IIT–JAM Mathematical Statistics (MS) 2005

1. Let

$$\boldsymbol{P} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 1 \\ 2 & 3 & 4 & 8 & 6 & 3 \\ 2 & 4 & 6 & 7 & 10 & 3 \\ 4 & 7 & 10 & 14 & 16 & 7 \end{bmatrix}$$

Then the rank of the matrix P is

(a) 1
(b) 2
(c) 3
(d) 4

Solution: (c)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 1 \\ 0 & -1 & -2 & 0 & -4 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & -2 & -2 & -4 & 3 \end{bmatrix}$$

$$R_2' = R_2 - 2R_1$$

$$R_3' = R_3 - 2R_1$$

$$R_4' = R_4 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 1 \\ 0 & -1 & -2 & 0 & -4 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 2 \end{bmatrix} \qquad R_4' = R_4 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 1 \\ 0 & -1 & -2 & 0 & -4 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_4' = R_4 - 2R_3$$

$$Rank = 3$$

2. Consider the following system of linear equations:

$$\boldsymbol{x + y + z = 3,}$$

$$\boldsymbol{x + az = b,}$$

$$\boldsymbol{y + 2z = 3}$$

This system has infinite number of solutions if

(a) $\boldsymbol{a = -1, b = 0}$
(b) $\boldsymbol{a = 1, b = 2}$

(c) $\boldsymbol{a = 0, b = 1}$
(d) $\boldsymbol{a = -1, b = 1}$

Solution: (a)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 0 & a & b \\ 0 & 1 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & a-1 & b-3 \\ 0 & 1 & 2 & 3 \end{array}\right] \quad R_2' = R_2 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & a-1 & b-3 \\ 0 & 1 & a+1 & b \end{array}\right] \quad R_3' = R_3 + R_2$$

$$\Rightarrow a + 1 = 0; \qquad b = 0$$

$$\Rightarrow a = -1, \qquad b = 0$$

3. Six identical fair dice are thrown independently. Let S denote the number of dice showing even numbers on their upper faces. Then the variance of the random variable S is

(a) ½
(b) 1
(c) 3/2
(d) 3

Solution: (c) $P(S = x) = \binom{6}{x}\left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{6-x}; \ x = 0, 1, 2, 3, 4, 5, 6$

$$\Rightarrow S \ follows\ binomial\ dstribution\ with\ n = 6, \qquad p = 1/2$$

$$\Rightarrow variance \ = np(1-p) = 6 \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{2}$$

4. Let $\boldsymbol{X_1, X_2, \dots, X_{21}}$ be a random sample from a distribution having the variance 5. Let

$$\boldsymbol{\bar{X} = \frac{1}{21}\sum_{i=1}^{21} X_i \ \ and\ \ S = \sum_{i=1}^{21} (X_i - \bar{X})^2}$$

Then the value of $\sum(S)$ is

(a) 5
(b) 100
(c) 0.25
(d) 105

Solution: (b)

$$E\left[\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2\right] = \sigma^2 \ \ \Rightarrow E\left[\frac{1}{20}\sum_{i=1}^{21}(x_i - \bar{x})^2\right] = 5 \ \ \Rightarrow E(S) = 20 \times 5 = 100$$

5. **Let X and Y be independent standard normal random variables. Then the distribution of**

$$\boldsymbol{U = \left(\frac{X - Y}{X + Y}\right)^2 \ is}$$

(a) **chi–square with 2 degrees of freedom**
(b) **chi–square with 1 degree of freedom**
(c) **F with (2, 2) degrees of freedom**
(d) **F with (1, 1) degrees of freedom**

Solution: (d)

$$\frac{x^2_{(1)}}{x^2_{(1)}} = F(1,1)$$

6. **In three independent throws of a fair dice, let X denote the number of upper faces showing six. Then the value of $\boldsymbol{E(3 - X)^2}$ is**
(a) **20/3**
(b) **2/3**
(c) **5/2**
(d) **5/12**

Solution: (a)

$$When\ X = 1, \qquad probability\ is\ \binom{3}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2 = 25/72$$

$$When\ X = 2, \qquad probability\ is\ \binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right) = \frac{5}{72}$$

$$When\ X = 3, \qquad probability\ is\ \binom{3}{3}\left(\frac{1}{6}\right)^3 = \frac{1}{216}$$

$$When\ X = 0, \qquad probability\ is\ \binom{3}{0}\left(\frac{5}{6}\right)^3 = \frac{125}{216}$$

$$E(3 - X)^2 = \sum (3 - x)^2 P(X = x)$$

$$= \frac{100}{72} + \frac{5}{72} + 0 + \frac{125}{24} = \frac{160}{24} = \frac{20}{3}$$

7. Let

$$\boldsymbol{P} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1}+\boldsymbol{x} & \mathbf{1}+\boldsymbol{x} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1}+\boldsymbol{x} & \mathbf{0} & \mathbf{1}+\boldsymbol{x} \\ \mathbf{1} & \mathbf{1}+\boldsymbol{x} & \mathbf{1}+\boldsymbol{x} & \mathbf{0} \end{bmatrix}$$

Then the determinant of the matrix P is

- **(a)** $\mathbf{3}(\boldsymbol{x}+\mathbf{1})^3$
- **(b)** $\mathbf{3}(\boldsymbol{x}+\mathbf{1})^2$
- **(c)** $\mathbf{3}(\boldsymbol{x}+\mathbf{1})$
- **(d)** $(\boldsymbol{x}+\mathbf{1})(\mathbf{2}\boldsymbol{x}+\mathbf{1})$

Solution: (a)

$\begin{vmatrix} 1 & 0 & 1+x & 1+x \\ 0 & 1 & 1 & 1 \\ 0 & 1+x & -(1+x) & 0 \\ 0 & 1+x & 0 & (1+x) \end{vmatrix}$ Expanding this w.r.t. 1st column, we will get a 3×3 matrix, then perform the given operations:

$R_3{}' = R_3 - R_1$ & $R_4{}' = R_4 - R_1$

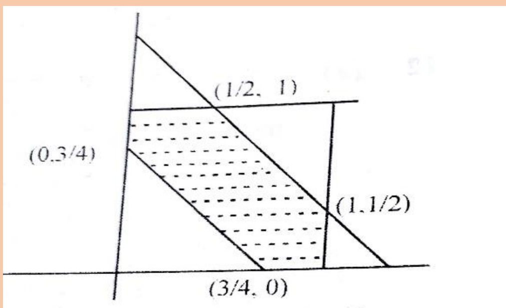
$$\det(P) = \begin{vmatrix} 1 & 1 & 1 \\ 1+x & -(1+x) & 0 \\ 1+x & 0 & -(1+x) \end{vmatrix} = 3(1+x)^2$$

8. The area of the region

$$\left\{(\boldsymbol{x},\boldsymbol{y}) : \mathbf{0} \le \boldsymbol{x},\boldsymbol{y} \le \mathbf{1}, \quad \frac{\mathbf{3}}{\mathbf{4}} \le \boldsymbol{x}+\boldsymbol{y} \le \frac{\mathbf{3}}{\mathbf{2}}\right\} \boldsymbol{is}$$

- **(a) 9/16**
- **(b) 7/16**
- **(c) 13/32**
- **(d) 19/32**

Solution: (d) $Required\ area = 1 \times 1 - \frac{1}{2} \times \frac{3}{4} \times \frac{3}{4} - \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = 1 - \frac{9}{32} = \frac{1}{8} = \frac{19}{32}$



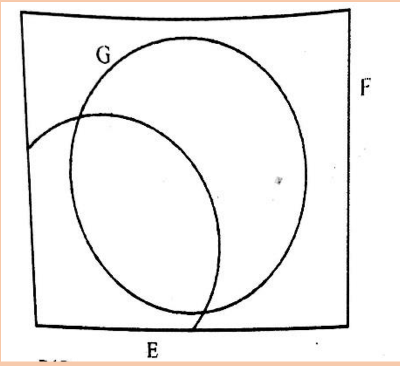
9. **Let E, F and G be three events such that the events E and F are mutually exclusive,**

$$\boldsymbol{P(E \cup F) = 1, P(E \cap G) = \frac{1}{4}\ and\ P(G) = \frac{7}{12}}$$

Then $\boldsymbol{P(E \cup F)}$equals

- **(a) 1/12**
- **(b) ¼**
- **(c) 5/12**
- **(d) 1/3**

Solution: (d)



$P(F \cap G) = P(G) - P(E \cap G)\ = \frac{7}{12} - \frac{1}{4} = \frac{1}{3}$

10. **Let X and Y have the joint probability mass function**

$$\boldsymbol{P(X = x, Y = y) = \frac{1}{3x}, y = 1, 2, \ldots, x; x = 1, 2, 3.}$$

Then the value of the conditional expectation $\boldsymbol{E(Y|X = 3)}$ is

- **(a) 1**
- **(b) 2**
- **(c) 1.5**
- **(d) 2.5**

Solution: (b)

$$E(Y|X = 3) = \sum y\ .\ \frac{P(Y = y; X = 3)}{P(X = 3)}$$

$$P(X = x) = \frac{1}{3x}(1 + 1 + \cdots + x\ times) = \frac{1}{3}$$

$$= \sum_{y=1}^{3} y \frac{1}{3} = \frac{1 + 2 + 3}{3}\ = 2\ .$$

11. Let X_1 *and* X_2 be independent random variables with respective moment generating function

$$M_1(t) = \left(\frac{3}{4} + \frac{1}{4}e^t\right)^3 \text{ and } M_2(t) = e^{2(e^t-1)}, \quad -\infty < t < \infty$$

Then the value of $P(X_1 + X_2 = 1)$ is

(a) $\frac{81}{64}e^{-2}$
(b) $\frac{27}{64}e^{-2}$
(c) $\frac{11}{64}e^{-2}$
(d) $\frac{27}{32}e^{-2}$

Solution: (a)

$$E[e^{tx}] = \left(\frac{3}{4} + \frac{1}{4}e^t\right)^3$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{tx} f(x)dx = \left(\frac{3}{4} + \frac{1}{4}e^t\right)^3$$

$$P(X_1 = 0;\, X_2 = 1) + P(X_1 = 1;\, X_2 = 0)$$

$$= \left(\frac{3}{4}\right)^3 \frac{2^1 e^{-2}}{1!} + 3_{C_1}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)\frac{e^{-2}2^0}{0!} = \left(\frac{54}{64} + \frac{27}{64}\right)e^{-2} = \frac{81}{64}e^{-2}$$

12.

$$\lim_{n\to\infty}\left[\frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\int_{n-\sqrt{2n}}^{\infty} e^{-\frac{t}{2}} t^{\frac{n}{2}-1} dt\right] \text{ equals}$$

(a) 0.5
(b) 0
(c) 0.0228
(d) 0.1587

Solution: (a)

$\lim_{n\to\infty} \Rightarrow \text{chi square} \Rightarrow \text{normal}$

$\text{mean to } \infty = 0.5$

$n - \sqrt{2n} = n - 1$. Standard deviation.

13. **Let X_1 *and* X_2 be two independent random variables having the same mean θ. Suppose that $E(X_1-\theta)^2 = 1$ *and* $E(X_2-\theta)^2 = 2$. For estimating θ, consider the estimators $T_\alpha(X_1,X_2) = \alpha X_1 - (1-\alpha)X_2, \alpha \in [0,1]$. The value of $\alpha \in [0,1]$, for which the variance of $T_\alpha(X_1,X_2)$ is minimum, equals**
    - **(a) 2/3**
    - **(b) ½**
    - **(c) ¼**
    - **(d) ¾**

Solution: (a)

$$V(\alpha X_1 - (1-\alpha)X_2) = \alpha^2 V(X_1) + (1-\alpha)^2 V(X_2) = \alpha^2 + 2(1-\alpha)^2 = 3\alpha^2 - 4\alpha + 2$$

$$= 3\left[\alpha^2 - \frac{4}{3}\alpha + \frac{2}{3}\right] = 3\left[\left(\alpha - \frac{2}{3}\right)^2 + \frac{2}{9}\right]$$

$$\Rightarrow \alpha = \frac{2}{3}\ for\ minimum\ variance.$$

14. **Let $x_1 = 3, x_2 = 4, x_3 = 3.5, x_4 = 2.5$ be the observed values of a random sample from the probability density function**

$$f(x|\theta) = \frac{1}{3}\left[\frac{1}{\theta}e^{\frac{x}{\theta}} + \frac{1}{\theta^2}e^{-\frac{x}{\theta^2}} + e^{x}\right], x > 0, \theta > 0$$

Then the method of moments estimate (MME) of θ is

- **(a) 3.5**
- **(b) 4**
- **(c) 2.5**
- **(d) 1**

Solution: (b)

$$L(x,\theta) = \frac{1}{3}\left[\frac{1}{\theta}e^{\frac{x}{\theta}} + \frac{1}{\theta^2}e^{-\frac{x}{\theta^2}} + e^{x}\right]$$

$$\frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \frac{-1}{\theta^2}e^{x/\theta} - \frac{x}{\theta^3}e^{x/\theta} - \frac{2}{\theta^3}e^{-x/\theta^2}$$

$$= \frac{-e^{\frac{x}{\theta}}}{\theta^3}(x+\theta) + \frac{2}{\theta^5}e^{-\frac{x}{\theta^2}}(x-\theta^2)$$

Which is increasing, so $\theta = x^{max} = 4$

15. **Let $x_1 = -2, x_2 = 1, x_3 = 3, x_4 = -4$ be the observed values of a random sample from the distribution having the probability density function.**

$$f(x|\theta) = \frac{e^{-z}}{e^{\theta} - e^{-\theta}}, -\theta \le x \le \theta, \theta > 0.$$

Then the maximum likelihood estimate of θ is

- **(a) 3**
- **(b) 0.5**
- **(c) 4**
- **(d) Any value between 1 and 2**

Solution: (c)

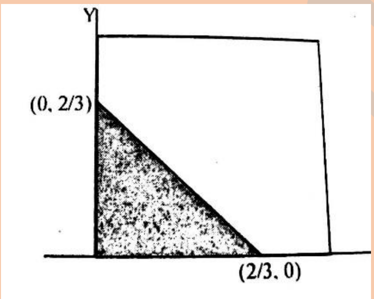
$$L(x,\theta) = \sum \frac{e^{-x_i}}{e^{\theta} - e^{-\theta}} \text{ ; which is minimum but for } -\theta \le x \le \theta, i.e., \theta = 4$$

16. **Let X and Y be independent and identically distributed uniform random variables over the interval (0,1) and let $S = X + Y$. Find the probability that the quadratic equation $9x^2 + 9Sx + 1 = 0$ has no real root.**

Solution:

$$81S^2 - 36 < 0 \quad gives \quad S^2 < \left(\frac{6}{9}\right)^2$$

$$\Rightarrow -\frac{2}{3} < S < \frac{2}{3}$$



$$Prob. = \frac{\frac{1}{2} \times \frac{2}{3} \times \frac{2}{3}}{1 \times 1} = \frac{2}{9}$$

17. **Find the number of real roots of the polynomial $f(x) = x^5 + x^3 - 2x + 1$.**

Solution: $f(x) = x^5 + x^3 - 2x + 1.$

Using Descartes rule:

f(x) = + + − + At most 2 positive.

$f(-x) = - - + \ +$ At most 1 negative

$$f'(x) = 5x^4 + 3x^2 - 2 = 5\left[x^4 + \frac{3}{5}x^2 - \frac{2}{5}\right] = 5\left[\left(x^2 + \frac{3}{10}\right)^2 - \left(\frac{7}{10}\right)^2\right]$$

$Putting\ f'(x) = 0$

$$\Rightarrow x^2 = \frac{4}{10} = \frac{2}{5} x = \pm\sqrt{\frac{2}{5}}$$

We know complex roots occur in pairs. So, the given equation has 3 real roots.

18. Consider the $\boldsymbol{n \times n}$ matrix

$$\boldsymbol{P} = \begin{bmatrix} \frac{2}{n+1} & \frac{1}{n+1} & \frac{1}{n+1} & \cdots & \frac{1}{n+1} \\ \frac{1}{n+1} & \frac{2}{n+1} & \frac{1}{n+1} & \cdots & \frac{1}{n+1} \\ \frac{1}{n+1} & \frac{1}{n+1} & \frac{2}{n+1} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+1} & \frac{1}{n+1} & \cdots & \frac{2}{n+1} \end{bmatrix}$$

Let I be the $\boldsymbol{n \times n}$ identity matrix and let J be the $\boldsymbol{n \times n}$ matrix whose all entries are 1. Express the matrix P as $\boldsymbol{aI + bJ}$, for suitable constants a and b. It is known that the inverse of P is of the form $\boldsymbol{cI - dJ}$, for some constants c and d. Find the constants c and d in terms of n.

Solution:

$$P = \frac{1}{n+1}I + \frac{1}{n+1}J$$

$$PP^{-1} = (aI + bJ)(cI - dJ)$$

$$I = acI + (bc - ad)J - bdJ^2 = acI + (bc - ad)J - (n+1)bdJ$$

$$\Rightarrow ac = 1 \,\&\, bc - ad - (n+1)bd = 0$$

$$\Rightarrow C = (n+1)$$

$$\frac{1}{n+1}(n+1) - \frac{1}{(n+1)}d - (n+1)\frac{1}{n+1}d = 0$$

$$\Rightarrow 1 = \frac{n+2}{n+1}d \quad \Rightarrow \quad d = \frac{n+1}{n+2}$$

$$\Rightarrow\ P^{-1} = (n+1)I - \frac{(n+1)}{(n+2)}J$$

19. Let $f(x) = (x+1)|x^2 - 1|, -\infty < x < \infty$. **Verify the differentiability of the function f(.) at the points $x = -1$ and $x = 1$.**

Solution:

$$f(x) = (x+1)(x^2-1)\,; \quad x \geq 1 \; or \; x \leq -1$$

$$= (x+1)(1-x^2)\,; \; -1 \leq x \leq 1$$

$$f'(x) = 3x^2 + 2x - 1\,; \qquad x \geq 1 \; or \; x \leq -1$$

$$= -3x^2 - 2x + 1 \;\; ; \quad -1 \leq x \leq 1$$

$$\Rightarrow f(x) is \; not \; differentiable \; at \; x = -1 \; and \; x = 1$$

20. **There are four urns labeled U_1, U_2, U_3 and U_4, each containing 3 blue and 5 red balls. The fifth urn, labeled U_5, contains 4 blue and 4 red balls. An urn is selected at random from these five urns and a ball is drawn at random from it. Given that the selected ball is red, find the probability that it came from the urn U_5.**

Solution:

$$\frac{\frac{1}{5} \times \frac{4}{8}}{\frac{1}{5} \times \frac{4}{8} + \frac{4}{5} \times \frac{5}{8}} = \frac{4}{24} = \frac{1}{6}$$

21. **Let**

$$F(x) = \begin{cases} 0, & if \; x > 0 \\ \frac{x^2}{10} & if \; 0 \leq x < 1 \\ \frac{x+2}{8} & if \; 1 \leq x < 2 \\ \frac{c(6x - x^2 - 1)}{2} & if \; 2 \leq x \leq 3 \\ 1, & if \; x > 3 \end{cases}$$

Find the value of c for which F(.) is a cumulative distribution function of a random variable X. Also evaluate $P(1 \leq X \leq 2)$.

Solution:

$$f(x) = \begin{cases} 0 & ; x < 0 \\ \frac{x}{5} & ; 0 \le x < 1 \\ \frac{1}{8} & ; 1 \le x < 2 \\ c(3-x) & ; 2 \le x \le 3 \\ 0 & ; x > 3 \end{cases}$$

$$\Rightarrow \int_0^1 \frac{x}{5} dx + \int_1^2 \frac{1}{8} dx + \int_2^3 C(3-x) dx = 1$$

$$\Rightarrow \frac{1}{10} + \frac{1}{8} + C\left(3 - \frac{5}{2}\right) = 1$$

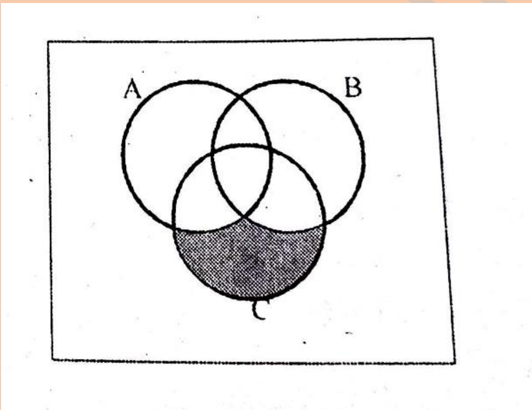
$$\Rightarrow \frac{C}{2} = 1 - \frac{1}{8} - \frac{1}{10} = \frac{31}{40}$$

$$\Rightarrow C = \frac{31}{20}$$

$$P(1 \le x < 2) = \int_1^2 \frac{1}{8} dx = \frac{1}{8}$$

22. Let A, B and C be pair wise independent events such that $P(A \cap B) = 0.1$ *and* $P(B \cap C) = 0.2$. Show that $P\left(A^C \cup C\right) \ge 7/8$.

Solution:



$P(A)P(B) = 0.1$

$P(B)P(C) = 0.2$

$P(A^C \cup C) = P(A^C) + P(C) - P(A^C)P(C)$

$= 1 - P(A) + P(C) - P(C)$

$= 1 - P(A) + P(A)P(C)$

$= 1 - P(A)[1 - P(C)]$

$= \ 1 - P(A)[1 - 2P(A)]$

$= \ 1 - P(A) + 2[P(A)]^2$

$= 2\left[\left(P(A)\right)^2 - \frac{1}{2}P(A) + \frac{1}{2}\right]$

$= 2\left[\left\{P(A) - \frac{1}{4}\right\}^2 + \frac{7}{16}\right]$

$= 2\left[P(A) - \frac{1}{4}\right]^2 + \frac{7}{8} \ \geq \ \frac{7}{8}$

23. Let the random variables X and Y have the joint probability density function

$$\boldsymbol{f(x, y) = \begin{cases} ce^{-(x+y)} & , y > x > 0 \\ 0, & otherwise \end{cases}}$$

Evaluate c and $\boldsymbol{E(Y|X = 2)}$

Solution:

$$\int_0^\infty \int_0^y ce^{-(x)}e^{-y}dx\, dy = \int_0^\infty c(1 - e^{-y})e^{-y}dy \ = c\left[\int_0^\infty e^{-y}dy - \int_0^\infty e^{-2y}dy\right] = c\left[1 - \frac{1}{2}\right]$$

$So, \qquad \frac{c}{2} = 1$

$\Rightarrow c = 2$

$$f(x) = \int_x^\infty 2e^{-x}e^{-y}dy$$

$$= 2e^{-2x} \quad x > 0$$

$$E(Y|X = 2) = \int_2^\infty \frac{2ye^{-(2+y)}}{2e^{-4}}dy = \int_0^\infty ye^{-(y-2)}dy = -ye^{-(y-2)} - e^{-(y-2)} \Big|_0^\infty = 2 - 1 = 1$$

24. Find the area of the region enclosed between the curves

$$\boldsymbol{y = (x - 2)^2 \text{ and } y = 4 - x^2}$$

Solution:

$$(x - 2)^2 = 4 - x^2 \ \Rightarrow 2x^2 - 4x = 0 \ \Rightarrow 2x(x - 2) = 0 \Rightarrow x = 0, 2$$

$$A = \int_0^2 (x - 2)^2 - 4 + x^2\, dx = \frac{(x - 2)^2}{3} - 4x + \frac{x^3}{3} \Big|_0^2 \ = Answer$$

25. Let $\bar{X}$ be the sample mean of a random sample of size 10 from a distribution having the probability density function

$$\boldsymbol{f(x|\theta) = \frac{1}{\theta} e^{\frac{x}{\theta}}, x > 0, \theta > 0.}$$

Using Chebyshev's inequality, show that

$$\boldsymbol{P(0 < \bar{X} \le 2\theta) \ge 0.9}$$

Solution:

$f(x|\theta) = \frac{1}{\theta} e^{-x/\theta}$; $\theta > 0; x > 0$

$\Rightarrow$ *it is exponential distribution with parameter* $\frac{1}{\theta}$,

hence mean will be $\frac{1}{1/\theta}$ *and, variance* $\frac{1}{1/\theta^2}$ *i.e.,* θ *and* θ^2

So, mean $= \theta$ *and s.d.* $= \theta$

So, by Chebyshev's inequality

$P(0 < \bar{X} < 2\theta) > 1 - \frac{1}{10} = 1 - 0.1$

$\Rightarrow P(0 < \bar{X} \le 2\theta) \ge 0.9$

26. Let X be a continuous random variable having the probability density function

$$\boldsymbol{f(x) = \begin{cases} \frac{2}{25}(x+2) & , -2 \le x \le 3 \\ 0, & otherwise \end{cases}}$$

Find the cumulative distribution function $Y = X^2$. Hence derive the expression for probability density function of Y.

Solution:

Let $F(x) = P(Y = X^2 \le x) = P(X^2 \le x) = P(-\sqrt{x} \le X \le \sqrt{x})$

$$F(x) = \begin{cases} 0 & x < 0 \\ \int_{-\sqrt{x}}^{0} \frac{2}{25}(x+2)dx + \int_{0}^{\sqrt{x}} \frac{2}{25}(x+2)dx & ; 0 \le x < 4 \\ \int_{-2}^{\sqrt{x}} \frac{2}{25}(x+2)dx & 4 \le x < 9 \\ 1 & x \ge 9 \end{cases}$$

$$\Rightarrow F(x) = \begin{cases} 0; & x < 0 \\ \dfrac{8\sqrt{x}}{25} & 0 \le x < 4 \\ \dfrac{x-4}{25} + \dfrac{4(\sqrt{x}+2)}{25} & ; 4 \le x < 9 \\ 1; & x \ge 9 \end{cases}$$

$$p.d.f.\,of\ Y = X^2$$

$$f(x) = \begin{cases} 0; & x \ge 0 \\ \dfrac{4}{25\sqrt{x}} & 0 \le x \le 4 \\ \dfrac{1}{25} + \dfrac{2}{25\sqrt{x}} & ; 4 \le x \le 9 \\ 0; & x \ge 9 \end{cases}$$

27. Let X be a single observation from the probability density function

$$\boldsymbol{f(x|\theta) = (\theta + 1)x^{\theta}}\ , \mathbf{0} < x < 1,$$

Where $\boldsymbol{\theta} \in \{\mathbf{1}, \mathbf{2}\}$ is unknown. Find the most powerful test of level

$$\boldsymbol{\alpha = \frac{13}{49}\ for\ testing\ H_0 : \ \theta = 1\ against\ H_1 : \ \theta = 2}$$

What is the power of the test?

Solution: $\alpha = p(reject\ |H_0\ is\ true)$

$H_0\ is\ true\ \Rightarrow \theta = 1 \Rightarrow f(x|1) = 2x,\ \ 0 < x < 1$

Let critical region be (k, 1), then

$$\int_k^1 f(x)dx = \alpha = \int_k^1 2x\,dx = \alpha$$

$$\Rightarrow 1 - k^2 = \frac{13}{49}$$

$$\Rightarrow k^2 = \frac{36}{49}$$

$$\Rightarrow k = \frac{6}{7}$$

So, acceptance region is (0, 6/7)

$\beta = p\ (accepts\ |H_1\ is\ true)$

$H_1\ is\ true\ gives\ \theta = 2$

$\Rightarrow f(x|2) = 3x^2; 0 < x < 1$

$\Rightarrow \beta = \int_0^{6/7} 3x^2 dx = \frac{216}{343}$

$\Rightarrow power\ of\ test, 1 - \beta = 1 - \frac{216}{343} = \frac{133}{343}.$

28. Let $X_1, X_2, \ldots, X_n$ be a random sample from a population having the probability density function

$$f(x|\theta,\sigma) = \begin{cases} \frac{1}{\sigma} e^{\frac{-(x-\theta)}{\sigma^2}} & , x > 0 \\ 0, & otherwise \end{cases}$$

$$-\infty < \theta < \infty, \sigma > 0$$

Find the method of moments and maximum likelihood estimators of θ and σ.

Solution: Likelihood function,

$$L(x,\theta) = \prod_{i=1}^{n} \frac{1}{\sigma} e^{-\frac{(x_i-\theta)}{\sigma^2}} = \frac{1}{\sigma^n} e^{-\frac{(\sum x_i - n\theta)}{\sigma^2}}$$

$$\Rightarrow \log L = -n \log \sigma - \frac{\sum x_i + n\theta}{\sigma^2}$$

$$\Rightarrow \frac{1}{L} \cdot \frac{\partial L}{\partial \theta} = \frac{n}{\sigma^2} > 0\ (always)$$

$$\Rightarrow given\ function\ is\ increasing.$$

Hence $\hat{\theta} = x_{(n)}$.

29. Let $f(x) = 3(x-2)^3 - (x-2), 0 \le x \le 20$.

Let x_0 and y_0 be the points of the global minima and global maxima, respectively, of f(.) in the interval [0, 20]. Evaluate $f(x_0) + f(y_0)$.

Solution:

$f(x) = 3(x-2)^3 - (x-2)$

$\Rightarrow f'(x) = 9(x-2)^2 - 1 = 9x^2 - 36x + 8$

$Now\ \ f'(x) = 0$

$\Rightarrow 9x^2 - 36x + 8 = 0$

$$\Rightarrow x = \frac{36 \pm \sqrt{1296 - 9 \times 8 \times 4}}{18}$$

$$= \frac{36 \pm 12\sqrt{7}}{18} = 2 \pm \frac{2\sqrt{7}}{3}$$

$f''(x) = 18x - 36 = 18(x - 2)$

$$f''\left(2 + \frac{2\sqrt{7}}{3}\right) > 0$$

$$\Rightarrow x = 2 + \frac{2\sqrt{7}}{3}\ is\ local\ minima\ point$$

$$f''\left(2 - \frac{2\sqrt{7}}{3}\right) < 0$$

$$\Rightarrow x = 2 - \frac{2\sqrt{7}}{3}\ is\ local\ maxima\ point.$$

Clearly $x = 0$ & $x = 20$ are global minima & global maxima point (given in the question).

$\Rightarrow x_0 = 0\ \ \& \, y_0 = 20$

$f(x_0) = f(0) = 3(-2)^3 - \ (-2) = -22$

$f(y_0) = f(20) = 3(18)^3 - (18) = 18[3(18)^2 - 1]\ = 17478$

$$So,\ \ f(x_0) + f(y_0) = f(0) + f(20) = 22 + 17478 = 17456$$