# IIT–JAM Mathematical Statistics (MS) 2008

**1. Consider the following two series**

$$S_1 = \sum_{k=1}^{\infty} \frac{1}{(k+1)(k+3)}, \quad S_2 = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}\sqrt{k+3}}$$

**Then**

**(a) $S_1$ $and$ $S_2$ converge**
**(b) $S_1$ diverges and $S_2$ converges**
**(c) $S_1$ converges and $S_2$ diverges**
**(d) $S_1$ $and$ $S_2$ diverge**

**Solution: (c)**

$$S_1 = \sum \frac{1}{(k+1)(k+3)}$$

Which is of the form

$$\sum \frac{1}{(n+1)(n+3)}$$

And when compared with $\sum \frac{1}{n^2}$ it has finite limit, so it is convergent but

$$\sum \frac{1}{\sqrt{k+1}\sqrt{k+3}}$$

has finite limit with $\sum \frac{1}{n}$ , so it is divergent.

**2. Let f(x) = $[x]$ $and$ $g(x) = \begin{cases} x, & 0 \le x < 1 \\ x-1 & 1 \le x < 2 \\ x-2 & 2 \le x < 3 \\ 0, & x = 3 \end{cases}$**

**$for$ $x \in [0,3]$. Then f(x) + g(x) is**

**(a) Discontinuous at points 1 and 2**
**(b) Continuous on [0,3] but not differentiable on (0,3)**
**(c) Differentiable once but not twice on (0,3)**
**(d) Twice differentiable on (0,3)**

**Solution: (d)**

$$f(x) + g(x) = \begin{cases} x\,; & 0 \le x < 1 \\ x\,; & 1 \le x < 2 \\ x\,; & 2 \le x < 3 \\ 0\,; & x = 3 \end{cases}$$

So it is twice differentiable.

**3. The area of the region enclosed by the curve $y = x^2$ and the straight line $x + y = 2$ is**

**(a) 3**
**(b) 27/2**
**(c) 9/2**
**(d) 9**

**Solution: (c)** $y = x^2 \; and \; x + y = z$

Intersects at $x^2 = 2 - x \;\; \Rightarrow x^2 + x - 2 = 0 \;\; \Rightarrow x = -2,1$

So, points of intersection is (1, 1) and (−2, 4). So, area enclosed between them is

$$\int_{-2}^{1} x^2 - (2 - x)dx = \frac{x^3}{3} - 2x + \frac{x^2}{2}\Big|_{-2}^{1} = \left(\frac{1}{3} - 2 + \frac{1}{2}\right) - \left(-\frac{8}{3} + 4 + 2\right) = -\frac{9}{2}$$

$$\Rightarrow Area \; = \frac{9}{2}$$

**4. If $\int_0^x f(t)dt \; = x^2 \sin x + x^3 \; then \; f\left(\frac{\pi}{2}\right) \; is$**

**(a)** $\left(\frac{\pi}{2}\right)^2 + \left(\frac{\pi}{3}\right)^3$
**(b)** $\pi + \frac{3\pi^2}{4}$
**(c)** $\pi - \frac{3\pi^2}{4}$
**(d) 0**

**Solution: (b)**

$$\int_0^x f(t)dt = x^2 \sin x + x^3 \; \ldots . (i)$$

Differentiating both sides with respect to x, we get

$$f(x) = 2x \sin x + x^2 \cos x + 3x^2$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \frac{2\pi}{2} + 3\left(\frac{\pi}{2}\right)^2 = \pi + \frac{3\pi^2}{4}$$

5. **If A is a $3 \times 3$ non–zero matrix such that $A^2 = 0$, then the number of non–zero eigen values of A is**
   **(a) 0**
   **(b) 1**
   **(c) 2**
   **(d) 3**

**Solution: (a)** $A^2 = 0$

$\Rightarrow A$ is an idempotent matrix, so all of it's eigen values will be zero.

6. **The solution of the differential equation**

$$\frac{dy}{dx} = \frac{x(x^2 + y^2 - 10)}{y(x^2 + y^2 + 5)}, \qquad y(0) = 1 \text{ is}$$

   **(a)** $x^4 - 2x^2y^2 - y^4 - 20x^2 - 10y^2 + 11 = 0$
   **(b)** $x^4 + 2x^2y^2 + y^4 + 20x^2 + 10y^2 - 11 = 0$
   **(c)** $x^4 + 2x^2y^2 - y^4 + 20x^2 - 10y^2 + 11 = 0$
   **(d)** $x^4 + 2x^2y^2 - y^4 + 20x^2 + 10y^2 - 11 = 0$

**Solution: (d)**

$$\frac{dy}{dx} = \frac{x(x^2 + y^2 - 10)}{y(x^2 + y^2 + 5)}$$

$$\Rightarrow x(x^2 + y^2 - 10)dx + y(x^2 + y^2 + 5)dy = 0 \ldots..(i)$$

$$Here, M = x(x^2 + y^2 - 10)$$

$$and\ N = y(x^2 + y^2 + 5)$$

$$\Rightarrow \frac{\partial M}{\partial N} = 2xy \ \& \ \frac{\partial N}{\partial x} = 2xy$$

$$As\ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So, differential equation is exact. So, its solution is

$$\frac{x^4}{4} + \frac{x^2y^2}{2} = \frac{10x^2}{2} + \frac{y^4}{4} + \frac{5y^2}{2} + c \ldots.(ii)$$

$$y(0) = 1 \quad \Rightarrow \frac{1}{4} + \frac{5}{2} - c = \frac{11}{4}$$

So, solution is

$$x^4 + 2x^2y^2 - 20x^2 + y^4 + 10y^2 - 11 = 0$$

7. **Let X be Poisson (2) and Y be Binomial (10, ¾) random variables. If X and Y independent, then P(XY= 0) is**

(a) $e^{-2}+\left(\frac{1}{4}\right)^{10}(1-e^{-2})$

(b) $e^{-2}+\left(\frac{1}{4}\right)^{10}(1-2e^{-2})$

(c) $e^{-2}\left(\frac{1}{10}\right)^{10}$

(d) $e^{-2}+1-\left(\frac{1}{10}\right)^{10}$

**Solution: (a)**

$$P(X=x)=e^{-2}\frac{2^x}{x!}; \quad x=0,1,2,\dots$$

$$P(Y=y)=\binom{10}{1}\left(\frac{3}{4}\right)^y\left(\frac{1}{4}\right)^{10-y}$$

$$P(XY=0)=P(X=0)P(Y\neq 0)+P(X\neq 0)$$

$$P(Y=0)+P(X=0)P(Y=0)$$

$$=e^{-2}\left(1-\left(\frac{1}{4}\right)^{10}\right)+(1-e^{-2})\left(\frac{1}{10}\right)^4+(e^{-2})\left(\frac{1}{4}\right)^{10}=e^{-2}\left(\frac{1}{4}\right)^{10}(1-e^{-2})$$

8. **Let $X_1,X_2,\dots,X_n$ $(n>1)$ be a random sample from Exp (1). Then the distribution of $(2n\bar{X})$ is**

(a) Exp (1/2)

(b) Exp (2n)

(c) $x_n^2$

(d) $x_{2n}^2$

**Solution: (d)** $X_1,X_2,\dots X_n$ (n > 1) be a random sample from Exp(1). The distinct of

$$\sum_{i=1}^{n}X_i\sim G(1,n) \quad or \quad nX\sim G(1,n)$$

$$y=nX\sim G(1,n) \quad so \quad M_y(t)=(1-t)^{-n}$$

So, the distinct of $2n\bar{X}=2y$

$$M_{2y}(t)=M_y(2t)=(1-2t)^{-n}$$

$$M_{2y}(t)=(1-2t)^{-2n/2}$$

which is of the m.g.f. of $x^2$ with 2n degree of freedom by uniqueness properly of M.G.F. theorem.

$So, 2n\bar{X}\sim x^2_{(2n)}$

9. **Suppose X is a random variable with finite variance. For $0 < \theta < 1$ and n>3, let $X_1 = X, X_2 = \theta X_1,\ X_3 = \theta X_2, \dots, X_n = \theta X_{n-1}$. Then $Corr\ (X_1, X_n)$ is**
   **(a) 1**
   **(b) 1**
   **(c) $\theta^{n-2}$**
   **(d) $\theta^{n-3}$**

**Solution: (a)**

X is a r.v. with finite variance.

Let $X_1 = X$

$X_2 = \theta X_1 = \theta X$

$X_3 = \theta X_2 = \theta(\theta X) = \theta^2(X) \dots$

$X_n = \theta X^{n-1} = \theta(\theta^{n-2}X) = \theta^{n-1}X$

$E(X_1) = E(X) = \mu$

$E(X_n) = E(\theta^{n-1}X) - \theta^{n-1}E(X) = \theta^{n-1}m$

$E[X_1, X_n] = E[X, \theta^{n-1}X] = \theta^{n-1}E[X^2]$

$cov\ (X_1, X_n) = E(X_1, X_n) - E(X_1)E(X_n)$

$= \theta^{n-1}E(X^2)\ \ = \theta^{n-1}\{E(X)\}^2$

$= \theta^{n-1}\{E(X^2) - E((X)^2\}\ = \theta^{n-1}V(X)$

$= V(X_1) = V(X)V(X_n) = V(\theta^{n-1}X) = \theta^{2n-2}V(X)$

$$r = \rho_{x_1, x_n} = \frac{cov\ (X_1, X_n)}{\sqrt{V(X_1)V(X_n)}} = \frac{\theta^{n-1}V(X)}{\sqrt{(\theta^{n-1})^2 V(X)V(X)}}$$

$Corr(X_1, X_n) = 1$

10. **Let $X_1, X_2, X_3$ be independent random variables with $X_k\ (k = 1, 2, 3)$ having the probability density function**

$$f_k(x) = \begin{cases} k\theta e^{-k\theta x} & , 0 < x < \infty \\ 0, & otherwise \end{cases} \quad where\ \theta > 0$$

**Then a sufficient statistic for $\theta$ is**

   **(a) $X_1 + X_2 + X_3$**
   **(b) $3X_1 + 2X_2 + X_3$**
   **(c) $X_1 + X_2 + 3X_3$**
   **(d) $X_1 + 2X_2 + 3X_3$**

**Solution: (d)**

Let $X_1, X_2, X_3$ be independent r.v. with $X_k(k = 1,2,3)$ having p.d.f.

$$f_k(x) = \begin{cases} k\theta e^{-k\theta x} & ,0 < x < \infty \\ 0, & otherwise \end{cases}$$

$$h(x,\theta) = f(x_1), f(x_2) f(x_3) = \theta e^{-\theta x_1} 2\theta e^{-2x_2}$$

$$h(x,\theta) = \frac{6\theta^3 e^{-\theta(x_1+2x_2+3x_3)}}{g(t\ 1,\theta)} .1$$

So, $(x_1 + \ 2x_2 + 3x_3)$ is sufficient statistics for $\theta$.

**11. Let X be a random variable such that $E|X| < \infty$ and**

$$P\left(x > \frac{1}{2} + x\right) = P\left(X < \frac{1}{2} < x\right) \ for \ all \ x \in R$$

**Then**

**(a) $E(X) = 1/2$ and Median (X) $= 1/2$**
**(b) E(X)= $1/2$ and Median (X) > ½**
**(c) E(X) <1/2 and Median (X) $= 1/2$**
**(d) E(X) < ½ and Median (X) > ½**

**Solution: (a)** $P\left(x \geq \frac{1}{2} + x\right) = P\left(X \leq \frac{1}{2} - x\right)$

$\Longrightarrow$ x follows normal distribution with first parameter ½.

So, $E(X) = \frac{1}{2}$ and Median (X) =1/2.

**12. Let $X_1, X_2, \ldots$ be a sequence of independent random variables. Suppose, for $k = 1, 2, \ldots$**
**$P(X_{2k-1} \quad 1) = P(X_{2k-1} \quad 1)$ ½ and the probability density function of $X_{2k}$ is**

$$f(x) = \frac{1}{\sqrt{2}} e^{-\sqrt{2|x|}}, \infty < x < \infty \ then$$

$$\lim_{n\to\infty} P\left|\frac{X_1 + X_2 + \cdots + X_{2n}}{\sqrt{2n}} > 1\right| \ is$$

**(a) $\phi(1)$**
**(b) 1/2**
**(c) $\phi(1)$**
**(d) 1**

**Solution: (b)** Asymptotic distribution tends to have probability ½.

13. Let $X_1, X_2, \dots, X_n$ be a random sample from $N(\theta, 1)$, where $-\infty < x < \infty$. Consider the following two tests for testing $H_0 : \theta = 2.5 \text{ against } H_1 : \theta = 4$

**Test 1 : Reject** $H_\theta \; if \; X_1 > 4$

**Test 2 : Reject** $H_\theta \; if \; \overline{X} > 3$

Suppose $\alpha_k \text{ and } \beta_k$ are the probabilities of Type I and Type II errors, respectively, for Test k (k= 1,2). Then

(a) $\alpha_1 = \alpha_2 \text{ and } \beta_1 = \beta_2$
(b) $\alpha_1 = \alpha_2 \text{ and } \beta_1 > \beta_2$
(c) $\alpha_1 > \alpha_2 \text{ and } \beta_1 = \beta_2$
(d) $\alpha_1 = \alpha_2 \text{ and } \beta_1 < \beta_2$

**Solution: (b)** $\alpha_0 = P\left[x \in \frac{w}{H_0}\right]$

$$\alpha_1 = P\left[x_1 > \frac{4}{H_0}\right] = P\left[\frac{x_1 - 2.5}{1} > \frac{4 - 2.5}{1}\right]$$

$$\alpha_1 = P[Z > 1.5]$$

$$\alpha_2 = P[\bar{x}, 3] = P\left[\frac{\bar{x} - 2.5}{\frac{1}{3}} > \frac{3 - 2.5}{\frac{1}{3}}\right]$$

$$\alpha_1 = P[Z > 1.5] so\ \alpha_1 = \alpha_2$$

$$1 - \beta_1 = P\left[x_1 > \frac{4}{H_1}\right] = P\left[\frac{x_1 - 4}{1} > \frac{4 - 4}{1}\right]$$

$$1 - \beta_1 = P[Z > 0 = 1/2$$

$$so, \beta_1 = 1/2$$

$$1 - \beta_2 = P\left[\bar{x} > \frac{3}{H_1}\right] = P\left[\frac{\bar{x} - 4}{\frac{1}{3}} > \frac{3 - 4}{\frac{1}{3}}\right]$$

$$1 - \beta_2 = P\left[Z > -\frac{1}{3}\right]$$

$$\Rightarrow \beta_2 = 1 - P\left[Z > -\frac{1}{3}\right]$$

$$\beta_2 = P\left[Z < -\frac{1}{3}\right]$$

So, $P\left[Z < -\frac{1}{3}\right] < 1/2$

So, $\beta_1 < \beta_2,\ \alpha_1 = \alpha_2 \ \text{and}\ \beta_1 > \beta_2$

14. Let $X_1, X_2, \ldots, X_n$ $(n > 1)$ be a random sample from $U(\theta, \theta+1)$. Consider the following three estimators for $\theta, \theta \in R$

$$T_1 = X_{(n)}$$

$$T_2 = \frac{X_{(1)} + X_{(n)}}{2}$$

$$T_3 = \frac{X_{(1)} + X_{(n)}}{2} - \frac{1}{2} \; Then$$

(a) $T_1$ and $T_3$ both are maximum likelihood estimators of q while $T_2$ is not
(b) $T_3$ is the unique maximum likelihood estimator of $\theta$
(c) $T_3$ is a maximum likelihood estimator of $\theta$ while $T_1$ and $T_2$ are not
(d) $T_1$ and $T_2$ both are maximum likelihood estimators of $\theta$ *while* $T_3$ is not

**Solution:** $X_1, X_2, \ldots, X_n$ be random sample from $U(\theta, \theta+1)$

$$so, f(x,\theta) = \frac{1}{\theta + 1 - \theta} = 1; \quad \theta < x < \theta + 1$$

$$L(x,\theta) = 1 \quad ; \theta < x < \theta + 1$$

So, $L(x,\theta)$ is maximum when

$$\theta < x_{(1)} \; and \; x_{(n)} < \theta + 1$$

$$\theta < x_{(n)} \; and \; x_{(n)} - 1 < \theta$$

$$x_{(n)} - 1 < \theta < x_{(1)}$$

So, $x_{(1)}$ *and* $x_{(n-1)}$ is joint MLE o/$\theta$.

$$so, T_2 = \frac{x_{(1)} + x_{(n)}}{2} \; and \; T_3 = \frac{x_{(1)} + x_{(n)}}{2} - \frac{1}{2}$$

Is only MLE of $\theta$.

15. A nonempty subset P is formed by selecting elements at random and without replacement from a set B consisting of n(>1) distinct elements. Another nonempty subset Q is formed in a singular fashion from the original set B consisting of the same n elements. Then the probability that P and Q do not have any common element is

(a) $\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\binom{n}{i}\binom{n-i}{j}}{\sum_{i=1}^{n}\sum_{j=1}^{n}\binom{n}{i}\binom{n}{j}}$

(b) $\frac{\sum_{i=0}^{n}\sum_{j=0}^{n-1}\binom{n}{i}\binom{n-i}{j}}{\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}\binom{n}{j}}$

**(c)** $\frac{\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\binom{n}{i}\binom{n-i}{j}}{\sum_{i=1}^{n}\sum_{j=1}^{n}\binom{n}{i}\binom{n}{j}}$

**(d)** $\frac{\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\binom{n}{i}\binom{n-i}{j}}{\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}\binom{n}{j}}$

**Solution:** **(b)** The sample space will be

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}\binom{n}{j}$$

The number of elements in favour from sample space will be

$$\sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}$$

So required probability

$$=\frac{\sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}}{\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}\binom{n}{j}}$$

**16. (a) One coin is selected at random from two coins. The probability of obtaining head let one of them is 1/3 and for the other it is ½. If the selected coin is tossed and the head shows up. What is the probability that it is the fair coin?**

**(b) Let p denote the probability that the weather (either wet or dry) tomorrow will be as that of today. If the weather is dry today, show that $P_n$, the probability that it will be dry n days later, satisfies.**

$$\boldsymbol{P_n = (2p-1)P_{n-1} + (1-p), n^3\ \ 1}$$

**Hence or otherwise determine the value of $P_{50}$ $for\ p = 3/4$.**

**Solution:**

**(a)** Required probability will be

$$P(A|B) = P(A \cap B)/P(B)$$

$$=\frac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{3}}=\frac{3}{5}$$

$A \equiv$ fair coin was chosen

$B \equiv$ head was turned up.

**(b)** The transition probability matrix of the given problem will be

$$P_1 = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

$$\Rightarrow p_n = P_1^n = (2p-1)P_{n-1} + (1-p)$$

$$\Rightarrow p_{50} = (2p-1)p_{49} + (1-p) \ldots\ldots (1)$$

$$p_{49} = (2p-1)p_{48} + (1-p) \ldots\ldots (2)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$p_2 = (2p-1)p + (1-p) - 49$$

Equation $2 \times (2p-1)$ *equation* $(3) \times (2p-1)^2 + \cdots +$ *equation of* $49 \times (2p-1)^{48}$ gives.

$$p_{50} = (2p-1)^{49}p + (1-p)[1 + (2p-1) + \cdots + (2p-1)^{48}]$$

$$p(2p-1)^{49} + (1-p)\frac{[1-(2p-1)^{49}]}{1-(2p-1)}$$

$$\Rightarrow p_{50} = p(2p-1)^{49} + \frac{[1-(2p-1)^{49}]}{2}$$

$$= (2p-1)^{49}\left[p - \frac{1}{2}\right] + \frac{1}{2}$$

At $p = 3/4\,;\; p_{50} = \left(\frac{1}{2}\right)^{49} + \left(\frac{1}{4}\right) + \frac{1}{2} = \frac{1}{2} + \left(\frac{1}{2}\right)^{51}$.

**17. (a) Let X and Y have the joint probability mass function**

$$P(X = x, Y = y) = \begin{cases} \dfrac{e^{-z}}{x!\,(y-x)!} & , x = 0, 2, \ldots, y, y = 0, 1, 2, \ldots \\ 0, & otherwise \end{cases}$$

**Determine M(t$_1$ , t$_2$), the joint moment generating function of (X, Y).**

**(b) The conditional probability density function of X given**

$Y = y\ (> 0)$ **is**

$$f(x|y) = \begin{cases} ye^{-yx} & , x > 0 \\ 0, & otherwise \end{cases}$$

**And the marginal probability density function of Y is**

$$g(y, \alpha) = \begin{cases} \alpha e^{-\alpha y} & , y > 0 \\ 0, & otherwise \end{cases}$$

**Derive the conditional probability density function of Y given $X = x$.**

**Solution: 17.** $P(X = x, Y = y) =$

$$= \frac{e^{-2}}{x!\,(y-x)!}$$

$$x = 0, 1, 2, \dots$$

$$y = 0, 1, 2, \dots$$

$$Mx^4[t_1, t_2] = e^{-2} \sum_{y=0}^{x} \sum_{x=0}^{y} \frac{1}{x!\,(y-x)!}$$

$$= e^{-2} \sum_{y=0}^{\infty} \frac{1}{y!} \sum_{x=0}^{y} \binom{y}{x}$$

$$= e^{-2} \sum_{y=0}^{\infty} \frac{1}{y!} \sum_{x=0}^{\infty} \frac{y!}{x!\,(y-x)!} e^{xt_1} e^{t_2 y}$$

$$= e^{-2} \sum_{y=0}^{\infty} \frac{1}{y!} e^{t_2 y} \sum_{x=0}^{\infty} \binom{y}{x} (e^{t_1})^x$$

$$= e^{-2} \sum_{y=0}^{\infty} \frac{1}{y!} e^{t_2 y} (1 + e^{t_1})^y$$

$$= e^{-2} \sum_{y=0}^{\infty} \frac{\{e^{t_2}(1 + e^{t_1})\}^y}{y!}$$

$$= e^{-2} e^{e^{t_2}(1+e^{t_1})}$$

$$f\left(\frac{x}{y}\right) = \frac{f(x,y)}{f(y)}$$

$$\Rightarrow f(x,y) = \alpha y e^{-yx} e^{-\alpha y} \quad x > 0\,; y > 0$$

$$f(x) = \alpha \int_0^{\infty} y e^{-yx} e^{-\alpha y} dy \ = \alpha \int_0^{\infty} y^{2t} e^{-y(x+\alpha)} dy$$

$$f(x) = \frac{\alpha\sqrt{2}}{(x+\alpha)^2} \; f(x) = \frac{\alpha}{(x+\alpha)^2} \quad x > 0$$

$$f\left(\frac{y}{x}\right) = \frac{f(x,y)}{f(x)} = \frac{\alpha y e^{-xy} e^{-\alpha y}}{\alpha(x+\alpha)^2}$$

$$= (x+\alpha)^2 y e^{-y(x+\alpha)}$$

$$y > 0$$

**18. Let $X_1, X_2, \dots, X_n$ be a random sample from a distribution having the probability density function**

$$f(x,\theta) = \begin{cases} \frac{2x}{\theta^2} & ,0 < x \le \theta \\ 0, & otherwise \end{cases}$$

**(a) Find the maximum likelihood estimator $\hat{\theta}$ of q and determine the constant c such that $E(c\hat{\theta}) = q$**

**(b) Find the method of moments estimator of q.**

**Solution:** $f(x,\theta) = 2x/\theta^2 \qquad 0 < x \le \theta$

$$L = \left(\frac{2}{\theta^2}\right)^n \frac{\sigma/x_i}{\mu} \dots\dots\dots\dots\dots..(i)$$

$$\frac{\partial L}{\partial \theta} = -\frac{2n}{\theta} = 0 \quad or\ \theta = \infty\ obviously\ absurd\ result.$$

In this case we locate M.L.E. as follows, we have to choose $\theta$ so that L in equation (i) maximum. Now L is maximum $iy\ \theta$ is minimum.

Let $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ be ordered random M.L.E. of $\theta = x_{(n)}$

$$E(c\hat{\theta}) = \theta$$

$$\Rightarrow E[cx_{(n)}] = cE[x_{(n)}]$$

$$P(x) = \frac{2}{\theta^2}\int_0^x x dx = \frac{2}{\theta^2}\left[\frac{x^2}{2}\right]_0^x = \frac{x^2}{\theta^2} = \left(\frac{x}{\theta}\right)^2$$

$$so, F(x_{(n)}) = [F(x)]^n = \left(\frac{x^2}{\theta^2}\right)^n = \frac{x^{2n}}{\theta^{2n}}$$

$$f(x_n) = \frac{2nx^{2n-1}}{\theta^{2n}} \quad 0 < x < \theta$$

$$E(x_{(n)}) = \frac{2n}{\theta^{2n}}\int_0^\theta x^{2n} dx = \frac{2n}{\theta^{2n}}\left[\frac{\theta^{2n+1}}{2n+1}\right] = \frac{2n\theta}{(2n+1)}$$

$$so, E[cx_{(n)}] = c\frac{2n\theta}{(2n+1)} = \theta$$

$$\Rightarrow c = \frac{2n+1}{2n}$$

$$f(x,\theta) = \frac{2x}{\theta^2}; \quad 0 < x \le \theta$$

$$E(x) = \frac{2}{\theta^2}\int_0^\theta x^2 dx = \frac{2}{\theta^2}\left[\frac{\theta^3}{3}\right] = \frac{2}{3}\theta$$

$$E\left(\frac{3}{2}x\right) = \theta$$

$$\Rightarrow T(x_1, x_2, \dots, x_n) = \frac{3}{2}\bar{x}$$

19. Let $X_1, X_2, \dots, X_n$ be a random sample from a distribution having the probability density function

$$f(x, \theta) = \begin{cases} \theta 2^{\theta}\, x^{-(\theta+1)}, x > 1 \\ 0, \ \ otherwise \end{cases}$$

**Where**

$$\sum_{k=1}^{n} ln\, X_k$$

(a) Show that $\sum_{k=1}^{n} ln\, X_k$ is sufficient and complete for q.

(b) Find the Cramer–Rao lower bound of the variance of an unbiased estimator of $(ln\, q)$

**Solution:**

(a) $f(x,\theta) = \theta^{2\theta} x^{(\theta+1)} \quad x > 2$

$$L = \frac{\theta^n 2^{n\theta} \prod_{i=1}^{n} x_i^{-(\theta+1)}}{g(t,\theta)} \cdot \frac{1}{h(x)}$$

$$so, \prod_{i=1}^{n} x_i = \sum \log x_k \ is\ sufficient\ statistic\ is\ \theta.$$

20. Let $X_1$ and $X_2$ be a random sample from a distribution having the probability density function f(x). Consider the resting of $H_\theta : f(x) = f_0(x)$ against $H_1$, $f(x) = f_1(x)$ based on $X_1$ and $X_2$, where

$$f_0(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & otherwise \end{cases}$$

$$and\ f_1(x) = \begin{cases} 4x^3 & ,0 < x < 1 \\ 0, & otherwise \end{cases}$$

(a) For given level, show that the critical region of the most powerful test for testing $H_0$ against $H_1$ is of the form $\{(x_1, x_2) : ln\, x_1 + ln\, x_2 > c\}$ for some constant c.

(b) Determine c in terms of a suitable cutoff point of a Chi–square distribution when the level is a.

**Solution:**

**(a)** For any given level, the critical region of the most powerful test for testing $H_2\ against\ H_1$ should be one of the form.

$$(x_1, x_2) : \ x_1 x_2 > k$$

Where k should be a positive constant, so

$$\ln x_1 x_2 > \ln k$$

$$or, \qquad \ln x_1 + \ln x_2 > c \, , for\ some\ constant\ c$$

**(b)**

$$\int_R \int \ln x_1 + \ln x_2 > \ln c$$

Gives the upper limit for c which is used to evaluate the values of c at different $\alpha$.

**21. (a) The random variable X has the distribution function**

$$\boldsymbol{F(x)} = \begin{cases} \mathbf{0}, & \boldsymbol{x} < 0 \\ \dfrac{\mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \mathbf{1}}{\mathbf{10}} & \mathbf{0} \leq \boldsymbol{x} < 1 \\ \dfrac{\mathbf{4}}{\mathbf{5}}, & \mathbf{1} < x < 2 \\ \dfrac{(\boldsymbol{x} + \mathbf{2})^{\mathbf{4}} + \mathbf{16}}{\mathbf{20}} & \mathbf{2} < x < 3 \\ \mathbf{1}, & \boldsymbol{x} > 3 \end{cases}$$

**Find the values of $P$ and P(0.5 < X <3).**

**(b) Let X be a normal random variable with mean 0 and variance 1. Show that**

**$P(X \geq c) \leq e^{-ct+\frac{t^2}{2}}$ for c > 0.**

**Solution:**

**(a)** $P[1 < X < 2.5] = F(2.5)\ F(1)$ Implying $\frac{(0.5)^4}{20} + \frac{16}{20} + \frac{4}{5} = \frac{1}{320}$

$P[0.5 < X < 3] = F(3) - F(0.5) = 1 - \frac{2(0.5)^5}{10} + 1 = 1 - \frac{1.5}{10} = \frac{10-1.5}{10} = \frac{8.5}{10} = \frac{85}{100} = \frac{13}{20}$

**(b)** $P[X \geq 0] < e^{-ct} M_x^{(t)} = e^{-ct}\ e^{t^2/2}$

$$So, \qquad P[X \geq c] < e^{-ct+t^2/2}$$

**22. (a) Let $a_1 = 0$ and**

$$a_{n+1} = \frac{a_n^2 + 3}{2(a_n + 1)} \; for \; n > 1$$

**Show that the sequence $\{a_n\}$ converges and find its limit.**

**(b) Let**

$$a_{2n-1} = \frac{9^{n-1}}{16^{n-1}} \; and \; a_{2n} = \frac{9^{n-1}}{16^n} \; for \; n^3 \; 1$$

**Test whether $\sum_{n=1}^{\infty} a_n$ is convergent.**

**Solution:**

**(a)** The given sequences is bounded and monotonic, let it's limit be 'l'

$$so, \quad \lim a_{n+1} = \lim \frac{a_n^2 + 3}{2(a_n + 1)}$$

$$\Rightarrow l = \frac{l^2 + 3}{2(l + 1)}$$

$$\Rightarrow 2l^2 + 2l = l^2 + 3$$

$$\Rightarrow l^2 + 2l - 3 = 0$$

$$\Rightarrow (l + 3)(l - 1) = 0$$

$$\Rightarrow l = 1, -3$$

As sequence has positive terms only, so limit = 1.

**(b)** $a_{2n-1} = \frac{9^{n-1}}{16^{n-1}}$ and $a_{2n} = \frac{9^{n-1}}{16^n}$ $gives$

$$\sum_{n=1}^{\infty} a_n = 1 + \frac{1}{16} + \frac{9}{16} + \frac{9}{16^2} + \frac{9^2}{16^2} + \cdots$$

$$= \left(1 + \frac{9}{16} + \frac{9^2}{16^2} + \cdots\right) + \frac{1}{16}\left(1 + \frac{9}{16} + \frac{9^2}{16^2} + \cdots\right)$$

$$= \frac{17}{16}\left(1 + \frac{9}{16} + \frac{9^2}{16^2} + \cdots\right)$$

$$= \frac{\frac{17}{16}}{1 - \frac{9}{16}} = \frac{17}{7}$$

$$so, \sum_{n=1}^{\infty} a_n \; is \; convergent.$$

**23. (a) Let $f[a,b] \to R$ be a four times differentiable function such that $f^{iv}(x) > 0$ $for\ x \in (a,b)$. $If\ c \in (a,b)$ is such that $f'(c) = f^2(c) = f^{2'}(c) = 0$, then show that f has a minimum at c.**

**(b) Let f(x,y) $= \begin{cases} \frac{x(x^2-y^2)}{x^2+y^2} & ,(x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$**

**(i) Find the first order partial derivatives of f at the point (0, q), if they exist.**

**(ii) Check for the continuity and the differentiability of f at the point (0,0)**

**Solution:**

**(a)** As even order derivative is not zero, so $x = c$ is extreme point.

Further,

$$f^{iv}(x) > 0 \quad for\ x \in (a,b)$$

$$so, f^{iv}(c) > 0$$

Hence f has minimum at c.

**(b)**

$$f(x,y) = \begin{cases} \dfrac{x(x^2-y^2)}{x^2+y^2} & ;(x,y) \neq (0,0) \\ 0; & ;(x,y) = (0,0) \end{cases}$$

(i) $\frac{\partial t}{\partial x} = \frac{(3x^2-y^2)(x^2+y^2)-2x^2(x^2-y^2)}{(x^2+y^2)^2} = \frac{x^4-y^4+4x^2y^2}{(x^2+y^2)^2}$

Along $y = mx$

$$\underset{x \to 0}{Lt} \frac{\partial f}{\partial x} = \frac{1+4m^2-m^4}{(1+m^2)^2}$$

Which does not exist.

$$\frac{\partial f}{\partial y} = \frac{-2xy(x^2+y^2)-2xy(x^2-y^2)}{(x^2+y^2)^2} = \frac{-4x^3y}{(x^2+y^2)^2}$$

Along $y = mx$

$$\underset{x \to 0}{Lt} \frac{\partial f}{\partial y} = \frac{-4m}{(1+m^2)^2}\ does\ not\ exist.$$

(ii) As $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$

So, f(x,y) is continuous at (0,0)

But none of the partial derivative exist at (0,0), so f is not differentiable at (0,0).

**24. (a) Find the volume of the region bounded by the planes**

$$x = 0, y = 0, z = 0 \;\; and \;\; 6x + 4y + 3z = 12$$

**(b) Let**

$$A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & 6 & 9 & 18 \\ 1 & 2 & 6 & 12 \end{bmatrix}$$

**Find a basis for the null space of A.**

**Solution:**

**(a)** Volume required

$$= \int_0^4 \int_0^{\frac{12-3z}{4}} \int_0^{\frac{12-4y-3z}{6}} dxdydz$$

$$= \int_0^4 \int_0^{\frac{12-3z}{4}} \left(\frac{12-4y-3z}{6}\right) dydz$$

$$= \int_0^4 \left[2y - \frac{y^2}{3} - \frac{yz}{2}\right]_0^{\frac{12-3z}{4}} dz$$

$$= \int_0^4 \left[\left(6 - \frac{3z}{2}\right) - \left(\frac{9 - \frac{9}{16}z^2 - \frac{9}{2}z}{3}\right) - z\left(\frac{3}{2} - \frac{3z}{8}\right)\right] dz$$

$$= \int_0^4 \left(\frac{9}{16}z^2 - \frac{3}{2}z + 3\right) dz$$

$$= \frac{3}{16}z^3 - \frac{3}{4}z^2 + 3z \,\Big|_0^4$$

$$= 12 - 12 + 12 = 12$$

**(b)** AX= 0 $\;\; if \; X = (x, y, z, w)^T$

$$\Rightarrow x + 2y + 3z + 6w = 0 \ldots..(i)$$

$$2x + 6y + 9z + 18w = 0 \ldots \ldots.(ii)$$

$$x + 2y + 6z + 12w = 0 \ldots \ldots.(iii)$$

From (i) & (ii) $x = 0$

Now from (i) & (iii) $y = 0 \; and \; z + 2w = 0$

So, a basis for null space will be (0, 0, 1, −1/2)

**25. Solve the following differentiable equation**

$$\boldsymbol{\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-x} \sin\left(\frac{\pi}{3} - x\right),}$$

$$\boldsymbol{y(0) = y'(0) = 0}$$

**Solution:**

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-x}\sin\left(\frac{\pi}{3} - x\right)$$

$$\Rightarrow (D^2 + 4D + 4)y = e^{-x}\left[\frac{\sqrt{3}\cos x}{2} - \frac{\sin x}{2}\right]$$

$$so, c.f. is\ y = (C_1 + C_2x)e^{-2x}$$

$$P.I. is\ \frac{1}{(D+2)^2}e^{-x}\left[\frac{\sqrt{3}\cos x - \sin x}{2}\right]$$

$$= e^{-x}\frac{1}{D^2 + 2D + 1}\left[\frac{\sqrt{3}\cos x - \sin x}{2}\right]$$

$$= e^{-x}\frac{1}{2D}\left[\frac{\sqrt{3}\cos x - \sin x}{2}\right]$$

$$= \frac{e^{-x}}{2}\left[\frac{\sqrt{3}\sin x + \cos x}{2}\right]$$

$$= \frac{e^{-x}(\sqrt{3}\sin x + \cos x)}{4}$$

So, the general solution is

$$y = (C_1 + C_2x)e^{-2x} + \frac{e^{-x}(\sqrt{3}\sin x + \cos x)}{4}$$

$$\Rightarrow y' = -2(C_1 + C_2x)e^{-2x} + C_2e^{-2x} + \frac{e^{-x}}{4}(-\sqrt{3}\sin x - \cos x + \sqrt{3}\cos x - \sin x)$$

Now, $y'(0) = 0 \Rightarrow -2C_1 + C_2 + \frac{1}{4}(\sqrt{3} - 1) = 0$

$And\ \ y(0) = 0\ \Rightarrow C_1 + \frac{1}{4} = 0$

$$\Rightarrow C_1 = -\frac{1}{4}\ \ and\ \ C_2 = \frac{-(\sqrt{3}+1)}{4}$$

$$\Rightarrow y = -\left(1 + \left(\frac{\sqrt{3}+1}{4}\right)x\right)e^{-2x} + \frac{e^{-x}(\sqrt{3}\sin x + \cos x)}{4}$$