

Uncovering patterns in heavy-tailed networks : A journey beyond scale-free

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Networks: Definition and Properties

A network:

is presented by a graph $G(V,E)$: V = nodes, E = edges (link node pairs).

Properties:

- Static properties:
 - Small world effect
 - **Degree distribution**
 - Triad and clustering coefficient
 - Network resilience
 - Community structure
- Temporal / Dynamic properties:
 - Densification
 - Shrinking diameter

Degree distribution

- In network theory, the degree of a node in a network is the number of connections it has to other nodes and the degree distribution is the probability distribution of these degrees over the whole network.
- Real-world heavy-tailed networks are claimed to be scale-free, meaning that the degree distributions follow the classical power-law.

What is Power-Law?

A random variable x is said to follow a power-law if it has a PDF(probability density function) $p(x)$ of the form

$$p(x) \propto x^{-\alpha}$$

Where α is the exponent of the power-law.

A brief history of fitting power-law

- **A. L. Barabasi, et al., Science, 2000:**
 - Estimates the exponent of power-law. But did not provide the statistical significance of the fit.
- **M. Goldstein and S. Morris, European physical journal, 2004:**
 - Authors gave emphasis on checking the fit of the distribution and found that power-law is not fitting properly.
- **A. Clauset, et. al., SIAM review, 2009:**
 - Provided different ways of estimating parameters. power-law is shown to be fitting some data sets and failing to fit some data sets.

Closer inspection of the degree distribution

- However, most of these heavy-tailed real-world networks do not follow power-law degree distributions.
- This is due to the existence of an identifiable non-linearity in the entire degree distribution in a log-log scale.

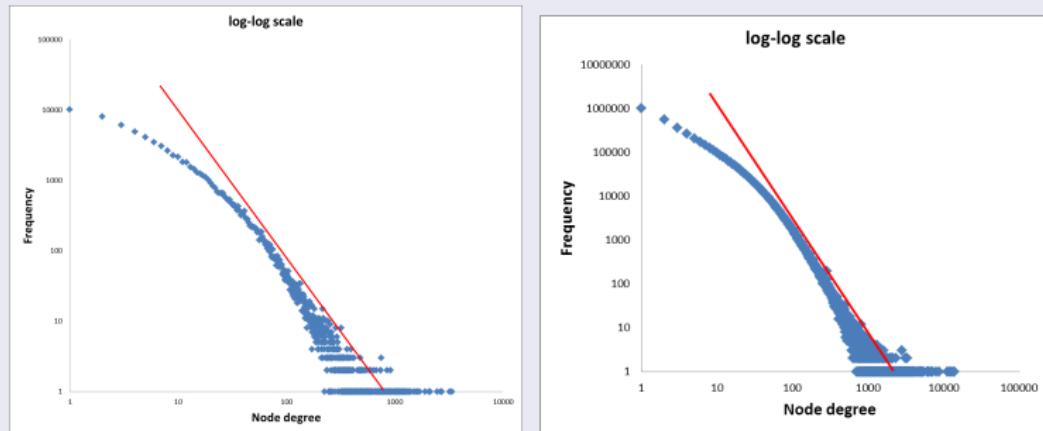


Figure: Twitter and LiveJournal Network

Recent controversy and our objectives

- **Stumpf MP, Porter MA., Science, 2012**
 - Critical truths about power-laws.
- **Broido AD, Clauset A., Nature Communication, 2019.**
 - Scale-free networks are rare.
- **Holme P., Nature Communication, 2019.**
 - Rare but everywhere: Perspectives on scale-free networks.
- **Our Observation:** Degree distributions of real-world heavy-tailed networks can't always be fitted using a single power-law distribution.
- **Our Goal:** Finding a new heavy-tailed distribution that can better-fit the real-world heavy-tailed complex networks.

Lomax Distribution

- Lomax distribution (**K S Lomax, JASA, 1954**) is popularly used as an alternative to power-law, exponential, gamma and weibull distribution for modeling heavy-tailed data in the domain of business, economics, actuarial science, queueing theory and Internet traffic modeling.
- **Definition:** A random variable Z follows Lomax distribution with parameters α and σ if the CDF and PDF are of the form:

$$F(z) = 1 - \left(1 + \frac{z}{\sigma}\right)^{-\alpha}; \quad f(z) = \frac{\alpha}{\sigma} \left(1 + \frac{z}{\sigma}\right)^{-\alpha-1}; \quad z \geq 0;$$

where $\alpha (> 0)$ is the shape parameter and $\sigma (> 0)$ is the scale parameter.

Modified Lomax Model

- Sometimes the Lomax distribution does not provide great flexibility in modeling heavy-tailed data sets in its whole range.
- This paper proposes a new class of lomax models where the non-negative shape parameter (α) is assumed to be expressible as a nonlinear function of the data for fitting these heavy-tailed networks.
- We propose a modified Lomax (MLM) distribution derived from a newly introduced hierarchical family of Lomax distributions for efficient modeling the entire degree distribution.

Definition: A continuous random variable X follows a family of heavy-tailed Lomax (HLM) distributions if and only if it has the following CDF:

$$F(x) = 1 - (1 + x)^{-m(x)}; \quad x \geq 0$$

and $F(x) = 0$ if $x < 0$, where $m : (0, \infty) \rightarrow \mathbb{R}^+$ is a real, continuous, positive function which is differentiable on $(0, \infty)$ and satisfies the following conditions:

- ① The function $m(\cdot)$ is strictly positive and have finite limit at infinity, i.e., $\lim_{x \rightarrow \infty} m(x) = \alpha (> 0)$.
- ② $\lim_{x \rightarrow 0^+} (1 + x)^{m(x)} = 1$ and $\lim_{x \rightarrow \infty} (1 + x)^{m(x)} = \infty$.
- ③ $\frac{m'(x)}{m(x)} \geq -\frac{1}{(1 + x) \log(1 + x)}, \quad x > 0$.

MLM Contd..

The PDF of this new family of heavy-tailed Lomax distribution is of the form:

$$f(x) = (1+x)^{-m(x)} \left[\frac{m(x)}{(1+x)} + m'(x) \log(1+x) \right], \quad x > 0.$$

$$f(x) = 0 \quad x \leq 0.$$

There can be a wide variety of choices of $m(x)$ satisfying

$$\lim_{x \rightarrow \infty} m(x) = \alpha (> 0)$$

Note that the simplest choice of $m(x) = \alpha$ and $x = \frac{z}{\sigma}$ corresponds to the Lomax distribution.

- There are several choices of $m(x)$ we studied.
- Here, we choose a nonlinear function $m(.)$ that adds a nonlinear exponents as follows:

$$m(x) = \alpha \left(\frac{\log(1+x)}{1 + \log(1+x)} \right)^\beta.$$

- The chosen $m(x)$ approaches to α from below if $-1 < \beta < 0$ as $x \rightarrow \infty$ and approaches to α from above for $\beta > 0$ as $x \rightarrow \infty$.

Definition: (Modified Lomax Distribution) A continuous random variable X follows $MLM(\alpha, \beta, \sigma)$ distribution with $\alpha (> 0)$ and $\beta (> -1)$ as the shape parameters and $\sigma (> 0)$ as the scale parameter if the CDF takes the following form:

$$F(x) = 1 - \exp \left[-\alpha \frac{\log^{\beta+1}(1 + x/\sigma)}{[1 + \log(1 + x/\sigma)]^\beta} \right]; \quad x > 0$$

and $F(x) = 0$ if $x \leq 0$. The corresponding PDF is given by,

$$f(x) = \frac{\alpha [\beta + 1 + \log(1 + \frac{x}{\sigma})] [\log(1 + \frac{x}{\sigma})]^\beta}{\sigma (1 + \frac{x}{\sigma}) [1 + \log(1 + \frac{x}{\sigma})]^{\beta+1}} \exp \left[-\alpha \frac{[\log(1 + \frac{x}{\sigma})]^{\beta+1}}{[1 + \log(1 + \frac{x}{\sigma})]^\beta} \right]$$

; $x > 0$ and $f(x) = 0$ if $x \leq 0$.

Plot of Lomax Distribution Function

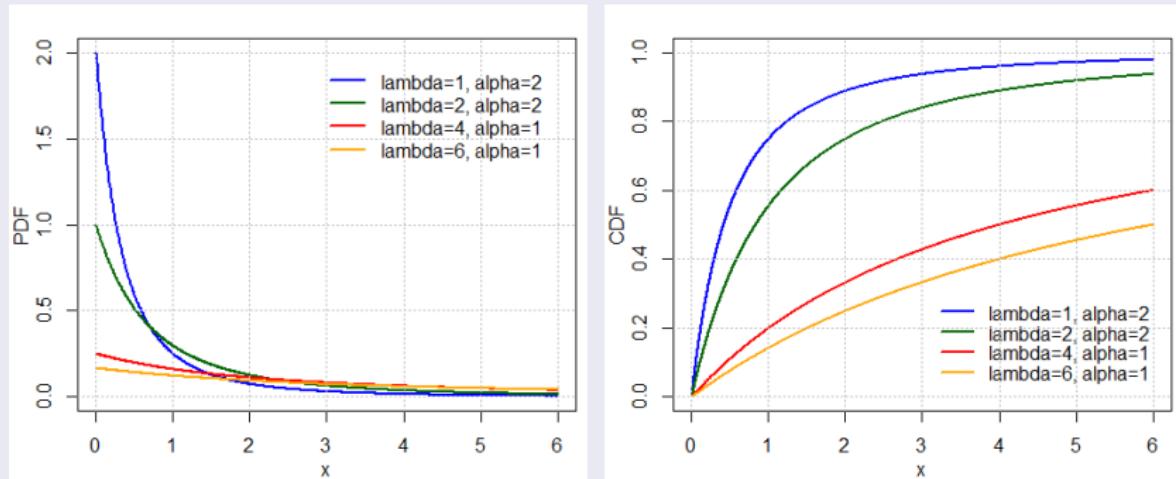


Figure: Lomax PDF and CDF

Plot of Modified Lomax Distribution Function

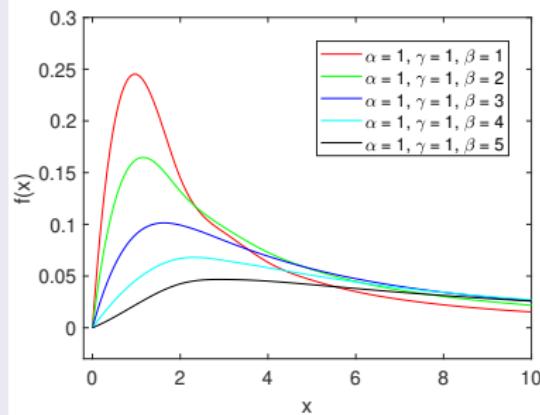
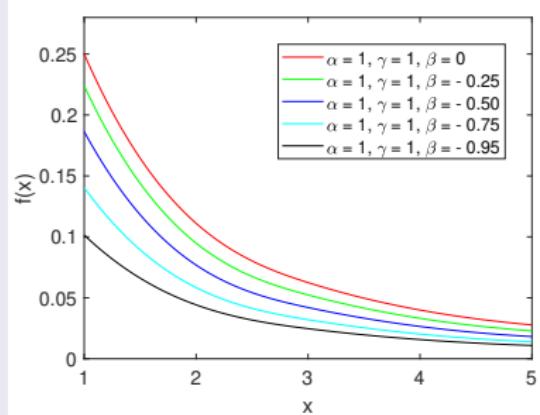


Figure: Modified Lomax PDF

Plot of Modified Lomax Distribution Function

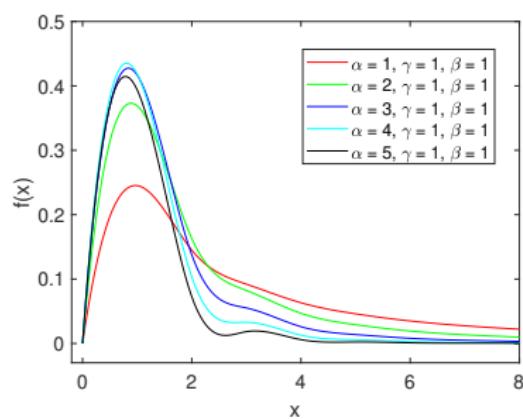
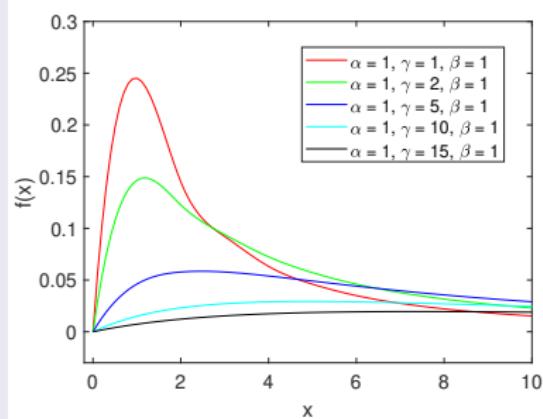


Figure: Modified Lomax PDF

MLE of parameters for MLM model

The log-likelihood function for the vector of parameters $\Theta = (\alpha, \beta, \sigma)^T$ is given by

$$\begin{aligned}\ell \equiv \ell(x; \alpha, \beta, \sigma) = & n \log(\alpha) - \sum_{i=1}^n \log(\sigma + x_i) + \sum_{i=1}^n \log \left[\beta + 1 + \log \left(1 + \frac{x_i}{\sigma} \right) \right] \\ & + \beta \sum_{i=1}^n \log \left[\log \left(1 + \frac{x_i}{\sigma} \right) \right] - (\beta + 1) \sum_{i=1}^n \log \left[1 + \log \left(1 + \frac{x_i}{\sigma} \right) \right] \\ & - \alpha \sum_{i=1}^n \frac{\left[\log \left(1 + \frac{x_i}{\sigma} \right) \right]^{\beta+1}}{\left[1 + \log \left(1 + \frac{x_i}{\sigma} \right) \right]^\beta}\end{aligned}$$

The maximum likelihood estimate for the parameters α, β , and σ are given by $\hat{\alpha}, \hat{\beta}$, and $\hat{\sigma}$, are obtained by maximizing the above likelihood function. The above likelihood function for the MLM distribution has a global maximum for the samples x_i at a finite point when the coefficient of variation (CV) > 1 (proof is given in the paper).

Extreme Value Properties

- In extreme value theory, one focuses on extreme and rare events.
- There exists a well elaborated statistical theory for extreme values.
- It applies to (almost) all (univariate) extremal problems.

Some extreme value properties of Modified Lomax Distribution are as follows (detailed proofs are given in the paper):

- MLM is tail-equivalent to Power-law distribution.
- MLM is a long-tailed distribution.
- MLM F belongs to the maximum domain of attraction (MDA) of the Frechet family of distribution (heavy-tailed).

Experimental Results

- **Data:** We consider standard data sets from different disciplines, namely Social Networks, Collaboration Networks, Citation Networks, Web Graphs, Product Co-purchasing Networks and Communication Networks.
- These are most standard network data sets which have heavy-tailed behaviors and used for modeling in the statistical paradigm.
- Previous studies focused on using standard single statistical distributions, namely power-law, Lomax (Pareto Type-II), Exponential, Log-normal for modeling this wide variety of network data sets.
- But these models fail in capturing the lower-degree nodes while modeling the degree distributions. To overcome the drawback, we apply the proposed MLM distribution on these data sets.

Experimental Results

Data sets		No. of nodes	No. of edges	Stat. Prop.			Estimated parameters			Bootstrap chi-square value (p)
				s	μ	$\frac{s}{\mu}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	
Social Networks	ego-Twitter(ln)	81,306	1,768,149	57.965	21.747	2.6654	1.9922	-0.3591	30.543	0.9920
	ego-Gplus(ln)	107,614	13,673,453	1404.8	283.42	4.9568	0.7108	-0.4983	23.077	0.9963
	soc-Slashdot	70,068	358,647	35.069	10.237	3.426	0.8663	-0.6228	1.0461	0.9955
	soc-Delicious(ln)	536,108	1,365,961	39.826	10.673	3.7312	1.3630	-0.6819	5.3709	0.9960
	soc-Digg(ln)	770,799	5,907,132	166.61	46.584	3.5765	0.7931	-0.6928	5.5163	0.9890
	soc-Academia	200,169	1,398,063	48.297	14.259	3.3871	2.7429	-0.3737	36.644	0.6087
	LiveJournal(ln)	4,847,571	68,993,773	44.969	15.368	2.926	2.6892	-0.7272	51.933	0.8983
	Dogster-Friendship	426,821	8,546,581	284.06	40.033	7.095	1.5634	0.3108	14.057	0.9500
	Higgs-Twitter(ln)	456,626	14,855,842	350.91	54.786	6.4051	1.6797	-0.0347	36.204	0.9870
	Artist-Facebook	50,615	819,307	63.427	32.366	1.9596	2.0117	-0.1445	39.337	0.9812
Citation Networks	cit-HepTh(ln)	27,770	352,807	43.139	15.220	2.8342	1.8410	-0.3093	16.416	0.8730
	cit-HepPh(ln)	34,546	421,578	27.286	14.933	1.8271	2.5553	-0.3622	34.349	0.9900
	cit-Patents(ln)	3,774,768	16,518,948	6.9125	5.0687	1.3637	4.4822	-0.2534	21.689	0.8080
	cit-Citeseer(ln)	227,320	814,134	9.8260	5.4322	1.8088	2.2630	-0.2788	7.4150	0.6350
Collaboration Networks	ca-CondMat	23,133	93,497	10.671	8.0189	1.3308	3.1068	0.3615	10.5353	0.9896
	ca-AstroPh	18,772	198,110	30.568	21.103	1.4484	16.434	37.276	0.0101	0.9990
	ca-GrQc	5,242	14,496	7.9186	5.5284	1.4322	2.2624	3.5861	0.6765	0.7849
	ca-HepPh	12,008	118,521	46.654	19.696	2.3687	0.9798	2.8780	0.6791	0.8163
	ca-HepTh	9,877	25,998	6.1867	5.2618	1.1757	2.9417	5.2791	0.4825	0.9332
Web Graphs	Google(ln)	875,713	5,105,039	43.320	7.1444	6.0634	1.1999	-0.6399	2.0429	0.9780
	BerkStan(ln)	685,230	7,600,595	300.08	12.316	24.364	1.4129	1.8449	0.7592	0.6250
	Wikipedia2009(ln)	1,864,433	4,507,315	12.846	4.8903	2.6268	1.3988	-0.6291	1.9658	0.9891
	WikipediaLinkFr(ln)	4,906,478	113,122,279	1864.4	48.608	38.356	1.0988	-0.7123	9.8888	0.9152
	Hudong(ln)	1,984,484	14,869,483	199.28	16.467	12.101	1.1567	10.921	0.0013	0.9883
Product co-purchasing networks	amazon0601(ln)	403,394	3,387,388	15.279	8.3989	1.8191	3.8261	-.7137	19.522	0.6010
	amazon0505(ln)	410,236	3,356,828	15.313	8.1826	1.8714	3.8367	-0.8006	19.984	0.6880
	amazon0312(ln)	400,727	3,200,444	15.073	7.9865	1.8873	3.7631	-0.8179	18.747	0.5890
Communication Networks	Email-Enron	36,692	183,831	36.100	10.021	3.6027	1.2417	-0.1275	2.9045	0.9641
	Wiki-Talk(ln)	2,394,385	5,021,410	12.259	2.1195	5.7844	1.5167	-0.2846	0.0016	0.9900
	Rec-Libimseti(ln)	220,970	17,359,346	413.71	102.85	4.0227	2.5008	-0.8496	331.18	0.9670

Experimental Results

Table of different statistical measures of different competitive models over real world networks

Data sets		MLM			Lomax			Power-law			Pareto		
		RMSE	KLD	MAE	RMSE	KLD	MAE	RMSE	KLD	MAE	RMSE	KLD	MAE
Social Networks	ego-Twitter(In)	16.800	0.00819	1.3498	29.366	0.01354	2.4701	204.35	0.1831	10.847	354.25	0.2857	15.603
	ego-Gplus(In)	1.6115	0.05601	0.1825	10.491	0.06444	0.3033	53.064	0.2299	0.9221	86.955	0.3113	1.1847
	soc-Slashdot	31.527	0.01365	2.3951	32.065	0.014102	2.4658	247.87	0.1007	10.074	247.84	0.1007	10.073
	soc-Delicious(In)	79.809	0.00839	3.7730	91.993	0.01326	4.8060	349.66	0.2021	14.867	471.02	0.1349	17.874
	soc-Digg(In)	13.634	0.02182	0.8440	24.841	0.02391	1.0269	208.01	0.1601	4.2185	212.87	0.1601	4.2312
	soc-Academia	16.323	0.00351	0.5705	48.951	0.01019	1.6178	229.54	0.2027	6.3889	440.15	0.274	10.464
	LiveJournal(In)	243.99	6.13e-04	5.4026	1764.9	0.02111	54.400	5025.2	0.1614	127.98	8100.9	0.1785	164.18
	Dogster-Friendship	32.203	0.01328	0.8502	36.449	0.01700	1.0755	358.27	0.2926	5.6815	549.57	0.4618	7.6734
	Higgs-Twitter(In)	19.821	0.00785	0.4710	20.609	0.00793	0.4621	260.32	0.2492	4.8938	524.96	0.4806	7.6938
Citation Networks	Artist-Facebook	11.708	0.01079	2.1381	12.923	0.01199	2.6173	100.49	0.1643	14.552	350.05	0.4010	26.467
	cit-HepTh(In)	3.2640	0.01354	0.5071	7.9393	0.01585	0.7585	73.531	0.1741	4.0821	122.79	0.2566	5.997
	cit-HepPh(In)	9.6810	0.00821	1.9016	21.303	0.01317	3.1135	128.55	0.1825	13.234	257.41	0.2689	21.445
	cit-Patents(In)	445.80	1.61e-04	47.603	2577.5	0.00192	230.35	27.5K	0.2266	2049.5	34.8K	0.2366	2533.2
Collaboration Networks	cit-CiteSeer(In)	40.728	0.00228	3.3778	28.032	0.00278	3.3902	889.88	0.3308	49.467	1156.2	0.2916	62.026
	ca-CondMat	14.830	0.00479	4.1570	36.094	0.00814	7.3904	107.86	0.1025	26.092	469.12	0.3738	63.075
	ca-AstroPh	23.890	0.02756	5.9796	32.799	0.03457	7.4448	92.255	0.1753	15.158	251.03	0.3816	27.707
	ca-GrQc	15.850	0.03055	7.2247	35.935	0.04013	12.286	124.24	0.2554	27.137	202.33	0.2741	44.221
	ca-HepPh	13.944	0.06959	4.1919	19.607	0.07266	4.7763	75.071	0.1769	8.0906	144.39	0.2569	14.668
Web Graphs	ca-HepTh	23.280	0.00896	10.851	61.797	0.01353	20.391	268.91	0.2346	66.108	437.19	0.2829	106.36
	Google(In)	360.62	0.01368	13.845	337.68	0.01546	14.201	1809.1	0.124	45.023	1809.2	0.124	45.023
	BerkStan(In)	71.819	0.03116	0.9478	105.20	0.0346	1.1962	615.03	0.1863	4.0722	615.01	0.1863	4.0721
	Wikipedia2009(In)	86.510	0.00169	7.7498	103.94	0.00197	8.5289	4371.9	0.1352	164.58	4371.9	0.1352	164.58
	WikipediaLinkFr(In)	124.98	0.01776	0.3174	146.14	0.03082	0.4465	248.09	0.1518	0.7857	397.39	0.1521	1.0815
	Hudong(In)	8.0517	0.00433	0.2508	25.163	0.00525	0.4600	587.21	0.0868	4.6828	587.22	0.0868	4.6828
Product co-purchasing networks	amazon0601(In)	94.347	0.00374	8.1863	147.602	0.00695	10.928	1495.4	0.2708	70.281	2539.8	0.4022	114.59
	amazon0505(In)	109.95	0.00412	9.1836	94.048	0.00499	8.7882	1572.9	0.2463	73.003	2494.5	0.3711	111.56
	amazon0312(In)	100.89	0.00430	8.5465	92.525	0.00495	8.5742	1564.4	0.2425	71.875	2462.9	0.3686	109.21
Communication Networks	Email-Enron	74.667	0.03523	5.2075	76.155	0.03531	5.2347	246.51	0.1779	14.886	245.25	0.1778	14.859
	Wiki-Talk(In)	670.47	0.00356	25.871	671.76	0.00357	25.898	9669.4	0.3376	293.63	9669.4	0.3376	293.63
	Rec-Libimseti(In)	23.341	0.02163	0.4953	66.434	0.09978	1.7923	77.081	0.2198	2.1486	133.91	0.2096	2.7441

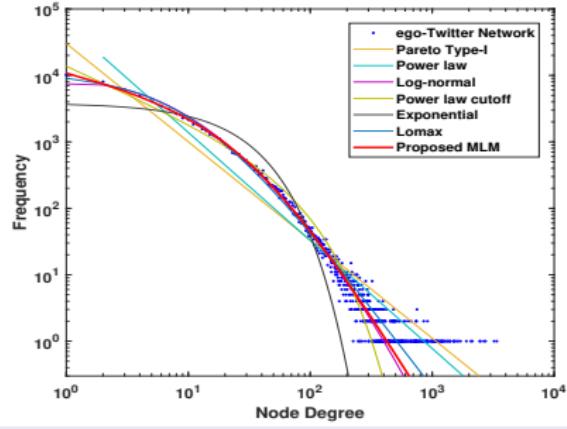
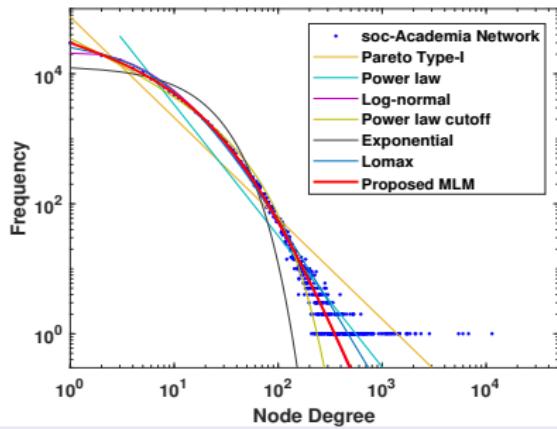
Experimental Results

Table of different statistical measures of different competitive models over real world networks

Data sets		Log-normal			Poisson			Power-law Cutoff			Exponential		
		RMSE	KLD	MAE	RMSE	KLD	MAE	RMSE	KLD	MAE	RMSE	KLD	MAE
Social Networks	ego-Twitter(In)	53.863	0.0169	2.9494	410.93	10.452	36.645	68.004	0.0397	4.1974	157.98	0.2733	11.567
	ego-Gplus(In)	10.155	0.0678	0.2523	95.967	25.317	3.0371	30.925	0.1475	0.6821	50.098	1.3131	1.8328
	soc-Slashdot	237.63	0.1058	10.549	684.36	10.069	42.407	19.598	0.0075	1.3599	434.25	0.6381	22.275
	soc-Delicious(In)	281.34	0.0579	10.781	957.82	6.8432	56.634	66.896	0.0185	4.2366	535.11	0.4626	25.304
	soc-Digg(In)	69.438	0.0552	1.9087	323.59	21.541	15.134	65.713	0.0441	1.9042	204.50	0.8907	8.0015
	soc-Academia	91.003	0.0169	2.0921	542.38	7.2349	22.153	62.376	0.0255	1.9845	198.11	0.1924	6.6739
	LiveJournal(In)	3473.6	0.0355	70.64	13.61K	9.1120	481.38	808.79	0.0101	24.501	7017.9	0.3449	186.01
	Dogster-Friendship	42.539	0.0272	1.1459	494.49	14.575	15.309	182.47	0.1862	4.1579	165.19	0.4765	5.4623
	Higgs-Twitter(In)	41.955	0.0134	0.5995	448.91	16.309	14.753	118.23	0.0914	2.8051	134.68	0.3163	4.4689
	Artist-Facebook	24.071	0.0154	3.026	323.46	12.537	53.920	56.801	0.0458	6.6452	88.351	0.1799	13.623
Citation Networks	cit-HepTh(In)	22.59	0.0255	2.331	153.39	8.0679	13.774	25.42	0.0464	2.816	58.74	0.2778	4.5286
	cit-HepPh(In)	44.951	0.0189	4.4405	303.32	7.9234	46.775	36.887	0.0221	4.5287	107.32	0.1801	14.145
	cit-Patents(In)	9612.7	0.0192	725.71	38.2K	1.6549	3657.1	2424.5	0.0061	271.89	13.2K	0.0659	1147.5
	cit-Citeser(In)	353.26	0.0299	21.921	1507.6	2.566	109.15	195.67	0.0131	13.877	629.02	0.1486	44.301
Collaboration Networks	ca-CondMat	42.665	0.0082	6.5746	378.65	2.7263	80.781	62.929	0.0287	13.362	64.985	0.0472	16.873
	ca-AstroPh	28.209	0.0312	6.8565	229.81	9.8703	55.51	50.604	0.0384	7.4579	68.185	0.1235	14.361
	ca-GrQc	30.184	0.0515	12.148	193.94	2.3256	63.61	58.305	0.0659	18.259	69.169	0.1418	25.759
	ca-HepPh	29.993	0.1011	6.8958	185.48	11.609	39.673	50.717	0.1128	8.2477	89.936	0.5187	17.589
	ca-HepTh	55.618	0.0178	21.032	370.15	1.5051	133.68	89.425	0.0245	27.613	109.96	0.0551	43.882
Web Graphs	Google(In)	1514.5	0.0878	40.067	4442.6	4.712	154.92	188.01	0.0157	9.6549	2589.4	0.4419	76.441
	BerkStan(In)	185.04	0.1002	2.0198	993.01	7.0379	11.9628	322.63	0.1037	2.8203	595.53	0.7438	6.6185
	Wikipedia2009(In)	2720.9	0.0798	116.43	8425.7	3.6475	398.72	781.87	0.0082	35.531	4727.1	0.3431	213.66
	WikipediaLinkFr(In)	240.18	0.0543	0.5234	762.72	25.726	4.0278	121.31	0.0622	0.5006	534.61	1.0217	2.0471
Product co-purchasing networks	Hudong(In)	746.47	0.1593	6.5493	1975.73	11.088	25.837	75.362	0.0063	0.8323	1494.8	1.1798	15.836
	amazon0601(In)	286.61	0.0102	16.881	2064.6	2.7267	140.46	297.39	0.0382	22.199	308.32	0.0574	24.114
	amazon0505(In)	358.59	0.0125	19.123	2172.5	3.0551	144.34	260.85	0.0342	20.136	390.13	0.0628	26.178
Communication Networks	amazon0312(In)	338.03	0.0116	17.742	2131.9	2.6839	140.75	273.39	0.0352	20.381	383.82	0.0639	26.299
	Email-Enron	121.47	0.0873	8.445	426.39	6.8601	38.373	95.468	0.0689	7.664	230.41	0.5405	18.139
	Wiki-Talk(In)	7978.6	0.1902	246.26	21.9K	1.2506	646.54	672.32	0.0036	25.905	16.5K	0.4879	542.31
	Rec-Libimseti(In)	87.472	0.0755	1.4021	281.18	30.222	8.0019	28.059	0.0359	0.6971	166.18	0.8547	3.9402

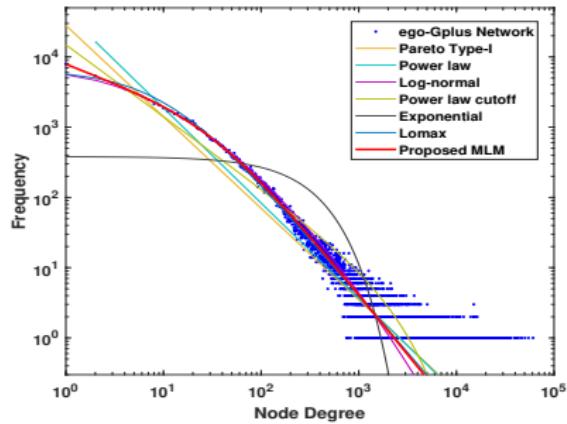
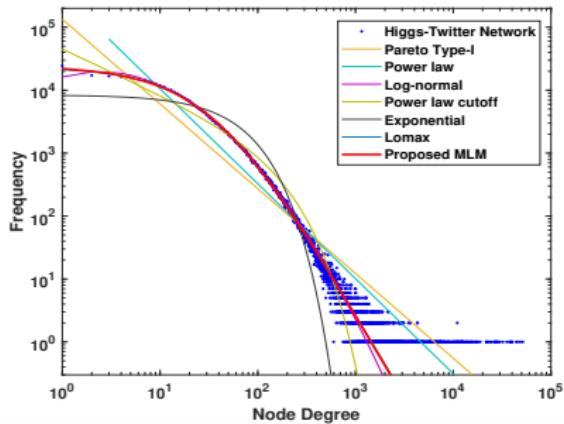
Experimental Results

Degree distribution of soc-Academia and ego-Twitter networks in log-log scale



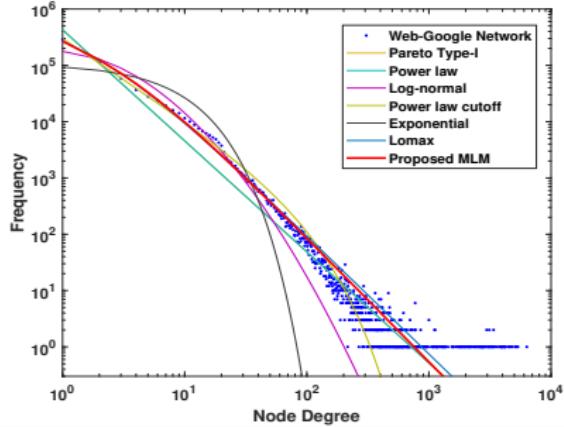
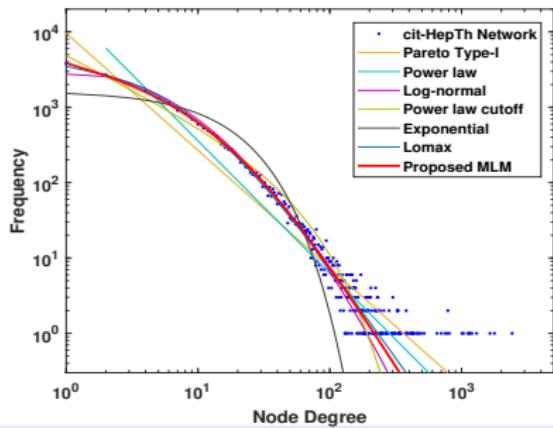
Experimental Results

Degree distribution of Higgs-Twitter and ego-Gplus networks in log-log scale



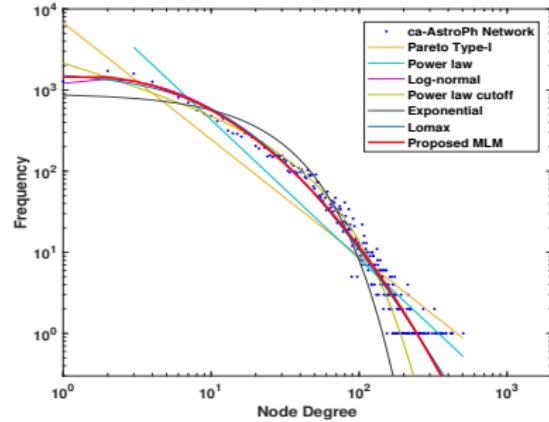
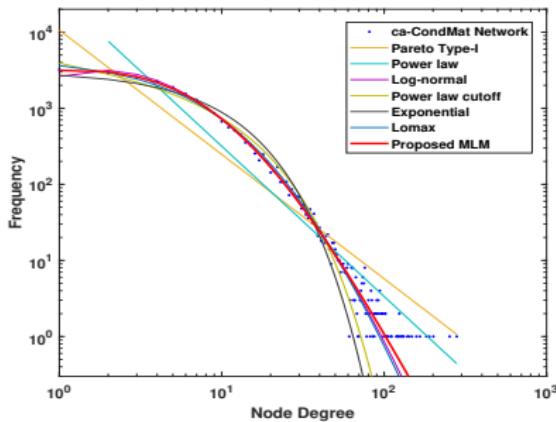
Experimental Results

Degree distribution of cit-HepTh and web-Google networks in log-log scale



Experimental Results

Degree distribution of ca-CondMat and ca-AstroPh networks in log-log scale



Concluding Remarks

- We have proposed a new distribution to model the entire degree distribution of real world networks.
- Proposed Modified Lomax distribution results in better modeling the entire degree distribution of a real world network.
- A sufficient condition for the existence of the MLE for the parameters of MLM distribution using the notion of coefficient of variation (CV).
- By simulating the parameters of a proposed fit MLM distribution, one can easily capture the spatial structure and dynamical pattern of a real-world network as the network evolves over time.
- The dynamic pattern analysis of such groups in real world networks is a future scope of research.

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The word "THANK YOU" is written in large, bold, 3D letters. Each letter is filled with a photograph of a person's face, specifically focusing on the eyes and nose area, all of whom are wearing white surgical masks. The letters are oriented diagonally, with "THANK" on the left and "YOU" on the right.