BSTAT/BMATH 2012 SOLVED PAPER

1. (B) Let coordinate of P is (h, k)

Let coordinate of A is (0, a) and coordinate of B is (b, 0)

Now, AP : PB = 1 : 2

- ⇨ $h = (0\times2 + 1\times b)/3$
- ⇨ $h = b/3$
- ⇨ $k = (a\times2 + 0\times1)/3$
- ⇨ $k = 2a/3$

Now, length of the rod remains constant as the rod slides on the wall.

- ⇨ $\sqrt{(a^2 + b^2)} = c$
- ⇨ $(3k/2)^2 + (3h)^2 = c^2$
- ⇨ $4h^2 + k^2 = c_1^2$
- ⇨ **The locus of P is** $4x^2 + y^2 = c_1^2$

2. (A) Now, $x^2 + y^2 = 2007$ **= odd.**

⇨ **One of x and y is even and another is odd.**

⇨ **Let x is even and y is odd (without loss of generality) Now,** $x^2 \equiv 0 \pmod 4$ **(as x is even)**

And, $y \equiv \pm1 \pmod 4$

⇨ $y^2 \equiv (\pm1)^2 = 1 \pmod 4$

Now, dividing the equation by 4 we get,

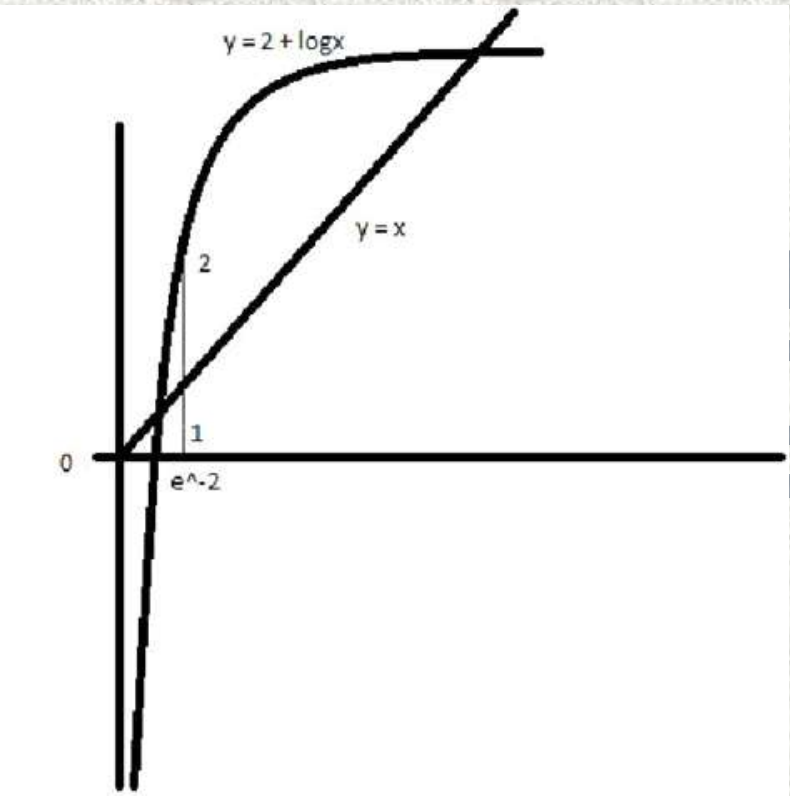
$0 + 1 \equiv 3 \pmod 4$

⇨ $1 \equiv 3 \pmod 4$

Which is impossible.

⇨ **No solution.**

3. **(C)**



Clearly they will intersect at to points. One in (0, 1) and another in (e, e^2) as $f_1(e) < f_2(e)$ and $f_1(e^2) > f_2(e^2)$.

4. (B) Now, $u_n = 1/2 + 2/2^2 + 3/2^3 + 4/2^4 + \ldots + n/2^n$

⇨ $(1/2)u_n = 1/2^2 + 2/2^3 + 4/2^4 + \ldots + (n-1)/2^n + n/2^{(n+1)}$

Now, $u_n - (1/2)u_n = 1/2 + (2/2^2 - 1/2^2) + (3/2^3 - 2/2^3) + (4/2^4 - 3/2^4) + \ldots + \{n/2^n - (n-1)/2^n\} - n/2^{(n+1)}$

⇨ $(1/2)u_n = (1/2 + 1/2^2 + 1/2^3 + 1/2^4 + \ldots + 1/2^n) - n/2^{(n+1)}$

⇨ $u_n = (1 + 1/2 + 1/2^2 + 1/2^3 + \ldots + 1/2^{n-1}) - n/2^n$

⇨ $u_n = 1 * \{1-(1/2)^n\}/(1 - 1/2) - n/2^n$

⇨ $u_n = 2(1 - 1/2^n) - n/2^n$

⇨ $\lim u_n$ as $n \to \infty = 2$

5. (D) Now, $1/(z - 3) + 1/(z - 3w) + 1/(z - 3w^2)$

$= \{(z - 3w)(z - 3w^2) + (z - 3)(z - 3w^2) + (z - 3)(z - 3w)\}/(z - 3)(z - 3w)(z - 3w^2)$

$= \{z^2 - 3wz - 3w^2z + 9w^3 + (z - 3)(z - 3w^2 + z - 3w)\}/(z - 3)(z^2 - 3wz - 3w^2z + 9w^3)$

$= \{z^2 + 3z + 9 + (z - 3)(2z + 3)\}/(z - 3)(z^2 + 3z + 9)$

$= (z^2 + 3z + 9 + 2z^2 + 3z - 6z - 9)/(z^3 - 27)$

$= 3z^2/(z^3 - 27)$

6. (C) The sample space looks like below,
{n, f(n)} = {(1, 1); (2, 2); (3, 3); (4, 4)}; {(1, 1); (2, 2); (3, 4); (4, 3)}; {(1, 1); (2, 4); (3, 3); (4, 2)}; {(1, 1); (2, 4); (3, 2); (4, 3)}; {(1, 2); (2, 1); (3, 4); (4, 3}; {(1, 2); (2, 3); (3, 4); (4, 1)}; {(1, 3); (2, 2); (3, 1); (4, 4)}; {(1, 3); (2, 2); (3, 4); (4, 1)}; {(1, 3); (2, 4); (3, 1); (4, 2)}; {(1, 3); (2, 4); (3, 2); (4, 1)}; {(1, 4); (2, 1); (3, 2); (4, 3)}; {(1, 4); (2, 3); (3, 2); (4, 1)}

⇨ **Number of such functions = 12.**

7. (D)

Now, lim $[\{f(x) - f(0)\}/(x - 0)]$ as $x \to 0+$

$= \lim e^{-1/x}/x$ as $x \to 0+$

Let, $z = 1/x$. As $z \to \infty$ as $x \to 0$

So, lim (z/e^z) as $z \to \infty$

$= \lim 1/e^z$ as $z \to \infty$ (Applying L"Hospital rule)

$= 0$

Now, lim$[\{f(x) - f(0)\}/(x - 0)\}$ as $x \to 0-$

$= 0$

⇨ **f is differentiable.**

Now, $f'(x) = e^{-1/x}/x^2$ for $x > 0$

$= 0$ for $x \le 0$

Now, lim f(x) as $x \to 0+$

$= \lim e^{-1/x}/x^2$

Let, $z = 1/x$, As $z \to \infty$ as $x \to 0+$

So, lim z^2/e^z as z -> ∞
= lim $2z/e^z$ as z -> ∞ (Applying L"hospital rule)
= lim $2/e^z$ as z -> ∞ (Again applying L"Hospital rule)
= 0

And lim f'(x) as x -> 0-
= 0

Also f'(0) = 0.

8. (B) Now, last digit of 9! is 0.

Now, $3^2 \equiv -1 \pmod{10}$

- ⇨ $(3^2)^{4983} \equiv (-1)^{4983} \pmod{10}$
- ⇨ $3^{9966} \equiv -1 \pmod{10}$
- ⇨ $3^{9966} \equiv 9 \pmod{10}$

9. (C) $f''(x) = \{(4x + 3)(2x - 1) - (2x^2 + 3x + 1)\times 2\}/(2x - 1)^2 = 0$

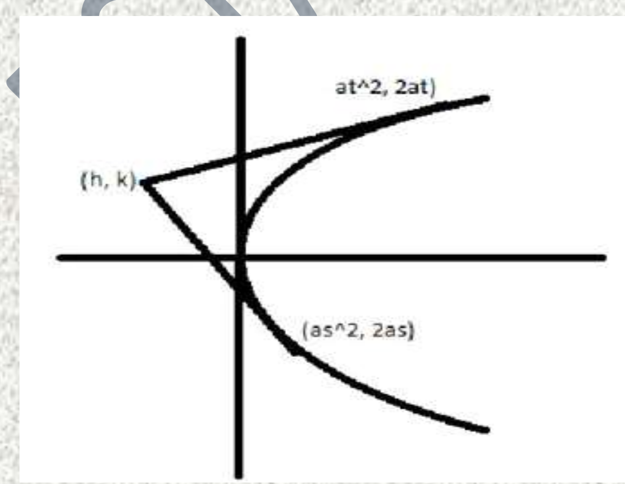
- ⇨ $8x^2 - 4x + 6x - 3 - 4x^2 - 6x - 2 = 0$
- ⇨ $4x^2 - 4x - 5 = 0$
- ⇨ $x = \{4 \pm \sqrt{(16 + 4\times 4\times 5)}\}/2\times 4$
- ⇨ $x = (4 \pm 4\sqrt{6})/8$
- ⇨ $x = (1 \pm \sqrt{6})/2$
- ⇨ **For no value of x in [2, 3] f(x) attains maximum or minimum value.**

Now, f'(x) also is not less than 0 for all values of x in (2, 3).

⇨ **f(x) is not decreasing in (2, 3).**

Now, f(2) = 5 and f(3) = 28/5

10. (D)



Now, the angle between the two tangents is 90°.

Now, the angle between the two tangents and their respective normal is 90°.

The two normal meets at focus (a, 0)

Now, the quadrilateral of four points (h, k), (at^2, 2at), (as^2, 2at), (a, 0) have total angle 360°

- **The normal meet at 90° at focus (360° - 90° - 90° - 90° = 90°)**
- **The quadrilateral is a square because the length of the tangents from a point are equal.**

So, we have, $(at^2 - a)^2 + (2at - 0)^2 = (as^2 - a)^2 + (2as - 0)^2$ (distances of (at^2, 2at) and (as^2, 2as) from focus are same as it is a square)

- $(t^2 - 1)^2 + 4t^2 = (s^2 - 1)^2 + 4s^2$
- $(t^2 + 1)^2 = (s^2 + 1)^2$
- $t^2 + 1 = s^2 + 1$
- $t^2 - s^2 = 0$
- $(t + s)(t - s) = 0$
- $t + s = 0$ (As $t \neq s$)
- $s = -t$

Now, the points are (at^2, 2at) and (at^2, -2at)

Now, the distances of these two points from P are equal.

- $(at^2 - h)^2 + (2at - k)^2 = (at^2 - h)^2 + (-2at - k)^2$
- $(2at + k)^2 = (2at - k)^2$
- $k = 0$
- **The locus of P is y = 0**
- **A straight line.**

11. (C)

12. (D) Now, $2^a - 5^b 7^c = 1$ where a, b, c are positive.

Minimum value of b, c = 1

- $a > 5$.

Dividing the equation by 8 we get,

$0 - (5 \text{ or } 1)(-1)^c \equiv 1 \pmod 5$ (If b is odd $5^b \equiv 5 \pmod 8$ and if even then $5^b \equiv 1 \pmod 8$)

Clearly to hold the equation, b must be even and c must be odd.

Now, dividing the equation by 5 we

get, $2^a - 0 \equiv 1 \pmod 5$

- a is divisible by 4 because $(2^2)^{2m} \equiv (-1)^{2m} \equiv 1 \pmod 5$

Now, dividing the equation by 3 we

get, $(-1)^a - (-1)^b \times 1^c \equiv 1 \pmod 3$

=> $1 - 1 \equiv 1 \pmod 3$ (As a, b both even)

Which is impossible.

The equation has no solution.

13. (A) Let $z = h + ik$

We have, $|h + ik - ia| = k + 1$

- ⇨ $\sqrt{\{h^2 + (k - a)^2} = k + 1$
- ⇨ $h^2 + k^2 - 2ak + a^2 = k^2 + 2k + 1$
- ⇨ $h^2 - 2k(a + 1) + (a^2 - 1) = 0$
- ⇨ Locus of z is $x^2 - 2y(a + 1) + a^2 - 1 = 0$
- ⇨ It's a parabola.

14. (B) Now, tan(sinx) never attains the value 1.

15. (D) Now, $f'(x) = \{2(x - 1) - 2x\}/(x - 1)^2 = -2/(x - 1)^2 < 0$

f(x) is strictly decreasing.
f(x) is one-one and onto.

16. (C) Now, $g(t) = \int f(x)dx$ (integration running from 0 to t)
$= \int f(c)dx$ (integration running from 0 to t)

(As $f(x + 1) = f(x) = tf(c)$)

- $g''(t) = f(c)$

Now, $h(t) = \lim g(t + n)/n$ as $n \to \infty$
$= \lim g'(t + n)/1$ as $n \to \infty$ (Applying L'Hospital rule)
$= \lim f(c)$ as $n \to \infty$
$= f(c)$.

So, h(t) is defined for all $t \in R$ and independent of t.

17. (A)

Now, $a_1 = 24^{1/3}$

Let $a_n < 3$

Now, $a_{n+1} = (a_n+24)^{1/3}$

We have, $a_n < 3$

- ⇨ $a_n + 24 < 27$
- ⇨ $(a_n + 24)^{1/3} < 3$
- ⇨ $a_{n+1} < 3$
- ⇨ $a100 < 3$
- ⇨ The integer part of a_{100} is 2.

18. (B) Now, $4/(4 - x^2) + 9/(9 - y^2)$

$= 4/(4 - x^2) + 9/(9 - 1/x^2)$

$= 4/(4 - x^2) + 9x^2/(9x^2 - 1)$

$= 4/(4 - x^2) + 1/(9x^2 - 1) + 1$

$= (36x^2 - 4 + 4 - x^2)/(4 - x^2)(9x^2 - 1) + 1$

$= 35x^2/(4 - x^2)(9x^2 - 1) + 1$

Let, $f(x) = 35x^2/(4 - x^2)(9x^2 - 1) + 1$

$f'(x) = 35[\{2x(4 - x^2)(9x^2 - 1) + 2xx^2(9x^2\ x^2)^2(9x^2 - 1)^2 = 0$

- ⇨ $36x^2 - 4 - 9x^4 + x^2 + 9x^4 - x^2 - 36x^2 + 9x^4 = 0$
- ⇨ $9x^4 - 4 = 0$
- ⇨ $x^2 = 2/3$

Now, $f(\pm\sqrt{(2/3)}) = 35\times(2/3)/(4 - 2/3)\{9\times(2/3) - 1\} + 1 = 12/5$

19. (A) Clearly the limit is e.

20. (D) $f(-1) = 0$

$f(x) = x(x^3 + 1) + (x^2 - 1)$

$= x(x + 1)(x^2 - x + 1) + (x + 1)(x - 1)$

$= (x + 1)(x^3 - x^2 + x + x - 1)$

$= (x + 1)(x^3 - x^2 + 2x - 1)$

Now, $x^3 - x^2 + 2x - 1 < 0$ for all $x < 0$

- ⇨ There cannot be any root for $x < 0$
- ⇨ Option (b) and (c) cannot be true.

Now, $f(x) = x^4 + x^2 + x - 1$

⇨ $f'(x) = 4x^3 + 2x + 1$

Now, $f'(-1) = -5$ and $f'(0) = 1$

The sign changes.

⇨ $f'(x)$ has at least one root in $(-1, 0)$

Now, $f''(x) = 12x^2 + 2 > 0$

⇨ f has a local minimum in $(-1, 0)$

Now, $f'(x) > 0$ for all x.

- ⇨ $f'(x)$ is strictly increasing.
- ⇨ $F'(x)$ has only one root and other two roots are complex.
- ⇨ f has exactly one local minima in $(-1, 0)$

21. (A) We take the arrangement A()()B,()()A()()B()()A()B()A()()B()()

We can put rest 6 A's in 4 places in ${}^{6+4-1}C_{4-1}$ ways = ${}^{9}C_{3}$ ways.

We can put rest 4 B's in 4 places in ${}^{4+4-1}C_{4-1}$ ways = ${}^{7}C_{3}$ ways.

So, this can be done in ${}^{9}C_{3} \times {}^{7}C_{3}$ ways.

Now, we have started with A. We can also start with B. So same another case appear.

- ⇨ Total number of ways = $2 \times {}^{9}C3 \times {}^{7}C3$

22. (C)

$f(x) = x^{n+1}$ if $x > 0$

$f(x) = -x^{n+1}$ if $x < 0$

$f(x) = 0$ if $x = 0$

$\lim [\{f(x) - f(0)\}/(x - 0)]$ as $x \to o+$
$= \lim \{(x^{n+1} - 0)/x\}$ as $x \to 0+$
$= \lim x^n$ as $x \to 0+$
$= 0$

lim [{f(x) – f(0)]/(x – 0)] as x -> 0-
= lim {($-x^{n+1}$ – 0)/x} as x -> 0-
= lim $-x^n$ as x -> 0-
= 0

⇨ **f is differentiable everywhere.**

23. (C) The equation of the given line is, 2x + 3y – k = 0

⇨ **x/(k/2) + y/(k/3) = 1**
⇨ **A = (k/2, 0) and B = (0, k/3)**

Mid-point of AB i.e., centre of the circle = (k/4, k/6)

Radius = $\sqrt{(k/2 - k/4)^2 + (0 - k/6)^2}$
= $\sqrt{k^2/16 + k^2/36}$ = (k/12)$\sqrt{9 + 4}$
= $(k\sqrt{13})/12$

Equation of the circle is, $(x - k/4)^2 + (y - k/6)^2$
= $\{(k\sqrt{13})/12\}^2$

i.e. $x^2 + y^2 - kx/2 - ky/3 = 13k^2/144 - k^2/16 - k^2/36$

i.e. $x^2 + y^2 - kx/2 - ky/3 = 0$

24. (D) If 0 < α < ½

Then, 2α < 1

⇨ **$(2\alpha)^n$ -> 0 as n -> ∞**
⇨ **x_n -> ∞**

If α > ½

Then 2α > 1

⇨ **$(2\alpha)^n$ -> ∞ as n -> ∞**
⇨ **x_n -> 0**

25. (C)

Now, option (a) cannot be true as θ Є (0, π/2) and sinθ Є (0, 1)

Now, option (b) cannot be true. Let θ = π/4, then cos(sinθ) > cosθ

Option (d) is same as option (a). Let cosθ = x then it becomes,

x < sinx

26. (A) Clearly shortest distance between A and B′ along the surface is $\sqrt{(5^2 + 2^2)} = \sqrt{29}$.

27. (B)

Let, $\Sigma(1/\sqrt{k})$ summation running from k = 2 to k = 9999 = S

Now, $1/\{2\sqrt{(k + 1)}\} < \sqrt{(k + 1)} - \sqrt{k}$

$1/2\sqrt{2} < \sqrt{2} - \sqrt{1}$

$1/2\sqrt{3} < \sqrt{3} - \sqrt{2}$

$1/2\sqrt{4} < \sqrt{4} - \sqrt{3}$

...

....

$1/2\sqrt{9999} < \sqrt{9999} - \sqrt{9998}$

Adding we get, $(1/2)\ \Sigma \sum_{k=2}^{9999}(1/\sqrt{k}) < \sqrt{9999} - \sqrt{1}$

$S < 2(\sqrt{9999} - 1)$

$S < 197.99$

Now,

$\sqrt{(k + 1)} - \sqrt{k} < 1/(2\sqrt{k})$

$\sqrt{3} - \sqrt{2} < 1/2\sqrt{2}$

$\sqrt{4} - \sqrt{3} < 1/2\sqrt{3}$

$\sqrt{5} - \sqrt{4} < 1/2\sqrt{4}$

...

...

$\sqrt{10000} - \sqrt{9999} < 1/2\sqrt{9999}$

Adding the inequalities we get,

$\sqrt{10000} - \sqrt{2} < \Sigma(1/2\sqrt{k})$ summation running from k = 2 to k = 9999

- ⇨ **$S > 2(100 - \sqrt{2})$**
- ⇨ **$S > 197.17$**
- ⇨ **The integer part of S is 197.**

28. (B)

Now, $x^6 + y^4 - (x^4 + y^2)$

$= x^4(x^2 - 1) + y^2(y^2 - 1)$

Now, x and y both less than or equal to 1.

- ⇨ **x^2 and y^2 both less than or equal to 1.**
- ⇨ **$x^6 + y^4 \le x^4 + y^2 \le 1$**
- ⇨ **All (x, y) that satisfies $x^4 + y^2 \le 1$ also satisfies $x^6 + y^4 \le 1$**
- ⇨ **A is a subset of B.**

29. (A) Now, AM ≥ GM

$({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + ... + {}^{10}C_{10})/11$
$> \{({}^{10}C_0)({}^{10}C_1)....({}^{10}C_{10})\}^{1/11}$

(Equality cannot hold as all the numbers are not equal)

- ⇨ **$(2^{10}/11)^{11} > \{({}^{10}C_1)({}^{10}C_9\}\{({}^{10}C_2)({}^{10}C_8)\}...\{({}^{10}C_4)({}^{10}C_6)\}({}^{10}C_5)$**
  $= ({}^{10}C_1)^2({}^{10}C_2)^2({}^{10}C_3)^2({}^{10}C_4)^2({}^{10}C_5)$

30. (D) Let the roots of the equation are s, sr, sr^2, sr^3

Now, $-a = s + sr + sr^2 + sr^3 = s(r^4 - 1)/(r - 1)$

$b = s*sr + s*sr^2 + s*sr^3 + sr*sr^2 + sr*sr^3 + sr^2*sr^3$

- ⇨ **$b = s^2r + s^2r^2 + 2s^2r^3 + s^2r^4 + s^2r^5 = s^2r(r^5 - 1)/(r - 1) + s^2r^3$**
- ⇨ **$c = s*sr*sr^2 + sr*sr^2*sr^3 + sr^2*sr^3*s + sr^3*s*sr$**
- ⇨ **$c = s^3r^3 + s^3r^6 + s^3r^5 + s^3r^4$**
- ⇨ **$c = s^3r^3(r^4 - 1)/(r - 1)$**
- ⇨ **$d = s\times sr\times sr^2\times sr^3$**
- ⇨ **$d = s^4r^6$**

Now, $c^2 = s^6r^6\{(r^4 - 1)/(r - 1)\}^4$
$= (s^4r^6)[s(r^4 - 1)/(r - 1)]^2$
$= d\times a^2$

- ⇨ **$c^2 = a^2d$**