IIT JAM : MATHEMATICAL STATISTICS (MS) – 2013

Question Paper with Answer Keys

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IMPORTANT NOTE FOR CANDIDATES

Question 1 – 10 (objective questions) carry *two* marks each, question 11 – 20 (fill in the blank questions) carry *three* marks each and questions 21 – 30 (descriptive questions) carry *five* marks each.

In objective questions there are negative marking. For each wrong answer you will be awarded – 0.5 mark. There is no negative marking in fill in the blank questions & descriptive type questions.

Q.1 Let E and F be two events with $P(E) = 0.7$, $P(F) = 0.4$ and $P(E \cap F^C) = 0.4$. Then $P(F | E \cup F^C)$ is equal to

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{1}{5}$

Solution: (B) $P(E \cup F^C) = P(E) + P(F) - P(E \cap F^C) = 0.7 + 0.6 – 0.4 = 0.9$

Also we have $P(E \cap F) = P(E) - P(E \cap F^C) = 0.7 – 0.4 = 0.3$

Now, $P(F | E \cup F^C) = \frac{P(F \cap (E \cup F^c))}{P(E \cup F^c)} = \frac{P(F \cap E)}{P(E \cup F^c)}$, since $P(F \cap F^C) = 0$.

$= \frac{3/10}{9/10} = \frac{1}{3}$.

Q.2 Let $\{a_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2}$.

Then $\lim_{n\to\infty} \frac{e^{a_n^2}+a_n}{4}$ is equal to

(A) ∞ (B) $\frac{e^{\frac{1}{4}}}{4}+\frac{1}{8}$ (C) $\frac{e^{\frac{1}{4}}}{4}$ (D) $\frac{1}{4}$

Solution: (D) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2}$ is given. Let us assume $a_n = \frac{1}{2^n}$.

Now, $\lim_{n\to\infty} \frac{e^{a_n^2}+a_n}{4} = \lim_{n\to\infty} \frac{e^{\frac{1}{2^{2n}}}+\frac{1}{2^n}}{4} = \frac{1}{4}\lim_{n\to\infty} e^{\frac{1}{2^{2n}}} + \lim_{n\to\infty} \frac{1}{2^{n+2}} = \frac{1}{4} + 0 = \frac{1}{4}$.

Q.3 Let $f : [0,\infty)$ be a twice differentiable and increasing function with $f(0) = 0$. Suppose that, for any $t \geq 0$, the length of the curve $y = f(x)$, $x \geq 0$ between $x = 0$ and $x = t$ is $\frac{2}{3}\left[(1+t)^{\frac{3}{2}} - 1\right]$. Then $f(4)$ is equal to

(A) $\frac{11}{3}$ (B) $\frac{13}{3}$ (C) $\frac{14}{3}$ (D) $\frac{16}{3}$

Solution: (D) Length of the arc = $\int_0^t \sqrt{1+[f'(x)]^2}\, dx$.

We know, $\int_0^t \sqrt{1+[f'(x)]^2}\, dx = \frac{2}{3}\left[(1+t)^{\frac{3}{2}} - 1\right]$

» $\int_0^t \sqrt{1+x}\, dx = \frac{2}{3}\left[(1+t)^{\frac{3}{2}} - 1\right]$ » $f'(x) = x$ » $f(x) = \frac{2}{3}x^{\frac{3}{2}}$ » $f(4) = \frac{16}{3}$.

Q.4 Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$f(x,y) = \frac{\sin(2(x^2+y^2))}{x^2+y^2} e^{3x.\sin\left(\frac{4}{y}\right)}, \quad if\ (x,y) \neq (0,0)$$

$$= \quad \alpha, \quad if\ (x,y) = (0,0)$$

where α is a real constant. If f is continuous at (0,0), then α is equal to

1 (B) 2 (C) 3 (D) 4

Solution: (B) $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\sin\left(2(x^2+y^2)\right)}{x^2+y^2} e^{3x.\sin\left(\frac{4}{y}\right)}$

$$= \lim_{(x,y)\to(0,0)} \frac{\sin\left(2(x^2+y^2)\right)}{x^2+y^2} . \lim_{(x,y)\to(0,0)} e^{3x.\sin\left(\frac{4}{y}\right)} = 2\lim_{u\to 0}\frac{sinu}{u} = 2.$$

Q.5 Let A be a 3× 3 real matrix with eigenvalues 1, 2, 3 and let B = $A^{-1} + A^2$. Then the trace of the matrix B is equal to

(A) $\frac{91}{6}$ (B) $\frac{95}{6}$ (C) $\frac{97}{6}$ (D) $\frac{101}{6}$

Solution: (B) Trace(B) = Sum of the eigenvalues of B

= (Sum of the eigenvalues of A^{-1}) + (Sum of the eigenvalues of A^2)

= (1 + ½ + 1/3) + ($1^2 + 2^2 + 3^2$)

= $\frac{95}{6}$.

Q.6 Let X_1, X_2,..... be a sequence of i.i.d random samples with variance 1. Then $\lim_{n\to\infty} p\left(\frac{(X_1-X_2)+(X_3-X_4)+\cdots+(X_{2n-1}-X_{2n})}{\sqrt{n}} \le x\right)$ is equal to

(A) $\Phi(x)$ (B) $\Phi(2x)$ (C) $\Phi(x\sqrt{2})$ (D) $\Phi\left(\frac{x}{\sqrt{2}}\right)$

Solution: (D) Let Z = $(X_1 - X_2) + (X_3 - X_4) + \cdots + (X_{2n-1} - X_{2n})$

Now it is clear that E(Z) = 0 & Var(Z) = 2n.

By Central Limit Theorem, $\frac{Z-0}{\sqrt{2n}} \sim N(0,1)$.

So, $\lim_{n\to\infty} p\left(\frac{(X_1-X_2)+(X_3-X_4)+\cdots+(X_{2n-1}-X_{2n})}{\sqrt{n}} \le x\right)$

$= \lim_{n\to\infty} p\left(\frac{(X_1-X_2)+(X_3-X_4)+\cdots+(X_{2n-1}-X_{2n})}{\sqrt{2n}} \le \frac{x}{\sqrt{2}}\right) = \Phi\left(\frac{x}{\sqrt{2}}\right).$

Q.7 Let $X_1, X_2, \ldots, X_{100}$ be a random sample from N(2,4) Population. Let $\bar{X} = \frac{1}{99}\sum_{i=1}^{99} X_i$,

$S = \sqrt{\frac{1}{98}\sum_{i=1}^{99}(X_i - \bar{X})^2}$ and W = $\frac{X_{100}-2}{S}$. Then the distribution of W is

(A) χ^2_{98} (B) χ^2_{99} (C) t_{98} (D) t_{99}

Solution: (C) W = $\frac{X_{100}-2}{S} = \frac{X_{100}-E(X_{100})}{\sqrt{\frac{1}{98}\sum_{i=1}^{99}(X_i - \bar{X})^2}}$; So, W ~ $\frac{N(O,1)}{\sqrt{\frac{\chi^2_{98}}{98}}}$, i.e. , W ~t_{98} .

Q.8 Let $X_1, X_2, \ldots, X_n, X_{n+1}$ be a random sample from N(μ,1) Population. Let $\overline{X_n} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and T = $\frac{1}{2}(\overline{X_n} + X_{n+1})$, then for estimating μ

(A) T is unbiased and consistent (b) T is biased and consistent

(C) T is unbiased and inconsistent (d) T is biased and inconsistent

Solution: (C) E(T) = E $\left[\frac{1}{2}(\overline{X_n} + X_{n+1})\right] = \frac{1}{2}E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] + \frac{1}{2}E[X_{n+1}] = \frac{n\mu}{2n} + \frac{\mu}{2}$ = μ .

So, T is unbiased. Now calculate Var(T)

Q.9. Let X be an observation from a population with density

$$f(x) = \lambda^2 x e^{-\lambda x}, if\ x > 0, \lambda > 0$$
$$= 0 \qquad , \text{elsewhere}$$

For testing $H_0 : \lambda = 2$ against $H_1 : \lambda = 1$, the most powerful test of size α is given by "Reject H_0 if X > c" , where c is given by

$\frac{1}{4}\chi^2_{4,\alpha}$ (B) $\frac{1}{4}\chi^2_{3,\alpha}$ (C) $\frac{1}{4}\chi^2_{2,\alpha}$ (D) $\frac{1}{4}\chi^2_{1,\alpha}$.

Solution: (A) Here $\frac{f_{\theta_1}(x)}{f_{\theta_0}(x)} = \frac{e^x}{4} > c$ » x > log4c.

Now you need to calculate $P_{H_0}[x > log4c]$. Your final answer will be $\frac{1}{4}\chi^2_{4,\alpha}$.

Q.10 A continuous random variable X has the density $f(x) = 2\varphi(x)\,\Phi(x),\ x \in \mathbb{R}$. Then

(A) $E(X) > 0$ (B) $E(X) < 0$ (C) $P[X \le 0] > 0.5$ (D) $P[X \ge 0] < 0.25$

Solution: (A) $E(X) = 2\int_{-\infty}^{\infty} x\varphi(x)\Phi(x)dx = 0 + 2\int_{-\infty}^{\infty} \varphi^2(x)dx > 0$.

Q. 11 If X has the probability density function

$f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1}e^{-x};\ x > 0, \alpha > 2$, then $\mathrm{Var}\left(\frac{1}{X}\right)$ *is equal to* .....

Solution: For Gamma distribution, $\mu_r' = E(X^r) = \frac{\Gamma(r+\alpha)}{\Gamma(\alpha)}$.

$$Var\left(\frac{1}{X}\right) = E\left(\frac{1}{X^2}\right) - E^2\left(\frac{1}{X}\right) = \frac{\Gamma(\alpha-2)}{\Gamma(\alpha)} - \frac{\Gamma(\alpha-1)}{\Gamma(\alpha)} = \frac{1}{(\alpha-1)^2(\alpha-2)}.$$

Q.12 Let the joint density function of (X,Y) be

$$f(x,y) = \begin{cases} c(x+y), if\ -x < y < x, 0 < x < 1 \\ 0, \qquad Otherwise. \end{cases}$$

Then the value of c is equal to .....

Solution: $\int_{-x}^{x}\int_{0}^{1} c(x+y)dxdy = 1.$ *Since* $f(x,y)$ *is a pdf* » $c = 1$ *, after integration.*

Q.13 Let X be an observation from a population with density function f(x). Then the power of the most powerful test of size $\alpha = 0.19$ for testing

$H_0: f(x) = \begin{cases} \frac{x}{2} \\ 0, Otherwise \end{cases}, if\ 0 < x < 2,$ vs. $H_0: f(x) = \begin{cases} \frac{3x^2}{8} \\ 0, otherwise \end{cases}, if\ 0 < x < 2,$

is equal to ....

$$Solution: \frac{f_{\theta_1}(x)}{f_{\theta_0}(x)} = \frac{3x}{4} > c \text{ » } x > \frac{4c}{3}.$$

$$P_{H_0}\left[x > \frac{4c}{3}\right] = \alpha = 0.19 \text{ » } \int_{\frac{4c}{3}}^{2} \frac{x}{2} dx = 0.19 \text{ » } c = \frac{27}{20}.$$

$$Now, power = P_{H_1}\left[x > \frac{4c}{3}\right] = \int_{\frac{9}{5}}^{2} \frac{3x^2}{8} dx = 0.271 .$$

Q.14 Bulbs produced by a factory F_i have lifetimes (in months) distributed as Exp$\left(\frac{1}{3^i}\right)$ for i = 1,2,3. A firm randomly produces 40% of its required bulbs from F_1 , 30% from F_2 and 30% from F_3 . A randomly selected bulb from the firm is found to be working after 27 months. The probability that it was produced by the factory F_3 is .....

Q.15 Let $X_1,X_2,.....,X_n$ be a random sample from a population with density

$$f(x,\mu) = \begin{cases} e^{\mu - x}, & if x > \mu, \\ 0, & Otherwise. \end{cases}$$

And let $X_{(1)} = \min\{ X_1,X_2,.....,X_n\}$. Then $\left(X_{(1)} - \frac{2}{n}\log_e 5, X_{(1)}\right)$ is a .... % confidence Interval for .

Solution: Steps at a glance: (i) Calculate the density of $X_{(1)}$.

(ii) Here $X_{(1)} - \frac{2}{n}\log_e 5 < \mu < X_{(1)}$.

(iii) Now, calculate the probability.

Q.16 Ten percent of bolts produced in a factory are defective. They are randomly packed in boxes such that each box contains 3 bolts. Four of these boxes are bought by a customer. The probability, taht the boxes that this customer bought have no defective bolt in them, is equal to ......

Solution:- Let the factory produces n bolts. Total defective items = n/10, non-defective items = 9n/10. Prob(12 bolts are non-defective) = $\frac{\binom{9n/10}{12}}{\binom{n}{12}}$.

$Q.17$ Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases} x^2, & if x\in\mathbb{R}, \\ \alpha x+\beta, & if\ x\in\mathbb{R}-\mathbb{Q}, \end{cases}$$

Where α and β are real constants. If f is differentiable at x = 1 then the value of $3\alpha+\beta$ is equal to ....

Solution:- $\lim_{x\to 1+} f(x) = \lim_{x\to 1-} f(x)\ gives\ \alpha+\beta=1.$

LHD = RHD gives f'(1-) = f'(1+) gives α = 2. So, β = -1. So, 3α+β = 5.

Q.18 Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that $|a_n| \leq \sqrt{n}, n = 1,2,$ ....
Then $\lim_{n\to\infty}\{e^{\frac{a_n}{n}}+\sqrt{n}\sin\left(\sqrt{\frac{2}{n}}\right)\}$ is equal to ......

Solution:- $\lim_{n\to\infty} |a_n| \leq \lim_{n\to\infty}\sqrt{n}$

$$\lim_{n\to\infty}\{e^{\frac{a_n}{n}}+\sqrt{n}\sin\left(\sqrt{\frac{2}{n}}\right)\} = \lim_{n\to\infty} e^{\frac{1}{\sqrt{n}}} + \lim_{n\to\infty}\sqrt{n}\sin\left(\sqrt{\frac{2}{n}}\right) = 1.$$

Q.19 Consider the linear system

x + y + 2z = α

x + 4y + z = 4

3y – z = γ

In the unknowns x, y and z. If the above system always has a solution then the value of $\alpha+\gamma$ is equal to ....

Solution:- The given system has a solution if Rank(A |b)=Rank(A)

Rank(A) = 2. Rank(A |b)=2 iff $4-\gamma-\alpha$)=0. So, $\alpha+\gamma$=4.

Q. 20 The general solution of the differential equation $(x^4-y)dx+(y^4-x)dy=0$ is equal to .....

Solution: Mdx + Ndy = 0 ; M = $x^4 - y$, N = $y^4 - x$.

$$\frac{dM}{dy} = -1, \qquad \frac{dN}{dx} = -1.$$

$$So, the\ given\ ODE\ is\ exact, hence\ the\ solution\ is$$

$$\int (x^4 - y)dx = c \quad » \quad \frac{x^5}{5} - yx = c \quad » \quad x^5 - 5yx = k\ is\ the\ general\ solution.$$

Q. 21 Consider the matrix P = $\begin{bmatrix} 1 & 1 & 1 \\ 0 & \alpha & \beta \\ \alpha & 0 & \beta \end{bmatrix}$. If P has eigenvalues 0 and 3 then determine the values of the pair (α,β).

Solution: Characteristic equation is $| P - \lambda I | = 0$. Here λ = 0 and 3.

Putting the values of λ in the characteristic equation.

So, |P| = 0 for λ = 0 » $\begin{vmatrix} 1 & 1 & 1 \\ 0 & \alpha & \beta \\ \alpha & 0 & \beta \end{vmatrix}$ = 0 » $2\alpha\beta - \alpha^2 = 0$ » Either $\alpha = 0$ or $\alpha = 2\beta$.

Also $| P - 3I | = 0$ for λ = 3

» $\begin{vmatrix} 1-3 & 1 & 1 \\ 0 & \alpha-3 & \beta \\ \alpha & 0 & \beta-3 \end{vmatrix}$ = 0 » $\alpha^2 - 9\alpha - 6\beta + \alpha\beta + 18 = 0$ ---- (*)

Now putting $\alpha = 0$ or $\alpha = 2\beta$ respectively in the equation (*), we get

For $\alpha = 0$ or $\beta = 3$,

For $\alpha = 2\beta$, we have $\beta = 1$, so $\alpha = 2$.

Hence, the values of the pair (α,) are (0,3) and (2,1).

Q. 22 Let a function f : [0,1]→ℝ be continuous on [0,1] and differentiable in (0,1). If f(0) = 1 and $[f(1)]^2 + 2f(1) = 5$, then prove that there exists a c ∈ (0,1) such that $f'(c) = \frac{2}{2+3[f(c)]^2}$.

Hints:- This is a simple application of Lagrange's Mean Value Theorem.

Q. 23 Let $\{a_n\}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Prove that the series $\sum_{n=1}^{\infty} \log(1 + {a_n}^4)$ converges.

Solution:- $\sum_{n=1}^{\infty} a_n$ converges absolutely => $\sum_{n=1}^{\infty} {a_n}^4$ converges absolutely.

Now we know log(1+x) ≤ x => $\log(1+x^4) \leq x^4$ => $|\log(1 + {a_n}^4)| \leq {a_n}^4$,

Now by the convergence of power series we know that $\sum_{n=1}^{\infty} {a_n}^4$ converges absolutely.

So, by Comparison Test, $\sum_{n=1}^{\infty} \log(1 + {a_n}^4)$ also converges absolutely.

Q. 24 Let D = $\{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq x \leq 1\}$ and let f : D→ ℝ} be defined by $f(x,y) = x^2 - 2xy + 2$, {(x,y)∈D. Then determine the maximum value of f in the region D.

Solution:- $f(x,y) = x^2 - 2xy + 2$, $f_x = 2x - 2y$, $f_y = -2x$

For minimum or maximum value of f, we put $f_x = 0$ & $f_y = 0$

So, (x,y) ≡ (0,0)

Also $f_{xx} = 2$, $f_{yy} = 0$ & $f_{xy} = -2$.

Now, $|D| = f_{xx} \cdot f_{yy} - {f_{xx}}^2 = -4 < 0$.

So, f(0,0) is the maximum value. F(0,0) = 2.

Q.25 Let X, Y and Z be independent random variables with respective moment generating functions $M_X(t) = \frac{1}{1-t}$, t < 1: $M_Y(t) = e^{\frac{t^2}{2}} = M_Z(t)$, t ∈ ℝ. Let $W = 2X + Y^2 + Z^2$ then determine the value of P(W>2).

Solution :- $M_X(t) = \frac{1}{1-t}$, t < 1: so, $f(x) = e^{-x}$: x≥ 0. So, X ~ Exp(1).

Same as, from MGF of Y & Z, we can see Y, Z ~ N(0,1).

So, $Y^2 + Z^2 \sim {\chi_2}^2$. Also, $2X \sim {\chi_2}^2$. So, $W \sim {\chi_4}^2$.

Now, calculate the probability using integration by yourself.

Q.26 Ram rolls a pair of fair dice. If the sum of the numbers shown on the upper faces is 5, 6, 10, 11 or 12 then Ram wins a gold coin. Otherwise, he rolls the pair of dice once again and wins a silver coin if the sum of the numbers shown on the upper faces in the second throw is the same as the sum of the numbers in the first throw. What is the probability that he wins a gold or silver coin?

Solution:- Prob(he wins a gold coin) = $\frac{15}{36}$ = Prob(he wins a silver coin)

Trials are independent of each other here.

So, the probability that he wins a gold or silver coin is $\frac{15}{36} + \frac{15}{36} = \frac{5}{6}$.

Q.27 Let $X_1, X_2, \ldots, X_n$ be a random sample from a uniform distribution on the interval [θ,2θ], θ>0. Find the method of moments estimator and the maximum likelihood estimator of θ. Further find the bias of the MLE.

Solution:- <u>Method of moments estimator:</u> $f(x_i,\theta) = \frac{1}{\theta}$, θ>0.

$$m_1' = \mu_1' = \bar{x} = \frac{\theta+2\theta}{2} = \frac{3\theta}{2}.$$

$$\text{So, } \hat{\theta} = \frac{2\bar{x}}{3}.$$

<u>Method of maximum likelihood estimator:</u> Likelihood function is given by

$$L(\theta|\underline{x}) = \frac{1}{\theta^n}, if\ \theta \le x_i \le 2\theta = \frac{1}{\theta^n}, if\ \theta \le x_{(1)} \le x_{(n)} \le 2\theta$$

For L to be maximum, $\frac{x_{(n)}}{2} \le \theta \le x_{(1)}$.

Hence, MLE of θ is $\hat{\theta} = \frac{x_{(n)}}{2}$.

Now, the bias of MLE is = $E[\hat{\theta}] = \frac{E[X_{(n)}]}{2} = \frac{n\theta}{2(n+1)}$.

Q.28 Let $(X_1,Y_1), (X_2,Y_2), \ldots$ be a sequence of i.i.d. bivariate normal random variables with $E(X_i) = 75$, $E(Y_i) = 25$, $Var(X_i) = 36$, $Var(Y_i) = 16$ and $Corr(X_i,Y_i) = 0.25$. Let $\overline{U} = \frac{1}{n}\sum_{i=1}^{n}(Xi, Yi)$. Find the maximum value of n so that $P(\overline{U} \le 104) \ge 0.99$.

Solution:- $X_i + Y_i \sim N(100,64)$

So, $\bar{U} = \frac{1}{n}\sum_{i=1}^{n}(Xi, Yi) \sim N(100,\frac{64}{n})$.

Now, $P(\bar{U} \leq 104) \geq 0.99 = \Phi(2.33)$, given in the table.

⇨ $P\left[\frac{\bar{U}-100}{\frac{8}{\sqrt{n}}} \leq \frac{104-100}{\frac{8}{\sqrt{n}}}\right] = \Phi(2.33)$,

⇨ $\Phi(\frac{\sqrt{n}}{2}) = \Phi(2.33)$

⇨ $n = 21.7 \cong 22$.

⇨ Q.29 The joint probability dencity function of (X,Y) is

⇨ $f(x,y) = \begin{cases} \frac{1}{2}e^{(1-x-y)}, if\ x+y>1, x>0, y>0 \\ 0\ , \qquad otherwise \end{cases}$

⇨ Find the probability density function of X and $E(Y \mid X = x)$, x>0.

Solution:- $f(x) = \int_0^\infty f(x,y)dy = \frac{1}{2}e^{1-x}, for\ 0 < x < \infty$.

PDF of Y given X = x is given by $f_{(Y\mid X)} = \frac{f(x,y)}{f(x)} = e^{-y}$;

$E(Y \mid X = x) = \int_0^\infty ye^{-Y}dy = y + 1\ ; x > 0$.

Q.30 Suppose the F is a cdf, where

$$F(x) = \begin{cases} 0, & if\ x < 0 \\ 1-e^{-x}, & if\ 0 \leq x < 1 \\ c, & if\ 1 \leq x < 2 \\ 1-e^{-x}, & if\ x \geq 2 \end{cases}$$

i. Find all possible values of c.
ii. Find P(0.5≤ $X \leq 2.5$) and P(X=1)+P(X=2).

Solution:- i. Since F(x) is a c.d.f. so it is right continuous.

F(x+0) = $\lim_{h\to 0+} F(0+h) = F(0)$ implying c = 0.

Similarly, $F(1) = c$ implying $c = 1 – e^{-1}$.

ii. $P(0.5 \leq X \leq 2.5) = F(2.5) - F(0.5^-)$ (Do yourself)

iii. $P(X=1)+P(X=2)=F(1) - F(1^-) + F(2) – F(2^-)=e^{-1} - e^{-2}$.