SOLVED PROBLEMS FOR MMATH (PMB)

1. Suppose $a_n \geq 0$ and $\sum a_n$ is convergent. Show that $\sum \frac{1}{n^2 a_n}$ is divergent.

Solution:- Using CS Inequality,

$$\left[(\sqrt{a_1})^2 + (\sqrt{a_2})^2 + \cdots + (\sqrt{a_n})^2 \right] \left[\left(\frac{1}{1.\sqrt{a_1}}\right)^2 + \left(\frac{1}{2\sqrt{a_2}}\right)^2 + \cdots + \left(\frac{1}{n\sqrt{a_n}}\right)^2 \right]$$

$$\geq \left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right]^2$$

RHS is divergent and $\sum a_n$ is convergent.

Hence, $\sum \frac{1}{n^2 a_n}$ is divergent.

2. Let $F : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$F(x_1, x_2, \ldots, x_n) = \max \{ |x_1|, |x_2|, \ldots, |x_n| \}.$$

Show that F is a uniformly continuous function.

Solution:- Take $X_1 = (x_{11}, x_{12}, x_{13}, \ldots, x_{1n})$, $X_2 = (x_{21}, x_{22}, \ldots, x_{2n})$.

$$|F(x_{11}, x_{12}, \ldots, x_{1n}) - F(x_{21}, x_{22}, \ldots, x_{2n})| = \max \{ |x_{11}|, |x_{12}|, \ldots, |x_{1n}| \} - \max \{ |x_{21}|, |x_{22}| \ldots, |x_{2n}| \}$$

$$\leq \left| \max \{ |x_{11} - x_{21}|, |x_{12} - x_{22}|, \ldots, |x_{1n} - x_{2n}| \} \right|$$

$$\leq \sqrt{(x_{11} - x_{21})^2 + (x_{12} - x_{22})^2 + \cdots + (x_{1n} - x_{2n})^2}$$

$$\leq |X_1 - X_2|$$

So, for any given $\epsilon > 0$, choose $\delta = \epsilon$, then $|F(X_1) - F(X_2)| < \epsilon$ $\forall$ $|X_1 - X_2| < \delta$.

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3. Let f be a continuous function on $[0,1]$. Evaluate

$$\lim_{n\to\infty} \int_0^1 x^n f(x)\,dx.$$

Solution:- Applying Mean Value theorem for integrals

$f: [0,1] \to \mathbb{R}$ and $g: [0,1] \to \mathbb{R}$ be two integrable functions and $g(x)$ has the same sign on $[0,1]$.

Then $$\int_a^b f(x)g(x)\,dx = \mu \int_a^b g(x)\,dx$$

Take $g(x) = x^n$; x^n doesnot change sign on $[0,1]$.

Hence $$\int_0^1 x^n f(x)\,dx = \mu \int_0^1 x^n\,dx$$

$$\Rightarrow \int_0^1 x^n f(x)\,dx = \frac{\mu}{n+1}$$

$$\Rightarrow \lim_{n\to\infty} \int_0^1 x^n f(x)\,dx = \lim_{n\to\infty} \frac{\mu}{n+1} = 0.$$

4. Show that the only field automorphism of $\mathbb{Q}$ is the identity. Using this prove that the only field automorphism of $\mathbb{R}$ is the identity.

Solution:- It's easy to see that as $\phi(1)$, any automorphism $\phi: \mathbb{Q} \to \mathbb{Q}$ as $n \in \mathbb{Z}^+ = \underbrace{1+1+\cdots+1}_{n} = n\,\phi(1) = n$.

$n \in \mathbb{Z}^-$ $\phi(-1) = \phi(1-1) = 0$

$\Rightarrow \phi(1) + \phi(-1) = 0$

$\Rightarrow \phi(1) = -\phi(1) = -1$.

Here all -ve integers are mapped to -ve integers.

p/q: p, q are integers.

$\phi(1/q)$, $\phi\left(q \cdot \frac{1}{q}\right) = \phi(q) \cdot \phi\left(\frac{1}{q}\right) = 1$

$\phi\left(\frac{1}{q}\right) = \frac{1}{\phi(q)} = \frac{1}{q}$.

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$$\phi\left(\frac{p}{q}\right) = \phi\left(\underbrace{\frac{1}{q}+\frac{1}{q}+\cdots+\frac{1}{q}}_{p \text{ times}}\right) = p\,\phi\left(\frac{1}{q}\right) = \frac{p}{q}.$$

Any field automorphism on $\mathbb{R}$ fixes $\oplus$ where it is restricted to $\oplus$

i.e., $\psi : \mathbb{R} \to \mathbb{R}$

$\psi|_{\mathbb{Q}} : \mathbb{Q} \to \mathbb{Q}$ is an identity map.

Now let's assume $a \in \mathbb{R} - \mathbb{Q}$ $\ni$

$$\phi(a) \neq a.$$

By archimedian property let us assume that $\exists$ q s.t. 'q' is in between a and $\phi(a)$ (q ↓ rational)

WLG $a \leq \phi(a)$ ——— ①

$$a \leq q < \phi(a)$$

$$a \leq q \Rightarrow \phi(a) \leq \phi(q) \Rightarrow \phi(a) \leq q \text{ ——— ②}$$

from ① & ②, $q = \phi(a)$

$$\Rightarrow \phi(a) = \phi(a)$$

$$\Rightarrow q = a.$$

then $\phi(a) = a$.

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5. Find the most general curve whose normal at each point passes through $(0,0)$. Find the particular curve through $(2,3)$.

Solution:- Let (α, β) be the point on the curve.

Let $y = f(x)$ be the curve ——— (1)

Slope of normal $= \dfrac{-1}{f'(x)}$.

Slope of normal at (α, β) is $(y-0) = \dfrac{-1}{f'(\alpha)}(x-0)$

i.e., $y = \dfrac{-x}{f'(\alpha)}$ is the equation of Normal at (α, β) ——— (2)

But (α, β) satisfies (1) and (2),

$$\beta = f(\alpha)$$

$$f(\alpha) f'(\alpha) = -\alpha.$$

Solving $\int f(t) f'(t)\, dt = -\int t\, dt$

$$\Rightarrow \frac{f^2(t)}{2} + \frac{t^2}{2} = c$$

general locus will be $x^2 + y^2 = d$.

Particular solution $4 + 9 = d \Rightarrow d = 13$.

i.e., $\boxed{x^2 + y^2 = 13}$.

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6 (a) Suppose the series $\sum_{n=1}^{\infty} a_n$ is convergent $(a_n \geq 0)$.

Is it true that $\sum_{n=1}^{\infty} \sqrt{a_n a_{n-1}}$ also convergent?

(b) Is the converse of the statement (a) true?

Solution:- (a) AM $\geq$ GM gives

$$\sqrt{a_n a_{n-1}} \leq \frac{a_n + a_{n-1}}{2}$$

If $\sum_{n=1}^{\infty} a_n$ is convergent then $\sum_{n=1}^{\infty} \frac{a_n + a_{n-1}}{2}$ is also convergent

Therefore $\sqrt{a_n a_{n-1}}$ is convergent.

(b) Take $a_n = n^2$ (for odd n); $a_n = n^{-102}$ (for even n).

Then $\sum_{n=1}^{\infty} a_n$ is divergent but $\sum_{n=1}^{\infty} \sqrt{a_n a_{n-1}}$ is convergent. (By p-series)

7. Evaluate the value of $\lim_{n\to\infty} \int_0^a e^{nx}\, dx$.

Solution:-

$$I = \lim_{n\to\infty} \int_0^a e^{nx}\, dx = \lim_{n\to\infty} \left[\frac{e^{nx}}{n}\right]_0^a$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[e^{na} - 1\right]$$

$$= \lim_{n\to\infty} a e^{na}.$$

$$I = \begin{cases} \infty & , a > 0 \\ 0 & , a \leq 0. \end{cases}$$

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8. For the following statements indicate true/false. If the statement is true then give a short proof. Otherwise give a counter example.

(a) There is no non-trivial group homomorphism from the symmetric group S_3 to $\mathbb{Z}_3$.

(b) If a group G is $\ni$ each proper subgroup of G is cyclic, then G is cyclic.

(c) Let R be a ring with unity. Consider $X^2 - 1 \in R[X]$. Then $X^2 - 1$ has at most two roots in R.

Solution:- (a) TRUE

$\phi : S_3 \to \mathbb{Z}_3$

$|S_3| = 6$ and $|\mathbb{Z}_3| = 3$.

$\Rightarrow \phi$ is not one-to-one $\Rightarrow$ non-trivial kernel.

$\mathbb{Z}_3$ has only two subgroups $\mathbb{Z}_3$ and $\{0\}$.

$\Rightarrow \phi$ is onto (as co-domain is a subgroup)

$S_3/\ker\phi \cong \mathbb{Z}_3$ [$\because \ker\phi$ is a normal subgroup]

$\Rightarrow |\ker\phi| = 2$.

But S_3 has no normal subgroup of order 2.

(b) FALSE

Counter Example: V_4 (klein four group).

(c) FALSE

Counter Example: Consider $\mathbb{R}^2$ with usual co-ordinate-wise addition and multiplication.

This is a ring and the equation $X^2 - 1$ has four solutions: $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$.

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[9]. Let $u, v : [a,b] \to \mathbb{R}$ be continuous. Define $f : [a,b] \to \mathbb{R}$ by

$$f(x) = \begin{cases} u(x) & \text{if } x \text{ is rational} \\ v(x) & \text{if } x \text{ is irrational} \end{cases}$$

Show that f is Riemann integrable on $[a,b]$ if and only if $u(x) = v(x)$ for all $x \in [a,b]$.

Solution: Now pick any partition 'P'.

Construct two Riemann sums

$$S(P, f, \xi) \text{ and } S'(P, f, \xi')$$

$S \to$ choose rationals as intermediate points.

$S' \to$ choose irrationals as intermediate points.

We know that $\lim_{\|P\|} S \to B$ and $\lim_{\|P\|} S' \to B$

as 'f' is Riemann integrable.

$$\Rightarrow \lim_{\|P\| \to 0} S - S' \to 0$$

$$\Rightarrow \lim_{\|P\| \to 0} u(x)(b-a) - v(x)(b-a) \to 0$$

$$\Rightarrow \lim_{\|P\| \to 0} u(x) - v(x) \to 0$$

$$\Rightarrow u(x) = v(x).$$

If $u(x) = v(x)$ for $x \in [a,b]$ and given u is continuous function, f is integrable (To show).

If 'f' is Riemann integrable then it satisfies the following property that for every partition 'P' and choice of intermediate points ξ,

$$\lim_{\|P\| \to 0} S(P, f) = B \rightsquigarrow \text{a real number, when } S(P, f, \xi) \text{ is the Riemann sum.}$$

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10. Let f be a bounded twice differentiable real valued function on $\mathbb{R}$ $\ni f''(x) \geq 0$ for all x. Show that f is a constant.

Solution:-

$$f'' \geq 0 \Rightarrow f \text{ is convex.}$$

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0).$$

Case I:- $f'(x_0) > 0$

$\Rightarrow f(x) \to \infty$ as $x \to \infty$

$\Rightarrow f(x)$ is constant [$\because f$ is bounded]

Case II:- $f'(x_0) < 0$

$\Rightarrow f(x) \to \infty$ as $x \to -\infty$

$\Rightarrow f(x)$ is constant as f is bounded.

Case III:- $f'(x_0) = 0$

$f'(x) = 0 \quad \forall \quad x \in \mathbb{R}$

$\Rightarrow f$ is a constant.

11. Find the gcd of $4X^4 - 2X^2 + 1$ and $-3X^3 + 4X^2 + X + 1$ in $\mathbb{Z}_7[X]$.

Solution:- gcd will be $X^2 + 2$.

Easy to check. Try Yourself.

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12 Find conditions on $a, b, c \in \mathbb{R}$ to ensure that the following system is consistent, and in that case, find the general solutions :

$$\begin{aligned} x + 3y - 2z &= a \\ -x - 5y + 3z &= b \\ 2x - 8y + 3z &= c \end{aligned}$$

Solution:-

$$A_{3\times 3}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

$$A = \begin{pmatrix} 1 & 3 & -2 \\ -1 & -5 & 3 \\ 2 & -8 & 3 \end{pmatrix} \xrightarrow{R_2' = R_1 + R_2} \begin{pmatrix} 1 & 3 & -2 \\ 0 & -2 & 1 \\ 2 & -8 & 3 \end{pmatrix}$$

$$\xrightarrow{R_3' = R_3 - 2R_1} \begin{pmatrix} 1 & 3 & -2 \\ 0 & -2 & 1 \\ 0 & -14 & 7 \end{pmatrix}$$

$$\xrightarrow{R_3' = R_3 - 7R_2} \begin{pmatrix} 1 & 3 & -2 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

So, $\text{Rank}(A) = 2$.

$$\bar{A} = \left(\begin{array}{ccc|c} 1 & 3 & -2 & a \\ -1 & -5 & 3 & b \\ 2 & -8 & 3 & c \end{array}\right) \xrightarrow{\text{Same operations}} \left(\begin{array}{ccc|c} 1 & 3 & -2 & a \\ 0 & -2 & 1 & a+b \\ 0 & 0 & 0 & c-9a-7b \end{array}\right)$$

For $\text{rank}(A) = \text{rank}(\bar{A}) \Rightarrow c - 9a - 7b = 0$

$\Rightarrow 9a + 7b = c$ is the required condition.

For general solution:

$$\left.\begin{aligned} x + 3y - 2z &= a \\ -2y + z &= a + b \end{aligned}\right\}$$

Take $z = t$, $y = -\left(\frac{a+b-t}{2}\right)$; $x = \frac{5a+3b+t}{2}$.

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13. Prove or disprove:

(a) $\exists$ a linear map $T: \mathbb{R}^2 \to \mathbb{R}^4$ such that

$$\text{Range}(T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\}.$$

(b) $\exists$ a linear map $T: \mathbb{R}^2 \to \mathbb{R}^3$ $\ni$

$$\text{Range}(T) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$$

Solution:- (a) No.

$$T: \mathbb{R}^2 \to \mathbb{R}^4$$

$$\dim(\mathbb{R}^2) = \dim(\text{Range}(T) + \ker(T))$$

$$\Rightarrow 2 = 3 + x$$

$$\Rightarrow \dim(\text{Range}(T)) = 3.$$

(b) $\dim(\text{Range}T) = 2.$

$$\dim(\mathbb{R}^2) = 2 + \dim(\ker T)$$

$$\Rightarrow \dim(\ker T) = 0,$$

Yes, linear tranformation is possible.

$$T: \mathbb{R}^2 \to \mathbb{R}^3$$

$$T(x_1, x_2) \longrightarrow (x_1, x_2, -x_1 - x_2).$$

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14. Suppose f is a group homomorphism from the set of all permutations of n elements, S_n to the multiplicative group C^* (set of all complex numbers without zero). Prove that $f(x) = +1$ or -1 for all x in S_n.

Solution:- $f: S_n \to C^*$

Recall, 'f' is a gp homomorphism,
so if $x \in S_n$ and $f(x) \in C^*$, then

$$O(f(x)) \mid O(x) \longrightarrow (*)$$

(O: Order)

Now, in C^* except $1, -1$; all elements are of infinite order.
S_n is a finite gp, hence the order of each element in S_n is finite.
So, for $(*)$ to hold $x \in S_n$, $f(x) = 1$ or -1.

15. Suppose C is a closed bounded polygon with area P. Consider

$$A(a) = \{P \cap R \ni R = |(x,y)| : y \geq a : a \in \mathbb{R}\}.$$

Prove that $\exists$ a real number a for which $A(a) = P/2$.

Solution:- For any give $\epsilon > 0$ and a_1 and a_2

$$|A(a_2) - A(a_1)| < \epsilon$$

$$|A(a_2) - A(a_1)| < l|a_2 - a_1|$$

For $\delta = \epsilon/l$; $|a_2 - a_1| < \delta$, we have

$$|A(a_2) - A(a_1)| < \epsilon.$$

$\Rightarrow A(a)$ is continuous at a_1.
$\Rightarrow A(a)$ is a constant function.
$\Rightarrow A(a)$ takes values from P to 0 as a goes from large negative value to a large positive value.
Hence by Intermediate value theorem, $A(a)$ attains value $P/2$.

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16. Prove that the polynomial $x^3+7ax^2+7bx+2013$ is irreducible in $\mathbb{Z}[x]$, the ring of integers.

Solution:- Recall Mod p Irreducibility test :-

Let 'p' be a prime and suppose $f(x) \in \mathbb{Z}[x]$ with $\deg(f(x)) \geq 1$. Let $f'(x)$ be the polynomial in $\mathbb{Z}_p[x]$ obtained from $f(x)$ by reducing its coefficient by modulo 'p'.

If $f'(x)$ is irreducible over $\mathbb{Z}_p[x]$ and $\deg(f'(x)) = \deg(f(x))$ then $f(x)$ is irreducible over $\mathbb{Q}$.

Given $x^3+7ax^2+7bx+2013 = f(x)$

$p=7$; $f'(x) = x^3+4$.

By verification we can see that $f'(x)$ is irreducible over $\mathbb{Z}_7[x]$ and $\deg(f(x)) = \deg(f'(x))$.

Hence, $f(x)$ is irreducible over $\mathbb{Z}[x]$.

17. Define $K = \left\{ \sum_{n=0}^{n} a_n 2^{n/3} ; a_n \text{ is from the field of rational numbers } \mathbb{Q} \right\}$.

Prove that K is of dimension 3.

Solution:-

Claim: $\{1, 2^{1/3}, 2^{2/3}\}$ is the basis of K.

Observe that $2^{n/3}$; $n \geq 3$ can be expressed as $2^k . 2^{i/3}$; $i = 0, 1, 2$.

Hence any arbitrary term $\sum_{n=0}^{n} a_n 2^{n/3}$ can be expressed as

$$\sum_{k=0}^{2} a_k 2^{k/3}.$$

Hence $L\{1, 2^{1/3}, 2^{2/3}\} = K$.

Observe none of $1, 2^{1/3}, 2^{2/3}$ is a scalar multiple of the other.

Hence, they are linearly independent.

So, $B = \{1, 2^{1/3}, 2^{2/3}\}$ (for $n \geq 3$).

Hence dimension of 'K' is 3. ($n \geq 3$).

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18. In each case, give an example of a function f, continuous on S and such that $f(S) = T$, or else explain why there can be no such f:

(a) $S = (0,1)$, $T = [0,1]$.

(b) $S = \mathbb{R}$, $T = \{$the set of rational numbers$\}$.

(c) $S = (0,1)$, $T = \mathbb{R}$.

(d) $S = [0,1]$, $T = (0,1)$.

Solution:- (a) $(0,1) \xrightarrow{f} (0,1]$

$(0,\frac{1}{2}]\,[\frac{1}{2},1)$

Map $(0,\frac{1}{2}]$ to $(0,1]$

$$f(x) = \begin{cases} 2x, & x \in (0,\frac{1}{2}] \\ 1, & x \in [\frac{1}{2},1) \end{cases}$$

(b) $\mathbb{R} \xrightarrow{f} \{$set of rational number$\}$

If possible let assume such f exists.

Using Intermediate theorem, $x_1, x_2 \in \mathbb{R}$

'f' takes all values between $f(x_1)$ & $f(x_2)$.

(c) $(0,1) \xrightarrow{f} \mathbb{R}$

$$f(x) = \tan\left[\pi\left(x - \tfrac{1}{2}\right)\right]$$

(d) $(0,1] \xrightarrow{f} (0,1)$

$$f(x) = \begin{cases} x, & x \in (0,1) \\ 1, & x = 1 \\ 0, & x = 0 \end{cases}$$

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19. Prove that a group of order 15 is cyclic.

Solution:- Using Sylow's theorem, we show that there are subgroups H and K of order 5 and 3 respectively (cyclic of course).

Due to $15 = 3 \times 5$.

$\mathbb{Z}/15 \cong \mathbb{Z}/3 \times \mathbb{Z}/5$ and since they are the only abelian groups of order 15, we can see that every abelian group of order 15 is cyclic.

Furthermore, since 3 does not divide $(5-1) = 4$, we can conclude that any group of order 15 is abelian and therefore cyclic.

20. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear tranformation. Show that there is a line L such that $T(L) = L$.

Solution:- We know that space of all tranformations from $\mathbb{R}^3$ to $\mathbb{R}^3$

$$L(\mathbb{R}^3, \mathbb{R}^3) \cong M_{3\times 3}.$$

Every transformation 'T' can be represented as a matrix M_T.

Every 3×3 matrix M_T has a corresponding ch. equation of deg. 3, hence has a real root $\Rightarrow$ at least one real eigenvalue α'

$\Rightarrow$ at least one eigenvector corresponding to α, $\underset{\sim}{x}$.

$$A\underset{\sim}{x} = \alpha \underset{\sim}{x}$$

So, if line $L = \alpha \underset{\sim}{x}$ is considered, then $T(L) = L$.

If $\alpha = 0$, then $T(L) \sqsubset L$.

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21. Consider a circle which is tangent to the y-axis at 0. Show that the slope at any point (x,y) satisfies

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}.$$

Solution:- General equation of such circle would be,

$$(x-r)^2 + y^2 = r^2 \quad \text{——— (*)}$$

$$\Rightarrow x^2 + y^2 - 2xr = 0 \quad \text{——— ①}$$

Differentiating (*), $2(x-r) + 2yy' = 0$

$$y' = \frac{r - x}{2y} \quad \text{——— ②}$$

Eliminating 'r' from ① & ② gives $y' = \frac{y^2 - x^2}{2xy}$.

22. Let G be a group which has only finitely many subgroups. Prove that G must be a finite group.

Solution:- Let G be infinite inspite of having finitely many subgroups.

Case I:- If G contains infinite gp's of finite order then done.

Case II:- Else G contains at least one element of infinite order, say 'a'.

$$\langle a \rangle = \{e, a, a^2, a^3, \ldots\}$$

This is an infinite cyclic group and hence is isomorphic to $(\mathbb{Z}, +)$

As $(\mathbb{Z}, +)$ contains infinite subgroups so does $\langle a \rangle$.

Then G contains infinite subgroups.

By contradiction, G is finite.

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23. Show that the fields $Q(\sqrt{2})$ and $Q(\sqrt{3})$ are isomorphic as Q-vector spaces but not as fields.

Solution:- All linear transformation between vector spaces are homomorphism. So, we need to define a linear tramformation b/w

$T : Q(\sqrt{2}) \to Q(\sqrt{3})$ and to prove it as bijective.

$$T : Q(\sqrt{2}) \to Q(\sqrt{3})$$
$$(1, \sqrt{2}) \longmapsto (1, \sqrt{3})$$

i.e., $T(1) = 1$, $T(\sqrt{2}) = \sqrt{3}$.

Claim: T is a bijective.

$$\ker(T) = T(a + b\sqrt{2}) = 0$$
$$= T(a \cdot 1) + T(b\sqrt{2}) = 0$$
$$= aT(1) + bT(\sqrt{2})$$
$$= a + b\sqrt{3} = 0$$

$\Rightarrow a = 0; \quad b = 0.$

So, $\ker(T) = \{0\}$

So, T is one-to-one.

Onto: Say $a + b\sqrt{3} \in Q(\sqrt{3})$

then $\exists \; a + b\sqrt{2} \in Q(\sqrt{2})$

s.t. $T(a + b\sqrt{2}) = T(a) + T(b\sqrt{2})$
$$= aT(1) + bT(\sqrt{2})$$
$$= a + b\sqrt{3}.$$

$\Rightarrow T$ is onto.

Then $Q(\sqrt{2})$ and $Q(\sqrt{3})$ are isomorphic Q vector spaces.

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24. For any irrational number α such that $\alpha^2 \in \mathbb{N}$, we define $\mathbb{Q}(\alpha) := \{a + b\alpha : a, b \in \mathbb{Q}\}$. Show that $\mathbb{Q}(\alpha)$ is a field.

Solution:- $\mathbb{Q}$ is a field $(\mathbb{Q}, +, \cdot)$.

Consider $\mathbb{Q}[x]$ → ring of polynomials with co-efficient from $\mathbb{Q}$.

Take the polynomial $x^2 - \alpha^2$; $\alpha^2 \in \mathbb{N}$ & α is an irrational

This polynomial has no roots in $\mathbb{Q}$.

Hence $\langle x^2 - \alpha^2 \rangle$ is irreducible in $\mathbb{Q}[x]$.

So, $\langle x^2 - \alpha^2 \rangle$ is a maximal ideal in $\mathbb{Q}[x]$.

Then $\mathbb{Q}[x]/\langle x^2 - \alpha^2 \rangle$ is a field.

$$\mathbb{Q}[x]/\langle x^2 - \alpha^2 \rangle = \{ ax + b + \langle x^2 - \alpha^2 \rangle \; ; \; a, b \in \mathbb{Q} \}$$

$$\phi : \mathbb{Q}[x]/\langle x^2 - \alpha^2 \rangle \longrightarrow \mathbb{Q}(\alpha)$$

$$\overline{ax + b} \longmapsto a + b\alpha$$

Easy to see that ϕ is an isomorphism, then $\mathbb{Q}(\alpha)$ is a field.

25. Write the class equation for a finite group. Use it to show that every nonabelian group of order 39 has a subgroup of order 13.

Solution:- The class equation says that $|G| = |Z| + \sum |C_i|$, where Z is the center of G and C_i are the nontrivial conjugcy classes. Thus $39 = |Z| + \sum |C_i|$. Suppose there is no subgroup of order 13. The centralizer of each element not in the center is a nontrivial proper subgroup of G, which must have order 3, hence index 13.

Hence $|C_i| = 13$ for every i. Then $39 = |Z| + n13$ for some n, hence $|Z|$ is divisible by 13. Since G is not abelian, Z is a proper subgroup of order divisible by 13, hence has order 13, a contradiction.

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26. Let K be a field and $K[x,y]$ denote the polynomial ring in the two variables x and y with coefficients from K. Prove that for any $a, b \in K$ the ideal generated by linear polynomials $x-a$ and $y-b$ is a maximal ideal of $K[x,y]$.

Solution:- Suppose $I = \langle x-a, y-b \rangle$ is not the maximal ideal. Then $\exists$ an ideal I' $\ni$ $I \subset I' \subseteq K[x,y]$. We will show $I' = K[x,y]$. Suppose I' contains I properly. $\exists$ an element $P(x,y)$ (a polynomial in x,y with coefficients from K), which is not in I but in I'.

Say $P(x,y) = \sum A_{m,n} x^m y^n$; $A_{m,n} \in K$.

Since $K[x,y]$ is a ring of polynomials over K which is a field, then $\exists$ unique $Q(x,y)$ $\ni$ $P(x,y) = Q(x,y)(x-a) + G(y)$ where $G(y)$ is a polynomial in y over K.

As $x-a$ is linear it will not leave any power of x in the remainder expression.

Again $G(y) = R(y)(y-b) + T$ where T is in K.

As $(y-b)$ is linear, the remainder won't have any power of y.

Therefore, $P(x,y) = Q(x,y)(x-a) + R(y)(y-b) + T$ in I'

$$\Rightarrow T = P(x,y) - (Q(x,y)(x-a) + R(y)(y-b)) \in I'$$

But T is a constant polynomial in $K[x,y]$.

Hence $T \times T^{-1} = 1 \in I'$.

Since I' contains multiplicative identity element of field K, it will absorb all polynomials from $K[x,y]$, implying $I' = K(x,y)$ [Proved]

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27. Show that every isometry of a compact metric space into itself is onto.

Solution:- Let X be a compact metric space and f be an isometric map from X to X. Then we are to show f is a surjective map.

Suppose $\exists\ x \in X \setminus f(X)$. Then x has positive distance d from the compact set $f(X)$. Now consider the recursively defined sequence

$$x_0 := x, \quad x_n := f(x_{n-1}) \ \forall\ n > 0.$$

We have $d(x_0, x_n) \geq d \ \forall\ n > 0$, by assumption on x.
This implies that we also have $d(x_k, x_{k+n}) = d(x_0, x_n) \geq d$ $\forall\ k$ and $n > 0$ (Here we use that f is an isometry).
Therefor $d(x_n, x_m) \geq d \ \forall\ m \neq n$, which is contradiction to sequential compactnes of X.

28. A function f is said to be periodic with period p if $f(x+p) = f(x)$ for all real x. Prove that if $f(x)$ is continuous and periodic then it is uniformly continuous on the reals.

Solution:- first, since it is continuous on the closed interval $[0,p]$ it is uniformly continuous on $[0,p]$:

$$x, y \text{ in } [0,p], \quad |x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

Second, if x and y are numbers with $|x-y| < \delta$ then there is an integer j such that $x + jp$ and $y + jp$ are in $[0,p]$, and

$$|x - y| < \delta \Rightarrow |(x+jp) - (y+jp)| < \delta$$
$$\Rightarrow |f(x) - f(y)| = |f(x+jp) - f(y+jp)| < \epsilon$$

i.e. $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$

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29. Show that if $a_n = \sqrt{n^2+2} - n$, then $a_n \to 0$ as $n \to \infty$.

Solution:- $0 < a_n = \sqrt{n^2+2} - n = \dfrac{2}{\sqrt{n^2+2}+n} \leq \dfrac{1}{n} \quad \forall\ n \in \mathbb{N}$

We know $\lim\limits_{n\to\infty} \frac{1}{n} = 0$. So, by Sandwich theorem, $\lim\limits_{n\to\infty} a_n = 0$.

30. If $a_{n+2} = (a_{n+1} + a_n)/2$ for all $n \geq 1$, then show that $a_n \to (a_1 + 2a_2)/3$.

Solution:- Choose $b_n = a_{n+1} - a_n$

So, $b_{n+1} = -\dfrac{b_n}{2}$

We have $b_{n+1} = \left(-\frac{1}{2}\right)^n b_1 \longrightarrow 0$ as $n \to \infty$ ----------(*)

[Given $a_{n+2} = \dfrac{a_{n+1}+a_n}{2}$

$\Rightarrow a_{n+2} - a_{n+1} = \frac{1}{2}(a_n - a_{n+1})$

Consider, $a_{n+2} - a_2 = \sum_{k=2}^{n+1} b_k = -\frac{1}{2}\sum_{k=1}^{n} b_k = \left(\frac{-1}{2}\right)(a_{n+1} - a_1)$

$\Rightarrow b_n + \left(\dfrac{3a_{n+1}}{2}\right) = \dfrac{a_1 + 2a_2}{2}$

So, we have $a_n \to (a_1+2a_2)/3$ by (*).

31. If $|a_n| < 2$ and $|a_{n+2} - a_{n+1}| \leq \frac{1}{8}|a_{n+1}^2 - a_n^2| \quad \forall\ n \geq 1$, prove that $\{a_n\}$ converges.

Solution:- Let $a_{n+1} - a_n = b_n$, then we have

$|b_{n+1}| \leq \frac{1}{8}|b_n||a_{n+1} + a_n| \leq \frac{1}{2}|b_n|$, since $|a_n| < 2 \ \forall\ n \geq 1$.

So, we have $|b_{n+1}| \leq \left(\frac{1}{2}\right)^n |b_1|$.

Let $m > n$, now consider —

$$\begin{aligned} |a_m - a_n| &= \left|(a_m - a_{m-1}) + (a_{m-1} - a_{m-2}) + \cdots + (a_{n+1} - a_n)\right| \\ &\leq |b_{m-1}| + \cdots + |b_n| \\ &\leq |b_1|\left[\left(\tfrac{1}{2}\right)^{m-2} + \cdots + \left(\tfrac{1}{2}\right)^{n-1}\right] \end{aligned}$$

Then $\{a_n\}$ is a Cauchy sequence since $\sum\left(\frac{1}{2}\right)^k$ converges. Hence, $\{a_n\}$ is a convergent sequence.

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32. If $0 < x_1 < 1$ and if $x_{n+1} = 1 - \sqrt{1-x_n}$ $\forall\ n \geq 1$, prove that $\{x_n\}$ is a decreasing sequence with limit 0. Prove that $x_{n+1}/x_n \to \frac{1}{2}$.

Solution:- We can see that $0 < x_{K+1} = 1 - \sqrt{1-x_n} < 1$; which implies $0 < x_n < 1$ $\forall\ n \in \mathbb{N}$.

$$x_{n+1} - x_n = (1-x_n) - \sqrt{1-x_n} = \frac{x_n(x_n-1)}{(1-x_n)^2 + (1-x_n)} < 0$$

since $0 < x_n < 1$.

So, $x_{n+1} < x_n$ $\forall\ n$.

$\Rightarrow \{x_n\}$ is a decreasing sequence and bounded also.

So, $\{x_n\}$ is a convergent sequence.

So, $\lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} \left(1 - \sqrt{1-x_n}\right)$

$\Rightarrow l = 1 - \sqrt{1-l}$

$\Rightarrow l(l-1) = 0$

Since $\{x_n\}$ is a decreasing sequence with $0 < x_n < 1$ $\forall\ n \in \mathbb{N}$, we finally have $l = 0$.

Now, $\lim\limits_{l\to 0} \dfrac{1-\sqrt{1-l}}{l} = \dfrac{1}{2}$

So, $\dfrac{x_{n+1}}{x_n} = \dfrac{1-\sqrt{1-x_n}}{x_n} \to \dfrac{1}{2}$

33 Show that the sequence $\lim\limits_{n\to+\infty} \dfrac{(2n)!!}{(2n+1)!!} = 0$.

Solution:- $f(n) = (2n)!$, $\dfrac{(2n)!!}{(2n+1)!!} = \dfrac{f(n)!}{(f(n)(2n+1))!}$

$$\leq \frac{1}{2n+1} \to 0.$$

[Let a and b be +ve integers satisfying $a \geq b > 1$, then we have $a!\,b \leq a!\,b! \leq (a+b)! \leq (ab)!$]

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34 Show that $a_n = \left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{[\sqrt{n}]}{n}\right) \to e^{1/2}$ as $n\to\infty$, where $[x]$ means Gauss symbol.

Solution:- Since $x - \frac{1}{2}x^2 \le \log(1+x) \le x \quad \forall\ x \in (-1,1)$

We have $$\sum_{k=1}^{[\sqrt{n}]} \frac{k}{n} - \frac{1}{2}\left(\frac{k}{n}\right)^2 \le \log a_n = \sum_{k=1}^{[\sqrt{n}]} \log\left(1+\frac{k}{n}\right) \le \sum_{k=1}^{[\sqrt{n}]} \frac{k}{n}$$

Consider $i^2 < n \le (i+1)^2$, then by Sandwich theorem, we know that $\lim_{n\to\infty} \log a_n = 1/2$, which implies that $a_n \to e^{1/2}$ as $n\to\infty$.

35 Show that $(n!)^{1/n} \ge \sqrt{n}$ for all $n \in \mathbb{N}$.

Solution:- We prove it by a special method following Gauss' method.

Consider $$n! = 1 \cdot \ldots \cdot k \cdot \ldots \cdot n$$
$$= n \cdot \ldots (n-k+1) \cdot \ldots 1$$

and thus let $f(k) := k(n-k+1)$, it is easy to show that $f(k) \ge f(1) = n$ for all $k = 1, 2, \ldots, n$.

So, we have proved that $(n!)^2 \ge n^n$

which implies that $(n!)^{1/n} \ge \sqrt{n}$.

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36

Given $f(x,y) = \frac{1}{x}\sin(xy)$ if $x \neq 0$, $f(0,y) = y$.

Show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Solution:-

(i) Since $(x \neq 0)$

$$\lim_{x\to 0} f(x,y) = \lim_{x\to 0} \frac{1}{x}\sin(xy) = y$$

So, $\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = 0$.

(ii) Since $(y \neq 0)$

$$\lim_{y\to 0} f(x,y) = \begin{cases} \lim_{y\to 0} \frac{1}{x}\sin(xy) = 0 & \text{if } x \neq 0, \\ \lim_{y\to 0} y = 0 & \text{if } x = 0, \end{cases}$$

we have $\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] = 0$.

(iii) Case: $(x,y) \neq (0,0)$.

Let $x = r\cos\theta$, $y = r\sin\theta$, where $0 \leq \theta < 2\pi$, and note that $(x,y) \to (0,0) \Leftrightarrow r \to 0$. Then

$$f(x,y) = \begin{cases} \frac{1}{r\cos\theta}\sin(r^2\cos\theta\sin\theta) & \text{if } x = r\cos\theta \neq 0, \\ r\sin\theta & \text{if } x = r\cos\theta = 0. \end{cases}$$

$$\to \begin{cases} 0 & \text{if } r \to 0, \\ 0 & \text{if } r \to 0. \end{cases}$$

So, we have $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

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37 If $x \in [0,1]$. Prove that the following limit exists,

$$\lim_{m\to\infty}\left[\lim_{n\to\infty}\cos^{2n}(m!\,\pi x)\right],$$

and that its value is 0 or 1, according to whether x is irrational or rational.

Solution:- If x is rational, say $x=\frac{q}{p}$, where g.c.d. $(q,p)=1$, then $p!\,x \in N$. So, $\lim_{n\to\infty}\cos^{2n}(m!\,\pi x) = \begin{cases} 1 & \text{if } m \geq p, \\ 0 & \text{if } m < p. \end{cases}$

Hence, $\lim_{m\to\infty}\left[\lim_{n\to\infty}\cos^{2n}(m!\,\pi x)\right] = 1.$

If x is irrational, then $m!\,x \notin N \;\forall\; m \in N$.
So, $\cos^{2n}(m!\,\pi x) < 1$ for all irrational x.

Hence, $\lim_{n\to\infty}\cos^{2n}(m!\,\pi x) = 0 \Rightarrow \lim_{m\to\infty}\left[\lim_{n\to\infty}\cos^{2n}(m!\,\pi x)\right] = 0.$

38 Let f be continuous on $[a,b]$ and let $f(x)=0$ when x is rational. Prove that $f(x)=0$ for every $x \in [a,b]$.

Solution:-

Given any irrational number x in $[a,b]$, and thus choose a sequence $\{x_n\} \subseteq \mathbb{Q} \ni x_n \to x$ as $n \to \infty$.
Note that $f(x_n)=0 \;\forall\; n$. Hence,

$$\begin{aligned} 0 &= \lim_{n\to\infty} 0 \\ &= \lim_{n\to\infty} f(x_n) \\ &= f\left(\lim_{n\to\infty} x_n\right) \quad \text{by continuity of } f \text{ at } x \\ &= f(x). \end{aligned}$$

Since x is arbitrary, we have shown $f(x)=0 \;\forall\; x \in [a,b]$.
That is, f is constant 0.

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39 Define $f(x) = e^{-1/x^2}$ if $x \neq 0$, $f(0) = 0$. Show that f is continuous for all x.

Solution:-

Since $\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(\frac{1}{e}\right)^{1/x^2} = 0 = f(0)$,

we know that f is continuous at 0.

40 Given a function f defined and having a finite derivative f' in the interval $0 < x \leq 1$ and $\exists\ |f'(x)| < 1$. Define $a_n = f(1/n)$ for $n = 1, 2, 3, \ldots$, and show that $\lim_{n \to \infty} a_n$ exists.

Solution:- Consider $n \geq m$, and by Mean Value theorem,

$$|a_n - a_m| = |f(1/n) - f(1/m)| = |f'(p)| \left|\frac{1}{n} - \frac{1}{m}\right|$$
$$\leq \left|\frac{1}{n} - \frac{1}{m}\right|$$

then $\{a_n\}$ is a Cauchy sequence. Hence, since $\{1/n\}$ is a Cauchy sequence we know that $\lim_{n \to \infty} a_n$ exists.

41 Assume f has a finite derivative in (a, b) and is continuous on $[a, b]$ with $f(a) = f(b) = 0$. Prove that for every real λ, there is some c in (a, b) $\exists$ $f'(c) = \lambda f(c)$.

Solution:- Consider $g(x) = f(x)e^{-\lambda x}$, then by Rolle's theorem,

$$g(a) - g(b) = g'(c)(a - b), \text{ where } c \in (a, b)$$
$$= 0$$

which implies that $f'(c) = \lambda f(c)$.

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42 Show that $e^{\pi} > \pi^e$.

Solution:- Since $\log x$ is strictly increasing on $(0, \infty)$, in order to show $e^{\pi} > \pi^e$, it suffices to show that

$$\pi \log e = \log e^{\pi} > \log \pi^e = e \log \pi.$$

which implies that $\frac{\log e}{e} > \frac{\log \pi}{\pi}$.

Consider $f(x) = \frac{\log x}{x} : [e, \infty)$, we have

$$f'(x) = \frac{1 - \log x}{x^2} < 0 \text{ where } x \in (e, \infty).$$

So, we know that $f(x)$ is strictly decreasing on $[e, \infty)$. Hence

$$\frac{\log e}{e} > \frac{\log \pi}{\pi}; \text{ i.e., } \pi^e < e^{\pi}.$$

43 Show that $e^x \geq 1 + x \ \forall \ x \in \mathbb{R}$.

Solution:- By Taylor's remainder theorem,

$$e^x = 1 + x + \frac{e^c}{2} x^2, \text{ for some } c.$$

So, $e^x \geq 1 + x \ \forall \ x \in \mathbb{R}$.

44 Does $\int_0^{\infty} \sin(t^2)\, dt$ converge?

Solution:-

$$\left| \int_x^{x'} \sin^2 t \, dt \right| = \left| \frac{1}{2} \int_x^{x'} \frac{\sin u}{\sqrt{u}} \, du \right|$$

$$= \left| \frac{1}{2} \left[\frac{1}{x} \int_x^y \sin u \, du + \frac{1}{x'} \int_y^{x'} \sin u \, du \right] \right|$$

[Using MVT 2]

$$= \frac{1}{2} \left[\frac{2}{x} + \frac{2}{x'} \right]$$

$$< \frac{2}{x}$$

which implies the integral exists. Ans. is YES.

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45 If $a_1 = 2, a_2 = 8$, $a_{2n+1} = \frac{1}{2}(a_{2n} + a_{2n-1})$, $a_{2n+2} = \frac{a_{2n}a_{2n-1}}{a_{2n+1}}$ then show that $L = 4$, where $L = \lim_{n\to\infty} a_{2n} / \lim_{n\to\infty} a_{2n-1}$.

Solution:-

$$a_{2n+1} = \frac{a_{2n} + a_{2n-1}}{2} \geq \sqrt{a_{2n}a_{2n-1}} \quad [\because AM \geq GM]$$

$$a_{2n+2} = \frac{a_{2n}a_{2n-1}}{a_{2n+1}} \leq \sqrt{a_{2n}a_{2n-1}} \leq a_{2n+1} \quad \forall\, n \in \mathbb{N}$$

By Mathematical induction, it is easy to show that

$$a_4 \leq a_6 \leq \cdots \leq a_{2n+2} \leq \cdots \leq a_{2n+1} \leq \cdots \leq a_5 \leq a_3$$
$$\forall\, n \in \mathbb{N}.$$

It implies that both of $\{a_{2n}\}$ and $\{a_{2n-1}\}$ converge, say

$$\lim_{n\to\infty} a_{2n} = x \quad \text{and} \quad \lim_{n\to\infty} a_{2n-1} = y.$$

With help of $a_{2n+1} = \frac{1}{2}(a_{2n} + a_{2n-1})$, we know that $x = y$.
In addition, by $a_{2n+2} = \frac{a_{2n}a_{2n-1}}{a_{2n+1}}$, $a_1 = 2$, and $a_2 = 8$,
we know that $x = 4$.

46 Let $a_1 = \frac{-3}{2}$, $3a_{n+1} = 2 + a_n^3$, $L = 1$. Modify a_1 to make $L = -2$.

Solution:- By Mathematical Induction, it is easy to show that

$$-2 \leq a_n \leq 1 \quad \forall\, n.$$

So, $3(a_{n+1} - a_n) = a_n^3 - 3a_n + 2 \geq 0$

Hence, $\{a_n\}$ is an increasing sequence with a upper bound 1.
So, $\{a_n\}$ is a convergent sequence with limit L.
So, by $3a_{n+1} = 2 + a_n^3$,

$$L^3 - 3L + 2 = 0$$

$$\Rightarrow L = 1 \text{ or } -2.$$

So, $L = 1$ since $a_n \uparrow$ and $a_1 = -\frac{3}{2}$.
In order to make $L = -2$, it suffices to let $a_1 = -2$,
then $a_n = -2 \quad \forall\, n$.

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47 Let $a_1 = 3$, $a_{n+1} = \dfrac{3(1+a_n)}{3+a_n}$, then show that $L = \sqrt{3}$.

Solution:- By Mathematical Induction, it is easy to show that

$$a_n \geq \sqrt{3} \quad \forall\, n. \quad \cdots\cdots\cdots (*)$$

So, $$a_{n+1} - a_n = \frac{3 - a_n^2}{3 + a_n^2} \leq 0$$

which implies that $\{a_n\}$ is a decreasing sequence. So, $\{a_n\}$ is a convergent sequence with limit L by $(*)$.

Hence, $$L = \frac{3(1+L)}{3+L}$$

which implies that $L = \pm\sqrt{3}$.

So, $L = \sqrt{3}$ since $a_n \geq \sqrt{3}\ \forall\, n$.

48 Let $a_n = \dfrac{b_{n+1}}{b_n}$, where $b_1 = b_2 = 1$, $b_{n+2} = b_n + b_{n+1}$, show that $L = \dfrac{1+\sqrt{5}}{2}$.

Solution:- By Mathematical induction, it is easy to show that

$$b_{n+2}\, b_n - b_{n+1}^2 = (-1)^{n+1} \quad \forall\, n \quad \text{and} \quad b_n \geq n \text{ if } n > 4.$$

Thus, (Note that $b_n \neq 0\ \forall\, n$)

$$|a_{n+1} - a_n| = \left|\frac{b_{n+2}}{b_{n+1}} - \frac{b_{n+1}}{b_n}\right| = \left|\frac{(-1)^{n+1}}{b_n\, b_{n+1}}\right| \leq \frac{1}{n(n+1)} < \frac{1}{n^2} \quad \text{if } n > 4.$$

So, $\{a_n\}$ is a Cauchy sequence.

In other words, $\{a_n\}$ is a convergent sequence, say $\lim_{n\to\infty} b_n = L$.

Then by $b_{n+2} = b_n + b_{n+1}$, we have

$$\frac{b_{n+2}}{b_{n+1}} = \frac{b_n}{b_{n+1}} + 1$$

which implies that $$L = \frac{1}{L} + 1.$$

$$\Rightarrow L = \frac{1 \pm \sqrt{5}}{2}.$$

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49. Let A be $n \times n$

(i) If $A^2 = 0$, prove that A is singular.

(ii) If $A^2 = A$, and $A \neq I_n$, prove that A is singular.

Solution:- (i) $A^2 = 0$

If A^{-1} exists, then deduce that $A^{-1}(AA) = A^{-1}0$.

$\Rightarrow A = 0$.

This is a contradiction, as the zero matrix is singular.

We conclude that A doesn't have an inverse.

(ii) $A^2 = A$ & A^{-1} exists. Then

$$A^{-1}AA = A^{-1}A,$$

which gives $A = I_n$.

Here also, A does not have an inverse.

50. Let $A \in M_{n\times n}(F)$. If $A^t = -A$, prove that $\det A = 0$ if n is odd and $1+1 \neq 0$ in F.

Solution:- $A^t = -A$, $A \in M_{n\times n}(F)$, where n is odd.

$$|A^t| = |-A|$$

$$|A| = (-1)^n |A| = -|A|$$

Hence $(1+1)\det(A) = 0$ and consequently $|A| = 0$ if $1+1 \neq 0$ in F.

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51. Let f be a bounded twice differentiable function (real valued) on $\mathbb{R} \ni f''(x) \geq 0 \ \hat{\forall} x$. Then show that f is constant.

Solution:- Given that $f'' > 0$, f is convex.
This implies f lies weakly above its tangent line at any point.
Formally, for any $x_0 \in \mathbb{R}$,

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0) \quad \text{---------} (*)$$

Now, if $f'(x_0) > 0$, then RHS of $(*)$ goes to ∞ as $x \to \infty$, which contradicts f being bounded.
Similarly, if $f'(x_0) < 0$, then RHS of $(*)$ goes to $+\infty$ as $x \to -\infty$, which implies $\lim_{x \to -\infty} f(x) = \infty$, another contradiction.
It follows that $f'(x_0) = 0$.
Since this is true for any $x_0 \in \mathbb{R}$, so that gives f is constant.

52. Let $f: [a,b] \to \mathbb{R}$ be twice differentiable and assume $f(a) = f(b) = 0$ and $f''(x) < 0 \ \forall x$ in (a,b). Show that $f(x) > 0$ on $[a,b]$.

Solution:- Assume $f(x) \leq 0$ for some $x \in (a,b)$.
Then for some $\xi \in (a,x)$ we have $f'(\xi) = \dfrac{f(x) - f(a)}{x - a} \leq 0$
and for some $\eta \in (a,b)$ we have $f'(\eta) = \dfrac{f(b) - f(x)}{b - x} \geq 0$.

Then for some $\theta \in (\xi, \eta)$, we have

$$f''(\theta) = \frac{f'(\eta) - f'(\xi)}{\eta - \xi} > 0, \text{ contradiction.}$$

Therefore $f(x) > 0 \ \forall x \in (a,b)$.

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53. Consider the set consisting of all cauchy sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \mathbb{N}$ $\forall$ n. Is the set S countable?

Solution:-

Let S be the set of Cauchy sequences of natural numbers. Any such sequence is eventually constant. (Why?)

Hence, we write $S = \bigcup_{n=1}^{\infty} S_n$

where S_n is the set of sequences $\{a_k\}$ of natural numbers $\ni$ $a_k = a_n$ $\forall$ $k \geq n$.

Now, there is a bijection $\mathbb{N}^n \to S_n$, hence S_n is countable.

So, S is a countable union of countable sets, hence is countable.

54. Let X be a metric space. If $A, B \in X$ are disjoint, if A is compact, and if B is closed, then $\exists$ $\delta > 0 : |\alpha - \beta| \geq \delta$ $\forall$ $\alpha \in A$, $\beta \in B$.

Solution:- Assume the contrary. Let $\alpha_n \in A$, $\beta_n \in B$ be chosen such that $|\alpha_n - \beta_n| \to 0$ as $n \to \infty$.

Since A is compact, $\exists$ a convergent subsequence of α_n $(n \in \mathbb{N})$ and α_{n_m} $(m \in \mathbb{N})$, which converges to $\alpha \in A$.

We have $|\alpha - \beta_{n_m}| \leq |\alpha - \alpha_{n_m}| + |\alpha_{n_m} - \beta_{n_m}| \to 0$ as $m \to \infty$.

Hence α is a limit point of B and since B is closed, $\alpha \in B$, contradiction.

55. Prove that $\exists$ a constant $c > 0$ $\ni$ for all $x \in [1, \infty)$,

$$\sum_{n \geq x} \frac{1}{n^2} \leq \frac{c}{x}.$$

Solution:-

$$\frac{1}{n^2} \leq \frac{1}{n^2 - n} = \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

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56 Let $f: G \to G'$ be a surjective homomorphism. Prove that if G is cyclic, then G' is also cyclic.

Solution:- Suppose that $G = \langle x \rangle$, let $y = f(x)$ and $h \in H$. Since f is onto, $\exists$ a $g \in G$ with $f(g) = h$. Since G is cyclic, $\exists$ an integer n with $x^n = g$.

But then $h = f(g) = f(x^n) = f(x)^n = y^n$

So, $h \in \langle y \rangle$. As h was arbitrary, we see that $H = \langle y \rangle$ as desired. Hence the proof.

57. Let $i = \sqrt{-1}$ and $\alpha = i + \sqrt{2}$. Construct a polynomial $f(x)$ with integer coefficients such that $f(\alpha) = 0$.

Solution:- $f(\alpha) = 0$, where $\alpha = i + \sqrt{2}$.

$f(x)$ is a polynomial with integer coefficients.

$\Rightarrow f(x) = ax^4 + bx^2 + c$

So, $f(\alpha) = a\alpha^4 + b\alpha^2 + c = a(4\sqrt{2}i - 7) + b(2\sqrt{2}i + 1) + c = 0$

implying $a = 1, b = 2, c = 5$.

So, $f(x) = x^4 + 2x^2 + 5$.

58. Find a third degree polynomial with real coefficients so that $P(1-i) = 2+i$ and $P(i) = 1-2i$.

Solution:- $P(x) = ax^3 + bx^2 + cx + d$; $a, b, c, d \in \mathbb{R}$

$$\begin{cases} (1-i)^2 = 1 - 2i + i^2 = -2i \\ (1-i)^3 = -2i + 2i^2 = -2i - 2 \end{cases}$$

$$\begin{cases} i^2 = -1 \\ i^3 = -i \end{cases}$$

Then by substitution we will get four equations with four unknowns;

$$d - b = 1 \quad \text{---} \quad (1)$$
$$c - a = -2 \quad \text{---} \quad (2)$$
$$c + d - 2a = 2 \quad \text{---} \quad (3)$$
$$-2a - 2b - c = 1 \quad \text{---} \quad (4)$$

Solving these we will get the answer.

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59. Let A be a 2×2 matrix with real entries $\ni$ $Tr(A)=0$ and $|A|=-1$. Is there a basis of $\mathbb{R}^2$ containing the eigenvectors of A.

Solution:- Let $x, y \neq 0$ be eigenvectors of A with eigenvalues λ and μ. Let x and y be linearly dependent, meaning that $x = \alpha y$ for some $\alpha \in \mathbb{R}$.

WLG, we can assume $\lambda \neq 0$.

Then $x = \frac{1}{\lambda}(\lambda x) = \frac{1}{\lambda}Ax = \frac{1}{\lambda}A(\alpha y) = \frac{\alpha}{\lambda}Ay = \frac{\mu}{\lambda}x$,

$= \frac{\alpha}{\lambda}\mu y \quad [\because x = \frac{\mu}{\lambda}x]$

$\Rightarrow \lambda = \mu$, a contradiction.

So, there is a basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

60. Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear tranformation $\ni T^2 = 0$. If r denotes the rank of T (i.e. $r = \dim(\text{Image}(T))$), then show that $r \leq \frac{n}{2}$.

Solution:-

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61. Let V be a finite dimensional vector space over $\mathbb{R}$ and $T: V \to V$ be a linear transformation with dimension of Range of $T = k$. Show that T can have at most $k+1$ distinct eigenvalues.

Solution:- If T is not injective, then $\lambda_0 = 0$ is an eigenvalue. If there are at least $k+1$ other distinct eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$, then we can find eigenvectors $v_1, v_2, \ldots, v_{k+1}$ $\ni T(v_i) = \lambda v_i$ for $i \le k+1$. We know $v_1, \ldots, v_{k+1}$ are LIN. However,

$$A = \{\lambda_1 v_1, \ldots, \lambda_{k+1} v_{k+1}\} \subset \text{range}(T).$$ So, A is also LIN.

However, no set of $(k+1)$ vectors in range(T) can be linearly independent since $\dim(\text{range}(T)) = k$.
Thus we can have at most $k+1$ distinct eigenvalues of T.

62. Let f is differentiable everywhere on (a,b), and that there is a number M with $|f'(x)| < M \ \forall \ x \in (a,b)$. Prove that f is uniformly continuous on (a,b).

Solution:- Let x, y be two number in (a,b). By Mean Value Theorem (MVT), there is a z between x and y $\ni$

$$f(x) - f(y) = f'(z)(x-y)$$

Thus, $|f(x) - f(y)| = |f'(z)||x-y| \le M|x-y|$.

Let $\epsilon > 0$ be given. Let $\delta < \frac{\epsilon}{M}$. For all $x, y \in (a,b) \ni$ $|x-y| < \delta$ we have $|f(x) - f(y)| < M \cdot \frac{\epsilon}{M} = \epsilon$.
This shows that f is uniformly continuous on (a,b).

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63. Let C be a subset of a compact metric space (X, d). Assume that, for every continuous function $h : X \to \mathbb{R}$, the restriction of h to C attains a maximum on C. Prove that C is compact.

Solution:- If C is finite, then C is compact.

So, we may assume the case C is infinite.

Since C is an infinite subset of the compact space X, it has a limit point in X, say q. Consider the function $h : X \to \mathbb{R}$ defined by $h(x) = \frac{1}{d(x,q)+1}$.

This function attains its maximum at $d(x,q) = 0$ (i.e., $x = q$)

Since q is a limit point of C, there is a sequence $\{x_n\}_{n=0}^{\infty}$ of points in C which converge to q. Since h, when restricted to C, must obtain its maximum by assumption, we have that $q \in C$.

64. If $f(x,y) = \sqrt[3]{x^3+y^3}$, then is f differentiable at $(0,0)$?

Solution:-

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65. Let R be a commutative ring, and let P be a prime ideal of R. Suppose that P has no nontrivial zero divisors in it. Show that R is an integral domain.

Solution:- If $ab = 0$, then from $0 \in P$ we get $a \in P$ or $b \in P$. Since P doesn't contain non-zero divisors so we get $a = 0$ or $b = 0$.

66. If $\{a_n\}$ is a sequence in $(0,1)$, then show that $\frac{1}{n}\sum_{k=1}^{n} a_k \to 0$ if and only if $\frac{1}{n}\sum_{k=1}^{n} a_k^2 \to 0$.

Solution:-

67. Prove that the largest possible number of 1's in $n \times n$ invertible matrix with all entries 0 or 1 is $n^2 - n + 1$.

Solution:-

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68. Show that the constant function $f(x)=1$ on $[0,1]$ is Riemann integrable and $\int_0^1 1\,dx = 1$.

Solution:- Let $P = \{I_1, I_2, \ldots, I_n\}$ be any partition of $[0,1]$ with end points $\{0, x_1, x_2, \ldots, x_{n-1}, 1\}$.

Since f is constant, $M_k = \sup_{I_k} f = 1$, $m_k = \inf_{I_k} f = 1$ $\forall$ $k = 1, 2, \ldots, n$.

Therefore, $U(f;P) = L(f;P)$

$$= \sum_{k=1}^{n} (x_k - x_{k-1}) = x_n - x_0 = 1.$$

$U(f) = \inf_{P \in \Pi} U(f;P) = \inf\{1\} = 1$, $L(f) = \sup\{1\} = 1$ are equal and the integral of f is 1.

69. Show that $\int_0^\infty \left|\frac{\sin x}{x}\right| dx = \infty$.

Solution:- Let $N \in \mathbb{N}$, $N > 1$, we have

$$\int_0^{2\pi N} \left|\frac{\sin x}{x}\right| dx = \sum_{n=0}^{N-1} \int_{2\pi n}^{2\pi(n+1)} \left|\frac{\sin x}{x}\right| dx$$

$$\geq \sum_{n=0}^{N-1} \frac{1}{2\pi(n+1)} \int_{2\pi n}^{2\pi(n+1)} |\sin x|\, dx$$

$$= \sum_{n=0}^{N-1} \frac{1}{2\pi(n+1)} \int_0^{2\pi} |\sin x|\, dx$$

$$= \sum_{n=0}^{N-1} \frac{2}{\pi(n+1)}$$

As $N \to \infty$, this sum diverges, and so does the original integral.

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70. Consider an $n \times n$ matrix $A = (a_{ij})$ with $a_{12} = 1$, $a_{ij} = 0$ $\forall\ (i,j) \neq (1,2)$. Prove that there is no invertible matrix P $\ni$ PAP^{-1} is a diagonal matrix.

Solution:- If there's some invertible $P \ni$

$$P^{-1}AP = D \text{ is diagonal then}$$

$$PA^2P^{-1} = D^2$$

$$\Rightarrow P0P^{-1} = D^2$$

$$\Rightarrow D = 0 \quad [\text{Because } D \text{ is diagonal}]$$

So, $A = 0$

a contradiction.

71. Suppose that f is uniformly continuous on a set $S \subset \mathbb{R}$ and $\{x_n\}$ is a Cauchy sequence in S. Prove that $\{f(x_n)\}$ is a Cauchy sequence.

Solution:- Let $\epsilon > 0$. Since f is uniformly continuous on S,

$\exists\ \delta > 0 \ni |x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon\ \forall\ x, y \in S$.

For this $\delta > 0\ \exists\ N \in \mathbb{Z}^+ \ni$ for $n, m > N$,

$|x_n - x_m| < \delta$.

By above it follows that for $n, m > N$,

$|f(x_n) - f(x_m)| < \epsilon$. Hence $\{f(x_n)\}$ is cauchy.

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72. Let $A = (a_{ij})$ be an $n \times n$ matrix $\ni$ $a_{ij} = 0$ whenever $i \geq j$. Prove that A^n is the zero matrix.

Solution:- Eigen values of A is $0 := \lambda$.

$\lambda^n = 0 \Rightarrow A^n$ is a zero matrix.

73. Determine the integers n for which $\mathbb{Z}_n$, the set of integers modulo n, contains elements x, y so that

$x + y = 2, \quad 2x - 3y = 3.$

Solution:-

$$\exists\, x, y : \begin{matrix} 2x - 3y = 3 \\ x + y = 2 \end{matrix}$$

$$\Leftrightarrow \exists\, x, y : \begin{matrix} 5y = 1 \\ x + y = 2 \end{matrix}$$

$$\Leftrightarrow \exists\ y : 5y = 1$$

$$\Leftrightarrow \gcd(5, n) = 1.$$

74. Let a_1, b_1 be arbitrary positive real numbers. Define

$a_{n+1} = \frac{a_n + b_n}{2}$ and $b_{n+1} = \sqrt{a_n b_n} \quad \forall\ n \geq 1.$

Show that a_n and b_n converge to a common limit.

Solution:-

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75. Let $f: \mathbb{R} \to \mathbb{R}$ be differentiable and $|f'(x)| \leq K < 1$ for all $x \in \mathbb{R}$. Let $a_0 \in \mathbb{R}$. Define the numbers $a_n = f(a_{n-1})$.

(a) Show that $|a_{n+1} - a_n| \leq K^n |a_1 - a_0| \quad \forall\, n \in \mathbb{N}$

(b) Show that for all $m, n \in \mathbb{N} \ni m > n$ we have

$$|a_m - a_n| \leq \frac{K^n}{1-K}.$$

Solution:-

(a) By the MVT there is a $c_1 \in (a_1, a_0) \ni$ $f(a_1) - f(a_0) = f'(c_1)(a_1 - a_0)$. Thus $|a_2 - a_1| \leq K|a_1 - a_0|$ since $|f'(c_1)| \leq K$.

Likewise, we can write $|a_3 - a_2| \leq K|a_2 - a_1|$

$$\leq K^2 |a_1 - a_0|.$$

Now suppose that $|a_n - a_{n-1}| \leq K^n |a_1 - a_0|$.

Then $|a_{n+1} - a_n| \leq K|a_n - a_{n-1}| \leq K^{n+1} |a_1 - a_0|$.

(b) Let $m, n \in \mathbb{N} \ni m > n$.

We have then $|a_m - a_n| \leq |a_m - a_{m-1}| + \cdots\cdots$

$$\leq |a_1 - a_0| \sum_{i=n}^{m} K^i$$

$$= \frac{K^n}{1-K} |a_1 - a_0|.$$

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