IIT–JAM Mathematical Statistics (MS) 2012 Solved Paper

1. An eigenvector of the matrix M= $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ is

(a) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

(b) $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(d) $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

Solution: (a) Eigenvalue of M $= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ is 2.

So, let $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the eigenvector.

$\Rightarrow (M - 2I)X = 0$

$\Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow y = 0, z = 0$

Hence $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector.

2. The volume of the solid of revolution generated by revolving the area bounded by the curve $y = \sqrt{x}$ and the straight lines $x = 4$ *and* $y = 0$ about the x–axis, is

(a) 2π
(b) 4π
(c) 8π
(d) 12π

Solution: (c)

Volume of solid revolution about x–axis

$$v = \int_0^4 \pi\, y^2 dx = \int_0^4 \pi x dx = \frac{\pi x^2}{2}\Big|_0^4 = 8\pi\,.$$

3. **Let $I = \int_0^1 \int_{x^2}^{2-x} xydxdy$. The change of order of integration in the integral gives I as**

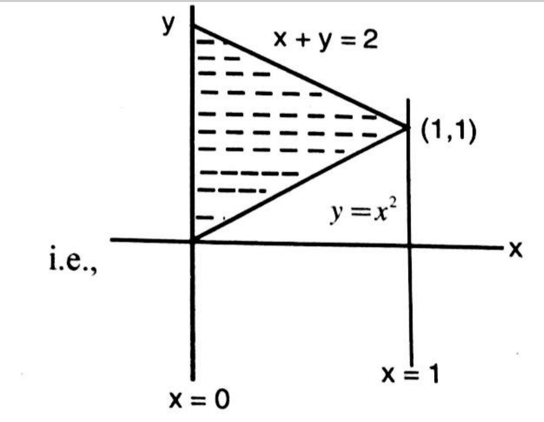
(a) $I = \int_0^1 \int_0^{\sqrt{y}} xydxdy + \int_1^2 \int_0^{2-y} xydxdy$

(b) $I = \int_0^1 \int_0^{2-y} xydxdy + \int_1^2 \int_0^{2-y} xydxdy$

(c) $I = \int_0^1 \int_0^{\sqrt{y}} xydxdy + \int_1^2 \int_0^{2-y} xydxdy$

(d) $I = \int_0^1 \int_0^{2-y} xydxdy + \int_1^2 \int_0^{\sqrt{y}} xydxdy$

Solution: (a) In the given integration, region bounded is from $y = x^2$ to $y = 2 - x$ and $x = 0$ to $x = 1$.



By changing the order of integration, we get

$$\int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xydxdy + \int_{y=1}^{2} \int_{x=0}^{2-y} xydxdy$$

4. **Let $L = \lim_{n \to \infty} n \left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{k}{n}\right) - kf(0) \right]$**

where k is a positive integer, if f(x)= $sin\ x$, then L is equal to

(a) $\frac{(k+1)(k+2)}{6}$

(b) $\frac{(k+1)(k+2)}{2}$

(c) $\frac{k(k+1)}{2}$

(d) $k(k+1)$

Solution: (c)

$$L = \lim_{n \to \infty} n \left[f\left(\frac{1}{h}\right) + f\left(\frac{2}{h}\right) + \cdots + f\left(\frac{k}{n}\right) - kf(0) \right]$$

$$= \lim_{n \to \infty} n \left[\sin\frac{1}{h} + \sin\frac{2}{h} + \cdots + \sin\frac{k}{n} - k\sin 0 \right]$$

$$L = \lim_{n\to\infty} \frac{\sin\frac{1}{h}}{\frac{1}{h}} + \frac{\sin\frac{2}{h}}{\frac{2}{n}} \times 2 + \cdots + \frac{\sin\frac{k}{n}}{\frac{k}{n}} \times k$$

$$L = 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

5. Let

$$\boldsymbol{f(x,y) = \begin{cases} \sqrt{x^2+y^2}\, sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & if\ (x,y) \neq (0,0) \\ 0 & if\ (x,y) = (0,0) \end{cases}}$$

Then at the point (0, 0)

- **(a) f is continuous and $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ exist.**
- **(b) f is continuous and $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ do not exist.**
- **(c) f is not continuous and $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ exist.**
- **(d) f is not continuous and $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ don't exist.**

Solution : (b)

$$\underset{(x,y)\to(0,0)}{Lt} f(x,y) = \underset{r\to 0}{Lt} f(r\cos\theta, r\sin\theta)$$

$$= \underset{r\to 0}{Lt}\, r\sin\frac{1}{r} = 0 = f(0,0)$$

So, f(x,y) is continuous at (0,0)

$$\frac{\partial t}{\partial x} at\ (0,0) = \underset{h\to 0}{Lt} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{|h|\sin\frac{1}{|h|}}{h}$$

Which does not exist.

$$\frac{\partial t}{\partial x} at\ (0,0) = \lim_{k\to 0} \frac{f(0,0) + k - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{|k|\sin\frac{1}{|k|}}{k}\ does\ not\ exist.$$

6. **Let $\{a_n\}$ be a real sequence converging to a, where a > 0. Then**

(a) $\sum_1^\infty a_n$ *converges but* $\sum_1^\infty \frac{a_n}{n}$ **diverges**

(b) $\sum_1^\infty a_n$ *diverges but* $\sum_1^\infty \frac{a_n}{n}$ **converges**

(c) **Both** $\sum_1^\infty a_n$ *and* $\sum_1^\infty \frac{a_n}{n}$ **converge**

(d) **Both** $\sum_1^\infty a_n$ *and* $\sum_1^\infty \frac{a_n}{n}$ **diverge**

Solution: (d) As, $\lim a_n = a > 0$

So, $\sum_{n=1}^\infty a_n$ diverges as $\lim a_n = 0$ is necessary condition for convergence.

Also $\sum_{n=1}^\infty \frac{a_n}{n}$ diverges as $\lim n\left(\frac{a_n}{n}\right) = a > 0$ and by Śleszyński–Pringsheim theorem $\lim nu_n = 0$ is necessary for convergence of positive term series.

7. **A four digit number is chosen at random. The probability that there are exactly two zeroes in that number is**

(a) **0.73**

(b) **0.973**

(c) **0.027**

(d) **0.27**

Solution: (c) Number of 4 digit numbers $= 9 \times 10^3 = 9000$

Number of 4 digit numbers with exactly 2 zeroes $= 9C_1 \times 3C_2 \times 1 \times 9 = 243$

So, required probability $= \frac{243}{9000} = 0.027$

8. **A person makes repeated attempts to destroy a target. Attempts are made independent of each other. The probability of destroying the target in any attempt is 0.8. Given that he fails to destroy the target in the first five attempts, the probability that the target is destroyed in the 8th attempt is**

(a) **0.128**

(b) **0.032**

(c) **0.160**

(d) **0.064**

Solution: (b) Probability that target will be hit in 8th attempt is equal to product of probability of failure in the 6th and 7th attempt and success in the 8th attempt.

$$P = (0.2)(0.2)(0.8) = 0.032$$

9. Let the random variable $X \sim B(5, p)$ such that $P(X = 2) = 2P(X = 3)$. Then the variance of X is

(a) 10/3

(b) 10/9

(c) 5/3

(d) 5/9

Solution: (b) $P(x = 2) = 2P(x = 3)$

$$\Rightarrow \binom{5}{2} p^2(1-p)^3 = 5_{c_3} p^3(1-p)^2$$

$$\Rightarrow 1 - p = 2p \Rightarrow 3p = 1 \Rightarrow p = \frac{1}{3}$$

Variance of $x = np(1-p) = 5 \times \frac{1}{3} \times \frac{2}{3} = \frac{10}{3}$.

10. Let $X_1, \dots, X_8$ be i.i.d. $M(0, \sigma^2)$ random variables. Further, let $U = X_1 + X_2$ and $V = \sum_{i=1}^{8} X_i$. The correlation coefficient between U and V is

(a) 1/8

(b) ¼

(c) ¾

(d) ½

Solution: (d) $E(X_i) = 0;\ V(X_i) = \sigma^2;\ E(X_i^2) = \sigma^2; E(UV) = E(X_1^2 + X_2^2) = 2\sigma^2$

$E(U)E(V) = 0;\ \ Var\ (U) = 2\sigma^2;\ \ Var\ (V) = 8\sigma^2;\ \ Cov\ (U, V) = E(UV) - E(U)E(V) = 2\sigma^2$

Correlation coefficient U and V is

$$= \frac{COV\ (U, V)}{\sqrt{var\ (U), Var\ (V)}} = \frac{2\sigma^2}{\sqrt{2\sigma^2, 8\sigma^2}} = \frac{1}{2}$$

11. Let $X \sim F_{8.15}$ and $Y \sim F_{15.8}$. If P(X>4) =0.01and $P(Y \le k) = 0.01$, then the value of k is

(a) 0.025

(b) 0.25

(c) 2

(d) 4

Solution: (b) $X \sim F_{8,\ 15}$ and $Y \sim F_{15,\ 8}$ $\Rightarrow Y = \frac{1}{X}$;

$P(X > 4) = P\left(\frac{1}{X} \le \frac{1}{4}\right) = P\left(Y \le \frac{1}{4}\right) = P(Y \le 0.25)$.

12. **Let $X_1, \dots, X_n$ be i.i.d. Exp (1) random variables and $S_n = \sum_{i=1}^{n} X_i$. Using the central limit theorem, the value of $\lim_{n\to\infty} P(S_n > n)$ is**

(a) **0**
(b) **1/3**
(c) **½**
(d) **1**

Solution: (c) $X_i \sim \exp(1) \Rightarrow E(X_i) = 1;\ V(X_i) = 1$

$$\frac{S_n - n\mu}{\sqrt{n}} \to \xi, where\ \xi\ follows\ N(0, \sigma^2)$$

Where $\sigma^2\ is\ v(X_i)$

$$\lim_{n\to\infty} P(S_n > n) = \lim_{n\to\infty} P\left(\frac{S_n - n.1}{\sqrt{n}} > 0\right)$$

$$\Rightarrow \ln N(0,1) P(x > 0) = 1/2$$

13. **Let the random variable $X \sim U\ (5, 5 + \theta)$. Based on a random sample of size 1, say X_1, the unbiased estimator of θ^2 is**

(a) $3(X_1 - 5)^2$
(b) $\frac{X_1^2 - 5}{12}$
(c) $3(X_1 + 5)^2$
(d) $\frac{X_1^2 + 5}{12}$

Solution: (a)

$$E(X^2) = \int_5^{5+\theta} x^2 . \frac{1}{\theta} dx = \frac{(5+\theta)^3 - 5^3}{3\theta} = \frac{\theta^2}{3} + 5\theta + 25$$

$$E(X) = \int_5^{5+\theta} x . \frac{1}{\theta} dx = \frac{(5+\theta)^2 - 5^2}{2\theta} = \frac{\theta}{2} + 5$$

$$E(X-5)^2 = E(X^2) - 10E(X) + 25 = \frac{\theta^2}{3}$$

$$\Rightarrow E[3(X-5)^2] = \theta^2$$

So, $3(X-5)^2$ is unbiased estimator of θ^2.

14. **Let $X_1, \dots, X_n$ be a random sample of size n from $N(\mu, 16)$ population. If a 95% confidence interval for μ is $[\bar{X} - 0.98, \bar{X} + 0.98]$, then the value of n is**

(a) 4

(b) 16

(c) 32

(d) 64

Solution: (d) 95% confidence interval for μ is

$$\left[\bar{X} - \frac{1.96\sigma}{\sqrt{n}}, \bar{X} + \frac{1.96\sigma}{\sqrt{n}}\right]$$

$$\Rightarrow \frac{1.96\sigma}{\sqrt{n}} = 0.98$$

$$\Rightarrow \sqrt{n} = 2\sigma = 2(4) = 8$$

$$\Rightarrow n = 64$$

15. **A coin is tossed 4 times and p is the probability of getting head in a single trial. Let S be the number of head(s) obtained. It is decided to test**

$$H_0: p = \frac{1}{2} \ against \ H_1: p \neq \frac{1}{2}$$

Using the decision rule: Reject H_0 if S is 0 to 4. The probabilities of Type I error (α), and Type II error (β) when $p = \frac{3}{4}$, are

(a) $\alpha = \frac{1}{4}, \beta = \frac{87}{128}$

(b) $\alpha = \frac{1}{8}, \beta = \frac{87}{128}$

(c) $\alpha = \frac{1}{8}, \ \beta = \frac{41}{256}$

(d) $\alpha = \frac{1}{4}, \ \beta = \frac{41}{256}$

Solution: (b) $\alpha = p$ (reject H_0/H_0 is true)

i.e., $p = 1/2$ & Rejection H_0 means $S = 0,4$

$$\alpha = P(S = 0 \ or \ 4 | P = 1/2) \ = 4C_0\left(\frac{1}{2}\right)^4 + 4C_4\left(\frac{1}{2}\right)^4 = \frac{1}{8}$$

$$\beta = P(accept \ H_0 | H_1 \ is \ true) = P(S = 1,2,3 \ |P = 3/4)$$

$$= 4C_1\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^3 + 4C_2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^2 + 4C_3\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)$$

$$= \frac{12 + 54 + 108}{256} = \frac{174}{256} = \frac{87}{128}.$$

16. (a) Find the value(s) of λ for which the following system of linear equations

$$\begin{pmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(i) Has a unique solution

(ii) Has infinitely many solutions

(iii) Has no solution

(b) Let $a_1 = 2, b_1 = 1$ $and\ for\ n \geq 1$,

$$a_{n+1} = \frac{a_n + b_n}{2}, b_{n+1} = \frac{2a_n b_n}{a_n + b_n}$$

Show that

(i) $b_n \leq a_n$, for all n

(ii) $b_{n+1} \leq b_n$ for all n,

(iii) The sequences $\{a_n\}$ and $\{b_n\}$ converge to the same limit $\sqrt{2}$.

Solution: (a)

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 1 & \lambda & 1 & 1 \\ \lambda & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 0 & \lambda - 1 & 1 - \lambda & 0 \\ 0 & 1 - \lambda & 1 - \lambda^2 & 1 - \lambda \end{array}\right]$$

By $R_2 \leftarrow R_2 - R_1$ & $R_3 \leftarrow R_3 - \lambda R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 0 & \lambda - 1 & 1 - \lambda & 0 \\ 0 & 0 & 2 - \lambda - \lambda^2 & 1 - \lambda \end{array}\right]$$

By $R_3 \leftarrow R_3 + R_2$

(i) For unique solution $(\lambda - 1)(2 - \lambda - \lambda^2) \neq 0$

$\Rightarrow -(\lambda - 1)^2(\lambda + 2) \neq 0 \quad \Rightarrow \lambda \neq 1\ or\ -2$

(ii) For $\lambda = 1$, we have infinitely many solutions

(iii) For $\lambda = -2$, we have no solution.

(b) As a_{n+1} is arithmetic mean and b_{n+1} is harmonic mean between $a_n\ and\ b_n,\ so\ a_{n+1} \geq b_{n+1}, \forall n\ as\ A.M. \geq G.M.$

Hence, we have (i) $b_n \leq a_n$, for all n

Also $b_1 \leq b_2 \leq b_3$… monotonically decreasing and bounded below by b_1. So it is convergent and similarly $\{b_n\}$ is monotonically increasing and bounded above by a_1. So it is also convergent.

Let $Lt\ a_n = l\ and\ Lt\ b_n = m$

$$\Rightarrow Lt\ a_{n+1} = \lim \frac{a_n + b_n}{2} \Rightarrow l = \frac{l+m}{2} \Rightarrow l = m\,.$$

So, $\{a_n\} and\ \{b_n\}$ converges to same limit.

Let l be the common limit of $\{a_n\}\ and\ \{b_n\}$

$$a_{n+1}b_{n+1} = a_n b_n = a_{n-1}b_{n-1} = \cdots = a_1 b_1$$

$$\Rightarrow a_n b_n = a_1 b_1 = 2$$

$$\Rightarrow \lim a_n b_n = 2 \Rightarrow l^2 = 2 \Rightarrow l = \sqrt{2}$$

Hence $\{a_n\}\ and\ \{b_n\}$ converges to same limit = $\sqrt{2}$.

17. (a) Solve: $\boldsymbol{(x^2y^3 + xy)dy = dx}$

(b) Find the general solution of the differential equation

$$\boldsymbol{(D^2 - 4D + 4)y = x\ sin\ 2x, where\ D \equiv \frac{d}{dx}}$$

Solution: (a) $(x^2y^3 + xy)dy = dx \Rightarrow \frac{dx}{dy} = xy + x^2y^3$

$$\Rightarrow \frac{1}{x^2}\frac{dx}{dy} = \frac{1}{x}y + y^3; let\ \frac{1}{x} = z$$

$$\Rightarrow -\frac{1}{x^2}\frac{dx}{dy} = \frac{dz}{dy}$$

$$\Rightarrow -\frac{dz}{dy} = yz + y^3 \Rightarrow \frac{dz}{dy} + yz = -y^3$$

$$\Rightarrow ze^{y2} = \int e^{y2}(-y^3)dy + e; Let\ y^2 = t$$

$$\Rightarrow 2ydy = dt$$

$$\Rightarrow ze^t = \int -e^t\frac{t}{2}dt + c = -\frac{1}{2}[te^t - e^t] + c$$

$$\Rightarrow z = -\frac{1}{2}[t-1] + ce^{-t}$$

$$\Rightarrow \frac{1}{x} = -\frac{1}{2}[y^2 - 1] + ce^{-y2}\ is\ the\ solution.$$

(b) $(D^2 - 4D + 4)y = x \sin 2x$

$$\Rightarrow (D-2)^2 y = x \sin 2x$$

Auxiliary equation is $(m-2)^2 = 0$

$$\Rightarrow m = 2, 2$$

So, complementary function is

$$y = (c_1 + c_2 x)e^{2x}$$

Particular integral is

$$y = \frac{1}{(D-2)^2}\ x \sin 2x$$

$$= \left[x - \frac{2(D-2)}{(D-2)^2}\right]\frac{1}{(D-2)^2}\sin 2x$$

$$= \left[x - \frac{2}{D-2}\right]\frac{1}{D^2 - 4D + 4}\sin 2x$$

$$= \left[x - \frac{2}{D-2}\right]\frac{1}{-4D}\sin 2x$$

$$= \left[x - \frac{2}{D-2}\right]\frac{\cos 2x}{8}$$

$$= \frac{x\cos 2x}{8} - \frac{1}{4} \times \frac{D+2}{D^2-4}(\cos 2x)$$

$$= \frac{x\cos 2x}{8} + \frac{1}{32}(-2\sin 2x + 2\cos 2x)$$

$$= \frac{x\cos 2x}{8} + \frac{\cos 2x - \sin 2x}{16}$$

$$= \frac{1}{16}[(1+2x)\cos 2x - \sin 2x]$$

So, the complete solution is

$$y = (c_1 + c_2 x)e^{2x} + \frac{1}{16}[(1+2x)\cos 2x - \sin 2x]$$

18. (a) Find all the critical points of the function $f(x,y) = x^3 + y^3 + 3xy$ and examine those points for local maxima and local minima

(b) If f is a continuous real–valued function on [0,1], show that there exists a point $c \in (0,1)$ such that

$$\int_0^1 xf(x)dx = \int_c^1 f(x)dx.$$

Solution: (a) $f(x,y) = x^3 + y^3 + 3xy$

$$f_x = 3x^2 + 3y\,;\; f_y = 3y^2 + 3x$$

$$f_{xx} = 6x\,;\; f_{yy} = 6y$$

$$f_{xy} = 3$$

For critical points, we have

$$f_x = 0 \; and \; f_y = 0 \implies x^2 + y = 0 \; and$$

$$y^2 + x = 0$$

$$y = -x^2 \implies x^4 + x = 0$$

$$\implies x(1 + x^3) = 0 \implies x = 0, -1$$

At $x = 0, y = 0; at\ x = -1, y = -1$

So, critical points are (0,0) and (−1, −1)

$$f_{xx}f_{yy} - \left(f_{xy}\right)^2 = 36xy - 9 = 9(4xy - 1)$$

At (0,0); $f_{xx}f_{yy} - \left(f_{xy}\right)^2 = -9 < 0$. So, (0, 0) is a saddle point.

At (−1, −1); $f_{xx}f_{yy} - \left(f_{xy}\right)^2 = 9(4 - 1) = 27 > 0$

And f_{xx} at (−1, −1) = $-6 < 0. So(-1, -1)$ is local maxima point.

(b)

$$\int_0^1 xf(x)dx \;\; lies\ between\ 0 \int_0^1 f(x)dx\ and\ 1.\int_0^1 f(x)dx$$

Now as f(x) is a continuous function in the interval [0,1], so

$$\int_1^0 f(x)dx \;\leq \int_0^1 f(x)dx$$

Now, as f(x) is continuous function, so integral

$$\int_c^1 f(x)dx\ exist\ for\ each\ c \in [0,1].$$

Hence,

$$\int_0^1 f(x)dx = \int_c^1 f(x)dx\ for\ a\ point\ c \in (0,1)$$

19. (a) Evaluate the triple integral

$$\int_{z=0}^{z=4}\int_{x=0}^{x=2\sqrt{z}}\int_{y=0}^{y=\sqrt{4z-x^2}} dydxdz$$

(b) Let

$$M=\begin{pmatrix}1 & 2 & 0\\ 0 & 2 & 1\\ 1 & 0 & 1\end{pmatrix}.$$

If $M^{-1}=\frac{5}{4}I+kM+\frac{1}{4}M^2$, where I is the identity matrix of order 3, find the value of k. hence or otherwise, solve the system of equations:

$$M\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$

Solution: (a)

$$\int_{z=0}^{z=4}\int_{x=0}^{x=2\sqrt{z}}\int_{y=0}^{y=\sqrt{4z-x^2}} dydxdz=\int_{z=0}^{z=4}\int_{x=0}^{x=2\sqrt{z}}\sqrt{4z-x^2}\,dxdz$$

$$=\int_{z=0}^{z=4}\left[\frac{x}{2}\sqrt{4z-x^2}+\frac{4z}{2}sin^{-1}\frac{x}{2\sqrt{z}}\right]_{x=0}^{x=2\sqrt{z}}dz=\int_{z=0}^{z=4}\pi zdz=8\pi\,.$$

(b) By Cayley Hamilton theorem, every square matrix satisfies it's characteristic equation, so,

$|M-\lambda I|=0$ is satisfied by M.

$$|M-\lambda I|=\begin{vmatrix}1-\lambda & 2 & 0\\ 0 & 2-\lambda & 1\\ 1 & 0 & 1-\lambda\end{vmatrix}=(1-\lambda)[2+\lambda^2-3\lambda]+2$$

$$=-\lambda^3+4\lambda^2-5\lambda+4$$

Hence, $-M^3+4M^2-5M+4I=0\ \Rightarrow -M^2+4M-5I+4M^{-1}=0$

(By taking product by M^{-1})

$$\Rightarrow M^{-1}=\frac{1}{4}[M^2-4M+5I]=\frac{1}{4}M^2-M+\frac{5}{4}I$$

$$\Rightarrow K=-1$$

$$Now, M=\begin{pmatrix}1 & 2 & 0\\ 0 & 2 & 1\\ 1 & 0 & 1\end{pmatrix}\Rightarrow M^2=\begin{pmatrix}1 & 6 & 2\\ 1 & 4 & 3\\ 2 & 0 & 1\end{pmatrix}$$

$$\Rightarrow M^{-1} = \frac{1}{4}[M^2 - 4M + 5I] = \frac{1}{4}\begin{pmatrix} 2 & -2 & 2 \\ 1 & 1 & -1 \\ 2 & 0 & 2 \end{pmatrix}$$

$$\Rightarrow M^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix}$$

$$M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ -\frac{1}{4} \end{pmatrix}$$

$$\Rightarrow x = \frac{1}{2}; y = \frac{1}{4} \,\&\, z = -\frac{1}{4}$$

20. (a) Let N be a random variable representing the number of fair dice thrown with probability mass function

$$\boldsymbol{P(N = i) = \frac{1}{2^i}, i = 1, 2, \ldots\ldots}$$

Let S be the sum of numbers appearing on the faces of the dice. Given that $S = 3$, what is the probability that 2 dice were thrown?

(b) Let $X \sim N(0,1)$ and $Y = X + |X|$. Find $E(Y^3)$.

Solution: (a) Prob. (2 dice were thrown $|sum\ on\ the\ face\ = 3$)

$$= \frac{Prob.(2\ dice\ were\ thrown\ \cap sum\ on\ the\ face = 3)}{Prob.(sum\ on\ the\ face = 3)}$$

$$= \frac{P(2|s = 3)}{P(1|s = 3) + P(2|s = 3) + P(3|s = 3)}$$

$$= \frac{\frac{1}{2^2} \times \frac{2}{36}}{\frac{1}{2} \times \frac{1}{6} + \frac{1}{2^2} \times \frac{2}{36} + \frac{1}{2^3} \times \frac{1}{216}}$$

$$= \frac{\frac{1}{72}}{\frac{1}{12} + \frac{1}{72} + \frac{1}{1728}} = \frac{\frac{1}{72}}{\frac{144 + 24 + 1}{178}}$$

$$= \frac{24}{169}$$

(b) $X \sim N(0,1)$

$$Y = X + |X| = 2X; X > 0$$

$$= 0\,; X \leq 0$$

$$Y^3 = 8X^3 \quad X > 0$$

$$= 0 \quad ; X \leq 0$$

$$E(Y^3) = \int_0^\infty 8x^3 . \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

Let $z = \frac{x^2}{2} \implies dz = dx$

$$\implies E(Y^3) = \frac{1}{\sqrt{2\pi}} \int_0^\infty 16 z e^{-z} dz = \frac{1}{\sqrt{2\pi}} \times 16 = 8\sqrt{\frac{2}{\pi}}$$

21. Let Y $\sim N(\mu, \sigma_y^2)$ and $Y = \ln X$.

(a) Find the probability density function of the random variable X and the median of X.

(b) Find the maximum likelihood estimator of the median of the random variable X based on a random sample of size n.

Solution: (a) $Y \sim N(\mu_y, \sigma_y^2)$; $Y = \ln X$ is log normal distribution whose PDF, f(x) is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}x} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}; \; x > 0$$

Median of $X = M$, then we have

$$\int_0^M f(x) dx = \frac{1}{2} \implies M = e^{\mu}$$

(b) Likelihood function

$$f_L(x : \mu, \sigma) = \prod_{i=1}^{n} \left(\frac{1}{x_i}\right) f_N(\ln x;\, \mu, \sigma)$$

f_L is pdf of log normal and f_N is pdf of normal

$$\Rightarrow l_L = (\mu, \sigma \left| x_1, x_2, \dots, x_n \right) = -\sum_k \ln x_k + l_N(\mu, \sigma | \ln x_1, \ln x_2, \dots, \ln x_n)$$

$$= constant + l_N(\mu, \sigma | \ln x_1, \ln x_2, \dots, \ln x_n)$$

So, by using the concept of Normal distribution, we have

$$\hat{\mu} = \frac{\sum_k \ln x_k}{n} \ and \ \hat{\sigma}^2 = \frac{\sum_k (\ln x_k - \hat{\mu})^{\ 2}}{n}$$

22. (a) A random variable X has probability density function

$$\boldsymbol{f(x) = \alpha x e^{-\beta^2 x^2}, x > 2, \alpha > 2, \beta > 0}$$

If $E(X) = \frac{\sqrt{\pi}}{2}$, determine α and β.

(b) Let X and Y be two random variables with join probability density function

$$\boldsymbol{f(x, y) = \begin{cases} e^{-y} \ if & 0 \le x \le y \le \infty \\ 0 & otherwise \end{cases}}$$

(i) Find the marginal density functions of X and Y

(ii) Examine whether X and Y are independent

(iii) Find $Cov\ (X, Y)$

Solution: (a)

$$E(X) = \int_0^\infty x f(x) dx + \int_0^\infty \alpha x^2 e^{-\beta^2 x^2} dx$$

Let $y = (\beta x)^2 \ \Rightarrow dy \ = 2\beta^2 x dx$

$$\Rightarrow E(X) = \int_0^\infty \frac{\alpha}{2\beta^2} . \frac{y^{1/2}}{\beta} e^{-y} dy \ = \frac{\alpha}{2\beta^3} \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{\alpha}{4\beta^3} \sqrt{\pi} = \frac{\sqrt{\pi}}{2}$$

$$\Rightarrow \alpha = 2\beta^3 \dots \dots \dots \dots (1)$$

$$Also, \int_0^\infty f(x) dx = 1$$

$$\Rightarrow \int_0^\infty \alpha x e^{-\beta^2 x^2} dx = 1$$

$$\Rightarrow \int_0^{\infty} \frac{\alpha}{2\beta^2} e^{-y} dy = 1$$

$$\Rightarrow \frac{\alpha}{2\beta^2} = 1$$

$$\Rightarrow \alpha = 2\beta^2 \ldots\ldots (2)$$

From (1) & (2) we get $\beta = 1 \;\Rightarrow \alpha = 2$

(b) (i) Marginal density function of X,

$$f(x) = \int_2^{\infty} e^{-y} dy; 0 \leq x < \infty$$

$$\Longrightarrow f(x) = e^{-x}; 0 \leq x < \infty$$

Marginal density function of Y,

$$f(y) = \int_0^{y} e^{-y} dx = ye^{-y} = ye^{-y};\; 0 \leq y < \infty$$

(ii) As $f(x)f(y) = e^{-x}(ye^{-y}) = ye^{-(x+y)}\; and\; f(x,y) = e^{-y}$

So, $f(x,y) \neq f(x)f(y)$ hence X and Y are not independent.

(iii)

$$E(X) = \int_0^{\infty} xe^{-x} dx = [-xe^{-x} - e^{-x}]_0^{\infty} = 1$$

$$E(Y) = \int_0^{\infty} y^2 e^{-y} dy \; = [(-y^2 - 2y - z)e^{-y}]_0^{\infty} = 2$$

$$E(XY) = \int_{y=0}^{\infty} \int_{x=0}^{y} xye^{-y} dx dy = \int_{y=0}^{\infty} \frac{y^3}{2} e^{-y} dy$$

$$= \frac{1}{2}[-y^3 - 3y^2 - 6y - 6]e^{-y} \int_0^{\infty} = 3$$

$$Cov\,(X,Y) = E(XY) - E(X)E(Y) = 3 - (1)(2) = 3 - 2 = 1.$$

23. (a) Let $X_1, \ldots, X_n$ be a random sample from $Exp\left(\frac{1}{\theta}\right)$ population. Obtain the Cramer–Rao lower bound for the variance of an unbiased estimator of θ^2.

(b) Let $X_1, \ldots, X_n$ (n>4) be a random sample from a population with mean μ and variance σ^2.

Consider the following estimators of μ

$$U = \frac{1}{n}\sum_{i=1}^{n} X_i,$$

$$V = \frac{1}{8}X_1 + \frac{3}{4(n-2)}(X_2 + \cdots + X_{n-1}) + \frac{1}{8}X_n$$

(i) Examine whether the estimates U and V are unbiased

(ii) Examine whether the estimates U and V are consistent

(iii) Which of these two estimates is more efficient? Justify your answer.

Solution: (a) If T(X) is unbiased estimator of a function $\Psi(\theta)$ of the parameter θ, then Cramer Rao bound of variance of $\Psi(\theta)$ is given by

$$V(T(X)) \geq \frac{[\Psi'(\theta)]^2}{I(\theta)}$$

Where $I(\theta)$ is Fisher's information.

$$I(\theta) = E\left[\left(\frac{\partial L(x,\theta)}{\partial \theta}\right)^2\right] = -E\left[\frac{\partial^2 L(x;\, \theta)}{\partial \theta^2}\right]$$

$$L(x,\theta) = \log f(x,\theta)$$

$$f(x_{i=1}\theta) = \frac{1}{\theta}e^{-x/\theta};\ \theta > 0, x > 0$$

$$f(x,\theta) = \prod_{i=1}^{n}\frac{1}{\theta}e^{-xi/\theta} = \frac{1}{\theta^n}e^{-\sum xi/\theta}$$

$$L(x,\theta) = \log f(x,\theta) = \log \theta^{-n} e^{-(\sum xi)/\theta}$$

$$= -n\log\theta - \frac{(\sum xi)}{\theta}$$

$$= I(\theta) = E\left[\left(-\frac{n}{\theta} + \frac{\sum x_i}{\theta^2}\right)^2\right]$$

Given $\Psi(\theta) = \theta^2 \Rightarrow \Psi'(\theta) = 2\theta$

$$\Rightarrow Var\,[T(x)] \geq \frac{4\theta^2}{E\left[\left(-\frac{n}{\theta}+\frac{\sum xi}{\theta^2}\right)^2\right]}$$

So, lower bound for variance is

$$\frac{4\theta^2}{E\left[\left(-\frac{n}{\theta}+\frac{\sum xi}{\theta^2}\right)^2\right]}$$

(b)

$$U = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$V = \frac{1}{8}X_1 + \frac{3}{4(n-2)}(X_2 + \cdots + X_{n-1}) + \frac{1}{8}X_n$$

(i)

$$E(U) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{n\mu}{n} = \mu$$

$$E(X) = \frac{\mu}{8} + \frac{3}{4}\frac{(n-2)}{(n-2)}\mu + \frac{1}{8}\mu$$

$$= \mu\left(\frac{1}{8}+\frac{3}{4}+\frac{1}{8}\right) = \mu$$

Both U and V are unbiased estimator of μ as $E(U) = E(V) = \mu$

(ii) $\lim_{n\to\infty} U = \mu \; and \; \lim_{n\to\infty} V = \mu$

So, both U and V are consistent estimator of μ.

(iii)

$$Var(U) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}[Var\,X_1 + Var\,X_2 + \cdots + Var\,X_n]$$

$$= \frac{1}{n^2}[\sigma^2 + \sigma^2 + \cdots + \sigma^2] = \frac{\sigma^2}{n}$$

$$Var\,(V) = \frac{1}{64}\sigma^2 + \frac{9}{16(n-2)^2}(n-2)\sigma^2 + \frac{1}{64}\sigma^2$$

$$= \frac{1}{32}\sigma^2 + \frac{9}{16(n-2)}\sigma^2$$

As, Var(U) < Var (V)

Here U is more efficient estimator than V.

24. Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli population with parameter p,
(a) (i) Find a sufficient statistic for p

(ii) Consider an estimator $U(X_1, X_2)$ of $\frac{P(1-p)}{n}$ given by

$$U(X_1, X_2) = \begin{cases} \frac{1}{2n} \; if & X_1 + X_2 = 1 \\ 0 & otherwise \end{cases}$$

Examine whether $U(X_1, X_2)$ is the an unbiased estimator

(b) Using the results obtained in (a) above and Rao−Blackwell theorem, find the uniformity minimum variance unbiased estimator (UMVUE) of

$$\frac{p(1-p)}{n}.$$

Solution: (a)

(i) $T(X) = X_1 + X_2 + \cdots + X_n$ is sufficient statistics for p.

As $P_r = (X = x) = \Pr\{X_1 = x_1, X_2 = x_2, \ldots., X_n = x_n\}$

$$= P^{x_1}(1-p)^{1-x_1} p^{x_2}(1-p)^{1-x_2} \ldots P^{x_n}(1-p)^{1-x_n}$$

$$= P^{(x_1+x_2+\cdots+x_n)(1-p)^{n-(x_1+x_2+\cdots+x_n)}}$$

$$= p^{\sum x_i}(1-p)^{n-\sum x_i}$$

$$= P^{T(X)}(1-P)^{n-T(X)}$$

Hence $T(x) = \sum x_i$

(ii) $X_1 + X_2 = 1$

Evaluate $U(X_1, X_2)$ and if

$$E\big(U(X_1, X_2)\big) = \frac{P(1-p)}{n} \; then \; U \; is \; unbiased$$

$$E(U) = \frac{1}{2n}(X_1 = 0, X_2 = 1) + \frac{1}{2n}(X_1 = 1, X_2 = 0)$$

$$= \frac{p(1-p)}{2n} + \frac{p(1-P)}{2n} = \frac{p(1-p)}{n}$$

Hence U is an unbiased estimator of $p(1-p)$.

(b) Now by Rao−Blackwell theorem, we obtain UMVUE, which requires the statistics to be sufficient.

Also, mean square error of Rao−Blackwell estimator does not exceed that of the original estimator.

Rest of the calculations are left for you as an exercise.

25. (a) Let $X_1, \dots, X_n$ be a random sample from the population having probability density function

$$f(x,\theta) = \begin{cases} \frac{2x}{\theta^2} e^{-\frac{x^2}{\theta^2}} & if\ x > 0 \\ 0 & otherwise \end{cases}$$

Obtain the most powerful test for testing $H_0: \theta = \theta_0\ against\ H_1: \theta = \theta_1 (\theta_1 < \theta_0)$

(b) Let $X_1, \dots, X_n$ be a random sample of size n from $N(\mu, 1)$ population. To test $H_0 :\ \mu = 5\ against\ H_1: \mu = 4$, the decision rule is : Reject H_0 if $\overline{x} \leq c$. If $\alpha = 0.05\ and\ \beta = 0.10$, determine n (rounded off to an integer) and hence c.

Solution: (a) Likelihood function,

$$L(x,\theta) = \prod_{i=1}^{n} \frac{2x_i}{\theta^2} e^{-x_i^2/\theta^2}$$

Consider $H_0: \theta = \theta_0\ \&\ H_1: \theta = \theta_1$

The best critical region, using Neyman−Pearson Lemma is given by

$$= \frac{1}{\theta_1^{2n}} \prod_{i=1}^{n} 2x_i e^{-x_i^2/\theta_i^2} \geq k\ \frac{1}{\theta_0^{2n}} \prod_{i=1}^{n} 2x_i e^{-x_i^2/\theta_0^2}$$

$$\Rightarrow K \leq \left(\frac{\theta_0}{\theta_1}\right)^{2n} \prod_{i=1}^{n} e^{-x_i^2\left(\frac{1}{\theta_1^2} - \frac{1}{\theta_0^2}\right)}$$

$$\Rightarrow \log K \leq 2n(\log\theta_0 - \log\theta_1) + \sum_{i=1}^{n}\left(\frac{1}{\theta_0^2} - \frac{1}{\theta_1^2}\right)x_i^2$$

$$Now\ P\left(\sum_{i=1}^{n}\left(\frac{1}{\theta_0^2} - \frac{1}{\theta_1^2}\right)x_i^2 \geq \log k - 2n(\log\theta_0 - \log\theta_1)\ |H_0 is\ true\right) = \alpha$$

So it is used for most powerful test.

(b) Same as above. Do yourself.