B. Stat. (Hons.) Admission Test : 2009

Group A (Each of the following questions has exactly one correct option and you have to identify it)

1. If k times the sum of the first n natural numbers is equal to the sum of the squares of the first n natural numbers, then $\cos^{-1}\{(2n - 3k)/2\}$ is
   (a) $5\pi/6$
   (b) $2\pi/3$
   (c) $\pi/3$
   (d) $\pi/6$

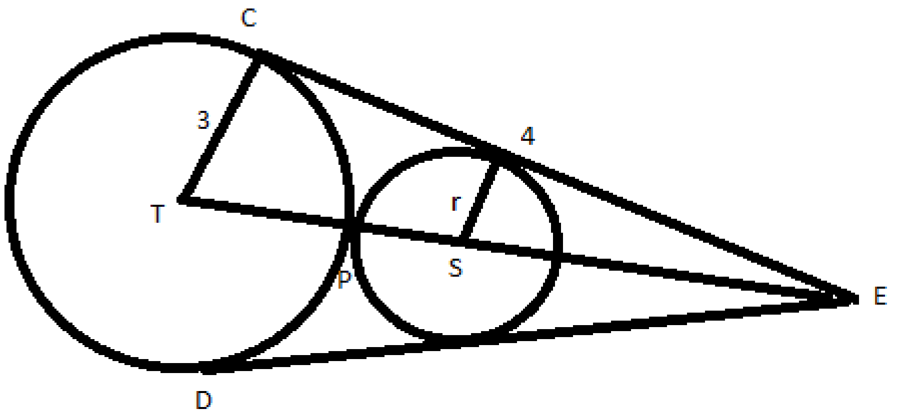
Solution :

We have $kn(n + 1)/2 = n(n + 1)(2n + 1)/6$

- ⇨ $3k = 2n + 1$
- ⇨ $(2n - 3k)/2 = -1/2$
- ⇨ $\cos^{-1}\{(2n - 3k)/2\} = \cos^{-1}(-1/2) = 2\pi/3$.
- ⇨ Option (b) is correct.

2. Two circles touch each other at P. The two common tangents to the circles, none of which passes through P, meet at E. They touch the larger circle at C and D. The larger circle has radius 3 units and CE has length 4 units. Then the radius of the smaller circle is
   (a) 1
   (b) 5/7
   (c) ¾
   (d) ½.

Solution :



Clearly, TE = $\sqrt{(3^2 + 4^2)} = 5$

TS = 3 + r

- ⇨ SE = 5 – 3 – r = 2 – r

Now, r/3 = SE/TE (triangles are similar)

- ⇨ r/3 = (2 – r)/5
- ⇨ 5r = 6 – 3r
- ⇨ r = 6/8 = ¾
- ⇨ Option (c) is correct.

3. Suppose ABCDEFGHIJ is a ten digit number, where the digits are all distinct. Moreover, A > B > C satisfy A + B + C = 9, D > E > F are consecutive even digits and G > H > I > J are consecutive odd digits. Then A is
   (a) 8
   (b) 7
   (c) 6
   (d) 5.

Solution :

Now, G > H > I > J are consecutive odd digits.

- ⇨ Either (G, H, I, J) = (7, 5, 3, 1) or (9, 7, 5, 3) i.e. 3, 5, 7 must be there.

Now, A + B + C = 9.

- ⇨ All of A, B, C cannot be even because RHS is 9 i.e. odd.

Now, odd numbers that are available is 1 or 9.

Now, A or B or C cannot be 9 otherwise sum of the rest two will be 0 and all are distinct.

- ⇨ A or B or C is 1.

Now, A cannot be 1 as there is no option 1.

Let C = 1.

- ⇨ A + B = 8

Now, A cannot be 5 or 7 because all the odd digits are occupied.

- ⇨ Option (b) and (d) cannot be true.

Let A = 6, then B = 2

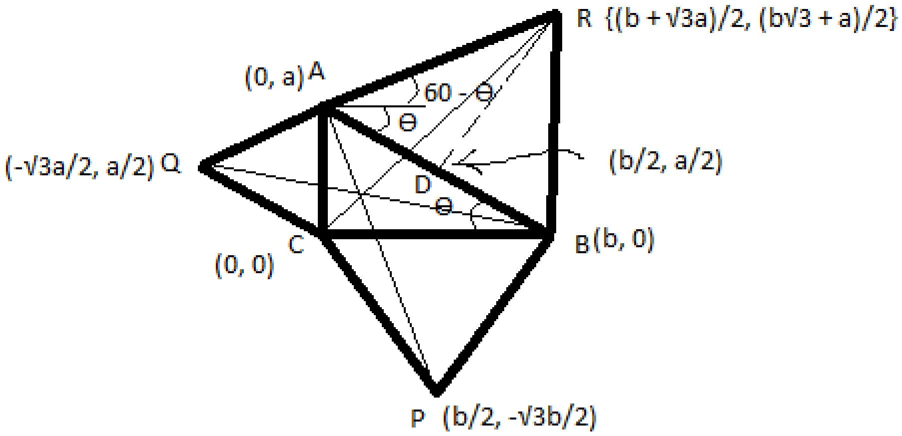
Now, D > E > F are consecutive even digits.

So, the above case cannot hold true.

- ⇨ Option (c) cannot be true.
- ⇨ Option (a) is correct.

4. Let ABC be a right angled triangle with AB > BC > CA. Construct three equilateral triangle BCP, CQA and ARB, so that A and P are on opposite sides of BC; B and Q are on opposite sides of CA; C and R are on opposite sides of AB. Then
   (a) $CR > AP > BQ$
   (b) $CR < AP < BQ$
   (c) $CR = AP = BQ$
   (d) $CR^2 = AP^2 + BQ^2$.

Solution :



We have defined the co-ordinate system as shown in figure.

Now, BCP is an equilateral triangle. So, it's median and perpendicular from P to BC is same. Thus we get co-ordinate of P $(b/2, -\sqrt{3}b/2)$

Now, $AP^2 = (b/2 - 0)^2 + (-\sqrt{3}b/2 - a)^2 = a^2 + b^2 + \sqrt{3}ab$

Similarly, we get, $BQ^2 = (b + \sqrt{3}a/2)^2 + (0 - a/2)^2 = a^2 + b^2 + \sqrt{3}ab$

Let, angle CBA = Θ

- ⇨ Angle BAS = Θ (S is any point on the line passing through A and parallel to x-axis)
- ⇨ Angle RAS = $60° - \Theta$ (As angle RAB = 60°)
- ⇨ The slope of the line AR = $\tan(60° - \Theta) = (\tan 60° - \tan\Theta)/(1 + \tan 60° \tan\Theta)$

Now, $\tan\Theta = a/b$ (From triangle ABC)

- ⇨ The slope of AR = $(a/b - \sqrt{3})/(1 + \sqrt{3}b/a)$
- ⇨ Equation of AR is, $y - a = (a/b - \sqrt{3})/(1 + \sqrt{3}b/a)x$

Now, perpendicular from R to AB is the median also as ARB is equilateral.

- ⇨ Co-ordinate of D = $(b/2, a/2)$

Slope of AB = $-a/b$

- ⇨ Slope of DR = a/b (As DR and AB are perpendicular to each other)
- ⇨ Equation of DR is, $y - a/2 = (a/b)(x - b/2)$

Solving equation of AR and DR we get, co-ordinate of R $\{(b + \sqrt{3}a)/2, (b\sqrt{3} + a)/2\}$

- ⇨ $CR^2 = \{(b + \sqrt{3}a)/2\}^2 + \{(b\sqrt{3} + a)/2\}^2 = a^2 + b^2 + \sqrt{3}ab$
- ⇨ CR = AP = BQ

⇨ Option (c) is correct.

5. The value of $(1 + \tan 1°)(1 + \tan 2°)....(1 + \tan 44°)$ is
   (a) 2
   (b) A multiple of 22
   (c) Not an integer
   (d) A multiple of 4.

Solution :

Now, $\tan(45° - \Theta) = (\tan 45° - \tan\Theta)/(1 + \tan 45°\tan\Theta) = (1 - \tan\Theta)/(1 + \tan\Theta)$

⇨ $1 + \tan(45° - \Theta) = 2/(1 + \tan\Theta)$
⇨ $\{1 + \tan(45° - \Theta)\}(1 + \tan\Theta) = 2$

When $\Theta = 1°$ we get, $(1 + \tan 44°)(1 + \tan 1°) = 2$

When $\Theta = 2°$, we get, $(1 + \tan 43°)(1 + \tan 2°) = 2$

...

...

When $\Theta = 22°$, we get, $(1 + \tan 23°)(1 + \tan 22°) = 2$

So, the expression = 2^{22} (a multiple of 4)

⇨ Option (d) is correct.

6. Let y = x/(1 + x), where x = w^(2009^2009^2009^...2009 times) and w is a complex cube root of 1. Then y is
   (a) w
   (b) $-w$
   (c) w^2
   (d) $-w^2$

Solution :

Now, $2009 \equiv -1 \pmod 3$

⇨ $2009^m \equiv (-1)^m = -1 \pmod 3$ if m is odd.

Now, 2009^2009^.... = odd as 2009 is odd.

⇨ $x = w^{3r - 1} = (w^3)^r(1/w) = 1*w^3/w = w^2$

- ⇨ $y = w^2/(1 + w^2) = w^2/(-w) = -w$
- ⇨ Option (b) is correct.

7. The number of solutions of Θ in the interval [0, 2π] satisfying $\{\log_{\sqrt{3}}(\tan\Theta)\sqrt{\{\log_{\tan\Theta}(3) + \log_{\sqrt{3}}(3\sqrt{3})\}} = -1$ is
   (a) 0
   (b) 2
   (c)4
   (d) 6

Solution :

Now, $\{\log_{\sqrt{3}}(\tan\Theta)\sqrt{\{\log_{\tan\Theta}(3) + \log_{\sqrt{3}}(3\sqrt{3})\}} = -1$

- ⇨ $\{\log(\tan\Theta)/\log(\sqrt{3})\}\sqrt{\{\log(3)/\log(\tan\Theta) + \log_{\sqrt{3}}(\sqrt{3})^3\}} = -1$
- ⇨ $\{2\log(\tan\Theta)/\log(3)\}\sqrt{\{\log(3)/\log(\tan\Theta) + 3\}} = -1$

Let, $\log(\tan\Theta) = x$ and $\log(3) = a$

The equation becomes, $(2x/a)\sqrt{\{(a + 3x)/x\}} = -1$

- ⇨ $2\sqrt{\{x(a + 3x)\}} = -a$
- ⇨ $4x(a + 3x) = a^2$
- ⇨ $12x^2 + 4xa - a^2 = 0$
- ⇨ $x = \{-4a \pm \sqrt{(16a^2 + 48a^2)}\}/24$
- ⇨ $x = \{-4a \pm 8a)/24$
- ⇨ $x = -a/2 , a/6$
- ⇨ $\log(\tan\Theta) = -\log(3)/2$
- ⇨ $\log(\tan\Theta) = \log(1/\sqrt{3})$
- ⇨ $\tan\Theta = 1/\sqrt{3}$
- ⇨ $\Theta = \pi/6$

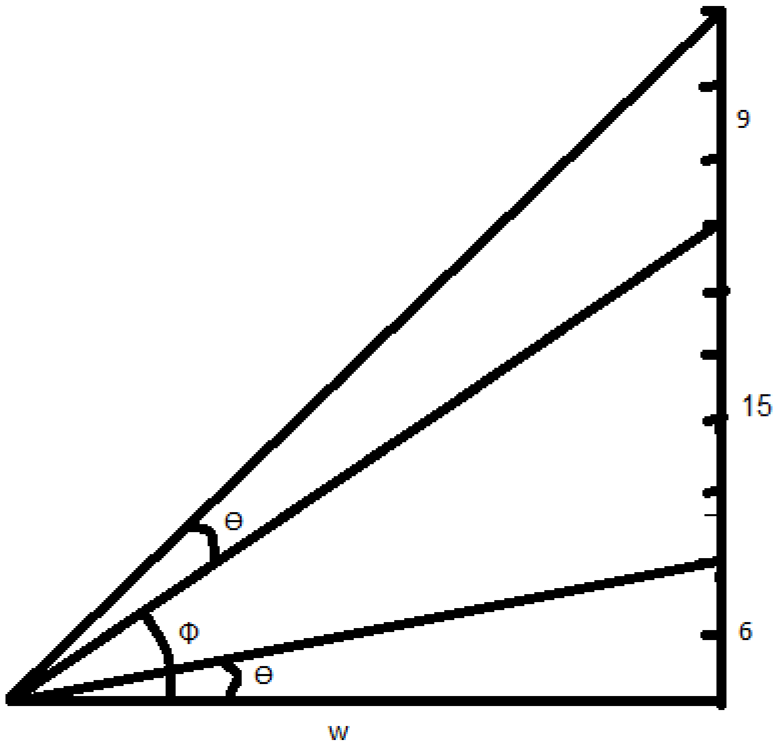
Now, $x = a/6$

- ⇨ $\log(\tan\Theta) = \log(3)/6$
- ⇨ $\log(\tan\Theta) = \log(3^{1/6})$
- ⇨ $\tan\Theta = 3^{1/6}$
- ⇨ $\Theta = \tan^{-1}(3^{1/6})$
- ⇨ 2 solutions.
- ⇨ Option (b) is correct.

8. A building with ten storeys, each storey of height 3 metres, stands on one side of a wide street. From a point on the other side of the street directly opposite to the building, it is observed that the three

uppermost storeys together subtend an angle equal to that subtended by the two lowest storeys. The width of the street is
(a) $6\sqrt{35}$ metres
(b) $6\sqrt{70}$ metres
(c) 6 metres
(d) $6\sqrt{3}$ metres

Solution :



Let width of the street is w.

Clearly, from figure, $\tan\Theta = 6/w$ and $\tan\Phi = 21/w$

Now, also from figure, $\tan(\Phi + \Theta) = 30/w$

- ⇨ $(\tan\Phi + \tan\Theta)/(1 - \tan\Phi\tan\Theta) = 30/w$
- ⇨ $\{(21/w) + (6/w)\}/\{1 - (21/w)(6/w) = 30/w$
- ⇨ $(27/w)/\{(w^2 - 126)/w^2\} = 30/w$
- ⇨ $9w^2/(w^2 - 126) = 10$
- ⇨ $9w^2 = 10w^2 - 1260$
- ⇨ $w^2 = 1260$
- ⇨ $w = 6\sqrt{35}$
- ⇨ Option (a) is correct.

9. A collection of black and white balls are to be arranged on a straight line, such that each ball has at least one neighbour of different colour. If there are 100 black balls, then the maximum number of white balls that allow such an arrangement is
   (a) 100
   (b) 101
   (c) 202
   (d) 200

Solution :

The hundred black balls are put into the line. Now, we can put maximum 2 white balls in between 2 black balls. Now, there are 100 black balls. So there are 99 gaps between the hundred balls. So number of white balls we can keep in between the 100 black balls = 2*99 = 198. Now, put one white ball before the left most black ball and put another white ball after the right most black ball i.e. the row starts with a white ball and also ends with a white ball.

So maximum number of white balls = 198 + 1 + 1 = 200.

⇨ Option (d) is correct.

10. Let f(x) be a real-valued function satisfying $af(x) + bf(-x) = px^2 + qx + r$, where a and b are distinct real numbers and p, q and r are non-zero real numbers. Then f(x) = 0 will have real solution when
   (a) $\{(a + b)/(a - b)^2 \leq q^2/4pr$
   (b) $\{(a + b)/(a - b)\}^2 \leq 4pr/q^2$
   (c) $\{(a + b)/(a - b)^2 \geq q^2/4pr$
   (d) $\{(a + b)/(a - b)\}^2 \geq 4pr/q^2$

Solution :

Now, $af(x) + bf(-x) = px^2 + qx + r$

Putting x = -x we get, $af(-x) + bf(x) = px^2 - qx + r$

Adding the above two equations, we get, $a\{f(x) + f(-x)\} + b\{f(x) + f(-x) = 2px^2 + 2r$

⇨ $f(x) + f(-x) = (2px^2 + 2r)/(a + b)$

Now, subtracting the above two equations, we get, $a\{f(x) - f(-x)\} - b\{f(x) - f(-x)\} = 2qx$

- ⇨ $f(x) - f(-x) = 2qx/(a - b)$

Now, adding the evaluated two equations we get, $2f(x) = (2px^2 + 2r)/(a + b) + 2qx/(a - b)$

- ⇨ $f(x) = (px^2 + r)/(a + b) + qx/(a - b)$

Now, $f(x) = 0$

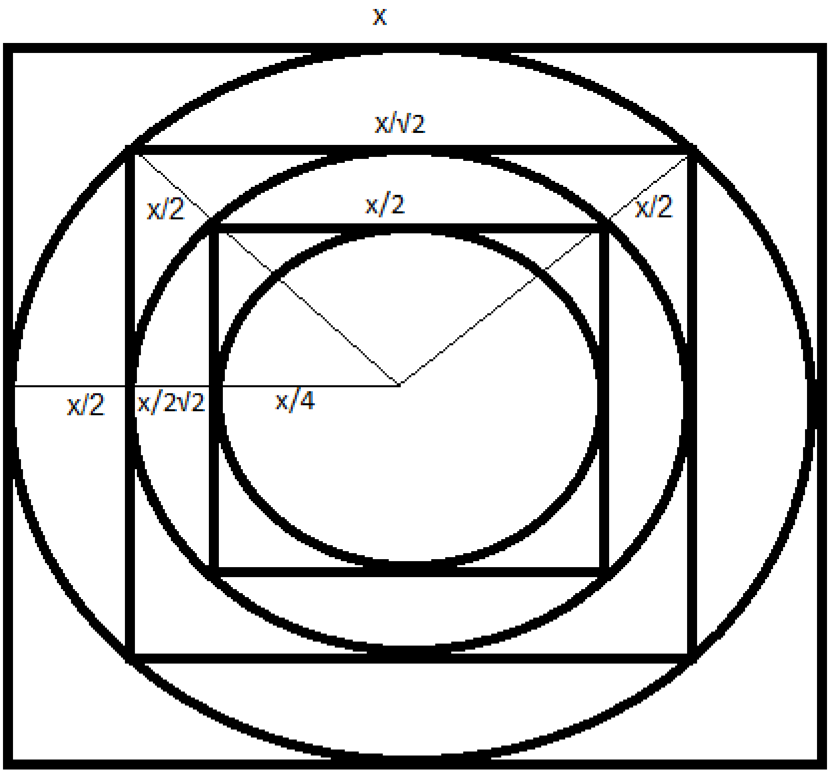
- ⇨ $(px^2 + r)/(a + b) + qx/(a - b) = 0$
- ⇨ $(a - b)px^2 + qx(a + b) + r(a - b) = 0$

Now, this equation has real roots if $\{q(a + b)\}^2 - 4(a - b)p*r(a - b) \geq 0$

- ⇨ $\{(a + b)/(a - b)\}^2 \geq 4pr/q^2$
- ⇨ Option (d) is correct.

11. A circle is inscribed in a square of side x, then a square is inscribed in that circle, a circle is inscribed in the latter square, and so on. If S_n is the sum of the areas of the first n circles so inscribed, then $\lim(S_n)$ as $n \to \infty$ is
(a) $\pi x^2/4$
(b) $\pi x^2/3$
(c) $\pi x^2/2$
(d) πx^2.

Solution :



Clearly from the picture, radius of first circle is x/2, then $x/2^{3/2}$, then $x/2^2$, then $x/2^{5/2}$, ...

The power of 2 makes an arithmetic progression, 1, 3/2, 2, 5/2,...

So,n^{th} term is 1 + (n − 1)(1/2) = (n + 1)/2

So, radiuses are x/2, $x/2^{3/2}$, $x/2^2$, $x/2^{5/2}$, ..., $x/2^{(n+1).2}$.

So areas are $\pi(x/2)^2$, $\pi(x/2^{3/2})^2$, $\pi(x/2^2)^2$, $\pi(x/2^{5/2})^2$, ..., $\pi(x/2^{(n+1)/2})^2$

This is a geometric progression with first term $\pi(x/2)^2$ and common ratio 1/2

- ⇨ $S_n = \pi(x/2)^2\{1 - (1/2)^n\}/(1 - ½)$
- ⇨ $S_n = (\pi x^2/2)\{1 - (1/2)^n\}$
- ⇨ lim S_n as n -> ∞ = $(\pi x^2/2)(1 - 0) = \pi x^2/2$
- ⇨ Option (c) is correct.

12. Let 1, 4, ... and 9, 14, ... be two arithmetic progressions. Then the number of distinct integers in the collection of first 500 terms of each of the progressions is
    (a) 833
    (b) 835
    (c) 837
    (d) 901

Solution :

Clearly 19 is the first common term of both the series.

The common term will repeat after 15.

Now, 500^{th} term of the first progression is 1 + (500 – 1)*3 = 1498

Let there are r number of common terms in both the sequence.

Therefore, 19 + (r – 1)*15 ≤ 1498

- ⇨ (r – 1)*15 ≤ 1479
- ⇨ (r – 1) ≤ 98.6
- ⇨ r ≤ 99.6
- ⇨ r = 99

So, 99 terms common.

- ⇨ Number of distinct integers = 2*500 – 99 = 901
- ⇨ Option (d) is correct.

13. Consider all the 8-letter words that can be formed by arranging the letters in BACHELOR in all possible ways. Any two such words are called *equivalent* if those two words maintain the same relative order of the letters A, E, O. For example, BACOHELR and CABLROEH are equivalent. How many words are there which are equivalent to BACHELOR?
    (a) $^8C_3$*3!
    (b) $^8C_3$*5!
    (c) 2*$(^8C_3)^2$
    (d) 5!*3!*2!

Solution :

Now, there can be 3! Arrangement of the letters A, E and O.

Each combination will have same number of words arranged.

Now, total number of possible cases is 8!

- ⇨ For any particular of arrangement of A, E and O will have 8!/3! Equivalent words.
- ⇨ The answer is 8!/3! = {8!/(5!*3!)}*5! = $^8C_3$*5!
- ⇨ Option (b) is correct.

14. The limit
lim {1/6 + 1/24 + 1/60 + 1/120 + ... + $1/(n^3 - n)$}
equals
(a) 1
(b) ½
(c) ¼
(d) 1/8

Solution :

Let S = {1/6 + 1/24 + 1/60 + 1/120 + ... + $1/(n^3 - n)$}

$t_n = 1/(n^3 - n) = 1/n(n - 1)(n + 1) = (1/2)\{(n + 1) - (n - 1)\}/n(n - 1)(n + 1) =(1/2)\{1/n(n - 1) - 1/n(n + 1)\}$

- ⇨ $2t_n = 1/n(n - 1) - 1/n(n + 1)$

$2t_2 = 1/(1*2) - 1/(2*3)$

$2t_3 = 1/(2*3) - 1/(3*4)$

$2t_3 = 1/(3*4) - 1/(4*5)$

...

...

$2t_n = 1/n(n - 1) - 1/n(n + 1)$

Adding we get, $2S = 1/(1*2) - 1/n(n + 1)$

- ⇨ S = (1/4) – 1/2n(n + 1)
- ⇨ lim S as n -> ∞ = ¼ - 0 = ¼
- ⇨ Option (c) is correct.

15. Let a and b be two real numbers satisfying $a^2 + b^2 \neq 0$. Then the set of real numbers c, such that the equations al + bm = c and $l^2 + m^2 = 1$ have real solutions for l and m is
(a) $[-\sqrt{(a^2 + b^2)}, \sqrt{(a^2 + b^2)}]$
(b) [-|a + b|, |a + b|]
(c) $[0, a^2 + b^2]$
(d) (-∞, ∞).

Solution :

Let l = sinA, then m = cosA

Now, al + bm = c

- ⇨ asinA + bcosA = c
- ⇨ $\sqrt{(a^2 + b^2)}[\{a/\sqrt{(a^2 + b^2)}\}\sin A + \{b/\sqrt{(a^2 + b^2)}\}\cos A] = c$

Let, $a/\sqrt{(a^2 + b^2)} = \cos B$, then $b/\sqrt{(a^2 + b^2)} = \sin B$

The equation becomes, $\sqrt{(a^2 + b^2)}[\sin A\cos B + \cos A\sin B] = c$

- ⇨ $\sqrt{(a^2 + b^2)}\sin(A + B) = c$

Now, $-1 \le \sin(A + B) \le 1$

- ⇨ $-\sqrt{(a^2 + b^2)} \le \sqrt{(a^2 + b^2)}\sin(A + B) \le \sqrt{(a^2 + b^2)}$
- ⇨ $-\sqrt{(a^2 + b^2)} \le c \le \sqrt{(a^2 + b^2)}$
- ⇨ Option (a) is correct.

16. Let f be an onto and differentiable function defined on [0, 1] to [0, T], such that f(0) = 0. Which of the following statements is necessarily true?
    (a) f′(x) is greater than or equal to T for all x
    (b) f′(x) is smaller than T for all x
    (c) f′(x) is greater than or equal to T for some x
    (d) f′(x) is smaller than T for some x.

Solution :

Let us take an example, $f(x) = \sin^{-1}(x)$ for $0 \le x \le 1$

Range is [0, π/2]. Here T = π/2

Now, $f'(x) = 1/\sqrt{(1 - x^2)}$

Clearly for some x Є [0, 1], f′(x) > π/2

- ⇨ Option (c) is correct.

17. The area of the region bounded by $|x| + |y| + |x + y| \le 2$ is
    (a) 2
    (b) 3
    (c) 4
    (d) 6

Solution :

Let, x > 0, y > 0; then $x + y + x + y \le 2$

⇨ $x + y \le 1$

Let, $x < 0, y < 0$; then $-x - y - (x + y) \le 2$

⇨ $x + y \ge -1$

Let, $x > 0, y < 0, x + y > 0$; then $x - y + x + y \le 2$

⇨ $x \le 1$

Let, $x > 0, y < 0, x + y < 0$; then $x - y - x - y \le 2$

⇨ $y \ge -1$

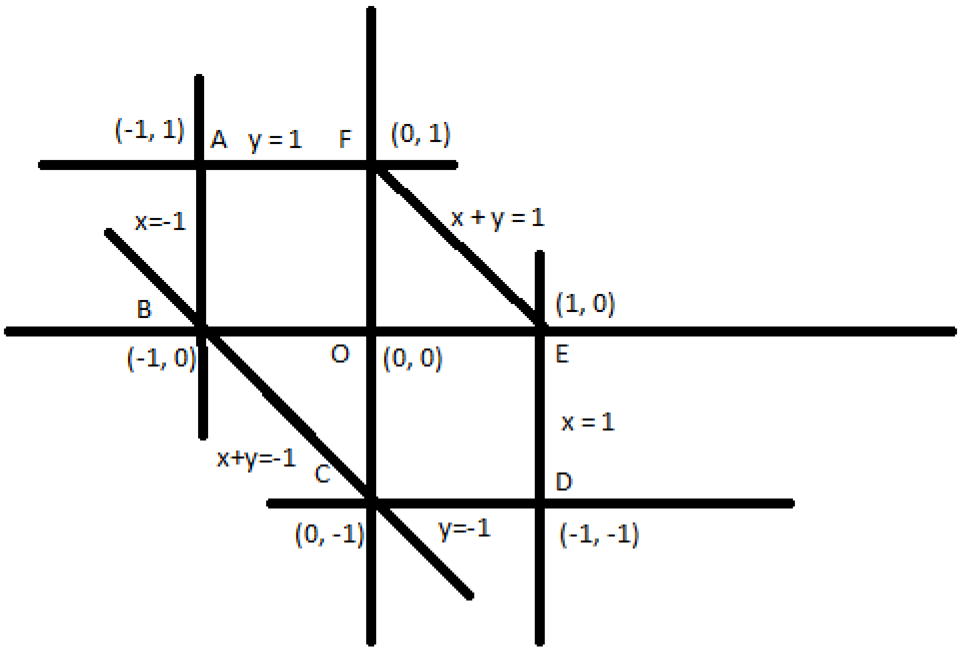
Let, $x < 0, y > 0, x + y > 0$; then $-x + y + x + y \le 2$

⇨ $y \le 1$

Let, $x < 0, y > 0, x + y < 0$; then $- x + y - x - y \le 2$

⇨ $x \ge -1$

The enclosed portion is shown in the figure,



Clearly, from figure ABCDEFA is the enclosed region.

Now, area of triangle OFE = (1/2)*1*1 = ½

Similarly area of triangle OCB = ½

Area of square ABOF = 1*1 = 1

Similarly, area of square OCDE = 1

Therefore, area of ABCDEFA = ½ + ½ + 1 + 1 = 3

- ⇨ Option (b) is correct.

18. Let f and g be two positive valued functions defined on [-1, 1], such that f(-x) =1/f(x) and g is an even function with ∫g(x)dx = 1 (integration running from -1 to 1). Then I = ∫f(x)g(x)dx (integration running from -1 to 1) satisfies
    (a) $I \geq 1$
    (b) $I \leq 1$
    (c) $1/3 < I < 3$
    (d) $I = 1$.

Solution :

Let $f(x) = e^x$ (f(x) is positive valued, f(-x) = 1/f(x) | all criteria satisfied)

Let g(x) = |x|

Then g(x) is even.

g(x) = -x, x < 0

g(x) = x, x > 0

∫g(x)dx (integration running from -1 to 1) = ∫g(x)dx (integration running from -1 to 0) + ∫g(x)dx (integration running from 0 to 1)

= -∫xdx (integration running from -1 to 0) + ∫xdx (integration running from 0 to 1)

$= - x^2/2|_{-1}^{0} + x^2/2|_{0}^{1}$

= 1

So, all criteria are satisfied.

Now, ∫f(x)g(x)dx (integration running from -1 to 1)

= ∫f(x)g(x)dx (integration running from -1 to 0) + ∫f(x)g(x)dx (integration running from 0 to 1)

= $-\int xe^x dx$ (integration running from -1 to 0) + $\int xe^x dx$ (integration running from 0 to 1)

$= -xe^x + e^x|_{-1}^{0} + (xe^x - e^x)|_{0}^{1}$

$= e^0 - e^{-1} - e^{-1} + e - e + e^0$

$= 2 - 2e^{-1}$

$= 2(1 - e^{-1}) > 1$

⇨ Option (a) is correct.

19. How many possible values of (a, b, c, d) with a, b, c, d real, are there such that abc = d, bcd = a, cda = b, dab = c?
(a) 1
(b) 6
(c) 9
(d) 17

Solution :

Now, (abc)(bcd)(cda)(dab) = abcd

⇨ $(abcd)^3 - (abcd) = 0$
⇨ $abcd\{(abcd)^2 - 1\} = 0$
⇨ abcd = 0, abcd = 1, abcd = -1

(a, b, c, d) = (0, 0, 0, 0); (-1, -1, 1, 1); (-1, 1, -1, 1); (-1, 1, 1, -1); (1, -1, -1, 1); (1, -1, 1, -1); (1, 1, -1, -1); (-1, -1, -1, -1); (1, 1, 1, 1)

So, 9 values are possible.

⇨ Option (c) is correct.

20. What is the maximum possible value of a positive integer n, such that for any choice of seven distinct elements from {1, 2, ..., n}, there will exist two numbers x and y satisfying $1 < x/y \le 2$?
(a) 2*7
(b) $2^7 - 2$
(c) $7^2 - 2$
(d) $7^7 - 2$.

Solution :

If we take the below numbers then the condition doesn't hold for minimum n.

1

1*2 + 1 = 3

3*2 + 1 = 7

7*2 + 1 = 15

15*2 + 1 = 31

31*2 + 1 63

63*2 + 1 = 127

So, maximum value of n to hold the condition = 126 = $2^7 - 2$.

- ⇨ Option (b) is correct.

Group B (Each of the following questions has either one or two correct options and you have to identify all the correct options.)

21. Which of the following are roots of the equation $x^7 + 27x = 0$?
    - (a) -√3i
    - (b) (√3/2(-1 + √3i)
    - (c)−(√3/2)(1 + i)
    - (d) (√3/2)(√3 − i)

Solution :

Now, $x^7 + 27x = 0$

- ⇨ $(x^2)^3 + 3^3 = 0$
- ⇨ $(x^2 + 3)(x^4 - 3x^2 + 9) = 0$
- ⇨ $(x^2 + 3) = 0$
- ⇨ x = -√3i is a root

Now, $x^4 - 3x^2 + 9 = 0$

- ⇨ $x^2 = [3 - \sqrt{\{(-3)^2 - 4*1*9\}}]/(2*1)$
- ⇨ $x^2 = (3/2)(1 - i\sqrt{3})$
- ⇨ $x^2 = (3/4)(2 - 2\sqrt{3}i)$
- ⇨ $x^2 = (3/4)\{(\sqrt{3})^2 - 2*\sqrt{3}*i + i^2\}$
- ⇨ $x^2 = (3/4)(\sqrt{3} - i)^2$
- ⇨ x = (√3/2)(√3 − i)
- ⇨ Option (a) and (d) are correct.

22. The equation $|x^2 - x - 6| = x + 2$ has
    - (a) Two positive roots
    - (b) Two real roots

(c)Three real roots
(d) None of the above.

Solution :

Now, $|x^2 - x - 6| = (x + 2)$

- ⇨ $(x + 2)^2(x - 3)^2 = (x + 2)^2$
- ⇨ $(x + 2)^2\{(x - 3)^2 - 1\} = 0$
- ⇨ $(x + 2)^2(x - 2)(x - 4) = 0$
- ⇨ Two positive roots, x = 2, 4
- ⇨ Three real roots x = -2, 2, 4
- ⇨ Option (a) and (c) are correct.

23. If $0 < x < \pi/2$, then
(a) cos(cosx) > sinx
(b) sin(sinx) > sinx
(c)sin(cosx) > cosx
(d) cos(sinx) > sinx

Solution :

Clearly (b), (c), (d) cannot be true as they are of the form sin(A) > A, cos(A) > A

- ⇨ Option (a) is correct.

24. Suppose ABCD is a quadrilateral such that the coordinates of A, B and C are (1, 3), (-2, 6) and (5, -8) respectively. For which choices of the coordinates of D will ABCD be a trapezium?
(a) (3, -6)
(b) (6, -9)
(c)(0, 5)
(d) (3, -1)

Solution :

Slope of AB = (6 – 3)/(-2 – 1) = -1

If we take D as option (a) then slope of opposite side CD = (-6 + 8)/(3 – 5) = -1

So, option (a) is correct.

Now, slope of BC = (-8 – 6)/(5 + 2) = -2

If we take D as option (d) then slope of opposite side AD = (-1 – 3)/(3 – 1) = -2

So, option (d) is correct.

- ⇨ Options (a) and (d) are correct.

25. Let x and y be two real numbers such that 2log(x – 2y) = log(x) + log(y) holds. Which of the following are possible values of x/y?
    (a) 4
    (b) 3
    (c) 2
    (d) 1.

Solution :

Now, 2log(x – 2y) = log(x) + log(y)

- ⇨ $(x - 2y)^2 = xy$
- ⇨ $\{(x/y) - 2\}^2 = (x/y)$ (Dividing both sides by y^2)

Clearly x/y = 4 satisfies the equation.

- ⇨ Option (a) is correct.

26. Let f be a differentiable function satisfying f′(x) = f′(-x) for all x. Then
    (a) f is an odd function
    (b) f(x) + f(-x) = 2f(0) for all x
    (c) (½)f(x) + (½)f(y) = f{(1/2){x + y)} for all x, y
    (d) If f(1) = f(2), then f(-1) = f(-2).

Solution.

Now, f′(x) = f′(-x)

- ⇨ ∫f′(x)dx = ∫f′(-x)dx + c
- ⇨ f(x) = -f(-x) + c
- ⇨ f(x) + f(-x) = c

Putting x = 0 we get, c = 2f(0)

⇨ f(x) + f(-x) = 2f(0)

Putting x = 1 we get, f(1) + f(-1) = 2f(0)

Putting x = 2 we get, f(2) + f(-2) = 2f(0)

⇨ f(1) + f(-1) = f(2) + f(-2)

Now, iff(1) = f(2) then f(-1) = f(-2)

⇨ Options (b) and (d) are correct.

27. Consider the function f(x) = max{x, 1/x}/min{x, 1/x} when x ≠ 0; f(x) = 1 when x = 0. Then
(a) lim f(x) = 0 as x -> 0+
(b) lim f(x) = 0 as x -> 0-
(c)f(x) is continuous for all x ≠ 0
(d) f is differentiable for all x ≠ 0.

Solution :

f(x) = (1/x)/x = $1/x^2$ when x < -1

f(x) = x/(1/x) = x^2 when -1 ≤ x < 0

f(x) = (1/x)/x = $1/x^2$ when 0 < x ≤ 1

f(x) = x/(1/x) = x^2 when x > 1

Now, lim f(x) as x -> 0+ = lim ($1/x^2$) as x -> 0+ = ∞

lim f(x) as x -> 0- = lim (x^2) as x ->0- = 0

lim f(x) as x -> 1+ = lim (x^2) as x -> 1+ = 1

lim f(x) as x -> 1- = lim ($1/x^2$) as x -> 1- = 1

f(1) = 1

So, f(x) is continuous at x = 1.

Now, lim f(x) as x -> -1+ = lim (x^2) as x -> -1+ = 1

lim f(x) as x -> -1- = lim ($1/x^2$) as x -> -1- = 1

f(-1) = 1

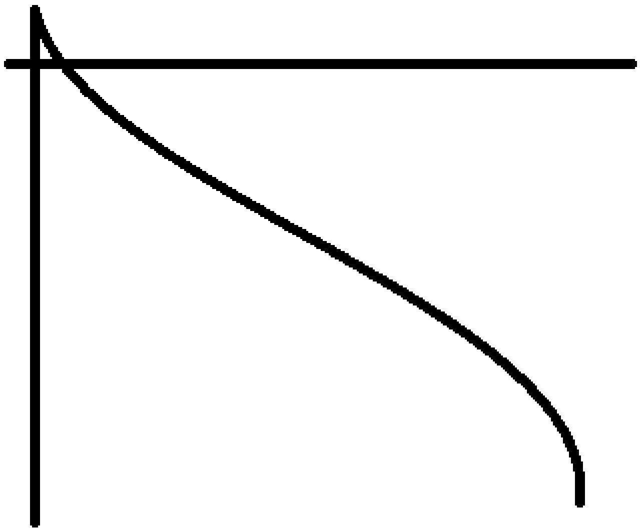
⇨ f(x) is continuous for all x ≠ 0

Now, lim [{f(x) – f(1)}/(x – 1)] as x -> 1+ = lim {(x^2 – 1)/(x – 1)} as x -> 1+ = lim (x + 1) as x -> 1+ = 2 lim [{f(x) – f(1)}/(x – 1)} as x -> 1- = lim {(1/x^2 – 1)/(x – 1)} as x -> 1- = lim [(1 – x^2)/{x^2(x – 1)}] as x -> 1- = lim –(1 + x)/x^2 as x -> 1- = -2

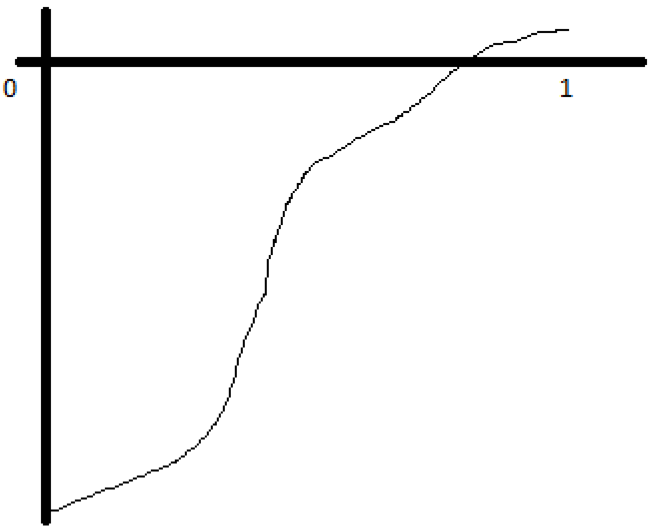
So, f(x) is not differentiable at x = 1.

⇨ Options (b) and (c) are correct.

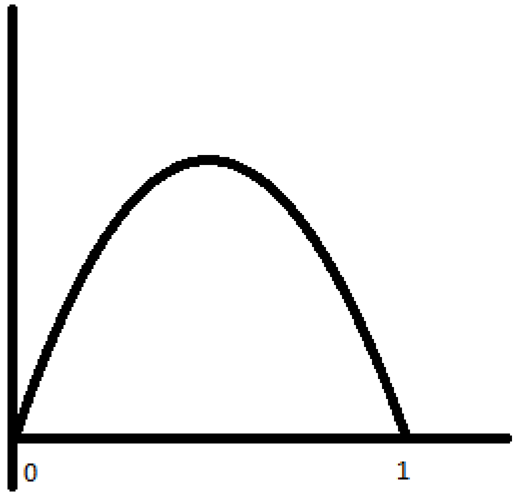
28. Which of the following graphs represent functions whose derivatives have a maximum in the interval (0, 1)?
(a)



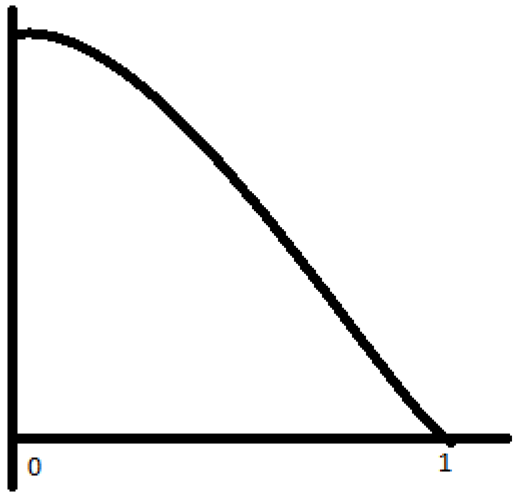
(b)



(c)



(d)



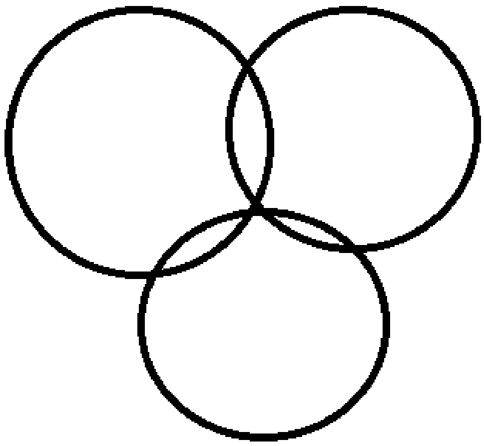
Solution :

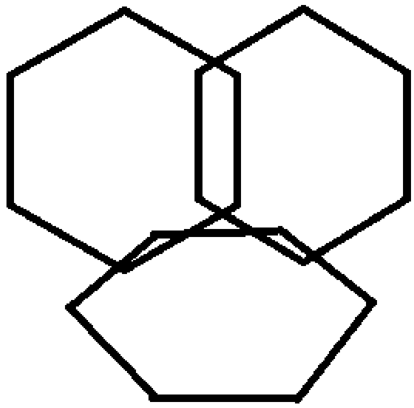
Options (a) and (d) are correct.

29. A collection of geometric figures is said to satisfy *Helly property* if the following condition holds :
For any choice of three figures A, B, C from the collection satisfying $A\cap B \neq \Phi$, $B\cap C \neq \Phi$ and $C\cap A \neq \Phi$, one must have $A\cap B\cap C \neq \Phi$.
Which of the following collections satisfy Helly property?

(a) A set of circles
(b) A set of hexagons
(c) A set of squares with sides parallel to the axes
(d) A set of horizontal line segments.

Solution :





Clearly from the above figure option (a) and (b) is not true.

And it is clear that options (c) and (d) are correct.

30. Consider an array of m rows and n columns obtained by arranging the first mn integers in some order. Let b_i be the maximum of the numbers in the i-th row and c_j be the minimum of the numbers in the j-th column. If
$b = \min(b_i)$ where $1 \le i \le m$ and $c = \max(c_j)$ where $1 \le j \le n$,
then which of the following statements are necessarily true?
(a) $m \le c$
(b) $n \ge b$
(c) $c \ge b$
(d) $c \le b$.

Solution :

If we put 1, 2, 3, ..., n in i-th row then b_i has the minimum value at i-th column and it is n

So, b = n and c_n is maximum at c_n and it is equal to n.

So, c = n (if n < m then c < m, so option (a) cannot be true)

⇨ b = c

If we put mn, mn – 1, mn – 2, ..., mn – m in j-th column, then c_j has the maximum value and it is equal to mn – m.

So, c = mn – m and b_m has the minimum value and it is equal to mn – m

So, b = mn – m (option (b) cannot be true)

⇨ b = c

Now, let us take m = 4 and n = 3 and put the numbers in random from 1 to 12 as shown in the picture below,

11	6	8
7	1	10
4	9	12
2	5	3

Here, b_1 = 11, b_2 =10, b_3 = 12, b_4 = 5

⇨ b = 5

c_1 = 2, c_2 = 1, c_3 = 3

⇨ c = 3
⇨ c < b
⇨ Option (c) cannot be true.
⇨ Option (d) is correct.

B. Math. (Hons.) Admission Test : 2009

1. The domain of definition of $f(x) = -\log(x^2 - 2x - 3)$ is :
   (a) (0, ∞)
   (b) (-∞, -1)
   (c)(-∞, -1)∪(3, ∞)
   (d) (-∞, -3)∪(1, ∞).

Solution :

$x^2 - 2x - 3 > 0$