IIT–JAM Mathematical Statistics (MS) 2007

1. Let the random variable X have binomial distribution with parameters 3 and θ. A test of hypothesis $H_0 : \theta = 3/4$ against $H_1 : \theta = 1/4$ rejects H_0 if X < 1. Then the test has

(a) Size $= \frac{5}{32}$ $and\ power\ = 27/32$

(b) size$= \frac{5}{32}$ $and\ power\ = 18/32$

(c) size$= \frac{15}{32}$ $snd\ power = 27/32$

(d) size$= \frac{1}{32}$ $and\ power = 31/32$

Solution: (a) $X \sim B(3, q)$

$$So, P(X = x) = \binom{3}{x}(q)x\ (1-q)^{3-x}$$

$$\alpha = P[x \le 1|H_0] = P\left[X \le 1 \middle| \theta = \frac{3}{4}\right] = P[X = 0] + P[x = 1]$$

$$\alpha = \left(\frac{1}{4}\right)^3 + \binom{3}{1}\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^2$$

$$\alpha = \frac{1}{64} + 3 \times \frac{3}{4} \times \frac{1}{16} = \frac{5}{32}$$

$$1 - \beta = P[X \le 1|H_1] = P\left[X \le 1|\theta = \frac{1}{4}\right]$$

$$= P[X = 0] + P[X = 1]$$

$$1 - \beta = \left(\frac{3}{4}\right)^3 + \binom{3}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^2$$

$$\Rightarrow 1 - \beta = \frac{27}{64} + \frac{27}{64}$$

$$\Rightarrow 1 - \beta = \frac{27}{32}$$

So, $\alpha = \frac{5}{32}$ $and\ power = 27/32$

2. Let X be a random variable having probability density function

$$f(x; x_0, \alpha) = \begin{cases} \dfrac{\alpha x_0^\alpha}{x^{\alpha+1}}, & x > x_0 \\ 0, & x \le x_0 \end{cases}$$

$$where\ \alpha > 0, x_0 > 0. If\ Y = ln\left(\frac{X}{x_0}\right), \quad then\ P(Y > 3) is$$

$(a) e^{-3\alpha x_0}$ $\quad$ $(b) 1 - e^{-3\alpha x_0}$ $\quad$ $(c)\ e^{-3\alpha}$ $\quad$ $(d)\ 1 - e^{-3\alpha}$

Solution: (c)

$f(x, x_0, \alpha) = \frac{\alpha x_0^\alpha}{x^{\alpha+1}} \quad x > x_0$

P[Y > 3] $= P\left[\log\left(\frac{x}{x_0}\right) > 3\right] = P[X > e^3 x_0] = P[X > x_0 e^3]$

$$= \alpha \int_{x_0 e^3}^{\infty} \frac{x_0^\alpha}{x^{\alpha+1}} dx \;= \alpha x_0^\alpha \left[-\frac{1}{\alpha x^\alpha}\right]_{x_0 e^3}^{\infty}$$

$$P[Y > 3] = \frac{1}{e^{3\alpha}} = e^{-3\alpha}$$

3. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by T(x, y,z) $= (x + y, x - z)$. Then the dimension of the null space of T is

(a) 0

(b) 1

(c) 2

(d) 3

Solution: (b)

$T(x, y, z) = (x + y, x - z)$

Now, $(x + y, x - z) = (0,0)$

$$\Rightarrow x + y = 0; x - z = 0$$

$$\Rightarrow x = -y = z$$

$$\Rightarrow N(T) = \{(a, -a, a) | a \in R\}$$

i.e., nullity = 1, as only one variable can have freedom.

4. Let $X_1, X_2, \dots, X_{2n}$ be random variables such that $V(X_i) = 4, i = 1, 2, \dots, 2n \text{ and } Cov\left(X_i, X_j\right) = 3, 1 \le i \ne j \le 2n$. Then $V(X_1 - X_2 + X_3 -$ *X4+…+X2n−1−X2n* is

(a) n

(b) 2n

(c) 3n−2

(d) n+1

Solution: (b) As, $V(X_1 - X_2 + \cdots + X_{2n-1} - X_{2n}) = \sum_{i=1}^{2n} V(X_i) + 2\sum cov(x_i, x_j)$

$$= 8n - 2(3n) = 2n$$

5. **Let X_1 *and* X_2 be independent random variables, each having exponential distribution with parameter λ. Then, the conditional distribution of X_1 *given* $X_1 + X_2 = 1$ is**
   (a) **Exponential with mean 2**
   (b) **Beta with parameters $\frac{\lambda}{2}$ *and* $\frac{\lambda}{2}$**
   (c) **Uniform on the interval (0,1)**
   (d) **Gamma with mean 2λ**

Solution: (c)

$$f(x_1|x_1 + x_2 = 1) = \frac{f(x_1=x, x_2=1-x)}{f(x_1+x_2=1)} = \frac{f(x_1=x)f(x_2=1-x)}{f(x_1+x_2=1)}$$

$\because x_1, x_2$ are independent.

$$f(x_1 \Big| x_1 + x_2 = 1) = \frac{\lambda e^{-\lambda x}\lambda e^{-\lambda(1-x)}}{\lambda^2 e^{-\lambda}} = 1$$

So, $f(x_1|x_1 + x_2 = 1) = 1 \quad if \quad 0 < x_1 < 1$

Which is uniform distribution over (0,1).

6. **Let $X_1, X_2, \dots, X_n$ be a random sample from a uniform distribution on the interval $(0, \theta)$. Then the uniformly minimum variance unbiased estimator (UMVUE) of θ is**
   (a) $\left(\frac{n+1}{n}\right)X_{(n)}$
   (b) $X_{(1)} + X_{(n)}$
   (c) $2\overline{X}$
   (d) $X_{(n)}$

Solution: (a)

Let $X_1, X_2, \dots, X_n$ be random sample.

Let $T = X_{(n)} = \max X_i, 1 \le i \le n$ is sufficient and complete statistics for θ

$$Also, E(T) = E\left(X_{(n)}\right) = \left(\frac{n}{n+1}\right)\theta$$

$$\Rightarrow E\left(\frac{n+1}{n}X_{(n)}\right) = \theta$$

So, $\frac{n+1}{n}X_{(n)}$ is an UMVUE of θ.

7. **Let A be a 4×4 nonsingular matrix and B be the matrix obtained from A by adding to its third row twice the first row. Then $det\,(2A^{-1}B)$ equals**
   (a) 2
   (b) 4
   (c) 8
   (d) 16

Solution: (d) $|2A^{-1}B| = 16|A^{-1}B| = 16|A^{-1}||B| = 16$

$$\left(As\ |B| = |A|\ and\ |A^{-1}| = \frac{1}{|A|}\right)$$

8. **Independent trails consisting of rolling a fair die are performed, the probability that 2 appears before 3 or 5 is**
   (a) ½
   (b) 1/3
   (c) ¼
   (d) 1/5

Solution: (b) P(3 or 5)=$\frac{2}{6}$; $P(2) = 1/6$

$$Required\ Probability = \frac{1}{6} + \frac{1}{2} \times \frac{1}{6} + \left(\frac{1}{2}\right)^2 \frac{1}{6} + \cdots$$

$$= \frac{\frac{1}{6}}{1 - \frac{1}{2}} = \frac{1}{3}$$

9. **Let $X_1, X_2, \ldots, X_6$ be independent random variables such that**

$$P(X_i = -1) = P(X_i = 1) = \frac{1}{2}, \qquad i = 1, 2, \ldots, 6$$

Then $P\left[\sum_{i=1}^{6} X_i = 4\right]$ *is*

   (a) 3/32
   (b) ¾
   (c) 3/64
   (d) 3/16

Solution: (a) $P(X_1 + X_2 + X_3 + X_4 + X_5 + X_6 = 4)$

$$= \binom{6}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right) = \frac{6}{64} = \frac{3}{32}$$

10. Let 1, x and x^2 be the solutions of a second order linear non–homogeneous differential equation on −1< x < 1. Then its general solution, involving arbitrary constants C_1 *and* C_2, can be written as

(a) $C_1(1-x)+C_2(x-x^2)+1$

(b) $C_1x+C_2x^2+1$

(c) $C_1(1+x)+C_2(1+x^2)+1$

(d) $C_1+C_2x+x^2$

Solution: (a)

x *and* x^2 will be solution of complementary function and 1 will be particular solution, so general solution is

$$y=C_1x+C_2x^2+1$$

11. Let $f(x)=\begin{cases} x^2 \sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$. *Then*

(a) $f'(x)$ is continuous at $x=0$

(b) $f''(x)$ is continuous at $x=0$

(c) $f'(0)$ exists

(d) $f''(0)$ exists

Solution: (c) $f'(0)=0$

$$f'(0)=\underset{h\to 0}{Lt}\frac{f(0+h)-f(0)}{h}=\underset{h\to 0}{Lt}\frac{h^2\sin\frac{1}{h}}{h}=\underset{h\to 0}{Lt}\, h\sin\frac{1}{h}=0$$

12. Let E and F be two events such that 0 < P(E) < 1 and $P(E|F)+P(E|F^C)=1$. Then

(a) E and F are mutually exclusive

(b) $P(E^C|F)+P(E^C|F^C)=1$

(c) E and F are independent

(d) $P(E|F)+P(E^C|F^C)=1$

Solution: (a)

$$\frac{P(E\cap F)}{P(F)}+\frac{P(E\cap F^C)}{P(F^C)}=1$$

$$implies\ \frac{P(E\cap F)}{P(F)}+\frac{P(E)-P(E\cap F)}{1-P(F)}=1$$

$\Longrightarrow$ E & F are mutually exclusive.

13. Let $X_1, X_2, \dots, X_n$ be a random sample from an exponential distribution with mean $1/\lambda$, the maximum likelihood estimator of the median of the distribution is

(a) $\frac{\bar{X}}{(ln\,2)}$

(b) $\bar{X}(ln\,2)$

(c) $\frac{ln\,2}{\bar{X}}$

(d) $ln\,(2\bar{X})$

Solution: (b) As mean of exponential distribution is $1/\lambda$

$$P(X = x) = \lambda e^{-\lambda x}; x > 0, \lambda > 0$$

Median $= M$

$$\Rightarrow \int_0^M f(x)dx = \frac{1}{2}$$

$$\Rightarrow \int_0^M \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_0^M = 1 - e^{-\lambda M} = \frac{1}{2}$$

$$\Rightarrow \lambda M = \ln 2$$

$$\Rightarrow M = \ln 2/\lambda = \bar{X}\ln 2$$

14.

$$\lim_{n\to\infty} \frac{1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-2n)}{\sqrt{n^2+1} + \sqrt{n^2-1}}\ equals$$

(a) ∞

(b) ½

(c) 0

(d) −1/2

Solution: (d)

$$\lim_{n\to\infty} \frac{-n}{\sqrt{n^2+1} + \sqrt{n^2-1}}$$

$$= \lim_{n\to\infty} \frac{-1}{\sqrt{1+\frac{1}{n^2}} + \sqrt{1-\frac{1}{n^2}}} = -\frac{1}{2}$$

15. By changing the order of integration, the integral

$$\int_0^1 \int_1^{e^x} f(x,y)dy\,dx$$

Can be expressed as

(a) $\int_0^1 \int_1^{\ln y} f(x,y)dx\,dy$
(b) $\int_0^1 \int_0^{\ln y} f(x,y)dx\,dy$
(c) $\int_1^e \int_1^{e^y} f(x,y)dx\,dy$
(d) $\int_1^e \int_{\ln y}^1 f(x,y)dx\,dy$

Solution: (d) $1 \le y \le e^x$ and $0 \le x \le 1$

$$\Rightarrow \int_1^e \int_{\ln y}^1 f(x,y)dx\,dy$$

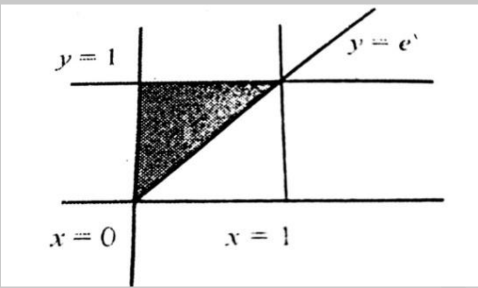


Figure: Region of the given area.

16. (a) Let $f(x) = x^3 + 3x - 2, x \in \mathbb{R}$. Show that the equation $f(x) = 0$ has only one real root. Also, find x_0 in the interval (0,1) such that the tangent to the curve $y = f(x)$ at the point $(x_0, f(x_0))$ is parallel to the line joining the points (0, −2) and (1, 2).

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation with $T(1,1) = (0,0,1)$ and $T(1,2) = (0,1,1)$. Then find the linear transformation T(x, y). Also, find the associated matrix referred to the standard bases.

Solution:

(a) $f(x) = x^3 + 3x - 2 \implies f'(x) = 3x^2 + 3$

So, f'(x) is increasing function, hence f(x) has only one root in (0,1)

As, f(0) < 0 and f(1) > 0

16.(b) $T(x,y) = (2x - y)(1,1) + (y - x)(1,2)$

$$\implies T(x,y) = (2x - 1)(0,0,1) + (y - x)(0,1,1) = (0, y - x, x)$$

17. (a) Find the volume of the solid whose base is the region in the xy−plane that is bounded by the parabola $y = 2 - x^2$ and the line $y = x$, while the top of the solid is bounded by the plane $z = x + 2$.

(b) Find all the values of x for which the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{\left(n+\frac{1}{n}\right)} \; converges$$

Solution:

(a) $y = 2 - x^2 \;\;\&\;\; y = x$

$\Rightarrow x = 2 - x^2 \Rightarrow x^2 + x - 2 = 0 \Rightarrow (x+2)(x-1) = 0 \Rightarrow x = -2, 1$

$$Required\ Volume = \int_{x=-2}^{1}\int_{y=x}^{2-x^2}\int_{z=0}^{x+2} dz\, dy\, dx$$

$$= \int_{-2}^{1}\int_{y=x}^{2-x^2} (x+2) dy\, dz$$

$$= \int_{-2}^{1} (x+2)(2-x^2-x)dx$$

$$= \int_{-2}^{1} (-x^3 - 3x^2 + 4)dx$$

$$= -\frac{x^4}{4} - x^3 + 4x \Big|_{-2}^{1}$$

$$= \frac{15}{4} - 9 + 4(3) = 3 + \frac{15}{4} = \frac{27}{4}$$

(b)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{\left(n+\frac{1}{n}\right)}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1}\frac{nx^n}{(n^2+1)}$$

$$\underset{n\to\infty}{Lt} \frac{nx^n}{(n^2+1)} = 0 \;\; if\ |x| \le 1$$

So, the given series converges if $|x| \le 1$

18. The cumulative distribution function of a random variable X is

$$\boldsymbol{F(x)} = \begin{cases} \mathbf{0}, & \boldsymbol{x} < 0 \\ \dfrac{\boldsymbol{x+k}}{\mathbf{5}}, & \boldsymbol{k \le x} < k + \mathbf{1}, \boldsymbol{k} = \mathbf{0,1,2} \\ \mathbf{1}, & \boldsymbol{x \ge 3} \end{cases}$$

Find:

(a) $\boldsymbol{P(X = j)}$ **for all non−negative integers j**
(b) $\boldsymbol{P(X} > 2)$
(c) $\boldsymbol{P}(\mathbf{-1} \le \boldsymbol{X} < 1)$

Solution:

$$F(x) = \begin{cases} \dfrac{x+k}{5} & k \le x < k+1 \\ 1 & x \ge 3 \end{cases}$$

$$P[X = 0] = P[X \le 0] = 0$$

$$P[X = 1] = P[X \le 1] - P[X \le 0] = \frac{2}{5} - 0 = \frac{2}{5}$$

$$P[X = 2] = P[X \le 2] - P[X \le 1] = \frac{4}{5} - \frac{2}{5} = \frac{2}{5}$$

$$P[X = 3] = P[X \le 3] - P[X \le 2] = 1 - \frac{4}{5} = \frac{1}{5}$$

$$P[X > 2] = 1 - P[X \le 2] = 1 - \frac{4}{5} = \frac{1}{5}$$

$$P[-1 \le x \le 1] = P[-1 \le x \le 0] + P[0 \le x \le 1] = 0 + \tfrac{1}{5} = \tfrac{1}{5}$$

19. Let $\boldsymbol{X_1, \ldots, X_n}$ be independent random variables with $\boldsymbol{X_k}$ having normal distribution with mean $\boldsymbol{k\theta}$ and variance $\boldsymbol{\sigma^2}$ for $\boldsymbol{k = 1, 2, \ldots, n}$. Find the maximum likelihood estimator of $\boldsymbol{\theta}$ based on $\boldsymbol{X_1, \ldots, X_n}$. Show that it is an unbiased and consistent estimator of $\boldsymbol{\theta}$.

Solution: $X_k \sim N(k\theta, \sigma^2)$

$$So, f(x_k) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{e\sigma^2}(x_k - k\theta)^2}$$

$$\sigma > 0\,, \qquad -\infty < x_k < \infty$$

$$\alpha(x_1, x_2, x_3) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n - e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{n}(x_k - k\theta)^2}$$

$$\log L(x) = -n\log\left(\sigma\sqrt{2\pi}\right) - \frac{1}{2\sigma^2}\sum_{k=1}^{n}(x_k - k\theta)^2$$

$Differentiating\ \ w.r.t.\theta\ \&\ equating\ with\ 0\ gives$

$$\frac{\partial \log L(x)}{\partial \theta} = \frac{1}{\sigma^2}\sum_{k=1}^{n} k(x_k - k\theta) = 0$$

$$=> \ \sum_{k=1}^{n} k(x_k - k\theta) = 0$$

$$=> \sum_{k=1}^{n} kx_k - \theta \sum_{k=1}^{n} = 0$$

$$\hat{\theta} = \frac{1}{\sum_{i=1}^{n} k^2}\sum_{k=1}^{n} kx_k = \frac{1}{n(n+1)(2n+1)}\sum_{k=1}^{n} kx_k \ is\ M.L.E.of\ \theta.$$

$$E(\hat{\theta}) = E\left[\frac{6}{n(n+1)(2n+1)}\sum_{k=1}^{n} kx_k\right]$$

$$= \frac{6}{n(n+1)(2n+1)}\sum_{k=1}^{n} kE(x_k)$$

$$= \frac{6}{n(n+1)(2n+1)}\sum_{k=1}^{n} k^2\theta = \frac{6}{n(n+1)(2n+1)}\frac{n(n+1)(2n+1)}{6} = \theta$$

So, $E(\hat{\theta}) = \theta$, i.e., θ is unbiased estimates of θ.

$$V(\hat{\theta}) = V\left\{\frac{6}{n(n+1)(2n+1)}\sum_{k=1}^{n} kx_k\right\}$$

$$V(\hat{\theta}) = \frac{36}{n^2(n+1)^2(2n+1)^2}\sum_{k=1}^{n} k^2 v(X_k)$$

$$= \frac{36}{n^2(n+1)^2(2n+1)^2}\sum_{k=1}^{n} k^2\sigma^2$$

$$V(\hat{\theta}) = \frac{36\sigma^2}{n^2(n+1)^2(2n+1)^2}\frac{n(n+1)(2n+1)}{6}$$

$$V(\hat{\theta}) = \frac{6\sigma^2}{n(n+1)(2n+1)} = \frac{3\sigma^2}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)}$$

When $n \to \infty$

$$E(\hat{\theta}) = \theta\ and\ V(\hat{\theta}) \to 0$$

$So, \hat{\theta}$ is consistent estimator of θ.

20. Let the joint probability mass function of random variables X and Y be given by

$$\boldsymbol{P(X = m, Y = n) = \frac{e^{-1}}{(n-m)!\,m!\,2^n}}$$

$$\boldsymbol{m = 0, 1, 2, \ldots n; n = 0, 1, 2, \ldots}$$

Find the marginal probability mass functions of X and Y. Also, find the conditional probability mass function of X given $Y = 5$, and that of Y given $X = 6$.

Solution:

$$P[X = m, Y = n] = \frac{e^{-1}}{(n-m)!\,m!\,2^n}$$

$$m = 0, 1, 2, \ldots, n$$

$$n = 0, 1, 2, \ldots., n$$

$$P[X = m] = \frac{e^{-112}}{2^m m!} \sum_{n=m}^{\infty} \frac{(112)^{n-m}}{(n-m)!} = \frac{e^{-1}}{2^m m!} e^{112}$$

$$P[X = m] = \frac{e^{-112}}{2^m m!} \quad m = 0, 1, 2, \ldots$$

$$P[Y = n] = \frac{e^{-1}}{n!\,2^n}$$

$$\sum_{m=0}^{n} \binom{n}{m} = \frac{e^{-1}}{n!} \sum_{m=0}^{n} \left(n_{C_m} \left(\frac{1}{2}\right)^m \left(\frac{1}{2}\right)^{n-m} \right)$$

$$P[Y = n] = \frac{e^{-1}}{n!}; \qquad n = 0, 1, \ldots$$

$$P[X|Y = n] = \frac{e^{-1} n!}{(n-m)!\,m!\,2^n e^{-1}} = n_{C_m} \left(\frac{1}{2}\right)^n$$

$$m = 0, 1, \ldots, 5$$

$$P[X|Y] = 5_{C_m} \left(\frac{1}{2}\right)^5$$

$$m = 0, 1, \ldots, 5$$

$$P[Y|X = m] = \frac{e^{-1} 2^m m!}{(n-m)!\,2^m m!\; e^{-112}} = \frac{2^m e^{-112}}{(n-m)!}$$

$$P[Y|X = 6] = \frac{2^6 e^{-112}}{(n-6)!} = \frac{642 e^{-112}}{(n-6)!}$$

$$n = 0,1,2, \ldots., \infty$$

21. Let $X_1, \ldots, X_n (n \geq 2)$ be a random sample from a distribution having the probability mass function $P(X = x) = \theta(1-\theta)^x, x = 0, 1, 2, \ldots$ where $0 < \theta < 1$. Show that

$$T = \sum_{i=1}^{n} X_i$$

is a complete sufficient statistic. Find the uniformly minimum variance unbiased estimator (UMVUE) of θ.

Solution: $P[X = x] = \theta(1-\theta)^x \quad x = 0, 1, 2, \ldots$

$$\alpha(x_1, x_2, \ldots, x_n) = \theta^n (1-\theta)^{\sum_{i=1}^{n} x_i}$$

$$since\ E(X) = \frac{(1-\theta)}{\theta} = p$$

$$Hence\ \theta = \frac{1}{(p+1)}$$

$$a(x, p) = \left(\frac{1}{p+1}\right)^n \left(\frac{p}{p+1}\right)^{\sum_{i=1}^{n} x_i}$$

By Neyman factorization theorem.

$\sum_{i=1}^{n} X_i$ is a sufficient statistics for p.

Since $X_i \sim G(\theta)$then

$$S = \sum_{i=1}^{n} X_i \sim NB(n, \theta)$$

Now, show that S is complete

$$E_p[g(x)] = \sum_{x=0}^{\infty} g(x) \binom{n+x-1}{x} \theta^n (1-\theta)^x = 0$$

$$Now \quad \theta^n \sum_{x=1}^{n} g(x) \binom{n+x-1}{x} (1-\theta)^x = 0$$

Is true only when $g(x) = 0\ for\ x = 0, 1, 2, \ldots$ It follows that S is complete and also sufficient.

S is complete sufficient statistics.

$$Now\ E(S) = \frac{n(1-\theta)}{\theta} = np$$

$Hence\ E\left(\frac{S}{n}\right) = p, this\ show\ that\ \ T = \left(\frac{S}{n}\right) = \bar{X}\ is\ an\ unbiased\ estimator\ of\ t(p) = p.$

It follows by Lehmann scheffe theorem; $\bar{X}$ is an UMVUE of θ.

22. Find the continuous solution of

$$\boldsymbol{\frac{dy}{dx} + y = g(x), 0 \le x < \infty; y(0) = 2}$$

Where

$$\boldsymbol{g(x) = \begin{cases} 3, & 0 \le x < \pi/2 \\ \cos x, & x \ge \pi/2 \end{cases}}$$

Solution:

$$\frac{dy}{dx} + y = g(x)$$

$$\Rightarrow \frac{dy}{dx} + y = 3; \quad 0 \le x < \frac{\pi}{2}$$

$$\Rightarrow ye^x = \int 3e^x + c$$

$$\Rightarrow y = 3 + ce^{-x}$$

$$y(0) = 2$$

$$\Rightarrow 2 = 3 + c$$

$$\Rightarrow c = -1$$

$$\Rightarrow y = 3 - e^{-x}; 0 \le x < \frac{\pi}{2}$$

$$\frac{dy}{dx} + y = \cos x; \quad x \ge \frac{\pi}{2}$$

$$\Rightarrow ye^x = \int \cos x e^x \, dx + c$$

$$= \int e^x \left(\frac{1}{2}(\cos x + \sin x) + \frac{1}{2}(\cos x - \sin x)\right) dx + c$$

$$= \frac{e^x(\cos x + \sin x)}{2} + c$$

$$\Rightarrow y = \frac{\cos x + \sin x}{2} + ce^{-x}; \quad x > \frac{\pi}{2}$$

For continuity of solution $a + \frac{\pi}{2}$

$$3 - e^{-\pi/2} = \frac{0 + 1}{2} + ce^{-\pi/2}$$

$$\Rightarrow \frac{5}{2} - e^{-\pi/2} = ce^{-\pi/2}$$

$$\Rightarrow c = \frac{5}{2}e^{\pi/2} - 1$$

So, general solution is

$$y = 3 - e^{-x}; 0 \leq x < \pi/2$$

$$i.e., y = \left(\frac{5}{2}e^{\frac{\pi}{2}} - 1\right)e^{-x} + \frac{\cos x + \sin x}{2}; \quad x \geq \pi/2$$

23. Let $X_1, X_2, \ldots,$ be a sequence of independent and identically distributed random variables each with mean 4 and variance 4. Show that for large n,

$$\mathbf{0.5 \leq P\left[16n - 12\sqrt{n} \leq \sum_{i=1}^{n} X_{2i}X_{2i-1} \leq 16n + 12\sqrt{n}\right] \leq 0.9}$$

Solution:

$E(X_1 + X_2 + \cdots + X_n) = 4n$

$V(X_1 + X_2 + \cdots + X_n) = 4n$

$$As, X_1^2 + X_2^2 + \cdots + X_n^2 = \sum X_i^2 = \left(\sum X_i\right)^2 - 2\sum X_i X_j i_{7j}$$

$$Also\ we\ know\ \ E\sum_{i=1}^{n} X_{2i}X_{2i-1} = 16n$$

So, for large, n

$0.7 - 0.2 \leq P\left(\left|\sum(X_{2i} - X_{2i-1}) = 16n\right| \leq 12\sqrt{n}\right) \leq 0.7 + 0.2$

$\Rightarrow 0.5 \leq P\left[\left|\sum(X_{2i} - X_{2i-1}) - 16n\right| \leq 12\sqrt{n}\right] \leq 0.9$

$\Rightarrow 0.5 \leq P\left[16n - 12\sqrt{n} \leq \sum(X_{2i} - X_{2i-1}) \leq 16n + 12\sqrt{n}\right] \leq 0.9$

24. An urn contains ten balls of which M (an unknown number) are white. To test the hypothesis $H_0 : M = 3\ against\ H_1 : M = 7$, three balls are drawn at random from the urn without replacement. If X is the number of white balls drawn, show that the most powerful test rejects $H_0\ if\ X \geq k$, where k is a constant. Find the power, if the size of this test is 11/60.

Solution : Size of the test,

$$\alpha = \frac{11}{60}$$

$$As\ \alpha = P(reject\ |H_0 is\ true) = P(reject\ |M = 3)$$

$M = 3$ gives

$$P(X = x) = \frac{\binom{3}{x}.\binom{7}{3-x}}{\binom{10}{3}} \; ; x = 0, 1, 2, 3$$

$$P(X = 0) = \frac{\binom{7}{3}}{\binom{10}{3}} = \frac{35}{120}$$

$$P(X = 1) = \frac{63}{120}$$

$$P(X = 2) = \frac{21}{120}$$

$$P(X = 3) = \frac{1}{120}$$

As, $P(X = 2) + P(X = 3) = 11/60$

So, critical values are {2, 3}

Acceptance region = {0, 1}

H_1 is true implies $M = 7$

$\beta = P(accepts \; |H_1 \; is \; true)$

$$\Rightarrow \beta = P(X = 0, 1 \; |7 \; white \; balls) = \frac{\binom{3}{3}.\binom{7}{0} + \binom{3}{2}.\binom{7}{1}}{\binom{10}{3}} = \frac{1 + 21}{120} = \frac{22}{120} = \frac{11}{60}$$

$$\Rightarrow Power, 1 - \beta = 1 - \frac{11}{60} = \frac{49}{60}$$

25. (a) Evaluate the integral

$$\iint_R e^{(x^2+y^2)/2} dxdy$$

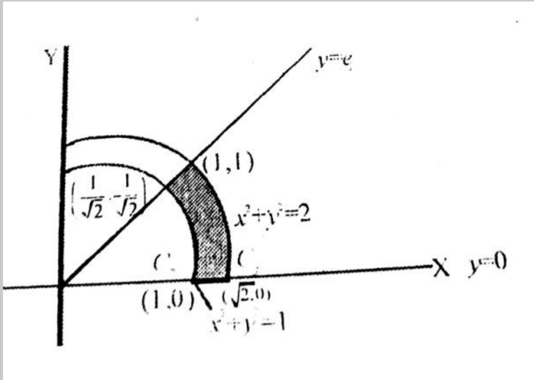
Where R is the region bounded by the lines $y = 0 \; and \; y = x$, and the arcs of the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$.

(b) Let

$$f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x^2 + y^2} & , (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Determine whether the function is continuous and differentiable at (0,0).

Solution: (a)



$$\Rightarrow \iint_R e^{(x^2+y^2)/2} dx\, dy$$

$$= \iint_{C_1} e^2 dx\, dy - \iint_{C_2} e^{\frac{1}{2}} dx\, dy$$

$$= \frac{\frac{\pi}{4}}{2\pi} e^1 \left(\pi\left(\sqrt{2}\right)^2\right) - \frac{\frac{\pi}{4}}{2\pi} e^{\frac{1}{2}} (\pi(1)^2)$$

$$= \frac{\pi e}{4} - \frac{\pi e^{1/2}}{8}$$

$$= \frac{\pi e^{\frac{1}{2}}}{8}\left(2e^{\frac{1}{2}} - 1\right)$$

(b)

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$f(x,mx) = \underset{x \to 0}{Lt} \frac{x^3 + (mx)^3}{x^2 + (mx)^2} \quad along \quad y = mx$$

$$= \underset{x \to 0}{Lt} \frac{x(1+m^3)}{1+m^2} = 0$$

This is true for any curve passing through the origin (0,0).

$$\Rightarrow \underset{(x,y) \to (0,0)}{Lt} f(x,y) = 0 = f(0,0)$$

So, f(x, y) is continuous at (0,0)

$$f_x = \underset{h \to 0}{Lt} \frac{h^3}{h^2} = \underset{h \to 0}{Lt} h = 0$$

$$f_y = \underset{h \to 0}{Lt} \frac{h^3}{h^2} = \underset{h \to 0}{Lt} h = 0$$

$$\underset{(h,k) \to (0,0)}{Lt} \frac{f(0+h, 0+k) - f(0,0)}{(h,k)}$$

$$= \underset{h \to 0}{Lt} \frac{\frac{h^3}{h^2} - 0}{h} = 1; \textit{ along x axis in positive direction}$$

$$= \underset{h \to 0}{Lt} \frac{\frac{-h^3}{h^2}}{h} = 1; \textit{ along x axis in negative direction}$$

As f_x *and* f_y are continuous, so derivative exists at (0,0).