# Data Analytics

Course Taught at IIFT

***Day 3: Sampling Distributions and Hypothesis Testing***

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# Today's Topics……

- Basics of sampling distribution
- Central limit theorem and its Applications
- Standard Sampling distributions
- Principle of Statistical Inference (SI)
- Hypotheses testing procedures
- Case Study 1: Coffee Sale
- Case Study 2: Machine Testing
- Summary

# Basic terminologies

Some basic terminology which are closely associated to the above-mentioned tasks are reproduced below.

- **Population:** A population consists of the totality of the observation, with which we are concerned.
- **Sample:** A sample is a subset of a population.
- **Random variable:** A random variable is a function that associates a real number with each element in the sample.
- **Statistic:** Any function of the random variable constituting random sample is called a statistic.
- **Statistical inference:** It is an analysis basically concerned with generalization and prediction.

# Statistical Inference

There are two facts, which are key to statistical inference.

1. Population parameters are fixed number whose values are usually unknown.
2. Sample statistics are known values for any given sample, but vary from sample to sample, even taken from the same population.

- In fact, it is unlikely for any two samples drawn independently, producing identical values of sample statistics.
- In other words, the variability of sample statistics is always present and must be accounted for in any inferential procedure.
- This variability is called sampling variation.

**Note:**

A sample statistic is random variable and like any other random variable, a sample statistic has a probability distribution.

# Sampling Distribution

More precisely, sampling distributions are probability distributions and used to describe the variability of sample statistics.

**Definition : Sampling distribution**

The sampling distribution of a statistics is the probability distribution of that statistic.

- The probability distribution of sample mean (hereafter, will be denoted as $\bar{X}$) is called the sampling distribution of the mean (also, referred to as the distribution of sample mean).
- Like $\bar{X}$, we call sampling distribution of variance (denoted as $S^2$).
- Using the values of $\bar{X}$ and $S^2$ for different random samples of a population, we are to make inference on the parameters $\mu$ and $\sigma^2$ (of the population).

# Sampling Distribution

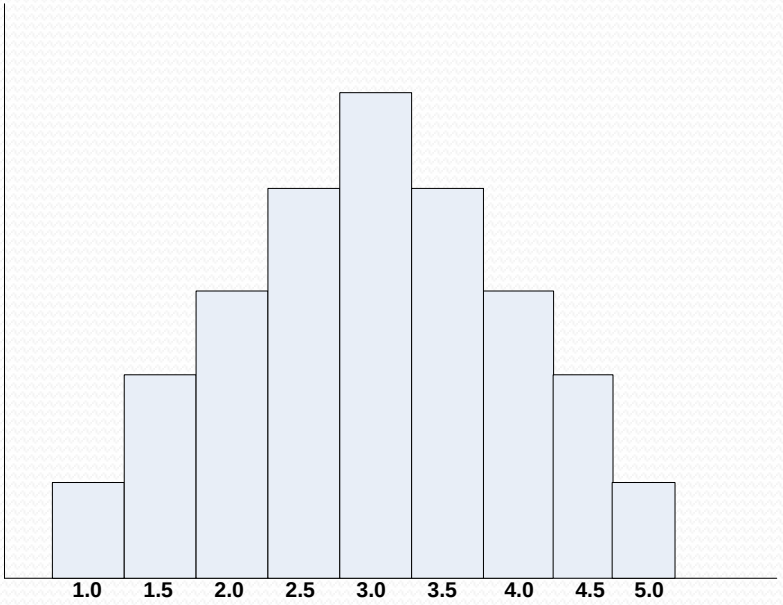
**Example 1:** Consider five identical balls numbered and weighting as 1, 2, 3, 4 and 5. Consider an experiment consisting of drawing two balls, replacing the first before drawing the second, and then computing the mean of the values of the two balls. Following table lists all possible samples and their mean.

| Sample ($X$) | Mean ($\bar{X}$) | Sample ($X$) | Mean ($\bar{X}$) | Sample ($X$) | Mean ($\bar{X}$) |
|---|---|---|---|---|---|
| [1,1] | 1.0 | [2,4] | 3.0 | [4,2] | 3.0 |
| [1,2] | 1.5 | [2,5] | 3.5 | [4,3] | 3.5 |
| [1,3] | 2.0 | [3,1] | 2.0 | [4,4] | 4.0 |
| [1,4] | 2.5 | [3,2] | 2.5 | [4,5] | 4.5 |
| [1,5] | 3.0 | [3,3] | 3.0 | [5,1] | 3.0 |
| [2,1] | 1.5 | [3,4] | 3.5 | [5,2] | 3.5 |
| [2,2] | 2.0 | [3,5] | 4.0 | [5,3] | 4.0 |
| [2,3] | 2.5 | [4,1] | 2.5 | [5,4] | 4.5 |
| | | | | [5,5] | 5.0 |

# Sampling Distribution

**Sampling distribution of means**

| $\bar{X}$ | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 |
|---|---|---|---|---|---|---|---|---|---|
| $f(\bar{X})$ | $\frac{1}{25}$ | $\frac{2}{25}$ | $\frac{3}{25}$ | $\frac{4}{25}$ | $\frac{5}{25}$ | $\frac{4}{25}$ | $\frac{3}{25}$ | $\frac{2}{25}$ | $\frac{1}{25}$ |



# Issues with Sampling Distribution

1. In practical situation, for a large population, it is infeasible to have all possible samples and hence probability distribution of sample statistics.

2. The sampling distribution of a statistic depends on
   - the size of the population
   - the size of the samples and
   - the method of choosing the samples.

?

# Theorem on Sampling Distribution

Theorem 1: **Sampling distribution of mean and variance**

The sampling distribution of a random sample of size $n$ drawn from a population with mean $\mu$ and variance $\sigma^2$ will have mean $\bar{X} = \mu$ and variance $S^2 = \frac{\sigma^2}{n}$

**Example 2:** With reference to data in Example 1

For the population, $\mu = \frac{1+2+3+4+5}{5} = 3$

$$\sigma^2 = \frac{(25-1)}{12} = 2$$

Applying the theorem, we have $\bar{X} = 3 \; and \; S^2 = 1$

Hence, the theorem is verified!

# Central Limit Theorem

Theorem 1 is an amazing result and in fact, also verified that if we sampling from a population with unknown distribution, the sampling distribution of $\bar{X}$ will still be approximately normal with mean $\mu$ and variance $\frac{\sigma^2}{n}$ provided that the sample size is large.

This further, can be established with the famous "central limit theorem", which is stated below.

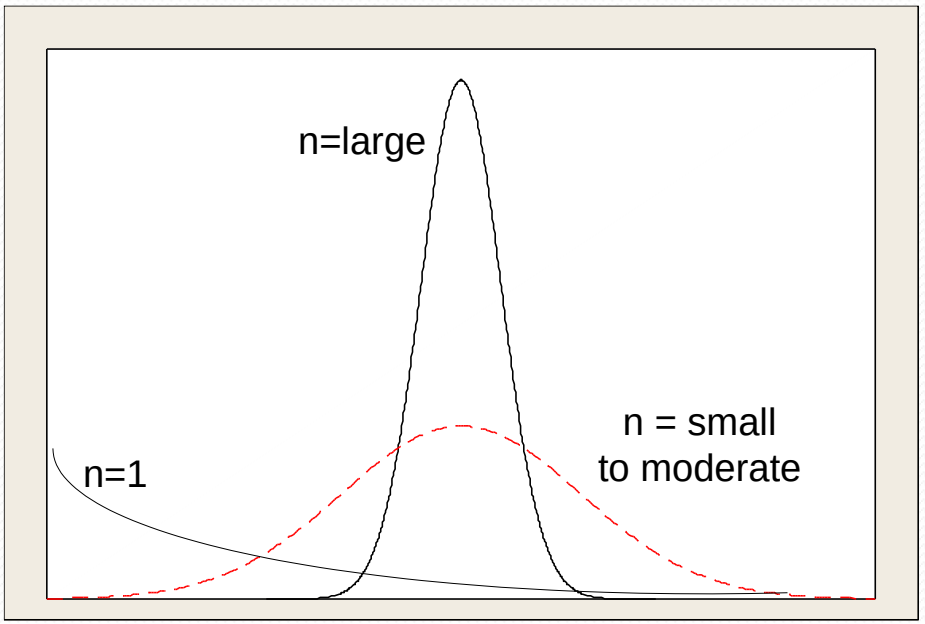
**Theorem 2: Central Limit Theorem**

If random samples each of size $n$ are taken from any distribution with mean $\mu$ and variance $\sigma^2$, the sample mean $\bar{X}$ will have a distribution approximately normal with mean $\mu$ and variance $\frac{\sigma^2}{n}$.

The approximation becomes better as $n$ increases.

# Applicability of Central Limit Theorem

- The normal approximation of $\overline{X}$ will generally be good if $n \geq 30$
- The sample size $n = 30$ is, hence, a guideline for the central limit theorem.
- The normality on the distribution of $\overline{X}$ becomes more accurate as $n$ grows larger.




- One very important application of the **Central Limit Theorem** is the determination of reasonable values of the population mean $\mu$ and variance $\sigma^2$.

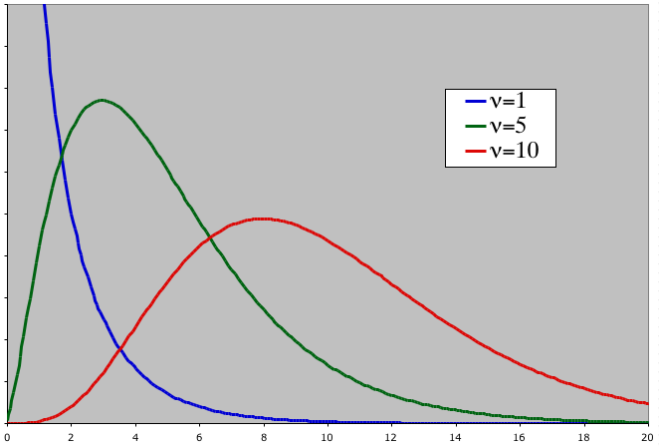
- For standard normal distribution, we have the z-transformation

$$Z = \frac{\overline{X} - \mu}{S} = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$$

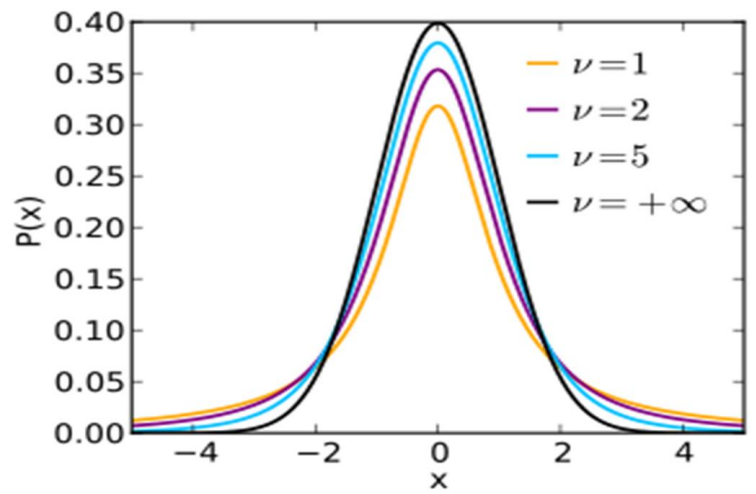
# Standard Sampling Distributions

- Apart from the standard normal distribution to describe sampling distribution, there are some other quite different sampling, which are extensively referred in the study of statistical inference.
    - $\chi^2$: Describes the distribution of variance.
    - $t$: Describes the distribution of normally distributed random variable standardized by an estimate of the standard deviation.
    - F: Describes the distribution of the ratio of two variables.

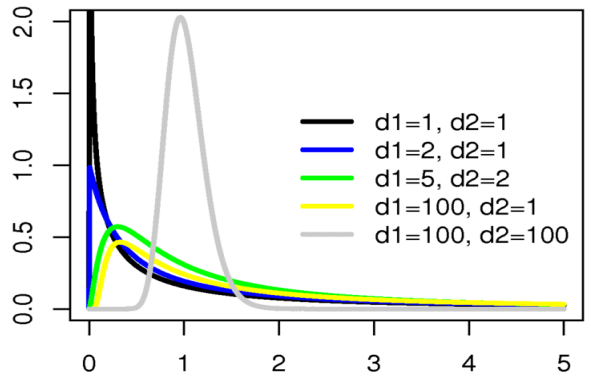
# Standard Sampling Distributions




$\chi^2$ - (Chi-Square) distribution curve




t- distribution curve




F – distribution curve

# The $\chi^2$ Distribution

A common use of the $\chi^2$ distribution is to describe the distribution of the sample variance.

**Definition 1: $\chi^2$ distribution**

If $x_1, x_2 \ldots\ldots\ldots x_n$ are independent random variables having identical normal distribution with mean $\mu$ and variance $\sigma^2$, then the random variable

$$Y = \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma}\right)^2$$

has a Chi squared distribution with $n-1$ degrees of freedom.

**Note:** To **calculate degrees of freedom**, subtract the number of relations from the number of observations. For **determining** the **degrees of freedom** for a sample variance, you need to subtract one (1) from the number of observations, *n*.

# The $\chi^2$ Distribution

**Note:** Each of the $n$ independent random variable $\left(\frac{x_i-\mu}{\sigma}\right)^2, i = 1, 2, 3, \ldots\ldots n$ has Chi-squared distribution with 1 degree of freedom.

Now we can derive $\chi^2$- distribution for sample variance.
We can write

$$\sum_{i=1}^{n}(x_i - \mu)^2 = \sum_{i=1}^{n}[(x_i - \overline{x}) + (\overline{x} - \mu)]^2$$

$$= \sum_{i=1}^{n}(x_i - \overline{x})^2 + n.(\overline{x} - \mu)$$

or, $$\frac{1}{\sigma^2}\sum(x_i - \mu)^2 = \frac{(n-1)S^2}{\sigma^2} + \frac{(\overline{x}-\mu)^2}{\sigma^2/n}$$

Chi-square distribution with n-degree

Chi-square distribution with (n-1) degree of freedom

Chi-square distribution with 1 degree of freedom $[= Z^2]$

# The $\chi^2$ Distribution

**Definition 2: $\chi^2$-distribution for Sampling Variance**

If $S^2$ is the variance of a random sample of size *n* taken from a normal population having the variance $\sigma^2$, then the statistics

$\chi^2 = \frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^{n}\left(\frac{x_i-\bar{x}}{\sigma}\right)^2$ has a chi-squared distribution with $v = n - 1$ degrees of freedom.

This way $\chi^2$- distribution is used to describe the sampling distribution of $S^2$.

# Chi-Squared Distribution

Definition 3: **Chi-squared distribution**

The continuous random variable $x$ has a Chi-squared distribution with $v$ degrees of freedom, is given by

$$f(x:v)=\begin{cases}\dfrac{1}{2^{\frac{v}{2}}\,\Gamma(v/2)}\,x^{v/2-1}e^{-\frac{x}{2}}, & x>0\\ 0 & Otherwise\end{cases}$$

where $v$ is a positive integer.

- The Chi-squared distribution plays an important role in statistical inference .
- The mean and variance of Chi-squared distribution are:

$$\mu = v \text{ and } \sigma^2 = 2v$$

# The $t$ Distribution

- **The $t$ Distribution**

1. To know the sampling distribution of mean we make use of Central Limit Theorem with $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$
2. This require the known value of $\sigma$ a priori.
3. However, in many situation, $\sigma$ is certainly no more reasonable than the knowledge of the population mean $\mu$.
4. In such situation, only measure of the standard deviation available may be the sample standard deviation $S$.
5. It is natural then to substitute $S$ for $\sigma$. The problem is that the resulting statistics is not normally distributed!
6. The $t$ distribution is to alleviate this problem. This distribution is called $student's\ t$ or simply $t - distribution$.

# The $t$ Distribution

**Definition: $t$ −distribution**

The $t$ −distribution with $v$ degrees of freedom actually takes the form

$$t(v) = \frac{Z}{\sqrt{\frac{\chi^2(v)}{v}}}$$

where $Z$ is a standard normal random variable, and $\chi^2(v)$ is $\chi^2$ random variable with $v$ degrees of freedom.

# The *t* Distribution

**Corollary:** Let $X_1, X_2, \ldots\ldots, X_n$ be independent random variables that are all normal with mean $\mu$ and standard deviation $\sigma$.

Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

Using this definition, we can develop the sampling distribution of the sample mean when the population variance, $\sigma^2$ is unknown.

That is,

$Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ has the standard normal distribution.

$\chi^2 = \frac{(n-1)S^2}{\sigma^2}$ has the $\chi^2$ distribution with $(n-1)$ degrees of freedom.

Thus, $T = \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2/\sigma^2}{n-1}}}$ or $T = \frac{\bar{X}-\mu}{S/\sqrt{n}}$

This is the $t - distribution$ with $(n-1)$ degrees of freedom.

# The $F$ Distribution

**Definition : *F* distribution**

The statistics F is defined to be the ratio of two independent Chi-Squared random variables, each divided by its number of degrees of freedom. Hence,

$$\mathrm{F}(v_1, v_2) = \frac{\chi^2(v_1)/v_1}{\chi^2(v_2)/v_2}$$

**Corollary :** Recall that $\chi^2 = \frac{(n-1)S^2}{\sigma^2}$ is the Chi-squared distribution with $(n-1)$ degrees of freedom.

Therefore, if we assume that we have sample of size $n_1$ from a population with variance ${\sigma_1}^2$ and an independent sample of size $n_2$ from another population with variance ${\sigma_2}^2$, then the statistics

$$F = \frac{{S_1}^2/{\sigma_1}^2}{{S_2}^2/{\sigma_2}^2}$$

**Note:** The $F$ distribution finds enormous applications in comparing sample variances.

# Basic Inferential Approaches

**Approach 1: Hypothesis testing**

- We conduct **test on hypothesis**.
  - We hypothesize that one (or more) parameter(s) has (have) some specific value(s) or relationship.
- Make our decision about the parameter(s) based on one (or more) sample statistic(s)
- Accuracy of the decision is expressed as the probability that the decision is incorrect.

**Approach 2: Confidence interval measurement**

- We estimate one (or more) parameter(s) using sample statistics.
  - This estimation usually done in the form of an interval.
- Accuracy of the decision is expressed as the **level of confidence** we have in the interval.

# Hypothesis Testing

Statistical inference

Null hypothesis

Sample

Alternative hypothesis

A **hypothesis** about the value of a population parameter is an **assertion** about its value.

# Statistical Hypothesis

- If the hypothesis is stated in terms of population parameters (such as mean and variance), the hypothesis is called statistical hypothesis.
- Data from a sample (which may be an experiment) are used to test the validity of the hypothesis.
- A procedure that enables us to agree (or disagree) with the statistical hypothesis is called a **test of the hypothesis.**

**Example :**

1. To determine whether the wages of men and women are equal.
2. A product in the market is of standard quality.
3. Whether a particular medicine is effective to cure a disease.

# The Hypotheses

- The main purpose of statistical hypothesis testing is to choose between two competing hypotheses.

**Example :** One hypothesis might claim that wages of men and women are equal, while the **alternative** might claim that men make more than women.

- Hypothesis testing start by making a set of two statements about the parameter(s) in question.
- The hypothesis actually to be tested is usually given the symbol $H_0$ and is commonly referred as the **null hypothesis.**
- The other hypothesis, which is assumed to be true when null hypothesis is false, is referred as the **alternate hypothesis** and is often symbolized by $H_1$
- The two hypotheses are exclusive and exhaustive.

# The Hypotheses

**Example:**

Ministry of Human Resource Development (MHRD), Government of India takes an initiative to improve the country's human resources and hence set up **23 IIT's** in the country.

To measure the engineering aptitudes of graduates, MHRD conducts GATE examination for a mark of 1000 in every year. A sample of 300 students who gave GATE examination in 2020 were collected and the mean is observed as 220.

In this context, statistical hypothesis testing is to determine the mean mark of the all GATE-2020 examinee.

The two hypotheses in this context are:

$$H_0: \mu = 220$$

$$H_1: \mu < 220$$

# The Hypotheses

**Note:**

1. As null hypothesis, we could choose $H_0: \mu \leq 220$ or $H_0: \mu \geq 220$
2. It is customary to always have the null hypothesis with an equal sign.
3. As an alternative hypothesis there are many options available with us.

**Examples:**

I. $H_1: \mu > 220$

*II.* $H_1: \mu < 220$

*III.* $H_1: \mu \neq 220$

4. The two hypothesis should be chosen in such a way that they are **exclusive** and **exhaustive**.
   - One or other must be true, but they cannot both be true.

# The Hypotheses

**One-tailed test**

- A statistical test in which the alternative hypothesis specifies that the population parameter lies entirely above or below the value specified in $H_0$ is called a one-sided (or one-tailed) test.

Example.

$$H_0: \mu = 100 \qquad H_1: \mu > 100$$

**Two-tailed test**

- An alternative hypothesis that specifies that the parameter can lie on their sides of the value specified by $H_0$ is called a two-sided (or two-tailed) test.

Example.

$$H_0: \mu = 100 \qquad H_1: \mu <> 100$$

# Hypothesis Testing Procedures

The following five steps are followed when testing hypothesis

1. Specify $H_0$ and $H_1$, the null and alternate hypothesis, and an **acceptable level of $\alpha$.**
2. Determine an appropriate sample-based test statistics and the **rejection region** for the specified $H_0$.
3. Collect the sample data and calculate the test statistics.
4. Make a decision to either reject or fail to reject $H_0$.
5. Interpret the result in common language suitable for practitioners.

# Hypothesis Testing Procedure

- In summary, we have to choose between $H_0$ and $H_1$
- The standard procedure is to assume $H_0$ is true.
  **(Just we presume innocent until proven guilty)**
- Using statistical test, we try to determine whether there is sufficient evidence to declare $H_0$ false.
- We reject $H_0$ only when the **chance is small** that $H_0$ is true.
- The procedure is based on probability theory, that is, there is a chance that we can make errors.

# Errors in Hypothesis Testing

In hypothesis testing, there are two types of errors.

**Type I error:** A type I error occurs when we incorrectly reject $H_0$ (i.e., we reject the null hypothesis, when $H_0$ is true).

**Type II error:** A type II error occurs when we incorrectly fail to reject $H_0$ (i.e., we accept $H_0$ when it is not true).

| | Observation | |
|---|---|---|
| **Decision** | $H_0$ is true | $H_0$ is false |
| $H_0$ is accepted | Decision is correct | Type II error |
| $H_0$ is rejected | Type I error | Decision is correct |

# Probabilities of Making Errors

## Type I error calculation

**$\alpha$:** denotes the probability of making a Type I error

$$\alpha = \mathbf{P}(\text{Rejecting } H_0 | H_0 \text{ is true})$$

## Type II error calculation

**$\beta$:** denotes the probability of making a Type II error

$$\beta = \mathbf{P}(\text{Accepting } H_0 | H_0 \text{ is false})$$

**Note:**

- $\alpha$ and $\beta$ are not independent of each other as one increases, the other decreases
- When the sample size increases, both to decrease since sampling error is reduced.
- In general, we focus on Type I error, but Type II error is also important, particularly when sample size is small.

# Calculating $\alpha$

Assuming that we have the results of random sample. Hence, we use the characteristics of sampling distribution to calculate the probabilities of making either Type I or Type II error.

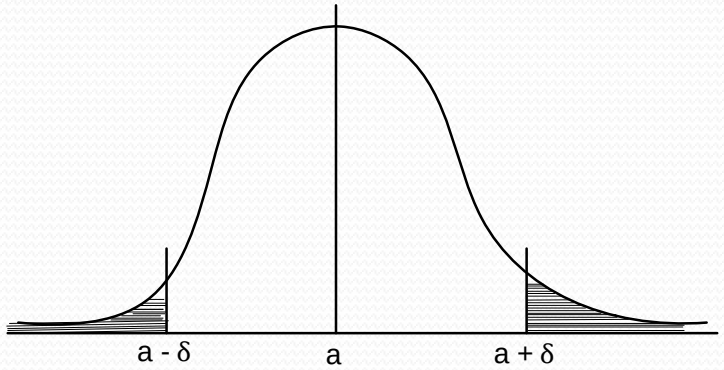
**Example :**

Suppose, two hypotheses in a statistical testing are:

$$H_0: \mu = a$$
$$H_1: \mu \neq a$$

Also, assume that for a given sample, population obeys normal distribution. A threshold limit say $a \pm \delta$ is used to say that they are significantly different from a.

# Calculating $\alpha$




Here, shaded region implies the probability that, $\overline{X} < a - \delta$ *or* $\overline{X} > a + \delta$

Thus the null hypothesis is to be rejected if the mean value is less than $a - \delta$ or greater than $a + \delta$.

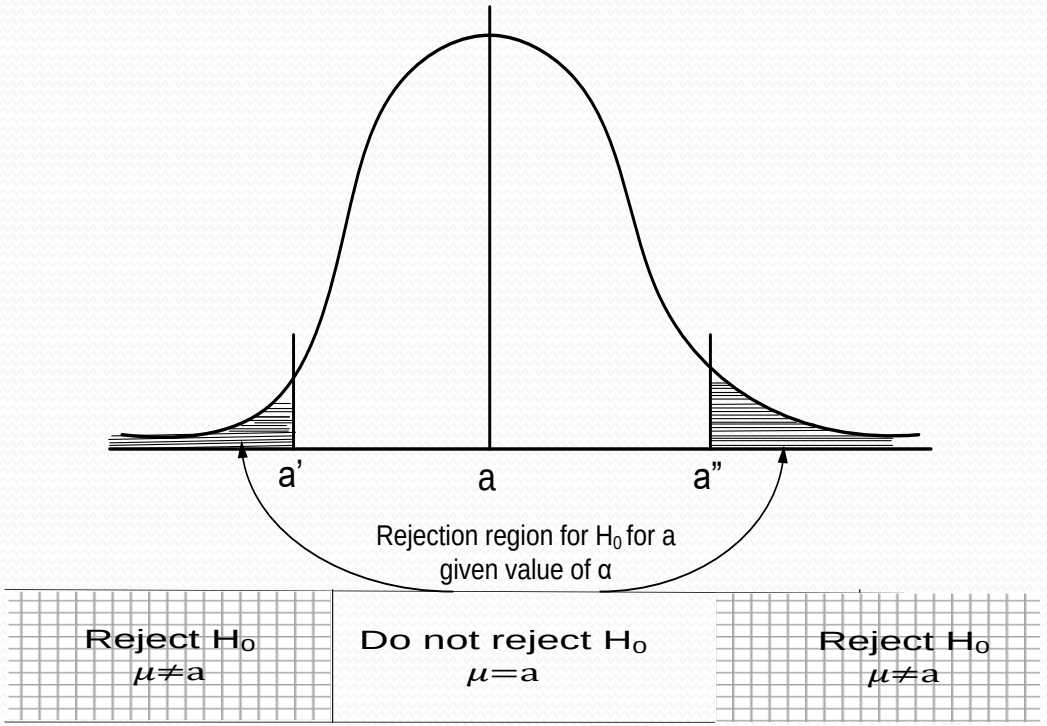
If $\overline{X}$ denotes the sample mean, then the Type I error is

$$\alpha = P(\overline{X} < a - \delta \; or \; \overline{X} > a + \delta, \qquad when \; \mu = a \, , \qquad \text{i.e.}, \; H_0 \text{ is true})$$

# The Rejection Region

The rejection region comprises of value of the test statistics for which

1. The probability when the null hypothesis is true is less than or equal to the specified $\alpha$.
2. Probability when $H_1$ is true are greater than they are under $H_0$.



# Two-Tailed Test

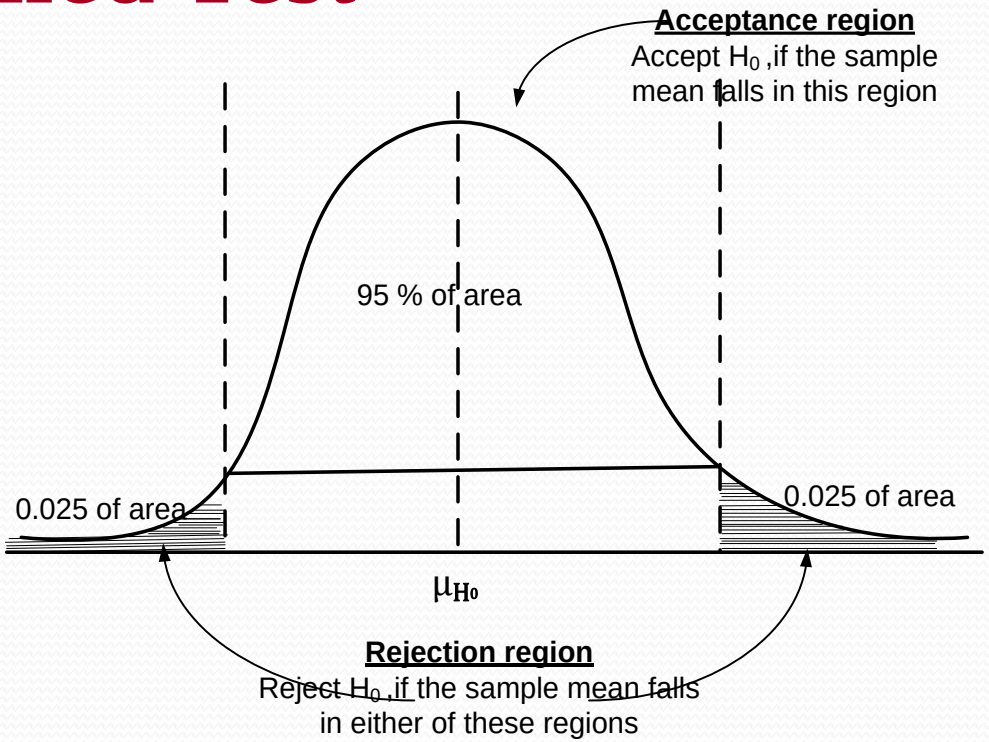
For two-tailed hypothesis test, hypotheses take the form

$$H_0: \mu = \mu_{H_0}$$
$$H_1: \mu \neq \mu_{H_0}$$

In other words, to reject a null hypothesis, sample mean $\mu > \mu_{H_0}$ or $\mu < \mu_{H_0}$ under a given $\alpha$.

Thus, in a two-tailed test, there are two rejection regions (also known as critical region), one on each tail of the sampling distribution curve.

# Two-Tailed Test




Acceptance and rejection regions in case of a two-tailed test with 5% significance level.

# One-Tailed Test

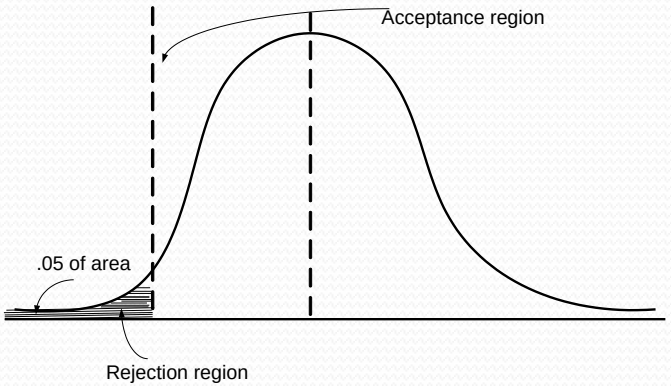
A one-tailed test would be used when we are to test, say, whether the population mean is either lower or higher than the hypothesis test value.

Symbolically,

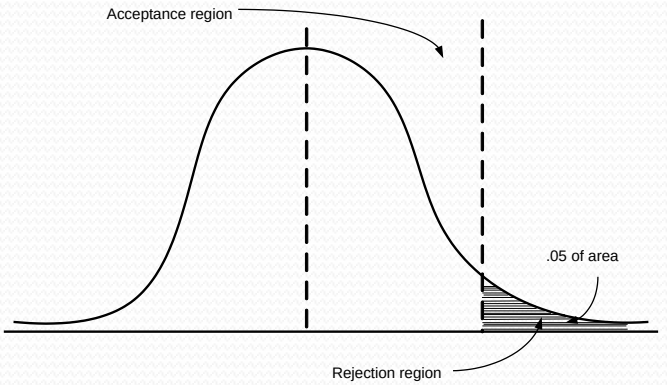
$$H_0: \mu = \mu_{H_0}$$

$$H_1: \mu < \mu_{H_0} \qquad [or\ \mu > \mu_{H_0}]$$

Wherein there is one rejection region only on the left-tail (or right-tail).




Left – tailed test




Right – tailed test

# Example : Calculating $\alpha$

Consider the two hypotheses are

The null hypothesis is

$$H_0 : \mu = 8$$

The alternative hypothesis is

$$H_1 : \mu \neq 8$$

Assume that given a sample of size 16 and standard deviation is 0.2 and sample follows normal distribution.

# Example : Calculating $\alpha$

We can decide the rejection region as follows.

Suppose, the null hypothesis is to be rejected if the mean value is less than 7.9 or greater than 8.1. If $\overline{X}$ is the sample mean, then the probability of Type I error is

$$\alpha = P(\overline{X} < 7.9 \; or \; \overline{X} > 8.1, \text{when } \mu = 8)$$

Given $\sigma$, the standard deviation of the sample is 0.2 and that the distribution follows normal distribution.

Thus,

$$P(\overline{X} < 7.9) = P\left[Z = \frac{7.9 - 8}{0.2/\sqrt{16}}\right] = P[Z < -2.0] = 0.0228$$

and

$$P(\overline{X} > 8.1) = P\left[Z = \frac{8.1 - 8}{0.2/\sqrt{16}}\right] = P[Z > 2.0] = 0.0228$$

Hence, $\alpha = 0.0228 + 0.0228 = 0.0456$

# Example : Calculating $\alpha$ and $\beta$

There are two identically appearing boxes of chocolates. Box A contains 60 red and 40 black chocolates whereas box B contains 40 red and 60 black chocolates. There is no label on the either box. One box is placed on the table. We are to test the hypothesis that "Box B is on the table".

To test the hypothesis an experiment is planned, which is as follows:

- Draw at random five chocolates from the box.
- We replace each chocolates before selecting a new one.
- The number of red chocolates in an experiment is considered as the sample statistics.

**Note:** Since each draw is independent to each other, we can assume the sample distribution follows binomial probability distribution.

# Example : Calculating $\alpha$

Let us express the population parameter as $p =$ the number of red chocolates in Box $B$.

The hypotheses of the problem can be stated as:

$H_0: p = 0.4$ // Box B is on the table

$H_1: p = 0.6$ // Box A is on the table

***Calculating $\alpha$***:

In this example, the null hypothesis $(H_0)$ specifies that the probability of drawing a red chocolate is $0.4$. This means that, lower proportion of red chocolates in observations $(i.e., sample)$ favors the null hypothesis. In other words, drawing all red chocolates provides sufficient evidence to reject the null hypothesis. Then, the probability of making a $Type\ I$ error is the probability of getting five red chocolates in a sample of five from Box B. That is,

$$\alpha = P(X = 5 \ \ when\ p = 0.4)$$

Using the binomial distribution

$$= \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}\ where\ n = 5, x = 5$$

$$= (0.4)^5 = 0.01024$$

Thus, the probability of rejecting a true null hypothesis is $\approx 0.01$. That is, there is approximately 1 in 100 chance that the box B will be mislabeled as box A.

# Example : Calculating $\beta$

The *Type II* error occurs if we fail to reject the null hypothesis when it is not true. For the current illustration, such a situation occurs, if Box A is on the table but we did not get the five red chocolates required to reject the hypothesis that Box B is on the table.

The probability of *Type II* error is then the probability of getting four or fewer red chocolates in a sample of five from Box A.

That is,

$$\beta = P(X \leq 4 \qquad \text{when } p = 0.6)$$

Using the probability rule:

$$P(X \leq 4) + P(X = 5) = 1$$

$$\text{That is, } P(X \leq 4) = 1 - P(X = 5)$$

$$\text{Now, } P(X = 5) = (0.6)^5$$

$$\text{Hence, } \beta = 1 - (0.6)^5$$

$$= 1 - 0.07776 = 0.92224$$

That is, the probability of making *Type II* error is over 92%. This means that, if Box A is on the table, the probability that we will be unable to detect it is 0.92.

# Case Study 1: **Coffee Sale**

A coffee vendor nearby Howrah railway station has been having average sales of 500 cups per day. Because of the development of a bus stand nearby, it expects to increase its sales. During the first 12 days, after the inauguration of the bus stand, the daily sales were as under:

550 570 490 615 505 580 570 460 600 580 530 526

On the basis of this sample information, can we conclude that the sales of coffee have increased?

Consider 5% level of confidence.

# Hypothesis Testing : 5 Steps

The following five steps are followed when testing hypothesis

1. Specify $H_0$ and $H_1$, the null and alternate hypothesis, and an **acceptable level of $\alpha$.**
2. Determine an appropriate sample-based test statistics and the **rejection region** for the specified $H_0$.
3. Collect the sample data and calculate the test statistics.
4. Make a decision to either reject or fail to reject $H_0$.
5. Interpret the result in common language suitable for practitioner.

# Case Study 1: **Step 1**

## Step 1: Specification of hypothesis and acceptable level of α

Let us consider the hypotheses for the given problem as follows.

$H_0: \mu = 500$ cups per day

The null hypothesis that sales average 500 cups per day and they have not increased.

$H_0: \mu > 500$

The alternative hypothesis is that the sales have increased.

Given the acceptance level of $\alpha = 0.05$ $(i.e., 5\%\ level\ of\ significance)$

# Case Study 1: **Step 2**

## Step 2: Sample-based test statistics and the rejection region for specified $H_0$

Given the sample as

550 570 490 615 505 580 570 460 580 530 526

Since the sample size is small and the population standard deviation is not known, we shall use $t - test$ assuming normal population. The test statistics $t$ is

$$t = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

To find $\bar{X}$ and $S$, we make the following computations.

$$\bar{X} = \frac{\sum X_i}{n} = \frac{6576}{12} = 548$$

# Case Study 1: **Step 2**

| *Sample #* | $X_i$ | $X_i - \bar{X}$ | $(X_i - \bar{X})^2$ |
|---|---|---|---|
| 1 | 550 | 2 | 4 |
| 2 | 570 | 22 | 484 |
| 3 | 490 | −58 | 3364 |
| 4 | 615 | 67 | 4489 |
| 5 | 505 | −43 | 1849 |
| 6 | 580 | 32 | 1024 |
| 7 | 570 | 22 | 484 |
| 8 | 460 | −88 | 7744 |
| 9 | 600 | 52 | 2704 |
| 10 | 580 | 32 | 1024 |
| 11 | 530 | −18 | 324 |
| 12 | 526 | −22 | 484 |
| $n = 12$ | $\sum X_i = 6576$ | | $\sum (X_i - \bar{X})^2 = 23978$ |

# Case Study 1: **Step 2**

$$S = \sqrt{\frac{\sum(X_i - \bar{X})^2}{n-1}} = \sqrt{\frac{23978}{12-1}} = 46.68$$

Hence, $t = \frac{\bar{X}-\mu}{S/\sqrt{n}} = \frac{48}{46.68/\sqrt{12}} = \frac{48}{13.49} = 3.558$

**Note:**

Statistical table for t-distributions gives a *t*-value given *n*, the degrees of freedom and α, the level of significance and vice-versa.

# Case Study 1: **Step 3**

**Step 3: Collect the sample data and calculate the test statistics**

$$Degree\ of\ freedom = n - 1 = 12 - 1 = 11$$

As $H_1$ is one-tailed, we shall determine the rejection region applying one-tailed in the right tail because $H_1$ is more than type ) at 5% level of significance.

Using table of $t - distribution$ for 11 degrees of freedom and with 5% level of significance,

$$R: t > 1.796$$

# Case Study 1: **Step 4**

**Step 4: Make a decision to either reject or fail to reject $H_0$**

The observed value of $t = 3.558$ which is in the rejection region and thus $H_0$ is rejected at 5% level of significance.

# Case Study 1: **Step 5**

## Step 5: Final comment and interpret the result

We can conclude that the sample data indicate that coffee sales have increased.

# Case Study 2: **Machine Testing**

A medicine production company packages medicine in a tube of 8 ml. In maintaining the control of the amount of medicine in tubes, they use a machine. To monitor this control a sample of 16 tubes is taken from the production line at random time interval and their contents are measured precisely. The mean amount of medicine in these 16 tubes will be used to test the hypothesis that the machine is indeed working properly.

# Case Study 2: **Step 1**

## Step 1: Specification of hypothesis and acceptable level of α

The hypotheses are given in terms of the population mean of medicine per tube.

The null hypothesis is

$$H_0: \mu = 8$$

The alternative hypothesis is

$$H_1: \mu \neq 8$$

We assume $\alpha$, the significance level in our hypothesis testing $\approx 0.05$.
(This signifies the probability that the machine needs to be adjusted less than 5%).

# Case Study 2: **Step 2**

**Step 2: Sample-based test statistics and the rejection region for specified $H_0$**

**Rejection region:** Given $\alpha = 0.05$**,** which gives $|Z| > 1.96$ (obtained from standard normal calculation for $n(Z: 0,1) = 0.025$ for a rejection region with two-tailed test).

# Case Study 2: **Step 3**

**Step 3: Collect the sample data and calculate the test statistics**

Sample results: $n = 16,\ \bar{x} = 7.89, \sigma = 0.2$

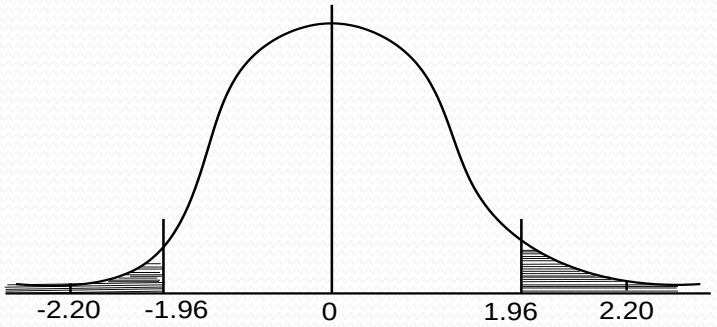
With the sample, the test statistics is

$$Z = \frac{7.89-8}{0.2/\sqrt{16}} = -2.20$$

Hence, $|Z| = 2.20$

# Case Study 2: **Step 4**

**Step 4: Make a decision to either reject or fail to reject $H_0$**




Since $Z > 1.96$, we reject $H_0$

# Case Study 2: **Step 5**

## Step 5: Final comment and interpret the result

We conclude $\mu \neq 8$ and recommend that the machine be adjusted.

# Case Study 2: **Alternative Test**

Suppose that in our initial setup of hypothesis test, if we choose $\alpha = 0.01$ instead of 0.05, then the test can be summarized as:

1. $H_0: \mu = 8$ , $H_1: \mu \neq 8$ $\alpha = 0.01$
2. Reject $H_0$ if $Z > 2.576$
3. Sample result n =16, $\sigma$ = 0.2, $\overline{X}$=7.89, Z $= \frac{7.89-8}{0.2/\sqrt{16}} = -2.20$, $|Z| = 2.20$
4. $|Z| < 2.20$, we fail to reject $H_0$= 8
5. We do not recommend that the machine be readjusted.

# Hypothesis Testing Strategies

- The hypothesis testing determines the validity of an assumption (technically described as null hypothesis), with a view to choose between two conflicting hypothesis about the value of a population parameter.

- There are two types of tests of hypotheses
  - ✓ Non-parametric tests (also called distribution-free test of hypotheses)
  - ✓Parametric tests (also called standard test of hypotheses).

# Parametric Tests : Applications

- Usually assume certain properties of the population from which we draw samples.
  - Observation come from a normal population
  - Sample size is small
  - Population parameters like mean, variance, etc. are hold good.
  - Requires measurement equivalent to interval scaled data.

# Parametric Tests

**Important Parametric Tests**

The widely used sampling distribution for parametric tests are

- $Z - test$
- $t - test$
- $\chi^2 - test$
- $F - test$

**Note:**

All these tests are based on the assumption of normality (i.e., the source of data is considered to be normally distributed).

# Parametric Tests : Z-test

***Z – test*:** This is most frequently test in statistical analysis.

- It is based on the normal probability distribution.
- Used for judging the significance of several statistical measures particularly the mean.
- It is used even when *binomial distribution* or $t-distribution$ is applicable with a condition that such a distribution tends to normal distribution when *n* becomes large.
- Typically it is used for comparing the mean of a sample to some hypothesized mean for the population in case of large sample, or when **population variance** is known.

# Parametric Tests : t-test

***t – test*:** It is based on the t-distribution.

- It is considered an appropriate test for judging the significance of a sample mean or for judging the significance of difference between the means of two samples in case of
  - small sample(s)
  - **population variance is not known** (in this case, we use the variance of the sample as an estimate of the population variance)

# Parametric Tests : $\chi^2$ -test

**$\chi^2 - test$:** It is based on Chi-squared distribution.

- It is used for comparing a sample variance to a theoretical population variance.

# Parametric Tests : $F$ -test

**$F - test$:** It is based on F-distribution.

- It is used to compare the variance of two independent samples.
- This test is also used in the context of analysis of variance (ANOVA) for judging the significance of more than two sample means.

# Hypothesis Testing : Assumptions

**Case 1:** Normal population, population infinite, sample size may be large or small, variance of the population is known.

$$z = \frac{\bar{X} - \mu_{H_0}}{\sigma/\sqrt{n}}$$

**Case 2:** Population normal, population finite, sample size may large or small……variance is known.

$$z = \frac{\bar{X} - \mu_{H_0}}{\sigma/\sqrt{n}[\sqrt{(N-n)/(N-1)}]}$$

**Case 3:** Population normal, population infinite, sample size is small and variance of the population is unknown.

$$t = \frac{\bar{X} - \mu_{H_0}}{S/\sqrt{n}} \qquad with\ degree\ of\ freedom = (n-1)$$

and

$$s = \sqrt{\frac{\sum(X_i - \bar{X})^2}{(n-1)}}$$

**Case 4:** Population finite

$$t = \frac{\bar{X} - \mu_{H_0}}{\sigma/\sqrt{n}[\sqrt{(N-n)/(N-1)}]}\ with\ degree\ of\ freedom = (n-1)$$

**Note:** If variance of population $(\sigma)$ is known, replace $S$ by $\sigma$. Population normal, population infinite, sample size is small and variance of the population is unknown.

# Hypothesis Testing : Non-Parametric Test

- ***Non-Parametric tests***
  - ✓Does not under any assumption
  - ✓Assumes only nominal or ordinal data

**Note:** Non-parametric tests need entire population (or very large sample size)

# References


- Probability and Statistics for Engineers and Scientists (8th Ed.) by Ronald E. Walpole, Sharon L. Myers, Keying Ye (Pearson), 2013.

# Any question?

You may also send your question(s) at tanujitisi@gmail.com