IIT-JAM Mathematical Statistics (MS) 2010 Solved Paper

1. Let E and F be two events with P(E) > 0, P(F/E) $= \mathbf{0.3}$ *and* $\boldsymbol{P(E \cap F^C) = 0.02}$. Then P(E) equals
(a) 1/7
(b) 2/7
(c) 4/7
(d) 5/7

Solution: (b)

$$P(E \cap F^C) = 0.2 \quad \Rightarrow P(E) - P(E \cap F) = 0.2$$

$$And\ we\ have\quad P\left(\frac{F}{E}\right) = 0.3 \text{ gives}$$

$$\frac{P(E \cap F)}{P(E)} = 0.3$$

$$\Rightarrow \frac{P(E) - 0.2}{P(E)} = 0.3$$

$$\Rightarrow 0.7 \times P(E) = 0.2$$

$$\Rightarrow P(E) = 2/7$$

2. Let $X_1, \dots, X_n$ be i.i.d. random variables with the probability density function

$$\boldsymbol{f\left(\frac{x}{\theta}\right) = \begin{cases} \dfrac{2\theta^2}{x^3}, & x \geq \theta \\ 0, & otherwise, \end{cases}}$$

where $\boldsymbol{\theta}(> 0)$ is unknown. Then the maximum likelihood estimator of q is

(a) $\left(\prod_{i=1}^{n} X_i\right)^{3/2}$
(b) $\frac{1}{n}\sum_{i=1}^{n} \ln X_i$
(c) $\boldsymbol{max}\,\{X_1, \dots, X_n\}$
(d) $\boldsymbol{min}\,\{X_1, \dots., X_n\}$

Solution: (c) Likelihood function,

$$L(X, \theta) = \prod_{i=1}^{n} \frac{2\theta^2}{X_i^3} = \frac{2^n \theta^{2n}}{x_1^3\, x_2^3 \dots x_n^3}$$

$$\log L = n \log 2 + 2n \log q - 3 \sum_{i=1}^{n} \log x_i$$

$$\Rightarrow \frac{1}{L}.\frac{\partial L}{\partial \theta} = \frac{2n}{\theta}.$$

So L is increasing with respect to n. So, for M.L.E, $\hat{\theta} = \max(x_1, x_2, \dots, x_n)$

3. The value of $\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{y} e^{-\frac{1}{2}(x^2+y^2)} dxdy$ equals

(a) $\frac{\pi}{4}$

(b) $\frac{1}{2\pi}$

(c) ¼

(d) ½

Solution: (d)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{y} e^{-\frac{1}{2}(x^2+y^2)}\, dxdy$$

$$= \frac{1}{2\pi}\int_{\pi/4}^{5\pi/4}\int_{0}^{\infty} e^{-\frac{r^2}{2}}\, rdr\, d\theta$$

$$= \frac{1}{2\pi}\int_{\pi/4}^{5\pi/4}\left[-e^{-\frac{r^2}{2}}\right]_0^{\infty} d\theta$$

$$= \frac{1}{2\pi}\int_{\pi/4}^{5\pi/4} d\theta = \frac{1}{2}$$

4. The value of $\int\int e^{-(x+y)} dxdy$, where $S\{(x,y)^S : 0 < x < 1, y > 0, 1 < x+y < 2\}$ equals

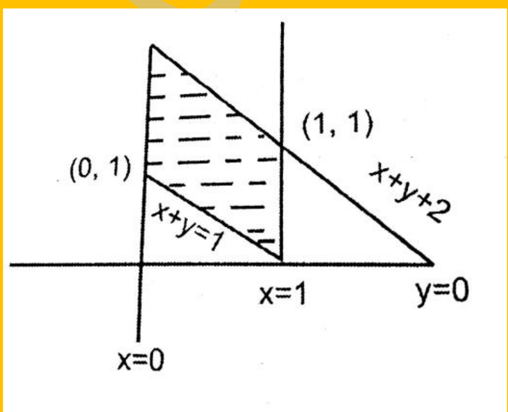
(a) 1

(b) 2

(c) $e^{-1} - e^{-2}$

(d) $e^2 - e$

Solution : (c)



$$\int\int_S e^{-(x+y)}dx\,dy = \int_0^1\int_{1-x}^{2-x} e^{-x}e^{-y}\,dydx$$

$$\int_0^1 e^{-x}(-e^{-y})\Big|_{1-x}^{2-x} dx = \int_0^1 e^{-x}[e^{x-1}-e^{x-2}]dx$$

$$= \int_0^1 (e^{-1}-e^{-2})dx = (e^{-1}-e^{-2})$$

5. **Two coins with probability of heads u and v, respectively, are independent. If P(both coins show up tails) $=$ P (both coins show up heads), the u+v equals**

(a) **¼**

(b) **½**

(c) **¾**

(d) **1**

Solution: **(d)** P(both tails) = P(both heads)

$$\Longrightarrow (1-u)(1-v) = uv$$

$$\Longrightarrow u+v = 1$$

6. **Let f be a thrice differentiable function on $\left[-\frac{\pi}{6},\frac{\pi}{6}\right]$ such that $f'(x) = 1 + \left(f(x)\right)^2$. If $f(0) = 1$, then the coefficient of x^2 in Taylor's expansion of f about zero is**

(a) **2**

(b) **3**

(c) **4**

(d) **5**

Solution: **(a)** $f'(x) = 1 + \left(f(x)\right)^2$

$$\Rightarrow f''(x) = 2 + f(x)f'(x)$$

$$f(0) = 1$$

$$\Rightarrow f'(0) = 2$$

$$\Rightarrow f''(0) = 4$$

$$Coefficient\ of\ x^2 = \frac{f''(0)}{2!} = \frac{4}{2} = 2\,.$$

7. **Let X be a discrete random variable with**

$$\boldsymbol{P(X=k)=\frac{2e^{-1}}{3k!}+\left(\frac{1}{3}\right)^{k+1}\frac{2}{3}\ ,k=0,1,2,\ldots}$$

Let E {0, 2, 4,…}. Then P$(X\in E)$ equals

(a) $\frac{5}{12}+\frac{2}{3}e^{-2}$
(b) $\frac{5}{12}+\frac{1}{3}e^{-2}$
(c) $\frac{7}{12}-\frac{1}{3}e^{-2}$
(d) $\frac{7}{12}+\frac{1}{3}e^{-2}$

Solution: (d)

$$P(X=k)=\frac{2e^{-1}}{3k!}+\left(\frac{1}{3}\right)^{k+1}.\frac{2}{3}$$

$$P(X\in E)=\frac{2e^{-1}}{3}\left[1+\frac{1}{2!}+\frac{1}{4!}+\cdots\right]+\frac{2}{3}\left[\frac{1}{3}+\left(\frac{1}{3}\right)^3+\left(\frac{1}{3}\right)^5+\cdots\right]$$

$$=\frac{2e^{-1}}{3}\left[\frac{e+e^{-1}}{2}\right]+\frac{2}{3}.\frac{\frac{1}{3}}{1-\frac{1}{9}}\quad\frac{1+e^{-2}}{3}+\frac{1}{4}=\frac{7}{12}+\frac{e^{-2}}{3}$$

8. **Let X and Y have the joint probability mass function**

$$\boldsymbol{P(X=n,Y=k)=\left(\frac{1}{2}\right)^{n+2k+1}\quad;n=-k,-k+1,\ldots;k=1,2,\ldots}$$

Then E(Y) equals

(a) 1
(b) 2
(c) 3
(d) 4

Solution: (b) $P(X=n,Y=k)=\left(\frac{1}{2}\right)^{n+2k+1}\quad;n=-k,-k+1,\ldots\ldots;k=1,2,\ldots$

$$P(Y=k)=\left(\frac{1}{2}\right)^k\ ;k=1,2,\ldots$$

$$E(Y)=\sum_{k=1}^{\infty}k\left(\frac{1}{2}\right)^k=1.\left(\frac{1}{2}\right)+2.\left(\frac{1}{2}\right)^2+3\left(\frac{1}{2}\right)^3+\cdots=2$$

9. If X_1 and X_2 are i.i.d. N(0, 1) random variables, then $P(X_1^2 + X_2^2 \leq 2)$ equals

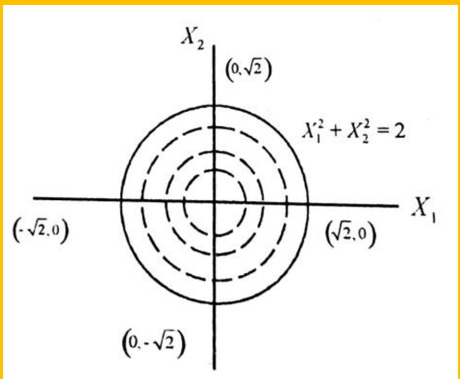
(a) e^{-1}

(b) e^{-2}

(c) $1 - e^{-1}$

(d) $1 - e^{-2}$

Solution: (d)



$$P(X_1^2 + X_2^2 < 2) = \frac{1}{2\pi}\int_{-\sqrt{2}}^{\sqrt{2}}\int_{\sqrt{2-x_2^2}}^{\sqrt{2-x_2^2}} e^{-\frac{1}{2}(x_1^2+x_2^2)} dx_1\, dx_2$$

$$X_1 = r\cos\theta \;\; and\; X_2 = r\sin\theta$$

$$\Rightarrow dx_1\, dx_2 = r\, dr\, d\theta$$

$$So, P(x_1^2 + x_2^2 < 2)$$

$$= \frac{1}{2\pi}\int_{\theta=0}^{2\pi}\int_{r=0}^{2} e^{-r^2/2} r\, dr\, d\theta = 1 - e^{-2}$$

10. Let $f : R \to R$ be a function defined by

$$f(x) = \begin{cases} x^{3/2}, & x \geq 0, \\ -|x|^{3/2}, & x < 0 \end{cases}$$

Then f is

(a) Continuous everywhere, but not differentiable at $x = 0$

(b) Differentiable everywhere except at $x = 0$

(c) Differentiable everywhere, but not twice differentiable at $x = 0$.

(d) Differentiable infinitely many times everywhere.

Solution: (c)

$$f(x) = \begin{cases} x^{3/2} & ; x \geq 0 \\ -\left(-x^{\frac{3}{2}}\right) & ; x < 0 \end{cases}$$

$$f'(x) = \begin{cases} \frac{3}{2}x^{1/2} & ; x \geq 0 \\ \frac{3}{2}(-x)^{1/2} & ; x < 0 \end{cases}$$

But second derivative onwards will not exist at $x = 0$

11. Let $P = \frac{\underset{\sim}{x}\,\underset{\sim}{x}^T}{\underset{\sim}{x}^T\underset{\sim}{x}}$ be an $n \times n$ (n>1) matrix, where $\underset{\sim}{x}$ is a non zero column vector. Then which one of the following statement is FALSE?

(a) P is idempotent

(b) P is orthogonal

(c) P is symmetric

(d) Rank of P is one

Solution: (b) As x is of rank 1 and x^T is also of rank 1, so xx^T cannot be of rank more than 1. So P is not orthogonal, as orthogonal matrix of order n should be of rank 'n'.

12. Let X and Y be i.i.d. binomial random variables with parameters n and ½ and let Z be another binomial random variable with parameters 2n and ½. Then $P(X = Y)$ equals

(a) $P(Z = 0)$

(b) $P(Z = n)$

(c) $P(Z = 2n - 1)$

(d) $P(Z = n + 1)$

Solution: (b)

$$P(X = Y) = \sum_{k=0}^{n} \left[\binom{n}{k}\left(\frac{1}{2}\right)^n\right]^2 = \binom{2n}{n}\left(\frac{1}{2}\right)^n = P(Z = n)$$

13. Let X be a random variable with the probability density function

$$f\left(\frac{x}{\theta}\right) = \begin{cases} 2\theta x + 1 - \theta, 0 < x < 1 & -1 \leq \theta \leq 1 \\ 0, & otherwise \end{cases}$$

Based on a sample of size one, the most powerful critical region (rejection region) for testing $H_0 : \theta = 0 \; against \; H_1 : \theta = 1 \; at \; level \; \alpha = 0.2$ is given by

(a) $x > \frac{4}{5}$

(b) $x \leq \frac{2}{5}$

(c) $x > \frac{8}{5}$

(d) $x < \frac{4}{5}$

Solution: (d) $\alpha = P(x \in \omega | H_0 \text{ is true})$

$$\alpha = P(x \in \omega \big| \theta = 0) = \int_{x \in \omega} 1 dx$$

As, at $q = 0, f(x) = 1$

If $X > \frac{4}{5}$ *then*

$$\int_{4/5}^{1} 1\, dx = -\frac{4}{5} = \frac{1}{5} = 0.2$$

14. Let $P = (P_{ij})$ be a 50×50 matrix where $P_{ij} = min\,(i,j)$; $i,j = 1, \dots, 50$. Then the rank of P equals

(a) 1

(b) 2

(c) 25

(d) 50

Solution: (d)

$$P = \begin{bmatrix} 1 & 1 & 1 \dots 1 \\ 1 & 2 & 2 \dots 2 \\ 1 & 2 & 3 \dots 3 \\ \vdots & \vdots & \vdots \\ 1 & 2 & 3 \dots 50 \end{bmatrix}$$

By $R_1 \leftarrow R_i - R_{i-1}; i = 50, 49, 48, \dots .2$

We get

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \dots & 1 \\ 0 & 1 & 1 & 1 \dots & 1 \\ 0 & 0 & 1 & 1 \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \dots & 1 \\ 0 & 0 & 0 & 0 \dots & 1 \end{bmatrix}$$

Which is non–singular, so rank is 50.

15. If X is a $G\left(2, \frac{1}{2}\right)$ random variable, then $P(X > 4)$ equals

(a) $3e^{-2}$

(b) $2e^{-2}$

(c) e^{-2}

(d) $0.5e^{-2}$

Solution: (a) $X \sim G\left(2, \frac{1}{2}\right)$

$$\Rightarrow p.d.f.\, is\ given\ by \quad f(x) = \frac{\left(\frac{1}{2}\right)^2}{\sqrt{(2)}} x^{2-1} e^{-x/2}; \quad x > 0$$

$$P(x > 4) = \int_4^{\infty} \frac{1}{4} x e^{-x/2}\, dx$$

$$= \left[-\frac{1}{2} x e^{-\frac{x}{2}} - e^{-\frac{x}{2}}\right]_4^{\infty} = 2e^{-2} + e^{-2} = 3e^{-2}\,.$$

16. (a) Let X_1 and X_2 denote the lifetimes (in months) of bulbs produced at factories F_1 and F_2 respectively. The random variables X_1 and X_2 are Exp(1/8) and Exp(1/2) respectively. A shop procures 80% of its supply of bulbs from factory F_1 and 20% from factory F_2. A randomly selected bulb from the shop is put on testing and is found to be working after 4 months. What is the probability that it was procured from factory F_2?

(b) Let X be a $G(4, \lambda), \lambda > 0$, random variable. Find the value of λ that minimizes E(Y), where

$$\boldsymbol{Y = X + \frac{3}{4X}}.$$

Solution: (a)

P(Bulb from factory F, to the working after 4 months)

$$= \int_4^{\infty} \frac{1}{8} e^{-\frac{x}{8}} dx = \left[-e^{-\frac{x}{8}}\right]_4^{\infty} = e^{-1/2}$$

P(Bulb from factory F_2 to be working after 4 months)

$$\int_4^{\infty} \frac{1}{2} e^{-x/2} dx = \left[-e^{-\frac{x}{2}}\right]_4^{\infty} = e^{-2}$$

P(Factory F_2 of bulb given lifetime is more than 4 months),

$$\frac{P(F_2 \cap t > 4)}{P(t > 4)} = \frac{\frac{1}{5} \times e^{-2}}{\frac{4}{5} \times e^{-1/2} + \frac{1}{5} e^{-2}} = \frac{e^{-2}}{4e^{-1/2} + e^{-2}}$$

(b) $X \sim G(4, \lambda)$

$$\Rightarrow f(x) = \frac{\lambda^4}{\Gamma(4)} x^{4-1} e^{-\lambda x}; x > 0$$

$$Y = X + \frac{3}{4X}$$

$$E(Y) = \int_0^{\infty} yf\,(y) = \int_0^{\infty}\left(x+\frac{3}{4x}\right)\frac{\lambda^4}{\Gamma(4)}x^3e^{-\lambda x}dx$$

$$= \int_0^{\infty}\frac{\lambda^4}{6}.\left[x^4+\frac{3x^2}{4}\right]e^{-\lambda x}dx$$

$$= \frac{\lambda^4}{6}\left(\frac{\Gamma(5)}{\lambda^5}+\frac{3}{4}\frac{\Gamma(3)}{\lambda^3}\right) = \frac{\lambda^4}{6}\left(\frac{24}{\lambda^5}+\frac{3}{2\lambda^3}\right) = \frac{4}{\lambda}+\frac{\lambda}{4}$$

$$\Rightarrow E(Y) = \frac{4}{\lambda}+\frac{\lambda}{4}$$

For optimum value: $E' = 0$

$$\Rightarrow -\frac{4}{\lambda^2}+\frac{1}{4} = 0$$

$$\Rightarrow \lambda^2 = 16$$

$$\Rightarrow \lambda = 4\ (as\ \lambda > 0)$$

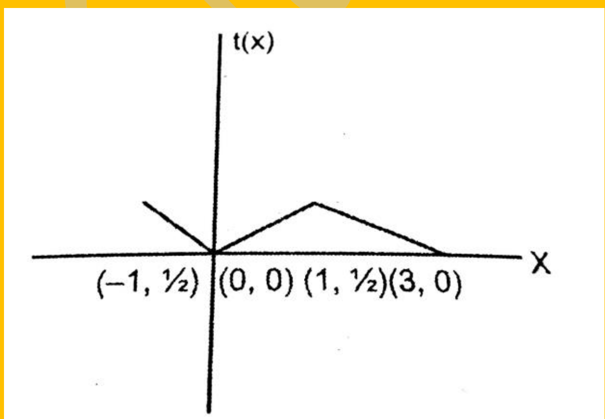
So, E(Y) is minimum if $\lambda = 4$.

17. The probability density function of a random variable X is

$$f(x) = \begin{cases} \frac{|x|}{2}, & -1 \le x \le 1, \\ \frac{3-x}{4}, & 1 < x \le 3, \\ 0, & otherwise \end{cases}$$

Find the cumulative distribution function and the probability density function $Y = |X|$. Also, find the median of the distribution of Y.

Solution:



For $y = |x|$, $f(x)\ for\ 0 \le x \le 1$ will be doubled and for $1 < x \le 3$ it will be retained.

i.e., for $Y = |X|$, pdf will be

$$f(y) = \begin{cases} y & ; 0 \le y \le 1 \\ \dfrac{3-y}{4} & ; 1 < y \le 3 \end{cases}$$

p.d.f. for Y will be given by

$$F(Y) = \int_0^y ydy \; if \; 0 \le y < 1$$

$$= \int_0^1 ydy + \int_1^y \frac{3-y}{4} dy \quad if \; 1 \le y < 3$$

$$i.e., F(y) = \begin{cases} \dfrac{y^2}{2} \; if & 0 \le y < 1 \\ \dfrac{1}{2} + \dfrac{\left(3y - \frac{y^2}{2}\right) - \left(\frac{5}{2}\right)}{4} & ; 1 \le y \le 3 \end{cases}$$

Median of the distribution of Y is 1, as $P(Y < 1) = P(Y \ge 1) = 1/2$

18. Let the joint probability density function of continuous random variables X and Y be given by

$$\boldsymbol{f(x,y) = \begin{cases} \dfrac{1}{\alpha\beta} e^{-\left(\frac{x}{\alpha}+\frac{y-x}{\beta}\right)}, \\ 0, \end{cases}}$$

$$\boldsymbol{0 < x \le y < \infty, \alpha > 0, \quad \beta > 0, \quad \alpha \ne \beta \; otherwise}$$

Find the conditional probability density function of Y given $X = x (x > 0)$ and the correlation coefficient between X and Y.

Solution:

$$f(x,y) = \frac{1}{\alpha\beta} e^{-\left(\frac{x}{\alpha}+\frac{y-x}{\beta}\right)}; \; 0 < x \le y < \infty; \; \alpha > 0, \beta > 0, \alpha \ne \beta$$

Marginal density function of X

$$f(x) = \int_x^\infty \frac{1}{\alpha\beta} e^{-\frac{x}{\alpha}+\frac{x}{\beta}} e^{-\frac{y}{\beta}} dy = \frac{e^{x\frac{(\alpha-\beta)}{\alpha\beta}}}{\alpha\beta} \frac{e^{-\frac{y}{\beta}}}{-1/\beta} \Big|_x^\infty$$

$$= \frac{e^{\frac{x(\alpha-\beta)}{\alpha\beta}}}{\alpha} e^{-x/\beta}$$

$$or, f(x) = \frac{1}{\alpha} e^{-\frac{x}{\beta}}; a > 0; x > 0$$

Marginal density function of g

$$f(y) = \int_0^y \frac{1}{\alpha\beta}\, e^{-\frac{y}{\beta}} e^{x\left(\frac{\alpha-\beta}{\alpha\beta}\right)} dx = \frac{e^{-y/\beta}}{(\alpha-\beta)} e^{x(\alpha-\beta)/\alpha\beta} \Big|_0^y$$

$$= \frac{e^{-y/\beta}\left[e^{y\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)} - 1\right]}{(\alpha-\beta)} = \frac{e^{-y/\alpha} - e^{-y/\beta}}{(\alpha-\beta)}; y > 0$$

$$P(Y = y \,|\, X = x) = \frac{P(Y = y; X = x)}{P(X = x)} = \frac{f(x,y)}{f(x)}$$

$$= \frac{\frac{1}{\alpha\beta}\, e^{-\left(\frac{x}{\alpha}+\frac{y-x}{\beta}\right)}}{\frac{1}{\alpha} e^{-\left(\frac{x}{\alpha}\right)}} = \frac{1}{\beta}\, e^{-\left(\frac{x-y}{\beta}\right)}; \quad y \geq x$$

(this is the conditional p.d.f. of g)

$$E(X) = \alpha\, ; v(X) = \alpha^2$$

$$E(Y) = \frac{\frac{1}{\alpha^2} - \frac{1}{\beta^2}}{(\alpha-\beta)}\, ;$$

$$Var(Y) = \frac{\alpha^3 + \beta^3}{(\alpha-\beta)\alpha^3\beta^3}$$

$$E(XY) = \int_{y=0}^{\infty}\int_{x=0}^{y} xy.\frac{1}{\alpha\beta} e^{-\left(\frac{x}{\alpha}+\frac{y-x}{\beta}\right)} dx\, dy$$

$$Cov\,(X,Y) = E(XY) - E(X)E(Y)$$

$$r = \frac{Cov\,(x,\gamma)}{\sqrt{v(x)v(\gamma)}}$$

19. (a) Let X_1, X_2, X_3, be i.i.d. random variables with the probability density function

$$f(x\,|\,\theta) = \geq \begin{cases} \frac{1+\theta x}{2}, & -1 < x < 1;\ -1 < \theta < 1 \\ 0, & otherwise \end{cases}$$

Find the method of moments estimator (T) of θ. Let $S = X_1 + 2X_2$ be a another estimator of θ. Find the efficiency of T relative to S.

(b) Let $X_1, \ldots, X_n$ be independent random variables with X_i having a $N\left(\theta, \frac{1}{i}\right)$ distribution, $\theta \in R; i = 1, \ldots., n$. Find the maximum likelihood estimator of θ.

Solution:

(a)

$$L = \prod_{i=1}^{3} f(x_i|\theta) = \prod_{i=1}^{3}\left(\frac{1+\theta x_i}{2}\right)$$

$$\log L = \sum_{i=1}^{3} \log\left(\frac{1+\theta x_i}{2}\right)$$

$$\Longrightarrow \frac{1}{L}\frac{\partial L}{\partial \theta} = \sum_{i=1}^{3} \frac{1}{\left(\frac{1+\theta x_i}{2}\right)} x_i = \sum_{i=1}^{3} \frac{2x}{1+\theta x_i}$$

$$\frac{\partial L}{\partial \theta} = 0$$

$$\Longrightarrow \frac{x_1}{1+\theta x_1} + \frac{x_2}{1+\theta x_2} + \frac{x_3}{1+\theta x_3} = 0$$

$$\Longrightarrow \sum x_1(1+\theta x_2)(1+\theta x_3) = 0$$

$$\Longrightarrow 3\theta^2 x_1 x_2 x_3 + 2\sum x_1 x_2^0 + x_1 + x_2 + x_3 = 0$$

$$\Longrightarrow \theta = \frac{-2\sum x_1 x_2 + 2\sqrt{\left(\sum x_1 x_2\right)^2 - 3(x_1 x_2 x_3)(x_1 + x_2 + x_3)}}{6 x_1 x_2 x_3}$$

$$As\ x_1 x_2 x_3\ are\ i.i.d.\ So, \qquad T = x_1 + x_2 + x_3$$

$$V(T) = 3\sigma^2\ ; V(S)\ = 5\sigma^2$$

Efficiency of T relative to $S = 5\sigma^2/3\sigma^2 = 5/3$

(b) Likelihood function

$$L = \prod_{L=1}^{n} \frac{1}{\sqrt{2\pi/i}}\, e^{-\frac{(x_i-\theta)^2 i^2}{2}}$$

$$L = \prod_{L=1}^{n} \frac{\sqrt{i}}{\sqrt{2\pi}} e^{-\frac{i^2(x_i-\theta)^2}{2}}$$

$$\Rightarrow \log L = \sum \log\sqrt{i} - \sum \log\sqrt{2\pi} - \sum \frac{i^2(x_i-\theta)^2}{2}$$

Differentiating we get

$$\Rightarrow \frac{1}{L}\frac{\partial L}{\partial \theta} = -\sum i^2(x_i - \theta) = 0$$

$$\Rightarrow \theta . \frac{n(n+1)(2n+1)}{6} = \sum i^2 x_i$$

$$\Rightarrow \hat{\theta} = \frac{6[x_1 + 2^2x_2 + 3^2x_3 + \cdots + n^2x_n]}{n(n+1)(2n+1)}$$

So, maximum likelihood estimator of θ is

$$\hat{\theta} = \frac{6(x_1 + 2^2x_2 + 3^2x_3 + \cdots + n^2x_n)}{n(n+1)(2n+1)}$$

20. (a) Consider the simple linear regression model

$$\boldsymbol{y_i = \alpha + \beta x_i + \varepsilon_i, i = 1, \dots, n,}$$

where ε_i's are i.i.d. random variables with mean 0 and variance σ^2. Suppose that we have a data set $(x_1, y_1), \dots, (x_n, y_n)$ with $n = 10$,

$$\sum_i x_i = 50, \quad \sum_i y_i = 40, \quad \sum_i x_i^2 = 500,$$

$$\sum_i y_i^2 = 400 \text{ and } \sum_i x_i y_i = 400.$$

Find the least square estimates of a and b. Also find an unbiased estimate of σ^2.

(b) Let {1, 2, 3, 4, 5, 6, 7, 8, 9} be the set of observed values of a random sample from a interval for m and σ^2 are unknown. Find a 95% confidence interval for μ.

(You may take the value of $\sqrt{5/6}$ as 0.9)

Solution:

(a) Linear curve fitted over the given data set will be

$$\begin{vmatrix} y & x & 1 \\ \sum y_i & \sum x_i & n \\ \sum x_i y_i & \sum x_i^2 & \sum x_i \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} y & x & 1 \\ 40 & 50 & 10 \\ 400 & 500 & 50 \end{vmatrix} = 0$$

$$\Rightarrow -2500y + 2000x = 0$$

$$\Rightarrow y = \frac{4}{5}x$$

$$\Rightarrow a = \frac{4}{5} \text{ and } b = 0 \text{ unbiased estimator for } \mu^2 \text{ will be}$$

$$= \frac{1}{n-1}\left(\sum x_i^2 \left(\sum x_i\right)^{\frac{2}{n}}\right)$$

$$= \frac{1}{9}\left(500 - \frac{(50)^2}{10}\right) = \frac{250}{9} = \frac{5}{3}\sqrt{10}$$

(b) $X = \{1, 2, 3, 4, 5, 6, 7, 9\}$

$$\sum x_i = 1 + 2 + \cdots + 9 = 45;$$

$$\sum x_i^2 = 1^2 + 2^2 + \cdots + 9^2$$

$$\frac{9 \times 10 \times 19}{6} = 285\bar{x} = 5$$

$$S^2 = \frac{1}{8}\sqrt{(x-5)^2} = \frac{1}{8}(2)(4^2 + 3^2 + 2^2 + 1^2) = \frac{15}{2}$$

95% confidence interval for m will be

$$(\bar{x} - 1.965, \bar{x} + 1.965)$$

$$= \left(5 - 1.96\sqrt{\frac{15}{2}}, 5 + 1.96\sqrt{\frac{15}{2}}\right)$$

$$= \big(5 - (1.96)(2.7), 5 + (1.96)(2.7)\big)$$

$$= (0.292, 10.292)$$

21. Let $X_1, \ldots, X_5$ be i.i.d. random variables with the probability density function

$$f\left(x \middle| \theta\right) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & x > 0, \theta > 0, \\ 0, & \text{otherwise} \end{cases}$$

Find the power of the most powerful test for testing $H_0 : \theta = 1$ $against$ $H_1 : \theta = 2$ $at\ the\ level$ $\alpha = 0.05$.

Solution: $\alpha = P\left(Accept\,\frac{c}{H_0}\,is\ true\right) = 0.05\ (given)$

$$\beta = P\left(Accept\,\frac{c^c}{H_1}\,is\ true\right)$$

$$L = \prod_{i=1}^{5} f\left(\frac{x_i}{\theta}\right) = \frac{1}{\theta^5}\, e^{-(\sum x_i)/\theta}$$

Consider $H_1 : \theta = 2$

The best critical region, using Neyman–Pearson Lemma is

$$\frac{1}{32} e^{-(\sum x_i)/2} \geq k e^{-\sum x_i}\,; k > 0$$

$$\Rightarrow e^{(\sum x_i)/2} \geq 32k$$

$$\Rightarrow \frac{\sum x_i}{2} \geq \log(32k)$$

$$\Rightarrow \sum x_i \geq 2\log(32k)$$

$$P\left(\sum x_i \geq (2\log(32k)|H_0)\right) = \alpha \quad \ldots\ldots(1)$$

$$Now, M\left[\left(\frac{2}{\theta}\right)\sum x_i\right](t) = (1-2t)^{-n}$$

[As it follows chi–Square distribution]

$$\Rightarrow W_0 = \left\{x : 2\sum x_i \geq x^2_{1-\alpha,10}\right\}$$

$$W_0 = \left\{x : 2\sum x_i \geq x^2_{0.95,10}\right\}$$

Power of the test

$$1-\beta = P[x \in W_0|H_i]$$

$$= P\left[\sum_{i=1}^{5} x_i \geq \frac{1}{2} x^2_{1-\alpha.2n}\,|H_1\right]$$

$$= P\left[\sum_{i=1}^{5} x_i \geq \frac{1}{2} x^2_{0.95,10}|H_1\right]$$

$$= P\left[x^2_{(10)} \geq \frac{1}{2} x^2_{0.95,10}\right]$$

22. (a) Evaluate $\lim\limits_{x\to 0}\left[\frac{1}{x^2} - \frac{1}{x\tan x}\right]$.

(b) Let $\{a_n\}_{n\geq 1}$ **be a sequence of real numbers such that** $a_1 = 1$ *and* $\lim\limits_{n\to\infty} a_n = 3$**. Find the value of**

$$\sum_{n=1}^{\infty}\left[a_{n+1}^2 - a_n^2\right].$$

Solution:

(a)

$$\lim_{x\to 0}\left[\frac{1}{x^2}-\frac{1}{x\tan x}\right]=\lim_{x\to 0}\frac{\tan x - x}{x^2\tan x}$$

$$=\lim_{x\to 0}\frac{\left(x+\frac{x^3}{3}+\cdots\right)-x}{x^2\left(x+\frac{x^3}{3}+\cdots\right)}=\frac{1}{3}$$

(b)

$$\sum_{n=1}^{\infty}[a_{n+1}^2-a_n^2]$$

$$=\lim_{n\to\infty}(a_2^2-a_1^2)+(a_3^2-a_2^2)+\cdots(a_{n+1}^2-a_n^2)$$

$$=\lim_{n\to\infty}(a_{n+1}^2-a_1^2)=3^2-1=8$$

23. (a) Test for the convergence of the series

$$\sum_{n=1}^{\infty}\left(1-e^{-\frac{1}{n}}\right)ln\left(1+\frac{1}{n}\right)$$

(b) Determine whether the function

$$f(x,y)=\begin{cases}\frac{x^4+y^4}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

is continuous at (0, 0).

Solution:

(a) $u_n=\left(1-e^{-\frac{1}{n}}\right)\ln\left(1+\frac{1}{n}\right)=\left[1-\left\{1-\frac{1}{n}+\frac{1}{2!}\cdot\frac{1}{n^2}-\frac{1}{3!}\cdot\frac{1}{n^3}+\cdots\right\}\right]\left[\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}+\cdots\right]$

$$=\frac{1}{n^2}\left[1-\frac{1}{2!\,n}+\frac{1}{3!\,n^2}\cdots\right]\left[1-\frac{1}{2n}+\frac{1}{3n^2}+\cdots\right]$$

Let us take convergent series $\sum V_n$ where

$$V_n=\frac{1}{n^2}$$

$$Further\ \lim_{n\to\infty}\frac{u_n}{V_n}=1$$

$$So,\sum u_n=\sum\left(1-e^{-\frac{1}{n}}\right)\ln\left(1+\frac{1}{n}\right)\ \ is\ convergent.$$

(b) $(x, y) = (r\cos\theta, r\sin\theta); (x, y) \to (0,0)$

$$\underset{(x,y)\to(0,0)}{Lt} f(x,y) = \underset{(x,y)\to(0,0)}{Lt} \frac{x^4 + y^4}{x^2 + y^2}$$

$$= \underset{r\to 0}{Lt} \frac{r^4(cos^4\theta + sin^4 0)}{r^2}$$

$$= \underset{r\to 0}{Lt} r^2(cos^4\theta + sin^4\theta) = 0$$

Also, $f(0,0) = 0$

As, $\underset{(x,y)\to(0,0)}{Lt} f(x,y) = f(0,0)$

So, f(x, y) is continuous at (0, 0).

24. (a) Find the area of the smaller of the two regions enclosed between

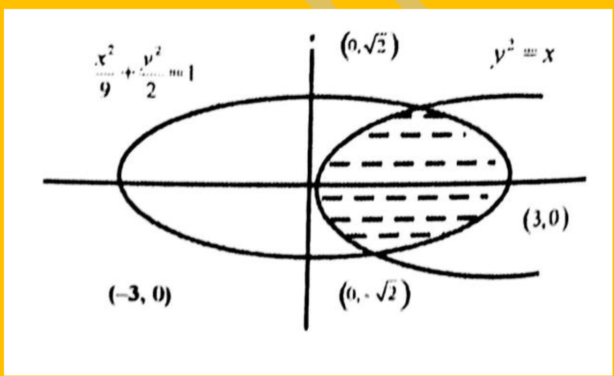
$$\frac{x^2}{9} + \frac{y^2}{2} = 1 \ and\ y^2 = x$$

(b) Evaluate

$$\int_1^\infty \int_0^{y-1} e^{-\frac{y}{x+1} - x}\, dxdy$$

Solution :

(a)



$$\frac{x^2}{9} + \frac{y^2}{2} = 1 \ and\ y^2 = x$$

$$\Rightarrow \frac{x^2}{9} + \frac{x}{2} = 1$$

$$\Rightarrow 2x^2 + 9x - 18 = 0$$

$$\Rightarrow x = \frac{-9 \pm \sqrt{81+144}}{4}$$

$$\Rightarrow x = \frac{-9 \pm 15}{4} = -6, \frac{3}{2}$$

As x can be positive only so, $x = 3/2$

So, from the figure the shaded area will be

$$2\left[\left|\int_0^{\frac{3}{2}} \sqrt{x}dx\right| + \left|\int_{\frac{3}{2}}^{3} \sqrt{2\left(1-\frac{x^2}{9}\right)}\,dx\right|\right]$$

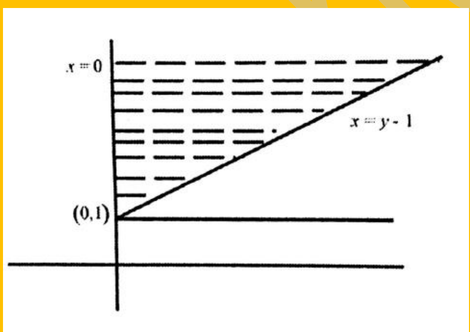
$$= 2\left[\frac{x^{\frac{3}{2}}}{3/2}\int_0^{\frac{3}{2}} + \sqrt{\frac{2}{9}}\left\{\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}.sin^{-1}\frac{x}{3}\right\}\Big|_{3/2}^{3}\right]$$

$$= 2\left[\sqrt{\frac{3}{2}} + \frac{\sqrt{2}}{3} - \left\{\frac{9}{2}.\frac{\pi}{2} - \frac{3}{4}.\frac{3}{2}\sqrt{3} - \frac{9}{2}.\frac{\pi}{6}\right\}\right]$$

$$= 2\left[\sqrt{\frac{3}{2}} + \frac{\sqrt{2}}{3}\left\{\frac{9\pi}{6} - \frac{9\sqrt{3}}{8}\right\}\right]$$

$$= \sqrt{6} + \sqrt{2}\pi - \frac{3\sqrt{6}}{4}$$

(b)



$$\int_1^{\infty}\int_0^{y-1} e^{-\frac{y}{x+1}-x}\,dxdy = I(let)$$

The shaded region is the required region, so, by change of order of integration, we get

$$I = \int_0^{\infty}\int_{x+1}^{\infty} e^{-x}e^{-\frac{y}{x+1}}\,dxdy$$

$$= \int_0^\infty e^{-x} \frac{e^{-\frac{y}{x+1}}}{\left(-\frac{1}{x}+1\right)} \int_{x+1}^\infty dx$$

$$= \int_0^\infty (x+1)e^{-(x+1)}dx$$

$$= -(x+1)e^{-1(x+1)} - e^{-(x+1)}\Big|_0^\infty = 2/e$$

25. (a) Solve the differential equation

$$(\boldsymbol{x}+\boldsymbol{y}+\mathbf{2})\boldsymbol{dy} - (\boldsymbol{y}+\mathbf{2})\boldsymbol{dx} = \mathbf{0}$$

(b) Consider the linear transformation

$$\boldsymbol{L} : \boldsymbol{R}^3 \to \boldsymbol{R}^3 \text{ is given by}$$

$$\boldsymbol{L}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = (\boldsymbol{kx}+\boldsymbol{y}+\boldsymbol{z}, \boldsymbol{x}+\boldsymbol{ky}+\boldsymbol{z}, \boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z})$$

Find the matrix P associated with the above transformation with respect to the standard basis. Further, find the values of k for which P has zero as one of its eigenvalues.

Solution:

(a) $(x+y+2)dy - (y+2)dx = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{(y+2)}{x+y+2}$$

$$\Rightarrow \frac{dx}{dy} = \frac{x+(y+2)}{(y+2)}$$

$$\Rightarrow \frac{dx}{dy} - \frac{1}{y+2}x = 1$$

Which is a linear differentiable equation. So,

$$xe^{\int -\frac{1}{y+2}dx} = \int e^{\int -\frac{1}{y+2}dy}.1dy + c$$

$$\Rightarrow \frac{x}{y+2} = +\int \frac{1}{y+2}dy + c = \ln(y+2) + c$$

$$\Rightarrow x = (y+2)[\ln(y+2) + c]$$

(b) $L(x, y, z) = (kx + y + z, x + ky + z, x + y + kz)$

Matrix P associated with the given transformation is

$$P\begin{bmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{bmatrix}$$

One of the eigenvalue of P will be zero if $\det(P) = |P| = 0$

$$i.e., \det\begin{bmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{bmatrix} = 0$$

$$\Rightarrow k(k^2 - 1) - (k - 1) + (1 - k) = 0$$

$$\Rightarrow k^3 - 3k + 2 = 0$$

$$\Rightarrow (k - 1)(k^2 + k - 2) = 0$$

$$\Rightarrow (k - 1)^2(k + 2) = 0$$

$$\Rightarrow k = -2, 1$$

So, for $k = 1 \; or \; -2$ at least one eigenvalue is zero.