IIT-JAM Mathematical Statistics (MS) 2011 Solved Paper

1. Let the function f : $[0, \infty) \to R$ be given by $f(x) = x^2 e^{-x}$. Then the maximum value of f is

(a) e^{-1}

(b) $4e^{-2}$

(c) $9e^{-1}$

(d) $16e^{-4}$

Solution: (b) $f(x) = x^2 e^{-x}$

$$Or, f'(x) = e^{-x}(2x - x^2) = xe^{-x}(2 - x)$$

$$f''(x) = e^{-x}(-2x + x^2 + 2 - 2x)$$

$$= e^{-x}(x^2 - 4x + 2)$$

$$f'(x) = 0 \Rightarrow x = 0, 2$$

$$f''(2) = e^{-2}(-2) < 0$$

So, at $x = 2$ maxima occurs.

So, maximum value is $4e^{-2}$.

2. An eigenvector of the matrix $\begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}$ is

(a) $(1, 2)^T$

(b) $(5, 0)^T$

(c) $(0, 2)^T$

(d) $(1, 1)^T$

Solution: (b)

$$A = \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}; eigen\ values\ of\ A\ is\ 1, 1$$

$$If\ X = \begin{pmatrix} x \\ y \end{pmatrix}\ is\ an\ eigen\ vector, then$$

$$(A - I)X = 0 \quad \begin{bmatrix} 0 & 8 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow 8y = 0$$

$$\Longrightarrow y = 0,$$

So $\binom{a}{0}$ $a \neq 0$ is an eigen vector, so one such value is $[5 \quad 0]^T$.

3. **Define $f, g : R \to R$ by**

$$f(x) = \begin{cases} x, & if\ x \in Q \\ sin\,(x), & if\ x \in R - Q \end{cases} \quad and$$

$$g(x) = \begin{cases} x\, sin\,(x) sin\,(1\backslash x), & if\ x \neq 0 \\ 0, & if\ x = 0 \end{cases}$$

At $x = 0$,

- **(a) Both f and g are differentiable**
- **(b) f is differentiable but g is NOT differentiable**
- **(c) g is differentiable but f is NOT differentiable**
- **(d) neither f nor g are differentiable**

Solution: (a)

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h}{h} = 1 \;\; or \;\; \lim_{h \to 0} \frac{\sin h}{h} = 1$$

So, f(x) is differentiable at $x = 0$.

$$\lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{h \sin h \sin\left(\frac{1}{h}\right)}{h} = 0$$

So, g(x) is differentiable at $x = 0$

4. **Consider the series S_1 and S_2 given by**

$$S_1 : \sum_{n=1}^{\infty} \frac{n^2 + n + 1}{n(n+1)} \quad and\ S_2 : \sum_{n=1}^{\infty} \frac{n^2 + 1}{n^2(n+1)}$$

Then

- **(a) Both S_1 and S_2 converge**
- **(b) S_1 converges and S_2 converges**
- **(c) S_2 converges and S_1 converges**
- **(d) Both S_1 and S_2 diverge**

Solution: (d)

$$\frac{n^2 + n + 1}{n(n+1)} \approx 1 \;\; and \;\; \frac{n^2 + 1}{n^2(n+1)} \approx \frac{1}{n}$$

So, both S_1 and S_2 are divergent.

5. **The equation $x^{13} - e^{-x} + x - \sin(x) = 0$ has**

(a) **No real root**
(b) **More than two real roots**
(c) **Exactly two real roots**
(d) **Exactly one real root**

Solution: (d) $x^{13} - e^{-x} + x - \sin x = f(x) \quad (let)$

$$\Rightarrow f'(x) = 13x^{12} + e^{-x} + 1 - \cos x > 0$$

So, f(x) is increasing function. $f(-\infty) < 0 \; and \; f(1) > 0$

So, there is exactly one root of

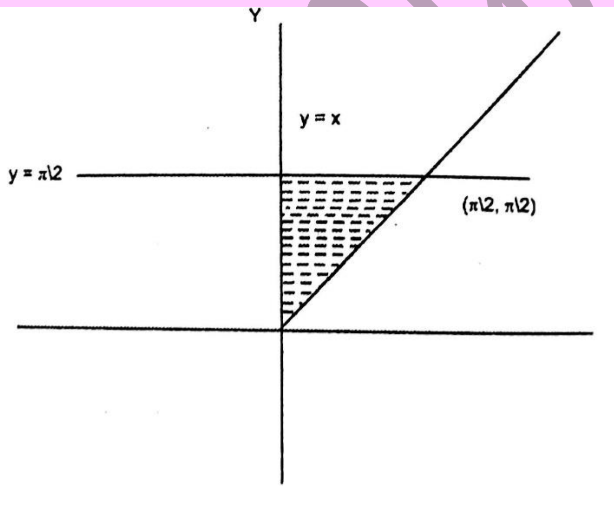
$$x^{13} - e^{-x} + x - \sin x = 0$$

6. **Let D be the triangle bounded by the y–axis, the line $2y = \pi$ and the line $y = x$. Then the value of the integral**

$$\int\int_D \frac{\cos(y)}{y} dxdy \text{ is}$$

(a) **½**
(b) **1**
(c) **3/2**
(d) **2**

Solution: (b)



$$\int\int_D \frac{\cos y}{y} dxdy = \int_{y=0}^{\pi/2} \int_{x=0}^{y} \frac{\cos y}{y} dxdy = \int_0^{\pi/2} \cos y \, dy = 1$$

7. **Let X be a random sample of size one from $U(\theta, \theta+1)$ distribution, $\theta \in R$. For testing $H_0: \theta = 1$ against $H_1: \theta = 2$, the critical region $\{x : x > 1\}$ has**

(a) **Power $= 1$ and size$= 1$**
(b) **Power $= 0$ and size $= 1$**
(c) **Power $= 1/2$ and size$= 1$**
(d) **Power $= 1$ and size $= 0$**

Solution: (a)

$$H_0 \text{ is true, then } X \sim U(1,2)$$

$$\Rightarrow f(x) = \begin{cases} 1; & 1 \le x \le 2 \\ 0; & otherwise \end{cases}$$

$$\alpha = P(x \in \omega | H_0 \text{ is true}) = P(x > 1) = 1$$

$$H_1 \text{ is true, then } X \sim U(2,3)$$

$$\Rightarrow f(x) = \begin{cases} 1; & 2 \le x \le 3 \\ 0; & otherwise \end{cases}$$

$$\beta = P(x \in \bar{\omega} | H_1 \text{ is true}) = P(x \le 1) = 0$$

Power $= 1 - b = 1, Size = a = 1$

8. **Let $X_1, \ldots, X_n$ be i.i.d. $B(1, \theta)$ random variables, $0 < \theta < 1$. Then, as an estimator θ,**

$$T(X_1, \ldots, X_n) = \frac{\sum_{i=1}^{n} X_i + \frac{\sqrt{n}}{2}}{n + \sqrt{n}} \text{ is.}$$

(a) **Both consistent and unbiased**
(b) **Consistent but NOT unbiased**
(c) **Unbiased but NOT consistent**
(d) **Neither unbiased nor consistent**

Solution: (d)

$$T(X_1, X_2, \ldots, X_n) = \frac{\sum_{i=1}^{n} X_i + \frac{\sqrt{n}}{2}}{n + \sqrt{n}} = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i + \frac{1}{2\sqrt{n}}}{1 + \frac{1}{\sqrt{n}}}$$

$$\Rightarrow T = \frac{\bar{X} + \frac{1}{2\sqrt{n}}}{1 + \frac{1}{\sqrt{n}}}$$

$$E(T) = E\left(\frac{\bar{X} + \frac{1}{2\sqrt{n}}}{1 + \frac{1}{\sqrt{n}}}\right) = \frac{1}{1 + \frac{1}{\sqrt{n}}}\left(\frac{1}{\theta + 1} + \frac{1}{2\sqrt{n}}\right)$$

Which is not equal to θ. So T is not unbiased.

Further

$$\underset{h\to\infty}{Lt}\ E(T) = \underset{h\to\infty}{Lt}\frac{1}{1+\frac{1}{\sqrt{n}}}\left(\frac{1}{\theta+1}+\frac{1}{2\sqrt{n}}\right)=\frac{1}{\theta+1}$$

So, T is consistent estimator of $\frac{1}{\theta+1}$ not q. So it is not consistent.

9. $\boldsymbol{X_1, X_2, X_3}$ **be i.i.d.** $\boldsymbol{N(0,\theta^2)}$ **random variables,** $\boldsymbol{\theta > 0}$**. Then the value of k for which the estimator**

$$\left(k\sum_{i=1}^{3}|X_i|\right)$$

is an unbiased estimator of $\boldsymbol{\theta}$ **is**

(a) $\frac{1}{3\pi}$

(b) $\sqrt{\frac{9\pi}{2}}$

(c) $\sqrt{\frac{\pi}{18}}$

(d) $\frac{2}{3\pi}$

Solution : (b) Estimator,

$$T = \left(K\sum_{i=1}^{3}|X_i|\right)$$

$$E|X_i| = \int_{-\infty}^{\infty}|X_i|\,P(X = x_i)dx_i$$

$$2\int_0^{\infty} x\,\frac{1}{\sqrt{2\pi}\theta}\,e^{-\frac{x^2}{2\theta^2}}\,dx = I(say)$$

$$Let\ \frac{x^2}{2\theta^2} = y \Rightarrow \frac{2x}{2\theta^2}dx = dy \Rightarrow \frac{x}{\theta^2}dx = dy$$

$$\Rightarrow I = 2\int_0^{\infty}\frac{\theta}{\sqrt{2\pi}}e^{-y}dy = \frac{2\theta}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}\theta$$

$$E(T) = KE\left(\sum_{i=1}^{3}|x_i|\right) = 3K\sqrt{\frac{2}{\pi}}\theta = K\sqrt{\frac{2}{9\pi}}\theta$$

So, the value of k for which T is unbiased estimate is $\sqrt{\frac{9\pi}{2}}$.

10. Let the random variables of X and Y have the joint probability mass function

$$\boldsymbol{P(X=x, Y=y) = e^{-2}\left(x_{c_y}\right)\left(\frac{3}{4}\right)^{y}\left(\frac{1}{4}\right)^{x-y}\frac{2^x}{x!}; y = 0, 1, \dots, x; x = 0, 1, 2, \dots}$$

Then E(Y)=

- **(a) ½**
- **(b) 1**
- **(c) 3/2**
- **(d) 2**

Solution: (c) $P(X = x\,; Y = y)$

$$= e^{-2}\binom{x}{y}\left(\frac{3}{4}\right)^{y}\left(\frac{1}{4}\right)^{x-y}\frac{2^x}{x!} \quad ; y = 0, 1, 2, \dots, x \quad x = 0, 1, 2, \dots$$

$$\Rightarrow P(Y = y) = \sum_{x=y}^{\infty} e^{-2}\binom{x}{y}\left(\frac{3}{4}\right)^{y}\left(\frac{1}{4}\right)^{x-y}\frac{2^x}{x!}$$

$$E(Y) = \sum_{y=0}^{\infty} y \sum_{y=0}^{\infty} e^{-2}\binom{x}{y}\left(\frac{3}{4}\right)^{y}\left(\frac{1}{4}\right)^{x-y}\frac{2^x}{x!}$$

$$P(Y = y) = \sum_{x=y}^{\infty} e^{-2}\binom{x}{y}$$

$$= \sum_{x=0}^{\infty}\frac{e^{-2}2^x}{x!\;4^x}.x4^{x-1}.3 = \frac{3\,e^{-2}}{4}\sum_{x=0}^{\infty}\frac{x2^x}{x!}$$

$$= \frac{3e^{-2}}{4} \times 2\sum_{x-1=0}^{\infty}\frac{2^{x-1}}{x-1!} = \frac{3}{2}$$

11. Let X_1 and X_2 be i.i.d. Poisson random variables with mean 1. Then $\boldsymbol{P}(max\,(X_1, X_1) > 1) =$

- **(a)** $\boldsymbol{1 - e^{-2}}$
- **(b)** $\boldsymbol{1 - 2e^{-2}}$
- **(c)** $\boldsymbol{1 - 3e^{-2}}$
- **(d)** $\boldsymbol{1 - 4e^{-2}}$

Solution: (d) $P(\max(x_1, x_2) > 1) \; = 1 - P(\max(x_1, x_2) \le 1) \; = 1 - P\{(\max(x_1, x_2)) = 0 \; or \; 1\}$

$-P(x_1 = 1)P(x_2 = 0) - P(x_1 = 1)P(x_2 = 1) = 1 - e^{-2} - e^{-2} - e^{-2} - e^{-2} = 1 - 4e^{-2}$

12. A fair dice is rolled 3 times. The conditional probability of 6 appearing exactly once, given that it appeared at least once, equals

(a) $\frac{3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)}{1-\left(\frac{5}{6}\right)^3}$

(b) $\frac{\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2}{1-\left(\frac{5}{6}\right)^3}$

(c) $\frac{3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2}{1-\left(\frac{5}{6}\right)^3}$

(d) $\frac{\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)}{1-\left(\frac{5}{6}\right)^3}$

Solution: (c) $A = 6$ appearing exactly once

$B = 6$ appearing at least once

$$A \cap B = A$$

$$P(B) = 1 - \left(\frac{5}{6}\right)^3 ; P(A) = \binom{3}{1} . \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{3.\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2}{1-\left(\frac{5}{6}\right)^3}$$

13. Let $X \sim B\left(2, \frac{1}{2}\right)$. Then $E\left(\frac{2}{1+X}\right) =$

(a) 7/6

(b) 1

(c) 6/7

(d) 2/5

Solution: $x \sim B\left(2, \frac{1}{2}\right)$

$$\Rightarrow P(X = x) = \frac{1}{B\left(2, \frac{1}{2}\right)} . \frac{x^{2-1}}{(1+x)^{2+\frac{1}{2}}} ; 0 \le x < \infty$$

$$E\left(\frac{2}{1+x}\right) = \frac{2}{B\left(2, \frac{1}{2}\right)} \int_0^\infty \frac{x^{2-1}}{(1+x)^{2+\frac{3}{2}}} dx$$

$$= \frac{2}{B\left(2, \frac{1}{2}\right)} . B\left(2, \frac{3}{2}\right) = 2 \frac{\Gamma(2) \times \Gamma\frac{3}{2} \times \Gamma\frac{5}{2}}{\Gamma\frac{7}{2} \times \Gamma 2 \times \Gamma\frac{1}{2}} = \frac{2}{5}.$$

14. **The moment generating function of an integer valued variable X is given by**

$$M_x(t) = \frac{1}{10}(2 + e^t + 4e^{2t} + 3e^{3t})e^{-t}$$

Then $P(2X + 5 < 7) =$

(a) 3/10
(b) 7/10
(c) 1
(d) 4/10

Solution: (a) $M_x(t) = \frac{1}{10}(2 + e^t + 4e^{2t} + 3e^{3t})e^{-t}$

$$P(2x + 5 < 7) = P(x < 1) = P(x = 0) + P(x = -1)$$

$$= coefficient\ of\ e^{-t} + constant\ term = \frac{2}{10} + \frac{1}{10} = \frac{3}{10}$$

15. **Let X_1 and X_2 be i.i.d. Exp (3) random variables. Then** $P(X_1 + X_2 > 1) =$

(a) $2e^{-3}$
(b) $3e^{-3}$
(c) $4e^{-3}$
(d) $5e^{-3}$

Solution: (c) $X \sim Exp\ (3) \Rightarrow f(x) = 3e^{-3x}; x > 0$

$$P(x_1 + x_2 > 1) = 1 - P(x_1 + x_2 \le 1)$$

$$= 1 - \int_{x_1=0}^{1} \int_{x_2=0}^{1-x} 3e^{-3x_1}.3e^{-3x_2}\, dx_2 dx_1 = 1 - \int_0^1 -3x^{-3x_1}e^{-3x_2} \int_0^{1-x_1} dx_1$$

$$= 1 - \int_0^1 3(e^{-3x_1} - e^{-3})\, dx_1 = 1 - [-e^{-3x_1} - 3e^{-3x}x_1]_0^1$$

$$= 1 - [1 - e^{-3} - 3e^{-3}] = 4e^{-3}$$

16. **(a) Student population of a University has 30% Asian, 40% American, 20% European and 10% African students. It is known that 40% of all Asian students, 50% of all American students, 60% of all European students and 20% of all African students are girls. Find the probability that a girl chosen at random from the University is an Asian.**

(b) Let A_1, A_2 *and* A_3 be pairwise independent events with $P(A_i) = \frac{1}{2}, i = 1, 2, 3$. Suppose that A_3 and $A_1 \cup A_2$ are independent. Find the value of $P(A_1 \cap A_2 \cap A_3)$.

Solution:

(a) Probability that a girl chosen at random is an Asian i.e.,

$$P\left(\frac{Asian}{Girl}\right) = \frac{P(Asian\ Girl)}{P(Girl)}$$

$$= \frac{30\% \times 40\%}{(30\% \times 40\%) + (40\% \times 50\%) + (20\% \times 60\%) + (10\% \times 20\%)}$$

$$= \frac{12}{12 + 20 + 12 + 2}$$

$$= \frac{12}{46} = \frac{6}{23}$$

(b) $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$ as it is given that they are pairwise independent.

So, $P(A_1 \cap A_2 \cap A_3) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

17. Let X be a random variable with cumulative distribution function

$$\boldsymbol{F(x)} = \begin{cases} \mathbf{0}, & \boldsymbol{if\ x} < 0 \\ \boldsymbol{x^2} + \frac{\mathbf{1}}{\mathbf{4}}, & \boldsymbol{if\ 0 \le x} < \frac{\mathbf{1}}{\mathbf{2}} \\ \boldsymbol{x} + \frac{\mathbf{1}}{\mathbf{8}} & \boldsymbol{if} \frac{\mathbf{1}}{\mathbf{2}} \le \boldsymbol{x} < \frac{\mathbf{3}}{\mathbf{4}} \\ \frac{\boldsymbol{x} + \mathbf{1}}{\mathbf{2}} & \boldsymbol{if} \frac{\mathbf{3}}{\mathbf{4}} \le \boldsymbol{x} < 1 \\ \mathbf{1}, & \boldsymbol{if\ x} \ge \mathbf{1} \end{cases}$$

Find the values of $\boldsymbol{P\left(0 \le X < \frac{1}{4}\right), P\left(0 \ge \frac{3}{4}\right)}$ and $\boldsymbol{P\left(X = \frac{1}{2}\right)}$

Solution: From the given cumulative distribution function, probability density function is

$$f(x) = \frac{d}{dx}F(x)\ i.e.,$$

$$f(x) = \begin{cases} 0 & if\ x > 0 \\ 2x & if\ 0 \le x < \frac{1}{2} \\ 1 & if \frac{1}{2} \le x < \frac{3}{4} \\ \frac{1}{2} & if \frac{3}{4} \le x < 1 \\ 0 & if\ x \ge 1 \end{cases}$$

$$P\left(0 \le x < \frac{1}{4}\right) = \int_0^{1/4} 2x dx = x^2 \int_0^{1/4} = \frac{1}{16}$$

$$P\left(x \ge \frac{3}{4}\right) = \int_{3/4}^{1} \frac{1}{2} dx = \frac{x}{2} \int_{3/4}^{1} = \frac{1}{8}$$

$P\left(x = \frac{1}{2}\right)$
$= 0, as\ in\ continuous\ probability\ of\ probability\ random\ variable\ taking\ a\ single\ value\ is\ 0.$

18. (a) Let X be a continuous random variable with probability density function

$$\boldsymbol{f(x) = \frac{1}{2}e^{-|x-1|}};\ -\infty < x < \infty$$

Find the value of $P(1 < |X| < 2)$.

(b)Let X and Y be i.i.d U(0,1) random variable.

Find the value of $P\left(\frac{1}{4} \leq X^2 + Y^2 \leq 1\right)$

Solution:

(a) $P(1 < |x| < 2) = P(-2 < x < -1) + P(1 < x < 2)$

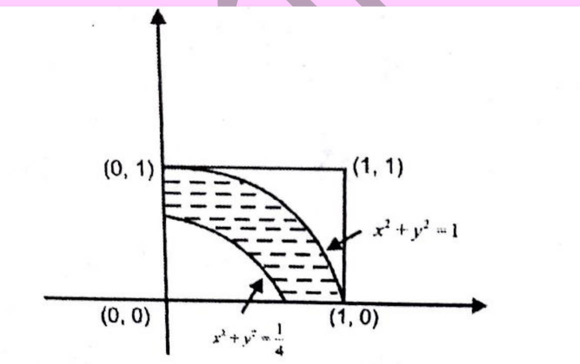
$$= \int_{-2}^{-1} \frac{1}{2} e^{-|x-1|} dx + \int_{1}^{2} \frac{1}{2} e^{-|x-1|} dx$$

$$= \int_{-2}^{-1} \frac{1}{2} e^{x-1} dx + \int_{1}^{2} \frac{1}{2} e^{-(x-1)} dx$$

$$= \frac{e^{x-1}}{2} \int_{-2}^{-1} + \left(-\frac{e^{-(x-1)}}{2}\right) \int_{1}^{2} = \frac{1}{2}(e^{-2} - e^{-3}) + \frac{1}{2}(-e^{-1} + 1)$$

$$= \frac{1}{2}(1 - e^{-1} + e^{-2} - e^{-3})$$

(b)



The square in the above figure is the sample space and shaded portion is the favourable area.
Shaded area is area of sector between circles of radius 1 and ½ unit.

$$Required\ Probability = \frac{\frac{1}{4}\left(\pi - \pi\left(\frac{1}{2}\right)^2\right)}{1}\theta = \frac{3\pi}{16}.$$

19. (a) Let the random X_1 and X_2 have joint probability density function

$$f(x_1, x_2) = \begin{cases} \dfrac{x_1 e^{-x_1 x_2}}{2}, & if\ 1 < x_1 < 3, x_2 > 0 \\ 0, & otherwise \end{cases}$$

Find the covariance between X_1 and X_2.

(b) Let $X\ X_1, \ldots, X_{100}$ be i.i.d. U(−0.5, 0.5) random variables and let T= $X_1 + \cdots + X_{100}$. Using Chebyshev's inequality show that

$$P(T^2 \geq 25) \leq \frac{1}{3}.$$

Solution:

(a)

$$f(x_1, x_2) = \begin{cases} \dfrac{x_1 e^{-x_1 x_2}}{2}; & 1 < x_1 < 3, x_2 > 0 \\ 0\,; & otherwise \end{cases}$$

$$f(x_1) = \int_0^\infty \frac{x_1}{2} e^{-x_1 x_2} dx_2 = \left[-\frac{e^{-x_1 x_2}}{2}\right]_0^\infty = \frac{1}{2}$$

$$\Rightarrow f(x_1) = \frac{1}{2} \qquad ; 1 < x_1 < 3$$

$$f(x_2) = \int_1^3 \frac{x_1}{2} e^{-x_1 x_2}\, dx_1$$

$$f(x_2) = \frac{1}{2}\left[-\frac{x_1 e^{-x_1 x_2}}{x_2} - \frac{e^{-x_1 x_2}}{x_2^2}\right]_1^3$$

$$\Rightarrow f(x_2) = \frac{1}{2}\left[\frac{-3e^{-3x_2}}{x_2} - \frac{e^{-3x_2}}{x_2^2} + \frac{e^{-x_2}}{x_2} + \frac{e^{-x_2}}{x_2^2}\right]$$

$$\Rightarrow f(x_2) = \frac{(e^{-x_2} - e^{-3x_2})}{2x_2^2} + \frac{(e^{-x_2} - 3e^{-3x_2})}{2x_2};\ x_2 > 0$$

$$\Rightarrow E(x_1) = \int_1^3 x_1 \frac{1}{2} dx_1 = 2$$

$$\Rightarrow E(x_2) = \int_0^\infty \frac{e^{-x_2} - e^{-3x_2}}{2x_2} + \frac{e^{-x_2} - 3e^{-3x_2}}{2} dx_2$$

$$\Rightarrow E(x_1 x_2) = \int_{x_1=1}^3 \int_{x_2=0}^\infty \frac{x_1 e^{-x_1 x_2}}{2} dx_2 dx_1$$

$Cov.(x_1, x_2) = E(x_1, x_2) - E(x_1)E(x_2)$. Rest you can easily do yourself.

(b) For U(−0.5, 0.5),

$$E(X) = \int_{-1/2}^{1/2} x dx = 0$$

$$E(x^2) = \int_{-1/2}^{1/2} x^2 dx = \frac{2}{3}\left(\frac{1}{8}\right) = \frac{1}{12}$$

$$V(x) = \sigma^2 = \frac{1}{12}$$

$$\Rightarrow E(T) = E\left(\sum x_i\right) = 0; \quad V(T) = 100.\frac{1}{12} = \frac{25}{3}$$

$$\Rightarrow \sigma(T) = \frac{5}{\sqrt{3}}$$

By Chebyshev's inequality

$$P(|T - E(T)| \geq K\sigma(T) \leq \frac{1}{K^2}$$

$$\Rightarrow P\left(T \geq \frac{5k}{\sqrt{3}}\right) \leq \frac{1}{K^2} \quad (Using\ K = \sqrt{3}\,)$$

$$\Rightarrow P(T \geq 5) \leq \frac{1}{3}$$

$$\Rightarrow P(T^2 \geq 25) \leq \frac{1}{3}$$

20. (a) Let $X_1, \ldots, X_n$ be a random sample from a population having a probability density function

$$f(x\,|\,\theta) = \begin{cases} \frac{4}{\theta} x^3 e^{-\frac{x^4}{3}}, & if\ x > 0 \\ 0, & otherwise \end{cases}$$

Where q > 0. Find the uniformly minimum variance unbiased estimator of q.

(b) Let $\bar{x} = 9\ and\ s_x^2 = 6$ be the sample mean and sample variance, respectively, based on a random sample of size 3 from $N(\mu_1, \sigma^2)$. Also let $\bar{y} = 7$ and $s_y^2 = 4$ be the sample mean and sample variance, respectively, based on a random sample of size 3 from $N(\mu_2, 2\sigma^2)$. Where $\mu_1, \mu_2 \in R\ and\ \sigma^2 > 0$ are unknown. Find a 95% confidence interval for $\mu_1 - \mu_2$.

Solution:

(a) $f(x\,|\,\theta) = \frac{4}{\theta} x^3\, e^{-\frac{x^4}{\theta}};\ \ x > 0$

Likelihood function

$$L(x\,|\theta) = \prod_{i=1}^{n} \frac{4}{\theta} x_i^3 e^{-\frac{x_i^4}{\theta}} = \frac{4}{\theta^n}(x_1 x_2 \dots . x_n)^3\, e^{-\frac{\sum x_i^4}{\theta}}$$

$$\log L = \log 4 - n\log\theta + 3\sum \log x_i - \frac{\sum x_i^4}{\theta}$$

$$\frac{1}{L}\frac{\partial L}{\partial \theta} = -\frac{n}{\theta} + \frac{\sum x_i^4}{\theta^2} = 0$$

$$\Longrightarrow \theta = \frac{1}{n}\sum x_i^4$$

$$\Longrightarrow \hat{\theta} = \frac{x_1^4 + x_2^4 + \cdots + x_n^4}{n} \; is\ the\ most\ likelihood\ estimator\ of\ q.$$

(b) Test statistics of $m_1 - m_2$ will be

$$\frac{(\bar{x}-\bar{y}) - E(\bar{x}-\bar{y})}{\sqrt{var\ (\bar{x}-\bar{y}}} = \frac{(\bar{x}-\bar{y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{1}{3}\left(S_x^2 + S_y^2\right)}}$$

$$= \frac{9 - 7 - (\mu_1 - \mu_2)}{\sqrt{\frac{1}{3}(6+4)}} = \frac{2 - (\mu_1 - \mu_2)}{\sqrt{\frac{10}{3}}}$$

For 95% confidence interval, we should have,

$$\left|\frac{2 - (\mu_1 - \mu_2)}{\sqrt{\frac{10}{3}}}\right| \le 1.96$$

$$\Rightarrow |2 - (\mu_1 - \mu_2)| \le \left(\sqrt{\frac{10}{3}}\right)1.96$$

$$\Rightarrow -\sqrt{\frac{10}{3}}(1.96) \le 2 - (\mu_1 - \mu_2) \le \sqrt{\frac{10}{3}}(1.96)$$

$$\Rightarrow 2 - \sqrt{\frac{10}{3}}(1.96) \le \mu_1 - \mu_2 \le 2 + \sqrt{\frac{10}{3}}(1.96)$$

21. **Let $X_1, \dots, X_{10}$ be a random sample of size 10 from a population having a probability density function**

$$f(x|\theta) = \begin{cases} \dfrac{\theta}{x^{\theta+1}}, & if\ x > 0 \\ 0, & otherwise \end{cases}$$

where $\theta > 0$. For test $H_0 : \theta = 2\ against\ H_1: \theta = 4$ at the level of significance $\alpha = 0.05$, find the most powerful test. Also find the power of this test.

Solution:

$$f(x \mid \theta) = \begin{cases} \dfrac{\theta}{x^{\theta+1}}, & if\ x > 1 \\ 0, & otherwise \end{cases}$$

Likelihood function $L(x, \theta)\ is$

$$\prod (x \mid \theta) = \prod_{i=1}^{10} \frac{\theta}{x_i^{\theta+1}} = \theta^{10} \prod_{i=1}^{10} \frac{1}{x_i^{\theta+1}}$$

Consider $H_1 : \theta = 4\ \ and\ H_0 : \theta = 2$

The best critical region, using Neyman–Pearson Lemma is given by

$$(4)^{10} \prod_{i=1}^{10} \frac{1}{x_i^5} \geq K(2)^{10} \prod_{i=1}^{10} \frac{1}{x_i^3}$$

$$\Rightarrow (2)^{10} \prod_{i=1}^{10} \frac{1}{x_i^2} \geq K \Rightarrow K \leq (2)^{10} \prod_{i=1}^{10} \frac{1}{x_i^2}$$

$$\Rightarrow \log K \leq 10 \log 2 + \sum_{i=1}^{10} -\log x_i$$

$$\Rightarrow 2 \sum_{i=1}^{10} \log x_i \leq 10 \log 2 - \log K$$

Now as,

$$P\left(2 \sum_{i=1}^{10} \log x_i \leq 10 \log 2 - \log k | H_0\ is\ true\right) = \alpha$$

And it is given that $a = 0.05$ we evaluate the most powerful test.

Now, $\beta = p(x \in \overline{W} | H_1\ is\ true)$

$$So, \beta = P\left(2 \sum_{i=1}^{10} \log x_i > 10 \log 2 - \log k | \theta = 4\right)$$

And hence the power of the test will be

$$1-\beta = P\left(2\sum_{i=1}^{10}\log x_i \le 10\log 2 - \log k \mid \theta = 4\right)$$

22. Let $\boldsymbol{f}: [-\mathbf{1}, \mathbf{1}] \to \boldsymbol{R}$ be a continuous function.

(a) Show that

$$\sum_{n=1}^{\infty}\left(f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right)\right) \textit{ is a convergent series.}$$

(b) Further, if f is differentiable on (0,1) and $|f'(x)| < 1$ *for all* $x \in (0,1)$, then show that

$$\sum_{n=1}^{\infty}\left(f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right)\right) \textit{ is a convergent series.}$$

Solution: (a)

$$\sum_{n=1}^{\infty}\left(f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right)\right)$$

$$= \left[f(1) - f\left(\frac{1}{2}\right)\right] + \left[f\left(\frac{1}{2}\right) - f\left(\frac{1}{3}\right)\right] + \left[f\left(\frac{1}{3}\right) - f\left(\frac{1}{4}\right)\right] + \cdots + \left[f\left(\frac{1}{n-1}\right) - f\left(\frac{1}{n}\right)\right] + \left[f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right)\right]$$

$$= f(1) - f\left(\frac{1}{n+1}\right)$$

Now as f is a continuous function in closed interval [−1, 1], so it is finite for every value in it (bounded).

$$So, f(1) - f\left(\frac{1}{n+1}\right) \; is\; finite$$

Hence the series is convergent.

(b)

$$\frac{f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right)}{\frac{1}{n} - \frac{1}{n+1}} = f'(\xi_n); \quad \frac{1}{n+1} < \xi_n < \frac{1}{n}$$

$$\Rightarrow f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right) = \frac{1}{n(n+1)} f'(\xi_n)$$

$$\Rightarrow \left|f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right)\right| = \frac{1}{n(n+1)} |f'(\xi)|$$

$$\Rightarrow \sum_{n=1}^{\infty} \left| f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right) \right| = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} |f'(\xi)|$$

$$< \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$$

$$So, = \sum_{n=1}^{\infty} \left| f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right) \right| \text{ is convergent series.}$$

23. (a) Let $f: [0,1] \to R$ be a continuous function such that

$$\int_0^1 f(t)dt = 1.$$

Then show that there exists a point $c \in (0,1)$ such that $f(c) = 3c^2$.

(b) Find the general solution of

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = -e^x$$

where it is given that $y = xe^x$ is a particular solution.

Solution:

(a) Let $\frac{d}{dx}F(x) = f(x)$

$$So, \int_0^1 f(t)dt = 1 \text{ gives } F(1) - F(0) = 1$$

Now let F(1)–F(0) = 1 and $G(1) - G(0) = 1$

By Cauchy generalized mean value theorem, we get

$$\frac{f(c)}{3c^2} = \frac{F(1) - F(0)}{G(1) - G(0)} \text{ for } c \in (0,1)$$

(b)

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = -e^x$$

$$\Rightarrow (D^2 - 3D + 2)y = -e^x \ldots\ldots (i)$$

Auxiliary equation is $m^2 - 3m + 2 = 0$

$$\Rightarrow (m-2)(m-1) = 0$$

$$\Rightarrow m = 1, 2$$

So, complementary function is

$$y = c_1 e^x + c_2 e^{2x} \dots\dots (ii)$$

As, particular solution is $y = xe^x$. So, general solution is $y = c_1 e^x + c_2 e^{2x} + xe^x$.

24. (a) Let $f: R \times R \to R$ be defined by $f(x, y) = x^2 + xy + y^2 - x - 100$. Find the points of local maximum and local minimum, if any of f.

(b) Find

$$\lim_{n\to\infty} \frac{4^{3n} \sin(n)}{3^{4n}}$$

solution:

(a)

$$f(x, y) = x^2 + xy + y^2 - x - 100$$

$$\text{So, } f_x = 2x + y - 1$$

$$f_y = x + 2y;\ f_{xx} = 2;\ f_{yy} = 2;\ f_{xy} = 1$$

$$f_x = 0 \ \ and\ f_y = 0 \ \Rightarrow x = \frac{2}{3}\ and\ y = -\frac{1}{3}$$

$$so, \left(\frac{2}{3}, -\frac{1}{3}\right)\ is\ the\ critical\ point.$$

$$f_{xx} f_{yy} - \left(f_{xy}\right)^2 = 2.2 - (1)^2 = 3 > 0\ and\ f_{xx} > 0$$

$$So, \left(\frac{2}{3}, -\frac{1}{3}\right)\ is\ local\ minima\ point.$$

(b)

$$\lim_{n\to\infty} \frac{4^{3n} \sin(n)}{3^{4n}} = \lim_{n\to\infty} \frac{(4^3)^n \sin(n)}{(3^4)^n}$$

$$= \lim_{n\to\infty} \left(\frac{64}{81}\right)^n \sin(n) \ \ = 0 \times (finite\ value) = 0$$

25. (a) Consider the following matrix

$$\boldsymbol{A}\begin{pmatrix} \frac{\alpha}{\sqrt{11}} & \frac{1}{\sqrt{66}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{11}} & -\frac{4}{\sqrt{66}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{11}} & \frac{7}{\sqrt{66}} & \frac{\beta}{\sqrt{6}} \end{pmatrix}$$

Find $\alpha\ and\ \beta$ so that A becomes an orthogonal matrix. Using these values of $\alpha\ and\ \beta$, solve the system of equations.

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

(b) Let

$$A = \begin{pmatrix} 1 & 5 & 7 & 9 \\ 0 & 1 & 3 & 5 \\ 1 & 6 & 10 & 14 \\ 1 & 4 & 4 & \gamma \end{pmatrix}$$

Find γ so that the rank of A is two.

Solution:

(a) For A to be orthogonal matrix $AA^T = I$. Hence each row vector is unit vector and dot product of any two row vector should be zero.

$$\Rightarrow \frac{\alpha^2}{11} + \frac{1}{66} + \frac{1}{6} = 1$$

$$\Rightarrow 6\alpha^2 + 12 = 66$$

$$\Rightarrow \alpha^2 = 9$$

$$also\ \frac{1}{11} + \frac{49}{66} + \frac{\beta^2}{6} = 1$$

$$\Rightarrow 55 + 11\beta^2 = 66$$

$$\Rightarrow \beta^2 = 1$$

$$and\ \frac{\alpha}{11} - \frac{4}{66} + \frac{2}{6} = 0$$

$$\Rightarrow \alpha = -3$$

$$and\ \frac{1}{11} - \frac{28}{66} + \frac{2\beta}{6} = 0$$

$$\Rightarrow \beta = 1$$

$$A\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix}\ \begin{pmatrix}x\\y\\z\end{pmatrix}=A^{-1}\begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$A^T\begin{pmatrix}1\\0\\1\end{pmatrix}\ \ [As\ here\ A^{-1}=A^T]$$

$$\Rightarrow\begin{pmatrix}-\frac{3}{\sqrt{11}} & \frac{1}{\sqrt{11}} & \frac{1}{\sqrt{11}}\\ \frac{1}{\sqrt{66}} & -\frac{4}{\sqrt{66}} & \frac{7}{\sqrt{66}}\\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\Rightarrow x=-\frac{2}{\sqrt{11}}\ ;y=\frac{8}{\sqrt{66}}\ ;z=\frac{2}{\sqrt{6}}$$

(b) For

$$A=\begin{pmatrix}1 & 5 & 7 & 9\\ 0 & 1 & 3 & 5\\ 1 & 6 & 10 & 12\\ 1 & 4 & 4 & \gamma\end{pmatrix}$$

First and second rows are independent so from Rank of A equals to 2, third and fourth row should be dependent on them. Now $R_3=R_1-R_2\ and\ R_4$ should be R_1-R_2 for Rank 2.

Hence $\gamma=9-5=4$