\ 0 < \in < x_0$, we then have

$$\begin{aligned} f_{(1+x_0)} &= \frac{f(x_0)^2-1}{x_0} \\ &= \frac{(1+x_0-\in)^2-1}{x_0}\ x_0 - 2 \\ &\in +2+\frac{\in^2 - 2\in}{x_0} \le\ x_0-2 \\ &\in +2+\in -2 =\ x_0 - \in < x_0 \end{aligned}$$

Using this bound we get

$$f_{(x_0+2)} = \frac{f(x_0+1)^2-1}{x_0+1} < \frac{{x_0}^2-1}{x_0+1} < x_0 - 1.$$

This is turn implies that

$$\begin{aligned} f_{(x_0+3)} &= \frac{f(x_0+2)^2-1}{x_0+2} < \frac{(x_0-1)^2-1}{x_0+2} \\ &= \frac{x_0(x_0-2)}{x+2} < x_0 - 2 \end{aligned}$$

By an easy induction, we see that $f_{(x_0+k)} < x_0 - k + 1$

If k is large enough, then $f_{(x_0+k)} < 1$. This contradiction forces $f_{(x)} = 1+x\ for\ all\ x\ \in [1,\infty)$

In some cases the functional relation may reveal some useful information about the function.

GEOMETRY

1) The length of a rectangle is increasing by 60%. By what percent would the width have to be decreased to maintain the same area?

(a) 37.5% (b) 37% (c) 75% (d) none

Sol.: Let the length = x and breadth = y

$\therefore$ Area $= xy$

New length $= \frac{160x}{100} = \frac{8x}{5}$

Let the new breadth be y

$\therefore\ \frac{8x}{5} \times c = xy.$

$or,\ \ \frac{8x}{5} \times c = xy$

$\therefore$ decreases in breadth $= y - \frac{5y}{8} = \frac{3y}{8}$

$\therefore Decrease\ \% = \frac{3y}{8} \times \frac{1}{y} \times 100 = 37.5\%$

2) On the two square fields, the area of one is 1 heal are, while the other one is border by 2%. The different in their areas is

(a) $400m^2$; (b) $404m^2$; (c) 410 m^2 ; (d) none

Sol.: Each side of a square field is $\sqrt{1}\ hectare\ \sqrt{10{,}000}\ m^2 = 100m.$

$\therefore$ each side of the second square field is $(100 + 100 \times 1\%)m = 102m.$

Thus the area is $(102m)^2 = 10{,}404m^2$

$\therefore$ Required area $= (10{,}404 - 10{,}000)m^2 = 404m^2.$

3) **The diagonal of a square A is (x+y). The diagonal of a square B with twice the area of A is**

   (a) $\sqrt{(x+y)}$; (b) $\sqrt{(2x+y)}$; (c) $\sqrt{(x+2y)}$; (d) none

Sol.: If a is a side of the square A, then $2a^2 = (x+y)^2 \ldots\ldots..(i)$

$\therefore$ Area of the square $B = 2 \times$ Area of square A $= 2 \times (2a^2) = 4a^2. But\ 4a^2 = 2(x+y)^2$ from (i)

$\therefore$ Side of the square B $= \sqrt{2(x+y)}$

4) **If the base of a rectangle is increased by 10% and the area is unchanged, then the corresponding altitude must be decreased by**

   (a) 10%; (b) $9\left(\frac{1}{11}\right)$%; (c) 11% ; (d) none

Sol.: Area of the rectangle is xy where x is base and y is itself. Second time area is $\left(x + \frac{x}{10}\right)y'$ where y' is height.

$$\frac{1x}{10}.y' = \frac{10}{11}y$$

$\therefore$ Altitude decreased $= y - \frac{10}{11}y = \frac{9}{11}y\ i.e. \frac{9}{11}$% is decreased.

5) **The interior angles of a hexagon are in the ratio 1 :2 :2: 3: 2: 2, then the largest angle is**

   (a) 150°; (b) 170°; (c) 180°; (d) none

Sol.: The sum is interior angles of a hexagon $= (2 \times 6 - 4)90° = 720°$

Now let its angles be x, 2x, 2x, 3x, 2x, 2x respectively

$\therefore x + 2x + 2x + 3x + 2x + 2x = 720.$

$\therefore 12x = 720°, x = 60°$

$\therefore The\ largest\ angle = 3x = 3 \times 60° = 180°$

6) **The difference between the interior and exterior angles of a regular polygon is 60°. The polygon is**

   (a) Hexagon; (b) Octagon; (c) Decagon; (d) none

Sol.: Let the polygon be n-sided.

Thus, per the condition given

$$\frac{(2n-4) \times 90}{n} - \frac{360}{n} = 60$$

$$or, \quad (2n-4)90 - 360 = 60n\ or\ 180n - 720 = 60n$$

$$or, \quad n = 6$$

$\therefore$The polygon is a hexagon.

7) As ABC and DBC are on the same base BC. AL ⊥BC and DM ⊥BC. Then area Δ ABC : Area ΔDBC is

(a) AO : AD; (b) $\boldsymbol{AO^2}: \boldsymbol{OD^2}$; (c) AO : OD ; (d) none

Sol.: $\frac{Area\ (\Delta ABC)}{Area\ (\Delta DBC)} = \frac{\frac{1}{2}.BC.AL}{\frac{1}{2}.BC.DM} = \frac{AL}{DM} = \frac{AO}{OD}$

(as ΔALO Δ DMO are similar)

∴ The needed ratio = AO: OD.

8) If in ΔABC and Δ DEF $\frac{\boldsymbol{AB}}{\boldsymbol{DE}} = \frac{\boldsymbol{BC}}{\boldsymbol{EF}} = \frac{\boldsymbol{AC}}{\boldsymbol{DF}} = \frac{\mathbf{2}}{\mathbf{3}}$, then area ΔABC : area ΔDEF is

(a) 2: 3; (b) 4: 9; (c) 3: 2; (d) none

Sol.: Since ΔABC ~ ΔDEF

∴ $\frac{Area\ of\ \Delta ABC}{Area\ of\ \Delta DEF} = \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{AC^2}{DF^2} = \frac{4}{9}$

9) In an equilateral ΔABC, if AD ⊥ BC, then

(a) $\boldsymbol{3AB^2 = 4AD^2}$; (b) $\boldsymbol{3AB^2 = 2AD^2}$; (c) $\boldsymbol{2AB^2 = AD^2}$; (d) none

Sol.: Let its side be a

$$\therefore AD^2 = a^2 - \left(\frac{a}{2}\right)^2 = \frac{3a^2}{4} = \frac{3AB^2}{4}$$

$$\therefore 3AB^2 = 4AD^2$$

10) The parallel sides of a trapezium are a and b. Then the line joining the mid-points of its non-parallel sides will be

(a) $\boldsymbol{\frac{a+b}{2ab}}$; (b) $\boldsymbol{\frac{a-b}{2}}$; (c) $\boldsymbol{\sqrt{ab}}$; (c) none

Sol.: By rule, $\frac{a+b}{2}$

11) The radius of a circle is 5 cm. Two chords of length 6 cm and 8 cm, respectively are drawn parallel to each other. Then the distance between the chord is

(a) 1cm; (b) 3 cm; (c) 4 cm; (d) none

Sol.: In the figure, OC = 5 cm = OA

CM $= \frac{1}{2}CD = \frac{1}{2}8cm = 4\ cm$

$AN = \frac{1}{2}AB = \frac{1}{2}6\ cm = 3cm$

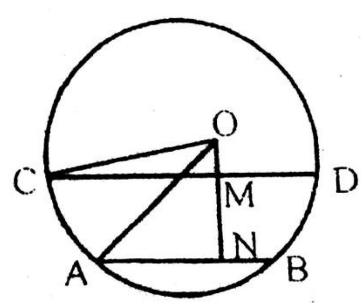
$\therefore ON^2 = OA^2 - AN^2 = 5^2 - 3^2 = 25 - 9 = 16$

$\Longrightarrow ON = 4$

$Again, \Longrightarrow OM^2 = OC^2 - CM^2 = 5^2 - 4^2 = 25 - 16 = 9$

$\therefore OM = 3cm$

∴ MN = The distance between the chords = ON –OM = 4 -3 = 1 cm.



12) If AB and AC are tangents to the circle with centre O, if ∠CAB = 60° then ∠BDC = is

(a) 60°; (b) 70°; (c) 80 °; (d) none

Sol.: Here $\angle ACO = 90°; \angle CAO = \frac{1}{2} \times 60° = 30°$

$\therefore \angle COA = 180° - 90° - 30° = 60°$

$Again\ \angle COA = \angle BOA = 60°.$

$Thus, \angle COB = \angle ODA + \angle BOA = 60° + 60° = 120°.$

$Also, \angle BDC = \frac{1}{2}\angle COB = \frac{1}{2} \times 120° = 60°.$

13) Let A, B, C are three points on a circle with centre 0. If ∠AOB = 90° $\boldsymbol{and\ \angle BOC = 130°, then\ \angle ABC}$ is

(a) 45°; (b) 55°; (c) 65°; (d) none

Sol.: $\angle AOC = 360° - (90° + 130°) = 140°$

$\therefore$Also $\angle ABC = \frac{1}{2}\angle AOC = \frac{1}{2} \times 140° = 70°$

14) Let AD, AE, BC are tangents to the circle at D, E, F respectively, then

(a) AD= AB+ BC+ CA; (B) 2AD= AB+BC+ CA; (C) 3AD = AB+BC+CA; (D) none

Sol.: Knowing that the tangents drawn to a circle from a point outside are equal, on get

AD = AE, BD = BF, CF = CE

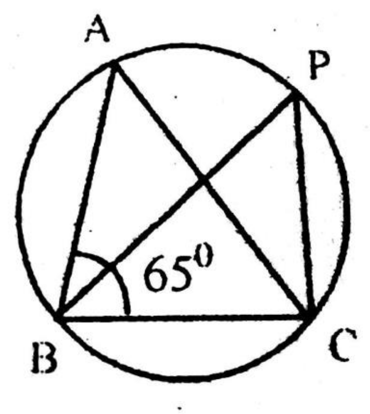
∴ AD = AB+BD= AB+BF

Also AD= AE = AC + CE = AC +CF

∴ 2AD = AB+ AC+BF+ CF = AB +AC +BC.

15) In the given fig. ∠ABC = 65° and AB = AC, then the measure of ∠BPC is

(a) 115°; (b) 30°; (c) 70°; (d) none



Sol.: In Δ ABC; AB = AC

∴ ∠ACB = ∠ABC = 65°

∴ ∠BAC = 180° − (65° + 65°) = 50°

But ∠BPC = ∠BAC = 50°

(SUBJECTIVE TYPE)

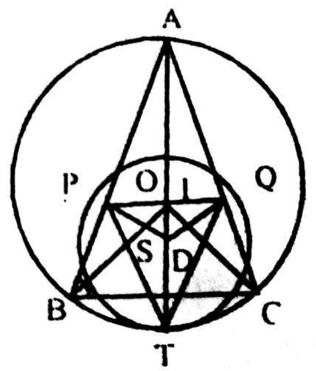
1) In a Δ ABC, AB = AC. A circle is drawn touching the circum circle of ΔBC internally and also, touching the sides AB and Ac at P and Q respectively. Show the mid. Point of PQ is the in centre of ΔABC.

Sol.: Let $\angle ABC = \angle ACB = \beta$.

ATS the angle bisector of ∠A. I is the mid-point of PQ. Now AP = AQ as the smaller circle touches AB and AC at P and Q respectively. The centre of the circle PQT lies on the angle bisector of ∠A, namely AT; since

PQ is the chord of contact of the circle PQT , PQ ⊥ AT and the midpoint 1 of PQ lies on AT.

Now to prove that I is the in centre of ΔABC, it is enough to prove that BI is the angle bisector of ∠B. and CI is the angle bisector of ∠C respectively. By symmetry ∠PTI = ∠QTI = $\alpha°$



Now ∠ABT = 90° (∵AT is diameter of ⨀ABC)

∴ ∠PBT = 90° Also ∠PIT = 90°

∴ PBTI is cyclic

∴ ∠PBI = ∠PTI = α°(angle in the same segment)

∴ ∠IBD = ∠ABD -∠ABI = $\beta - \alpha$

∠TBC = ∠TAC = 90° − β

$\therefore \angle I\beta T = \angle IBD + \angle DBT = \beta - \alpha + 90° - \beta = 90° - \alpha$

$Since\ PBTI\ is\ cyclic, \angle IPT = \angle IBT = 90° - \alpha \ldots\ldots\ldots (1)$

$\angle BPT = 180° - \angle TPA = 180° - \angle API - \angle IPT \quad = 180° - \beta - 90° + \alpha = 90° + \alpha - \beta \ldots\ldots\ldots. (2)$

But APT is a tangent to circle PQT ∠BPT = ∠PQT = ∠IQT from (1) and (2),

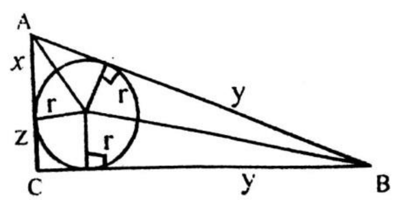
we get $90° + \alpha - \beta = 90° - \boldsymbol{\alpha} \;\therefore 2\alpha = \beta$

$\therefore \angle IBD = \beta - \angle PBI = 2\alpha - \alpha = \alpha \quad \therefore \angle IBD = \angle PBI$

∴ BI is the angle bisector of ∠B. Hence the result.

2) ABC is a rt. Angled triangle with ∠ C = 90°. The centre and the radius of the inscribed circle is I and r. Show that $\boldsymbol{AI} \times \boldsymbol{BI} = \sqrt{\mathbf{2}} \times \boldsymbol{AB} \times \boldsymbol{r}$.

Sol.:



Area of the right angled Δ ACD = $\frac{1}{2}\, AC \times BC$

$= \frac{1}{2}(x + r)(y + r) = \frac{1}{2}\{xy + r(x + y) + r^2\}$

$\Rightarrow AC \times BC = xy + rAB + r^2 \Rightarrow xy = AC \times BC - rAB - r^2$

$Now\ AI^2 \times BI^2 = (x^2 + r^2)(y^2 + r^2) = x^2y^2 + r^2(x^2 + y^2) + r^4$

$= x^2y^2 + r^2[(x+y)^2 - 2xy] + r^4$
$= x^2y^2 + r^2[(AB^2 - 2xy)] + r^4$

$= x^2y^2 + r^2 AB^2 - r^2 + 2xy + r^4$
$= x^2 AB^2 + (r^2 - xy)^2$

$= r^2AB^2 + [r^2 - AC \times BC + rAB + r^2]^2$
$= r^2AB^2 + [2r^2 - AC \times BC + rAB]^2$

$Area\ of\ the\ \Delta ABC = r(r + x + y)$
$= r(r + AB) = r^2 + rAB$
$= \frac{1}{2} AC.BC$

$\Longrightarrow AC \times BC = 2r^2 + 2r.AB$

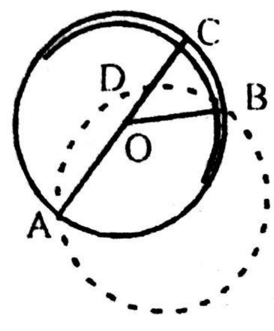
$\therefore AI^2 \times BI^2 = r^2AB^2 + [2r^2 - 2r^2 - 2rAB + rAB]^2 = r^2AB^2 + r^2AB^2 = 2r^2.AB^2$

$\therefore AI.BI = \sqrt{2r^2.AB^2} = \sqrt{2r}.AB.$

3) **Let A and B be two points on a circle k. Suppose that on arc k' of a another circle 1 connects A with B and divides the area inside the circle k into two equal parts. Show that arc 'k' is longer than the diameter k.**

Sol.: Since arc 'k' bisector the area of the circle k, k cannot entirely lie on the one side of any diameter of circle k.

Hence every diameter ok k intersects k' Let AC be one such diameter and k' intersects AG at D, say. Now the centre O of the circle k lies inside the circle L, and hence the radius AO of circle k lies inside L and now D lies on the radius OC.



Length of arc ABD > AD+ DB

As we have to prove that arc ABD > AC = AD +DC, we should show that DB> DC.

Now the circle k'' with centre D and radius DC, is a circle touching k internally and B lies outside this circle k'', So the radius of k'' is less than DB i.e. DC <DB or DB >DC.

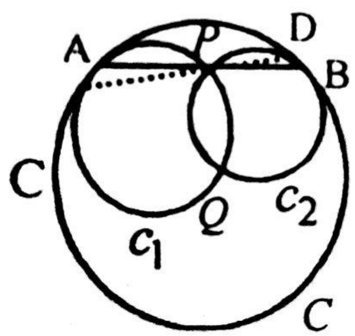
$\Longrightarrow$ arc ADB > AD +BD > AD+ DC = AC

$\Longrightarrow$ arc ADB > the diameter of k.

Note that O lies inside the circle k' since every diameter of k meets the circle k' (i.e. arc AB) as k' bisects area in k.

4) **Two given circles intersects in two points P and Q. Show how to construct a segment AB passing through P and terminating on the two circles such that AP. PB is a maximum.**

Sol.: Let c_1, c_2 be two circles. We first show that if APB is a straight line such that there is a circle c touching c_1 at A and c_2 at B, then A, AB is segment giving the required maximum.



Let A'P and P'B be any other chords so that A'PB' may be collinear and the extension of these chords meet the circle c at C and D.

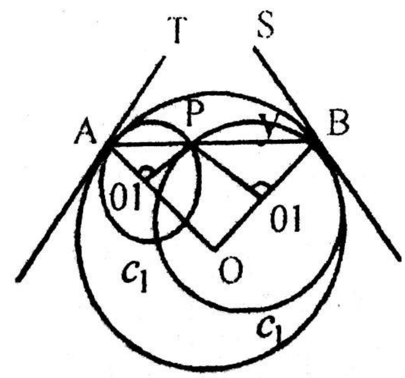
CP. CD = AP. PB > A'P × PB'

∴ AP. PB is maximum. Now we need to construct such a chord APB. For this we need to construct a circle c touching c_1 *and* c_2 at points A and B. So that APB are collinear. Let us find the properties of the points A and B.

Let O be the centre of the circle C and O_1 *and* O_2 be the centres of the circles c_1 *and* c_2. Now C and c_1 touch at A.

∴ A O_1O are collinear. Similarly B O_2O are collinear. Let AT, BS be the common tangents to circles C and c_1 and C and c_2 respectively.

Let ∠PAT = x and ∠PBS = y since AT is tangent to circle c.



$\angle PAT = x = \frac{1}{2}\angle AOB$ (angle in the alternate segment theorem). Since BS is tangent to circle c.

$\angle PBS = y = \frac{1}{2}\angle AOB$. ∴ x = y since AT is tangent to circle c_1, we get $\angle PAT = x = \frac{1}{2}\angle AO_1P$

Similarly since BS is tangent to circle c_2, we get $\angle PBS = y = \frac{1}{2}\angle BO_2P = x \; \therefore \; \angle A\,O_1P = \angle AOB = \angle BO_2P$

$\therefore \Delta AO_1P \sim \Delta PO_2B$

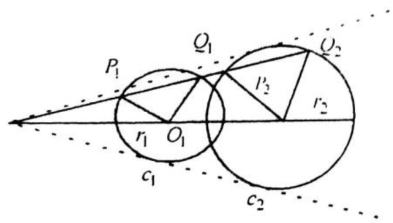
$\therefore \dfrac{AP}{PB} = \dfrac{AO_1}{PO_2} = \dfrac{r_1}{r_2}.$

There for the line segment AB must be such that P divides AB internally in the ratio $r_1 : r_2$

Further $PO_2 \parallel OO_1$ *and* $PO_1 \parallel OO_2$.

So join PO_1 *and* PO_2. Through O_2 draw a line parallel to PO_1 to must the circle c_2 *in B*. Now these two parallel lines drawn meet at O. If we drawn a circle with O as centre and radius OA = OB, then the circle touches c_1 at A and

c_2 at B. By retracting the arguments we can prove that APB are collinear and AB is the required chord.



Note→ in the previous problem the line AB and O_1O_2 meet in a point S_1 say. This point S_1 divides O_1O_2 externally in the ratio $r_1 : r_2$. The point S_1 is called the external centre of similitude of 2 circles c_1 *and* c_2. If we draw any line 1 through S_1 meeting c_1 *in* P_1, Q_1 *and* c_2 *in* P_2, Q_2 *then* $O_1P_1 \parallel O_2P_2$ *and* $O_1Q_1 \parallel O_2Q_2$.

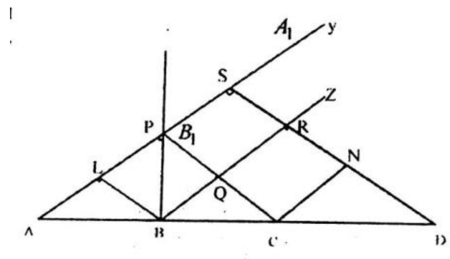
Moreover the direct common tangents to the two circles c_1 *and* c_2 meet at S_1.

5) **Let A, B, C , D be four given points on a line. Contract a square such that two of its parallel on a line. Contract a square such that two of its parallel sides or their extensions go through A and B respectively and the other two sides (or their extensions) go through C and D respectively.**

Sol.: Draw BB' ⊥ r' to l and BB' = CD. Join AB' and extend it to y. Through C and D draw perpendiculars to meeting Ay at P and S. Through B draw BZ perpendicular to CP and SD meeting them at Q and R respectively. PQRS is the required square.

Proof: Draw BL and CN ⊥r to AS and SD respectively.

$$\Delta LBB' \equiv \Delta\, NCD \text{ as } \angle LBB' = 90° - \angle ABL = \angle LAB = \angle NCD \text{ and } BB' = CD \text{ and } \angle BLB' = \angle CND = 90°. PQ = LB = CN = QR = PS$$

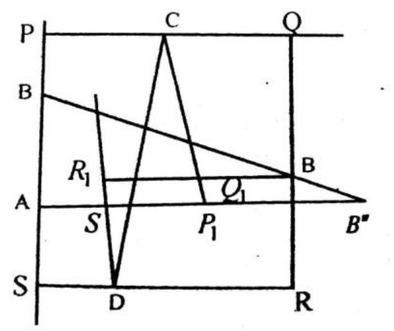


Thus the adjacent sides of the rectangle PQRS are equal and hence it is a square.

If B' is constructed on the opposite half-plane, we get P'Q'R'S', the reflection on the PQRS about the line 1 and lying on the opposite half plane.

This construction exactly follows the same procedure BB'(BB) is perpendicular to CD and equal to CD. Join AB'(or AB")

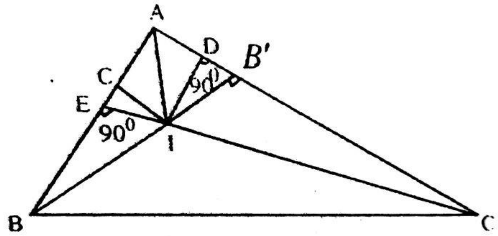
Draw CP . DS⊥ r to AB' produced and extend PC and SD. Through B draw BR and BQ perpendicular to SR and PQ. PQRS is the required square.



Draw $CP_1, DS_1 \perp r\ to\ AB''$ and through B draw ⊥ r to cp_1 $and\ D\ S_1$produced meeting them at Q and r_1 respectively. $P_1Q_1R_1S_1$ is the required square. Thus there are two solutions(The proof is similar to that to the former one).

6) In Δ ABC, AB ≠ AC. The bisector of ∠ B and ∠C meet their opposite sides AC and AB at B' and C respectively. The Two bisectors intersects at I. Show that, if IB' = IC', then ∠BAC = 60°.

Sol.: Let us first prove that the perpendiculars from 1 to AB and AC. i.e. IE and ID lie on opposite sides of the bisectors CC' i.e. (points E and D lie on opposite, sides of the bisectors CC' , where IE and ID are the perpendiculars drawn from the in centre I to AB and Ac.) In the figure D and E lie on the same side of CC'. IE = ID (in radii of ΔABC)



IC' = IB' (given)

∠IEC' = ∠IDB' = 90°,

$so\ \ \Delta^s\ IEC'and\ \ IDA'$ are congruent ∠IC'E = ∠IB'D ……..(1)

$In\ \ \Delta AIC' and\ \Delta AIB'$. AI is the bisector of ∠A

$\therefore \angle IAC' = \angle IAB'$

$\therefore \angle IC'A = \angle IB'A$

$[From\ (1)] \ldots\ldots\ldots.(2)$

AI is common. So $\Delta^s AIC'and\ AIB'$ are argument from(2)

∠BC'I = $180° - \angle IC'A = 180° - \angle IB'A = \angle IB'C$

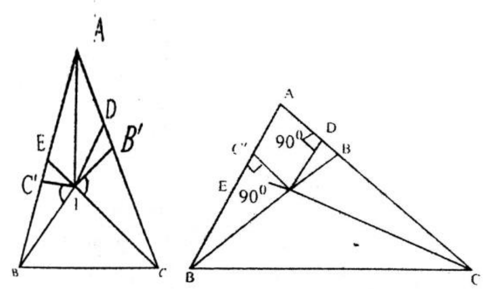
$\angle C'IB = \angle B'IC$ (vertically opposite ∠s)

$IC' = IB'(given)$

∴BC' = CB'

∴AB = BC'+C'A= CB'+B'A =AC

But by hypothesis AB ≠ AC and hence, the assumption that E and D are on the same side of CC' is false.



In the figure, we have taken AB < AC, and E and D lie on opposite sides of CC'.

If AB> AC, then E and D lie on opposite sides of BB'

Now, for the solution of the main problem.

In Δ^s $IB'D and\ IC'E. IC' = IB', ID = IE$ and hence Δ IB'D = ΔIC'E (congruence of right angled triangles)

∠EC'I= ∠BC'I = ∠BC'C = $180° - \left(B + \frac{1}{2} + C\right)$

∠IB'D = ∠BB'D = C+$\frac{1}{2}B$

But ∠IB'D = ∠IC'E (since the ΔIB'D = IC' E) proved.

$\therefore 180° - \left(B + \frac{1}{2}C\right) = C + \frac{1}{2}B$

$\Rightarrow \frac{3}{2}(B + C) = 180°,$

$B + C = \frac{2}{3} \times 180° = 120°$

$\Rightarrow A = 180° - 120° = 60°$

7) **A circle passes through the vertex c of a rectangle ABCD and touches the sides AB and AD at M and N respectively. If the distance from c to the line segment MN is equal to b units, find the area of the rectangles.**

Sol.: Let 'O' be the centre of the circle.

OM= ON = OC each being the radius of the same circle, end AMON is a square.

∠CMP = ∠CMN = ∠CND (angle in the alternate segment)

∠CNP = ∠CNM = ∠CMB (angle in the alternate segment)

∴∠CMP+∠CNP=∠CMP+∠CMB= ∠PMB = 180° − 45° = 135°

∠MCN $= \frac{1}{2}\angle MON = \frac{1}{2}. 90° = 45°$ (∴∠AMN = ∠OMN = 45° in the square ANON)

Δ^s CPN and CBM are similar for ∠CPN = ∠CBM = 90° and ∠CNP = ∠CMB …………(1)

Again Δ^s CPM and CDN are similar for ∠CPM = ∠CDN = 90° and ∠CMP = ∠CND $\frac{CP}{CB} = \frac{CN}{CM}$ (from 1), $= \frac{CD}{CP}$ $(from\ 2)$

$\therefore \frac{CP}{CB} = \frac{CD}{CP} \Rightarrow CP^2 = CD.CB \Rightarrow 5^2 = CD.CB,$ i.e., area of the rectangle is CB.CD = 25 sq. units.

8) **ABCD is a convex pentagon inscribed in a circle of radius 1 unit with AE as diameter. It AB = a, BC = b, CD = c, DE = d, Prove that $\boldsymbol{a^2 + b^2 + c^2 + d^2 + abc + bcd < 4}$**

Sol.: Since AE is the diameter ∠ACE = 90° and $Ac^2 + cE^2 = AE^2 = 2^2 = 4.$

By cosine formula (for Δ ABC)

$AC^2 = a^2 + b^2 - 2ab\cos(180° - \theta) = a^2 + b^2 + 2ab\cos\theta$

$Similarly\ in\ \Delta CED, CE^2 = c^2 + d^2 - 2cd\cos(90° + \theta) = c^2 + d^2 + 2cd\sin\theta$

$\therefore Ac^2 + CE^2 = a^2 + b^2 + c^2 + d^2 + 2ab\cos\theta + 2cd\sin\theta$

$In\ \Delta ACE, \dfrac{AC}{AE} = \sin\theta \Rightarrow AC = 2\sin\theta > b(\because AE = 2) \ldots\ldots\ldots.(1)$

$and\ \dfrac{CE}{AE} = \cos\theta\ (AE = 2)$

$\Rightarrow CE = 2\cos\theta > c$ …….(2)

(Because in Δ^s ABC and CDE, ∠B and ∠D are obtain angles and AC is the greatest side of ΔABC and CE is the greatest side of ΔCDE)

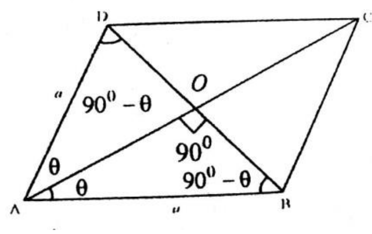
$\therefore AC^2 + CE^2 = a^2 + b^2 + c^2 + d^2 + 2ab\cos\theta + 2cd\cos\theta = 4$

$\Rightarrow\ a^2 + b^2 + c^2 + d^2 + ab.2\cos\theta + cd.2\sin\theta = 4$

$\Rightarrow a^2 + b^2 + c^2 + d^2 + abc + bcd < 4\ \ [by\ (1) and\ (2)]$

9) A rhombus has half the area of the square with the same side length. Find the ratio of the longer diagonal to that the shortest one.

Sol.: If a is the side of the rhombus, then area of the rhombus is $\frac{1}{2}a^2\sin 2\theta \times 2$.



By hypothesis, this area is equal to $\frac{1}{2}a^2 = a^2\sin\theta \Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = 30°\ or\ 150°$

$\Rightarrow \theta = 15°\ or\ 75°$

[If the acute angle of the rhombus is 30°, the other angle which is obtuse is 150°]

By sine formula, $\frac{BD}{\sin 2\theta} = \frac{AB}{\sin(90-\theta)}\ (In\ \Delta ABD)$

$\Rightarrow BD = \frac{a\times 2\sin\theta\cos\theta}{\cos\theta} = 2a\sin\theta$

$Again\ \ \dfrac{AC}{\sin(180-2\theta)} = \dfrac{a}{\sin\theta}\ (In\ \Delta ABC)$

$AC = \dfrac{a\sin 2\theta}{\sin\theta} = \dfrac{2a\sin\theta\cos\theta}{\sin\theta} = 2a\cos\theta$

$AC : BD = \cos\theta : \sin\theta\ [if\ \theta = 15°, then\ AC > BD\ \ and\ \theta = 75°, BD > AC].$

$AC{:}BD = \cos 15° : \sin 15° = \sin 75° : \sin 15° = \sin(45° + 30°) : \sin(45° - 30°)$

$= \sin 45°\cos 30° + \cos 45°\sin 30° : \sin 45°\cos 30° - \cos 45°\sin 30°$

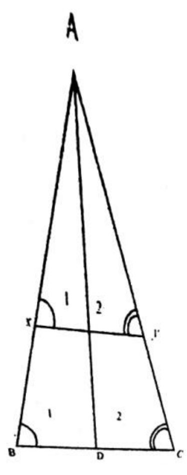
$= \frac{1}{2}(\sqrt{3} + 1) : \frac{1}{2}(\sqrt{3} - 1)$

$= (\sqrt{3} + 1) : (\sqrt{3} - 1)$

$$or, \quad \frac{AC}{BD} = \frac{\sqrt{3}+1}{\sqrt{3}-1} : (2+\sqrt{3}).$$

10) From a point E on the median AD of Δ ABC, the perpendiculars EF is dropped to the sides BC. From a point M on EF perpendiculars MN and MP are drawn to the sides AC and AB respectively. If N, E, P are collinear. Show that M lies on the internal bisector of ∠BAC.

Sol.: Before proving the main problem, let us prove the following: If in ΔABC, AD is the median xy is a line segment parallel to BC intersecting the median AD at E, then AE is the median of ΔAXY, or in other words XE = YE.



Δ AXE similar to ΔABD ………………..(1) and Δ AYE similar to Δ ACD ………………(2)

$\frac{AX}{AB} = \frac{AE}{AD} = \frac{XE}{BD} \ldots \ldots \ldots \ldots (3) and$

$$\frac{AY}{AC} = \frac{AE}{AD} = \frac{EY}{DC} \ldots \ldots \ldots . (4)$$

From (3) and (4) $\frac{XE}{BD} = \frac{AE}{AD} = \frac{EY}{DC}$

$$\Rightarrow \frac{XE}{BD} = \frac{EY}{DC} \Rightarrow \frac{BD}{DC} = \frac{XE}{EY}.$$

But D is the midpoint of BC and hence BD = DC ⟹XE = EY i.e., XE = YE ……….(5).

Now draw XY parallel to BC through E. join AM join the collinear points P, E, N. MPAN is a cyclic quadrilateral as ∠MPA+∠MNA = 90° + 90° = 180°.

Since EF is perpendicular to BC and XY is drawn parallel to BC. ∠XEM = ∠EFB = 90°. In the quadrilateral MPXE. ∠MPX+∠MEX = 90°+90° = 180° and hence MPXE is a cyclic quadrilateral and in the quadrilateral MENY. ∠MEY = ∠MNY = 90° …………..(6)

So MENY is a cyclic quadrilateral, since ∠MEY and ∠MNY are subtended by MY at E and N and they are equal by (6). In Δ^s MEX and MEY, XE = YE.

∠MEX = ∠MEY = 90° ME is common and hence ΔMEX ≡ ΔMEY

∴ ∠MEX = ∠MYE …………(7)

∠PAM = ∠PNM (angle on the same segment, in the cyclic quadrilateral MPAN) = ∠ENM = ∠EYM (angle on the same segment in quadrilateral EMYN by (7)) = ∠EXM = ∠EPM (angle on the same segment in cyclic quadrilateral MPEX) = ∠NPM = ∠NAM (cyclic quadrilateral APMN)

That is AM bisects the vertical angle A of Δ ABC. That is M lies on the bisector of ∠A.

11) Δ ABC is an isosceles triangle and XY is drawn parallel to the base cutting the sides in X and T. Show that is four points B, C, X, Y lie on a circle.

Sol.: Since xy ∥BC, and AB meets them, therefore,

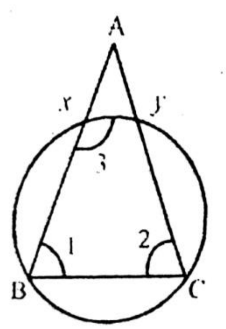
∠BXY +∠XBC = 2rt ∠S ……….(1)

Also, since AB = AB, ∠B = ∠C …………(2)

From (1) and (2), we find that

∠BXY +∠BCY = 2rt. ∠S

Since a pair of opposite angles of the quadrilateral BCYX is supplementary, therefore it is cyclic. i.e. the points B, C, X, Y lie on a circle.

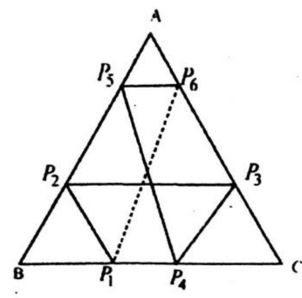


12) Take any pointP_1 on one side BC of a triangle ABC and draw the following chain of lines : P_1P_2 parallel to AC; P_2P_3 parallel to BC : P_3P_4 parallel to AB; P_4P_5 parallel to CA; P_5P_6 parallel to BC. Here P_2, P_5 lie on AB; P_3, P_6 on CA; and P_4 on BC. Show that P_6P_1 is parallel to AB.

Sol.: Suppose B$P_1 = k\ BC$, (so thatP_1 divides BC in the ratio k: 1-k).

Since $P_1P_2 \parallel AC, and\ BP_1 : BC = k : 1$

Therefore, from similar triangles BP_1P_2 and BCA, B$P_2 = k, BA, P_2A = (1-k)BA$.



Since $P_2P_3 \parallel BC,\ and\ P_2A = (1-k)BA$, therefore from similar triangles $P_2AP_3\ and\ BAC, we\ have$

$AP_3 = (1-k)AC, P_3C = k\ AC$

Since $P_3P_4 \parallel AB, and\ CP_3 = k.AC$

Therefore from similar triangles C$P_3P_4\ and\ CAB$.

$CP_4 = kCB. P_4B = (1-k)CB.$

$Since\ P_4P_5 \parallel CA, and\ P_4B = (1-k)CB,$ therefore B$P_5 = (1-k)BA.$

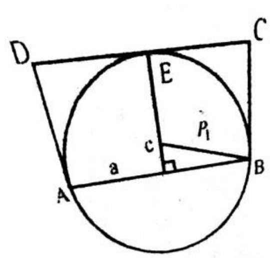
$Since\ P_5P_6 \parallel BC,$ and A$P_5 : AB = k : 1,$ therefore similar triangle A$P_5P_6, ABC, AP_6 = k$AC, $P_6C = (1-k)AC$. In ΔCAB P_1 divides CB in the ratio $1-k; k$ and P_6 divides CA in the ratio 1-k: M; i.e. $CP_6: P_6A = CP_1: P_1B$.

$Therefore\ P_6P_1 \parallel AB$

(Δs CP_6P_1 and CAB are similar and hence equi-triangular; consequently $\angle CP_6P_1 = \angle CAB$ which are corresponding angles)

13) Let ABCD be a rectangle with AB = a and BC = b. Suppose r_1 is the radius of the circle passing through A and B and touching CD; and similarly r_2 is the radius of the circle passing through B and C and touching AD. Show that $r_1 + r_2 \geq \frac{5}{8}(a + b)$

Sol.: Let O be the centre of the circle which touches CD and passes through the points A and B, E the point of tangency and F the point at which OE meets AB. Since OE ⊥ OC, and AB ∥DC, therefore OF ∥ AB, Also OF = b-r_1 $and\ FB = \frac{1}{2}a$



In right angled triangle OFB, $OB^2 = OF^2 + FB^2, show\ that\ {r_1}^2 = (b - r_1)^2 + \left(\frac{a}{2}\right)^2$

$$i.e., \qquad r_1 = \left(\frac{1}{2b}\right)\left(\frac{a^2}{4} + b^2\right) = \frac{1}{2}b + \frac{a^2}{8b}$$

Similarly, $r_1 = \frac{1}{2}a + \frac{b^2}{8a}$

Now $r_1 + r_2 = \left(\frac{1}{2}b + \frac{a^2}{8b}\right) + \left(\frac{1}{2}a + \frac{b^2}{8b}\right) = \frac{1}{2}(a+b) + \frac{a^2}{8b} + \frac{b^2}{8b}, = \frac{1}{2}(a+b) + \frac{a^3+b^3}{8ab}$

$$= \frac{1}{2}(a+b) + \frac{(a+b)[(a-b)^2 + ab]}{8ab}$$
$$\geq \frac{1}{2}(a+b) + \frac{1}{8}(a+b)$$
$$= \frac{5}{8}(a+b).$$

$$Thus\ \ r_1 + r_2 \geq \frac{5}{8}(a+b).$$

14) A rigid square plate ABCD of unit side rotates in its own plane about the middle point of CD unit the new position of A coincide with the old position of B. How far is the new position of B from the old position of A?

Sol: Let O be midpoint of CD. Since the new position of A coincides with the old position of B. Therefore, the rotation is in the counter clockwise sense (in fig) about O, through the angle AOB

Let OB and AB' intersects at P. Then ∠BOB' = ∠AOB. Also OA = OB = OB'. Therefore OB is the internal bisector of ∠AOB' of isosceles triangle AOB'. Therefore OP ⊥ AB' and AP = PB'

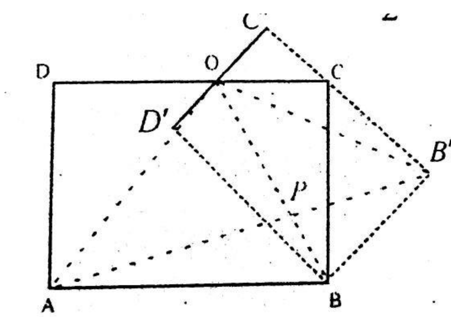
Also, $\angle AOB = \frac{1}{2} \times area\ of\ square\ ABCD = \frac{1}{2}$.

$$But, [AOB] = \frac{1}{2} AP.OB, OB = \sqrt{(OC^2 + CB^2)}$$

$$= \frac{\sqrt{5}}{2}$$

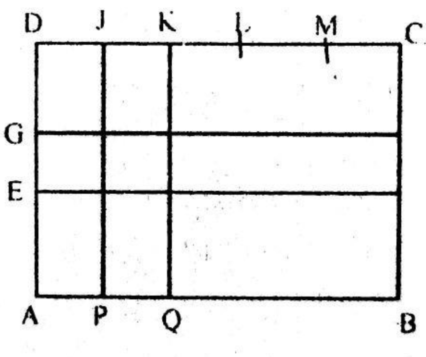
$$\therefore AP = \frac{2}{\sqrt{5}}, so\ that$$

$$AB' = 2AP = \frac{4}{\sqrt{5}} = \frac{4\sqrt{5}}{5}$$



15) Show how will you cut a rectangular sheet of paper along two lines segments parallel to a side and two parallel to an adjacent side, into five pieces whose areas are in the ratio 1 : 2 : 3 :4 : 5.

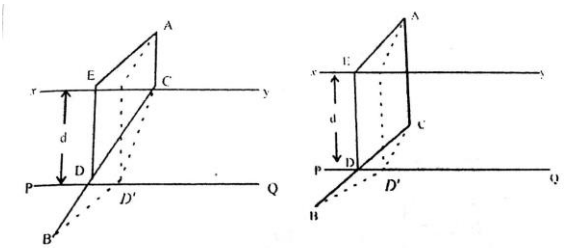
Sol.: Since 1+2+3+4+5 = 15, therefore really speaking the rectangles has to be divided into 15 equal parts, and then we have to take 1, 2, 3, 4, and 5 parts respectively to get the five pieces. We take two points E and G in AD so that AE = EG = GD and draw GH, EP parallel to AB. Also take points J. K, L, M in DC such that DJ = JK = KL = LM = MC and draw JP, KQ parallel to CD. Clearly areas of rectangles DJRG, GSTE, SHFT, JCHR, EFBA are in the ratio 1 : 2 : 3 : 4 :5. The division has been made by two lines GH. EP parallel to DC, and two lines JP, KQ parallel to DA.



16) A river flows between two houses A and B, the house standing some distance away from the banks , where should a bridge be built on the river. So that a person going from A to B using the bridge to cross the river may do so by the shortest path? Assume that the banks of the river are straight and parallel, and the bridge must be perpendicular to the banks.

Sol.: Let xy and PQ denote the two banks of the river (parallel to each other) at a distance d from each other and let A and B be the situated on opposite sides of the river as shown in the figure.

Take a point C on the line through A perpendicular to the banks and towards the bank such that AC= L (two different cases arise according as the point C is between A and xy, or is between xy and PQ of course, it can even be on xy) join CB. Let CB meet PQ in D. It can be shown that if DE be the perpendicular from D and xy, then bridge should be built along DE. As is clear from fig (a) and (b), the position of the bridge does not depend on the position on C.



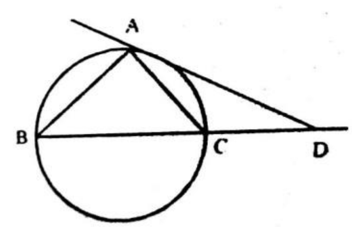
If the bridge is along ED, the distance required to be travelled for reaching from A to B = AE+ED+DB+= CD+AC+DB = L+CB.

In instead of building the bridge along ED, the bridge is built along some other line, say E', D' then the distance to be travelled would be AE'+E'D+D'B= CD'+AC+D'B= l+ C'D+D'B.

In ΔCD'B, CD'+D'B>CB, therefore if instead of building the bridge along DE, the bridge along some other line the distance required to be travelled would be greater.

17) ΔABC is scalene with ∠A having measure greater than 90°.Determine the set of points D on the extended line BC for which $|\boldsymbol{AD}| = \sqrt{\{(|\boldsymbol{BD}|)(|\boldsymbol{CD}|)\}}, \boldsymbol{where}\ |\boldsymbol{BD}|$ refers to the (positive) distance o0f B and D.

Sol.: We shall use the following well known result. If the tangent to a circle at a point. A meet a chord BC at D, then $AD^2 = |BD|.|CD|$



Draw the circle of the triangle ABC, and let the tangent to it at A meet BC produced at D. Then D is desired point. In order to prove the result stated above, we have only to observe that in Δs ABD and CAD, ∠ABC = ∠CAD (angles in the alternate segment): ∠ADB = ∠ADC so that the triangles are equiangular and hence similar consequently $\frac{AD}{BD} = \frac{CD}{AD}, where AD^2 = |BD|.|CD|$

18) Given any acute—angled Δ ABC, let points A'B' C' be located as follows: A' is the point where altitude A on BC meets the onwards –facing semicircle drawn on BC as diameter points B'. C' are located similarly. Show that $[\boldsymbol{BC'A}]^2 + [\boldsymbol{CAB'}]^2 + [\boldsymbol{ABC}]^2 = [\boldsymbol{ABC'}]^2$, $\boldsymbol{where}\ |\boldsymbol{ABC}|\boldsymbol{denotes\ the\ area\ of}\ \Delta\boldsymbol{ABC}$ etc.

Sol.: In right angled triangles are equiangular and hence similar. Consequently $\frac{BD}{A'D} = \frac{A'D}{DC}$, so that A'$D^2 = BD.DC\ since\ BD = A'Dcot\ B, DC = A'Dcot\ C$,

$Therefore\ A'^{D^2} = AD^2\ \cot B \cot C$

$$\therefore [BC'A]^2 = \left(\frac{1}{2}.BC.A'D\right)^2 = \frac{1}{4}BC^2.A'D^2$$

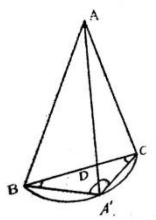
$$= \frac{1}{4}BC^2.AD^2 \cot B \cot C$$

$$= [ABC]^2 \cot B \cot C \ldots\ldots (1)$$

$Similarly, [CAB']^2$

$$= [ABC]^2 \cot C \cot A \ldots\ldots (2)$$

$[ABC]^2 = [ABC]^2 \cot A \cot B \ldots\ldots\ldots\ldots (3).$



Adding corresponding sides of (1), (2) and (3) and using the fact that in any angle.

$\cot B \cot C + \cot C \cot A + \cot A \cot B = 1,$

We get the desire result.

19) Given a ΔABC, define the equalities x, y, z as follows : x = $\tan\frac{B-C}{2}.\tan\frac{A}{2}; y = \tan\frac{C-A}{2}.\tan\frac{B}{2}; z = \tan\frac{A-B}{2}.\tan\frac{C}{2}$. *Prove that* $x + y + z + xyz = 0$.

Sol.: By Napier's analogies,

$$\tan\left[\frac{B-C}{2}\right] = \left[\frac{(b-c)}{(b+c)}\right]\cot\left(\frac{A}{2}\right)$$

$$So\ that\ x = \frac{(b-c)}{(b+c)}$$

$$Similarly, y = \frac{(c-a)}{(c+a)}, z = \frac{(a-b)}{(a+b)}$$

$$Now, x + y + z = \frac{b-c}{b+c} + \frac{c-a}{c+a} + \frac{a-b}{a+b},$$

$$= \left(-\frac{1}{p}\right)\sum(b-c)\{(c+a)(a+b)\}.where$$

$p = (b+c)(c+a)(a+b)$

$$Now \sum(b-c)(c+a)(a+b) = \sum a^2(b-c) + \sum a^2(b-c)$$

$\sum bc(b-c) + \sum a(b^2-c^2)$

$Now, \sum bc(b-c) = -(b-c)(c-a)(a-b),$

$\sum a(b^2-c^2) = -\sum a^2(b-c),$

$So\ that \sum bc(b-c) = -(b-c)(c-a)(a-b),$

$\sum a(b^2-c^2) = -\sum a^2(b-c).$

$So\ that\ x + y + z = \left(\frac{1}{p}\right)\left(-(b-c)(c-a)(a-b)\right) = -xyz$

$Hence\ x + y + x + xyz = 0.$

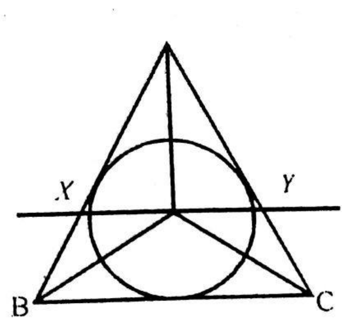
20) ΔABC has in centre I. Let x, y, z be located on the line segments that AB. AC respectively. So that Bx. AB= IB^2 *and* $Cy.AC = IC^2$. Given that the points x, I, y lie on a straight line, find the possible values the measure of ∠A.

Sol.: Since Bx. AB= IB^2, $therefore\ \frac{Bx}{BI} = \frac{BI}{BA}$ ……………(1)

In Δs B x I and BIA, ∠xBI = ∠ABI, and $\frac{Bx}{BI} = \frac{BI}{BA}$ from (1). Therefore Δs BxI and BIA are similar and hence equiangular.

$\therefore \angle BIx = \angle BAI = \frac{1}{2}\angle A$ ………….. (2)

Since AI is the interval bisector of ∠BAC (I being the in centre)



Similar by considering triangles $cyI\ and\ CIA$, we have $\angle CIY = \frac{1}{2}\angle A$ …………… (3)

$Also, \angle BIC = 180° - (\angle IBC + \angle ICB)$

$$= 180° - \left(\frac{1}{2}\angle B + \frac{1}{2}\angle\right)$$

$$= 180° - \left[90° - \frac{1}{2}\angle A\right],$$

$$= 90° + \frac{1}{2}\angle A \text{ ………..} (4)$$

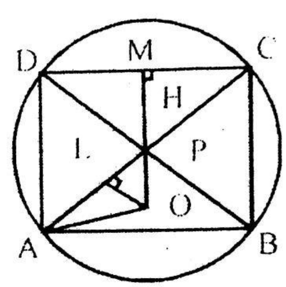
Since $xIY\ is\ a\ st.line, therefore\ \angle xIB + \angle BIC + \angle CIY = 180°$ ……..(5)

Adding corresponding sides of relations (2), (3) and (4), and using (5),

we have $\frac{1}{2}\angle A + \left(90° + \frac{1}{2}\angle A\right) + \frac{1}{2}\angle A = 180°\ So\ that\ \angle A = 60°$.

21) The diagonals AC and BD of a cyclic quadrilateral ABCD intersect at P. Let O be the circumcentre of ΔAPB and H be the orthocenter of ΔCPD. Show that the points H.P.O are collinear.

Sol.:



We shall show that if OP is produced to meet CD in M, then PM ⊥CD, so that PM is an altitude of ΔPCD. This will ensure that the orthocenter H of the triangle PCD lies on PM and consequently the points O, P, H are collinear. To complete the proof join AO, draw OL ⊥AP and consider Δs PLO and PMC. In these two triangles ∠LOP = $\frac{1}{2}\angle AOP$ (because O is the circumcentre of ΔABP, and OL ⊥ chord AP) = ∠APB (being the angle subtended by the chord AP at a point B of the circumcentre of the circle ABP)= ∠PCD(angles in the same segment of a circle).

Now in Δs PLO and PMC, ∠LOP = ∠PCM (proved), and ∠LOP = ∠CPM (vert opp ∠S), therefore ∠CMP= ∠PLO = 90° and consequently PM ⊥ CD, and the proof is complete.

22) Show that there exist convex hexagon in the plane such that (a) all its interior angles are equal (b) its sides are 1, 2, 3, 4, 5, 6 in some order.

Sol.: Suppose there exists a convex hexagon ABCDEF such that all its interior angles are equal, and the lengths of its sides are a, b, c, d, e, f respectively. Produced AB and DC meet at x, CD and EF to meet at y, and EF and BA to meet at z. It can easily be seen that Δxyz is equilateral. In fact since the hexagon ABCDEF us equiangular, each of its interior angles is 120°, consequently each of its interior angles is 60°be ∠XBC= ∠xcB = 60°, ∠EDY =∠DEY = 60°, ∠FAZ= ∠AFZ = 60°.

It follows that $\angle BxC, \angle DyE, \angle FzA$ are each equal to 60°. Since ΔBxC is equilateral, therefore, $Bx = xC = b$, similarly, $Dy = y\,E = d, Fz = zA = f$,

∴ zx = f+a+b, xy = b +c +d, yz = d+ e +f, since Δxyz is equilateral,

therefore f+ a +b = b +c +d = $d + e + f = \frac{1}{3}[(f + a + b) + (b + c + d) + (d + e + f)]$

$$= \frac{1}{3}(a + b + c + d + e + f) + \frac{1}{3}(b + d + f) \ldots\ldots\ldots..(1)$$

We are inserted in contacting a hexagon for which a, b, c, d, e, f are numbers 1, 2, 3, 4, 5, 6 in some order so, that $\frac{1}{3}(a + b + c + d + e + f) = 7$.

We shall choose values of b, d, f out of the given values in such a manner (to simplify the working) that b +d +f is a multiple of 3. This can be done in several ways. One with this choice, b +d+ f = 15, so that f +a +b= b +c +d = d+ e +f = 7 + $\frac{1}{3}$. 15 = 12

Using the values b = 4, d = 5, f = 6, we have a = 2, c = 3, e = 1. Thus we get a = 2, b = 4, c = 3, d = 5, e = 1, f= 6 we therefore have the following construction:

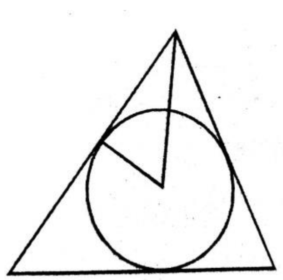
Construct an equilateral triangle xyz having each side equal to 12 units. Cut of P $xb = xc = 4$ units, $yD = yE = 5\ units, zA = zF = 6\ units$

Join the pairs of points B, C; D, E; F, A, ABCDEF is the desired hexagon.

Justification for the above construction has already been provided by the analysis of the problem.

23) Let ΔABC and circle C' be drawn lying inside the triangle touching the two sides AB and AC. Show that the radii of the circles C' and C is equal to $\boldsymbol{tan^2\left(\frac{\pi-A}{4}\right)}$

Sol.



:

Let I be the in centre, r the in radius and E the point of contact of the in circle with AB. Also let l' be the center of the circle touching AB., AC and the in circle, r' the radius of this circle and F its point of contact with AB. Since AB and AC both touch this circle and F its point of contact with AB. Since AB and AC both touch this circle, its centre must also lie on AI. From l' draw I'D $\perp$ IE in $\coprod'd$

$ID = r - r'll' = r + r'$

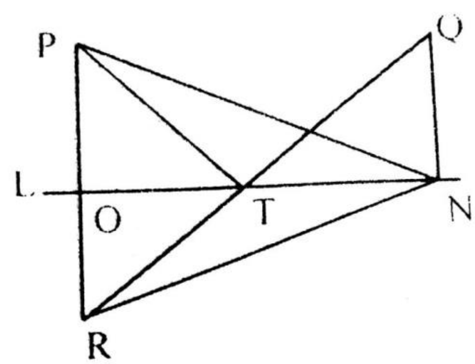
$\angle lDl' = \frac{\pi}{2}, \angle Dl'l = \frac{A}{2},$

$\therefore \frac{r-r'}{r+r'} = \sin\left(\frac{A}{2}\right) = \cos\theta\ ;\ \ where\ \theta = \frac{\pi - A}{2}.$

$$\therefore \frac{r'}{r} = \frac{1-\cos\theta}{1+\cos\theta} = \ tan^2\left(\frac{\theta}{2}\right) = \ tan^2\left(\frac{\pi - A}{4}\right).$$

24) Let λ be straight line and P.Q two distinct arbitrary points lying on one side of the line λ, but not lying on l. Determine with proof the point T on L such that the sum of the distance of P and Q from T shall be least.

Sol.:



Let R be the reflection of P in l, i.e. Let R be a point such that l is the perpendicular bisector of PR. Join QR and let T be its point of intersection with λ. We claim that T is the desired point.

Let N be the point on λ other than T. We shall show that since λ is the right bisector of PR, therefore every point on λ is equidistant from P and R. In particular TP = TR, NP = NR.

Therefore PT +QT = TR+QT= QR

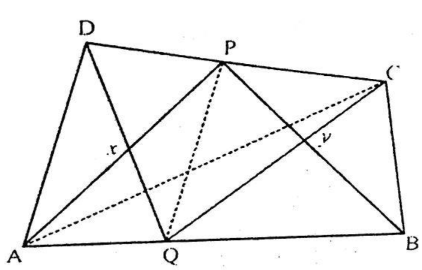
(Since Q, T, R are in a straight line).

Also PN +QN = QN+NR. Since two sides of a triangle are together greater than the third, therefore in Δ QNR, $Qn + NR > Q, i.e. PN + QN > PT + QT$.

Hence the sum of the distances of P and Q from T is the least.

25) ABCD is a quadrilateral and P, Q are mid-points of CD, AB, AP, DQ meet at x and BP, CQ meet at y. Show that ΔADx+ΔBCy= area of quadrilateral PxQy

Sol.: As usual we shall denote the area of a ΔABC by [ABC] and that of a quadrilateral ABCD by [ABCD] join AC and PQ. Since P is the midpoint of DC, therefore ΔAPD and ΔACP have equal bases PD and CP, and a common vertex A.



Consequently, [APD]≠[ACP] ………..(1) Again, Δs BQC and AQC have equal base QB and AQ (because Q is the mid points of AB), and a common vertex C. Consequently, [BQC]= [AQC] ………(2)

Adding corresponding sides of (1) and (2), we have [APD]+[BQC]= [ACP]+[BQC]= [AQCP], = [AQP]+[PQC]……(3)

Since Q is the midpoint of AB, therefore Δs AQP and BQP have equal bases AQ and QB, and a common vertex P. Consequently [AQP]=[QBP] …………..(4)

Again, since P is the midpoint of CD, therefore Δs PQC and QPD have equal bases CP and PD, and a common vertex Q. Consequently, [PCQ] = [QPD] …………..(5)

Adding corresponding sides of (4) and (5), we have [AQP]+[PQC]= [QBP]+[QPD]= [QBPD] …………..(6)

From (3) and (6), we have [APD]+[BQC]= [QBPD] ………..(7)

Since [APD]= [AxD] +[DxP] …………….(8)

[BQC] = [BYC]+[BYQ] …………….(9), [QBPD]= [QBY]+[PxQY]+[DxP] ………..(10)

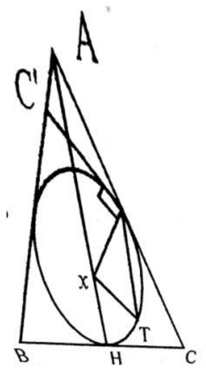
Substituting from (8), (9) and (10), we have

[AxD]+[DxP]+[ByC]+[ByQ]=[QBy]+[PxQy]+ [DxP] i.e. [AxD]+[ByC]= [PxQy] as described.

26) A Δ ABC has in centre I. It's in circle touches the side BC at T. The line through T parallel to IA meets the in circle again at S and the tangent to the in circle at S' meets the sides AB, AC at points C, B'

respectively. Show that ΔA'B'C' is similar to ΔABC.

Sol.:



Let Al produced meet BC in H. Since Δs ABC and AB'C' have ∠A in common, therefore in order to show that Δs ABC and ΔAB'C' are similar, we should compute ∠B' (or ∠C') in terms of ∠A, ∠B and ∠C, and show that ∠B' is equal to either ∠B or ∠C.

Let us denote the point of intersection of B'C' and AH by x. In ΔAB'x, ext.

$$\angle B'xl = \angle xAB' + \angle AB'C'$$

$$So\ that, \angle AB'C' = \angle B'xl - \frac{1}{2}\angle A$$

$$= (90° - \angle xLs) - \frac{1}{2}\angle A$$

$$= 90° - \angle IST - \frac{1}{2}\angle A,$$

$Since\ \angle xlS\ and\ \angle IST\ are\ all. \angle S\ since\ IT = IS = 90° - \angle ITS - \frac{1}{2}\angle A, = \angle STC - \frac{1}{2}\angle A$

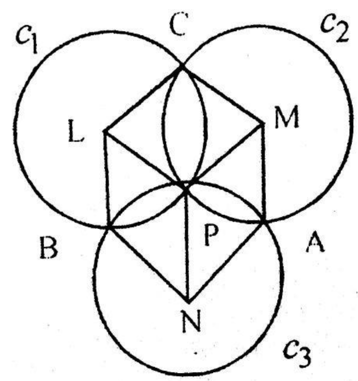
$$= \angle IHC - \frac{1}{2}\angle A (corres, \angle S)$$

$$= \left(\frac{1}{2}\angle A + \angle B\right) - \frac{1}{2}\angle A, = \angle B$$

Since ∠A is common and ∠B'= ∠B, therefore ΔAB'C' and ΔABC are equiangular and hence similar.

27) Let ABC be a triangular in a plane∑. Find the set of all points P (distinct from A, B, C) in the plane ∑ such that the circumcircles of Δs ABP, BCP, and CAP have the same radii.

Sol.:



If P lies on the circumcircle of ΔABC, then the circumcircles Δs BCP, CAP, and ABP coincide with the circumcircle of triangle ABC, and therefore they are all congruent. Therefore every point on the circumcircle of ΔABC satisfies the given condition.

Let P be a point not lying on the circumcircle of ΔABC and satisfying the condition that circumcircle of Δs PBC, PCA, and PAB are congruent.

Let us denote the circles PBC, PCA, PAB by c_1, c_2, c_3 respectively and their centres by L, M, N respectively, Since L and M are the centres of two congruent circles c_1 *and* c_2 respectively. And the points P, C lie on both these circles, therefore LC = LP = MC = MP.

Therefore P, L, C, M are the vertices of a rhombus. Similarly P, L, B, N are the vertices of a rhombus, and P, M, A., A, N are also the vertices of a rhombus.

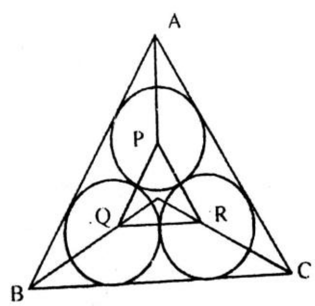
Now PL is equal and parallel to BN, and PL is also equal and parallel to CM. Therefore BN is equal and parallel to CM. Therefore BNMC is a parallelogram. Therefore BC is parallel to MN.

Now MN is the line joining the centres of the circles c_2 *and* c_3 and PA is the common chord of these circles. Therefore MN ⊥AP. Since BC ||MN, it follows that BC⊥ AP. Similarly CA⊥BP. AB ⊥CP consequently P is the orthocenter of ΔABC.

Thus we find that if P be a point such that the circles PBC, PCA, PAB are congruent, then P is either the orthocenter of ΔBC or it is a point on the circumcircle of ΔABC.

28) Three congruent circles have a common point O and lie inside a triangle such that each circle touches a pair of sides of the triangle. Show that the in centre and the circumcentre of the triangle and the point O are collinear.

Sol.: Suppose three congruent circles with centres P , Q, R lie inside a ΔABC, and are such that the circle with centre P touches AB and AC, that with centre Q touches BC, and AB, and that with centre R touches CA and BC. BC and AB, and that with centre R touches CA and BC.



Also let the circles pass through a common point O.

Since O lies on all the three circles, therefore PO = QO = RO. Therefore O is the circumcentre of ΔPQR. Let O' (not shown in the fig) be circumcentre of ΔABC.

Since BC is a tangent to the circles Q and R, the lengths of perpendiculars from Q and R on these circles are equal.

Therefore QR ||BC

Similarly RP ||CA, PQ||AB

Again , since AB and AC both touch the circle with centre P therefore P is equidistant from

AB and AC. Therefore P lies on the internal bisector of ∠A.

Similarly Q and R lie on the internal bisector of ∠B and ∠C respectively.

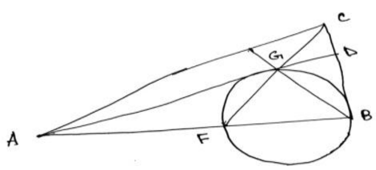
Therefore AP, BQ, CR when produced meet at the in centre l of ΔABC. Since QR ∥BC, RP ∥CA, PQ∥AB, it follows that l is also the in centre of ΔPQR.

29) Let G be the centriod of ΔABC in which the angle at C is obtuse AD, Fare the medians from A, C respectively on to the sides BC, AB. If the four points B, D, G, F are concyclic, show that $\frac{AC}{BC} > \sqrt{2}$. If further P is a point on the line BG extended such that AGCP is a parallelogram, Show that the Δs ABC and GAP are similar.

Sol.: Since chords BF and DG of the circle BDGF meet at A (outside the circle), therefore BA. FA = DA. GA so that $\frac{1}{2}AB^2 = \frac{2}{3}AD^2, i.e. AD^2 = \frac{3}{4}AB^2 \ldots\ldots\ldots\ldots..(i)$

Again, since chords FG and BD meet at C (outside the circle), therefore FC. GC= BC. DC.

So that $\frac{2}{3}FC^2 = \frac{1}{2}BC^2\ i.e. CP^2 = \frac{3}{4}BC^2 \ldots\ldots\ldots\ldots(ii)$



$CA^2 + CB^2 = 2CF^2 + 2AF^2$

So that, $\frac{1}{2}AB^2 = AC^2 + BC^2 - 2\left(\frac{3}{4}BC^2\right)$,

$i.e., AB^2 = 2AC^2 - BC^2$

Since ∠C is an obtuse angle, therefore $AB^2 > AC^2 + BC^2, so\ that\ 2AC^2 - BC^2 > AC^2 + BC^2$.

$i.e., AC^2 > 2BC^2\ or \frac{AC}{BC} > \sqrt{2}$

Since AGCP is an 11^{8m}, therefore

$\angle PAG = 180° - \angle CGA = 180° - \angle DGF = \angle CBA \ldots\ldots\ldots..(iii)$

Also $\frac{AG}{AP} = \frac{\left(\frac{2}{3}\right)AD}{\left(\frac{2}{3}\right)CF} = \frac{AB}{BC}$, by (i) and (ii) …………(iv)

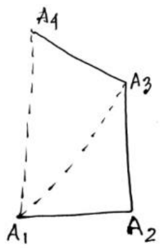
From (iii) and (iv), we find that ΔGAP and ΔABC are similar, the correspondence G→A, A→B, P→C being a similarity.

30) Let $A_1, A_2, A_3, \ldots, A_n$ is n sided regular polygon such that $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$. Determine n, the number of sides of the polygon.

Sol.: Let each side of the polygon be of length a since $\angle A_1A_2A_3 = \frac{(n-2)\pi}{n}, A_1A_2 = A_2A_3 = a$. therefore from $\Delta A_1A_2A_3, A_1A_3 = 2a\cos\left(\frac{\pi}{n}\right)$.

Also from $\Delta A_2A_3A_4, A_2A_4 = 2a\cos\left(\frac{\pi}{n}\right)$

Since the polygon $A_1\ A_2\ \ldots..A_n$ is regular, its vertices lie on a circle. In particular, the quadrilateral $A_1A_2A_3A_4$ is cyclic.



By Ptolemy's theorem$A_1A_2A_3A_4 + A_1A_4.A_2A_3 - A_1A_3 - A_2A_4$

$\therefore a^2 + A_1A_4.A_2A_3 = \left[2\,\mathrm{acos}\left(\frac{\pi}{n}\right)\right]^2 \quad ; \quad i.e.,$

$A_1A_4 = a\left(4cos^2\,\frac{\pi}{n} - 1\right)$

$$Since\ \frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$$

$$Therefore, \frac{1}{a} = \frac{1}{2\,\mathrm{acos}\left(\frac{\pi}{n}\right)} + \frac{1}{a\left[4cos^2\left(\frac{\pi}{n}\right) - 1\right]}$$

$$\Rightarrow 2\cos\left(\frac{\pi}{n}\right)\left[4cos^2\,\frac{\pi}{n} - 1\right]$$
$$= 4cos^2\,\frac{\pi}{n} - 1 + 2\,\mathrm{acos}\left(\frac{\pi}{n}\right)$$

$$\Rightarrow 8cos^3\,\left(\frac{\pi}{n}\right) - 4cos^2\,\left(\frac{\pi}{n}\right) - 4\cos\left(\frac{\pi}{n}\right) + 1 = 0$$

$\Rightarrow x = \mathrm{acos}\left(\frac{\pi}{n}\right)\ is\ a\ root\ of\ the\ equation$

$8x^3 - 4x^2 - 4x + 1 = 0 \ldots\ldots\ldots\ldots(1)$

We shall show that the roots of (1) are $\cos\left(\frac{\pi}{7}\right), \cos\left(\frac{3\pi}{7}\right), \cos\left(\frac{5\pi}{7}\right)$

$$Let\ 7\theta = (2n+1)\pi.$$

$So\ that\ 4\theta = (2n+1)\pi - 3\theta$

$\therefore \cos 4\theta = \cos[(2n+1)\pi - 3\theta] = -\cos 3\theta$

$\Rightarrow 2cos^2\,2\theta - 1 = (4cos^3\theta - 3\cos\theta\,),$

$\Rightarrow 2(2cos^2\theta - 1)^2 = 1 + 4cos^3\theta - 3\cos\theta = 0,$

$\Rightarrow 8cos^4\theta + 4cos^3\theta - 8cos^2\theta - 3\cos\theta + 1 = 0,$

$\Rightarrow (\cos\theta + 1)8cos^3\theta - 4cos^2\theta - 4\cos\theta + 1 = 0 \ldots\ldots\ldots..(2)$

Now (2) is satisfied by $\theta = \frac{\pi}{7}, \frac{3\pi}{7}, \frac{5\pi}{7}, \pi, \ldots\ldots$

Rejecting the factor $\cos\theta + 1$ which corresponds to 0= π and putting $\cos\theta$ =y,

We find that

$\cos\frac{\pi}{7}\ , \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}, are\ roots\ of\ 8y^3 - 4y^2 - 4y + 1 = 0 \ldots\ldots\ldots(3)$

Since equations (1) and (3) are the same, therefore the roots of (1) are $\cos\frac{\pi}{7}\ , \cos\frac{3\pi}{7}, and\ \cos\frac{5\pi}{7}$. But $\cos\frac{\pi}{n}$ is a root of (1). Therefore we must have n = 7.