CMI SAMPLE PROBLEMS & SOLUTIONS

1. Let a be a positive integer from set {2,3,....,9999}. Show that there are exactly two positive integers in that set such that 10000 divides a(a – 1).

Solution:

Now, $10000 = 2^4 \times 5^4$

Now, a and a – 1 are relatively prime.

⇨ 5^4 divides any one of them and another is divisible by 2^4.

So, we need to have $5^4 \times k \pm 1 \equiv 0 \pmod{16}$

⇨ $k \pm 1 \equiv 0 \pmod{16}$ (As $5^4 \equiv 1 \pmod{16}$)

So, k can be 1, 15, 17.

But, $5^4 \times 17 = 10625 > 9999$.

⇨ There can be only 2 values of k viz. 1, 15.

If k = 1, then $5^4 \times k - 1 = 624$ and it is divisible by 16.

If k = 15, then $5^4 \times k + 1 = 5^4 \times 15 + 1 = 9376$ and it is divisible by 16.

So, there are exactly two integers in the set which satisfies the given criteria viz. 625 and 9376.

2. P(x) is a polynomial. Show that lim $P(t)/e^t$ as t -> ∞ exists. Also show that the limit does not depend on the polynomial.

Solution :

Let, $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

Now, $\lim (a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0)/e^t$ as t -> ∞

This is ∞/∞ form. So, we can use L'Hospital rule.

And we will apply it till $P^{(n)}(t)$ where $P^{(n)}(t)$ is n-th derivative of P(t) w.r.t. t.

Then we will have the limit as, $\lim (a_n \times n!)/e^t$ as t -> ∞ = 0.

Therefore, the limit exists and also it is independent of the polynomial.

3. **We define function $f(x) = e^{-1/x}/x$ when $x < 0$; $f(x) = 0$ if $x = 0$ and $f(x) = e^{-1/x}/x$ when $x > 0$. Show that the function is continuous and differentiable. Find limit at $x = 0$.**

Solution:

Now, lim f(x) as x -> 0- = lim f(x) x -> 0+ as the function is same in both the cases.

So, lim f(x) as x ->0- = lim $e^{-1/x}/x$ as x -> 0-

Let, x = 1/z as x -> 0, z -> ∞

The limit becomes, lim z/e^z as z -> ∞

We can apply L'Hospital rule as it is ∞/∞ form.

So, lim $1/e^z$ as z -> ∞ = 0.

So, lim f(x) as x -> 0- = lim f(x) as x -> 0+ = f(0)

⇨ The function is continuous.

Now, lim {f(x) – f(0)}/(x – 0) as x -> 0- = lim $e^{-1/x}/x^2$ as x -> 0

Let, x = 1/z, x -> 0, z -> ∞

So, lim z^2/e^z as z -> ∞ = lim $2z/e^z$ as z -> ∞ (Applying L"Hospital rule)
= lim $2/e^z$ as z -> ∞
= 0

Similarly, lim {f(x) – f(0)}/(x – 0) as x -> 0 = 0.

⇨ The function is differentiable.

And already discovered above that lim f(x) as x -> 0 = 0.

4. p, q, r any real number such that $p^2 + q^2 + r^2 = 1$.
Show that $3(p^2q + p^2r) + 2(r^3 + q^3) \le 2$

Solution:

$3(p^2q + p^2r) + 2(r^3 + q^3)$
$= 3p^2(r + q) + 2(r + q)(r^2 - rq + q^2)$
$= (r + q)(3p^2 + 2r^2 - 2rq + 2q^2)$
$= (r + q)(3 - r^2 - 2rq - q^2)$ (As $p^2 = 1 - r^2 - q^2$)
$= (r + q)\{3 - (r + q)^2\}$
So, we have to prove, $(r + q)\{3 - (r + q)^2\} \le 2$ i.e. to prove, $(r + q)^3 - 3(r + q) + 2 \ge 0$
i.e. to prove, $(r + q - 1)^2(r + q + 2) \ge 0$ (Apply vanishing method and factorize)
Now, $(r + q - 1)^2$ always greater than 0.
So, we have to prove, $r + q + 2 > 0$
Now, $p^2 + q^2 + r^2 = 1$
⇨ $p^2 + q^2 = 1 - r^2 > 0$ (As $p^2 + q^2$ cannot be negative)
⇨ $r^2 < 1$
⇨ $-1 < r < 1$

Similarly, $-1 < q < 1$
Adding the two inequalities we get, $-2 < r + q < 2$
$r + q + 2 > 0$

5. Let g(n) is GCD of $(2n + 9)$ and $6n^2 + 11n - 2$ then find the greatest value of g(n).

Solution :

Now, $6n^2 + 11n - 2$

$= 6n^2 + 12n - n - 2$

$= 6n(n + 2) - (n + 2)$

$= (n + 2)(6n - 1)$

Now, let p divides both $(n + 2)$ and $(2n + 9)$

⇨ $n + 2 \equiv 0 \pmod{p}$

⇨ $n \equiv -2 \pmod{p}$

Also, $2n + 9 \equiv 0 \pmod{p}$

⇨ 2(-2) + 9 ≡ 0 (mod p)

⇨ 5 ≡ 0 (mod p)

⇨ p = 5.

So, maximum common factor between (n + 2) and (2n + 9) is 5.

Now, let q divides both (2n + 9) and (6n – 1)

⇨ 2n + 9 ≡ 0 (mod q)

⇨ 2n ≡ - 9 (mod q)

Also, 6n – 1 ≡ 0 (mod q)

⇨ 3×(-9) – 1 ≡ 0 (mod q)

⇨ -28 ≡ 0 (mod q)

⇨ 28 ≡ 0 (mod q)

⇨ 2^2×7 ≡ 0 (mod q)

⇨ 7 ≡ 0 (mod q) (As q cannot have a factor 2 because 2n + 9 and 6n – 1 both odd)

⇨ q = 7.

So, maximum common factor between (2n + 9) and (6n – 1) is 7.

So, maximum value of g(n) = 5×7 = 35 {occurs at n = 13m+35, m ∈ N}

6. **There are n parking spaces along a one –way road down which n drivers are travelling. Each driver goes to his favorite parking space and parks there if it is free; otherwise, he parks at the nearest free place down the road. If there is no free space after his favorite, he drives away. How many lists $a_1, \dots, a_n$ of favorite parking spaces are there which permit all of the drivers to park?**

Solution:

There are $(n+1)^{n-1}$ such lists. To each list of preferences $(a_1, \dots, a_n)$ which allows all drivers to park, associate the list $(b_2, \dots, b_n)$, where b_i is the difference mod n+ 1 between the numbers of the space driver i wants and the space the previous driver took. Clearly any two lists give rise to different sequences of b_i.

We now argue that any list of b_i comes from a list of preferences. Imagine that the n parking spaces are arranged in a circle with an extra phantom space put in at the end. Put the first driver in any space, then for i = 2, …, n, put driver i in the first available space after the space b_i away from the space taken by driver i-1; this gives a list of preferences if and only if the one space not taken at the end is the phantom space. However, by shifting the position of the first driver, we can always ensure that the phantom space is the space not taken. Thus the sequences of b_i are equal in number to the lists of preferences, so there are $(n+1)^{n-1}$ of each.

7. Find all positive integers n such that $\mathbf{3^{n-1}+5^{n-1}}$ $\boldsymbol{divides}$ $\mathbf{3^n+5^n}$.

Solution:

This only occurs for n = 1. Let $s_n = 3^n + 5^n$ and note that

$$s_n = (3+5)s_{n-1} - 3.5.\,s_{n-2}$$

So s_{n-1} must also divide $3.5.\,s_{n-2}$. If n >1, then s_{n-1} is co-prime to 3 and 5, so s_{n-1} must divide s_{n-2}, which is impossible since $s_{n-1} > s_{n-2}$.

8. Several positive integers are written on a blackboard. One can erase any two distinct integers and write their greatest common divisor and least common multiple instead. Prove that eventually the numbers will stop changing.

Solution:

If a, b are erased and c < d are written instead, we have c ≤ min (a, b) and d ≥ max (a, b); moreover, ab = cd. From this we may conclude $a+b \le c+d$ by writing $ab + a^2 = cd + a^2 \le ac + ad$ (the latter since $(d-a)(c-a) \le 0$) and dividing both sides by a. Thus the sum of the numbers never decreases, and it is obviously bounded (e.g. by n times the product of the numbers, where n is the number of numbers on the board); hence it eventually stops changing, at which time the numbers never change.

9. No three diagonals of a convex 1996 –gon meet in a point. Prove that the number of triangles lying in the interior of the 1996-gon and having sides on its diagonal is divisible by 11.

Solution:

There is exactly one such triangle for each choice of six vertices of the 1996-gon: if A, B, C, D, E, F are the six vertices in order, the corresponding triangle is formed by the lines AD, BE, CF. Hence the number of triangle is $\binom{1996}{6}$; since 1991 is a multiple of 11, so is the number of triangles.

10.Prove that for every polynomial $x^2 + px + q$ with integer coefficients, there exists a polynomial $2x^2 + rx + s$ with integer coefficients such that the sets of values of the two polynomials on the integers are disjoint.

Solution:

If p is odd, then $x^2 + px + q$ has the same parity as q for all integers x, and it suffices to choose r even and s of the opposite parity as q. If p = 2m is even, then $x^2 + px + q = (x + m)^2 + (q - m)^2$ which is congruent to $q - m^2$ or $q - m^2 + 1$ modulo 4. Now it suffices to choose r even and s congruent to $q - m^2 + 2$ modulo 4.

11.In a convex pentagon ABCDE, AB = BC, ∠ABE + ∠DBC = ∠EBD, and ∠AED + ∠BDC = π. Prove that the orthocenter of triangle BDE lies on AC.

Solution:

By the assumption ∠AEB + ∠BDC = π, there exists a point F on AC such that ∠AFB = ∠AEB and ∠BFC = ∠BDC; this means F is the second intersection of the circumcircles of BCD and ABE. The triangle ABC is isosceles, so ∠FCB = $\frac{(\pi - \angle ABC)}{2}$. Hence $\angle FDB = \angle FCB = \frac{\pi}{2} - \angle DBE$by the assumption ∠ABE + ∠DBC = ∠EBD, and so DF ⊥ BE. Similarly EF ⊥ BD, and so F is the orthocenter of BDE.

12. Sergey found 11 different solutions to the equation $f\left(19x - \frac{96}{x}\right) = 0$. Prove that if he had tried harder, he could have found at least one more solution.

Solution:

The equations $19x - \frac{96}{x} = t$ can be re-written $19x^2 - tx - 96 = 0$; $since\ t^2 + 19.96 > 0$, it always has two real roots. Therefore the number of zeroes of f (if finite) is an even integer, so Sergey can find at least one more zero.

13. Find all quadruples of polynomials $P_1(x), P_2(x), P_3(x), P_4(x)$ with real coefficients such that for each quadruple of integers x, y, z, t such that $xy - zt = 1$, one has

$$P_1(x)P_2(y) - P_3(z)P_4(t0 = 1.$$

Solution: If $P_1(1) = 0$, then $P_3(z)P_4(t) = -1$ for each pair of integers z, t, and so $P_3\ and\ P_4$ are constant functions; moreover, $P_1(x)P_2(y) = 0$, so one of $P_1\ and\ P_2$ is identically zero. Ignoring such cases, which are easily enumerated, we assume $P_i(1) \neq 0$ for all i.

We first note that $P_1(x)P_2(1) = P_1(1)P_2(x)$ for all nonzero integers x, so that $P_1\ and\ P_2$ are equal up to a scalar factor; similarly, $P_3\ and\ P_4$ are equal up to a scalar factor. Now note that $P_1(x)P_2(ay) = P_1(ax)P_2(y)$ for all nonzero a, x, y, so that the difference between the two sides is identically zero as a polynomial in a. In particular, that means no term in $P_1(x)P_2(y)$ has unequal exponent in x and y, and the same is true of $P_1(x)P_1(y)$ on the other hand, if $P_1(x)$ has term of more than one degree, then $P_1(x)P_1(y)$ contains a term with different degrees in x and y. Hence $P_1(x) = cx^k$ for some integer k and some constant c, and similarly $P_2(x) = dx^k, P_3(x) = ex^m, P_4(x) = fx^m$.

Thus we must determine when $cdx^ky^k - efz^mt^m = 1$ whenever $xy - zt = 1$ in integers. Clearly k = m since otherwise one of the two terms on the left dominates the other, and cd =1 by setting x = y = 1 and z = t = 0, and similarly ef = 1. Now note that $(xy)^k - (zt)^k = 1$ can only happen in general for k = 1, since for k >1, there are no consecutive perfect k-th powers. We conclude $P_1(x) = cx, P_2(x) = \frac{x}{c}, P_3(x) = ex, P_4(x) = \frac{x}{e}$ for some nonzero real numbers c, e.

14. Two players play the following game on a 100×100 board. The first player marks a free square, then the second player puts a 1×2 domino down covering two free squares, one of which is marked. This continuous until one player is unable to move. The first player wins if the entire board is covered, otherwise the second player wins. Which player has a winning strategy?

Solution:

The first player has a winning strategy. Let us say a position is stable if every square below or to the right of a free square is free. Then we claim the first player can always ensure that on his turn, either the position is stable or there is a free square with exactly one free neighbor (or both).

Let us label the square in the i-th row and j-th column as (i, j), with (1, 1) in the top left. We call a free square a corner if is not below or to the right of another free square. Let $(a_1, b_1), (a_2, b_2), \dots, (a_k, b_k)$ be the corners from top to bottom.

First notice that if (a, b) is a corner such that both $(a + 1, b - 1) and\ (a - 1, b + 1)$ are nonfree (or off the board), then the first player may mark (a, b), and however the second player moves, the result will be a stable position. More generally, if $(a, b), (a + 1, b - 1), \dots, (a + k, b - k)$ are corners and $(a - 1, b + 1) and\ (a + k + 1, b - k - 1)$ are both nonfree or off the board, the first player can be sure to return to a stable position.

To show this, first note that we cannot have both a = 1 and b –k = 1, or else the number of nonfree squares would be odd, which is impossible. Without loss of generality, assume that b –k ≠ 1 is not the final corner. The first player now marks (a, b). If the second player covers (a, b) and (a, b+1), the position is again stable. Otherwise, the first player marks $(a + 1, b - 1)$ and the second player is forced to cover it and $(a + 2, b - 1)$. Then the first player marks $(a + 2, b - 2)$ and the second player is forced to cover it and $(a + 3, b - 2)$, and so on. After $(a + k, b - k)$ is marked, the result is a stable position.

(Note that the assumption b –k ≠ 1 ensures that the moves described do not cross the edge of the board.) To finish the proof, we need to show that such a chain of corners must exist. Write the labels $(a_1, b_1), \dots, (a_k, b_k)$ in a row, and join two adjacent labels by a segment if they are of the form $(a, b), (a + 1, b - 1)$. If two adjacent labels $(a, b), (a + i, b - j)$ are not joined by a segment, then either i = 1 or j = 1 but not both. If i = 1, draw an arrow between the labels pointing towards $(a + i, b - j)$; otherwise draw the arrow the other way. Also draw arrows pointing to $(a_1, b_1) and\ (a_k, b_k)$. There is now one more chain of corners (joined by segments) than arrows, so some chain has two arrows pointing to it. That chain satisfies the condition above, so the first player can use it to create another stable position. Consequently, the first player can ensure victory.

15. Let BD be the bisector of angle B in triangle ABC. The circumcircle of triangle BDC meets AB at E, while the circumcircle of triangle ABD meets BC at F. Prove that AE = CF.

Solution:

By power-of-a-point. AE .AB = AD. AC and CF. CB = CD. CA, so $\frac{AE}{CF} = \left(\frac{AD}{CD}\right)\left(\frac{BC}{AB}\right)$. However, $\frac{AB}{CB} = \frac{AD}{CD}$ by the angle bisector theorem, so AE = CF.

16. A 10×10 table consists of positive integers such that for every five rows and five columns, the sum of the numbers at their intersections is even. Prove that all of the integers in the table are even.

Solution:

We denote the first five entries in a row as the “head” of that row. We first show that the sum of each head is even. We are given that the sum of any five heads is even; by subtracting two such sums overlapping in four heads, we deduce that the sum of any two heads is even. Now subtracting two such relations from a sum of five heads, we determine that the sum of any head is even.

By a similar argument, the sum of any five entries in a row is even.

By the same argument as above, we deduce that each entry is even.

17. Prove that there are no positive integers a and b such that for each pair p, q of distinct primes greater than 1000, the number $ap + bq$ is also prime.

Solution:

Suppose a, b are so chosen, and let m be a prime greater than a +b, by Dirichet's theorem, there exist infinitely many primes in any nonzero residue class modulo m; in particular, there exists a pair p, q such that p ≡ b (mod m), q ≡ -a (mod m), giving $ap + bq$ is divisible by m, a contradiction.

18. In triangle ABC, the angle A is $60°$. A point O is taken inside the triangle such that $\angle AOB = \angle BOC = 120°$. The points D and E are the midpoints of sides AB and AC. Prove that the quadrilateral ADOE is cyclic.

Solution:

Since $\angle OBA = 60° - \angle OAB = \angle OAC$, the triangles OAB and OCA are similar, so there is a spiral similarity about O carrying OAB to OCA. This similarity preserves midpoints, so it carries D to E, and therefore ∠AOD = ∠COE = 120° $- \angle AOE$. We conclude ∠DOE = 120° and so ADOE is cyclic.

19. There are 2000 towns in a country, each pair of which is linked by a road. The Ministry of Reconstruction proposed all of the possible assignments of one way traffic to each road. The ministry of Transportation rejected each assignment that did not allow travel from any town to any other town. Prove that more of half of the assignments remained.

Solution:

We will prove the same statement for n ≥ 6 towns. First suppose n = 6. In this case there are 2^{15} assignments, and an assignment is rejected only if either one town has road to all of the others in the same direction, or if there are two sets of three towns, such that within each town the roads point in a circle, but all of the roads from one set to the other point in the same direction. There are 5.2^{11} had assignments of the first kind and 20.8 of the second kind, so the fraction of good assignments is at least $\frac{5}{8}$.

For $n \geq 6$, we claim that the fraction of good assignments is at least

$$\frac{5}{8}\prod_{i=6}^{n-i}\left(1-\frac{1}{2^{i-1}}\right).$$

We show this by induction on a good assignment or r h -1 vertices can be extended to a good assignment on vertices simply by avoiding having all edges from the last vertex pointing in the same direction, which occurs in 2 cases out of 2^{n-1}.

Now it suffices to show that the above expression is more than $\frac{1}{2}$.

In fact,

$$\prod_{i=5}^{\infty}\left(1-\frac{1}{2}\right)^{-1} \leq 1+\sum_{i=5}^{\infty}\frac{i-4}{2^i}$$

$$= 1+\frac{1}{2^5}\sum_{i=0}^{\infty}\frac{i+1}{2^i}$$

$$= 1+\frac{1}{2^5}\sum_{i=0}^{\infty}\sum_{k=i}^{\infty}\frac{1}{2^i}$$

$$= 1+\frac{1}{2^5}\sum_{i=0}^{\infty}\frac{1}{2^i-1}$$

$$= 1+\frac{4}{2^5}=\frac{9}{8}$$

Thus the fraction of good assignments is at least $\left(\frac{5}{8}\right)\left(\frac{8}{9}\right)=\frac{5}{9}>\frac{1}{2}$.

20. A set of geometric figures consists of red equilateral triangles and blue quadrilaterals with all angles greater than 80° and less than 100°. A convex polygon with all of its angles greater than 60° is assembled from the figures in the set. Prove that the number of (entirely) red sides of the polygon is a multiple of 3.

Solution:

We first enumerate the ways to decompose various angles α into sums of 60° angles (T) and angles between 80° and 100° (Q):

$60° < \alpha < 180°$ $\alpha =$ T, 2T, T + Q, 2Q

$\alpha = 180°$ $\alpha =$3T, 2Q

$\alpha = 360°$ $\alpha =$ 6T, 3T+2Q, 4Q

The range for Q cannot be increased, since 3Q ranges from 240° to 300°; even including the endpoints would allow for additional combinations above.

The set of all of the vertices of all of the polygons can be divided into three categories, namely those which lie in the interior, on an edge, or at a vertex of the large polygon. The above computation shows that the number of T angles at interior or edge vertices is a multiple of 3; since the total number is 3 times the number of triangles, we deduce that the number of T angles at vertices of the large polygon is also multiple of 3.

Next note that every edge is entirely of one color, since we cannot have both at T and a Q at a 180° angle. Additionally, no vertex of the large polygon consists of more than two angles, and a T cannot occur by itself. All this means that the number of red sides is half the number of T angles at the vertices, which is a multiple of 3.