IIT–JAM Mathematical Statistics (MS) 2009

1. **For detecting a disease, a test gives correct diagnosis with probability 0.99. It is known that 1% of a population suffers from this disease. If a randomly selected individual from that population tests positive, then the probability that the select individual actually has the disease is**
   (a) 0.01
   (b) 0.05
   (c) 0.5
   (d) 0.99

Solution: (c) P(disease $|tests$ positive)

$$= \frac{P(disease \,\&\, tests\ positive}{P(tests\ positive)}$$

$$= \frac{1\%° \times (0.99)}{99\% \times (0.01) + 1\% \times (0.99)} = \frac{1}{2} = 0.5\,.$$

2. **Let X be any random variable with mean μ and variance 9. Then the smallest value of m such that $P(|X-\mu| < m) > 0.99$, is**
   (a) 90
   (b) $\sqrt{90}$
   (c) $\sqrt{100/11}$
   (d) 30

Solution: (d) By Chebyshev's inequality

$$P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

$$\Rightarrow P(|X-\mu| < 3k) \geq 1 - \frac{1}{10^2}$$

$$\Rightarrow P(|X-\mu| < 30) \geq 1 - \frac{1}{10^2}$$

$$\Rightarrow m = 30$$

3. If a random variable X has the cumulative distribution function

$$F(x) = \begin{cases} 0, & if\ x > 0 \\ \frac{1}{3}, & if\ x = 0 \\ \frac{1+x}{3} & if\ 0 < x < 1 \\ 1, & if\ x > 1 \end{cases}$$

Then E(X) equals

(a) 1/3
(b) 1
(c) 1/6
(d) ½

Solution: (c)

$$f(x) = \begin{cases} 0 & ; x \leq 0 \\ \frac{1}{3} & ; 0 < x < 1 \\ 0 & ; x \geq 0 \end{cases}$$

$$\Rightarrow E(X) = \int_0^1 x f(x) dx$$

$$= \frac{1}{3} \int_0^1 x dx = \frac{1}{6}$$

4. If

$$Y = \frac{ln\ U_1}{ln\ U_1 + ln\ (1 - U_2)}$$

Where U_1 *and* U_2 are independent U(0, 1) random variables, then variance of Y equals

(a) 1/12
(b) 1/3
(c) ¼
(d) 1/6

Solution: (a) As $Y \sim U(0,1)$, so

$$E(Y) = \int_0^1 y dy = \frac{1}{2}$$

$$E(Y^2) = \int_0^1 y^2 dy = \frac{1}{3}$$

$$\Rightarrow V(Y) = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$$

5. **If X is a Binomial (30, 0.5) random variable, then**

   (a) $P(X > 15) = 0.5$

   (b) $P(X < 15) = 0.5$

   (c) P(X>15)>0.5

   (d) P(X< 15) < 0.5

Solution: (d) P(X<15) $= P(X = 0) + \cdots + P(X = 14) < 0.5$

$$as\ P(X = 0) = P(X = 30)$$

$$P(X = 1) = P(X = 29)$$

$$\vdots \quad \vdots \quad \vdots$$

$$P(X = 14) = P(X = 16)$$

$$P(X < 15) = P(X = 0) + \cdots + P(X = 14)$$

$$= P(X = 16) + \cdots + P(X = 30)$$

$$= \frac{1 - P(X = 15)}{2}$$

$$= 0.5 - \frac{P(X = 15)}{2} < 0.5$$

6. **If the joint probability density function of (X, Y) is given by**

$$f(x, y) = \frac{1}{y} e^{-\frac{x}{y}}, x > 0, 0 < y < 1$$

Then

(a) $E(X) = 0.5\ and\ E(Y) = 0.5$

(b) $E(X) = 1.0\ and\ E(Y) = 0.5$

(c) $E(X) = 0.5\ and\ E(Y) = 1.0$

(d) $E(X) = 1.0\ and\ E(Y) = 1.0$

Solution: (b)

$$f(x, y) = \frac{1}{y} e^{-\frac{x}{y}}, x > 0, 0 < y < 1$$

$$\Rightarrow f(x) = \int_{y=0}^{1} \frac{1}{y} e^{-x/y} dy$$

$$Let \frac{x}{y} = t$$

$$\Rightarrow -\frac{x}{y^2} dy = dt$$

$$\Rightarrow f(x) = \int_{\infty}^{x} \frac{t}{x} e^{-t} \left[-\frac{x}{t^2}\right] dt$$

$$= \int_{\infty}^{x} -\frac{1}{t} e^{-t} dt$$

$$= \int_{x}^{\infty} \frac{1}{t} e^{-t} dt$$

$$\Rightarrow E(x) = \int_{0}^{\infty} x f(x) dx$$

$$= \int_{0}^{\infty} \int_{x}^{\infty} \frac{1}{t} e^{-t} dt \, dx = 1$$

$$f(y) = \int_{0}^{\infty} \frac{1}{y} e^{-x/y} dx = -e^{-x/y} \Big|_{0}^{\infty}$$

$$= 1 \; ; \; 0 < y < 1$$

$$E(y) = \int_{0}^{1} y f(y) dy = \int_{0}^{1} y \, dy$$

$$= \frac{y^2}{2} \Big|_{0}^{1} = \frac{1}{2} \, .$$

7. If X is an F(m, n) random variable, where m > 2, n>2, then E(X) $E\left(\frac{1}{X}\right)$ equals

(a) $\frac{n(n-2)}{m(m-2)}$

(b) $\frac{m(m-2)}{n(n-2)}$

(c) $\frac{mn}{(m-2)(n-2)}$

(d) $\frac{m(n-2)}{n(m-2)}$

Solution: (c) For F(m,n)

$$E(X) = \frac{n}{n-2} \; ; E\left(\frac{1}{X}\right) = \frac{m}{m-2}$$

$$\Rightarrow E(X) E\left(\frac{1}{X}\right) = \frac{mn}{(m-2)(n-2)} \, .$$

8. **Let X be a random variable having probability mass function**

$$f(x)=\begin{cases}\dfrac{2+4\alpha_1+\alpha_2}{6}, & if\ x=1\\ \dfrac{2-2\alpha_1+\alpha_2}{6}, & if\ x=2\\ \dfrac{1-\alpha_1-\alpha_2}{3}, & if\ x=3\end{cases}$$

Where, $\alpha_1 > 0\ and\ \alpha_2 > 0$ are unknown parameters such that $\alpha_1+\alpha_2 \leq 1$. For testing the null hypothesis $H_0 :\ \alpha_1+\alpha_2=1$, against the alternative hypothesis $H_1 :\ \alpha_1=\alpha_2=0$, suppose that the critical region is $C=\{2,3\}$. Then, this the critical region has

(a) **size= $\frac{1}{2}$ $and\ power\ =2/3$**
(b) **$size=\frac{1}{4} and\ power\ =2/3$**
(c) **size= $\frac{1}{2} and\ power=1/4$**
(d) **size= $\frac{2}{3} and\ power=1/3$**

Solution: (b) Size $=\alpha=P(reject\ |H_0\ is\ true)\ =\frac{2-2\alpha_1+\alpha_2}{6}+\frac{1-\alpha_1-\alpha_2}{3}=\frac{1}{4}$

$$\beta=P(accepts\ |H_1\ is\ true)\ =\frac{2}{6}=\frac{1}{3}$$

$$\Rightarrow Power=1-\beta=1-\frac{1}{3}=\frac{2}{3}$$

$$From\ \sum f(x)=1$$

$$\Rightarrow \frac{2+4\alpha_1+\alpha_2}{6}+\frac{2-2\alpha_1+\alpha_2}{6}+\frac{1-\alpha_1-\alpha_2}{3}=1$$

$$\Rightarrow \frac{3+3\alpha_1}{6}+\frac{3-3\alpha_1}{6}+0=1$$

For $\alpha_1+\alpha_2=1$

9. **The observed value of the mean of a random sample from $N(\theta,1)$ distribution is 2.3. If the parameter space is $\Theta=\{0,1,2,3\}$, then the maximum likelihood estimate of θ is**

(a) **1**
(b) **2**
(c) **2.3**
(d) **3**

Solution: (c) M.L.E. of $\theta=\hat{\theta}=$

$$\frac{x_1+x_2+\cdots+x_n}{n}$$

i.e., sample mean $=2\times 3$

10. **The series**

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{x^n\sqrt{n^2+1}}, x > 0$$

(a) **Converges for x>1 and diverges for $x \le 1$.**
(b) **Converges for $x \le 1$ and diverges for x>1.**
(c) **Converges for all x > 0**
(d) **Diverges for all x >0**

Solution: (a)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{x^n\sqrt{n^2+1}}$$

$$u_n = \frac{\sqrt{n}}{x^n\sqrt{n^2+1}}$$

For x < 1 the series is divergent and for x >1, it is convergent.

11. **Let f be a differentiable function defined on [0,1]. If $\xi \in (0,1)$ is such that $f(x) < f(\xi) = f(0)$ for all $x \in (0,1], x \neq \xi$, then**
(a) $f'(\xi) = 0$ *and* $f'(0) = 0$
(b) $f'(\xi) = 0$ *and* $f'(0) \le 0$
(c) $f'(\xi) > 0$ *and* $f'(0) \le 0$
(d) $f'(x) = 0$ *and* $f'(0) > 0$

Solution: (b) $f(x) < f(\xi)$ *for all* $x \in (0,1]$

$\Rightarrow \xi$ *is local maxima point.* $\Rightarrow f'(\xi) = 0$ *and as* $f(\xi) = f(0)$

So, $f''(0) \le 0$

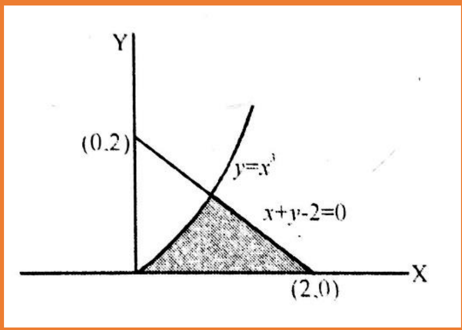
12. **The area of the region bounded by $y = x^3, x + y - 2 = 0$ *and* $y = 0$ is**
(a) **0.25**
(b) **0.5**
(c) **0.75**
(d) **1.0**

Solution: (c) $y = x^3$ & $x + y - 2 = 0$

$$\Rightarrow x^3 + x - 2 = 0$$

$$\Rightarrow (x-1)(x^2 + x + 2) = 0$$

$$\Rightarrow x = 1$$



So, the required area is

$$\int_0^1 x^3 dx + \int_1^2 (2-x)dx$$

$$= \frac{x^4}{4}\Big|_0^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^2 = \frac{1}{4} + \frac{1}{2} = 0.75$$

13. The system of equations

$$\boldsymbol{x + 3y + 2z = k}$$

$$\boldsymbol{2x + y - 4z = 4}$$

$$\boldsymbol{5x - 14z = 10}$$

- **(a) Has unique solution for $\boldsymbol{k = 2}$**
- **(b) Has infinitely many solutions for $\boldsymbol{k = 2}$**
- **(c) Has no solution for $\boldsymbol{k = 2}$**
- **(d) Has unique solution for any $\boldsymbol{k \neq 2}$**

Solution: (b) Given system of equations is

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & k \\ 2 & 1 & -4 & 4 \\ 5 & 0 & -14 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 2 & k \\ 0 & 5 & -8 & 4-2k \\ 0 & -15 & -24 & 10-5k \end{array}\right]$$

By $R_2 \leftarrow R_2 - 2R_1$ & $R_3 \leftarrow R_3 - 5R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 2 & k \\ 0 & -5 & -8 & 4-2k \\ 0 & 0 & 0 & -2+k \end{array}\right]$$

By $R_3 \leftarrow R_3 - 3R_2$

For $-2 + k = 0$ $i.e.$, $k = 2$, there are infinitely many solutions.

14. **Let $A = \left(\left(a_{ij}\right)\right)$ be an orthogonal matrix of order n such that $a_{1j} = \frac{1}{\sqrt{n}}, j = 1, \dots, n$**

$$\boldsymbol{If\ \bar{a} = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}, then\ \sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij} - \bar{a}\right)^2\ equals}$$

- **(a)** $\frac{n+1}{n}$
- **(b)** $\frac{n-1}{n}$
- **(c)** $\frac{n^2+1}{n}$
- **(d)** $\frac{n^2-1}{n}$

Solution: (d)

$$A = \begin{vmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \vdots & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & & \frac{1}{\sqrt{n}} \end{vmatrix}$$

For order 1, $A_1 = [1]$

Required sum $= 0$

For order 2,

$$A_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\bar{a} = \frac{1}{2\sqrt{2}}$$

$$\sum\sum\left(a_{ij} - \bar{a}\right)^2 = \frac{1}{8} \times 3 + \frac{9}{8} = \frac{12}{8} = \frac{3}{2}$$

$$\Rightarrow \sum_{j=1}^{n}\sum_{i=1}^{n}\left(a_{ij} - \bar{a}\right)^2 = \frac{n^2 - 1}{n}$$

15. The solution of the differential equation

$$\frac{dy}{dx} = \frac{y^2 \cos x + \cos y}{x \sin y - 2y \sin x}\ ; y\left(\frac{\pi}{2}\right) = 0\ is$$

(a) $y^2 \cos x + x \sin y = 0$
(b) $y^2 \sin x + x \cos y = \pi/2$
(c) $y^2 \sin x + x \sin y = 0$
(d) $y^2 \cos x + x \cos y = \pi/2$

Solution: (b)

$$\frac{dy}{dx} = \frac{y^2 \cos x + \cos y}{x \sin y - 2y \sin x}$$

$$\Longrightarrow (y^2 \cos x + \cos y)dx + (2y \sin x - x \sin y)dy = 0$$

$$M = y^2 \cos x + \cos y$$

$$N = 2y \sin x - x \sin y$$

$$\frac{\partial M}{\partial y} = 2y \cos x - \sin y\ \ \&\ \frac{\partial N}{\partial x} = 2 \cos x - \sin y$$

So, D.E. is exact. Hence the solution is

$$\int y^2 \cos x + \cos y dx = c$$

$$\Longrightarrow y^2 \sin x + x \cos y = c$$

$$y\left(\frac{\pi}{2}\right) = 0$$

$$\Longrightarrow c = \pi/2$$

$$\Longrightarrow y^2 \sin x + x \cos y = \pi/2\ \ is\ the\ solution.$$

16. Let $U_1, U_2, \dots, U_n$ be n urns such that urn U_k contains k white and k[2] black balls, $k = 1, \dots, n$. Consider the random experiment of selecting an urn and drawing a ball out of it at random. If the probability of selecting urn U_k is proportional to (k+1), then

(a) Find the probability that the ball drawn is black.
(b) Find the probability that urn U_n was selected, given that the ball drawn is white.

Solution: (a) As

$$P(U_k)\ \infty\ \frac{1}{k+1}$$

$$\Rightarrow P(U_k) = \frac{c}{k+1}\ \ (let)$$

$$Then\ c = \frac{1}{\left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n+1}\right)}$$

$$\Rightarrow P(Black\ ball\ is\ drawn)$$

$$= \sum_{k=1}^{n} P(U_k)\ P(Black\ ball\ from\ U_k)$$

$$= \sum_{k=1}^{n} \frac{c}{k+1} \times \frac{k^2}{(k^2+k)}$$

$$= \sum_{k=1}^{n} \frac{ck}{(k+1)^2} = c\left(\frac{1}{2^2} + \frac{2}{3^2} + \cdots + \frac{n}{(n+1)^2}\right)$$

$$= \frac{\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{n}{(n+1)^2}}{\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right)}$$

(b) $P(U_n\ |white\ ball\ drawn)$

$$= \frac{P(white\ ball\ drawn\ from\ U_n)}{P(White\ ball)}$$

$$= \frac{P(U_n)P(white\ from\ U_n)}{\sum_{k=1}^{n} P(U_k)P(white\ from\ U_k)}$$

$$= \frac{\frac{c}{n+1} \times \frac{n}{n(n+1)}}{\sum_{k=1}^{n} \frac{c}{k+1} \times \frac{k}{k(k+1)}}$$

$$= \frac{\frac{1}{(n+1)^2}}{\left[\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(n+1)^2}\right]}$$

17. Let the joint probability density function of (X, Y) be given by

$$\boldsymbol{f(x,y) = \frac{1}{4}(1 + x^3y^3), -1 \le x \le 1, -1 \le y \le 1.}$$

(a) Find the joint probability density function of (X^2, Y^2).

(b) Calculate the correlation coefficient between X and Y.

Solution: (a) $U = X^2\ and\ V = Y^2$

$$\Rightarrow \frac{\partial(U,V)}{\partial(x,y)} = \begin{vmatrix} 2X & 0 \\ 0 & 2Y \end{vmatrix} = 4XY$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(U,V)} = \frac{1}{4XY} = \frac{1}{4\sqrt{UV}}$$

$$As\ -1 \le \quad x \le 1, \qquad 0 \le x^2 \le 1$$

$$and\ -1 \le y \le 1, \qquad 0 \le y^2 \le 1$$

Joint density function of

$$(X^2, Y^2)\ i.e., (U, V)\ is$$

$$g(u,v) = \frac{1}{4}\left(1 + (uv)^{\frac{3}{2}}\right) \times \frac{1}{4(uv)^{1/2}}$$

$$0 \le u \le 1, \qquad 0 \le v \le 1$$

$g(u,v) = \frac{1}{16}\left[uv + (uv)^{-\frac{1}{2}}\right]$ is the joint distribution function of $X^2\ and\ Y^2$

Where $0 \le u \le 1,\ 0 \le v \le 1$

(b) Marginal p.d.f. of X,

$$f(x) = \int_{-1}^{1} \frac{1}{4}(1 + x^3y^3)dy$$

$$\Rightarrow f(x) = \frac{1}{2}\ ;\ -1 < x < 1.$$

Also

$$\Rightarrow f(y) = \frac{1}{2}\ ;\ -1 \le y \le 1$$

$$E(X) = \int_{-1}^{1} \frac{1}{2} x dx = 0$$

$$E(Y) = \int_{-1}^{1} \frac{1}{2} y\, dy = 0$$

$$E(XY) = \int_{-1}^{1}\int_{-1}^{1} \frac{1}{4}(xy + x^4y^4)dxdy$$

$$= 4.\frac{1}{4}\int_{0}^{1}\int_{0}^{1} x^4y^4 dx\, dy \ = \frac{1}{25}$$

$$E(X^2) = \int_{-1}^{1} x^2 \times \frac{1}{2} dx = \frac{1}{3}$$

$$E(Y^2) = \int_{-1}^{1} y^2 \times \frac{1}{2} dx = \frac{1}{3}$$

$$\Rightarrow V(X) = V(Y) = 1/3$$

$$cov\ (X,Y) = E(XY) - E(X)E(Y) = 1/25$$

$$\Rightarrow correlation\ coefficient\ between\ X\ \&\ Y.$$

$$\rho_{X,Y} \ = \frac{1/25}{\sqrt{\frac{1}{3} \times \frac{1}{3}}} = \frac{3}{25}$$

18. (a) Let X be a random variable having probability mass function

$$p(x) = \begin{cases} 2\theta, & if\ x = -1 \\ \theta^2 & if\ x = 0 \\ 1 - 2\theta - \theta^2 & if\ x = 1 \end{cases}$$

where $\theta \in [0, \sqrt{2} - 1]$. Show that there is one, and only, unbiased estimator of $(\theta + 1)^2$ based on a single observation.

(b) Let $X_1, \ldots, X_n$ be a random sample from a population having probability density function

$$f(x) = \frac{1}{\theta} e^{-\frac{(x-\theta)}{\theta}}, x \geq \theta$$

where $\theta \in (0, \infty)$. Find the maximum likelihood estimator of θ.

Solution:

(a) $E(\theta + 1)^2 = E(\theta^2 + 2\theta + 1) = E(\theta^2) + 2E(\theta) + 1 = V(\theta) + 2\theta + 1 + [E(\theta)]^2$

So, we have $E(\theta + 1)^2 = V(\theta) + \theta^2 + 2\theta + 1$

Hence it is unbiased estimator of $(\theta + 1)^2$ and also as $V(\theta)$ *and* θ are unique so it is unique.

(b) Likelihood function

$$L(x, \theta) = \frac{1}{\theta^n} e^{-\frac{\sum x_i - n\theta}{\theta}}$$

$$= \frac{1}{\theta^n} e^{-\left(\frac{\sum x_i}{\theta} - n\right)}$$

$$= \frac{1}{\theta^n} e^n - \frac{\sum x_i}{\theta}$$

$$\log L = -n \log \theta + n - \frac{\sum x_i}{\theta}$$

$$\Rightarrow \frac{1}{2}\frac{\partial L}{\partial \theta} = \frac{n}{\theta} + \frac{\sum x_i}{\theta^2}$$

$$\frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \frac{\sum x_i}{\theta^2} = \frac{n}{\theta}$$

$$\Rightarrow \theta = \frac{\sum x_i}{n}$$

$$\Rightarrow Maximum\ likelihood\ of\ \theta, \hat{\theta} = \frac{\sum x_i}{n}$$

19. (a) A box contains M white and 3−M black balls. To test the null hypothesis $H_0: M = 2$ against the alternative hypothesis $H_1 : M = 1$, five balls are drawn at random from the box with replacement. If X is the number of white balls drawn, then find the most powerful test of size $\alpha = \frac{11}{243}$. Also find its power.

(b) Let $X_1, \dots, X_5$ be a random sample from $Exp(\lambda)$ distribution, where $\lambda > 0$. Find a pair of constants (c_1, c_2) such that $P(c_1 \bar{X} \le \lambda \le c_2 \bar{X}) = 0.9$ where $\bar{X}$ denotes the sample mean.

(Useful data : If X has $\chi^2(10)$ distribution, then $P(X \le 3.94) = 0.05$ and $P(X \le 18.3) = 0.95$)

Solution:

(a)

$$\beta = P(accepts \mid H_1 \text{ is true}) \; ; \; \alpha = P(rejects \mid H_0 \text{ is true})$$

For H_0 is true, $M = 2$, so there are 2 white and 1 black ball

$$P(X = x) = \binom{5}{x} \left(\frac{2}{3}\right)^x \cdot \left(\frac{1}{3}\right)^{5-x}$$

$$= \binom{5}{x} \cdot \frac{2^x}{(243)} \; ; \quad x = 0,12,3,4,5$$

$$P(X = 0) + P(X = 1) = \frac{11}{243} = \alpha$$

So, $X = 0$ & $X = 1$ implies rejection and $X = 2,3,4,5$ implies acceptance.

For H_1 is true, $M = 1$, so there are 1 white and 2 black balls.

$$P(X = x) = \binom{5}{x} \frac{2^{5-x}}{243} \; ; x = 0, 1, 2, 3, 4, 5$$

$$\beta = P(accepts \mid H_1 \text{ is true})$$

$$= \sum_{x=2}^{5} \binom{5}{x} \frac{2^{5-x}}{243}$$

$$= \frac{1}{243} \left[\binom{5}{2} \times 2^3 + \binom{5}{3} \times 2^2 + \binom{5}{4} \times 2 + \binom{5}{5} \right]$$

$$= \frac{80 + 40 + 10 + 1}{243} = \frac{131}{243}$$

Power of the test,

$$1 - \beta = 1 - \frac{131}{243} = \frac{112}{243}$$

(b) $P(X \le 3.94) = 0.05$

And $P(X \le 18.3) = 0.95 \Rightarrow P(3.94 \le X \le 18.3) = 0.9$

Which when compared to

$$P(c_1\bar{X} \le \lambda \le c_2\bar{X}) = P\left(\frac{\lambda}{c_2} \le \bar{X} \le \frac{\lambda}{c_1}\right) = 0.9$$

$$gives\ \frac{\lambda}{c_2} = 3.94\ and\ \ \frac{\lambda}{c_1} = 18.3$$

20. (a) Using the Central Limit Theorem, evaluate

$$\lim_{n\to\infty} \sum_{j=0}^{n} \binom{j+n-1}{j} \frac{1}{2^{j+n}}$$

(b) Suppose that n balls are randomly placed in 2n cells. Let X denote the number of balls occupying the first cell. Find $\lim_{n\to\infty} P(X=5)$

Solution: (a)

$$\lim_{n\to\infty} \sum_{j=0}^{n} \binom{j+n-1}{j} \frac{1}{2^{j+n}}$$

$$= \lim_{n\to\infty} \left[\binom{n-1}{0} \times \frac{1}{2^n} + \binom{n}{1} \times \frac{1}{2^{n+1}} + \binom{n+1}{2} \times \frac{1}{2^{n+2}} \cdots + \binom{2n-1}{n} \times \frac{1}{2^{2n}}\right]$$

$$= \lim_{n\to\infty} \frac{1}{2^n}\left[\binom{n-1}{0} + \binom{n}{1} \times \frac{1}{2} + \binom{n+1}{2} \times \frac{1}{2^2} + \cdots + \binom{2n-1}{n} \times \frac{1}{2^n}\right] = 1\ By\ C.L.T.$$

(b) $P(X=5) = \frac{\binom{n}{5}\binom{n-5}{n-5}(2n)^{n-5}}{(2n)^n} = \frac{\binom{n}{5}}{(2n)^n} = \frac{n^4 - 10n^3 + 35n^2 - 50n + 24}{3840n^4}$

21. (a) Suppose that the joint probability density function of (X, Y) is given by

$$f(x,y) = \begin{cases} \frac{1}{2}, & if\ (x,y) \in R \\ 0, & otherwise \end{cases}$$

Where R is the region in xy plane bounded by the lines $y = x, y = x - 2, y = -x + 2$. Define

$$U = \frac{X+Y}{2}\ and\ V = \frac{X-Y}{2}$$

Show that U and V are independent U(0, 1) random variables.

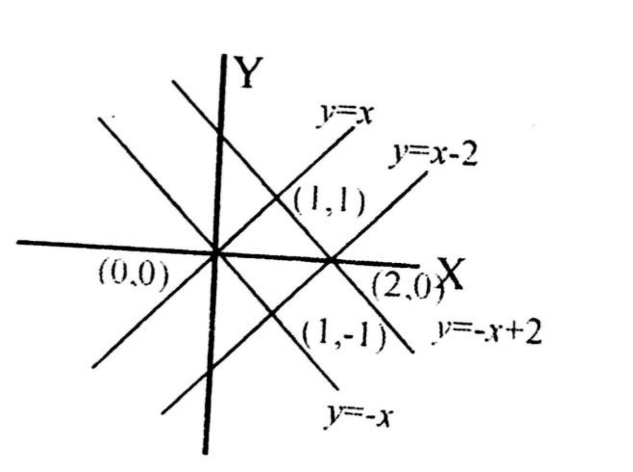
(b) The probability mass function p(x) of a discrete random variable X satisfies $p(x+1) = \lambda p(x), x = 1, 2, 3, \ldots$, where $0 < \lambda < 1$. For positive integers m and n, find $P(X \ge m + n | X \ge m)$.

Solution:

(a)

$$U = \frac{X+Y}{2}; V = \frac{X-Y}{2}$$

$$\Longrightarrow X = U + V \,; Y = U - V$$



$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

$$\Longrightarrow |J| = 2$$

$$\Longrightarrow g(u,v) = 1; V = 0, U = 0, V = 1, U = 1$$

$$\Longrightarrow g(U) = \int_0^1 1dv = 1; \;\; 0 \le v \le 1$$

$$g(v) = \int_0^1 1du \;\; = 1 \;\; ; \quad 0 \le u \le 1$$

So, both U and V are U(0, 1)

Also, $g(u,v) = g(u)g(v)$, so they are independent of each other.

(b) $p(x+1) = \lambda p(x)$

$$\Rightarrow p(2) = \lambda p(1); \; p(3) = \lambda p(2) = \lambda^2 p(1)$$

$$\Rightarrow \sum_{x=1}^{\infty} p(x) = p(1)[1 + \lambda + \lambda^2 + \cdots] = \frac{p(1)}{1-\lambda} = 1$$

$$\Rightarrow p(1) = 1 - \lambda \;\&\; p(x) = \lambda^{x-1}(1-\lambda)$$

$$p(X \ge m+n \,\Big|\, X \ge m) \;\; = \frac{p(X \ge m+n)}{p(X \ge m)}$$

$$= \frac{\lambda^{m+n-1}[1 + \lambda + \lambda^2 + \cdots]\,[1-\lambda]}{\lambda^{m-1}[1 + \lambda + \lambda^2 + \cdots]\,[1-\lambda]} = \lambda^n$$

22. **(a) Let f be a differentiable function on (0, 1) such that $|f'(x)| < 1 \; for \; all \; x \in (0, 1)$. Show that the sequence $\{a_n\}$ defined by**

$$a_n = f\left(\frac{1}{n}\right),$$

$$n = 1, 2, \ldots . \, is \; convergent.$$

(b) Let $\{a_n\} and \{b_n\}$ be sequences of real numbers such that $\lim_{n\to\infty} a_n = l \;\; and \; b_n \leq 1, for \; n = 1, 2, \ldots$. Define, $c_n = max\,(a_n, b_n), n = 1, 2, \ldots$. Show that $\lim_{n\to\infty} c_n = l$.

Solution:

(a) $|f'(x)| < 1$

$$a_n = f\left\{\frac{1}{n}\right\} \; ; n = 1, 2, \ldots$$

For $n = 1,2,3, \ldots$

$$\frac{1}{n} \in (0,1)$$

So, as $|f'(x)| < 1 \; for \; x \in (0,1)$ so rate of increase of f(x) w.r.t. x is less than that of x.

Hence, $a_n = f\left(\frac{1}{n}\right)$ is convergent.

(b) $c_n = \max(a_n, b_n)$

As $\lim_{n\to\infty} a_n = 1 \;\; and \;\; b_n \leq l \Rightarrow \max(\lim_{n\to\infty} a_n, b_n) \; = l \Rightarrow \lim_{n\to\infty} c_n = l.$

23. **Let A be a 3×3 real non–diagonal matrix with $A^{-1} = A$.**
Show that $tr(A) = -det\,(A) = \pm 1$.

Solution:

$A^{-1} = A \;\; \Rightarrow |A^{-1}| = |A| \Rightarrow \frac{1}{|A|} = |A| \; \Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1$

So, $A^{-1} = A \Rightarrow A^2 = 1$

$If \; \lambda_1, \lambda_2, \lambda_3$ are eigenvalue of A then $\lambda_1^2, \lambda_2^2, \lambda_3^2$ will be e.v. of A² i.e. of I.

$$\Rightarrow \lambda_1, \lambda_2, \lambda_3 = \pm 1$$

As A is non diagonal, so eigenvalues of A will be −1, −1 and 1 if $|A| = 1$ and 1, 1, −1 if $|A| = -1$

$\Rightarrow tr\;(A) - \det(A) = \pm 1$.

24. Let $\alpha(t)$ *and* $\beta(t)$ be differentiable functions on R $\ni \alpha(0) = 2$ *and* $\beta(0) = 1$.
If $\alpha(t) + \beta'(t) = 1$ *and* $\alpha'(t) + \beta(t) = 1$ *for all* $t \in [0, \infty)$, find the value of $\alpha(ln\, 2)$.

Solution:

$\alpha(t) + \beta'(t) = 1 \ldots\ldots (i)$

$\alpha'(t) + \beta(t) = 1 \ldots\ldots (ii)$

From equation (ii), we get

$$\alpha''(t) + \beta'(t) = 0$$

$$\Rightarrow \alpha''(t) - \alpha(t) = -1$$

$$(D^2 - 1)\big(\alpha(t)\big) = -1$$

$$\Rightarrow C.F.\, is\ \alpha(t) = c_1 e^t + c_2 e^{-t}$$

$$P.I.\, is\ \alpha(t) = \frac{1}{-1} \times -1 = 1$$

$$\Rightarrow \alpha(t) = c_1 e^t + c_2 e^{-t} + 1$$

$$\alpha'(t) = c_1 e^t - c_2 e^{-t}$$

$$\alpha(0) = 2\ \&\ \beta(0) = 1 - \alpha'(0) = .1$$

$$\Rightarrow \alpha'(0) = 0$$

$$\Rightarrow c_1 = c_2, c_1 + c_2 = 2$$

$$\Rightarrow c_1 = c_2 = 1$$

$$\Rightarrow \alpha(t) = e^t + e^{-t} + 1$$

$$\Rightarrow \alpha(\ln 2) = e^{\ln 2} + e^{-\ln 2} + 1 = 2 + \frac{1}{2} + 1 = \frac{7}{2}$$

25. (a) Find the global minimum value of the function

$$\boldsymbol{g(x, y) = 2(x - 2)^2 + 3(y - 1)^2}$$

In the region $R = \{(x, y) : 0 \le x \le y \le 3\}$. Also find the point(s) at which it is attained.

(b) Let f be a continuous function on [a, b]. If $\alpha \le x_1 < x_2 < \cdots < x_n \le b (n \ge 2)$, then show that there exists $\alpha\ \xi \in (a, b)$ such that

$$\boldsymbol{\sum_{k=1}^{n} \alpha_k f(x_k) = f(\xi), where\ \alpha_1 > 0, i = 1, 2, \ldots, n\ \ and\ \alpha_1 + \cdots + \alpha_n = 1}$$

Solution:

(a) $g(x, y) = 2(x - 2)^2 + 3(y - 1)^2$

$g_x = 4(x-2) = 0 \quad \Longrightarrow x = 2$

$g_y = 6(y-1) = 0 \Longrightarrow y = 1$

$$g_{xx} = 4$$

$$g_{yy} = 6$$

$$g_{xy} = 0$$

$$g_{xy} - g(xy)^2 = 24 > 0$$

$$and\ g_{xx} = 4 > 0$$

So, (2,1) is the point of local minima.

(b)

$$\sum_{k=1}^{n} \alpha_k\, f(x_k), where$$

$$\sum x_i = \alpha_1 + \alpha_2 + \cdots + \alpha_n = 1\ \ gives\ convex\ combination\ of\ f(x_1), f(x_2), \ldots, f(x_n)$$

Hence, each of such combination should lie in the range of f(a,b) so, there exists $a\ \xi \in (a,b)$ such that

$$\sum_{k=1}^{n} \alpha_k f(x_k) = f(\xi)$$