IIT JAM MATHEMATICAL STATISTICS

SYLLABUS

The Mathematical Statistics (MS) test paper comprises of Mathematics (40% weightage) and Statistics (60% weightage).

MATHEMATICS

Sequences and Series: Convergence of sequences of real numbers, Comparison, root and ratio tests for convergence of series of real numbers.

Differential Calculus: Limits, continuity and differentiability of functions of one and two variables. Rolle's theorem, mean value theorems, Taylor's theorem, indeterminate forms, maxima and minima of functions of one and two variables.

Integral Calculus: Fundamental theorems of integral calculus. Double and triple integrals, applications of definite integrals, arc lengths, areas and volumes.

Table of Contents:

Linear Algebra

Vector Spaces (Page No. 5 – 42)
Matrix & Determinants (Page No. 43 – 64)
Linear Transformation (Page No. 65 – 88)
System of Linear Equations (Page No. 89 – 104)
Eigenvalues & Eigenvectors (Page No. 105 – 116)
Quadratic Forms (Page No. 117 – 130)

Mathematical Analysis

Sequence & Series of Real Numbers (Page No. 131 – 163)
Sequence & Series of Functions (Page No. 164 – 190)
Improper Integrals (Page No. 191 – 204)
Differential Calculus (Page No. 205 – 228)
Function of Several Variables (Page No. 229 – 254)
Maxima & Minima (255 – 270)
Double & Triple Integral (Page No. 271 – 300)
ISI MSTAT MATHEMATICS SYLLABUS

Mathematics

Arithmetic, geometric and harmonic progressions. Trigonometry. Two dimensional coordinate geometry: Straight lines, circles, parabolas, ellipses and hyperbolas.

Elementary set theory. Functions and relations. Elementary combinatorics: Permutations and combinations, Binomial and multinomial theorem.

Theory of equations.
Complex numbers and De Moivre’s theorem.


Linear Algebra & Real Analysis are covered here.
Rest of topics are covered in Mathematics Book.
**V**ectors **A**nd **V**ector **S**paces

**Field**: Suppose there is a set $F$ of objects $x, y, z, \ldots$ and two operations on the elements of $F$ as follows. The first operation, called addition, associates with each pair of elements $x, y$ in $F$ an element $(x+y)$, the second operation called multiplication, associates with each pair $x, y$, an element $xy$, and these two operations satisfy the following properties:

**Properties of Addition**: 
(a) **Closure**: $x \in F, y \in F \Rightarrow x+y \in F$.
(b) **Commutative**: $x+y = y+x \Rightarrow x, y \in F$.
(c) **Associative**: $x+(y+z) = (x+y)+z, \forall x, y, z \in F$.
(d) **Neutral element**: There is an unique element $0 \in F$ such that $x+0 = x, \forall x \in F$.
(e) **Inverse**: To each $x \in F$, there corresponds an unique element $-x$ in $F$ such that $x+(-x) = 0$.

**Properties of Multiplication**: 
(a) **Closure**: $x \in F, y \in F \Rightarrow xy \in F$.
(b) **Commutative**: $xy = yx \Rightarrow x, y \in F$.
(c) **Associative**: $x(yz) = (xy)z \Rightarrow x, y, z \in F$.
(d) **Neutral element**: There is a unique non-zero element $1 \in F$ such that $x \cdot 1 = x, \forall x \in F$.
(e) **Inverse**: To each non-zero $x \in F$, there corresponds an unique element $x^{-1}$ (or $\frac{1}{x}$) in $F$ such that $x \cdot x^{-1} = 1$.

**Properties of Addition & Multiplication** (Distributivity)

Multiplication distributes over addition, i.e.,
$$x \cdot (y+z) = xy + xz, \forall x, y, z \in F$$

The set $F$ together with these two operations is called a field.

Ex: $F = \{0, 1\}$. 
A vector in elementary physics is a physical quantity having both magnitude and direction.

\[ \overrightarrow{OP} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k} \]

Instead of characterising a vector by magnitude and direction, an equally satisfactory description could be achieved by the terminal point of vector originating from the origin.

Hence, we write \( \vec{a} = (a_1, a_2, a_3) \), where \( a_i \) is the \( i \)th component.

An ordered array of numbers : \( \rightarrow \) An ordered array of numbers \( (a_1, a_2, a_3, \ldots, a_n) \) is said to be an ordered array of \( n \) numbers if

\[ (a_{i_1}, a_{i_2}, a_{i_3}, \ldots, a_{i_n}) \]

is not the same or equivalent to

\[ (a_{j_1}, a_{j_2}, a_{j_3}, \ldots, a_{j_n}) \]

where \( (i_1, i_2, \ldots, i_n) \) and \( (j_1, j_2, \ldots, j_n) \) are two different permutations of \( (1, 2, \ldots, n) \).

An ordered array of \( n \)-numbers \( (a_1, a_2, \ldots, a_n) \) will be called an ordered \( n \)-tuple.

Definition of Vectors : \( \rightarrow \)

1. An \( n \)-component vector \( \vec{a} \) is an ordered \( n \)-tuple written as a row \( (a_1, a_2, \ldots, a_n) \) or written as a column

\[
\begin{pmatrix}
  a_1 \\
  a_2 \\
  \vdots \\
  a_n
\end{pmatrix}
\]

2. ** An ordered \( n \)-tuple of real numbers specifies a point in an \( n \)-dimensional space is called an \( n \)-component vector.

3. An ordered set of elements of a field is called a vector; the elements are called components. A vector of \( n \) components is called an \( \text{'n-component vecor'} \) or simply an \( \text{'n-vector'} \). An \( n \)-vector can be expressed in a horizontal or vertical line and in accordance, a row or column vector will appear.
**TYPES:**

(a) **Null Vectors:**
\[ \mathbf{0} = (0, 0, \ldots, 0) \]
all of whose elements are zero.

(b) **Unit Vectors:**
\[ \mathbf{e}_1 = (1, 0, 0, \ldots, 0) \]
\[ \mathbf{e}_2 = (0, 1, 0, \ldots, 0) \]
\[ \mathbf{e}_i = (0, 0, 0, \ldots, 1, 0, \ldots, 0) \]
\[ \mathbf{e}_n = (0, 0, 0, \ldots, 0, 1) \]
are called the unit vectors.

(c) **Sum Vector:**
\[ \mathbf{1} = (1, 1, \ldots, 1) \]
all of whose components are unity.

**VECTOR OPERATIONS:**

(a) Equal.
\[ \mathbf{a} = (a_1, a_2, \ldots, a_n) \]
and \[ \mathbf{b} = (b_1, b_2, \ldots, b_n) \]
be two \( n \)-component vectors.

Then \( \mathbf{a} \) and \( \mathbf{b} \) are said to be equal iff,
\[ a_i = b_i \quad \forall \quad i = 1(n) \]
Then we can say, \( \mathbf{a} = \mathbf{b} \).

**NOTE:** The vectors \((1, 2)\) and \((1, 2, 0)\) are not equal. Two vectors can't be equal unless they have the same number of components.

(b) **Addition:**
The sum of \( \mathbf{a} \) and \( \mathbf{b} \) is defined as
\[ \mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \]

**NOTE:** This definition is applied only to the vectors which have equal number of components.

(c) **Scalar Multiplication:**
The product of a scalar \( \lambda \) and a vector \( \mathbf{a} \) is defined as
\[ \lambda \mathbf{a} = (\lambda a_1, \lambda a_2, \ldots, \lambda a_n) \]

(d) **Subtraction:**
\[ \mathbf{a} - \mathbf{b} = \mathbf{a} + (-1) \mathbf{b} \]
\[ = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n) \]
Some Geometrical Concepts:

(a) Scalar Product: The scalar product of two vectors \( \mathbf{a} = \left( a_1, a_2, \ldots, a_n \right) \) and \( \mathbf{b} = \left( b_1, b_2, \ldots, b_n \right) \) is defined to be scalar if

\[
\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i.
\]

Properties:

1. \( \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \)
2. \( (\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda (\mathbf{a} \cdot \mathbf{b}) \)
3. \( \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \)

(b) Distance: The distance of a vector (or a point) \( \mathbf{a} \) to the vector (or the point) \( \mathbf{b} \) is defined as the scalar,

\[
|\mathbf{a} - \mathbf{b}| = \sqrt{(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})} = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}.
\]

Properties:

1. \( |\mathbf{a} - \mathbf{b}| = |\mathbf{b} - \mathbf{a}| \)
2. \( |\mathbf{a} - \mathbf{b}| \geq 0 \)
3. \( |\mathbf{a} - \mathbf{b}| + |\mathbf{b} - \mathbf{c}| \geq |\mathbf{a} - \mathbf{c}| \)

Result: Prove that for any two vectors \( \mathbf{a} \) and \( \mathbf{b} \),

\[
(\mathbf{a} \cdot \mathbf{b})^2 \leq |\mathbf{a}|^2 \cdot |\mathbf{b}|^2. \quad \text{[Cauchy-Schwarz Inequality]}
\]

Proof: For any scalar \( \lambda \),

\[
|\lambda \mathbf{a} + \mathbf{b}| \geq 0
\]

\[
\Leftrightarrow |\lambda \mathbf{a} + \mathbf{b}|^2 \geq 0
\]

\[
\Leftrightarrow (\lambda \mathbf{a} + \mathbf{b}) \cdot (\lambda \mathbf{a} + \mathbf{b}) \geq 0
\]

\[
\Leftrightarrow \lambda^2 |\mathbf{a}|^2 + 2 \lambda \mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 \geq 0
\]

\[
\Leftrightarrow |\mathbf{a}|^2 \lambda^2 + 2 \lambda \mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 \geq 0
\]

\[
\Leftrightarrow |\mathbf{a}|^2 \left[ \lambda + \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right]^2 + |\mathbf{b}|^2 - \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right)^2 \geq 0
\]

\[
\Leftrightarrow |\mathbf{a}|^2 \left[ \lambda + \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right]^2 + |\mathbf{b}|^2 - \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right)^2 \geq 0
\]

For, \( \lambda = -\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \), then

\[
|\mathbf{a}|^2 \left[ \lambda + \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right]^2 + |\mathbf{b}|^2 - \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right)^2 \geq 0
\]

\[
\Rightarrow (\mathbf{a} \cdot \mathbf{b})^2 \leq |\mathbf{a}|^2 |\mathbf{b}|^2, \quad \text{provided } \mathbf{a} \text{ and } \mathbf{b} \text{ have finite length.}
It holds iff \( \|k\| = 0\) for some \( k \).

iff \( k = 0\).

iff \( k = -k_0\times\lambda\), for some \( \lambda \).

iff \( k\) and \( k_0\) are collinear.

Remark → C-S inequality:

\[
\left( \sum_{i=1}^{n} a_i b_i \right) \leq \left( \sum_{i=1}^{n} a_i^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} b_i^2 \right)^{\frac{1}{2}}
\]

iff \( b_i = -a_i; \forall i = 1(n) \),

iff \( b_i \parallel a_i; \forall i = 1(n) \).

Result → For any three vectors \( a, b \), and \( c \), \( |a - b| + |b - c| \geq |a - c| \)

Triangle Inequality

Proof →

\[
|a - b| = |(a - b) + (b - c)|
\]

\[
= |(a - k) + (b - k)|
\]

\[
= (a - k)(b - k) + 2(a - k)(k - b) + (b - k)(k - b)
\]

\[
= |a - k|^2 + 2|a - k||k - b| + |b - k|^2 \leq 1a - k|^2 + 2|a - k||k - b| + |b - k|^2 \]

[Applying C-S inequality]

\[
\Rightarrow |a - b| \leq \sqrt{|a - k| + |b - k|}
\]

⇒ \( |a - k| + |b - k| \geq |a - b| \)

iff \( b - k = \lambda(a - k) \) for some \( \lambda \).

iff \( a, b, k \) are collinear.

(c) Length (Norm) → The length of a vector \( a \) is the distance between \( a \) and the origin \( O \).

\[
|a| = |a - 0|
\]

\[
= \sqrt{a \cdot a}
\]

\[
= \sqrt{\sum_{i=1}^{n} a_i^2}
\]

\[
|a| = \sqrt{a \cdot a}
\]
(a) **Angle**: The angle \( \theta \) between two vectors \( \mathbf{a} \) and \( \mathbf{b} \), where \( \mathbf{a}, \mathbf{b} \neq \mathbf{0} \), is given by

\[
\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}|| ||\mathbf{b}||} = \frac{\sum a_i b_i}{\sqrt{\sum a_i^2} \sqrt{\sum b_i^2}}
\]

**Remark**: The angle \( \theta \) between two non-null vectors \( \mathbf{a} \) and \( \mathbf{b} \) is given by \( \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}|| ||\mathbf{b}||} \).

Here for any \( \theta \), \( |\cos \theta| \leq 1 \)

\[ \Rightarrow \cos^2 \theta \leq 1 \]

\[ \Rightarrow \left( \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}|| ||\mathbf{b}||} \right)^2 \leq 1 \]

\[ \Rightarrow (\mathbf{a} \cdot \mathbf{b})^2 \leq ||\mathbf{a}||^2 ||\mathbf{b}||^2 \]

Provided \( \mathbf{a} \) and \( \mathbf{b} \) have finite length.

It is the **Cauchy-Schwarz inequality**, here it holds if \( \mathbf{a} = \lambda \mathbf{b} \) for some scalar \( \lambda \).

**Some interpretations by vector operations**

Let \( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \) be \( n \) values of a variable \( \mathbf{x} \), then,

\[ \overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \quad \text{[where } \mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) \text{]} \]

The deviations \( \Rightarrow \mathbf{x}_1 - \overline{x}, \mathbf{x}_2 - \overline{x}, \mathbf{x}_3 - \overline{x}, \ldots, \mathbf{x}_n - \overline{x} \).

The deviation can be represented as

\[ \mathbf{x} - \overline{x} = (\mathbf{x}_1 - \overline{x}, \mathbf{x}_2 - \overline{x}, \ldots, \mathbf{x}_n - \overline{x}) \]

As a measure of dispersion, we take the distance between \( \mathbf{x} \) and \( \overline{x} \), i.e., the length of deviations vector

\[ |\mathbf{x} - \overline{x}| = \sqrt{(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + \ldots + (x_n - \overline{x})^2} \]

Let, \( (\mathbf{x}_1, \mathbf{y}_1), \ldots, (\mathbf{x}_n, \mathbf{y}_n) \) be \( n \) pairs of values on \( \mathbf{x} \) and \( \mathbf{y} \).

Then define the deviation vectors for \( \mathbf{x} \) and \( \mathbf{y} \):

\[ \mathbf{x} - \overline{x} = d_1 \quad \text{and} \quad \mathbf{y} - \overline{y} = d_2 \]
If \( \theta \) be the angle between \( \mathbf{d}_1 \) and \( \mathbf{d}_2 \), is 0 then \( x \) and \( y \) are on a line. If the smaller the angle, the more the vectors \( \mathbf{d}_1 \) and \( \mathbf{d}_2 \) closer to a line.

As a measure of linear relationship, we define \( \cos \theta \) as

\[
\cos \theta = \frac{\mathbf{d}_1 \cdot \mathbf{d}_2}{\| \mathbf{d}_1 \| \| \mathbf{d}_2 \|}
\]

\[
= \frac{(x-x_2)(x_1-x_2)}{\sqrt{(x-x_2)^2 (x_1-x_2)^2}}
\]

\[
= \frac{(x-x_2)(y_1-y_2)}{\sqrt{(x-x_2)^2 (y_1-y_2)^2}}
\]

\[
= r_{xy}, \text{ which is the correlation coefficient between } x \text{ and } y.
\]

**Property:** \(-1 \leq r_{xy} \leq 1\).

**Interpret the cases:**

1. \( r_{xy} = 0 \) where \( \theta = 90^\circ \)
2. \( r_{xy} = 1 \) where \( \theta = 0 \)
3. \( r_{xy} = -1 \) where \( \theta = 180^\circ \).
Linear Combination: An m-vector \( \mathbf{a} \) is said to be a linear combination of the m-vectors \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \) if for some scalars \( \lambda_1, \lambda_2, \ldots, \lambda_n \), \( \mathbf{a} \) can be written as \( \mathbf{a} = \lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_n \mathbf{a}_n \).

Euclidean Space: An n-dimensional Euclidean space is the collection of all n-component vectors such that:

i) For any \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^n \), \( \mathbf{a} + \mathbf{b} \in \mathbb{R}^n \). Closure property w.r.t. Addition.

ii) For any \( \mathbf{a} \in \mathbb{R}^n \) and for any scalar \( \lambda \), \( \lambda \mathbf{a} \in \mathbb{R}^n \). Closure property w.r.t. scalar multiplication.

iii) There is a non-negative quantity associated with any two vectors \( \mathbf{a} \) and \( \mathbf{b} \) called distance between \( \mathbf{a} \) and \( \mathbf{b} \).

In \( \mathbb{R}^n \), a vector \( \mathbf{a} \in \mathbb{R}^n \) is a point in n-dimension.

Linear Dependence: A set of vectors \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \) from \( \mathbb{R}^n \) is said to be linearly dependent if \( \exists \) scalars \( \lambda_i, i \in \{1, 2, \ldots, n\} \) not all zero, such that

\[
\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_n \mathbf{a}_n = \mathbf{0}
\]

Linear Independence: A set of vectors \( \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n \) is said to be linearly independent if the only solution of

\[
\lambda_1 \mathbf{b}_1 + \lambda_2 \mathbf{b}_2 + \cdots + \lambda_n \mathbf{b}_n = \mathbf{0}
\]

Examples:

Show that \((0, 0, 1), (0, 1, 1)\) and \((1, 1, 1)\) from \( \mathbb{R}^3 \) are linearly independent.

Solve:
Let, \( \lambda_1 (0, 0, 1) + \lambda_2 (0, 1, 1) + \lambda_3 (1, 1, 1) = \mathbf{0} \) \( \text{(*)} \)

\[
\begin{align*}
(0, 0, \lambda_1) + (0, \lambda_2, \lambda_2) + (\lambda_3, \lambda_3, \lambda_3) &= (0, 0, 0) \\
(\lambda_1 + \lambda_2 + \lambda_3, \lambda_2 + \lambda_2 + \lambda_3, \lambda_3 + \lambda_3 + \lambda_3) &= (0, 0, 0)
\end{align*}
\]

\[
\begin{align*}
\lambda_1 &= 0 \\
\lambda_2 + \lambda_3 &= 0 \\
\lambda_1 + \lambda_2 + \lambda_3 &= 0
\end{align*}
\]

Hence, \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \) is the only solution of \( \text{(*)} \).

Therefore, the vectors are linearly independent.
Theorem 1. A set of vectors $\{a_1, a_2, \ldots, a_k\}$ from $\mathbb{R}^n$ is linearly dependent if and only if one of these vectors can be written as a linear combination of the others.

Proof:

(1) Only if part. Let $a_1, a_2, \ldots, a_k$ be linearly dependent vectors from $\mathbb{R}^n$. Then $\exists$ scalars $\lambda_i$'s, not all zero, such that

$$\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_k a_k = 0 \quad (\ast)$$

Let a non-zero scalar be $\lambda_i$, so $\ast$ implies

$$-\lambda_i a_i = \lambda_1 a_1 + \cdots + \lambda_i-1 a_{i-1} + \lambda_{i+1} a_{i+1} + \cdots + \lambda_k a_k$$

$$\Rightarrow a_i = \left( -\frac{\lambda_1}{\lambda_i} \right) a_1 + \left( -\frac{\lambda_2}{\lambda_i} \right) a_2 + \cdots + \left( -\frac{\lambda_{i-1}}{\lambda_i} \right) a_{i-1} + \left( -\frac{\lambda_{i+1}}{\lambda_i} \right) a_{i+1} + \cdots + \left( -\frac{\lambda_k}{\lambda_i} \right) a_k$$

$$= \sum_{j \neq i}^{k} \left( -\frac{\lambda_j}{\lambda_i} \right) a_j$$

a linear combination of the vectors $a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k$.

Hence, one vector $a_i$ has been written as a linear combination of the others.

(2) If part. Let $a_i$ can be written as a linear combination of the other vectors, giving

$$a_i = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_{i-1} a_{i-1} + \lambda_{i+1} a_{i+1} + \cdots + \lambda_k a_k$$

for some scalars $\lambda_i$'s.

$$\Rightarrow \lambda_1 a_1 + \cdots + \lambda_i-1 a_{i-1} + (-1)a_i + \lambda_{i+1} a_{i+1} + \cdots + \lambda_k a_k = 0$$

$$\Rightarrow \lambda_1 a_1 + \cdots + \lambda_k a_k = 0$$

is satisfied for $\lambda_i = -1 \neq 0$,

a non-zero scalar.

So, $a_1, a_2, \ldots, a_k$ are linearly dependent.

Hence, the set of vectors is linearly dependent.
Theorem 2. If \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \) is linearly independent and \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \) is linearly dependent, then \( \mathbf{a} \) is a linear combination of \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \).

**Proof:** Since \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \) is linearly dependent, then \( \mathbf{a} \) is a linear combination of \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \).

\[ \lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_m \mathbf{a}_m = \mathbf{0} \quad \text{(1)} \]

If \( \lambda = 0 \), then (1) implies that
\[ \lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_m \mathbf{a}_m = \mathbf{0} \]
for at least one \( \lambda_i \neq 0 \), which shows that \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \) is linearly dependent. But this contradicts our given condition, so \( \lambda \) is always non-zero.

As \( \lambda \neq 0 \), then (1) gives
\[ \mathbf{a} = -\frac{\lambda_1}{\lambda} \mathbf{a}_1 - \frac{\lambda_2}{\lambda} \mathbf{a}_2 - \cdots - \frac{\lambda_m}{\lambda} \mathbf{a}_m \]

i.e., \( \mathbf{a} \) is a linear combination of \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \).

Hence, the proof.

Example: Show that the set of vectors:
\[
\begin{align*}
\mathbf{a}_1 &= (1, -1, 0, 0, \ldots, 0, 0) \\
\mathbf{a}_2 &= (1, 0, -1, 0, \ldots, 0, 0) \\
\mathbf{a}_3 &= (1, 0, 0, -1, \ldots, 0, 0) \\
& \vdots \\
\mathbf{a}_{n-1} &= (1, 0, 0, 0, \ldots, 0, -1) \\
\mathbf{a}_n &= (n-1, -1, -1, \ldots, -1, -1)
\end{align*}
\]

is linearly dependent. Also, find a linearly independent set of vectors and determine the maximum number of linearly independent vectors in the set.

**Solu.** Note that \( \mathbf{a}_n = \mathbf{a}_1 + \mathbf{a}_2 + \cdots + \mathbf{a}_{n-1} \)
\[
\Rightarrow \left\{ \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-1}, \mathbf{a}_n \right\} \text{ are linearly dependent.}
\]

Now, consider the equation
\[
\sum_{i=1}^{n-1} \lambda_i \mathbf{a}_i = \mathbf{0}
\]
\[
\Rightarrow \left\{ \begin{array}{c}
\frac{\lambda_1}{\lambda} = -1 \\
\frac{-\lambda_1}{\lambda} = -1 \\
\vdots \\
\frac{-\lambda_{n-1}}{\lambda} = -1 \\
\frac{-\lambda_{n-1}}{\lambda} = -1
\end{array} \right.
\Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0
\]
Hence, \( \{ \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \} \) is linearly independent.

Here, the collection of \( n \) vectors \( \{ \alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n \} \) is linearly dependent but the collection of \( (n-1) \) vectors \( \{ \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \} \) is linearly independent.

2) If \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are linearly independent and \( \sum_{i=1}^{n} \alpha_i \alpha_i = \sum_{i=1}^{n} \beta_i \beta_i \), then show that \( \alpha_i = \beta_i \) \( \forall \ i = 1(1)n \).

\[ \text{Soln.} \Rightarrow \sum_{i=1}^{n} \alpha_i \alpha_i = \sum_{i=1}^{n} \beta_i \beta_i \]
\[ \Rightarrow \sum_{i=1}^{n} (\alpha_i - \beta_i) \alpha_i = 0 \]
\[ \Rightarrow (\alpha_i - \beta_i) \alpha_i = 0 \ \forall \ i = 1(1)n \]

since \( \alpha_i \)'s are linearly independent.

\[ \Rightarrow \alpha_i = \beta_i \ \forall \ i = 1(1)n \]

3) If \( \alpha \) and \( \beta \) are linearly independent, then show that \( \alpha + \alpha \beta \) and \( \alpha + \beta \alpha \) are also linearly independent if \( \alpha \neq \beta \).

\[ \text{Soln.} \Rightarrow \text{Let, } L_1 (\alpha + \alpha \beta) + L_2 (\alpha + \beta \alpha) = 0 \]
\[ \Rightarrow (L_1 + L_2) \alpha + (L_2 \beta + L_1 \alpha) \beta = 0 \]
\[ \Rightarrow L_1 + L_2 = 0 \ \& \ L_2 \beta + L_1 \alpha = 0 \text{, since } \alpha \text{ and } \beta \text{ are linearly independent} \]
\[ \Rightarrow L_1 = -L_2 \ \& \ L_1 (\alpha - \beta) = 0 \]
\[ \Rightarrow L_1 = 0, \ L_2 = 0 \text{ since } \alpha \neq \beta. \]

Hence \( (\alpha + \alpha \beta) \) and \( (\alpha + \beta \alpha) \) are linearly independent if \( \alpha \neq \beta \).
Theorem 3. Show that a set of vectors containing a null vector can't be linearly independent.

Proof: Let \( \{ \mathbf{e}, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \} \) be a set containing a null vector.

For, \( \lambda = 5, \lambda_1 = 0, \lambda_2 = 0, \ldots, \lambda_n = 0 \), the equation
\[
5\mathbf{e} + \lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \ldots + \lambda_n \mathbf{e}_n = \mathbf{0}
\]

is satisfied.

Hence, \( \exists \) a non-zero \( \lambda \), so the set of vectors is linearly dependent.

\( \therefore \) A set of vectors containing the null vector can't be linearly independent.

Theorem 4. Any \( n \)-vector must be linearly dependent on the unit vectors \( \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \), which by themselves form an independent set.

Proof: Let \( \mathbf{a} \) be an \( n \)-component vector, given by
\[
\mathbf{a} = (a_1, a_2, \ldots, a_n)
\]

Since \( e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 0, 1) \).

We can write \( \mathbf{a} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \ldots + \alpha_n \mathbf{e}_n \).

This shows that \( \mathbf{a} \) is, indeed, linearly dependent on \( \mathbf{e}_i \) \( (i=1, n) \).

Since \( \mathbf{a} \) is arbitrarily chosen, the first part of the theorem is established.

Second part has been proved earlier, i.e., unit vectors are linearly independent.

Theorem 5. A null vector is linearly dependent on any other set of non-null vectors.

Proof:
Definition: A vector space \( V^n \) is a collection of \( n \)-component vectors which is closed under the operations of addition and scalar multiplication.

\[ \text{i.e. } \forall \mathbf{a}, \mathbf{b} \in V^n \text{ then } \mathbf{a} + \mathbf{b} \in V^n \text{ [closed under addition]} \]

\[ \forall \mathbf{a} \in V^n \text{ for any scalar } \lambda, \lambda \mathbf{a} \in V^n \text{ [closed under scalar multiplication]} \]

**Example:** \( V_2(\mathbb{R}) = \{ (x_1, x_2) : x_1, x_2 \in \mathbb{R} \} \) is a vector space.

**Remark:** The concept of distance, scalar product and angle are not defined in vector spaces. Clearly, \( \mathbb{R}^n \) satisfies the properties of a vector space, so \( \mathbb{R}^n \) is a vector space.

If we define length in \( V^n \) as in \( \mathbb{R}^n \), then \( V^n \) is identical with \( \mathbb{R}^n \). Although, \( \mathbb{R}^n \) is a vector space, it does not follow that every \( V^n \) is \( \mathbb{R}^n \).

**Example:** Consider the collection of vectors of the form \( (x_1, x_2, 0) \) from \( \mathbb{R}^3 \) on \( V^3 \). Show that the collection is a vector space.

**S/o/n:** Define \( S_3 = \{ (x_1, x_2, 0) : x_1, x_2 \in \mathbb{R} \} \)

**Let** \( \mathbf{x}, \mathbf{y} \in S_3 \). Then \( \mathbf{x} = (x_1, x_2, 0) \) and \( \mathbf{y} = (y_1, y_2, 0) \).

Now, \( \mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, 0) \in S_3 \)

and \( \lambda \mathbf{x} = (\lambda x_1, \lambda x_2, 0) \in S_3 \)

Hence, \( S_3 \) is closed under addition and scalar multiplication.

**Note:** It is important to note that there are several subsets of \( \mathbb{R}^n \) on \( V^n \), which are itself a vector space.
**Vector Subspace:**

**Definition:**

1. A vector subspace $S_n$ of an $n$-dimensional vector space $V_n$ is a subset of $V_n$ which is itself a vector space.
2. A non-empty subset $S_n$ of vectors from $n$-dimensional vector space $V_n$ is called a subspace of $V_n$ if $S_n$ is closed under:
   - Addition, i.e. $a \in S_n, b \in S_n \Rightarrow a + b \in S_n$ and
   - Scalar multiplication, i.e. $a \in \mathbb{R}, a \in S_n \Rightarrow \lambda a \in S_n$.

**Example:**

$S_2 = \{(x_1, x_2, x_3) : x_1 + x_2 \in \mathbb{R}\}$ is a vector subspace.

**Remark:**

(a) For a vector space $V_n$ if $v \in V_n$, then $\alpha v \in V_n$ for any scalar $\alpha \in \mathbb{R}$.
   For $\alpha = 0$, we get $0 \in V_n$. Hence, the null vector is a member of a vector space.

(b) Any subspace of $E^3$ is either $E^3$ itself, a plane through the origin or just the origin itself.

**Example:**

Show that $S_1$ is a vector subspace, where $S_1 = \{(x_1, x_2, x_3) : x_1 + x_2 = x_3\}$.

**Solution:**

Clearly, $S_1 \subseteq E^3$.
Considered vectors $\mathbf{x}$ and $\mathbf{y} \in S_1$.

$\mathbf{x} = (x_1, x_2, x_3)$, where $x_1 + x_2 = x_3$,
$\mathbf{y} = (y_1, y_2, y_3)$, where $y_1 + y_2 = y_3$.

Now, $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$, where $x_1 + x_2 = x_3, y_1 + y_2 = y_3$.

$\lambda \mathbf{x} = (\lambda x_1, \lambda x_2, \lambda x_3)$, where $\lambda x_1 = \lambda x_2$ and $\lambda x_3 = \lambda x_3$.

$\Rightarrow \mathbf{x} + \mathbf{y} \in S_1$, and for any scalar $\lambda$,

$\Rightarrow \lambda \mathbf{x} \in S_1$.

Hence, $S_1$ is a vector subspace of $E^3$. 

\[ \text{i.e.} \quad \lambda \mathbf{x} \in S_1. \]
Example: In each of the following find out whether the subsets given form subspaces of the vector spaces.

1. \( S = \{ (x_1, x_2) : x_1 \geq 0, x_2 \geq 0 \} \)
2. \( T = \{ (x_1, x_2) : x_1 x_2 > 0 \} \)

Solution:

1. For \( \alpha < 0 \), \( \alpha (x_1, x_2) = (\alpha x_1, \alpha x_2) \), where \( \alpha x_1 \leq 0, \alpha x_2 \leq 0 \).
   
   So, \( S \) is not a subspace.

2. Consider the vectors \((3, 2)\) and \((-2, -4)\) from \( T \).
   
   Note that \((3, 2) + (-2, -4) = (1, -2)\) for which
   
   \( n_1 n_2 < 0 \) \( \neq T \).

\[ \text{e.g. 2} \]

Consider a subset \( S \) of \( V_2(\mathbb{R}) \):

\[ S = \{ (x, y) \mid 0 < x, y < \infty \} \subset V_2(\mathbb{R}) \]

Is \( S \) a subspace of \( V_2(\mathbb{R}) \)?

Ans.: Consider any two vectors \( a_1 = (a_{11}, a_{12}) \) and \( a_2 = (a_{21}, a_{22}) \) from \( S \).

Now, \( a_1 + a_2 = (a_{11} + a_{21}, a_{12} + a_{22}) \in S \) because \( a_{11} + a_{21} \) and \( a_{12} + a_{22} \) are both positive, giving \( S \) is closed under addition.

But \( S \) is not closed under the operations of multiplication by a scalar \( \lambda \), because for any vector \( a_1 \in S \) and \( \lambda \leq 0 \), \( \lambda a_1 = (\lambda a_{11}, \lambda a_{12}) \notin S \).

\( \therefore \) \( S \) is not a subspace of \( V_2(\mathbb{R}) \).

\[ \text{e.g. 3} \]

The subset \( S = \{ (x, y) \mid -\infty < x, y < \infty, x \neq 0, y \neq 0 \} \)

Ans.: For any vector \((a_1, a_2) \in S \) and \( \lambda = 0 \), \( \lambda (a_1, a_2) = (0, 0) \notin S \),

So, \( S \) is not closed under multiplication by a scalar,

\[ \therefore \] giving that \( S \) is not a subspace.
The subset \( S = \{ (u, y) \mid y = a + bu, -\infty < u, y < \infty; a \neq 0, b \neq 0 \} \) is not a subspace.

**Ans:** Consider any two vectors \( \mathbf{a}_1 = (a_{11}, a_{12}) \) and \( \mathbf{a}_2 = (a_{21}, a_{22}) \) from \( S \).

Now, \( \mathbf{a}_1 + \mathbf{a}_2 = (a_{11} + a_{21}, a_{12} + a_{22}) \)

\[ y = a + bu \]
\[ a_{12} = a + ba_{11} \]
\[ a_{22} = a + ba_{21} \]

\[ \Rightarrow \mathbf{a}_1 + \mathbf{a}_2 \notin S. \]

So, giving \( S \) is not closed under addition. Hence, \( S \) is not a subspace.

Construction of a vector subspace:

**RESULT:** Consider a set \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \) of vectors from \( V_n \). Then show that the collection of all possible linear combinations of \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \) is a vector subspace of \( V_n \).

**Proof:** The collection of all possible linear combinations of \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \) is

\[ S = \left\{ \sum_{i=1}^{n} l_i \mathbf{a}_i : l_i \in \mathbb{R}, i = 1, 2, \ldots, n \right\} \]

Let, \( \mathbf{x}, \mathbf{y} \in S \).

\[ \mathbf{x} = \sum_{i=1}^{n} l_i \mathbf{a}_i \] and \( \mathbf{y} = \sum_{i=1}^{n} l_i \mathbf{a}_i \), for some \( l_i \)'s and \( \mathbf{a}_i \)'s.

\[ \Rightarrow \mathbf{x} + \mathbf{y} = \sum_{i=1}^{n} (l_i + l_i) \mathbf{a}_i \]
\[ = \sum_{i=1}^{n} l_i \mathbf{a}_i \in S. \]

And, \( \alpha \mathbf{x} = \alpha \left( \sum_{i=1}^{n} l_i \mathbf{a}_i \right) \]
\[ = \sum_{i=1}^{n} (\alpha l_i) \mathbf{a}_i \]
\[ = \sum_{i=1}^{n} l_i \mathbf{a}_i \in S. \]

Clearly, \( S \subseteq V_n \).

Hence, \( S \) is a vector subspace of \( V_n \).
Spanning Set

Definition: A set \( \{a_1, a_2, \ldots, a_n\} \) of vectors from a vector space \( V_n \) is said to be span on generate \( V_n \) if every vector in \( V_n \) can be written as a linear combination of \( a_1, a_2, \ldots, a_n \).

Then we say that \( \{a_1, a_2, \ldots, a_n\} \) is a spanning set on generating set of the vector space \( V_n \).

Remark: The collection of all possible linear combinations of \( \{a_1, a_2, \ldots, a_n\} \) is \( S = \sum_{i=1}^{n} \lambda_i a_i \) which is a vector space.

The spanning set of \( S \) is \( \{a_1, a_2, \ldots, a_n\} \) on the span of \( \{a_1, a_2, \ldots, a_n\} \) is \( S = \sum_{i=1}^{n} \lambda_i a_i \).

Example: For \( V_2 \), the unit vectors \( e_1, e_2 \) form a spanning set. Show that any two linearly independent vectors in \( V_2 \) form a spanning set of \( V_2 \). Also, show that \( e_1, e_2, (1, 1) \) forms a spanning set of \( V_2 \).

Solution: Let \( \{a_1, a_2\} \) be two linearly independent vectors from \( V_2 \).
If possible, let \( x = l_1 a_1 + l_2 a_2 \), where \( x \in V_2 \).
\( x = l_1 (a_{11}, a_{12}) + l_2 (a_{21}, a_{22}) \)
\( = (l_1 a_{11}, l_1 a_{12} + l_2 a_{21}, l_2 a_{12}) \)
\( \Rightarrow \begin{align*}
x_1 &= l_1 a_{11} + l_2 a_{21} \\
x_2 &= l_1 a_{12} + l_2 a_{22} \\
l_1 &= \frac{x_1 a_{22} - x_2 a_{21}}{a_{11} a_{22} - a_{12} a_{21}} \\
l_2 &= \frac{x_1 a_{12} - x_2 a_{11}}{a_{11} a_{22} - a_{12} a_{21}}
\end{align*} \)
\( \Rightarrow a_1 = (a_{11}, a_{12}) \) and \( a_2 = (a_{21}, a_{22}) \) are linearly independent.
\( \Rightarrow a_2 \neq a_1 \)
\( \Rightarrow \frac{a_{11}}{a_{21}} \neq \frac{a_{12}}{a_{22}} \)
\( \Rightarrow a_{11} a_{22} - a_{12} a_{21} \neq 0 \)
Hence, \( l_1 \) and \( l_2 \) \( \Rightarrow x = l_1 a_1 + l_2 a_2 \)
Hence, \( \{a_1, a_2\} \) spans \( V_2 \).

Again, for \( x \in V_2 \), \( x = x_1 e_1 + x_2 e_2 + 0(1, 1) \)
\( \Rightarrow \{e_1, e_2, (1, 1)\} \) spans \( V_2 \).
**Basis**

**Definition:**

1. A linearly independent set of vectors that generates a subspace, \( S \), is called a basis of the subspace.
2. A set of vectors \( \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \} \) is said to constitute a basis of a subspace \( S \) if:
   
   i. \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \) span \( S \);
   
   ii. \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \) is a set of linearly independent vectors.

**Example:**

The unit vectors \( \mathbf{e}_1 = \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right), \mathbf{e}_2 = \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right), \mathbf{e}_3 = \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \) form a basis of \( \mathbb{V}_3(\mathbb{R}) \) because they are linearly independent and span \( \mathbb{V}_3(\mathbb{R}) \).

**Problems:**

1. Do the vectors \( \mathbf{e}_1 = \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right), \mathbf{e}_2 = \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right), \mathbf{e}_3 = \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \) form a basis of \( \mathbb{V}_3(\mathbb{R}) \)?

   \[ S = \{ \left( \begin{array}{c} u \\ v \\ w \end{array} \right) \mid z = 0, -\infty < u, v, w < \infty \} \]

   **Ans:** No, because in \( \mathbb{V}_3 \), \( z \neq 0 \), so that does not define the definition of spanning set.

2. Does \( S = \{ \left( \begin{array}{c} u \\ v \end{array} \right) \mid z = 0, -\infty < u, v < \infty \} \) form a subspace of \( \mathbb{V}_3(\mathbb{R}) \)?

   **Ans:** Considering any two vectors \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) from \( S \).

   \[ \mathbf{e}_1 = (a_{11}, a_{12}, 0) \]
   
   \[ \mathbf{e}_2 = (a_{21}, a_{22}, 0) \]

   Now, \( \mathbf{e}_1 + \mathbf{e}_2 = (a_{11} + a_{21}, a_{12} + a_{22}, 0) \in S \),

   because \( a_{11} + a_{21} > -\infty \) and \( a_{12} + a_{22} < \infty \).

   So, \( S \) is closed under addition.

   Now, for any scalar \( \lambda \),

   \[ \lambda \mathbf{e}_1 = (\lambda a_{11}, \lambda a_{12}, 0) \in S \]

   So, \( S \) is closed under multiplication by a scalar.

   \( \therefore \) \( S \) is a subspace of \( \mathbb{V}_2(\mathbb{R}) \).
3) Show that \( S = \{ (x, y, z) : x+y+z=0 ; x, y, z \in \mathbb{R} \} \) is a subspace.

**Soln**

Consider a vector \((x, y, z) \in S.\)

Here, \(x+y+z = 0\)
\[\Rightarrow \lambda (x+y+z) = 0\]
\[\Rightarrow (\lambda x + \lambda y + \lambda z) = 0\]
\[\Rightarrow (\lambda x, \lambda y, \lambda z) \in S.\]

i.e., \(S\) is closed under scalar multiplication.

Consider \((x, y, z) \in S \& (u, v, w) \in S\)
\[\Rightarrow x+y+z = 0, \quad u+v+w = 0\]
\[\Rightarrow (x+y+z) = 0 \quad \& \quad (u+v+w) = 0\]
\[\Rightarrow (x+u, y+v, z+w) \in S\]

i.e., \(S\) is closed under vector addition.

Therefore \(S\) is a vector space. (Ans)

Show that all the vectors \((x_1, x_2, x_3)\) in a vector space \(V_3\) which obey \(x_1-x_2 = 0\) form a subspace \(V\) and find a basis of this subspace.

OR

Show that the set of vectors \(S = \{ (x_1, x_2, x_3) \mid x_1-x_2 = 0, x_1, x_2, x_3 \in \mathbb{R} \}\)
forms a subspace and find its basis. \(x_1, x_2, x_3 \in \mathbb{R}\)

**Soln**

i) Considering any two vectors \(a_1\) and \(a_2\) from \(S\).
\[ a_1 = (a_{11}, a_{12}, a_{13}) \quad \& \quad a_2 = (a_{21}, a_{22}, a_{23}) \]

As \(x_1-x_2 = 0\)
\[\Rightarrow x_1 = x_2.\]
Here, \(a_{11} = a_{12}\)
\[a_{21} = a_{22}\]

Scanned by CamScanner

Ctanujit Classes

Page No. 23
Now, \( \mathbf{a} + \mathbf{b} = (a_{11} + a_{21}, a_{12} + a_{22}, a_{13} + a_{23}) \in S \)

because \( a_{11} + a_{21} \in \mathbb{R} \) and \( a_{13} + a_{23} \in \mathbb{R} \).

So, \( S \) is closed under addition.

Now, for any scalar \( \lambda \),

\[ \lambda \mathbf{a} = (\lambda a_{11}, \lambda a_{12}, \lambda a_{13}) \in S. \]

So, \( S \) is closed under multiplication by a scalar.

So, \( S \) forms a subspace.

\[ S = \{ \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \mid m_1 - m_2 = 0, m_1, m_2, m_3 \in \mathbb{R} \} \]

\[ = \{ \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \mid m_1, m_3 \in \mathbb{R} \} \]

\[ = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mid m_1, m_3 \in \mathbb{R} \} \]

\( \Rightarrow \) \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) span \( S \).

So, \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) are linearly independent.

So, they are the basis of \( S \).

[\textbf{NOTE:} \Rightarrow \) Here \( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) are not spanning set

because \( e_1, e_2, e_3 \) are not in the subspace.]

5. The vectors \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) form a basis of \( S = \{ \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \mid m_1 - m_2 = 0, m_1, m_3 \in \mathbb{R} \} \).

Does another basis of \( S \) exists?

\[ \textbf{Ans:} \Rightarrow \{ \begin{pmatrix} a_{11} \\ b_{11} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \} \quad \text{if} \quad a_{11} \text{ and } b_{11} \in \mathbb{R} \}

\[ \text{If we put any another set then for this particular subspace the new set would be a linearly dependent set, which contradicts our definition of basis.} \]
**Dimension:**

**Definition:**
1. The number of vectors in a basis of a subspace is called the dimension of the subspace.
2. Dimension of a vector space \( V_n \) denoted by \( d(V_n) \) is defined as the number of linearly independent vectors required to span \( V_n \), i.e., the number of vectors in any basis of \( V_n \).

\[
\begin{align*}
1. & \quad d(\mathbb{R}^n) = n, \\
2. & \quad d(\mathbb{R}) \leq n, \\
3. & \quad d(V_n) \geq p \text{ if there are an arbitrary collection of } p \text{ linearly independent vectors in } V_n.
\end{align*}
\]

**Example:**
1. \( \dim(V_2(\mathbb{R})) = 2 \) because \( \{e_1 = (1), e_2 = (0)\} \) form a basis of \( V_2(\mathbb{R}) \).
2. Can you find \( \dim(S) \) where \( S = \{ (\alpha, \beta) | \alpha, \beta \in \mathbb{R} \} \)?

**Ans:** \( \alpha = \binom{\alpha_1}{\alpha_2} , \beta = \binom{\beta_1}{\beta_2} \) \( \in S \), \( \alpha + \beta = \binom{\alpha_1 + \beta_1}{\alpha_2 + \beta_2} \) \( \in S \).

Here, \( S \) is not closed under addition and thus \( S \) is not a subspace, so \( \dim(S) \) is not defined in this case.

3. \( S = \{ (\alpha, \beta) | \alpha, \beta \in \mathbb{R} \} \). Calculate \( \dim(S) \)?

**Solu:**
\( \alpha = \binom{a_1}{a_2} , \beta = \binom{b_1}{b_2} \) \( \in S \), \( \alpha + \beta = \binom{a_1 + b_1}{a_2 + b_2} \) \( \in S \).

So, \( S \) is a subspace.

\[
S = \{ (\alpha, \beta) | \alpha, \beta \in \mathbb{R} \} = \{ (1, 0) | \alpha \in \mathbb{R} \}
\]

So, \( (1, 0) \) forms a basis of \( S \), giving \( \dim(S) = 1 \).

4. \( S_1 = \{ (\alpha, 0) | \alpha \in \mathbb{R} \} \) and \( S_2 = \{ (0, \beta) | \beta \in \mathbb{R} \} \) are two subspaces of \( V_2(\mathbb{R}) \). Find \( \dim(S_1 + S_2) \)?

**Solu:**
\( S_1 + S_2 = \{ (\alpha, \beta) | \alpha, \beta \in \mathbb{R} \} \)

\[
\begin{array}{c|c}
\text{Subspace} & \text{Dimension} \\
\hline
S_1 & 1 \\
S_2 & 1 \\
S_1 + S_2 & 2 \\
\end{array}
\]
$S_1 = \{ (x) \mid x \in \mathbb{R} \}$ and $S_2 = \{ (y) \mid y \in \mathbb{R} \}$ are two subspaces of $V_2(\mathbb{R})$. Find $\dim(S_1 + S_2)$?

**Solution**

$S_1 + S_2 = \{ (x) + (y) \mid x, y \in \mathbb{R} \}$

$= \{ (x, y) \mid \lambda x, \lambda y \in \mathbb{R} \}$

$= \{ (x+y), (c), (x-y), (d) \mid x, y \in \mathbb{R} \}$

$= \lambda S_1 + \lambda S_1, \lambda \neq 0$ form a basis of $S_1 + S_2$.

**Conclusion**

$\dim(S_1 + S_2) = 2$.

2) Consider the vectors $x_1 = (1, 3, 2)$ and $x_2 = (-2, 4, 3)$ in $V_3$ on $\mathbb{R}^3$. Find the span of $x_1, x_2$.

**Solution**

$\text{Span} \{ x_1, x_2 \}$ is

$S = \{ (l_1, l_2, l_3) : l_1, l_2, l_3 \in \mathbb{R} \}$

$= \{ (l_1 - 2l_2, 3l_1 + 4l_2, 2l_1 + 3l_2) : l_1, l_2, l_3 \in \mathbb{R} \}$

Let, $\Phi \in S$, then $S = \{ (\Phi_1, \Phi_2, \Phi_3) : \Phi_1, \Phi_2, \Phi_3 \in \mathbb{R} \}$

$\Phi_1 = l_1 - 2l_2, \quad \Phi_2 = 3l_1 + 4l_2, \quad \Phi_3 = 2l_1 + 3l_2$

$\Phi_1 - 7\Phi_2 + 10\Phi_3 = l_1 - 2l_2 - 21l_1 - 28l_2 + 20l_1 + 30l_2$

$\Rightarrow \Phi_1 - 7\Phi_2 + 10\Phi_3 = 0$

$S = \{ (\Phi_1, \Phi_2, \Phi_3) : \Phi_1 - 7\Phi_2 + 10\Phi_3 = 0 \}$
Result: For two subspaces $W_1$ and $W_2$ of $V_n$, let $W_1 \cap W_2$ be the intersection of $W_1$ and $W_2$. Then $W_1 \cap W_2$ is also a subspace of $V_n$. Let $x, y \in V_n$, then $x + y \in V_n$. Therefore, $x + y \in W_1 \cap W_2$. Thus, $\dim(W_1 \cap W_2) \leq \min\{\dim(W_1), \dim(W_2)\}$.

Example:

$W_1 = \left\{ \left[ \begin{array}{c} x \\ 0 \end{array} \right] \mid x, y \in \mathbb{R} \right\}$,

$W_2 = \left\{ \left[ \begin{array}{c} x \\ 2y \end{array} \right] \mid x, z \in \mathbb{R} \right\}$

$W_1 \cap W_2 = \left\{ \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \right\}$

is also a subspace.

$\dim(W_1) = 2$,

$\dim(W_2) = 2$,

$\dim(W_1 \cap W_2) = 1$.

$\dim(W_1 \cap W_2) \leq \min\{\dim(W_1), \dim(W_2)\}$.

Similarly, $W_1 \cap W_2 \subseteq W_2$

$\Rightarrow \dim(W_1 \cap W_2) \leq \dim(W_2)$

$\Rightarrow \dim(W_1 \cap W_2) \leq \min\{\dim(W_1), \dim(W_2)\}$.

Remark: 1. $\dim(W_1 \cap W_2) \leq \sqrt{\dim(W_1) \dim(W_2)}$.

2. Let $V$ be a vector space over a field $F$ and $W$ be a subspace of $V$. Then $\dim(V/W) = \dim V - \dim W$.
Result: Prove that \( S_1 \cap S_2 \) is also a vector subspace of \( V_n \).

Let \( S_1 = \{ x_1 \} \) and \( S_2 = \{ y \} \). What is \( S_1 \cap S_2 \)?

**Proof:** Let \( x, y \in S_1 \cap S_2 \).

Then \( x \in S_1 \) and \( y \in S_2 \), \( x \in S_1 \) and \( y \in S_2 \).

\[ x + y \in S_1 \cap S_2 \] as \( S_1 \) is closed under vector addition.

\[ \alpha x \in S_1 \cap S_2 \] for any \( \alpha \in \mathbb{F} \), \( \alpha x \in S_1 \) and \( \alpha x \in S_2 \), as \( S_1 \) and \( S_2 \) are closed under scalar multiplication.

Hence, \( S_1 \cap S_2 \) is a vector subspace of \( V_n \).

\[ V_2 = \{ \alpha x_1 + \lambda x_2 : \alpha, \lambda \in \mathbb{F} \} = S_2 \]

\[ S_1 \cap S_2 = S_1 \cap V_1 = S_1 \cap (S_1 \cap S_2) = S_1 \cap S_2 \]

**Theorem 19.** For any two subspaces \( W_1 \) and \( W_2 \) of \( V_n (\mathbb{F}) \),

\[ \dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) \]

**Proof:** Let \( \{ v_1, v_2, \ldots, v_k \} \) be a basis of \( W_1 \cap W_2 \).

As this set is LIN & contains vectors from \( W_1 \) and \( W_2 \),

It can be extended so as to constitute a basis \( S = v_1, v_2, \ldots, v_k, w_1, w_2, \ldots \) of \( W_1 \).

For similar reasons

\[ S_2 = \{ w_1, w_2, \ldots, v_1, v_2, \ldots \} \]

is a basis of \( W_2 \), contains vectors \( v_i, w_i \) of \( W_1 \).

We now prove that \( S = \{ v_1, v_2, \ldots, v_k, w_1, w_2, \ldots \} \)

is a basis of \( W_1 + W_2 \).

Let \( d \in W_1 + W_2 \)

\[ d = d_1 + d_2 \]

where \( d_1 \in W_1 \) and \( d_2 \in W_2 \)

\[ d = \sum_{i=1}^{k} \lambda_i v_i + \sum_{i=1}^{k} \lambda_i w_i \]

\[ d = \sum_{i=1}^{k} \lambda_i v_i + \sum_{i=1}^{k} \lambda_i w_i \]

\[ \Rightarrow d \] is a spanning set \( q \) of \( W_1 + W_2 \).

To show that \( S \) is LIN, we consider the equation.
\[ \sum_{i=1}^{p} k_{i_1} \alpha_i + \sum_{i=1}^{r} k_{i_2} \beta_i + \sum_{i=1}^{n} k_{i_3} \gamma_i = 0 \quad \text{(k's are scalars)} \]

\[ \sum_{i=1}^{p} k_{i_1} \alpha_i + \sum_{i=1}^{r} k_{i_2} \beta_i = -\sum_{i=1}^{n} k_{i_3} \gamma_i \quad \text{(1)} \]

(1) shows that \[-\sum_{i=1}^{n} k_{i_3} \gamma_i \in W_1, \]

as it is a linear combination of the basis vectors of \( W_1 \), moreover being a linear combination of the basis vectors of \( W_2 \),

\[ \sum_{i=1}^{n} k_{i_3} \gamma_i \in W_2, \]

so \( \sum_{i=1}^{n} k_{i_3} \gamma_i \in (W_1 \cap W_2) \), and hence

\[-\sum_{i=1}^{n} k_{i_3} \gamma_i = \sum_{i=1}^{p} \delta_i \alpha_i \text{ for some scalars } \delta_i, \text{ as } \{ \alpha_1, \alpha_2, \ldots, \alpha_p \} \text{ is a basis of } W_1. \]

\[ \sum_{i=1}^{p} \delta_i \alpha_i + \sum_{i=1}^{r} k_{i_2} \beta_i = 0 \]

\[ \delta_i = 0 \text{ and } k_{i_2} = 0 \quad \forall i \text{ because the set } \{ \alpha_1, \alpha_2, \ldots, \alpha_p, \beta_1, \ldots, \beta_r \} \]

is LIN.

Thus,

\[ \sum_{i=1}^{p} k_{i_1} \alpha_i + \sum_{i=1}^{r} k_{i_2} \beta_i = 0 \]

\[ s_{k_1} = 0 \quad \forall i \in \{1 \} \quad \text{because } \{ \alpha_1, \alpha_2, \ldots, \alpha_p, \beta_1, \ldots, \beta_r \} \text{ is also LIN set of vectors,}\]

or the set \( S \) containing all the \((p+q+r)\) vectors is LIN.

So, \( S \) is a basis of \( W_1 + W_2 \).

Therefore,

\[ \dim (W_1 + W_2) = p + q + r \]

Now, \( \dim (W_1 + W_2) = (p+q) + (p+r) = p \]

\[ \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) \]

\[ \dim(V_n(F)) \]
Orthogonal Vectors:

Definition:

1. Two vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be orthogonal if $\mathbf{a} \cdot \mathbf{b} = 0$. We generally use the notation $\mathbf{a} \perp \mathbf{b}$ to mean that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal. Clearly, the null vector $\mathbf{0}$ is orthogonal to every vector.

2. A set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ from $\mathbb{R}^n$ is said to be mutually orthogonal if $\mathbf{a}_i \cdot \mathbf{a}_j = 0 \forall i \neq j$.

Note: It is noted that the unit vectors are orthogonal since $\mathbf{e}_i \cdot \mathbf{e}_j = 0 \forall i \neq j$.

Let $\theta$ be the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|},$$

where $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$, $\|\mathbf{b}\| = \sqrt{\mathbf{b} \cdot \mathbf{b}}$.

Now, $\mathbf{a}$ and $\mathbf{b}$ are said to be orthogonal, then $\mathbf{a} \cdot \mathbf{b} = 0$

i.e. $\cos \theta = 0$

i.e. $\theta = \frac{\pi}{2}$,

i.e. vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

Example:

(a) $\mathbf{a} = \left( \begin{array}{c} 2 \\ 0 \end{array} \right)$ and $\mathbf{b} = \left( \begin{array}{c} 0 \\ 6 \end{array} \right)$ are orthogonal vectors.

(b) The set containing $\mathbf{a}_1 = \left( \begin{array}{c} 1 \\ -1 \end{array} \right)$, $\mathbf{a}_2 = \left( \begin{array}{c} 0 \\ -1 \end{array} \right)$ and $\mathbf{a}_3 = \left( \begin{array}{c} -1 \\ 2 \end{array} \right)$ is orthogonal because

\[
\mathbf{a}_1 \cdot \mathbf{a}_2 = 1 \times 1 + 1 \times 1 = 2 = 0
\]

\[
\mathbf{a}_2 \cdot \mathbf{a}_3 = 0 \times 1 + (-1) \times (-1) = 0
\]

\[
\mathbf{a}_1 \cdot \mathbf{a}_3 = 1 \times (-1) + 1 \times 2 = 1 = 0
\]

(c) $\mathbf{a}_1 = \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ \ldots \end{array} \right)$,

$\mathbf{a}_2 = \left( \begin{array}{c} -1 \\ -1 \\ 0 \\ 0 \\ \ldots \end{array} \right)$,

$\mathbf{a}_3 = \left( \begin{array}{c} 1 \\ 1 \\ -2 \\ 0 \\ \ldots \end{array} \right)$

\[\vdots\]

$\mathbf{a}_n = \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ \ldots \end{array} \right)$

is a set of orthogonal vectors.
(d) Let $W_1$ be the subspace generated by the vectors \[ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \] and \[ \begin{pmatrix} 0 \\ -1 \end{pmatrix} \] and $W_2$ the subspace generated by \[ \begin{pmatrix} -5 \\ 2 \end{pmatrix} \] are orthogonal.

\[ W_1 = \left\{ \lambda_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{R} \right\} \]

\[ W_2 = \left\{ \lambda \begin{pmatrix} -5 \\ 2 \end{pmatrix} : \lambda \in \mathbb{R} \right\} \]

For, \( a = \lambda_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} \) and \( b = \lambda \begin{pmatrix} -5 \\ 2 \end{pmatrix} \),

\[ a \cdot b = 0. \]

**Orthonormal Vectors**

**Definition:** \( \{a_1, a_2, \ldots, a_n\} \) is said to be a set of orthonormal vectors if

\[ a_i \cdot a_j = 0 \quad \forall \ i \neq j \]

\[ a_i \cdot a_j = 1 \quad \forall \ i = j \] (with unit length)

If we define a set of vectors \( \{a_1, a_2, \ldots, a_n\} \), then \( \{b_1, b_2, \ldots, b_n\} \) will be a set of orthonormal vectors.

**Orthogonal Basis**

**Definition:** A basis of a subspace of \( \mathbb{R}^n \) is called orthonormal if the basis vectors are orthonormal. A basis is called orthonormal if the basis vectors are of unit length in addition to orthogonal.

**Example:** The vectors \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} -1 \\ 1 \end{pmatrix} \) form an orthogonal basis of \( \mathbb{R}^2 \) because \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \).

The vectors \( \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \) and \( \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \) being of unit length form an orthonormal basis of \( \mathbb{R}^2 \).

In particular, \( \{e_1, e_2, \ldots, e_n\} \) forms an orthonormal basis for \( \mathbb{R}^n \).
Theorem 20. A set of orthogonal vectors not containing the null vector is linearly independent.

Proof: Let \( \{a_1, a_2, \ldots, a_n\} \) be a set of mutually orthogonal non-null vectors from \( \mathbb{F}^n \). To show that these vectors are linearly independent, consider the equation

\[
\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n = 0
\]

by considering the scalar products of \( a_i \) and the equation (1), if \( i = 1(1)n \), we get

\[
\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n = 0
\]

\[
\Rightarrow \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n = 0
\]

\[
\Rightarrow \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n = 0
\]

\[
\Rightarrow \lambda_1 = 0, \lambda_2 = 0, \ldots, \lambda_n = 0 \text{ as } a_i's \text{ are non-null vectors which gives } a_i \neq 0, i = 1(1)n.
\]

\[
\Rightarrow \text{the set of vectors } \{a_1, a_2, \ldots, a_n\} \text{ is linearly independent.}
\]

Theorem 21. Any set of \( n \) mutually orthogonal non-null vectors from \( \mathbb{F}^n \) forms a basis for \( \mathbb{F}^n \).

Proof: Let \( \{a_1, a_2, \ldots, a_n\} \) be a set of mutually orthogonal non-null vectors from \( \mathbb{F}^n \). Consider the equation

\[
\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n = 0
\]

Note that

\[
\Rightarrow \lambda_1 (a_1 \cdot a_1) + \lambda_2 (a_2 \cdot a_2) + \cdots + \lambda_n (a_n \cdot a_n) = 0
\]

\[
\Rightarrow \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n = 0
\]

\[
\Rightarrow a_i = 0, \text{ since } a_i \neq 0 \text{ at } i = 1(1)n, \text{ i.e. } a_i's \text{ are non-null. Hence the set } \{a_1, a_2, \ldots, a_n\} \text{ is linearly independent.}
\]

We know any set of \( n \) linearly independent vectors forms a basis for \( \mathbb{F}^n \). Hence, \( \{a_1, a_2, \ldots, a_n\} \) forms a basis of \( \mathbb{F}^n \).
Result: \( \text{If } \xi_1, \xi_2, \ldots, \xi_{n-1}, \xi \text{ be a set of mutually orthogonal } n\text{-component vectors} \), \( \text{if } \xi \text{ and } \xi' \text{ two non-null vectors such that each of } \xi \text{ and } \xi' \text{ is orthogonal to any other vector belonging to the given set, then } \xi \text{ must be a scalar multiple of } \xi' \).

\[ \text{Note that, } \{ \xi_1, \xi_2, \ldots, \xi_{n-1}, \xi \} \text{ is an orthogonal basis of } \mathbb{E}_n. \text{ Now, } y(\neq 0) \in \mathbb{E}_n \]

\[ \sum_{i=1}^{n-1} \theta_i \xi_i = y, \quad \text{at least one } \theta_i \neq 0 \]

\[ y = \sum_{i=1}^{n-1} \theta_i \xi_i + \theta_n \xi' \]

\[ \Rightarrow y, \xi = \sum_{i=1}^{n-1} \theta_i \xi_i, \xi + \theta_n \xi', \xi \]

\[ \Rightarrow \theta_n \xi', \xi = 0 \]

\[ \Rightarrow \theta_n = 0 \text{ as } \xi', \xi \neq 0 \]

\[ \therefore \theta_1 = 0 \]

\[ \text{Similarly, it can be shown that: } \theta_2 = \theta_3 = \cdots = \theta_{n-1} = 0. \]

Thus, \( y = \theta_n \xi \). \[ \theta_n \neq 0 \]

\[ \text{Orthogonal Basis: } \]

**Definition:** \( \parallel \) If we consider a set of \( n \) mutually orthogonal non-null vectors from \( \mathbb{E}_n \), then it forms a basis for \( \mathbb{E}_n \). Hence, a set of \( n \) mutually orthogonal non-null vectors from \( \mathbb{E}_n \) is known as an Orthogonal Basis for \( \mathbb{E}_n \).

**Orthonormal Basis:** \( \parallel \) If \( a_i \) is a non-null vector, then

\[ v_i = \frac{a_i}{||a_i||} \] is a vector of unit length.

For a set of \( a_1, a_2, \ldots \) any of mutually orthogonal non-null vectors from \( \mathbb{E}_n \), we obtain a set \( u_i = \frac{a_i}{||a_i||}, i = 1(n) \). Then,

\[ u_i \cdot u_j = \frac{a_i \cdot a_j}{||a_i|| ||a_j||} \]

The set \( a_1, a_2, \ldots \) of mutually orthogonal vectors of unit length and forms a basis for \( \mathbb{E}_n \). This type of set of vectors from \( \mathbb{E}_n \) is known as an orthonormal basis for \( \mathbb{E}_n \).
GIRAM - SCHMIDT ORTHOGONALISATION PROCESS.

By this process a set of independent vectors from $E^n$ can be converted to an orthonormal basis by a procedure known as the Schmidt orthogonalization process. Let us suppose that $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_n$ are $n$ linearly independent vectors from $E^n$. We select any vector from this set, say $\mathbf{g}_1$, i.e. $\mathbf{g}_1 \neq 0$, and define the vector of unit length $\mathbf{v}_1$ as

$$\mathbf{v}_1 = \frac{\mathbf{g}_1}{|\mathbf{g}_1|}.$$ 

To obtain a vector $\mathbf{v}_2$ orthogonal to $\mathbf{v}_1$, we subtract from $\mathbf{g}_2$ a scalar multiple of $\mathbf{v}_1$, i.e. $\mathbf{v}_2$ is expressed as

$$\mathbf{v}_2 = \mathbf{g}_2 - \lambda_1 \mathbf{v}_1,$$

where $\lambda_1$ is determined so that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.

This can be done since $|\mathbf{v}_2| \neq 0$. Because

$$\mathbf{v}_2 = 0$$

whenever

$$\mathbf{v}_2 = (\mathbf{v}_1 \cdot \mathbf{g}_2) \mathbf{v}_1 - (\mathbf{v}_2 \cdot \mathbf{g}_2) \mathbf{v}_1$$

and

$$\lambda_1 = \mathbf{v}_2 \cdot \mathbf{g}_2 - \lambda_1 = 0$$

therefore, $\mathbf{v}_2 = \mathbf{g}_2 - (\mathbf{v}_1 \cdot \mathbf{g}_2) \mathbf{v}_1$.

A second unit length vector orthogonal to $\mathbf{v}_1$ is defined by

$$\mathbf{v}_2 = \frac{\mathbf{v}_2}{|\mathbf{v}_2|}.$$ 

It is possible only when $\mathbf{g}_1 \neq 0$, but $\mathbf{g}_1 \neq 0$ as $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_n$ are mon-zero linear vectors, so, $|\mathbf{v}_2| \neq 0$.

To obtain a vector $\mathbf{v}_3$ orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, we write

$$\mathbf{v}_3 = \mathbf{g}_3 - (\mathbf{v}_1 \cdot \mathbf{g}_3) \mathbf{v}_1 - (\mathbf{v}_2 \cdot \mathbf{g}_3) \mathbf{v}_2$$

The third unit vector which is orthogonal to $\mathbf{v}_1, \mathbf{v}_2$ is

$$\mathbf{v}_3 = \frac{\mathbf{v}_3}{|\mathbf{v}_3|}.$$ 

This is also valid as $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ are linear giving $|\mathbf{v}_3| \neq 0$. 

This process can be repeated to obtain orthogonal vectors $\mathbf{v}_4, \mathbf{v}_5, \ldots, \mathbf{v}_n$. 

Thus, the set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthonormal basis for $E^n$. 

Scanned by CamScanner

Ctanujit Classes
This procedure is continued until an orthonormal basis is obtained. In general,

\[ y_n = x_n - \sum_{i=1}^{n-1} (y_i', x_n) y_i, \]

\[ u_n = \frac{y_n}{|y_n|}, \]

**Example:** Using the Schmidt process, construct an orthonormal basis \( \{ u_1, u_2, \ldots \} \) from \( \{ x_1 = [2, 3, 0], x_2 = [2, 1, 0], x_3 = [0, 2, 4] \} \).

**Solution:** Let scalars are \( \lambda_1, \lambda_2, \lambda_3 \) such that

\[ \lambda_1 \cdot x_1 + \lambda_2 \cdot x_2 + \lambda_3 \cdot x_3 = (0, 0, 0), \]

\[ \Rightarrow \lambda_1 (2, 3, 0) + \lambda_2 (2, 1, 0) + \lambda_3 (0, 2, 4) = (0, 0, 0) \]

\[ \Rightarrow 2\lambda_1 + 6\lambda_2 = 0; \quad 3\lambda_1 + \lambda_2 + 2\lambda_3 = 0; \quad 4\lambda_3 = 0 \]

\[ \Rightarrow \lambda_3 = 0 \quad \Rightarrow 3\lambda_1 + \lambda_2 = 0, \lambda_1 + 3\lambda_2 = 0 \]

\[ \Rightarrow \lambda_2 = -3\lambda_1; \quad \lambda_1 + 3(-3\lambda_1) = 0 \Rightarrow \lambda_1 - 9\lambda_1 = 0 \Rightarrow \lambda_1 = 0 \]

\[ \Rightarrow \lambda_2 = 0 \]

Hence the vectors are LI.

\[ u_1 = \frac{x_1}{|x_1|} = \frac{[2, 3, 0]}{\sqrt{13}} = [0.554, 0.831, 0]; \]

\[ u_2 = x_2 - (u_1', x_2) u_1; \]

\[ (u_1', x_2) = (0.554, 0.831, 0)' (2, 1, 0) = 4.16 \]

\[ (u_1', x_2) u_1 = [2.83, 3.45, 0]; \]

\[ u_2 = \frac{x_2}{|x_2|} = \frac{[3.73, -2.45, 0]}{\sqrt{13}} \]

\[ u_3 = x_3 - (u_1', x_3) u_1 - (u_2', x_3) u_2; \]

\[ (u_1, x_3) = (0.554, 0.831, 0) (0, 2, 4) = 1.064, \]

\[ u_2', x_3 = -1.106; \]

\[ (u_1', x_3) u_1 = [0.921, -3.36, 0], (u_2', x_3) u_2 = [-0.921, 0.814, 0]; \]

\[ u_3 = [0, 0, 1]. \]
Example 2: Construct an orthogonal basis for $\mathbb{R}^3$ with $(1,1,1)$ as the starting vector.

Solve:

\[ a_1 = (1,1,1) \]
\[ u_1 = \frac{(1,1,1)}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}} (1,1,1) \]

Note that $\{e_1, e_2, e_3\}$ forms a basis for $\mathbb{R}^3$.

Now, $(1,1,1) = 1 \cdot e_1 + 1 \cdot e_2 + 1 \cdot e_3$.

\[ \{ (1,1,1), e_2, e_3 \} \] forms a basis for $\mathbb{R}^3$, by replacement theorem.

Now, $x_1 = (1, 1, 1)$ and $u_1 = \frac{x_1}{|x_1|} = \frac{1}{\sqrt{3}} (1,1,1)$

\[ x_2 = a_2 - (a_2 \cdot x_1) u_1 = (2, -1, 0) - \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} (1,1,1) \]
\[ = \left(0, \frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \]
\[ u_2 = \frac{\left(0, \frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)}{\sqrt{\frac{2}{3}}} \]
\[ = \frac{1}{\sqrt{6}} (1, -2, 1) \]
\[ x_3 = a_3 - (a_3 \cdot x_1) u_1 - (a_3 \cdot u_2) u_2 \]
\[ = (2, -1, 0) - \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} (1,1,1) + \frac{2}{\sqrt{6}} \cdot \frac{1}{\sqrt{3}} (1, -2, 1) \]
\[ = (0, -1, 1) \]
\[ u_3 = \frac{x_3}{|x_3|} = \frac{1}{\sqrt{2}} (0, -1, 1) \]

Hence, $\{ (1,1,1), \frac{1}{\sqrt{3}} (1,1,1), \frac{1}{\sqrt{6}} (1, -2, 1), \frac{1}{\sqrt{2}} (0, -1, 1) \}$ is an orthogonal basis for $\mathbb{R}^3$.

Also, $\{ \frac{1}{\sqrt{3}} (1,1,1), \frac{1}{\sqrt{6}} (1, -2, 1), \frac{1}{\sqrt{2}} (0, -1, 1) \}$ is an orthonormal basis for $\mathbb{R}^3$. 


Example 3) Construct an orthonormal basis for $\mathbb{E}^4$ orthogonally.

Given $\frac{1}{2}(1, 1, 1, 1)$ as the starting vector.

**Solution:**

\[ u_1 = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \]

\[ \mathbf{e}_1 = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \]

Note that, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ forms a basis for $\mathbb{E}^4$.

Now, by the replacement theorem, $\left\{ \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \right\}$ is also a basis for $\mathbb{E}^4$.

Since $\left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) = \frac{1}{2} \mathbf{e}_1 + \frac{1}{2} \mathbf{e}_2 + \frac{1}{2} \mathbf{e}_3 + \frac{1}{2} \mathbf{e}_4$.

Now, we shall apply orthogonalization process to the set

$\left\{ \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \right\}$

\[ u_1 = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \]

\[ \mathbf{v}_2 = \mathbf{e}_2 - (\mathbf{e}_2 \cdot u_1) u_1 \]

\[ = \mathbf{e}_2 - (0, 1, 0, 0) \cdot \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \]

\[ = (0, 1, 0, 0) - \frac{1}{2} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \]

\[ = \left( -\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{1}{4} \right) \]

\[ \mathbf{u}_2 = \frac{\left( -\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{1}{4} \right)}{\sqrt{\frac{1}{16} + \frac{9}{16} + \frac{1}{16} + \frac{1}{16}}} \]

\[ = \frac{1}{2} \sqrt{\frac{1}{16}} \left( 1, -3, 1, 1 \right) \]

\[ = \frac{1}{2} \sqrt{\frac{1}{2}} \left( 1, -3, 1, 1 \right) \]

\[ \mathbf{v}_3 = \mathbf{e}_3 - (\mathbf{e}_3 \cdot \mathbf{u}_1) \mathbf{u}_1 - (\mathbf{e}_3 \cdot \mathbf{u}_2) \mathbf{u}_2 \]

\[ = (0, 0, 1, 0) - \frac{1}{2} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) - \frac{1}{2} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \left( 1, -3, 1, 1 \right) \]

\[ = \left( -\frac{1}{3}, 0, \frac{2}{3}, -\frac{1}{3} \right) \]

\[ \mathbf{u}_3 = \frac{\left( -\frac{1}{3}, 0, \frac{2}{3}, -\frac{1}{3} \right)}{\sqrt{\frac{1}{9} + \frac{4}{9} + \frac{1}{9}}} \]

\[ = \pm \frac{1}{2} \left( 1, 0, -2, 1 \right) \]
\[ \begin{align*}
\mathbf{U}_4 &= \mathbf{a}_4 - (\mathbf{u}_1 \cdot \mathbf{a}_4) \mathbf{u}_1 - (\mathbf{u}_2 \cdot \mathbf{a}_4) \mathbf{u}_2 - (\mathbf{u}_3 \cdot \mathbf{a}_4) \mathbf{u}_3 \\
&= (0,0,0,1) - \frac{1}{2} (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) - \frac{1}{16} (1, -3, 1, 1) \\
&= (\frac{1}{2}, 0, 0, \frac{1}{2}) \\
\mathbf{v}_4 &= \frac{(\frac{1}{2}, 0, 0, \frac{1}{2})}{\sqrt{\frac{1}{4} + \frac{1}{4}}} = -\frac{1}{2} \sqrt{2} (1, 0, 0, -1) \\
\end{align*} \]

Now, \( \left\{ \frac{1}{2} (1, 1, 1, 1), \frac{1}{2} (1, -3, 1, 1), \frac{1}{16} (1, 0, -2, 1), \right\} \) is an orthonormal basis for \( \mathbb{E}^4 \).

**Result:** Let \( \mathbf{a}_1, \mathbf{a}_2, \mathbf{b} \) be two \( \text{LIN} \) vectors and \( \mathbf{b} \neq \mathbf{0} \) is orthogonal to \( \mathbf{a}_1 \) and \( \mathbf{a}_2 \). Show that \( \mathbf{a}_1, \mathbf{a}_2, \mathbf{b} \) are \( \text{LIN} \).

**Proof:** Let \( S(\mathbf{a}_1, \mathbf{a}_2) \) and \( S(\mathbf{b}) \) be two subspaces spanned by \( \mathbf{a}_1, \mathbf{a}_2 \) and \( \mathbf{b} \) respectively. As \( \mathbf{b} \neq \mathbf{0} \) is orthogonal to \( \mathbf{a}_1 \) and \( \mathbf{a}_2 \) both, \( S(\mathbf{a}_1, \mathbf{a}_2) \) and \( S(\mathbf{b}) \) are mutually orthogonal subspaces, implying

\[ S(\mathbf{a}_1, \mathbf{a}_2) \cap S(\mathbf{b}) = \{ \mathbf{0} \} \]

To show \( \mathbf{a}_1, \mathbf{a}_2, \mathbf{b} \) are linearly independent, consider

\[ \lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \lambda \mathbf{b} = \mathbf{0} \]

\[ \Rightarrow \lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 = -\lambda \mathbf{b} \]

\[ \Rightarrow \left\{ \begin{aligned}
\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 &= 0 \text{ as } S(\mathbf{a}_1, \mathbf{a}_2) \cap S(\mathbf{b}) = \{ \mathbf{0} \} \\
\lambda \mathbf{b} &= \mathbf{0} \\
\Rightarrow \lambda_1 &= 0, \lambda_2 = 0 \text{ as } \mathbf{a}_1, \mathbf{a}_2 \text{ are LIN}. \\
\Rightarrow \lambda &= 0 \text{ as } \mathbf{b} \neq \mathbf{0}. \\
\Rightarrow \mathbf{a}_1, \mathbf{a}_2, \mathbf{b} \text{ are LIN}. 
\end{aligned} \right. \]
2) Obtain the maximum collection of linearly independent vectors using the following:

\[
\begin{align*}
\mathbf{x}_1 &= (1, 0, 0, 0, 0, 1), \quad \mathbf{x}_2 = (1, 0, 0, 0, 0, 1), \\
\mathbf{x}_3 &= (1, 0, 0, 0, 0, 1), \quad \mathbf{x}_4 = (1, 0, 0, 0, 0, 1), \\
\mathbf{x}_5 &= (1, 0, 0, 0, 0, 1), \quad \mathbf{x}_6 = (1, 0, 0, 0, 0, 1),
\end{align*}
\]

Also find a vector which is orthogonal to each of vectors belonging to the maximal collection.

**Solution:**

For any \( \lambda \in \mathbb{R} \),

\[
\mathbf{x}_2 = \lambda \mathbf{x}_1,
\]

\[
\begin{align*}
\mathbf{x}_3 &= \lambda \mathbf{x}_2, \\
\mathbf{x}_4 &= \lambda \mathbf{x}_3
\end{align*}
\]

which is not possible if all \( \lambda \)'s are non-zero, then \( \mathbf{x}_2 \) and \( \mathbf{x}_1 \) are LI.

If possible, let

\[
\mathbf{x}_5 = \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2
\]

\[
\Rightarrow \begin{cases} 
\lambda_1 + \lambda_2 = 1 \\
\lambda_1 + \lambda_2 = 0
\end{cases}
\]

which is not possible.

\[
\Rightarrow \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \text{ is LI.}
\]

If possible, let

\[
\mathbf{x}_4 = \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \lambda_3 \mathbf{x}_3
\]

\[
\Rightarrow \begin{cases} 
\lambda_1 + \lambda_2 + \lambda_3 = 1 \\
\lambda_1 + \lambda_2 = 1 \\
\lambda_2 + \lambda_3 = 0
\end{cases}
\]

which is not possible.

\[
\Rightarrow \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \text{ is LI.}
\]

If possible, let

\[
\mathbf{x}_5 = \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \lambda_3 \mathbf{x}_3 + \lambda_4 \mathbf{x}_4
\]

\[
\Rightarrow \begin{cases} 
\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\
\lambda_4 = 1 \\
\lambda_2 + \lambda_3 = 0 \\
\lambda_1 + \lambda_2 + \lambda_4 = 0
\end{cases}
\]

\[
\Rightarrow \begin{cases} 
\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 0, \lambda_4 = 1
\end{cases}
\]

i.e., \( \mathbf{x}_5 \) is LD of \( \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \).

If possible, let

\[
\mathbf{x}_6 = \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \lambda_3 \mathbf{x}_3 + \lambda_4 \mathbf{x}_4
\]

\[
\Rightarrow \begin{cases} 
\lambda_1 = 1 \\
\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\
\lambda_1 + \lambda_2 + \lambda_3 = 0 \\
\lambda_1 + \lambda_2 + \lambda_4 = 0
\end{cases}
\]

\[
\Rightarrow \begin{cases} 
\lambda_1 = -1, \lambda_2 = 0, \lambda_3 = 1, \lambda_4 = 1
\end{cases}
\]

Hence, \( \mathbf{x}_6 \) is LD of \( \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \).
Hence, a maximal collection of linearly independent vectors will be: \( \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \).

The choice of this 4 Lin vectors is not unique.

Let the vector be \( \mathbf{x} = \{ \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_6 \} \).

We are to find \( x_i \)’s \( \in \mathbb{R}^1 \)’s \( \in \mathbb{R}^1 \)

\[ x_i'x_i = 0 \quad \forall \ i \in \{1, 2, 3, 4\} \]

\[ \begin{cases} x_1 + x_2 + x_4 + x_5 = 0 \\ x_1 + x_2 + x_4 + x_6 = 0 \\ x_1 + x_3 + x_5 + x_6 = 0 \\ x_1 + x_2 + x_3 + x_5 = 0 \end{cases} \]

\[ \Rightarrow \begin{cases} x_2 = x_3 \\ x_4 = x_5 = x_6 \end{cases} \]

\[ \mathbf{x} = \begin{pmatrix} -1 & -2 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix} \]

Choose, \( \alpha = 1, \beta = 1 \), then \( \mathbf{x} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \).

Consequently, this particular problem we have exactly 2 Lin vectors each of which is orthogonal to \( \mathbf{x}_i' \), \( i \in \{1, 2, 3, 4\} \).

A choice of such two vectors (Lin) are: \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \), \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

**Orthogonal Spaces:** A vector \( \mathbf{a} \) is orthogonal to every vector in \( \mathbf{v} \), i.e., if \( \mathbf{a}' \mathbf{v} = 0 \), \( \forall \mathbf{v} \in \mathbf{v} \).

So, two subspaces \( \mathbf{v}_1, \mathbf{v}_2 \) of \( \mathbb{E}^n \) are said to be orthogonal if \( \mathbf{a}' \mathbf{v} = 0 \), \( \forall \mathbf{a} \in \mathbf{v}_1 \) and \( \forall \mathbf{v} \in \mathbf{v}_2 \);

i.e., if any vector, \( \mathbf{a} \in \mathbf{v}_1 \), is orthogonal to \( \mathbf{v}_2 \), i.e., if

\[ \mathbf{a}' \mathbf{v} = 0 \]
1. If \( U = L \{(1, 2, 1), (2, 1, 2)\} \) and \( W = L \{(1, 0, 0), (0, 0, 1)\} \), show that
\( U \) and \( W \) are subspaces of \( \mathbb{R}^3 \). Determine \( \dim U, \dim W, \dim (U+W) \), deduce that \( \dim (U+W) = 3 \).

**Solution:**

Let \( \alpha = (1, 2, 1), \beta = (2, 1, 2), \gamma = (1, 0, 0), \delta = (0, 0, 1) \),
\( \{\alpha, \beta\} \) is linearly independent and therefore \( U \) is a subspace of \( \mathbb{R}^3 \) of dimension 2,
\( \{\gamma, \delta\} \) is linearly independent and therefore \( W \) is a subspace of \( \mathbb{R}^3 \) of dimension 2.

Let \( \lambda \) be a vector in \( U \cap W \), then \( \lambda = a \alpha + b \beta \) for some real \( a \) & \( b \).

Therefore, \( a(1, 2, 1) + b(2, 1, 2) = c(1, 0, 0) + d(0, 0, 1) \)
\[ \Rightarrow a + 2b = c, \quad 2a + b = 0, \quad a + 3b = d. \]

So, \( a = -\frac{1}{3}, \quad b = \frac{2}{3}, \quad c = \frac{3}{2}, \quad d = \frac{1}{2} \).

Thus, \( \lambda = \left(\frac{1}{2}, 0, \frac{1}{2}\right) \) is a vector in \( U \cap W \).

Hence \( U \cap W \) is a subspace of \( \mathbb{R}^3 \) with the dimension 1.

So, \( \dim (U+W) = \dim U + \dim W - \dim (U \cap W) = 2 + 2 - 1 = 3 \).

2. Find a basis and the dimension of the subspace \( W \) of \( \mathbb{R}^3 \), where
\[ W = \{(x, y, z) \in \mathbb{R}^3 : x + y = 0\}. \]

**Sol.**

Let \( \vec{e} = (1, 0, 0) \) and \( \vec{e} = (0, 1, 0) \) be in \( W \).
They are linearly independent in \( W = L\{(1, 0, 0), (0, 1, 0)\} \).

So, \( \{(1, 0, 0), (0, 1, 0)\} \) is a basis of \( W \) and \( \dim W = 2 \).

3. Find a basis and the dimension of the subspace \( W \) of \( \mathbb{R}^3 \), where
\[ W = \{(x, y, z) \in \mathbb{R}^3 : x + 2y + 2z = 0, 2x + y + z = 0\}. \]

**Sol.**

Let \( \vec{e} = (a, b, c) \) be in \( W \),
\[ a + 2b + c = 0, \quad 2a + b + 3c = 0; \quad a, b, c \in \mathbb{R}. \]

Solving, we have \( \frac{a}{5} = \frac{b}{-1} = \frac{c}{-3} = k \).

Thus, \( \vec{e} \) takes the form \( k(\frac{1}{5}, -1, -3) \), where \( k \) is an arbitrary real.

Therefore, \( W = L\{\vec{e}\} \), where \( \vec{e} = (5, -1, -3) \).
Since \( \vec{e} \) is a linearly independent set, \( \vec{e} \) is a basis of \( W \) and \( \dim W = 1 \).
1. Extend the set of vectors $s \{(2,3,-1), (1,-2,4)\}$ to an orthonormal basis of the Euclidean space $\mathbb{R}^3$ with standard inner product and then find the associated orthonormal basis.

Solution: Let $x_1 = (2,3,-1), x_2 = (1,-2,4)$. Let $x_3 = (0,0,1)$. Then $\{x_1, x_2, x_3\}$ is linearly independent because

\[
\begin{vmatrix}
2 & 3 & -1 \\
1 & -2 & 4 \\
0 & 0 & 1
\end{vmatrix} \neq 0.
\]

So, $\{x_1, x_2, x_3\}$ is a basis of $\mathbb{R}^3$.

Let $\beta = x_3 - c_1 x_1 - c_2 x_2$, where $c_1 = \frac{\langle x_3, x_1 \rangle}{\langle x_1, x_1 \rangle}, c_2 = \frac{\langle x_3, x_2 \rangle}{\langle x_2, x_2 \rangle}$.

Then $\beta$ is orthogonal to $x_1$ and $x_2$ and $L\{x_1, x_2, x_3\} = L\{x_1, x_2, \beta\}$.

Therefore, $\{x_1, x_2, \beta\}$ is an orthonormal basis of $\mathbb{R}^3$.

$c_1 = \frac{1}{6}, c_2 = \frac{1}{2}$, and therefore

$\beta = \frac{1}{6} (2,3,-1) + \frac{1}{2} (1,-2,4) = \left(\frac{1}{3}, -\frac{1}{6}, \frac{1}{2}\right)$.

Hence an extended orthonormal basis is $\{\frac{1}{3}, -\frac{1}{6}, \frac{1}{2}\}$.

and the associated orthonormal basis is $\left\{\frac{1}{\sqrt{14}} (2,3,-1), \frac{1}{\sqrt{21}} (1,-2,4), \frac{1}{\sqrt{6}} (2,3,-1)\right\}$.

5. Use Gram-Schmidt process to obtain an orthonormal basis from the basis set $\{x_1, x_2, x_3\}$ of the Euclidean space $\mathbb{R}^3$ with standard inner product.

Solution: Let $x_1 = (1,0,1), x_2 = (1,1,1), x_3 = (1,3,4)$.

Let $\beta_1 = x_1 - \alpha_1 x_2$, where $\alpha_1$ is the scalar component of $x_2$ along $\beta_1$.

Then $\beta_2$ is orthogonal to $\beta_1$ and

$L\{\beta_1, \beta_2\} = L\{\beta_1, x_2\} = L\{x_1, x_2\}$.

$\alpha_1 = \frac{\langle x_2, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} = 1$. Therefore, $\beta_2 = x_2 - \beta_1 = (0,1,0)$.

Let $\beta_3 = x_3 - \alpha_1 x_1 - \alpha_2 x_2$, where $\alpha_1, \alpha_2$ are scalar components of $x_3$ along $\beta_1, \beta_2$ respectively.

Then $\beta_3$ is orthogonal to $\beta_1, \beta_2$ and $L\{\beta_1, \beta_2, \beta_3\} = L\{x_1, x_2, x_3\}$.

$d_1 = \frac{\langle x_3, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} = \frac{\langle x_3, x_1 \rangle}{\langle x_1, x_1 \rangle} = \frac{5}{2}, d_2 = \frac{\langle x_3, \beta_2 \rangle}{\langle \beta_2, \beta_2 \rangle} = \frac{2}{1} = 2$.

Therefore, $\beta_3 = (1,3,4) - \frac{5}{2} (1,0,1) - 2 (0,1,0) = \frac{3}{2} (-1,0,1)$.

Therefore, an orthonormal basis is

$\left\{\frac{1}{\sqrt{14}}, \frac{1}{\sqrt{21}}, \frac{1}{\sqrt{6}}\right\}$.

END

\[\Box\]
**MATRIX AND DETERMINANTS**

A **matrix** of order \( m \times n \) is a rectangular array of \( mn \) elements in \( m \) rows and \( n \) columns. It is usually denoted by an upper case letter.

If \( a_{ij} \) be the element in the \( i^{th} \) row and \( j^{th} \) column of the matrix \( A \), we write:

\[
A = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\]

\[
= (\begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n
\end{bmatrix}, \begin{bmatrix}
    a_{ij}
\end{bmatrix})_{mx1}
\]

\[
= (\begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_m
\end{bmatrix}, \begin{bmatrix}
    a_{ij}
\end{bmatrix})_{nx1}
\]

\[
= (((a_{ij}))_{i=1(1)m}, i)
\]

\[
\beta_i = (a_{1i}, a_{2i}, \ldots, a_{ni})
\]

**Square Matrix:** A matrix is said to be a **square matrix** if no of rows of the matrix is equal to the no of columns.

A square matrix \( A \) of \( m \times n \) is denoted by \( A^n \) as \( m = n \), so \( A^n = ((a_{ij})) \)

Then \( a_{ii} \)'s are called diagonal elements of matrix \( A^n \) and \( a_{ij} \)'s are called off diagonal elements of \( A^n \).

**Diagonal Matrix:** If all the off diagonal matrix elements of a square matrix vanish then it reduces to a diagonal matrix.

A diagonal matrix of order \( n \times n \) consisting of elements \((\lambda_1, \lambda_2, \ldots, \lambda_n)\) is denoted by \( \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \).

**Scalar Matrix:** If all diagonal elements of a diagonal matrix become equal then it reduces to a scalar matrix.

\[
A = \text{diag}(\lambda, \lambda, \ldots, \lambda)
\]

If \( \lambda = 1 \), then \( A \) is said to be an identity matrix.

\[
I_n = ((\delta_{ij})), \text{ } \delta_{ij} = \text{Kronecker's 'δ'}
\]

**Triangular Matrix:** Let \( A \) be a square matrix of order \( n \) and \( A = (a_{ij}) \)

Then \( A \) is said to be upper triangular matrix if \( a_{ij} = 0, \forall i > j \).

And \( A \) is lower triangular matrix if \( a_{ij} = 0, \forall i < j \).

**Null Matrix:** A matrix is said to be null if all of its elements vanish, denoted by \( O_{m \times n} \).

**Sum Matrix:** A matrix is said to be a sum matrix if all of its elements become unity.
Matrix Addition: Two matrices are said to be conformable for addition if they are of same order.

Let, \( A = (a_{ij})_{mxn} \) and \( B = (b_{ij})_{mxn} \)

Then \( A + B = (a_{ij} + b_{ij})_{mxn} \).

Matrix addition is commutative and associative:
\( A + B = B + A \).
\((A + B) + C = A + (B + C)\).

Multiplication by a Scalar:
Let \( A = (a_{ij})_{mxn} \), \( \alpha = \text{Scalar} \)

\( \alpha A = (\alpha a_{ij})_{mxn} \)

Combining the above two operations, we have:
\( \alpha A + \beta B = (\alpha a_{ij} + \beta b_{ij}) \)
\( A = (a_{ij})_{mxn} \), \( B = (b_{ij})_{mxn} \)

If we choose \( \alpha = 1 \) and \( \beta = -1 \), we get matrix subtraction.

Matrix Multiplication: Two matrices \( A \) and \( B \) are said to be conformable for matrix multiplication if the number of columns of \( A \) equals the number of rows of \( B \).

Let \( A = (a_{ij})_{mxn} \) and \( B = (b_{ij})_{nxk} \)

Then \( AB \) is defined as:
\( AB = (\sum_{k=1}^{n} a_{ik}b_{kj})_{mxk} \)

If \( AB \) is defined, \( BA \) may not be so and if \( AB \) and \( BA \) are both defined then they may not be equal.

Idempotent Matrix: A is said to be an idempotent matrix if
\( A^2 = A \).

Now,
\[
\left( \begin{array}{c}
I_n - \frac{1}{n} 1_n 1_n^T
\end{array} \right) \left( \begin{array}{c}
I_n - \frac{1}{n} 1_n 1_n^T
\end{array} \right)
= \left( I_n - \frac{1}{n} 1_n 1_n^T \right) \left( I_n - \frac{1}{n} 1_n 1_n^T \right)
= \left( I_n - \frac{1}{n} 1_n 1_n^T \right).
\]

Trace over Matrix:
Trace over a matrix defined as the sum of the diagonal elements of \( A \) and denoted by \( \text{tr}(A) \).

Properties:
\( \text{tr}(A + B) = \text{tr}(A) + \text{tr}(B) \)
\( \text{tr}(AB) = \text{tr}(BA) \)
\( \text{tr}(I_n - \frac{1}{n} 1_n 1_n^T) = \text{tr}(I_n) - \frac{1}{n} \text{tr}(1_n 1_n^T) \)
\( = n - \frac{1}{n} \text{xn} \)
\( = n - 1. \)
Transpose of a Matrix:- The transpose of a matrix $A$ is denoted by $A'$ and it is defined as a matrix obtained by placing rows of $A$ as columns and columns of $A$ as rows.

Properties:-

$$ (A')' = A', \quad (A + B)' = A' + B' $$

Symmetric and Skew-symmetric Matrix:- A square matrix $A$ is said to be symmetric if $A' = A$ and will be skew-symmetric if $A' = -A$.

$$ A' = A \iff a_{ij} = a_{ji} \quad \forall i, j $$

$$ A' = -A \iff a_{ij} = -a_{ji} \quad \forall i, j $$

Here, $AA'$, $A'A$ and $(A \pm A')$ are also symmetric matrices. (Check)

Note :- Any square matrix can uniquely be written as a sum of symmetric and skew-symmetric matrices.

$$ A = \frac{1}{2} (A + A') + \frac{1}{2} (A - A') $$

Determinants:-

$$ A = \begin{vmatrix}
   a_{11} & a_{12} & a_{13} \\
   a_{21} & a_{22} & a_{23} \\
   a_{31} & a_{32} & a_{33}
\end{vmatrix} = a_{11} (a_{22}a_{33} - a_{23}a_{32}) - a_{12} (a_{21}a_{33} - a_{23}a_{31}) + a_{13} (a_{21}a_{32} - a_{22}a_{31}) $$

$$ = \sum_{1 \leq i \neq j \neq k \leq 3} (-1)^{i+j+k} a_{1i} a_{2j} a_{3k} $$

$$ = \sum_{1 \leq i, j \leq 3, i \neq j} (-1)^{i+j} a_{1i} a_{2j} a_{3i} $$

$$ = \left( \prod_{i=1}^{n} a_{ii} \right)^{n-1} \cdot \left( \prod_{1 \leq i < j \leq n} a_{ij} \right) $$

$$ \because \quad \det (A) = \prod_{1 \leq i < j \leq n} a_{ij} $$

where $N(i, \ldots, in) =$ No. of inversions in $(i, \ldots, in)$. Summation is taken over all possible permutations of $(1, 2, \ldots, n)$, i.e. no. of terms under the summation is $n!$.
Properties 1. \( \text{det}(A) = \text{det}(A^T) \)

**Sol.** Let \( A = ([a_{ij}])_{n \times n} \) and \( B = ([b_{ij}])_{n \times n}, \) then \( B = A^T. \)

Clearly, \( a_{ij} = b_{ji} \) \( \forall \ (i,j) \)

\[ |A|^1 = |B| = \sum_{1 \leq i_1, i_2, \ldots, i_n \leq n} (-1)^{N(i_1, i_2, \ldots, i_n)} a_{i_11} a_{i_22} \ldots a_{i_n n} = |A|. \]

\[ \text{det}(\lambda A) = \lambda^n \text{det}(A), \text{ where } A \text{ is a } n \times n \text{ matrix.} \]

**Sol.** Let \( A = ([a_{ij}])_{m \times n} \) and \( B = ([b_{ij}])_{n \times m} \)

Let \( A = ([a_{ij}])_{m \times n}, \) and \( B = ([b_{ij}])_{n \times m} \)

Clearly, \( a_{ij} + \sum_{k \neq 1} \lambda_k a_{kj} = b_{ij} + \sum_{k \neq 1} \lambda_k a_{kj} \)

\[ |B| = \sum_{1 \leq i_1, i_2, \ldots, i_m \leq m} (-1)^{N(i_1, i_2, \ldots, i_m)} b_{i_11} b_{i_22} \ldots b_{i_m m} \]

\[ = \sum_{1 \leq i_1, i_2, \ldots, i_m \leq m} (-1)^{N(i_1, i_2, \ldots, i_m)} (a_{i_11} + \sum_{k \neq 1} \lambda_k a_{i_1 k}) a_{i_22} \ldots a_{i_m m} + 0 = |A|. \]

**P.2.** Let \( A = \left( \begin{array}{c} x_1' \\ x_2' \\ \vdots \\ x_m' \end{array} \right) = ([a_{ij}])_{m \times n} \) and \( \beta = \left( \begin{array}{c} x_1 + \lambda x_2 \\ x_2 + \lambda x_3 \\ \vdots \\ x_m + \lambda x_1 \end{array} \right) \) then \( |B| = |A|. \)

**Sol.** Let \( B = ([b_{ij}])_{m \times n} \)

\[ b_{ij} = a_{ij} + \sum_{k \neq 1} \lambda_k a_{kj}, \] \( b_{ij} = a_{ij} + \sum_{k \neq 1} \lambda_k a_{kj} \)

\[ |B| = \sum_{1 \leq i_1, i_2, \ldots, i_m \leq m} (-1)^{N(i_1, i_2, \ldots, i_m)} b_{i_11} b_{i_22} \ldots b_{i_m m} \]

\[ = \sum_{1 \leq i_1, i_2, \ldots, i_m \leq m} (-1)^{N(i_1, i_2, \ldots, i_m)} a_{i_11} a_{i_22} \ldots a_{i_m m} + 0 = |A|. \]

**P.3.** \( |AB| = |A||B| \)

Let \( A = ([a_{ij}])_{m \times n} \) \( A \beta = ([\sum a_{ik} b_{kj}])_{n \times n} \)

\[ |AB| = \sum_{1 \leq i_1, i_2, \ldots, i_n \leq n} (-1)^{N(i_1, i_2, \ldots, i_n)} \sum_{k=1}^n a_{i_1 k} b_{k_1 1} a_{i_2 k} b_{k_2 2} \ldots a_{i_n k} b_{k_n n} \]

\[ = \sum_{1 \leq k_1, k_2, \ldots, k_n \leq n} (-1)^{N(k_1, k_2, \ldots, k_n)} b_{k_1 i_1} b_{k_2 i_2} \ldots b_{k_n i_n} \]

Rewrite \( B \) as \( B = B(k_1, x_2, \ldots, x_n) \)

\( B(k_1, x_2, \ldots, x_n) \) is matrix obtained from \( B \) replacing the \( \text{1st} \) row by \( k_1 \text{th} \) row and \( 2nd \) row by \( k_2 \text{th} \) row and so on, \( \forall i = 1, \ldots, n. \)
\[ |AB| = \sum_{1 \leq k_1, k_2, \ldots, k_n \leq n} a_{1k_1} a_{2k_2} \ldots a_{nk_n} |B(k_1, \ldots, k_n)| \]

\[ = \sum_{1 \leq k_1 \neq k_2 \neq \ldots \neq k_n \leq n} (-1)^{N(k_1, \ldots, k_n)} |B(1, 2, \ldots, n)| a_{1k_1} a_{2k_2} \ldots a_{nk_n}, \text{ if at least two } k_i's \text{ are equal then } |B(k_1, \ldots, k_n)| \text{ vanishes.} \]

\[ = |A| |B|. \]

**Minors, Cofactors, Inverse:**

Let \( A \) be a square matrix of order \( n \times n \), \( A = (a_{ij}) \). Then, the minor of \( a_{ij} \) is the determinant of the matrix obtained from \( A \) by omitting the \( i \)th row and \( j \)th column.

Cofactor of \( a_{ij} \) is \( A_{ij} \), say

\[ = (-1)^{i+j} \times \text{minor of } a_{ij}. \]

Adjoint of \( A \) is \( A^T \) on \( A^T(A) \)

\[ = (A_i)^T. \]

Now, \( A = (a_{ij})_{n \times n} \), \( A_{ij} = \text{cofactor of } a_{ij} \)

\[ \sum a_{ij} A_{ij} = \begin{cases} |A| & \text{if } i = i' \\
0 & \text{if } i \neq i', V j \end{cases} \]

\[ \text{or, } \sum a_{ij} A_{ij} = \begin{cases} |A| & \text{if } j = j' \\
0 & \text{if } j \neq j', V i \end{cases} \]

**Result:**

\[ A = (a_{ij})_{n \times n}, A_{ij} = \text{cofactor of } a_{ij}, A^T = (A_{ij}) \]

Then \( A^T A = A^2 = |A|I_n \),

**Proof:**

\[ A^T A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} \\
\end{pmatrix}
\begin{pmatrix}
a_{11} & a_{21} & \cdots & a_{n1} \\
a_{12} & a_{22} & \cdots & a_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1n} & a_{2n} & \cdots & a_{nn} \\
\end{pmatrix}
\]

\[ = \begin{pmatrix}
\sum a_{1k} a_{1k} & \sum a_{2k} a_{2k} & \cdots & \sum a_{nk} a_{nk} \\
\sum a_{1k} a_{2k} & \sum a_{2k} a_{3k} & \cdots & \sum a_{nk} a_{nk} \\
\vdots & \vdots & \ddots & \vdots \\
\sum a_{1k} a_{nk} & \sum a_{2k} a_{nk} & \cdots & \sum a_{nk} a_{nk} \\
\end{pmatrix}
\]

\[ = \text{diag} \left( |A|, |A|, \ldots, |A| \right) = |A|I_n. \]
Singular and Non-Singular Matrices:

A matrix (square) is said to be a non-singular matrix if \( |A| \neq 0 \), otherwise, it is said to be singular.

Let us assume that \( A \) is non-singular, i.e., \( |A| \neq 0 \).

\[
A \left( \frac{1}{|A|}, A^* \right) = \left( \frac{1}{|A|}, A^* \right) A = I_n \quad \text{[See P.T.O.]} \\
\Rightarrow \quad A \left( \frac{1}{|A|}, A^* \right) = \left( \frac{1}{|A|}, A^* \right) \cdot \frac{1}{|A|} = \frac{1}{|A|} \cdot A^* = I_n \quad \text{[\( |A| \neq 0 \)]}
\]

[Analogous to the feature of the real numbers (non-zero), \( |x|^{-1} = \frac{1}{x} \).

Here, \( B \) is the inverse matrix of \( A \) and usually denoted by \( A^{-1} \).

\( \text{Then } \quad AA^{-1} = A^{-1}A = I. \)

Result: Inverse of a square matrix \( A \) exists if and only if \( A \) is non-singular.

Proof: 

If Part: Let \( A \) be a non-singular matrix, \( |A| \neq 0 \).

\[
\Rightarrow \quad AA^* = A^*A = |A|I \\
\Rightarrow \quad A \left( \frac{1}{|A|}, A^* \right) = \left( \frac{1}{|A|}, A^* \right) \cdot \frac{1}{|A|} = \frac{1}{|A|} \cdot A^* = I_n \quad \text{[\( |A| \neq 0 \)]}
\]

Only If Part: Let \( A^{-1} \) exists.

\[
\Rightarrow \quad AA^{-1} = A^{-1}A = I \\
\Rightarrow \quad |AA^{-1}| = 1 \\
\Rightarrow \quad |A| \cdot |A^{-1}| = 1 \\
\Rightarrow \quad |A| \neq 0.
\]

Properties: Inverse of a square matrix is unique.

\[
\Rightarrow \quad \text{If possible let } \quad B \quad \text{and} \quad C \quad \text{be the inverse of } \quad A. \quad \text{Then}
\]

\[
AB = BA = I \quad \text{(Pre-multiply)} \quad \text{[1]} \\
AC = CA = I \quad \text{(Post-multiply)} \quad \text{[2]}
\]

\[
\begin{align*}
CAB &= CBA = C \\
ACB &= CAB = B \\
\Rightarrow \quad B &= C.
\end{align*}
\]

\[\text{If-2:} \quad (AB)^{-1} = B^{-1}A^{-1} \]

\[
\begin{align*}
(AB)^{-1} &= B^{-1}A^{-1} \\
(AB)^{-1} &= A(B^{-1})A^{-1} = AA^{-1} = I \\
(AB)^{-1}AB & = B^{-1}A^{-1} = (AB)^{-1} \\
\Rightarrow \quad I & = B^{-1}A^{-1} = (AB)^{-1} \\
\Rightarrow \quad (AB)^{-1} & = B^{-1}A^{-1}.
\end{align*}
\]
Result: Let $A_{p\times p}$ be a non-singular matrix and $B_{p\times q}$ and $C_{q\times p}$ such that $A + BC$ is also non-singular. Then,

$$[A_{p\times p} + B_{p\times q}C_{q\times p}]^{-1} = A^{-1} - A^{-1}B(I_p + CA^{-1}B)CA^{-1},$$

when the inverses exist.

Proof:

$$\Delta = A + BC$$

$$\Delta A^{-1} = (A + BC)A^{-1}$$

$$= AA^{-1} + BCA^{-1}$$

$$= I_p + BCA^{-1}$$

$$\Delta A^{-1}B = (I_p + BCA^{-1})B$$

$$= B + BCA^{-1}B$$

$$= B(I_q + CA^{-1}B)$$

$$\Rightarrow \Delta A^{-1}B(I_q + CA^{-1}B)^{-1} = B$$

$$\Rightarrow \Delta A^{-1}B(I_q + CA^{-1}B)^{-1}C = BC$$

$$\Rightarrow A + \Delta A^{-1}B(I_q + CA^{-1}B)^{-1}C = A + BC = \Delta$$

$$\Rightarrow \Delta (I_p - A^{-1}B(I_q + CA^{-1}B)^{-1}C) = A$$

$$\Rightarrow \Delta (I_p - A^{-1}B(I_q + CA^{-1}B)^{-1}C)A^{-1} = I_p$$

$$\Rightarrow \Delta^{-1}D \left( A^{-1} - A^{-1}B(I_q + CA^{-1}B)^{-1}CA^{-1} \right) = \Delta^{-1}I_p$$

$$\Rightarrow \Delta^{-1} = (A + BC)^{-1} = A^{-1} - A^{-1}B(I_q + CA^{-1}B)^{-1}CA^{-1}.$$
Rank of a Matrix:

\[ V_c(A) : \text{Column space of } A \text{ (vector space generated by the columns of } A). \]

\[ V_r(A) : \text{Row space of } A. \]

- \[ \text{Column rank of } A \text{ is defined as } \dim [V_c(A)] \text{ on number of columns of } A. \]
- \[ \text{Row rank of } A = \text{Column rank of } A = R(A) \leq \min(m,n), \text{ where } A \in \mathbb{R}^{m \times n}. \]

**Definition:** Rank of the matrix \( A \) is the order of the highest order non-vanishing minor of \( A \).

**Some Useful Results:**

1. \[ \text{rank}(AB) \leq \min \left[ \text{rank}(A), \text{rank}(B) \right] \]

**Proof:** Let \( A \) be a matrix of order \( m \times n \) and \( B \) be a matrix of order \( n \times p \).

Suppose \[ A = \left( \begin{array}{c} a_{11} \ a_{12} \ \cdots \ a_{1n} \\ a_{21} \ a_{22} \ \cdots \ a_{2n} \\ \vdots \ \vdots \ \cdots \ \vdots \\ a_{m1} \ a_{m2} \ \cdots \ a_{mn} \end{array} \right) \]

\[ B = \left( \begin{array}{c} b_{11} \ b_{12} \ \cdots \ b_{1p} \\ b_{21} \ b_{22} \ \cdots \ b_{2p} \\ \vdots \ \vdots \ \cdots \ \vdots \\ b_{n1} \ b_{n2} \ \cdots \ b_{np} \end{array} \right) \]

\[ AB = \left( \begin{array}{c} a_{11} \ b_{11} \ a_{12} \ b_{12} \ \cdots \ a_{1n} \ b_{1n} \\ a_{21} \ b_{21} \ a_{22} \ b_{22} \ \cdots \ a_{2n} \ b_{2n} \\ \vdots \ \vdots \ \cdots \ \vdots \\ a_{m1} \ b_{n1} \ a_{m2} \ b_{n2} \ \cdots \ a_{mn} \ b_{np} \end{array} \right) \]

Columns of \( AB \) are linear combinations of columns of \( A \).

\[ V_c(AB) \leq V_c(A) \]

\[ \Rightarrow \dim V_c(AB) \leq \dim V_c(A) \]

\[ \Rightarrow \text{rank}(AB) \leq \text{rank}(A) \quad \text{ \ldots (i)} \]

Similarly, we can show that \( \text{rank}(AB) \leq \text{rank}(B) \quad \text{ \ldots (ii)} \)

Combining (i) & (ii) we have, \( \text{rank}(AB) \leq \min \left[ \text{rank}(A), \text{rank}(B) \right] \).
2. If \( A \) be an idempotent matrix then \( \text{Rank} (A) = \text{Trace}(A) \).

**Proof:** Let \( A \) be a matrix of order \( nxn \), such that \( A^2 = A \) and \( \text{rank} (A) = n \).

By rank-factorisation theorem, we have

\[
A = B n x n C n x n \text{ where } \text{rank} (B) = \text{rank} (C) = n.
\]

\[
A^2 = A
\]

\[
\Rightarrow B C B C = B C
\]

\[
\Rightarrow B' B C B C C' = B' B C C'
\]

We know, \( R(B'B) = \text{Rank}(B) = n \), because \( B'B \) is of order \( p \).

\( \therefore B'B \) is non-singular \( \Rightarrow (B'B)^{-1} \) exists.

Similarly, \( (CC')^{-1} \) exists.

\[
(B'B)^{-1} B' B C C' (CC')^{-1}
\]

\[
= (B'B)^{-1} B' B C C' (CC')^{-1}
\]

\[
\Rightarrow CB = In
\]

\( \therefore \text{Trace} (CB) = \text{trace}(BC) = \text{trace}(A) = \text{trace}(In) = n = \text{rank}(A) \).

3. Rank of a matrix remain unaltered if it is pre-multiplied or post-multiplied by a non-singular matrix, \( R(AB) = R(ABQ) = R(ABQ) = R(A) \), where \( A_{mxn}, B_{nxn} \) are non-singular matrices.

**Proof:**

\[ R(AB) \leq R(A) \quad \ldots \quad (1) \]

Again, \( R(A) = R(\text{Im} A) \)

\[ = R(P^{-1}PA) \left[ P^{-1} \text{ exists, since } P \text{ is n.s.} \right] \]

\[ \leq R(AB) \quad \ldots \quad (2) \]

\( (1) \& (2) \) gives \( R(A) = R(AB) \).
Null Space & Nullity:

Let \( A \) be a matrix of order \( m \times n \).

Define, \( N(A) = \{ x \mid Ax = 0 \} \)

\( N(A) \) constitutes a vector space.

Suppose, \( A = \begin{pmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{pmatrix} \)

\( Ax = 0 \Rightarrow \begin{pmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{pmatrix} x = 0 \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \)

\( \Rightarrow \) i.e. \( x \) is orthogonal to any row of \( A \).

\( \Rightarrow \) \( N(A) \) and \( V_R(A) \) are orthogonal spaces.

i.e. \( N(A) \perp V_R(A) \).

Thus, \( R(A) = n \).

i.e., \( \dim V_R(A) = n \)

Then, \( \dim N(A) = n - r \).

\[ N(A) + V_R(A) = \mathbb{R}^n \quad \& \quad N(A) \cap V_R(A) = \{ 0 \} \]

Here \( N(A) \) is termed as null space of the matrix \( A \) and \( \dim(N(A)) \) is the nullity of \( A \).

Result: Let \( \dim \left[ N(A)_{mxn} \right] = n - R(A) \).

Proof: Let \( \{ x_1, x_2, \ldots, x_r \} \) be a basis of \( V_R(A) \)

and \( \{ y_1, y_2, \ldots, y_t \} \) be a basis of \( N(A) \).

By definition of \( N(A) \), we have,

\( x_j y_i = 0 \quad \forall \ i, j \).

If \( \{ x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_t \} \) be a set of \( \text{lin vectors} \)

n-component vectors can't contain more than \( n \) vectors.

\( \Rightarrow t + r = n \)

If possible, let \( t + r < n \).
at least one vector which is orthogonal to any one of

\[ \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_2, \mathbf{y}_1, \ldots, \mathbf{y}_t \]

Let \( \mathbf{z} \) be such a vector \( \mathbf{z}^\top \mathbf{x} = 0 \) \( \forall \mathbf{x} = \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_2, \mathbf{y}_1, \ldots, \mathbf{y}_t \).

\[ \Rightarrow \mathbf{z} \in N(A) \]
\[ \Rightarrow \mathbf{z} = \sum_{i=1}^{t} \theta_i \mathbf{y}_i \] for some \( \theta_i \).

\[ \mathbf{y}^\top \mathbf{y} = 0 \] \( \forall \theta_i = 0 \)
\[ \mathbf{x}^\top \mathbf{x} = 0 \] \( \forall \mathbf{x} \in \mathbf{R}^{m} \).
\[ \dim (Y(A)) + \dim (N(A)) = n 
\]
\[ \dim (N(A)) = n - \dim (R(A)) \]

Result: \( R(AA^\top) = R(A) = R(A^\top) = R(A^\top A) \).

Proof - Let \( \mathbf{z} \in N(A^\top A) \)
\[ \Rightarrow A^\top A \mathbf{z} = 0 \]
\[ \Rightarrow A \mathbf{z} = 0 \]
\[ \Rightarrow \mathbf{A} \mathbf{x} = 0 \]
\[ \Rightarrow \mathbf{y} = 0 \] \( \forall \mathbf{y} \).

\[ \mathbf{A} \mathbf{x} = 0 \] \( \Rightarrow \mathbf{z} = \begin{pmatrix} \mathbf{y}^\top \\ 0 \\ \vdots \\ 0 \end{pmatrix} \), assuming \( \mathbf{A} \) has order \( m \times n \).

\[ \Rightarrow \mathbf{z} \in N(A) \]
\[ N(A^\top A) \subseteq N(A) \] \( \cdots \) \( \cdots \) \( \cdots \) \( \cdots \) \( \cdots \)

List \( \mathbf{z} \in N(A) \)
\[ \Rightarrow A \mathbf{z} = 0 \]
\[ \Rightarrow A^\top A \mathbf{z} = 0 \]
\[ \Rightarrow \mathbf{z} \in N(A^\top A) \]
\[ N(A) \subseteq N(A^\top A) \]
\[ R(A) = R(A^\top A) \]

Now, let \( A^\top = B \), so the rank matrix \( B \), rank \( (B \mathbf{B}) = \mathbf{r}(B) \).

\[ \therefore R(AA^\top) = R(A^\top) = R(A) \subseteq R(A^\top A) \]
Reducing a matrix into Normal form:

Let $A$ be a matrix of order $m \times n$, suppose $r(A) = r < \text{min}(m,n)$. Then $\exists$ non-singular matrices $P$ and $Q$ such that

$$PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

Proof: Since $r(A) = r < \text{min}(m,n)$

$\exists$ a non-singular matrix $P$, such that $PA = E$, an echelon matrix with $r$ non-null rows.

Clearly, $E = \begin{pmatrix} E_1 \\ 0 \end{pmatrix}$, $E_1$ is also an echelon matrix.

Now $\text{Rank} \left( E_1 \right) = r = \text{Column rank of } E_1$.

$E_1$ has $n-r$ columns each of which is LD on the rest $r$ LIN columns. Hence through column operations those dependent columns can be reduced to null columns.

i.e., $E_1 \sim \begin{pmatrix} E_2 & 0 \\ 0 & 0 \end{pmatrix}$, where $E_2$ has full rank.

$\therefore$ a non-singular matrix $P_2$ satisfies

$$P_2AP_2 = \begin{pmatrix} E_2 & 0 \\ 0 & 0 \end{pmatrix}$$

As $E_2$ is a square matrix of full rank through row column operation, it can be reduced to $I_p$.

Hence $\exists$ non-singular matrices $P_3$ and $P_4$

$$P_3P_2AP_2P_4 = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix}$$

Let $P_3P_2 = P$ and $P_2P_4 = Q$.

Clearly, $P$ and $Q$ are non-singular matrices as they are product of non-singular matrices.

A square matrix of full rank is always non-singular.

Let $A$ be a square matrix of order $n$ possessing full rank, then $\exists$ a non-singular matrices $P$ and $Q$ such that

$$PAQ = I_n$$

$|P| |A| |Q| = 1$

$\Rightarrow |A| \neq 0$.
**Sylvester's Inequality:**

\[ R(\mathbf{A} \oplus \mathbf{B}) \geq R(\mathbf{A}) + R(\mathbf{B}) - n. \]

**Proof:**

Let \( R(\mathbf{A}) = r < n \)

\( \exists \) n.s. matrices \( \mathbf{P} \) and \( \mathbf{Q} \) such that

\[ \mathbf{P} \mathbf{A} \mathbf{Q} = \begin{pmatrix} \mathbf{I}_r & 0 \\ 0 & 0 \end{pmatrix} \]

Define a square matrix \( \mathbf{C} \) of the order \( nxn \) such that

\[ \mathbf{P} \mathbf{C} \mathbf{Q} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{I}_{n-r} \end{pmatrix} \]

Clearly, \( \mathbf{P} (\mathbf{A} + \mathbf{C}) \mathbf{Q} = \mathbf{I}_n \)

Therefore, \( |\mathbf{P} (\mathbf{A} + \mathbf{C}) \mathbf{Q}| = |\mathbf{I}_n| = 1 \)

\[ |\mathbf{P}| |\mathbf{A} + \mathbf{C}| |\mathbf{Q}| = 1 \]

\( \Rightarrow \mathbf{A} + \mathbf{C} \) is non-singular.

Hence,

\[ R(\mathbf{C}) = R\left( (\mathbf{A} + \mathbf{C}) \mathbf{B} \right) \]

\[ = R\left( \mathbf{A} \mathbf{B} + \mathbf{C} \mathbf{B} \right) \]

\[ < R(\mathbf{A} \mathbf{B}) + R(\mathbf{C} \mathbf{B}) \]

\[ \leq R(\mathbf{A} \mathbf{B}) + R(\mathbf{C}) \]

\[ = R(\mathbf{A} \mathbf{B}) + n - r \]

\[ = R(\mathbf{A} \mathbf{B}) + n - R(\mathbf{A}) \]

\( \Rightarrow R(\mathbf{A} \mathbf{B}) \geq R(\mathbf{A}) + R(\mathbf{B}) - n. \)
Orthogonal Transformation: A matrix transformation \( y = Ax \) is said to be orthogonal if \( A \) is orthogonal.

Here, \( A^T = A^{-1} \)

\[ |A| = \pm 1 \]

\( A \) is singular.

Now, a matrix transformation is said to be non-singular if \( A \) is non-singular.

i.e., \( y = Ax \) \( \Rightarrow x = A^{-1}y \), i.e., N.S. transformation is one-to-one.

An orthogonal transformation is an n.s. transformation hence it is one-to-one.

Note that:

\[ y^T A^T A y = \sum_{i=1}^{n} y_i^2 \]

i.e., the length of the vector is preserved under orthogonal transformation.

Helmert's Transformation:

\[ (x_1, x_2, \ldots, x_n) \rightarrow (y_1, y_2, \ldots, y_n) \]

\[ y_1 = \frac{x_1}{\sqrt{n}} + \frac{x_2}{\sqrt{n}} + \ldots + \frac{x_n}{\sqrt{n}} \]

\[ y_2 = \frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}} \]

\[ y_3 = \frac{x_1}{\sqrt{6}} + \frac{x_2}{\sqrt{6}} - \frac{2x_3}{\sqrt{6}} \]

\[ \vdots \]

\[ y_n = \frac{x_1}{\sqrt{n(n-1)}} + \frac{x_2}{\sqrt{n(n-1)}} + \ldots + \frac{x_{n-1}}{\sqrt{n(n-1)}} - \frac{(n-1)x_n}{\sqrt{n(n-1)}} \]

i.e., \( y = Px \), where \( P \) is an orthogonal matrix.

\[ \Rightarrow y^T P^T P x = x^T \text{In} x = x^T x \]

\[ \Rightarrow \sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} x_i^2 \]
Example:

\[ s^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 \]

\[ n s^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i=1}^{n} y_i^2 \]

Note that,

\[ y_1 = 4 \frac{\sum x_i}{\sqrt{n}} = \frac{n \bar{x}}{\sqrt{n}} = \sqrt{n} \bar{x} \]

Now,

\[ n s^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - n \bar{x}^2 \]

\[ = \sum_{i=1}^{n} y_i^2 - \frac{\sum y_i^2}{\sqrt{n}} \]

\[ = \sum_{i=2}^{n} \frac{y_i^2}{i} \]

\[ \Rightarrow n s^2 = \sum_{i=2}^{n} \left( \frac{i-1}{i} \left( \frac{i-1}{i} \alpha_i - \frac{\alpha_1}{i} - \frac{\alpha_2}{i} - \cdots - \frac{\alpha_{i-2}}{i} \right) \right)^2 \]

\[ = \sum_{i=2}^{n} \left( \frac{\alpha_1}{\sqrt{i(i-1)}} + \frac{\alpha_2}{\sqrt{i(i-1)}} + \cdots + \frac{\alpha_{i-1}}{\sqrt{i(i-1)}} - \frac{\alpha_i}{\sqrt{i(i-1)}} \right)^2 \]

\[ = \sum_{i=2}^{n} y_i^2 \]

Problem:

Show that the SD of a set of observation \( x_1, x_2, \ldots, x_i \) is given by

\[ n s^2 = \sum_{i=2}^{n} \frac{i-1}{i} (x_i - \bar{x})^2, \text{ where } \bar{x} = \frac{1}{j} \sum_{j=1}^{i} x_j / i. \]

Solution:

Note that,

\[ \sqrt{\frac{i}{i-1}} (\bar{x}_i - \bar{x}) \]

\[ = \sqrt{\frac{i}{i-1}} \left( \frac{\alpha_1 + \cdots + \alpha_{i-1} + \alpha_i - i \bar{x}}{i} \right) \]

\[ = \frac{\alpha_1 + \cdots + \alpha_{i-1} - (i-1) \alpha_i}{\sqrt{i(i-1)}}, \text{ } i = 2(1)n. \]
\[ y_1 = \frac{x_1 + x_2 + \ldots + x_n}{\sqrt{n}}, \]
\[ y_i = \frac{x_1 + \ldots + x_{i-1} - (i-1)x_i}{\sqrt{i(i-1)}}, \quad i = 2(1)n. \]
\[ y_1 = \frac{x_1}{\sqrt{n}} + \frac{x_2}{\sqrt{n}} + \ldots + \frac{x_n}{\sqrt{n}} \]
\[ y_2 = \frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}} = \frac{x_1 - x_2}{\sqrt{2}} \]
\[ y_3 = \frac{x_1}{\sqrt{3}} + \frac{x_2}{\sqrt{3}} - \frac{2x_3}{\sqrt{6}} = \frac{x_1 + x_2 - 2x_3}{\sqrt{2}} \]
\[ y_4 = \frac{x_1}{\sqrt{12}} + \frac{x_2}{\sqrt{12}} + \frac{x_3}{\sqrt{12}} - \frac{3x_4}{\sqrt{12}} = \frac{x_1 + x_2 + x_3 - 3x_4}{\sqrt{3} \cdot 4} \]
\[ \vdots \]
\[ y_{n-1} = \frac{x_1 + \ldots + x_{n-1} - (n-2)x_{n-1}}{\sqrt{(n-2)(n-1)}} \]
\[ y_n = \frac{x_1 + \ldots + (x_{n-1} - (n-1)x_n}{\sqrt{(n-2) \ldots 1}} \]
\[ \left( \begin{array}{c} y_1 \\ \vdots \\ y_n \end{array} \right) = \left( \begin{array}{cccc} 1/\sqrt{n} & 1/\sqrt{n} & \ldots & 1/\sqrt{n} \\ 1/\sqrt{2} & 0 & \ldots & 0 \\ 1/\sqrt{6} & 2/\sqrt{6} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1/\sqrt{n(n-1)} & \ldots & \ldots & 1/\sqrt{n(n-1)} \end{array} \right) \left( \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right) \]
\[ y^T y = A \alpha \]
\[ \text{where } A \text{ is an orthogonal matrix.} \]
\[ (A \alpha)'(A \alpha) = \alpha' A' A \alpha \]
\[ = \alpha' \alpha \]
\[ = \alpha_i^2 \]
\[ \Rightarrow \sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} \alpha_i^2 \]
\[ \Rightarrow \sum_{i=1}^{n} \alpha_i^2 - y_i^2 = \sum_{i=2}^{n} y_i^2 \]
\[ \Rightarrow \sum_{i=1}^{n} \alpha_i^2 - n \overline{\alpha}^2 = \sum_{i=2}^{n} y_i^2 \]
\[ \Rightarrow \sum_{i=1}^{n} (\alpha_i - \overline{\alpha})^2 = \sum_{i=2}^{n} \frac{i}{i-1} (\alpha_i - \overline{\alpha})^2 \]
\[ \Rightarrow n \delta^2 = \sum_{i=2}^{n} \frac{i}{i-1} (\alpha_i - \overline{\alpha})^2. \]
Problems on Matrices & Determinants

1. Diagonalise the following matrix using non-singular transformation and find the non-singular matrix which diagonalise.

\[
A = \begin{pmatrix}
1 & 1 & 2 \\
-2 & 2 & -1 \\
1 & 2 & -1
\end{pmatrix}
\]

Solution:

\[
A \sim \begin{pmatrix}
1 & 1 & 2 \\
0 & 4 & 3 \\
0 & 1 & -3
\end{pmatrix} \quad R_2 \leftrightarrow R_2 + 2R_1
\]

\[
\begin{pmatrix}
1 & 1 & 2 \\
0 & 4 & 3 \\
0 & 0 & -15/4
\end{pmatrix} \quad R_3 \leftrightarrow R_3 - \frac{R_2}{4}
\]

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 4 & 3 \\
0 & 0 & -15/4
\end{pmatrix} \quad C_2 \leftrightarrow C_2 - C_1
\]

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 4 & 0 \\
0 & 0 & -15/4
\end{pmatrix} \quad C_3 \leftrightarrow C_3 - \frac{3}{4}C_2
\]

\[
E_2 E_1 A E_3 E_4 = \Lambda
\]

\[
E_1 = \begin{pmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
-1 & 0 & 1
\end{pmatrix}
\]

\[
E_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -1/4 & 1
\end{pmatrix}
\]

\[
P = E_2 E_1 \Rightarrow \begin{pmatrix}
1 & 0 & 0 \\
2 & 0 & 0 \\
-3/2 & -1/4 & 1
\end{pmatrix}
\]

\[
E_3 = \begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
E_4 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -3/4 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
Q = E_3 E_4 = \begin{pmatrix}
1 & -1 & -5/4 \\
0 & 1 & -3/4 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
\therefore PAQ = \Lambda
\]
2) Find the rank of the matrix
\[
\begin{pmatrix}
1 & 2 & 1 & 2 \\
1 & 3 & 2 & 2 \\
2 & 4 & 3 & 4 \\
3 & 7 & 4 & 6
\end{pmatrix}
\]

Solution:
\[
R_2' = R_2 - R_1 \\
R_3' = R_3 - 2R_1 \\
R_4' = R_4 - 3R_1 \\
R_4' = R_4 - R_2
\]

\[
\text{Rank}(A) = 3.
\]

3) Find \( \det \left( \begin{pmatrix} q_1^2 & p_1 q_2 & \cdots & p_1 q_n \\ p_1 q_2 & q_2^2 & \cdots & p_2 q_n \\ \vdots & \vdots & \ddots & \vdots \\ p_1 q_n & p_2 q_n & \cdots & q_n^2 \end{pmatrix} \right); \quad q_i \neq 0 \ \forall i = 1 \ \text{to} \ n
\]

Solution:
\[
\Delta = q_1^2 q_2^2 \cdots q_n^2 \begin{vmatrix}
1 & P & \cdots & P \\
P & 1 & \cdots & P \\
\vdots & \vdots & \ddots & \vdots \\
P & P & \cdots & 1
\end{vmatrix}
\]
\[
= \prod_{i=1}^{n} q_i^2 \left\{ C_1 - P \right\}^{n-1} \left\{ 1 + (n-1)P \right\}
\]

Conclusion:
\[
\begin{pmatrix} 1 + \alpha & \alpha & \alpha \\
\alpha & 1 + \alpha & \alpha \\
\alpha & \alpha & 1 + \alpha
\end{pmatrix}
= (n+\alpha)
\]

\[
R_i' = \frac{R_i}{q_i} \\
R_i'' = \frac{R_i}{(n+\alpha)} \\
c_i = c_1 - c_1
\]

\[
= (n+\alpha), \alpha^{n-1} = \alpha^n + n\alpha^{n-1}.
\]
4. Find \( A^{n \times n} \), where \( A_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \)

\[ \text{Find } |A| \text{. [ISI M.STAT]} \]

\[ \text{Sol. } \Delta = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix} = 1 + (n-1) \cdot n \cdot (1-n)^{n-1}. \]

5. Find \( A = \begin{bmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{bmatrix} \)

\[ \text{Sol. } A = \begin{bmatrix} a & (n-1) \cdot b \\ 0 & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a-b \end{bmatrix} = \begin{bmatrix} (a+b) \cdot (a-b)^{n-1} \\ 0 \cdot (a-b)^{n-1} \\ \vdots \end{bmatrix} = (a-b)^{n-1} + nb(a-b)^{n-1}. \]
6. Vandermonde's Determinant:

Find:

\[
\begin{vmatrix}
1 & 1 & 1 & \ldots & 1 \\
a_1 & a_2 & a_3 & \ldots & a_n \\
a_1^2 & a_2^2 & a_3^2 & \ldots & a_n^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \ldots & a_n^{n-1}
\end{vmatrix}
\]

Solution:

\[
4 = \begin{vmatrix}
1 & 1 & 1 & \ldots & 1 \\
0 & (a_2-a_1) & (a_3-a_1) & \ldots & (a_n-a_1) \\
0 & a_2(a_2-a_1) & a_3(a_3-a_1) & \ldots & a_n(a_n-a_1) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & a_2^{n-2}(a_2-a_1) & a_3^{n-2}(a_3-a_1) & \ldots & a_n^{n-2}(a_n-a_1)
\end{vmatrix}
\]

\[
= \prod_{i=2}^{n} (a_i-a_1) \prod_{i=3}^{n} (a_i-a_2) \prod_{i=1}^{n} (a_i-a_n) 
\]

\[
= (-1)^{\frac{n(n-1)}{2}} \prod_{i<j}^{n} (a_i-a_j)
\]
\[
\text{Find } \begin{vmatrix}
1+q_i^2 & a_1q_i & a_1q_2 & \cdots & a_1q_n \\
a_2q_1 & 1+q_2^2 & a_2q_3 & \cdots & a_2q_n \\
a_3q_1 & a_3q_2 & 1+q_3^2 & \cdots & a_3q_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_nq_1 & a_nq_2 & a_nq_3 & \cdots & 1+q_n^2
\end{vmatrix}
\]

\[
\text{Sol: } \Delta = \begin{vmatrix}
1+q_i^2 & a_1q_i & a_1q_2 & \cdots & a_1q_n \\
a_2q_1 & 1+q_2^2 & a_2q_3 & \cdots & a_2q_n \\
a_3q_1 & a_3q_2 & 1+q_3^2 & \cdots & a_3q_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_nq_1 & a_nq_2 & a_nq_3 & \cdots & 1+q_n^2
\end{vmatrix}
\]

\[
= \begin{vmatrix} a_1^2 & a_2^2 & \cdots & a_n^2 \end{vmatrix} \left( \frac{1}{q_i^2} + 1 \right)^{\frac{1}{a_i^2}} \begin{vmatrix} 1 & \cdots & 1 \end{vmatrix} \text{adj} \begin{vmatrix} 1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1
\end{vmatrix}
\]

\[
R_i' = R_i - a_i^2 R_i \quad \forall i > 1.
\]
For \( n \geq 2 \), suppose \( x_1, x_2, \ldots, x_n \) are not all equal. Consider the orthogonal transformation of \((y_1, y_2, \ldots, y_n)\) to \((z_1, z_2, \ldots, z_n)\) such that

\[
R^{nx1} = \begin{pmatrix}
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\
\frac{x_1 - \bar{x}}{\sqrt{2}(x_1 - \bar{x})^2} & \frac{x_2 - \bar{x}}{\sqrt{2}(x_2 - \bar{x})^2} & \cdots & \frac{x_n - \bar{x}}{\sqrt{2}(x_n - \bar{x})^2} \\
a_{31} & a_{32} & \cdots & a_{3n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}_{nxn}
\]

Show that

\[
\sum_{i=3}^{n} x_i^2 = \left\{ \sum_{i=1}^{n} (y_i - \bar{y})^2 \right\} \left\{ 1 - \rho_{xy}^2 \right\},
\]

where \( \rho_{xy} \) is the correlation coefficient between \( x \) and \( y \).

Solution:

Here, \( R^{nx1} = A^{nxn} \cdot y^{nx1} \)

where, \( A \) is the given orthogonal matrix.

By property,

\[
x \cdot x' = y' \cdot y
\]

\[
\Rightarrow \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i^2
\]

\[
\Rightarrow \sum_{i=3}^{n} x_i^2 = \sum_{i=1}^{n} y_i^2 - 2 \sum_{i=1}^{2} y_i^2
\]

Here,

\[
x_1 = \frac{y_1 + y_2 + \cdots + y_n}{\sqrt{n}} = \bar{y}
\]

\[
x_2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})y_i}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2}}
\]

\[
= \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2}}
\]

\[
= \rho_{xy} \cdot \sqrt{\sum_{i=1}^{n} (y_i - \bar{y})^2}
\]

\[
\therefore \sum_{i=3}^{n} x_i^2 = \left( \sum_{i=1}^{2} y_i^2 - n \bar{y}^2 \right) - \rho_{xy}^2 \cdot \sum_{i=1}^{n} (y_i - \bar{y})^2
\]

\[
= \sum_{i=1}^{n} (y_i - \bar{y})^2 - \rho_{xy}^2 \cdot \sum_{i=1}^{n} (y_i - \bar{y})^2
\]

\[
= \sum_{i=1}^{n} (y_i - \bar{y})^2 \cdot (1 - \rho_{xy}^2)
\]
LINEAR TRANSFORMATION

Linear mapping on Linear Transformation:

Definition: Let $V$ and $W$ be vector spaces over the same field $F$. A mapping $T: V \rightarrow W$ is said to be a linear mapping if it satisfies the following conditions:

1. $T(\alpha + \beta) = T(\alpha) + T(\beta)$ for all $\alpha, \beta \in V$.
2. $T(c \alpha) = cT(\alpha)$ for all $c \in F$ and $\alpha \in V$.

These two conditions can be replaced by the single condition:

$$T(c \alpha + b \beta) = cT(\alpha) + bT(\beta) \quad \forall c, b \in F \text{ and all } \alpha, \beta \in V.$$

Note: 1. A linear mapping $T: V \rightarrow W$ is also a homomorphism of $V$ to $W$.
2. Generally, a linear mapping $T$ is a transformation from one vector space $V$ to another vector space $W$, both over the same field of scalars. But the co-domain space $W$ may be the space $V$ itself. In this case, $T$ is said to be a linear mapping on $V$.

Examples:

1. The identity mapping: The mapping $T: V \rightarrow V$ defined by $T(\alpha) = \alpha$ for all $\alpha \in V$, is a linear mapping. This is called the identity mapping on $V$ and is denoted by $I_V$.
2. The zero mapping: The mapping $T: V \rightarrow W$ defined by $T(\alpha) = 0$ (zero vector of $W$) for all $\alpha \in V$, is called the zero mapping.
3. Let $P$ be the vector space of all real polynomials. The mapping $D: P \rightarrow P$ defined by $D(p(x)) = \frac{d}{dx} p(x)$ for all $p(x) \in P$, is a linear mapping.
4. Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined by $T(x_1, x_2, x_3) = (y_1, y_2, y_3)$, where $x_1, x_2, x_3$ are the components of $x$.

Let $\alpha = (x_1, x_2, x_3)$ and $\beta = (y_1, y_2, y_3)$ in $\mathbb{R}^3$.

Then $\alpha + \beta = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$ and $\alpha - \beta = (x_1 - y_1, x_2 - y_2, x_3 - y_3)$.

For $c \in \mathbb{R}$, $c \alpha = (cx_1, cx_2, cx_3)$ and $T(c \alpha) = (cy_1, cy_2, cy_3)$.

Therefore, $T$ is a linear mapping.

Remark: $A: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$ is a linear mapping from $\mathbb{R}^n$ into $\mathbb{R}^m$ if $A(\alpha + \beta) = A\alpha + A\beta$ and $A(c \alpha) = c A\alpha$.
5. Let $V$ be the vector space of all real-valued functions continuous on the closed interval $[a, b]$ and let $T : V \rightarrow \mathbb{R}$ be defined by $T(f) = \int_a^b f(t) \, dt$, $f \in V$. Then $T$ is a linear functional.

6. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined by $T(x_1, x_2, x_3) = (x_1, x_2, x_3)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.

Let $\alpha = (x_1, x_2, x_3)$, $\beta = (y_1, y_2, y_3) \in \mathbb{R}^3$

Then $\alpha + \beta = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

$T(\alpha + \beta) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

$\neq T(\alpha) + T(\beta)$

Therefore, $T$ is not a linear mapping.

**Properties of a linear transformation:**

---

**Theorem:** Let $V$ and $W$ be vector spaces over a field $F$ and $T : V \rightarrow W$ be a linear mapping. Then

(i) $T(\mathbf{0}) = \mathbf{0}$, where $\mathbf{0}$ is the null vector in $V$ and $W$ respectively.

(ii) $T(-\alpha) = -T(\alpha)$, $\forall \alpha \in V$.

(iii) $T(\alpha + \beta) = T(\alpha) + T(\beta)$, $\forall \alpha, \beta \in V$.

(iv) $T(a_1 \alpha_1 + a_2 \alpha_2 + \cdots + a_n \alpha_n) = a_1 T(\alpha_1) + a_2 T(\alpha_2) + \cdots + a_n T(\alpha_n)$, $\forall \alpha_1, \alpha_2, \ldots, \alpha_n \in V$ and $a_1, a_2, \ldots, a_n \in F$.

**Proof:**

(i) Let $\alpha \in V$, then $T(\alpha) \in W$. We have

$T(\alpha) + \mathbf{0} = T(\alpha)$

Note, in the vector space $W$, we have

$T(\alpha) + \mathbf{0} = T(\alpha) + T(\mathbf{0})$

$\Rightarrow T(\mathbf{0}) = \mathbf{0}$

(ii) $T[\alpha + (-\alpha)] = T(\alpha) + T(-\alpha)$

$\Rightarrow T(\mathbf{0}) = T(\alpha) + T(-\alpha)$

$\Rightarrow \mathbf{0} = T(\alpha) + T(-\alpha)$

$\Rightarrow T(-\alpha) = -T(\alpha)$, $\forall \alpha \in V$.

(iii) $T(\alpha - \beta) = T[\alpha + (-\beta)]$

$= T(\alpha) + T(-\beta)$

$= T(\alpha) - T(\beta)$

(iv) By induction, the proof can easily be done.
Kernel of $T$ or Null Space of $T$:

Let $V$ and $W$ be vector spaces over a field $F$. Let $T : V \rightarrow W$ be a linear mapping. The set of all vectors $\alpha \in V$ such that $T(\alpha) = 0$, $\alpha$ being the null vector in $W$, is said to be the kernel of $T$ on the null space of $T$, and denoted by $\ker T$ or $N(T)$.

$$\ker T (N(T)) = \{ \alpha \in V : T(\alpha) = 0 \}$$

**Theorem:** $T : V \rightarrow W$ is a linear mapping, then $T$ is injective on one-to-one mapping if and only if $\ker T = \{ 0 \}$.

**Proof:** Only if part: Let $T$ be injective, $T(\alpha) = 0$ is the only time $\alpha = 0$, $\ker T = \{ 0 \}$.

If part: $\ker T = \{ 0 \}$, and $\alpha, \beta$ be two elements of $V \in W$.

$$0 = T(\alpha) - T(\beta)$$

$$\Rightarrow T(\alpha - \beta) = 0$$

$$\Rightarrow \alpha - \beta \in \ker T$$

Thus $T(\alpha) = T(\beta) \Rightarrow \alpha = \beta$ and therefore $T$ is injective.

**Note:** A $\alpha = 0$ has only trivial solution $\alpha = 0 \Rightarrow \text{rank}(A^{m \times n}) = n$.

**Theorem:** Let $V$ and $W$ be vector spaces over a field $F$. Let $T : V \rightarrow W$ be a linear mapping, then $\ker T$ is a subspace of $V$.

Image of a linear mapping:

Let $V$ and $W$ be vector spaces over a field $F$. Let $T : V \rightarrow W$ be a linear mapping. The image of the elements of $V$ under the mapping $T$ form a subset of $W$. This subset is said to be the image of $T$ and is denoted by $\text{Im} T$. Or, it is also called $\text{Range} T$ ($R(T)$).

$$\text{Im}(T) \subset R(T) = \{ T(\alpha) : \alpha \in V \}$$

**Theorem:** Let $V$ and $W$ be vector spaces over a field $F$. Let $T : V \rightarrow W$ be a linear mapping, then $\text{Im} T$ is a subspace of $W$.

**Proof:** Obviously $R(T)$ is a non-empty subset of $W$.

Let $\beta_1, \beta_2 \in R(T)$, then $\exists$ vectors $\alpha_1, \alpha_2$ in $V$ such that $T(\alpha_1) = \beta_1, T(\alpha_2) = \beta_2$.

Let $a, b$ be any element of the field $F$, we have

$$a\beta_1 + b\beta_2 = aT(\alpha_1) + bT(\alpha_2)$$

$$= T(a\alpha_1 + b\alpha_2)$$

[$T$ is a linear mapping]

Now, $V$ is a vector space, therefore $\alpha_1, \alpha_2 \in V$.

$a, b \in F \Rightarrow a\alpha_1 + b\alpha_2 \in V$.

Consequently, $T(a\alpha_1 + b\alpha_2) = a\beta_1 + b\beta_2 \in R(T)$.

Therefore $R(T)$ is a subspace of $W$.
Worked Examples:

1. A mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, 2x_1 + x_2 + 2x_3, x_1 + 2x_2 + x_3)$, for $(x_1, x_2, x_3) \in \mathbb{R}^3$. Show that $T$ is a linear mapping, find $\ker T$ and dimension of $\ker T$. Is $T$ one-one? Find $\text{Im} T$ and the dimension of $\text{Im} T$.

Solution:

1st Part:

Let $\alpha = (a_1, a_2, a_3) \in \mathbb{R}^3$, $\beta = (b_1, b_2, b_3) \in \mathbb{R}^3$.

Then $T(\alpha) = (a_1 + a_2 + a_3, 2a_1 + a_2 + 2a_3, a_1 + 2a_2 + a_3)$,

$T(\beta) = (b_1 + b_2 + b_3, 2b_1 + b_2 + 2b_3, b_1 + 2b_2 + b_3)$.

$\alpha + \beta = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.

$T(\alpha + \beta) = ((a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3), 2(a_1 + b_1) + (a_2 + b_2) + 2(a_3 + b_3), (a_1 + b_1) + 2(a_2 + b_2) + (a_3 + b_3))$

$= ((a_1 + a_2 + a_3) + (b_1 + b_2 + b_3), (2a_1 + 2a_2 + 2a_3) + (2b_1 + 2b_2 + 2b_3), (a_1 + 2a_2 + a_3) + (b_1 + 2b_2 + b_3))$

$= T(\alpha) + T(\beta)$

Let $c \in \mathbb{R}$, $c\alpha = (ca_1, ca_2, ca_3)$

$T(c\alpha) = (ca_1 + ca_2 + ca_3, 2ca_1 + ca_2 + 2ca_3, ca_1 + 2ca_2 + ca_3)$

$= c(a_1 + a_2 + a_3, 2a_1 + a_2 + 2a_3, a_1 + 2a_2 + a_3) = cT(\alpha)$.

$\therefore$ $T$ is a linear mapping.

2nd Part:

$\ker T = \{ \alpha : T(\alpha) = 0 \}$

$= (x_1, x_2, x_3) = (0, 0, 0)$

From 1st two equations,

$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$

$\therefore x_1 = x_2 = x_3 = k$. The last equation is satisfied.

$\therefore (x_1, x_2, x_3) = k(1, 0, 1)$, $k \in \mathbb{R}$

$\ker T = \{ k(1, 0, 1) : k \in \mathbb{R} \}$

$\therefore \dim \ker T = 1$.

As $\ker T \neq \{0\}$, so the linear mapping $T$ is not one-to-one.
If \( \{ e_1, e_2, e_3 \} \) be a basis of the domain space \( \mathbb{R}^3 \),
\( \text{Im} T \) is the linear span of the vectors \( T(e_1), T(e_2), T(e_3) \).
\( \text{Im} T \) is spanned by \( \{ e_1, e_2, e_3 \} \).
\( T(e_1) = (1, 0, 0), T(e_2) = (0, 1, 0), T(e_3) = (0, 0, 1) \) is a basis of \( \mathbb{R}^3 \).
\( \{ T(e_1), T(e_2), T(e_3) \} \) is a spanning set of \( \text{Im} T \).
Since \( T(e_1) = T(e_2) \), \( \text{Im} T \) is the linear span of the vectors \( (1, 0, 0) \) and \( (1, 1, 0) \).
\( \{ (1, 0, 0), (1, 1, 0) \} \) is linearly independent in \( \mathbb{R}^3 \).
Hence \( \text{dim}(\text{Im} T) = 2 \).

\begin{align*}
\text{Then } T(x) &= (x_1 + x_2 + x_3, 2x_1 + x_2 + 2x_3, x_1 + 2x_2 + x_3) \\
T(x) &= x_1(1, 2, 1) + x_2(1, 1, 2) + x_3(1, 2, 1) \\
\text{Hence } \text{Im} T &= \langle (1, 2, 1), (1, 1, 2) \rangle \\
\therefore \text{dim}(\text{Im} T) &= 2.
\end{align*}

2. A linear mapping \( T: \mathbb{R}^3 \rightarrow \mathbb{R}^4 \) is defined by
\( T(x_1, x_2, x_3) = (x_2 + x_3, x_2 + x_1, x_1 + x_2, x_1 + x_2 + x_3) \).
Find \( \ker T \). Verify that the set \( \{ T(e_1), T(e_2), T(e_3) \} \) is linearly independent in \( \mathbb{R}^4 \).
Find \( \text{Im} T \) and the dimension of \( \text{Im} T \).

**Solution:**

1st Part:
\[ \ker T = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : T(x_1, x_2, x_3) = (0, 0, 0, 0) \} \]

\[ \text{Let } (x_1, x_2, x_3) \in \ker T. \]
Then \( x_2 + x_3 = 0, x_2 + x_1 = 0, x_1 + x_2 = 0, x_1 + x_2 + x_3 = 0. \)
The solution is \( x_1 = x_2 = x_3 = 0. \)

2nd Part:
\[ \ker T = \{ 0 \} \]
\[ T(e_1) = (0, 1, 1, 1), T(e_2) = (1, 0, 1, 1), T(e_3) = (0, 0, 0, 1). \]
To examine linear independence of the set \( \{ T(e_1), T(e_2), T(e_3) \} \),
let \( c_1 T(e_1) + c_2 T(e_2) + c_3 T(e_3) = 0, c_1, c_2, c_3 \in \mathbb{R}. \)
Then \( c_1 (0, 1, 1, 1) + c_2 (1, 0, 1, 1) + c_3 (0, 0, 0, 1) = (0, 0, 0, 0) \)
\[ \Rightarrow c_2 + c_3 = 0, c_1 + c_2 = 0, c_1 + c_2 + c_3 = 0. \]
The solution is \( c_1 = c_2 = c_3 = 0. \)
This proves that \( \{ T(e_1), T(e_2), T(e_3) \} \) is linearly independent in \( \mathbb{R}^4 \).
2nd Part: \( \text{Im} T = \text{the linear span of the vectors } T(e_1), T(e_2), T(e_3) \), where \( e_1, e_2, e_3 \) is a basis of \( \mathbb{R}^3 \).

\[ \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1) \} \text{ is a basis of } \mathbb{R}^3. \]

\[ T(e_1) = (0,1,1), T(e_2) = (1,0,1), T(e_3) = (1,1,0) \]

Therefore, \( \text{Im} T = \text{lin} \{ (0,1,1), (1,0,1), (1,1,0) \} \).

The set \( \{ (0,1,1), (1,0,1), (1,1,0) \} \) is linearly independent.

Therefore, the dimension of \( \text{Im} T \) is 3.

3. Show that the transformation \( T: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) defined by \( T(x,y,z) = (x+y, y+z, 2x+2z) \) is a linear transformation. Also find the dimension of \( \ker(T) \) and show that the transformation is one-to-one. Find \( \text{Im} T \). Is it onto?

Solution: 1st Part: Let \( \alpha = (a_1, a_2, a_3) \in \mathbb{R}^3 \)

\[ T(\alpha a_1 + \beta a_2) = T(a_1 + a_2) \]

Let \( \alpha, \beta \in \mathbb{R}^3 \), then

\[ T(\alpha x + \beta y) = (\alpha (x_1 + x_2) + \beta (y_1 + y_2), \alpha (x_2 + x_3) + \beta (y_2 + y_3), \alpha (x_3 + x_1) + \beta (y_3 + y_1)) \]

\[ = \alpha (x_1 + x_2, x_1 + x_3, x_1 + x_3) + \beta (y_1 + y_2, y_2 + y_3, y_3 + y_1) \]

\[ = \alpha T(x) + \beta T(y) \]

\( \therefore T \) is a linear transformation.

2nd Part: \( \ker T = \{ x \in \mathbb{R}^3 : T(x) = 0 \} \).

\[ T(x,y,z) = 0 \Rightarrow x+y = 0, y+z = 0, 2x+2z = 0 \]

\[ \Rightarrow x = 0, y = 0, z = 0 \]

\( \therefore \ker T = \{ 0 \} \) and hence \( T \) is one-to-one.

\[ \dim(\ker T) = 0 \]

3rd Part: \( \{ e_1, e_2, e_3 \} \) is a basis for \( \mathbb{R}^3 \).

\[ \{ T(e_1), T(e_2), T(e_3) \} \text{ is a basis for } \text{Im}(T). \]

\[ \{ (1,0,1), (1,1,0), (0,1,1) \} \text{ is a basis for } \text{Im}(T). \]

\( \text{Im} T = \text{lin} \{ (1,0,1), (1,1,0), (0,1,1) \} \). \( \text{Im} T = \mathbb{R}^3 \) and therefore \( T \) is an onto mapping.
4. Determine \( T: \mathbb{R}^2 \to \mathbb{R}^2 \) which maps the basis vectors \( e_1, e_2, e_3 \) of \( \mathbb{R}^3 \) to \( (1,1), (2,3), (1,2) \) of \( \mathbb{R}^2 \). Find \( T(\mathbf{x}) \).

**Solution:**

Let \( \mathbf{x} \in \mathbb{R}^3 \)

Then, \( \mathbf{x} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3 \)

\[ T(\mathbf{x}) = x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) + x_3 T(\mathbf{e}_3) \]

\[ = x_1 (1,1) + x_2 (2,3) + x_3 (-1,2) \]

\[ = (x_1 + 2x_2 - x_3, x_1 + 2x_2 + 2x_3) \]

5. Determine the linear mapping \( T: \mathbb{R}^2 \to \mathbb{R}^2 \) which maps the basis vectors \( e_1, e_2, e_3 \) of \( \mathbb{R}^3 \) to the vectors \( (1,1), (2,3), (3,2) \), respectively.

(i) Find \( \ker T \) and \( \operatorname{Im} T \).

(ii) Prove that \( T \) is not one-to-one, but onto.

**Solution:**

Let \( \mathbf{y} = (x, y, z) \) be an arbitrary vector in \( \mathbb{R}^3 \)

\[ \mathbf{y} = x \mathbf{e}_1 + y \mathbf{e}_2 + z \mathbf{e}_3 \]

Since \( T \) is linear,

\[ T(\mathbf{y}) = x T(\mathbf{e}_1) + y T(\mathbf{e}_2) + z T(\mathbf{e}_3) \]

\[ = x (1,1) + y (2,3) + z (3,2) \]

\[ = (x + 2y + 3z, x + 2y + 2z) \]

So, \( T \) is defined by \( T(x, y, z) = (x + 2y + 3z, x + 2y + 2z) \), \( (x, y, z) \in \mathbb{R}^3 \).

(i) \( T(1,1,0) = (3,4) \); \( T(0,0,1) = (3,1) \).

(ii) Let \( (x,y,z) \in \ker T \), then \( T(x,y,z) = (0,0) \)

\[ \Rightarrow x + 2y + 3z = 0, \quad x + 3y + 2z = 0 \]

The solution is \( x = \frac{k}{5}, \quad y = \frac{2}{5}, \quad z = k \)

\[ \Rightarrow (x,y,z) = k (-5,1,1), \text{where } k \in \mathbb{R}. \]

Consequently, \( \ker T = L \{(-5,1,1)\}, \text{where } \mathbf{z} = (-5,1,1) \).

\( \operatorname{Im} T \) is the linear span of these vectors \( T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3) \)

where \( \{e_1, e_2, e_3\} \) is any basis of \( \mathbb{R}^3 \).

Since \( \{\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}\} \) is a basis of \( \mathbb{R}^3 \),

\[ \operatorname{Im} T = L \{(1,1), (2,3), (3,2)\} \]

(iii) Since \( \ker T \neq \{0\} \), \( T \) is not one-to-one.

\[ \operatorname{Im} T = L \{(1,1), (2,3), (3,2)\} \]

These vectors are linearly dependent in \( \mathbb{R}^2 \).

The subset \( \{(1,1), (2,3)\} \) is linear in \( \mathbb{R}^2 \).

Therefore, \( \dim \operatorname{Im} T = 2 \).

Since \( \operatorname{Im} T \) is a subspace of \( \mathbb{R}^2 \) and \( \dim \operatorname{Im} T = 2 \), \( \operatorname{Im} T = \mathbb{R}^2 \).

Therefore, \( T \) is onto.
Nullity and Rank of a Linear Mapping:

Let \( V \) and \( W \) be vector spaces over a field \( F \) and \( T : V \rightarrow W \) be a linear mapping.

Then \( \ker T \) is a subspace of \( V \), the dimension of \( \ker T \) is called the nullity of \( T \). \( \text{Im} T \) is a subspace of \( W \). The dimension of \( \text{Im} T \) is called the rank of \( T \).

\[
\text{Nullity} (T) = \dim (\ker T) = \dim N(T) \\
\text{Rank} (T) = \dim (\text{Im} T) = \dim R(T)
\]

**Note:** If \( V \) be a finite dimensional vector space then \( \ker T \), being a subspace of \( V \), is finite-dimensional. The number of vectors in a basis of \( V \) is finite. As \( \text{Im} T \) is generated by the images of the vectors in a basis of \( V \), \( \text{Im} T \) is also finite dimensional.

**Theorem:** Let \( V \) and \( W \) be vector spaces over a field \( F \) and \( V \) is finite dimensional. If \( T : V \rightarrow W \) be a linear mapping then

\[
\dim \ker T + \dim \text{Im} T = \dim V.
\]

**Proof:**

**Case I:** Let \( \ker T = \{0\} \), then \( \text{Im} T \) consists of \( \{0\} \), where \( 0 \) is the null element in \( W \). \( \dim \ker T = 0 \),

\[
\dim \ker T + \dim \text{Im} T = \dim V + 0 = \dim V.
\]

**Case II:** Let \( \ker T = \{0\} \), and let \( \{v_1, v_2, \ldots, v_n\} \) be a basis of \( V \).

Then \( \{T(v_1), T(v_2), \ldots, T(v_n)\} \) is a basis of \( \text{Im} T \).

\[
\dim \ker T = 0, \quad \dim V = n, \quad \dim \text{Im} T = n
\]

The theorem holds good.

**Case III:** Let \( \ker T \neq \{0\} \) and let \( \{v_1, v_2, \ldots, v_n\} \) be a basis of \( \ker T \).

Let \( \ker T \) be a non-trivial proper subspace of \( V \). Let \( \{x_1, x_2, \ldots, x_k\} \) be a basis of \( \ker T \). The basis of \( \ker T \) can be extended to a basis of \( V \) and the extended basis of \( V \) be \( \{x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_n\} \).

\( \text{Im} T \) is generated by the vectors \( T(x_1), T(x_2), \ldots, T(x_n) \).

As, \( T(x_1) = T(x_2) = \ldots = T(x_k) = 0 \), \( \text{Im} T \) is generated by the vectors \( T(x_{k+1}), T(x_{k+2}), \ldots, T(x_n) \).

Considering the equation \( c_1 T(x_{k+1}) + c_2 T(x_{k+2}) + \cdots + c_n T(x_n) = 0 \), where \( c_i \in F \).

\[
T(c_{k+1} x_{k+1} + c_{k+2} x_{k+2} + \cdots + c_n x_n) = 0
\]

\[
T(c_{k+1} x_{k+1} + \cdots + c_n x_n) = 0
\]

Since \( \{x_1, x_2, \ldots, x_n\} \) is a basis for \( \ker T \),

The linear independence of \( \{x_1, x_2, \ldots, x_k\} \) requiring \( c_{k+1} = c_{k+2} = \cdots = c_n = 0 \).

\( \{T(x_{k+1}), \ldots, T(x_n)\} \) is linearly independent and is a basis for \( \text{Im} T \).

So, \( \dim (\ker T) = n - k \) and \( \dim \ker T + \dim \text{Im} T = k + (n - k) = n = \dim V \).
Theorem: Let $V$ and $W$ be finite-dimensional vector spaces over a field $F$, and $T : V \rightarrow W$ be a linear mapping. Then $T$ is one-to-one if and only if $T$ is onto.

Proof:
1. Assume $T$ is one-to-one.
   \[ \ker(T) = \{0\} \]
   \[ \dim(\ker(T)) = 0 \]
   \[ \dim(\ker(T) + \dim(\text{Im}(T)) = \dim(V) \]
   \[ \dim(\ker(T)) = \dim(V) \]
   \[ \dim(\text{Im}(T)) = \dim(W) \]
   \[ \text{But Im}(T) \text{ is a subspace of } W \text{ and so Im}(T) = W \]
   Hence $T$ is onto.

Conversely, let $T$ be onto. Then $\text{Im}(T) = W$.
\[ \dim(\ker(T)) + \dim(\text{Im}(T)) = \dim(V) \]
\[ \dim(\ker(T)) + \dim(W) = \dim(V) \]
As $\dim(W) = \dim(V)$, \[ \dim(\ker(T)) = 0 \]
\[ \Rightarrow \ker(T) = \{0\} \]
\[ \Rightarrow T \text{ is one-to-one.} \]

Example 1: Let $V$ be a vector space and $T$ is a linear transformation from $V$ into $V$. Prove that the following two statements about $T$ are equivalent:
(i) $\text{R}(T) \cap \text{N}(T) = \{0\}$
(ii) $T[T(x)] = x \Rightarrow T(x) = x$.

Solution:
To show $(i) \Rightarrow (ii)$
\[ \text{R}(T) \cap \text{N}(T) = \{0\} \]
\[ \Rightarrow T(x) = x \]
\[ \Rightarrow T(x) \in \text{R}(T) \cap \text{N}(T) \]
\[ \Rightarrow T(x) \in \text{N}(T) \]
\[ \Rightarrow T(T(x)) = 0 \]

To show $(ii) \Rightarrow (i)$
\[ \alpha \in \text{R}(T) \cap \text{N}(T) \]
\[ \Rightarrow \alpha \in \text{R}(T), \alpha \in \text{N}(T) \]
\[ \Rightarrow T(\alpha) = 0 \]

Now let $T(\beta) = \alpha$.
\[ \Rightarrow T(T(\beta)) = T(\alpha) \]
\[ = 0 \]
But $T(\beta) = \alpha \neq 0$.

This contradicts the hypothesis.
\[ \Rightarrow \alpha \notin \text{R}(T) \cap \text{N}(T) \]
\[ \Rightarrow \text{R}(T) \cap \text{N}(T) = \{0\} \]
1. Let $T$ be a linear transformation on $V$ and let $\text{Rank}T^2 = \text{Rank}T$, then show that $\text{Range}T \cap \text{Ker}T = \{0\}$.

**Solution:**

- $T : V \to V$, $T^2 : V \to V$

- $\text{Rank}(T) = \dim V - \dim \text{Ker}T$
- $\text{Rank}(T^2) = \dim V - \dim \text{Ker}T^2$

$\Rightarrow \dim \text{Ker}T = \dim \text{Ker}T^2$.

We claim $\text{Ker}T = \text{Ker}T^2$.

- $x \in \text{Ker}T \Rightarrow T(x) = 0 \Rightarrow T^2(x) = T(0) = 0$
- $x \in \text{Ker}T^2 \Rightarrow \text{Ker}(T) \subseteq \text{Ker}(T^2)$

Now, $x \in \text{Range}T \cap \text{Ker}T$

$\Rightarrow x \in \text{Range}T$ and $x \in \text{Ker}T$

$\Rightarrow T(x) = 0$, $x = T(y)$, for some $y \in V$.

$\Rightarrow T(T(y)) = T^2(y) = 0$
$\Rightarrow T^2(y) = 0$
$\Rightarrow y \in \text{Ker}T^2 = \text{Ker}T$

$\Rightarrow T(y) = 0 \Rightarrow x = 0$

$\Rightarrow \text{Ker}T \cap \text{Range}T = \{0\}$.

2. Determine the linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ which maps the basis vectors $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ of $\mathbb{R}^3$ to $(1, 1, 1)$, $(1, 1, 1)$, $(1, 1, 1)$ respectively. Verify that $\dim \text{Ker}T + \dim \text{Im}T = 3$.

**Solution:**

$c_1(0, 1, 1) + c_2(1, 0, 1) + c_3(1, 1, 0)$.
$c_1 + c_2 + c_3 = 0$.
$c_1 + c_3 = 2$.
$c_2 = 2 - c_1 - c_3$.

Solving $c_1 = \frac{2 + x + y}{2}$, $c_2 = \frac{2 + x - y}{2}$, $c_3 = \frac{x + y - 2}{2}$.

Since $T$ is linear,

$T(a) = c_1T(0, 1, 1) + c_2T(1, 0, 1) + c_3T(1, 1, 0)$

$= c_1(1, 1, 1) + c_2(1, 1, 1) + c_3(1, 1, 1)$

$= (c_1 + c_2 + c_3, c_1 + c_2 + c_3, c_1 + c_2 + c_3)$

To find $\text{Ker}T$, let $T(x, y, z) = (0, 0, 0)$.

$x + y + z = 0$, let $y = c, z = d, x = -(c + d)$.

Hence, $\text{Ker}T = \{(x, y, z) : (x, y, z) \in \mathbb{R}^3 \land \text{and since (1, 1, 1), (1, 1, 1), (1, 1, 1) are LIN, dim KerT = 2.} \}$
1. Find a linear operator $T$ on $\mathbb{R}^3$ so that $\ker T$ is the subspace $U = \{ (x, y, z) \in \mathbb{R}^3 : x + y + z = 0 \}$ of $\mathbb{R}^3$.

**Solution:**

Let $\mathcal{E} = (a, b, c)$ be a vector in the subspace $U$.

Then $a + b + c = 0$.

So, $\mathcal{E} = (a, b, -a-b) = a(0, 1, -1) + b(1, 0, -1)$; $a, b \in \mathbb{R}$.

This shows that $\mathcal{E} \in L = \{(1, 0, -1), (0, 1, -1)\}$.

2. $\{(1, 0, -1), (0, 0, 1)\}$ is a basis of $U$.

Thus, $U = \ker T$, $T(1, 0, -1) = (0, 0, 0)$, $T(0, 1, -1) = (0, 0, 0)$. 

3. Determine the linear mapping $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ which maps the basis vectors $(0, 0, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ of $\mathbb{R}^3$ to the vectors $(0, 2, 0)$, $(0, 0, 2)$, $(0, 0, 2)$ respectively. Find $\ker T$ and $\text{Im} T$.

Verify that $\dim(\ker T) + \dim(\text{Im} T) = 3$.

**Solution:**

Let $(x, y, z) \in \mathbb{R}^3$.

Then $x + y + z = 0$.

So, $x = y = z = 0$.

Thus, $\ker T = \{0\}$, $\dim(\ker T) = 0$.

Since $T$ is linear,

$$T(x, y, z) = xT(0, 0, 1) + yT(1, 0, 1) + zT(1, 1, 0) = \frac{x + y}{2}(0, 2, 0) + \frac{x - y}{2}(0, 0, 2) + \frac{x + z}{2}(0, 0, 2)$$

Thus, $T(x, y, z) = (0, 2x, 2y) \in \mathbb{R}^3$.

Since $\{(0, 0, 1), (1, 0, 1), (1, 1, 0)\}$ is a basis of $\mathbb{R}^3$,

Then $\text{Im} T = L = \{(0, 2, 0), (0, 0, 2), (0, 0, 2)\}$.

Thus, $\dim(\text{Im} T) = 3$.

Hence, $\dim(\ker T) + \dim(\text{Im} T) = 0 + 3 = 3$.
The basis \( \{(1,0,-1), (0,1,-1)\} \) of \( U \) can be extended to a basis
\( \{(1,0,-1), (0,1,-1), (1,0,0)\} \) of \( \mathbb{R}^3 \).

Let \( T \) be the linear operator on \( \mathbb{R}^3 \) \( \forall \ T(1,0,-1) = (0,0,0), \)
\( T(0,1,-1) = (0,0,0), T(1,0,0) = (1,0,0) \), then \( \text{Ker} T = 0 \).

Let \( (x, y, z) \in \mathbb{R}^3 \), \( (x, y, z) = c_1 (1,0,-1) + c_2 (0,1,-1) + c_3 (1,0,0) \)
\( c_1 = -y - z, c_2 = y, c_3 = x + y + z \).

\( T(x, y, z) = (-y - z) T(1,0,-1) + y T(0,1,-1) + (x + y + z) T(1,0,0) \)
\( = (-y - z)(0,0,0) + y (0,0,0) + (x + y + z) (1,0,0) \)
\( = (x + y + z, 0, 0) \) \( \forall \ (x, y, z) \in \mathbb{R}^3 \).

Note:- As the image of the basis vector \( (1,0,0) \) can be chosen
arbitrarily (other than \( (0,0,0) \)), \( T \) is not unique.

5. Find a linear mapping \( T: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) \( \forall \ \text{Im} T \) is the subspace
\( U = \{(x, y, z) \in \mathbb{R}^3 : x + y = 0 \} \) of \( \mathbb{R}^3 \).

Solution:- Let \( \mathbb{E} = (a, b, c) \) be a vector in the subspace \( U \), then \( a + b - c = 0 \).
\( \mathbb{E} = (a, b, a + b) = a (1,0,1) + b(0,1,1) \)
\( \Rightarrow \mathbb{E} \in L \{(1,0,1), (0,1,1)\} \).
\( \{(1,0,1), (0,1,1)\} \) is a basis of \( U \) as \( (1,0,1) \) and \( (0,1,1) \) are linearly independent.

\( \text{Im} T \) is generated by the images of the vectors of a basis. Let
we take the standard basis \( \{(1,0,0), (0,1,0), (0,0,1)\} \) of \( \mathbb{R}^3 \) and let
\( T(1,0,0) = (1,0,0), T(0,1,0) = (0,1,0), T(0,0,1) = (0,1,1) \).

Then \( \text{Im} T = L \{(1,0,0), (0,1,1)\} = U \).

Let \( (x, y, z) \in \mathbb{R}^3 \).

\( T(x, y, z) = x T(1,0,0) + y T(0,1,0) + z T(0,0,1) \), since \( T \) is linear,
\( = x(1,0,1) + y (0,1,1) + z (0,1,1) \)
\( = (x, y + z, x + y + z) \) \( \forall \ (x, y, z) \in \mathbb{R}^3 \).

Note:- As the image of the basis vector \( (0,0,1) \) can be chosen
arbitrarily (as a scalar multiple of \( (0,1,1) \)), \( T \) is not unique.

Linear Operators (Definition):- Let \( \mathbb{V} \) be a vector space. A linear
operator on \( \mathbb{V} \) is a function \( T \) from \( \mathbb{V} \) into \( \mathbb{V} \) \( \forall \ T(ax + b \beta) = aT(x) + bT(\beta) \forall \ x, \beta \in \mathbb{V} \) and \( a, b \in \mathbb{R} \).

Thus \( T \) is a linear operator on \( \mathbb{V} \) if \( T \) is a linear transformation
from \( \mathbb{V} \) into \( \mathbb{V} \) itself.
Composition of linear mapping:

Let $V, W$ and $U$ be vector spaces over a field $F$ and let $T: V \rightarrow W$, $S: W \rightarrow U$ be linear mappings. The composite mapping $S \circ T: V \rightarrow U$ is defined by $S(T(x))$ for all $x \in V$.

The composite $S \circ T$ is generally denoted by $ST$ and it is also said to be the product mapping $ST$.

**Theorem:** Let $V, W$ and $U$ be vector spaces over a field $F$, $T: V \rightarrow W$, $S: W \rightarrow U$ be linear mappings. Then the composite mapping $ST: V \rightarrow U$ is linear.

**Proof:** Let $\alpha, \beta$ in $V$ and $c \in F$.

$$ST(\alpha + \beta) = S[T(\alpha + \beta)]$$

Since $T$ is linear,

$$= S[T(\alpha) + T(\beta)]$$

Since $S$ is linear,

$$= S(T(\alpha)) + S(T(\beta))$$

$$= ST(\alpha) + ST(\beta)$$

$$ST(c\alpha) = S[T(c\alpha)]$$

Since $T$ is linear,

$$= S[cT(\alpha)]$$

Since $S$ is linear,

$$= cS[T(\alpha)]$$

$$= cST(\alpha)$$

This proves that $ST$ is linear.

**Note:** The unique inverse of the mapping $T$ is denoted by $T^{-1}$.

**Theorem:** Let $V$ and $W$ be vector spaces over a field $F$. A linear mapping $T: V \rightarrow W$ is invertible if and only if $T$ is one-to-one and onto.

**Theorem:** Let $V$ and $W$ be vector spaces over a field $F$. If a linear mapping $T: V \rightarrow W$ is invertible, then the inverse mapping $T^{-1}: W \rightarrow V$ is linear.

**Proof:** Let $\alpha', \beta' \in W$ and $T^{-1}(\alpha') = \alpha, T^{-1}(\beta') = \beta$.

Then $\alpha', \beta' \in V$ and $T(\alpha) = \alpha', T(\beta) = \beta'$.

Since $T$ is linear,

$$T(\alpha + \beta) = T(\alpha) + T(\beta) = \alpha' + \beta'$$

Therefore,

$$T^{-1}(\alpha' + \beta') = \alpha + \beta = T^{-1}(\alpha') + T^{-1}(\beta')$$

Since $T$ is linear,

$$T(c\alpha) = cT(\alpha), c \in F.$$  

Therefore,

$$T^{-1}(c\alpha') = c\alpha = cT^{-1}(\alpha') \in \mathbb{F}.$$  

This proves $T^{-1}$ is linear.
Definition: A linear mapping $T: V \to W$ is said to be non-singular if $T$ is invertible.

Isomorphism: Let $V$ and $W$ be vector spaces over a field $F$. A linear mapping $T: V \to W$ is said to be an isomorphism if $T$ is both one-to-one and onto.

Since $T$ is both one-to-one and onto, $T$ is invertible and $T^{-1}: W \to V$ is also a linear mapping which is both one-to-one and onto.

Thus the existence of an isomorphism $T: V \to W$ implies the existence of another isomorphism $T^{-1}: W \to V$. In this case the vector spaces $V$ and $W$ are said to be isomorphic.

Theorem: Two finite dimensional vector spaces $V$ and $W$ over a field $F$ are isomorphic if and only if $\dim V = \dim W$.

Isomorphisms from $V$ to $F^n$.

Theorem: Let $V$ be a vector space of dimension $n$ over a field $F$.

Then $V$ is isomorphic to $F^n$.

Proof: An isomorphism between $V$ and $F^n$ can be established in many ways.

Let $(\beta_1, \beta_2, \ldots, \beta_n)$ be an ordered basis of $V$. Then any vector $\xi$ of $V$ can be expressed as

$$\xi = c_1 \beta_1 + c_2 \beta_2 + \cdots + c_n \beta_n,$$

where $c_1, c_2, \ldots, c_n$ are scalars in $F$.

Let us define a mapping $\phi: V \to F^n$ by

$$\phi(\xi) = \left(\begin{array}{c} c_1 \\ c_2 \\ \vdots \\ c_n \end{array}\right),$$

where $\xi = (c_1 \beta_1 + c_2 \beta_2 + \cdots + c_n \beta_n) \in V$.

Let $\alpha = a_1 \beta_1 + a_2 \beta_2 + \cdots + a_n \beta_n \in V$,

$\beta = b_1 \beta_1 + b_2 \beta_2 + \cdots + b_n \beta_n \in V$.

Then $\alpha + \beta = (a_1 + b_1) \beta_1 + (a_2 + b_2) \beta_2 + \cdots + (a_n + b_n) \beta_n \in V$.

$$\phi(\alpha + \beta) = \left(\begin{array}{c} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{array}\right) = \phi(\alpha) + \phi(\beta) \quad \text{(i)}$$

$$\phi(p \xi) = \left(\begin{array}{c} p c_1 \\ p c_2 \\ \vdots \\ p c_n \end{array}\right) = p \phi(\xi) \quad \text{(ii)}$$

From (i) and (ii), $\phi$ is a homomorphism, linear mapping.
To prove that \( \phi \) is one-to-one, let \( \alpha, \beta \in V \).
\[
\begin{align*}
\phi(\alpha) &= \phi(\beta) \\
\Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} &= \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \\
\Rightarrow a_1 &= b_1, a_2 = b_2, \ldots, a_n = b_n
\end{align*}
\]
\[
\Rightarrow \alpha = \beta.
\]
So, \( \phi \) is one-to-one.

To prove that \( \phi \) is onto, let \( \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \) be an element in \( F^n \).
\[
\begin{align*}
\text{Then } n_1 \beta_1 + n_2 \beta_2 + \cdots + n_n \beta_n &\in V. \\
\text{and } \phi(n_1 \beta_1 + n_2 \beta_2 + \cdots + n_n \beta_n) &= \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}
\end{align*}
\]
So, \( \phi \) is onto.

Since \( \phi \) is both one-to-one and onto, so \( \phi \) is an isomorphism.

Since \( \phi \) is isomorphism, \( V \) is isomorphic to \( F^n \).

**Ex.1:** A linear mapping \( T: \mathbb{R}^2 \to \mathbb{R}^3 \) is defined by \( T(x, y, z) = (x-y, x+2y, -x+3y) \).
\( (x, y, z) \in \mathbb{R}^3 \). Show that \( T \) is non-singular and determine \( T^{-1}(y, z) \).

**Solution:** \( T \) is a linear mapping. Let us find ker \( T \).
Let \( (a, b, c) \in \ker T \). Then \( (a-b, a+2b, b+3c) = (0, 0, 0) \).
Therefore \( a-b = 0, a+2b = 0, b+3c = 0 \).
This gives \( a = b = c = 0 \).
\( \ker T = \{0\} \). \( \therefore \) Therefore \( T \) is one-to-one.

Here \( V = \mathbb{R}^3, W = \mathbb{R}^3 \) and therefore \( \dim V = \dim W \).
Since \( T: V \to W \) is one-to-one, \( T \) is onto, since \( T \) is one-to-one and onto, so it is non-singular.

Let \( T^{-1}(x, y, z) = (a, b, c) \).
Then \( (x, y, z) = T(a, b, c) = (a-b, a+2b, b+3c) \).
Therefore \( a-b = x, a+2b = y, b+3c = z \).
\[
\Rightarrow a = \frac{1}{3}(2x+y), b = \frac{1}{3}(-x+y), c = \frac{1}{3}(x-y+3z)
\]
Therefore \( T^{-1}: \mathbb{R}^3 \to \mathbb{R}^3 \) is defined by
\[
T^{-1}(x, y, z) = \left( \frac{2}{3}x + \frac{1}{3}y - \frac{1}{3}z, \frac{1}{3}x - \frac{1}{3}y + \frac{1}{3}z, \frac{2}{3}z \right); \quad (x, y, z) \in \mathbb{R}^3.
\]
Ex. (2): A linear mapping \( \phi : \mathbb{R}^3 \to \mathbb{R}^3 \) maps the vectors \((1, 2, 3), (2, 3, 1), (3, 1, 2)\) to the vectors \((0, 1, 1), (1, 0, 1), (1, 1, 0)\) respectively. Show that \( \phi \) is an isomorphism.

**Solution:** The set of vectors \( \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\} \) is a linear set in \( \mathbb{R}^3 \), a vector space of dimension 3. Therefore, \( \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\} \) is a basis of \( \mathbb{R}^3 \). The set of vectors \( \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\} \) is a linear set in \( \mathbb{R}^3 \), a vector space of dimension 3. Therefore, \( \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\} \) is a basis of \( \mathbb{R}^3 \). The domain space and the codomain space of the linear mapping \( \phi \) are of the same dimension and \( \phi \) maps a basis of the domain space to a basis of the codomain space. So, \( \phi \) is an isomorphism.

Ex. (3): A linear mapping \( \phi : \mathbb{R}^3 \to \mathbb{R}^3 \) maps the vectors \((0, 1, 1), (1, 0, 1), (1, 1, 0)\) to the vectors \((1, 1, 1), (1, 1, 1), (1, 1, 1)\) respectively. Show that \( \phi \) is not an isomorphism.

**Solution:** The set of vectors \( \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\} \) is a linear set in \( \mathbb{R}^3 \), a vector space of dimension 3. Therefore, \( \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\} \) is a basis of \( \mathbb{R}^3 \). The images of this basis vectors are the vectors \((1, 1, 1), (1, 1, 1), (1, 1, 1)\) respectively. The set \( \{(1, 1, 1), (1, 1, 1), (1, 1, 1)\} \) is a linear set in \( \mathbb{R}^3 \) and therefore, it is not a basis of \( \mathbb{R}^3 \). The linear mapping \( \phi \) maps a basis of \( \mathbb{R}^3 \) to a set which is not a basis of \( \mathbb{R}^3 \). So, \( \phi \) is not an isomorphism.

Ex. (4): Let \( T : \mathbb{R}^2 \to \mathbb{R}^3 \) be a linear transformation defined by \( T(x, y, z) = (ax, ay, a^2) \); then the dimension of the null space of \( T \) is

(A) 0  (B) 1  (C) 2  (D) 3.

**Solution:**

(B) Null space of \( T = \{ (ax, ay, a^2); T(ax, ay, a^2) = (0, 0, 0) \} \)

\( \Rightarrow (ax, ay, a^2) = (0, 0, 0) \)

\( \Rightarrow a(x, y, a) = (0, 0, 0) \)

\( \Rightarrow a = 0, x = 0, y = 0 \)

\( \Rightarrow \alpha = -y = 2 \)

Let, \( \alpha = t \), \( \Rightarrow (x, y, z) = t(1, -1, 1) \)

Null space of \( T \) is

\( \{ t(1, -1, 1); t \in \mathbb{R} \} \)

\( \therefore \) dim \( N(T) = 1 \).
Matrix representation of a linear mapping:

$T$ is completely determined by the images $T(x_1), T(x_2), \ldots, T(x_n)$.

Each $T(x_i)$ in $W$ is a linear combination of the vectors $\beta_1, \beta_2, \ldots, \beta_m$.

Let $T(x_1) = a_{11}\beta_1 + a_{12}\beta_2 + \cdots + a_{1m}\beta_m$.

$T(x_2) = a_{21}\beta_1 + a_{22}\beta_2 + \cdots + a_{2m}\beta_m$.

$\vdots$

$T(x_n) = a_{n1}\beta_1 + a_{n2}\beta_2 + \cdots + a_{nm}\beta_m$.

where $a_{ij}$ are unique scalars in $F$ determined by the ordered basis $(\beta_1, \beta_2, \ldots, \beta_m)$.

The co-ordinate vector of $T(x_i)$ relative to the ordered basis $(\beta_1, \beta_2, \ldots, \beta_m)$ is given by the $i$th column of $A$.

The matrix $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ is said to be the matrix associated with the linear mapping $T$ relative to the chosen ordered bases of $V$ and $W$. $A$ is also called the matrix of $T$ relative to the chosen ordered bases.

**Worked Examples:**

1. A linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(x_1, x_2, x_3) = (3x_1 - 2x_2, x_1 - 3x_2 - 2x_3), \ (x_1, x_2, x_3) \in \mathbb{R}^3$.

   Find the matrix of $T$ relative to the ordered bases (i) $\{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$ and $\{(1,0),(0,1)\}$ of $\mathbb{R}^2$;
   (ii) $\{(1,0,1), (1,0,0), (0,0,1)\}$ of $\mathbb{R}^3$ and $\{(0,1),(1,0)\}$ of $\mathbb{R}^2$;
   (iii) $\{(1,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ and $\{(1,0),(1,0)\}$ of $\mathbb{R}^2$;

   **Solution:**

   (i) $T(1,0,0) = (3,1) = 3(1,0) + 1(0,1)$;
   $T(0,1,0) = (-2,-3) = -2(1,0) - 3(0,1)$;
   $T(0,0,1) = (1,-2) = 1(1,0) - 2(0,1)$.

   (ii) $T(0,1,0) = (-2,-3) = -3(0,1) + 3(1,0)$;
   $T(1,0,0) = (3,1) = 1(0,1) + 3(1,0)$;
   $T(0,0,1) = (1,-2) = 2(0,1) + 1(1,0)$.

   (iii) $T(0,1,1) = (-1,-5) = -1(1,0) - 5(0,1)$;
   $T(1,0,1) = (4,-1) = 4(1,0) - 1(0,1)$;
   $T(1,1,0) = (1,-2) = 1(1,0) - 2(0,1)$.

   (iv) $T(0,1) = (1,0)$.

2. The matrix of $T = \begin{pmatrix} 3 & -2 & 0 \\ 1 & -3 & -2 \end{pmatrix}$.

3. The matrix of $T = \begin{pmatrix} -3 & 1 & -2 \\ -2 & 3 & 1 \end{pmatrix}$.

4. The matrix of $T = \begin{pmatrix} -1 & 4 & -1 \\ -5 & -1 & -2 \end{pmatrix}$. 
2. Let \((x_1, x_2, x_3), (\beta_1, \beta_2)\) be ordered bases of the real vector spaces \(V\) and \(W\) respectively. A linear mapping \(T : V \rightarrow W\) maps the basis vectors as
\[
T(x_1) = \beta_1 + \beta_2, \quad T(x_2) = 3\beta_1 - \beta_2, \quad T(x_3) = \beta_1 + 3\beta_2.
\]
And the matrix of \(T\) relative to the ordered bases
(i) \((x_1, x_2, x_3)\) of \(V\) and \((\beta_1, \beta_2)\) of \(W\);
(ii) \((x_1 + x_2, x_2, x_3)\) of \(V\) and \((\beta_1 + \beta_2)\) of \(W\).

**Solution:**
(i) \(T(x_1) = \beta_1 + \beta_2\), \(T(x_2) = 3\beta_1 - \beta_2\), \(T(x_3) = \beta_1 + 3\beta_2\)

The matrix of \(T\) is
\[
\begin{pmatrix}
1 & 3 & 1 \\
1 & -1 & 3
\end{pmatrix}
\]

(ii) \(T(x_1 + x_2) = T(x_1) + T(x_2) = 4\beta_1 = 4\beta_1 + 0(\beta_1 + \beta_2)
\]
\(T(x_2) = 3\beta_1 - \beta_2 = 4\beta_1 - 1(\beta_1 + \beta_2)
\)
\(T(x_3) = \beta_1 + 3\beta_2 = -2\beta_1 + 3(\beta_1 + \beta_2)
\)

The matrix of \(T\) is
\[
\begin{pmatrix}
4 & 4 & -2 \\
0 & -1 & 3
\end{pmatrix}
\]

3. The matrix of a linear mapping \(T : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) relative to the ordered basis \(\{(0,1), (1,0)\}\) of \(\mathbb{R}^2\) and \(\{(1,0), (1,1)\}\) of \(\mathbb{R}^3\) is
\[
\begin{pmatrix}
1 & 2 \\
2 & 4
\end{pmatrix}
\]

**Solution:**
The ordered bases \(\{(0,1), (1,0)\}\) of \(\mathbb{R}^2\) and \(\{(1,0), (1,1)\}\) of \(\mathbb{R}^3\)

\[
\begin{pmatrix}
T(0,1) = 1(0,1) + 2(1,0) = (3,2) \\
T(1,0) = 2(0,1) + 1(1,1) = (3,1) \\
T(1,1) = 4(1,1) + 0(1,1) = (4,0)
\end{pmatrix}
\]

Let \((x, y, z) \in \mathbb{R}^3\) and let \((x, y, z) = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0))\) where \(c_1, c_2, c_3\) are scalars in \(\mathbb{R}\).

\[
\begin{align*}
T(x, y, z) &= c_1T(0,1,1) + c_2T(1,0,1) + c_3T(1,1,0) \\
&= c_1(3,2) + c_2(3,1) + c_3(4,0) \\
&= (3c_1 + 3c_2 + 4c_3, 2c_1 + c_2)
\end{align*}
\]

2nd Part:
\[
\begin{align*}
T(1,1,0) &= (4,0) = 4(1,1,1) - 4(0,1) \\
T(1,0,1) &= (3,1) = 3(1,1,1) - 2(0,1) \\
T(0,1,1) &= (3,2) = 3(1,1,1) - 1(0,1)
\end{align*}
\]

The matrix of \(T\) is
\[
\begin{pmatrix}
4 & 3 & 3 \\
-4 & -2 & -1
\end{pmatrix}
\]
The Transpose of a Linear Transformation:

1. Let $V$ and $W$ be vector spaces over the field $F$. For each linear transformation $T$ from $V$ into $W$, there is a unique linear transformation $T^t$ from $W^*$ into $V^*$ such that $(T^t g)(x) = g(T(x))$ for every $g \in W^*$ and $x \in V$.

2. Let $V$ and $W$ be vector spaces over the field $F$, and let $T$ be a linear transformation from $V$ into $W$. The null space of $T^t$ is the annihilator of the range of $T$; then $V$ and $W$ are finite dimensional, then

(i) $\text{rank}(T^t) = \text{rank}(T)$
(ii) the range of $T^t$ is annihilator of the null space of $T$.

Worked Examples:

1. Let $(\alpha_1, \alpha_2, \alpha_3)$ be an ordered basis of a real vector space $V$ and a linear mapping $T : V \to V$ is defined by

$T(\alpha_1) = \alpha_1 + \alpha_2 + \alpha_3$, $T(\alpha_2) = \alpha_1 + \alpha_2$, $T(\alpha_3) = \alpha_3$.

Show that $T$ is non-singular. Find the matrix of $T^{-1}$ relative to the ordered basis $(\alpha_1, \alpha_2, \alpha_3)$.

Solution: Let $m(T)$ be the matrix of $T$ relative to the ordered basis $(\alpha_1, \alpha_2, \alpha_3)$

Then $m(T) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$m(T)$ is non-singular and therefore $T$ is non-singular.

$T^{-1}(\alpha_1 + \alpha_2 + \alpha_3) = \alpha_1$, $T^{-1}(\alpha_1) + T^{-1}(\alpha_2) + T^{-1}(\alpha_3) = \alpha_1$

$T^{-1}(\alpha_1 + \alpha_2) = \alpha_2$, $T^{-1}(\alpha_1) + T^{-1}(\alpha_2) = \alpha_2$

$T^{-1}(\alpha_1) = \alpha_3$, $T^{-1}(\alpha_2) = \alpha_3$ ; since $T^{-1}$ is also linear.

We have $T^{-1}(\alpha_1) = \alpha_3$, $T^{-1}(\alpha_2) = \alpha_2 = \alpha_3$,

$T^{-1}(\alpha_2) = \alpha_1 - \alpha_2$.

Therefore, $m(T^{-1}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$

$\therefore T^{-1}$ is the matrix representation of $T^{-1}$ with respect to the basis $(\alpha_1, \alpha_2, \alpha_3)$.
Let $T$ be the linear transformation from $\mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1).$$

If $\beta = \{(1, 0, -1), (1, 1, 1), (1, 0, 0)\}$, $\beta' = \{(0, 1), (1, 0)\}$ be ordered bases of $\mathbb{R}^3, \mathbb{R}^2$, respectively. Then find the matrix of $T$ relative to $\beta, \beta'$, also find $\text{rank}(T)$ and $\text{nullity}(T)$.

Solution:

(i) \[ T(1, 0, -1) = (1, -3) = -3(0, 1) + 1(1, 0) \]
\[ T(1, 1, 1) = (2, 1) = 1(0, 1) + 2(1, 0) \]
\[ T(1, 0, 0) = (1, 0) = -1(1, 0) + 1(0, 1) \]

The matrix of $T$ is

$$\begin{bmatrix}
-3 & 1 & -1 \\
1 & 2 & 1
\end{bmatrix}$$

(ii) \[ \text{let } (x_1, x_2, x_3) \in \text{ker}(T). \]
\[ T(x_1, x_2, x_3) = (0, 0) = (x_1 + x_2, 2x_3 - x_1) \]
\[ \Rightarrow x_1 + x_2 = 0, \ 2x_3 - x_1 = 0 \]
\[ \Rightarrow x_1 = -x_2, \ x_1 = 2x_3 \]
\[ \Rightarrow x_1 = -x_2 = 2x_3 \quad \text{let } x_3 = t, \]
\[ \Rightarrow \text{ker}(T) = \{ (2, -2, 1) : t \in \mathbb{R} \} \]
\[ \Rightarrow \dim \text{ker}(T) = 1, \]
\[ \Rightarrow \text{nullity}(T) = 1. \]

We know, $\text{rank}(T) + \text{nullity}(T) = \dim \mathbb{R}^3 = 3$.

\[ \Rightarrow \text{rank}(T) = 3 - 1 = 2. \]

3) Show that $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (y, x)$ is a linear transformation. For the basis sets $S_1 = \{(1, 0, 0) \}$ of $\mathbb{R}^3$ and $S_2 = \{(0, 1), (2, 3)\}$ of $\mathbb{R}^2$. Find the matrix of $T$.

Solution:

\[ T(x, y, z) = (y, x) = x(1, 0) + z(2, 3) \]
\[ T(x, y, z) = (y, x) = x(1, 0) + y(2, 3) \]
\[ T(x, y, z) = (y, x) = x(1, 0) + y(2, 3) \]
\[ T(x, y, z) = (y, x) = x(1, 0) + y(2, 3) \]

Hence $T$ is a linear transformation.

$$
\begin{bmatrix}
T(e_1) & T(e_2) & T(e_3)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & -1 \\
1 & 0 & 2
\end{bmatrix}
$$

The matrix representation of $T$ is

$$
\begin{bmatrix}
-2/3 & 1 & 0 \\
1/3 & 0 & 0
\end{bmatrix}
$$
PROBLEMS:

1. Let \( T \) be an invertible linear operation on \( \mathbb{V}_2(\mathbb{R}) \) defined by
   \[
   T(x, y, z) = (2x, x - y, 2x + y + z) \forall (x, y, z) \in \mathbb{V}_2(\mathbb{R})
   \]
   Then \( T^{-1}(3, 0, 1) \) is equal to
   (A) \((4, 0, 3)\) (B) \((1, 1, 1)\) (C) \((1/2, 3)\) (D) \((0, 0, 0)\)

   **Sol.**
   If \( T(x, y, z) = (3, 0, 1) \)
   Then \( T^{-1}(3, 0, 1) = (x, y, z) \)

   Now, \( T(x, y, z) = (3, 0, 1) \)
   \[
   \Rightarrow (2x, x - y, 2x + y + z) = (3, 0, 1)
   \]
   \[
   \Rightarrow 2x = 3, x - y = 0, 2x + y + z = 1
   \]
   \[
   \Rightarrow x = 1, y = 1, z = 1
   \]
   \[
   \therefore T^{-1}(3, 0, 1) = (1, 1, 1)
   \]

2. The linear transformation \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) is defined as
   \[
   T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2)
   \]
   Then the null space of \( T \) is
   (A) \( \{(0, 0, 0), (1, 0, 0)\} \) (B) \( \{(0, 0, 0), (0, 0, 0)\} \) (C) \( \{(1, 0, 0), (0, 1, 0)\} \) (D) None

   **Sol.**
   \[
   T(x_1, x_2, x_3) = (0, 0, 0) = (x_1 - x_2 + 2x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2)
   \]
   \[
   \Rightarrow x_1 - x_2 + 2x_3 = 0, \quad 2x_1 + x_2 - x_3 = 0, \quad -x_1 - 2x_2 = 0
   \]
   Coefficient matrix
   \[
   \begin{bmatrix}
   1 & -1 & 2 \\
   2 & 1 & -1 \\
   -1 & 2 & 0
   \end{bmatrix}
   \]
   \[
   |A| = -9 \neq 0
   \]
   \[
   \therefore \text{Rank}(A) = 3
   \]
   Hence the equations have the only trivial solution \( x_1 = x_2 = x_3 = 0 \).

3. The linear transformation \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) defined as
   \[
   T(x, y, z) = (2x - 2y + z, x - 2y - 2z) \quad \text{then the range of} \ T \text{is}
   \]
   (A) \( \{(3, -2), (1, -3, -2)\} \) (B) \( \{(3, 1), (-2, -3), (1, 2)\} \)
   (C) \( \{(1, 0, 0), (0, 1, 0)\} \) (D) None of the above

   **Solution:**
   (B) \( \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \) is a basis for \( \mathbb{R}^3 \).

   1. \( T(1, 0, 1) = (3, 1) \)
   \[
   T(0, 1, 0) = (-2, -3)
   \]
   \[
   T(0, 0, 1) = (1, -2)
   \]
1. Let \( T: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be the linear transformation defined by 
\[
T(x, y, z) = (x + 2y - 2z, y + z, x - 2y + 2z).
\]
Then which of the following is not a null space of \( T \)?
 choice (A) \( (3, 1, -1) \) choice (B) \( (6, -4, 2) \) choice (C) \( (2, -2/3, 2/3) \) choice (D) \( (6, -2, 2) \)

**Sol.**

\[
\begin{align*}
T(3, 1, -1) &= (0, 0, 0) \\
T(6, -4, 2) &= (0, 0, 0) \\
T(2, -2/3, 2/3) &= (0, 0, 0) \\
T(6, -2, 2) &= (0, 0, 0)
\end{align*}
\]

But \( T(6, -2, 2) \neq (0, 0, 0) \).

9. With respect to the standard basis vectors, a linear transformation \( T: \mathbb{R}^4 \rightarrow \mathbb{R}^3 \) is given by the matrix
\[
\begin{bmatrix}
3 & -1 & -1 & 0 \\
1 & 2 & 2 & 2 \\
1 & 1 & -2 & -1
\end{bmatrix}
\]

Then dimension of \( \text{null}(T) \) is

(A) 1  (B) 2  (C) 3  (D) 4

**Sol.**

(A) \( \text{dimension (null} T) = \dim(\mathbb{R}^4) - \text{rank}(T) = 4 - 3 = 1 \).  

6. The coordinate of the vector \( (2, 1, -6) \) relative to the basis \( \{x_1, x_2, x_3, x_4\} \), where \( x_1 = (1, 1, 2), x_2 = (3, -1, 0), x_3 = (2, 0, 1) \) is

(A) \( (-7/3, -15/4, 17/4) \)  (B) \( (-7/3, -13/4, 17/4) \)  (C) \( (-7/2, -9/4, 15/4) \)  (D) None of these.

**Sol.**

Let \( a_1, b_1, c_1 \) be defined in \( \mathbb{R} \).
\[
\begin{align*}
(2, 1, -6) &= a_1 x_1 + b_2 x_2 + a_3 x_3 \\
(2, 1, -6) &= a(1, 1, 2) + b_2 (3, -1, 0) + c_3 (2, 0, 1) \\
(2, 1, -6) &= (a + 3b_2 + 2a_3, a - b_2, 2a_1 - a_3) \\
\Rightarrow a + 3b_2 + 2a_3 &= 2, a - b_2 = 1, 2a_1 - a_3 = -6
\end{align*}
\]
Solving these equations, we get
\[
\begin{align*}
a &= -7/3, b_2 = -13/4, a_3 = 17/4.
\end{align*}
\]

7. A linear transformation \( T: \mathbb{R} \rightarrow \mathbb{R} \) is defined as
\[
T(x, y, z) = (2x, 4y, 5z). \text{ Then find the matrix of } T \text{ with respect to the basis } \{\frac{1}{2}, 1, 0\}, \{0, \frac{1}{2}, 0\}, \{0, 0, 1\}.
\]

**Sol.**

\[
\begin{align*}
T(\frac{1}{2}, 0, 0) &= (\frac{1}{2}, 0, 0) = 2(\frac{1}{2}, 0, 0) \\
T(0, \frac{1}{2}, 0) &= (0, 1, 0) = 4(0, \frac{1}{2}, 0) \\
T(0, 0, 1) &= (0, 0, 1) = 5(0, 0, 1)
\end{align*}
\]
Hence the required matrix is
\[
\begin{bmatrix}
2 & 0 & 0 \\
0 & 4 & 0 \\
0 & 0 & 5
\end{bmatrix}
\]
8. Let $T$ be a linear transformation on $\mathbb{R}^2$, the matrix of which in the standard ordered basis is $A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$.

Prove that the only subspaces of $\mathbb{R}^2$ invariant under $T$ are $\mathbb{R}^2$ and $\{0\}$.

**Sol.** Characteristic polynomial of $A$ (or $T$) is $\det \begin{bmatrix} x-1 & -1 \\ 2 & x-2 \end{bmatrix} = x^2 - 3x + 4$, whose roots are not real.

A characteristic value of $A$ (or $T$) do not exist in $\mathbb{R}$. If $W$ is an invariant subspace of $\mathbb{R}^2$ s.t. $W \neq \{0\}$, $\dim W = 1$.

Let $W$ be spanned by $v$. Then $T(v) \in W \Rightarrow T(v) = \alpha v$, $v \neq 0$.

$\Rightarrow$ $v$ is an eigenvalue of $T$ ($\alpha \in \mathbb{R}$), a contradiction.

So, $0$ and $\mathbb{R}^2$ are the only invariant subspaces of $\mathbb{R}^2$.

9. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation defined by $T(x, y, z) = (x+y+z, y+z, z)$. Then $T^n(x, y, z)$ is $\ldots$ $(n \geq 1)$.

**Sol.**

$T^2(x, y, z) = (x+2y+3z, y+2z, z) \Rightarrow T^n(x, y, z) = (x+ny+\frac{n(n+1)}{2}z, y+nz, z)$

10. Let $V = \mathbb{R}^3$ and $T : V \rightarrow V$ is a linear map $\alpha$ the matrix of $T$ w.r.t. the standard basis is $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

Show that $T$ is an isomorphism of $V$ onto $V$. Find the matrix of $T^{-1}$ w.r.t. the same basis and verify that it is the inverse of $A$.

**Sol.**

$T(x, y, z) = \begin{bmatrix} x+y+z \\ y+z \\ z \end{bmatrix}$,

Then, $(x+y+z, y+z, z) = (0, 0, 0)$.

$\Rightarrow x+y+z = 0$.

Hence, $T$ is one-to-one, so $T$ is an isomorphism.

Let $e_1, e_2, e_3$ be the standard basis.

Since $T(e_1) = e_1 + e_3$, $T(e_2) = e_2 + e_3$, $T(e_3) = e_1 + e_2$

$\Rightarrow e_1 = \frac{1}{2} \left[ T(e_1) + T(e_3) - T(e_2) \right] = \frac{1}{2} T \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ [As $T$ is linear]

$\Rightarrow T^{-1} \left( e_1 \right) = \frac{1}{2} \left( e_1 + e_3 - e_2 \right)$

(Similarly, $T^{-1} \left( e_2 \right) = \frac{1}{2} \left( e_2 + e_3 - e_1 \right)$

and $T^{-1} \left( e_3 \right) = \frac{1}{2} \left( e_1 + e_2 - e_3 \right)$)

Hence the matrix of $T^{-1}$ is $\frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = M$.

Now, $AM = I_3$, so we get that it is also the inverse of $A$. 

---
Ex. Let $P$ be a matrix of order $> 1$ and has integer entries, then what are the set of possible values of $|P|$?

Sol. $P$ has integer entries,

\[ \Rightarrow \lambda + \lambda_2 + \cdots + \lambda_n = \text{trace}(P) = \text{integer}, \]

\[ \Rightarrow \sum_{i<j} \lambda_i \lambda_j = \sum \text{sum of minors of order} \]

\[ \Rightarrow \prod_{i=1}^n \lambda_i = |P| = \text{integer}. \]

Then the eigenvalues of $P$ are $\frac{1}{\lambda_i}$ and they are also integer.

\[ \Rightarrow \lambda_i = \pm 1 \]

\[ \Rightarrow |P| = \prod_{i=1}^n \lambda_i = \pm 1. \]

Ex. Let $XY$ be a bivariate normal vector $\mathcal{N}$. $E(X) = E(Y) = 0$ and $V(X) = V(Y) = 1$. Let $S$ be a subset of $\mathbb{R}^2$ and defined by

\[ S = \{(a, b) : (aX + bY) \text{ is independent of } Y \} \]

(i) Show $S$ is a subspace.

Sol. $S = \{(a, b) : (aX + bY) \text{ is independent of } Y \}$.

\[ (a_1, b_1), (a_2, b_2) \in S, \]

Then $a_1X + b_1Y$ is independent of $Y$.

\[ a_2X + b_2Y \text{ is independent of } Y, \]

\[ (a_1X + b_1Y) + (a_2X + b_2Y) \in S \]

\[ a_1X + b_1Y \in S \]

Hence, $S$ is a subspace.

(ii) $(a, b) \in S$.

\[ \Rightarrow aX + bY \text{ is independent of } Y. \]

\[ \Rightarrow \text{cov}[aX + bY, Y] = 0 \]

\[ \Rightarrow a \text{cov}[X, Y] + b \text{cov}[Y, Y] = 0 \]

\[ \Rightarrow a\mu + b\sigma_Y = 0 \]

\[ \Rightarrow b = -a\mu \]

\[ \Rightarrow \text{cov}[X, Y] = \text{Var}(Y) = 1 \]

\[ \Rightarrow \mu_X = \frac{1}{\sqrt{\mu_Y}} \]
SYSTEM OF LINEAR EQUATION

Suppose that there are $m$ equations and $n$ unknowns, say $x_1, x_2, \ldots, x_n$. We may put the equations in the form

$$\begin{align*}
& a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
& \quad \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{align*}$$

We may write the equations in the matrix form

$$Ax = b \quad (*)$$

where, $A = ([a_{ij}]_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\
& a_{21} & a_{22} & \cdots & a_{2n} \\
& \vdots & \vdots & \ddots & \vdots \\
& a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n}$, and

$$x = \begin{pmatrix} x_1 \\
x_2 \\
\vdots \\
x_n \end{pmatrix} = (x_1, x_2, \ldots, x_n) \quad \text{and} \quad b = \begin{pmatrix} b_1 \\
b_2 \\
\vdots \\
b_m \end{pmatrix} = (b_1, b_2, \ldots, b_m)$$

\[\begin{align*}
\]$$
\]$$

\[\begin{align*}
\]$$

**Definition:**

- **Consistent system and Inconsistent System:**

  By a solution of the system $(*)$, we mean any set of values $x_1, x_2, \ldots, x_n$ that satisfies the $m$ equations simultaneously. The system is said to be **consistent if it has at least one solution**, otherwise it is said to be **inconsistent**.

**Ex. 1.** The system of equations

\[\begin{align*}
& x_1 + 2x_2 = 5 \\
& 2x_1 + 4x_2 = 3
\end{align*}\]

- It is an **inconsistent system** since it does not have any solution. The term 'inconsistent' is appropriate to this situation, for the first equation implies $2x_1 + 4x_2 = 10$, which is incompatible with the second equation.
* Ex. 2. The system
\[\begin{align*}
\alpha_1 + 2\alpha_2 &= 5 \\
3\alpha_1 - \alpha_2 &= 1
\end{align*}\]
- it is a consistent system having a unique solution with
\[\alpha_1 = 1, \alpha_2 = 2.\]

* Ex. 3. The equation
\[\alpha_1 + 2\alpha_2 = 5\]
- it is a consistent system having infinite number of solutions.

Now we shall go for finding conditions under which a system will be consistent.

### Case of a homogeneous System:
If in a system each term in the R.H.S is zero or in the other word, \(b = 0\), then the system is said to be homogeneous system.

\[A\alpha = 0 \quad (**\)\]

- A system of homogeneous equation is necessarily consistent as it possesses a trivial solution \(\alpha = \mathbf{0}\).
- i.e. \(\alpha = \mathbf{0}\) is always a sol. of (**)\(, \) the matrix \(A\) may be \(, \) we call it the solution a trivial solution.

Thus it may be of interest to know whether the system has any non-trivial (non-zero) or, if possesses the trivial solution as the only solution. If there is at least one non-trivial soln, then it is of interest to find a set of maximum number of linearly independent non-null soln.

### Theorem:
The necessary and sufficient condition for a system
\[A\alpha = 0 \quad (mxn, nx1, mx1)\]

**Proof:**
- Lied \(\text{rank}(A) < n\).
- \(\iff\) the columns of \(A\), say \(\alpha_1, \alpha_2, \ldots, \alpha_n\) are linearly dependent.
- \(\iff\) the equation \(\alpha_1\alpha_1 + \alpha_2\alpha_2 + \ldots + \alpha_n\alpha_n = 0\) has at least one non-trivial solution.
- \(\iff\) \(A\alpha = 0\) has at least one non-trivial solution.
- \(\iff\) System is consistent.
Theorem: Let \( \text{rank}(A) = r < n \), then the system \( AX = \theta \) has exactly \((n - r)\) independent solutions.

Proof: Let \( S \) denote the solution space of \( AX = \theta \) and \( R \) denotes the row space of \( A \).

Obviously, \( S \) will be an orthocomplement of \( R \).

Let, \( x_1, x_2, \ldots, x_n \) be an orthogonal basis of \( R \). This can be extended to an orthogonal basis of \( \mathbb{R}^n \), say, \( r_1, r_2, \ldots, r_{n-r}, r_{n-r+1}, \ldots, r_n \). Here in contradiction, we prove that \( r_1, r_2, \ldots, r_{n-r} \) is an orthogonal basis of \( S \).

\[ \dim(S) = n - r. \]

Consider a vector \( x \in S \), which can't be written as a linear combination of \( r_1, r_2, \ldots, r_{n-r} \). Then the set \( r_1, r_2, \ldots, r_{n-r}, x \) will be linearly independent and generate the subspace \( S_2 \subseteq S \).

Now, the set \( \{x_1, x_2, \ldots, x_n, r_1, r_2, \ldots, r_{n-r}, x\} \) is a set of non-orthogonal vectors and this is a linearly independent set of \( \mathbb{R}^n \) vectors in \( \mathbb{R}^n \), which is impossible.

Therefore \( r_1, r_2, \ldots, r_{n-r} \) forms a basis of \( S \). So, \( \dim(S) = n - r \), i.e., the system has exactly \((n - r)\) independent solutions.

Solution Space: The vector space generated by the soln. of \( AX = \theta \) is the solution space of the equation, i.e., \( \{x : AX = \theta\} \) which is the null space of \( A \). \( N(A) \) is the null space of \( A \).

We know, \( \dim(N(A)) = (n - r) \), where \( r = \text{rank}(A) \).

Clearly, \( \dim(N(A)) = 0 \) if \( A \) is of full column rank, in that case, we have the trivial soln. as the only soln.

If \( \dim(N(A)) > 0 \), we may search for a basis of the soln. space, \( [ \text{basis is the set of maximum no. of LIN soln.} ] \).

Suppose \( x_1, x_2, \ldots, x_{n-r} \) be a basis of the soln. space. Then any soln. of \( AX = \theta \) can be written as a linear combination of the basis vectors, i.e., \( x = \sum \alpha_i x_i \).
NOTE: If there exists a non-null solution then it must be orthogonal to the non-null row of $A$.

Rewrite $A$ as

\[ A = (x_1, x_2, \ldots, x_n) \]

and

\[ x_i = (x_1, x_2, \ldots, x_n) \]

Then $Ax = 0$ reduces to

\[ \sum_{i=1}^{n} x_i x_i = 0 \quad (***) \]

to search for a non-trivial solution is equivalent to check whether $(***)$ holds good for at least one $x_i \neq 0$.

In other words, the system of equations defined by $Ax = 0$ possesses a non-trivial solution if column of $A$ are linearly dependent.

**Ex. 1.** Obtain the general soln to the system of equations:

\[ \begin{align*}
\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 &= 0 \\
\lambda_1 - 2\lambda_2 + 2\lambda_3 &= 0 \\
\lambda_1 + 7\lambda_2 - 3\lambda_3 - \lambda_4 &= 0
\end{align*} \]

**Soln.**

\[ Ax = 0 \]

\[ A = \begin{pmatrix}
1 & 1 & 1 & 1 \\
-2 & 0 & 2 & 0 \\
1 & 3 & -1 & 0
\end{pmatrix} = (x_1 x_2 x_3 x_4) \]

$x_1$ and $x_2$ are linearly independent.

If possible, let $x_3 = \lambda_1 x_1 + \lambda_2 x_2$

\[ \Rightarrow \begin{pmatrix}
\frac{1}{3} \\
\frac{2}{3}
\end{pmatrix} = \lambda_1 \begin{pmatrix}
1 \\
1
\end{pmatrix} + \lambda_2 \begin{pmatrix}
\frac{1}{2} \\
1
\end{pmatrix} = \begin{pmatrix}
\frac{\lambda_1 + \lambda_2}{\lambda_1 + 2\lambda_2} \\
\frac{\lambda_1 - 2\lambda_2}{\lambda_1 + 7\lambda_2}
\end{pmatrix} \]

\[ \Rightarrow \lambda_1 = \frac{2}{3}, \lambda_2 = \frac{1}{3} \]

i.e., $\frac{2}{3} x_1 + \frac{1}{3} x_2 - x_3 = 0$

\[ \Rightarrow \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix} \begin{pmatrix}
\frac{2}{3} \\
\frac{1}{3} \\
-1 \\
0
\end{pmatrix} = 0 \]

i.e., \( \begin{pmatrix}
\frac{2}{3} \\
\frac{1}{3} \\
-1 \\
0
\end{pmatrix} \) is a non-trivial solution.
If possible, let \( A \mathbf{x} = \mathbf{0} \) = \( \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 \)

\[
\begin{pmatrix}
\lambda_1 + \lambda_2 \\
\lambda_1 - 2\lambda_2 \\
\lambda_1 + 7\lambda_2
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} \\
-2 \\
-7
\end{pmatrix}
\]

\( \Rightarrow \lambda_2 = -\frac{1}{2}, \lambda_1 = \frac{4}{3} \)

Then another non-trivial independent solution is:

\[
\begin{pmatrix}
\frac{4}{3} \\
-\frac{1}{3} \\
0
\end{pmatrix}
\]

Here, \( R(A) = 2 \Rightarrow \text{dim} (N(A)) = 2 \).

A basis of the solution space is:

\[
\begin{pmatrix}
\frac{2}{3} \\
\frac{1}{3} \\
-1
\end{pmatrix}, \begin{pmatrix}
\frac{4}{3} \\
-\frac{1}{3} \\
0
\end{pmatrix}
\]

A basis of the solution space:

\[
\begin{pmatrix}
2x + 4z \\
2x - 3y \\
-3y_1 - 3y_3
\end{pmatrix}
\]

\[
A = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -2 & 0 & 2 \\
1 & 7 & 3 & -1
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 3 & 1 & -1 \\
0 & -6 & -2 & 2
\end{pmatrix}
\]

\( \Rightarrow R_2' = R_1 - R_2, \quad R_3' = R_1 - R_3 \)

\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & -\frac{1}{3} \\
0 & 0 & 0
\end{pmatrix} \quad \Rightarrow R_2' = R_2/3, \quad R_3' = 2R_2 + R_3
\]

\( \Rightarrow R(A) = 2 \)

Here, rank \( (A) = 2 \).

1. \( \begin{pmatrix}
-2 \\
0 \\
1 \\
1
\end{pmatrix} \)

\( \Rightarrow \mathbf{x} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 + \mathbf{u}_4 = 0 \)

and, \( \mathbf{x} = \mathbf{u}_2 + \frac{\mathbf{u}_3}{3} - \frac{\mathbf{u}_4}{3} = 0 \)

\[
\begin{pmatrix}
-2x + \frac{2}{3}y \\
-\frac{2}{3}y \\
\frac{2}{3}y - \frac{1}{3}z
\end{pmatrix}
\]

**Note:** If \( m \) exceeds \( n \), then at least \( m-n \) equation become redundant, thus we consider \( m \leq n \).
Ex. 2. Obtain the solution to the system of equations:

\[
\begin{align*}
\lambda_1 + 3\lambda_2 + 2\lambda_3 + 3\lambda_4 &= 0 \\
\lambda_2 + 2\lambda_3 + \lambda_4 - \lambda_1 &= 0 \\
3\lambda_1 + 4\lambda_2 + \lambda_3 + 4\lambda_4 &= 0
\end{align*}
\]

Solu.\rightarrow

\[
A\lambda = 0
\]

\[
\begin{bmatrix}
-\frac{1}{2} & 1 & 2 & 3 \\
-\frac{1}{2} & 1 & 1 & 4 \\
\end{bmatrix}
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4 \\
\end{bmatrix} = 0
\]

on, \((\lambda_1 \lambda_2 \lambda_3 \lambda_4)\lambda = 0\)

Here, \(R(A) = 2\).

\(\lambda_1\) and \(\lambda_2\) are linearly independent,

Note that \(\lambda_3 = \lambda_2 - \lambda_1\)

\[
(\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4 \\
\end{bmatrix}
\begin{bmatrix}
-1 \\
1 \\
\end{bmatrix} = 0
\]

i.e. \((\begin{bmatrix}
-1 \\
1 \\
\end{bmatrix})\) is a solution.

Again, \(\lambda_2 - \lambda_4 = 0\)

\[
(\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
1 \\
\end{bmatrix} = 0
\]

\[
(\begin{bmatrix}
0 \\
1 \\
\end{bmatrix})\] is another linearly independent solution.

A basis of the solution space is \(S = \left\{ \left( \begin{bmatrix}
-1 \\
1 \\
\end{bmatrix}, \begin{bmatrix}
0 \\
-1 \\
\end{bmatrix} \right) \right\}\)

A basis of the solution space is \(\left\{ \left( \begin{bmatrix}
\alpha \\
\beta - \alpha \\
\alpha - \beta \\
\end{bmatrix} \right) \right\}\)
Ex. 3. Consider the system of equations:
\[
\begin{align*}
\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 &= 0 \\
\kappa_1 + 2\kappa_2 + 3\kappa_3 + 4\kappa_4 &= 0 \\
2\kappa_1 + \kappa_2 - \kappa_4 &= 0
\end{align*}
\]

**Solution**
\[
A = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 \\
2 & 1 & 0 & -1
\end{pmatrix}, \quad \kappa = \begin{pmatrix}
\kappa_1 \\
\kappa_2 \\
\kappa_3 \\
\kappa_4
\end{pmatrix}
\]

The \text{Gauss-Jordan method} method gives:
\[
A \sim \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 2 & 3 & 4 \\
0 & -1 & -2 & -3
\end{pmatrix} \sim \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 2 & 3 & 4 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[= R, \text{ a row-reduced echelon matrix}\]

So, \text{Rank} \(A\) = 2.

\[
\begin{pmatrix}
\kappa_1 \\
\kappa_2 \\
\kappa_3 \\
\kappa_4
\end{pmatrix} \sim \begin{pmatrix}
\kappa_1 - \kappa_3 - 2\kappa_4 = 0 \\
\kappa_2 + 2\kappa_3 + 3\kappa_4 = 0
\end{pmatrix}
\]

\[= \begin{pmatrix}
\kappa_1 \\
\kappa_2 \\
\kappa_3 \\
\kappa_4
\end{pmatrix} \sim \begin{pmatrix}
\alpha + 2\beta \\
-2\alpha - 5\beta \\
\alpha + 3\beta \\
\beta
\end{pmatrix}
\]

Ex. 1. The system of equations:
\[
\begin{align*}
x - y + z &= 0, \\
x + 2y - z &= 0, \\
2x + y + 3z &= 0
\end{align*}
\]

(A) infinite solutions,  (B) trivial solution, (C) no solution.

**Solution (B)**
\[
A = \begin{pmatrix}
1 & -1 & 1 \\
1 & 2 & 1 \\
2 & 1 & 3
\end{pmatrix} \sim \begin{pmatrix}
1 & -1 & 1 \\
0 & 2 & -1 \\
0 & 0 & 3
\end{pmatrix} \sim \begin{pmatrix}
1 & -1 & 1 \\
0 & 2 & -1 \\
0 & 0 & 3
\end{pmatrix}
\]

\[
\begin{pmatrix}
\kappa_1 \\
\kappa_2 \\
\kappa_3 \\
\kappa_4
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

\[
\begin{align*}
x - y + z &= 0 \\
2y - 2z &= 0 \\
3z &= 0
\end{align*}
\]

\[x = 0, y = 0, z = 0 \text{ is the only solution of the given system of equations.}\]
Case of non-homogeneous system: If $k \neq 0$ for the system of equations $Ax = b$, then $x = 0$ can't be a solution. It means that if there is at least one solution, then it must be a non-trivial solution. We will now establish a necessary and sufficient condition for a non-homogeneous system to be consistent.

$$Ax = b, \ k \neq 0.$$ 

This can be written as $\sum_{i=1}^{n} x_i a_{i1} = b_1$.

Augmented Matrix: By the augmented matrix for the system we mean the matrix $(A \mid b)$ of order $m \times (n+1)$ obtained by adjoining one more column, viz. $b$. Thus:

$$(A \mid b) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2m} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}_{m \times (n+1)}$$

C.O.V. ***

Theorem: The system of equations $Ax = b$ is compatible (consistent) or has at least one solution if and only if $\text{rank}(A \mid b) = \text{rank}(A)$.

Due to Kronecker-Capelli

Proof:

If Part: Suppose $\text{rank}(A \mid b) = \text{rank}(A)$.

Then $b$ can be written as a linear combination of the columns of $A$,

$$b \in \text{Vec}(A)$$

Suppose $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$

Therefore there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$\Rightarrow \left( \begin{array}{c} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{array} \right) = \left( \frac{b_1}{b_1}, \frac{b_2}{b_2}, \ldots, \frac{b_m}{b_m} \right) = b.$$

Hence $\left( \begin{array}{c} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{array} \right)$ is a solution to the system of equations $Ax = b$.

Therefore, the system is consistent (compatible).
Only if \( \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0 \) is consistent and let \( (\lambda_2, \cdots, \lambda_n) \) be a solution of the system.

\[
A \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = 0
\]

\[
\Rightarrow \sum_{i=1}^{n} \lambda_i x_i = 0
\]

i.e. \( \mathbf{k} \) is linearly dependent on the columns of \( A \).

\[
\Rightarrow \mathbf{k} \in \text{col}(A)
\]

i.e. \( \text{rank}(A) = \text{rank}(A:b) \).

\[
\text{column rank of } A = \text{column rank of } A:b
\]

**Theorem:** A non-homogeneous system of \( n \)-linear equations in \( n \) unknowns \( \mathbf{x} = \mathbf{b} \) has the unique solution \( \mathbf{x} = A^{-1}\mathbf{b} \) if and only if \( \text{rank}(A) = n \) on \( A \) is non-singular.

**Proof:**

If \( \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0 \), then \( A^{-1} \) exists. Putting \( \mathbf{x} = A^{-1}\mathbf{b} \) in the L.H.S of the system,

\[
A \mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = \mathbf{b} = \text{R.H.S}
\]

\[
\Rightarrow \mathbf{x} = A^{-1}\mathbf{b}
\]

is a solution of the system.

To show that this is the unique solution, let \( \mathbf{x} = \mathbf{x}' \) be another solution, we can write \( \mathbf{x} = A^{-1}\mathbf{b} + \mathbf{b}' \), where \( \mathbf{b}' = \mathbf{x} - A^{-1}\mathbf{b} \).

\[
A \mathbf{x} = A(A^{-1}\mathbf{b} + \mathbf{b}') = \mathbf{b}
\]

\[
\Rightarrow (AA^{-1})\mathbf{b} + A\mathbf{b}' = \mathbf{b}
\]

\[
\Rightarrow \mathbf{b} + A\mathbf{b}' = \mathbf{b}
\]

\[
A\mathbf{b}' = \mathbf{0}
\]

If \( \mathbf{b}' = \mathbf{0} \), then this is obviously satisfied. On the other hand, if \( \mathbf{b}' \neq \mathbf{0} \), then this means that the columns of \( A \) will form a linearly dependent set. This means \( \text{rank}(A) < n \) which is a contradiction. So, \( \mathbf{x} = A^{-1}\mathbf{b} \) is the unique solution.
Only if both: Suppose \( x_i = A^{-1} b \) is a unique solution of \( A x = b \).

So, \( x_i = 0 \) is the unique solution of \( A x = 0 \).

\[ \sum_{i=1}^{n} a_i x_i = 0 \]

has the only solution \( x_i = 0 \), so, \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are linearly independent, hence \( \text{rank}(A) = n \), where \( \alpha_i \) is the column vector of \( A \).

Note:

\[ A \hat{x} = \hat{b} \quad (\hat{b} \neq 0) \]

\[ \iff \sum_{i=1}^{n} a_i x_i = b \]

If \( \alpha_i \) are linearly independent then the collection constitutes the basis of the column space, in that case the representation of \( \hat{b} \) in terms of columns of \( A \) will be unique, i.e., the choice of \( \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \) is unique. Hence the system possesses a unique solution.

On the other hand, if the columns of \( A \) are linearly independent, then the representation of \( \hat{b} \) will not be unique.

**General solution to a system of non-homogeneous equation:**

Let \( A \hat{x} = \hat{b} \) be a system of non-homogeneous equation where \( \hat{y} \) is a specific solution to the system of equations, and \( \hat{x} \) is the general solution to \( A \hat{x} = 0 \), then \( (\hat{y} + \hat{x}) \) will be the general solution to \( A \hat{x} = \hat{b} \).

**Ex. 1:** Obtain the general soln. to the following system of equations:

\[
\begin{align*}
4u_1 + 2u_2 + 2u_3 + u_4 &= 2 \\
u_1 - 2u_2 + 2u_3 &= 1 \\
u_1 - 3u_2 + 3u_3 - u_4 &= 8
\end{align*}
\]

**Soln.**

\[ A = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -2 & 0 & 2 \\
1 & 7 & 3 & -1
\end{pmatrix}, \quad \hat{b} = \begin{pmatrix}
2 \\
1 \\
8
\end{pmatrix} \]

\[ A \hat{x} = \hat{b} \]

\[ \iff P A \hat{x} = P \hat{b}, \quad \text{where } P \text{ is an non-singular matrix} \]

i.e., \( A \hat{x} = \hat{b} \)

\[ PA \hat{x} = P \hat{b} \]

Reducing \( (A | \hat{b}) \) to \( \begin{pmatrix}
1 & 2 & 0 & -1 & 2 \\
0 & 3 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \) \( (H | P \hat{b}) \).
\[ (A : b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 0 & 2 & -1 \\ 1 & 7 & 2 & -1 & 8 \end{pmatrix} \]

\[ \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 2 & 1 & 1 & 3 \\ 0 & -6 & -2 & 2 & -6 \end{pmatrix} \]

1. \( R_2' = R_1 - R_2 \)
2. \( R_3' = R_1 - R_3 \)

\[ \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1/3 & -1/3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]

\[ \Rightarrow R(A : b) = R(A) = 2 \]
Hence it is consistent.

Note that, \( b = \alpha_1 + \alpha_2 \)
\[ A = (\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4) \]

So, here
\[ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \]

The general solution of \( Ax = 2 \) is
\[ \begin{pmatrix} 2\alpha + 4\beta \\ \alpha - \beta \\ -3\alpha \\ -3\beta \end{pmatrix} \] [obtained before]

Thus the general solution of \( Ax = b \) is
\[ \begin{pmatrix} 2\alpha + 4\beta + 1 \\ \alpha - \beta + 1 \\ -3\alpha \\ -3\beta \end{pmatrix} \]

Note: In particular, if \( A \) is non-singular then the unique solution is \( A^{-1}b \).
Ex. 2. Obtain the general solution to the following system of equations:

\[ \begin{align*}
\alpha_1 + 3\alpha_2 + 2\alpha_3 + 3\alpha_4 &= -1 \\
\alpha_2 + 2\alpha_3 + \alpha_4 - \alpha_1 &= -3 \\
3\alpha_1 + 4\alpha_2 + \alpha_3 + 4\alpha_4 &= 2
\end{align*} \]

**Solution:** Let it be \( \mathbf{Ax} = \mathbf{b} \).

First of all, we note to show that the system is consistent i.e. \( \mathbb{R}(\mathbf{A}; \mathbf{b}) = \mathbb{R}(\mathbf{A}) \).

Here,
\[
\begin{pmatrix}
1 & 2 & 2 & 3 \\
-1 & 1 & 2 & 4 \\
3 & 4 & 1 & 4
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4
\end{pmatrix}
= 
\begin{pmatrix}
-1 \\
-3 \\
2
\end{pmatrix}
\]

It is to be noted that \( \lambda = \alpha_1 - \alpha_2 \).

Hence, a specific form is \( \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \).

Again, the general solution of \( \mathbf{Ax} = \mathbf{0} \) is \( \begin{pmatrix} \alpha \\ \beta - \alpha \\ -\alpha \\ -\beta \end{pmatrix} \) [Obtained earlier].

Thus, the general solution to this system will be \( \begin{pmatrix} \alpha + 1 \\ \beta - \alpha \\ -\alpha \\ -\beta \end{pmatrix} \).

Ex. 3. Show whether the system is consistent or not:

\[ \begin{align*}
\alpha_1 + \alpha_2 + \alpha_3 &= 4 \\
2\alpha_1 + 5\alpha_2 + 2\alpha_3 &= 3 \\
\alpha_1 + 7\alpha_2 - 7\alpha_3 &= 5
\end{align*} \]

**Solution:** \( \mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{pmatrix} \) and the method of sweep-out gives \( \mathbf{A} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -4/3 \\ 0 & 0 & 0 \end{pmatrix} \).

\( \mathbb{R}(\mathbf{A}) = 2 \).

Again, \( \mathbb{R}(\mathbf{A}; \mathbf{b}) = 3 \).

Hence, \( \mathbb{R}(\mathbf{A}; \mathbf{b}) > \mathbb{R}(\mathbf{A}) \), the system of equations is inconsistent, i.e., does not have a solution.
Problem 3: State a necessary and sufficient condition for the existence of a solution to \( A \mathbf{x} = \mathbf{b} \). Show that if a system of linear equations has two distinct solutions, then there exists an infinite number of solutions.

**Answer:**

\[
A = ([a_{ij}]_{m \times n}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}
\]

A necessary and sufficient condition for the existence of a solution to \( A \mathbf{x} = \mathbf{b} \) is \( \text{rank}(A) = \text{rank}(A:b) \).

Let \( A \mathbf{x} = \mathbf{b} \) be a system of linear equations having two distinct solutions \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \).

\[
\Rightarrow \quad A \mathbf{x}_1 = \mathbf{b} \quad \text{and} \quad A \mathbf{x}_2 = \mathbf{b}.
\]

\[
\Rightarrow \quad A(\mathbf{x}_1 - \mathbf{x}_2) = 0.
\]

\[
(\mathbf{x}_1 - \mathbf{x}_2) \neq 0 \quad \Rightarrow \quad (\mathbf{x}_1 - \mathbf{x}_2) \text{ is a solution of the corresponding homogeneous system } A\mathbf{x} = 0.
\]

\[
\Rightarrow \quad A(\mathbf{u}_1 - \mathbf{u}_2) \quad \text{is a solution of } A\mathbf{x} = 0 \quad \text{for any scalar } \lambda.
\]

\[
\Rightarrow \quad \mathbf{x}_1 + \lambda(\mathbf{u}_1 - \mathbf{u}_2) \quad \text{is a solution of } A\mathbf{x} = \mathbf{b} \quad \text{because}
\]

\[
A \{ \mathbf{x}_1 + \lambda(\mathbf{u}_1 - \mathbf{u}_2) \}
\]

\[
= A \mathbf{x}_1 + A(\lambda(\mathbf{u}_1 - \mathbf{u}_2))
\]

\[
= b + 0
\]

\[
= b
\]

\[
\Rightarrow \quad \text{infinitely many solutions: } \quad \mathbf{x}_1 + \lambda(\mathbf{u}_1 - \mathbf{u}_2) \quad \text{for varying } \lambda \quad \text{of } A\mathbf{x} = \mathbf{b} \text{ exists.} \]
4) Find the value of $c$ for which the following equations admit a solution:

\[ 4u_1 + 6u_2 = 1 \]
\[ 4u_3 - 2u_2 = 7 + c \]
\[ 2u_1 - u_2 + 5u_3 = 4 \]

**Ans:** Given that the system admits a solution for a value of $c$.

\[ \Rightarrow \text{rank } (A:b) = \text{rank } (A) \]

Here $A = \begin{pmatrix} 4 & 0 & 6 \\ 0 & -2 & 4 \\ 2 & -1 & 5 \end{pmatrix}$ and the method of sweep-out gives $A \rightarrow \begin{pmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

As such, $\text{rank } (A) = 2$

Again, $(A:b) = \begin{pmatrix} 4 & 0 & 6 & 1 \\ 0 & -2 & 4 & 7 + c \\ 2 & -1 & 5 & 4 \end{pmatrix}$

\[ \rightarrow \begin{pmatrix} 1 & 0 & 3/2 & 1/4 \\ 0 & -1 & 2 & 7/2 \\ 0 & -1 & 2 & 7/2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3/2 & 1/4 \\ 0 & 1 & -2 & -3/2 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \]

So, we know from the given assumption that $\text{rank } (A:b) = \text{rank } (A) = 2$

\[ \Rightarrow c/2 = 0 \]

\[ \Rightarrow c = 0 \]

So, for $c = 0$ the equations admit a solution.
Investigate for what values of \( \lambda \) and \( \mu \), the system of equations
\[
\begin{align*}
\alpha + \gamma + z &= 6 \\
\alpha + 2\gamma + 3z &= 10 \\
\alpha + 2\gamma + 4z &= \mu
\end{align*}
\]
has

i) no solution,

ii) a unique solution,

iii) an infinite no. of solution.

\[ \begin{pmatrix} A \mid b \end{pmatrix} = \begin{pmatrix}
1 & 1 & 1 & 6 \\
1 & 2 & 3 & 10 \\
1 & 2 & \lambda & \mu
\end{pmatrix} \]

\[ \begin{align*}
R_2' &= R_2 - R_1 \\
R_3' &= R_3 - R_1 \\
R_3'' &= R_3 - R_2
\end{align*} \]

i) No solution \( \Rightarrow \) \( \alpha(A) < \alpha(A \mid b) \)
\( \Rightarrow \lambda = 3 \) but \( \mu \neq 10 \).

\[ \begin{pmatrix} \alpha(A) = 2 < 3 = \alpha(A \mid b) \end{pmatrix} \]

ii) Unique solution \( \Rightarrow \alpha(A) = 3 \)
\( \Rightarrow \lambda \neq 3 \).

iii) An infinite no. of solution
\( \Rightarrow \alpha(A) = \alpha(A \mid b) < 3 \)
\( \Rightarrow \lambda = 3, \mu = 10 \).
1. Show that the three equations
\[ -2x + y + z = a \]
\[ x - 2y + z = b \]
\[ x + y - 2z = c \]

Have no solutions, unless \( a + b + c = 0 \).

**Solution:**

\[
\begin{bmatrix}
-2 & 1 & 1 \\
1 & -2 & 1 \\
1 & 1 & -2
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c + a + b + c
\end{bmatrix}
\]

If \( a + b + c = 0 \), then \( \text{r}(A) = \text{r}(A; b) = 2 \).

Then the system of equations has infinitely many solutions.

If \( a + b + c \neq 0 \), then \( \text{r}(A; b) = 3 \neq \text{r}(A) = 2 \).

Then these 2 equations have no solution.

2. Use matrices to find the solution set of
\[
2x + y + 2z = 1
\]
\[
x - 2y - 3z = 1
\]
\[
3x + 2y + 4z = 5
\]

**Solution:**

\[
\begin{bmatrix}
2 & 1 & 1 \\
1 & -2 & -3 \\
3 & 2 & 4
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
5
\end{bmatrix}
\]

\[
\lambda \begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
2 & 1 & 1 \\
1 & -2 & -3 \\
3 & 2 & 4
\end{bmatrix}^{-1}
\begin{bmatrix}
1 \\
5
\end{bmatrix}
\]

\[
\Rightarrow 
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 
-\frac{1}{\lambda}
\begin{bmatrix}
-2 & -2 & -1 \\
-13 & 5 & 7 \\
8 & -1 & -5
\end{bmatrix}
\begin{bmatrix}
1 \\
5
\end{bmatrix}
\]

\[
\Rightarrow 
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
-3 \\
2 \\
2
\end{bmatrix}
\]

Solution set is \( \{ (1, -3, 2) \} \).

3. For which value of \( \lambda \), the following system of equations is inconsistent?
\[
3x + 2y + 2z = 10
\]
\[
2x + 3y + 2z = 10
\]
\[
x + 2y + \lambda z = 10
\]

**Solution:**

The given system of equations is inconsistent if

\[
\begin{vmatrix}
3 & 2 & 1 \\
2 & 3 & 2 \\
1 & 2 & \lambda
\end{vmatrix}
= 0
\Rightarrow 5\lambda - 7 = 0
\Rightarrow \lambda = \frac{7}{5}
\]

END
EIGENVALUES & EIGENVECTORS

Differentiation with respect to a vector: => 

Let \( f = f(u_1, u_2, \ldots, u_n) \) be a function of the \( n \) real variables: \( u_1, u_2, \ldots, u_n \).

Suppose further that the partial derivatives \( \frac{\partial f}{\partial x_i} \) (\( i=1(n) \))
exists for \( u_1=a_1, u_2=a_2, \ldots, u_n=a_n \).

Then by the vector of partial derivatives, denoted by

\[
\left[ \frac{\partial f}{\partial x_1} \right]_{x=a} = \left[ \frac{\partial f}{\partial x_2} \right]_{x=a} = \ldots = \left[ \frac{\partial f}{\partial x_n} \right]_{x=a}
\]

E.g., Suppose \( f(x) = Q(x) = x^T A x \), where \( A \) is a symmetric matrix, \( A = (a_{ij})_{n \times n} \)

\[
f(x) = (a_{11}x^1 + a_{22}x^2 + \ldots + a_{nn}x^n) + 2x_1(a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n) + 2x_2(a_{23}x_3 + a_{24}x_4 + \ldots + a_{2n}x_n) + \ldots + 2x_{n-1}(a_{n-1,n}x_n)
\]

We have, \( \frac{\partial f}{\partial x_1} = 2a_{12}x_1 + 2a_{13}x_1 + \ldots + 2a_{1n}x_1 \)

In the same way,

\[
\frac{\partial f}{\partial x_2} = 2 \sum_{j=1}^{n} a_{2j}x_j = 2a_{22}x_2
\]

\[
\frac{\partial f}{\partial x_n} = 2 \sum_{j=1}^{n-1} a_{nj}x_j = 2a_{nn}x_n
\]

\[
\therefore \left[ \frac{\partial f}{\partial x} \right]_x = 2 \left( a_{11}x_1 + a_{22}x_2 + \ldots + a_{nn}x_n \right) = 2 A \cdot x
\]
Maxima and minima of a real function:

Suppose the function \( f \) has partial derivatives of both first and second orders in the neighbourhood of the point \( x = a \). Let us rewrite

\[
\frac{\partial f}{\partial x_i} \bigg|_{x=a} = g_i, \quad g_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} \bigg|_{x=a}
\]

Define,

\[
g = [g_1, g_2, \ldots, g_n] = \frac{\partial f}{\partial x} \bigg|_{x=a}
\]

\[G = \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix}
\]

so that \( G \) is a symmetric matrix.

By Taylor's series for \( f \) at \( a = a + h \), where \( h = [h_1, h_2, \ldots, h_n] \), one may, for sufficiently small \( h \), write

\[
f(a+h) = f(a) + \langle h, g \rangle + \frac{1}{2!} h^T G h + R,
\]

where \( R \) is a remainder term \( \Rightarrow R \to 0 \) as \( h \to 0 \).

A necessary condition for \( f \) to have a maximum or a minimum at \( a \) is that \( g = 0 \), since this ensures that \( f \) is stationary at \( a \). Whether this is a maximum or a minimum or neither a maximum nor a minimum depends on the quadratic form \( h^T G h \). One may then assert:

1) If \( G \) is a positive definite matrix, then \( f \) is a minimum at \( a \).
2) If \( G \) is a negative definite, then \( f \) is a maximum at \( a \).
3) If \( G \) is indefinite, then \( f \) is neither a maximum nor a minimum at \( a \).

**Method of Lagrangian Multipliers**

\[
f(x) = f(u_1, u_2, \ldots, u_n) + \lambda (g(u_1, u_2, \ldots, u_n) - c)
\]

\[
g(x) = c
\]

\[
\frac{\partial g}{\partial x_1} = 0, \ldots, \frac{\partial g}{\partial x_n} = 0
\]

\[
A \delta x + \lambda = 0
\]

And,

\[
\frac{\partial f}{\partial x} (A - \lambda I) \delta x = 0
\]

if \( \lambda \neq 0 \), then this implies \( \text{rank} (A - \lambda I) < n \)

\[
\Rightarrow |A - \lambda I| = 0
\]
An optimization problem : Suppose \( a^\prime A x \) is a quadratic form in the \( n \) variables \( x_1, x_2, \ldots, x_n \), which is to be maximized or minimized subject to the condition \( x^\prime x = 1 \).

Introducing a Lagrange multiplier \( \lambda \), we are then led to consider the quadratic form

\[
ax^\prime A x - \lambda x^\prime x = \lambda (A - \lambda I) x. \tag{1}
\]

To get a maximum or a minimum, we equate

\[
\frac{\partial}{\partial x_i} \left( \lambda (A - \lambda I) x \right) = 0, \quad \frac{\partial}{\partial x_i} \left( x^\prime (A - \lambda I) x \right) = 0
\]

This yields \( (A - \lambda I) x = 0 \) \( \tag{2} \)

One solution of the system of linear equations (2) is, of course, \( x = 0 \). But we are not interested in this trivial solution, since it does not meet the condition \( x^\prime x = 1 \). Here, a non-trivial solution exists if \( \text{rank} \ (A - \lambda I) < n \).

**Definitions:** The values of the scalar parameter \( \lambda \) satisfying \( |A - \lambda I| = 0 \) are called the *eigenvalues* or *latent roots* or *eigen roots* or *characteristic roots* of the square matrix \( A \).

Any non-null vector satisfying \( (A - \lambda I) x = 0 \) corresponding to an eigenvalue is called an *eigen vector* or *latent vector* or *characteristic vector* of \( A \).

One other words, let \( \lambda \) be an eigenvalue of \( A \) and \( x \) be a non-null vector satisfying \( \lambda x = (A - \lambda I) x = 0 \). We call \( \lambda \) and \( x \) as an eigenvalue-eigenvector pair of \( A \) corresponding to \( \lambda \).

Let \( \lambda \) be a characteristic root of \( A \).

If \( \lambda \) be the rank of \( (A - \lambda I) \), then there are \((n - \lambda)\) linearly independent characteristic eigenvectors of \( A \) corresponding to this \( \lambda \).
Example: Consider the matrix 

\[ A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \]

The characteristic equation of \( A \) is

\[ \det(A - \lambda I) = 0 \]

or,

\[ (4-\lambda)(1-\lambda) - 4 = 0 \]

or,

\[ \lambda(\lambda - 6) = 0 \]

As such, the eigenvalues are \( \lambda_1 = 0, \lambda_2 = 6 \).

To determine the eigenvectors corresponding to \( \lambda_1 = 0 \), we have to solve the linear homogeneous equations

\[ (A - \lambda_1 I) \mathbf{x} = 0 \]

i.e., the equations

\[ 4\mathbf{x}_1 + 2\mathbf{x}_2 = 0, \]

\[ 2\mathbf{x}_1 + \mathbf{x}_2 = 0 \]

These give

\[ \mathbf{x}_2 = -2\mathbf{x}_1 \]

Take \( \mathbf{x}_1 = \alpha \neq 0 \), so \( \alpha \mathbf{x}_2 = -2\alpha \).

Any vector of the form \( \begin{pmatrix} \alpha \\ -2\alpha \end{pmatrix}, \alpha \neq 0 \), is an eigenvector corresponding to \( \lambda_1 = 0 \).

To determine the eigenvector corresponding to \( \lambda_2 = 6 \), we have to solve the following linear homogeneous equations:

\[ (A - \lambda_2 I) \mathbf{x} = 0 \]

\[ \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \mathbf{x} = 0 \]

\[ \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix} \mathbf{x} = 0 \]

or, \( \mathbf{x}_1 = 2\mathbf{x}_2 \)

Take \( \mathbf{x}_1 = \beta \), \( \mathbf{x}_2 = \beta/2 \).

Any vector of the form \( \begin{pmatrix} \beta \\ \beta/2 \end{pmatrix}, \beta \neq 0 \) is an eigenvector corresponding to the eigenvalue \( \lambda_2 = 6 \).
Problem: Obtain the eigenvalues and hence the eigen vectors of the matrix \( A = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} \).

\[ \text{Soln.:} \quad \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \lambda \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \]

\[ \implies \begin{bmatrix} 2 - \lambda & \sqrt{2} \\ \sqrt{2} & 1 - \lambda \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 0 \]

\[ \begin{vmatrix} 2 - \lambda & \sqrt{2} \\ \sqrt{2} & 1 - \lambda \end{vmatrix} = 0 \]

\[ (2 - \lambda)(1 - \lambda) - 2 = 0 \]

\[ \implies \lambda^2 - 3\lambda = 0 \]

\[ \therefore \lambda(\lambda - 3) = 0 \]

\[ \therefore \lambda = 0, 3. \]

When \( \lambda = 0 \),

\[ \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ \implies \sqrt{2}\lambda_1 + \lambda_2 = 0 \tag{1} \]

\[ \therefore \lambda_1 + \lambda_2 = 0 \tag{2} \]

Let \( \lambda_2 = 1 \),

\[ \therefore \lambda_1 = -\frac{1}{\sqrt{2}} \quad \text{[from (1)]} \]

Putting these values in the equation (2),

\[ \left(\frac{1}{\sqrt{2}}\right) + 1 = 1 \]

\[ \implies a = \frac{1}{\sqrt{2}} \]

\[ \therefore \lambda_2 = \frac{1}{\sqrt{2}}, \quad \lambda_1 = - \frac{1}{\sqrt{2}} \]

When \( \lambda = 3 \),

\[ \begin{bmatrix} 2 - 3 & \sqrt{2} \\ \sqrt{2} & 1 - 3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ \implies -\lambda_1 + \sqrt{2}\lambda_2 = 0 \tag{3} \]

\[ \therefore \lambda_1 + \lambda_2 = 0 \tag{4} \]
Let \( \lambda_2 = b \), \( \lambda_1 = \sqrt{2b} \) \[ \text{From (2)} \]

Putting these values into Eq. (1), \( (\sqrt{2b})^2 + b = 1 \)
\( \Rightarrow b = \sqrt{\frac{1}{3}} \)
\( \Rightarrow \lambda_2 = \sqrt{\frac{1}{3}}, \lambda_1 = \sqrt{\frac{2}{3}} \)

Hence, the eigen values are 0 and \( \sqrt{\frac{1}{3}} \) and eigen vectors are
\[ (\begin{pmatrix} -\sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix}) \quad \begin{pmatrix} \sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} \end{pmatrix} \]

**Similar Matrices:**

**Definition:** Two square matrices \( A \) and \( B \) of the same order, say \( n \), are said to be similar if there exists a non-singular matrix of order \( n \), such that \( B = P^{-1}AP \).

\( \Rightarrow \) If \( A \) and \( B \) be two square matrices, are said to be similar if they have same characteristics equation and hence same set of eigen values.

**Result:** If \( A \) and \( B \) are similar matrices, then they must have the same eigenvalues.

**Proof:** Since \( A \) and \( B \) are similar, we have
\( B = P^{-1}AP \) for some non-singular matrix \( P \).

Now,
\[ B - \lambda I = P^{-1}A^{-1} - \lambda I \]
\[ = P(A - \lambda I)P^{-1} \]

Taking determinants of both sides, we have
\[ |B - \lambda I| = |P| |A - \lambda I| |P^{-1}| \]
\[ = |A - \lambda I| |P| |P^{-1}| \]
\[ = |A - \lambda I| \quad \text{Since} \ |P| |P^{-1}| = |P|^{-1} = |I| = 1. \]

Thus, the two matrices \( A \) and \( B \) have the same characteristic equation and hence the same eigen values.

\[ \text{OR} \]
\[ |A - \lambda I| = 0 \]
\[ \Rightarrow |P^{-1}A - \lambda I| = 0 \]
\[ \Rightarrow |P^{-1}B - \lambda I| = 0 \]
\[ \Rightarrow |B - \lambda I| = 0 \quad \text{[} \because B = P^{-1}A, \text{as} A \text{ and} B \text{ are similar matrices]} \]

Therefore \( A \) and \( P^{-1}B \) (or, \( B \)) give same characteristic equation.
**Important Theorems on Eigen Values & Eigen Vectors**

**Theorem:** If \( \lambda \) is an eigenvalue of the square matrix \( A \), then \( \lambda - k \) is an eigenvalue of \( A - kI \).

**Proof:** Since \( \lambda \) is an eigenvalue of \( A \), there exists \( \mathbf{x} \neq \mathbf{0} \) such that

\[
A \mathbf{x} = \lambda \mathbf{x}
\]

i.e., \( A \mathbf{x} = \lambda \mathbf{x} \)

\[
\Rightarrow A(k) \mathbf{x} = k \lambda \mathbf{x}
\]

\[
\Rightarrow (A - kI) \mathbf{x} = (\lambda - k) \mathbf{x}
\]

\[
\Rightarrow (\lambda - k) \text{ is an eigenvalue of } (A - kI).
\]

**Theorem:** If \( \lambda \) is an eigenvalue of \( A \), then \( \lambda^m \) is an eigenvalue of \( A^m \), for any positive integer \( m \).

**Proof:** Since \( \lambda \) is an eigenvalue of \( A \), there exists \( \mathbf{x} \neq \mathbf{0} \) such that

\[
A \mathbf{x} = \lambda \mathbf{x}
\]

\[
\Rightarrow A(A \mathbf{x}) = A(\lambda \mathbf{x})
\]

\[
\Rightarrow A^2 \mathbf{x} = \lambda^2 \mathbf{x}
\]

\[
\Rightarrow A^m \mathbf{x} = \lambda^m \mathbf{x}
\]

\[
\Rightarrow \lambda^m \text{ is an eigenvalue of } A^m.
\]

We have,

\[
A \mathbf{x} = \lambda \mathbf{x}
\]

\[
\Rightarrow A(A \mathbf{x}) = A(\lambda \mathbf{x})
\]

\[
\Rightarrow A^2 \mathbf{x} = \lambda^2 \mathbf{x}
\]

\[
\Rightarrow A^m \mathbf{x} = \lambda^m \mathbf{x}
\]

\[
\Rightarrow \lambda^m \text{ is an eigenvalue of } A^m.
\]

In the same way, we can prove that for any \( m = 2, 3, 4, \ldots \), \( \lambda^m \) is an eigenvalue of \( A^m \).

**Theorem:** If \( \lambda \) is a characteristic root of \( A \), then \( \lambda k \) is an eigenvalue of \( kA \).

**Proof:** As \( \lambda \) is a characteristic root of \( A \), then \( \exists \mathbf{x} \) non-zero such that

\[
A \mathbf{x} = \lambda \mathbf{x}
\]

\[
\Rightarrow kA \mathbf{x} = k\lambda \mathbf{x}
\]

\[
\Rightarrow (kA) \mathbf{x} = (k\lambda) \mathbf{x}
\]

\[
\Rightarrow k\lambda \text{ is an eigenvalue of } kA.
\]
Theorem: If \( \lambda \) is an eigen value of an m.s. matrix \( A \), then \( \frac{1}{\lambda} \) is an eigen value of \( \text{Adj}(A) \).

Proof: Since \( \lambda \) is a char. root of \( A \), then \( A - \lambda I \) is a char. eqn. of \( A \). By the next theorem, \( \frac{1}{\lambda} I \) is a char. root of \( \text{Adj}(A) \). So, \( \frac{1}{\lambda} \) is a char. root of \( \text{Adj}(A) \), giving \( \frac{1}{|A|} \) is an eigen value of \( \text{Adj}(A) \).

Theorem: Suppose \( A \) is m.s. and \( \lambda \) is an eigen value of \( A \), then \( \frac{1}{\lambda} \) is an eigen value of \( A^{-1} \). Further \( A \) and \( A^{-1} \) has the same set of eigen vectors.

Proof: Let \( A \) be an m.s. matrix and \( \lambda \) is an eigen value of \( A \), then there exists \( \mathbf{x} \neq \mathbf{0} \) such that \( A \mathbf{x} = \lambda \mathbf{x} \). 

Premultiplying both sides by \( A^{-1} \), we get

\[ A^{-1} (A \mathbf{x}) = A^{-1} (\lambda \mathbf{x}) \]
\[ \Rightarrow (A^{-1} A) \mathbf{x} = \lambda (A^{-1} \mathbf{x}) \]
\[ \Rightarrow \frac{1}{\lambda} \mathbf{x} = A^{-1} \mathbf{x} \quad \text{(2)} \]

This implies that \( \frac{1}{\lambda} \) is an eigen value of \( A^{-1} \).

From (1) & (2), we see that \( \mathbf{x} \) is an eigen vector of \( A \) corresponding to \( \lambda \) iff \( \mathbf{x} \) is an eigen vector of \( A^{-1} \) corresponding to \( \frac{1}{\lambda} \). Thus \( A \) and \( A^{-1} \) has the same set of eigen vectors.

Theorem: If \( \lambda \) be an eigen value of an \( n \times n \) matrix \( A \), then \( \lambda \) be the corresponding eigen vector of \( A \) and \( p \) be the corresponding eigen value of \( p \times p \) matrix \( A \).

Proof: Since \( \lambda \) is an eigen value of \( A \) and \( \mathbf{x} \) be the corresponding eigenvector, \( \mathbf{y} = p \mathbf{x} \) is an eigen vector of \( p \times p \) matrix \( p \mathbf{I} \).

Premultiplying both sides by \( p \mathbf{I} \), we get

\[ p \mathbf{I} \mathbf{y} = p \mathbf{I} \mathbf{x} \]
\[ \Rightarrow (p \mathbf{I}^2) \mathbf{y} = p \mathbf{I} \mathbf{x} \]
\[ \Rightarrow \lambda p \mathbf{I} \mathbf{y} = \lambda p \mathbf{I} \mathbf{x} \]
\[ \Rightarrow \lambda \mathbf{y} = \lambda \mathbf{x} \]

Thus, \( \lambda \) is an eigen value of \( p \mathbf{I} \mathbf{A} \) treating \( \mathbf{y} \) as the corresponding eigen vector where \( \mathbf{y} = p \mathbf{x} \).
Cayley Hamilton Theorem:

Every square matrix satisfies its own characteristic equations.

Let \( A \) be a square matrix of order \( n \), \( I \) be the identity matrix of order \( n \), and \( O \) be the zero matrix of order \( n \). The theorem states that if

\[
|A - \lambda I| = (-1)^{n-1} \left( \lambda^n + p_1 \lambda^{n-1} + p_2 \lambda^{n-2} + \cdots + p_n \right) = 0
\]

be the characteristic equation then \((A)^n + p_1 (A)^{n-1} + \cdots + p_n I = 0\).

Problem 1. \( A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \). Use C-H theorem to express \( A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2 \) as a linear polynomial in \( A \).

Proof:

Ch. eqn:\( (A - \lambda I) = 0 \)

\[
\Rightarrow \begin{vmatrix} 1 - \lambda & 2 \\ -1 & 3 - \lambda \end{vmatrix} = 0
\]

or \( (1 - \lambda)(3 - \lambda) + 2 = 0 \)

or \( \lambda^2 - 4\lambda + 5 = 0 \)

\( \therefore \) the matrix \( A \) satisfies \( A^2 - 4A + 5I = 0 \).

\[
\begin{align*}
A^2 &= 4A - 5I \\
A^3 &= 4A^2 - 5A \\
A^4 &= 4A^3 - 5A^2 \\
A^5 &= 4A^4 - 5A^3 \\
A^6 &= 4A^5 - 5A^4 \\
\end{align*}
\]

\( \therefore \) \( A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2 = -15A^4 \)

\( = -5A^4 + 8A^4 - 12A^3 + 14A^2 \)

\( = 3A^4 - 12A^3 + 14A^2 \)

\( = -15A^4 + 14A^2 \)

\( = -A^4 \)

\( = 5I - 4A \)
2. \( A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \). Find \( A^9 \) and \( A^{-1} \) by using Cayley-Hamilton theorem.

**Proof:**

\[
A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Ch. eqns.

\[
|A - \lambda I| = 0
\]

\[
\begin{vmatrix}
1 - \lambda & 0 & 2 \\
0 & 1 - \lambda & 1 \\
0 & 0 & 1 - \lambda
\end{vmatrix} = 0
\]

\[
\Rightarrow \lambda^3 - 2\lambda + 1 = 0
\]

The matrix \( A \) satisfies \( A^3 - 2A + I = 0 \). Hence, \( A^3 = 2A - I \).

\[
A^9 = (A^3)^3 = (2A - I)^3
\]

\[
= 8A^3 - 12A^2 + I + 6A
\]

\[
= 16A - 8I - 12A^2 - I + 6A
\]

\[
= 22A - 12A^2 - 9I
\]

\[
= \begin{bmatrix}
22 & 0 & 0 \\
0 & 22 & 0 \\
0 & 0 & 22
\end{bmatrix} - \begin{bmatrix}
12 & 24 & 24 \\
0 & -12 & 0 \\
0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
9 & 0 & 0 \\
0 & 9 & 0 \\
0 & 0 & 9
\end{bmatrix}
\]

\[
\text{Here, } |A| = -1 \neq 0, \text{ so } A^{-1} \text{ exists.}
\]

\[
A^3 - 2A + I = 0
\]

\[
\Rightarrow A^{-1} = 2I - A
\]

\[
= \begin{bmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{bmatrix} - \begin{bmatrix}
1 & 2 & 2 \\
0 & 2 & -1 \\
0 & -1 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & -2 & -2 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]
Result: 
* If $A$ be a symmetric matrix of order $n \times n$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ then, $\det(A) = \prod_{i=1}^{n} \lambda_i$.

Proof: 
Method 1. Since $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the matrix $A$, the characteristic polynomial of the matrix $A$ can be written in the form:

$$|A - \lambda I| = f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \cdots (\lambda - \lambda_n)$$

Putting $\lambda = 0$ in both sides of the above equation, we get

$$|A| = \prod_{i=1}^{n} \lambda_i$$

Method 2. We know that a symmetric matrix $A$ of order $n \times n$ having distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ can be diagonalized by the orthogonal matrix $P$ such that $P^T A P = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ where the columns of $P$ are the eigen vectors of length unity corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Here, $P^T A P = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ which implies

$$|P^T A P| = \prod_{i=1}^{n} \lambda_i$$

Since, $P^T = P^{-1}$, hence $P^T A P = \lambda I$, where $\lambda$ is the eigen value of $A$.

$\therefore |A| = \prod_{i=1}^{n} \lambda_i$, since $P^T = I$ as $P$ is orthogonal.

Result: Prove that the characteristic roots of Hermitian Mtx are real.

Solution: Let $A$ be a Hermitian matrix, $\lambda$ be an eigen root of $A$ and $x$ be the corresponding eigen vector, then,

$$A \bar{x} = \lambda \bar{x}$$

From multiplying both sides by $\bar{x}^T$, we get

$$\bar{x}^T A \bar{x} = \lambda \bar{x}^T \bar{x}$$

Taking conjugate transpose of the both sides, we get

$$\bar{x}^T A \bar{x} = \bar{x}^T \bar{x} \lambda$$

$$\Rightarrow \bar{x}^T A \bar{x} = \bar{x}^T \bar{x} \lambda$$

$\Rightarrow \lambda = \frac{\bar{x}^T A \bar{x}}{\bar{x}^T \bar{x}}$ is real, since $\bar{x}^T A \bar{x}$ and $\bar{x}^T \bar{x}$ are real.
Problem: Let \( A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \), \( P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \). If

\( A = P^{-1}BP \), then the matrix \( D \) is equal to

(A) \( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \)  
(B) \( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \)  
(C) \( \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix} \)  
(D) \( \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix} \)

Solution:

\( A \) and \( P \) are non-singular matrices.

If \( P = P^{-1}BP \),

then \( D \) be a diagonal matrix having eigen values as diagonal entry.

then we find eigen values of \( A \).

characteristic equation is \( |A - \lambda I| = 0 \)

\( \begin{vmatrix} 3 - \lambda & 0 & 0 \\ 0 & 1 - \lambda & 1 \\ 0 & 1 & 1 - \lambda \end{vmatrix} = 0 \)

\( \begin{vmatrix} 3 - \lambda & (1 - \lambda)^2 - 1 \\ 0 & 1 - \lambda & 1 \\ 0 & 1 & 1 - \lambda \end{vmatrix} = 0 \)

\( \lambda(3 - \lambda)(\lambda - 2) = 0 \)

\( \lambda = 0, 2, 3 \)

so, \( D \) becomes\( \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \)

2. A real quadratic form \( X'AX \) is positive definite if

(A) All eigen values of \( A \) > 0  
(B) All eigen values of \( A \) < 0  
(C) All eigen values of \( A \) = 0  
(D) None.

Solution: \( Q(X) = X'AX \) is positive definite

\( \Rightarrow Q(X) > 0 \; \; \; \; X \neq 0 \)

\( \Rightarrow \) All eigen values of \( \lambda > 0 \).
**Quadratic Forms**

**Linear Form:** An expression of the type $\Sigma_{i=1}^{n} a_i x_i$, where $a_i$ are constant coefficients and $x_i$ are variables, is called a linear form w.r.t. the variables. It may be denoted by $L(x)$, being looked upon as a function of the vector variable $\mathbf{x} = [x_1, x_2, \ldots, x_n]'$. Putting $A = [a_1, a_2, \ldots, a_n]'$, we may write $L(x) = A\mathbf{x}$.

**Quadratic Form:** An expression of the type $\Sigma_{i,j=1}^{n} a_{ij} x_i x_j$, where $a_{ij}$ are constant coefficients while $x_i$ and $x_j$ are variables, is called a quadratic form w.r.t. the variables. It is denoted by $Q(x)$, being looked upon as a function of the vector variable $\mathbf{x}$. Putting $A = (a_{ij})$, we may write $Q(x) = \mathbf{x}' A \mathbf{x}$.

**Definition:** A quadratic form in $x_1, x_2, \ldots, x_n$ is a second degree homogeneous function in $n$ variables $x_1, x_2, \ldots, x_n$, i.e.,

$$Q(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{i<j}^{n} a_{ij} x_i x_j$$

Define $\mathbf{x} = (x_1, x_2, \ldots, x_n)'$, and

$$A = (a_{ij})_{n \times n}$$

Then,

$$Q(x) = \mathbf{x}' A \mathbf{x} = \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{i<j}^{n} a_{ij} x_i x_j$$

for a quadratic form, $\mathbf{x}' A \mathbf{x}$. $A$ is said to be a matrix of the quadratic form. If the matrix of a quadratic form be not symmetric then it can be reduced to a symmetric matrix.

**Example:**

$n = 2$,

$$Q(x, y) = ax^2 + bxy + cy^2$$

for a quadratic form, $\mathbf{x}' A \mathbf{x}$. $A$ is said to be a matrix of the quadratic form. If the matrix of a quadratic form be not symmetric then it can be reduced to a symmetric matrix.

$$n = 2$$,

$$Q(x, y) = ax^2 + bxy + cy^2$$

Putting $A = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$, we get

$$Q(x, y) = \begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix}$$
Note: The square matrix $A$ in a quadratic form $Q(x)$ may, without loss of generality, be supposed to be a symmetric matrix. For, in case $A$ is not symmetric, we may take another matrix $B = (b_{ij})$ such that

$$b_{ij} = \frac{a_{ij} + a_{ji}}{2} \quad \text{for all } i, j.$$ 

which implies that $b_{ii} = a_{ii}$ for all $i$.

Then,

$$b_{ij} = b_{ji} \quad \text{for all } i, j,$$

so that

$$B = B^t \quad \text{i.e. } B \text{ is a symmetric matrix.}$$

Now,

$$w^t B u = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} w_i u_j$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \frac{a_{ij} + a_{ji}}{2} \right) w_i u_j$$

$$= \frac{1}{2} \left[ \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} w_i u_j + \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ji} w_i u_j \right]$$

$$= \frac{1}{2} \left[ w^t A u + a^t A^t a \right]$$

But $w^t A u$ is a scalar, so $\Rightarrow (w^t A u) = w^t A u$.

$$w^t A u = w^t A u.$$

In our discussion, then, we shall always assume $A$ is symmetric. We shall also assume that each element of $X$ can take only real values.

The determinant $|A|$ is said to be the discriminant of the quadratic form $w^t A x$.

**Ex. 1.**

$$x^2 - 2x_2 + 3x_3 + 4x_1 x_2 - 6x_1 x_2 + 8x_2 x_3$$

is a QRf, with

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & -2 & 5/2 \\ -3 & 5/2 & 3 \end{pmatrix}$$

**Ex. 2.**

$$x_1^2 + 4x_2 + 9x_3^2 + 12x_1 x_2 - 16x_2 x_3 + 8x_3 x_2$$

is a QRf, with

$$A = \begin{pmatrix} 1 & 6 & -4 \\ 6 & 4 & -8 \\ -4 & -8 & 9 \end{pmatrix}$$
Classification of Quadratic forms / Classification of the matrix of the Quadratic forms:

Every real quadratic form \(Q(x)\) can be put into one of the following 4 types, depending on the range of \(x\) values that \(Q(x)\) may assume, i.e. depending on the nature of \(A\):

1. **Non-negative definite Quadratic form:** If \(Q(x) \geq 0 \forall x \in \mathbb{R}^n\) then it is said to be a non-negative definite quadratic form.

   i.e. \(Q(x) = x'Ax\) is said to be a non-negative definite quadratic form if
   \[
   x'Ax \geq 0 \quad \forall \ x \neq 0
   \]
   \[
   = 0 \quad \text{if } \ x = 0
   \]
   \(Q(x)\) is a non-negative definite quadratic form means \(A\) is a non-negative definite matrix.

2. **Positive definite Quadratic form:** A quadratic form \(Q(x) = x'Ax\) is said to be a positive definite quadratic form if

   \[
   x'Ax > 0 \quad \forall \ x \neq 0
   \]
   \[
   = 0 \quad \text{if } \ x = 0
   \]
   \(Q(x)\) is a positive definite quadratic form means \(A\) is a positive definite matrix.

3. **Positive semi-definite Quadratic form:** A quadratic form \(Q(x) = x'Ax\) is said to be a positive semi-definite quadratic form if

   \[
   x'Ax \geq 0 \quad \forall \ x
   \]
   \(Q(x)\) is a positive semi-definite quadratic form means \(A\) is a positive semi-definite matrix.

   Moreover, An n.n.d. quadratic form which is not p.d.
   is said to be a positive semi-definite (or p.s.d) quadratic form.

4. **Non-positive definite Quadratic form:** A quadratic form \(Q(x) = x'Ax\) is said to be a non-positive definite quadratic form if

   \[
   x'Ax \leq 0 \quad \forall \ x
   \]
   i.e. if \(Q(x) \leq 0 \forall x \in \mathbb{R}^n\) then it is said to be a non-positive definite quadratic form.

   • A n.p.d. \(\iff\) A n.n.d.

   A non-positive definite quadratic form (at least negative semi definite) is either negative definite or negative semi definite quadratic form.
**Negative definite Quadratic form:**

A quadratic form \( Q(x) = x^T A x \) is said to be a negative definite (or n.d.) quadratic form if

\[
\begin{align*}
    x^T A x < 0 & \quad \forall \ x \neq 0 \\
    = 0 & \quad \text{iff} \ x = 0
\end{align*}
\]

- \( A : \text{n.d.} \iff -A : \text{pd.} \) matrix.

**Negative Semi-definite Quadratic Form:**

A quadratic form \( Q(x) = x^T A x \) is said to be an n.s.d. quadratic form if

\[
\begin{align*}
    x^T A x \leq 0 & \quad \forall \ x \neq 0 \\
    = 0 & \quad \text{for at least one} \ x \neq 0
\end{align*}
\]

- \( A : \text{nsd} \iff A : \text{psd} \)

An n.p.d. quadratic form which is not n.d. is said to be negative semi-definite (or n.s.d.).

**Indefinite Quadratic form:**

A quadratic form \( Q(x) = x^T A x \) is said to be indefinite if

\[
\begin{align*}
    Q(x) = x^T A x & \geq 0 \quad \text{for some} \ x \\
    < 0 & \quad \text{for some} \ x
\end{align*}
\]

A q.f. which is neither p.s.d. nor n.s.d. is called indefinite quadratic form.

**Example 1:**

\[
Q(x) = 5x_1^2 - 6x_1 x_2 + 4x_2^2
\]

\[
= 5\left(x_1^2 - x_1 x_2 + \frac{x_2^2}{4}\right) + \frac{1}{4}x_2^2
\]

\[
= 5\left(x_1 - \frac{x_2}{2}\right)^2 + \frac{1}{4}x_2^2
\]

\[
\therefore Q(x) \geq 0 \quad \forall \ x_1, x_2 \neq 0
\]

\[
Q(x) = 0 \iff \left(x_1 - \frac{x_2}{2}\right) = 0 \quad \text{and} \ x_2 = 0
\]

i.e. \( x_1 = x_2 = 0 \).

\[
\therefore Q(x) \text{ is a positive definite quadratic form.}
\]
Ex. 2. \( Q(u) = 3u_1^2 - 6u_1u_2 + 3u_2^2 \)
\[ = 3(u_1 - u_2)^2 \]

\[ \therefore Q(u) \geq 0 \ \forall \ u_1, u_2 \]

Here, \( Q(u) = 0 \Rightarrow u_1 = u_2 \)

\[ Q(u) = 0 \ \text{for at least one} \ u \neq 0. \]

\[ \Rightarrow \text{It is a positive semi-definite quadratic form.} \]

Ex. 2. \( Q(u) = 4u_1^2 - 7u_1u_2 - 2u_2^2 \)

Thus, \( Q(u) \) may assume values that may be positive, negative or zero. Thus, for instance, in case \( u_1 = 1, u_2 = 0 \),

\[ Q(u) = 4; \]

when \( u_1 = 0, u_2 = 1 \),

\[ Q(u) = -2; \]

and when \( u_1 = 2, u_2 = 1 \) or \( u_1 = u_2 = 0 \),

\[ Q(u) = 0. \]

As such, we now have an indefinite quadratic form.

**Theorem:** If \( \mathbf{u}'A\mathbf{u} \) is positive definite (p.d.), then \( \mathbf{u}'(-A)\mathbf{u} \) is negative definite (n.d.). Conversely, if \( \mathbf{u}'(-A)\mathbf{u} \) is negative definite (n.d.), then \( \mathbf{u}'A\mathbf{u} \) is positive definite (p.d.).

**Proof:**

We have, \( \mathbf{u}'(-A)\mathbf{u} = \sum_i \sum_j (-a_{ij})u_iu_j \)

\[ = \sum_i \sum_j a_{ij}u_iu_j \]

\[ = \mathbf{u}'A\mathbf{u}. \quad (1) \]

**Case I:** Now, let \( \mathbf{u}'A\mathbf{u} \) be p.d. \( \forall \mathbf{u} \).

Then for \( \mathbf{u} \neq 0 \), we have, \( \mathbf{u}'A\mathbf{u} > 0 \).

\[ \Rightarrow -\mathbf{u}'A\mathbf{u} < 0 \quad [\text{Applying (1)}] \]

if \( \mathbf{u} = 0 \), we have, \( \mathbf{u}'A\mathbf{u} = 0 \).

\[ \Rightarrow \mathbf{u}'(-A)\mathbf{u} = 0 \]

Hence, \( \mathbf{u}'(-A)\mathbf{u} \) must be n.d., \( \forall \mathbf{u} \).

**Case II:** Again, let \( \mathbf{u}'A\mathbf{u} \) be n.d. \( \forall \mathbf{u} \).

Then for \( \mathbf{u} \neq 0 \), we have, \( \mathbf{u}'A\mathbf{u} < 0 \).

\[ \Rightarrow \mathbf{u}'(-A)\mathbf{u} < 0 \]

For \( \mathbf{u} = 0 \), we have, \( \mathbf{u}'A\mathbf{u} = 0 \).

\[ \Rightarrow \mathbf{u}'(-A)\mathbf{u} = 0 \]

also, hence, \( \mathbf{u}'(-A)\mathbf{u} \) must be p.d., \( \forall \mathbf{u} \).
Canonical reduction of Quadratic Forms:

Let $A$ be a p.d. matrix, then $x^T A x = Q(x) > 0$ for all $x \neq 0$.

$$Q(x) = y^T \Lambda y = \sum_{i=1}^{n} \lambda_i y_i^2,$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and $y = (y_1, \ldots, y_n)$.

$$Q(y) = \sum_{i=1}^{n} \lambda_i,$$

Choose $x = py$, then we get:

$$Q(x) = x^T A x = y^T p^T A p y = y^T \Lambda y = \sum_{i=1}^{n} \lambda_i y_i^2.$$

Choose $x = \Delta^{1/2} x$, where $\Delta^{1/2} = \text{diag}(\frac{1}{\sqrt{\lambda_1}}, \ldots, \frac{1}{\sqrt{\lambda_n}})$.

$$Q(y) = \left| y = \Delta^{1/2} y \right|^2 = y^T \Delta y = \frac{1}{\Delta} \Delta^{1/2} \Delta^{1/2} y^T \Delta^{-1/2} \Delta^{1/2} y = \sum_{i=1}^{n} \lambda_i^{1/2} y_i^2,$$

where $\Delta^{1/2} = \text{diag}(\frac{1}{\sqrt{\lambda_1}}, \ldots, \frac{1}{\sqrt{\lambda_n}})$.

Defn. of canonical form:

If by any real non-singular linear transformation a real quadratic form be expressed as a sum and difference of the squares of the new variables, then this later expression is called the canonical form of the given form.
**Result**

If $A$ be a p.s.d. matrix of order $n \times n$ with rank $(A) = n \leq m$,
then $Q(x) = x'Ax \rightarrow Q(y) = \sum_{i=1}^{m} \lambda_i y_i^2$.

Every quadratic form $x'Ax$ can be reduced to a canonical form $\sum_{i=1}^{m} \lambda_i y_i^2 = y'\Lambda y$, where $\Lambda = \text{diag} (\lambda_1, \lambda_2, \ldots, \lambda_n)$ by an m.s. transformation of variables.

**Proof:**

Let, $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues and $u_1, u_2, \ldots, u_n$ be the corresponding eigen vectors.

We can take these vectors as orthogonal.

Take $Q = (u_1, u_2, \ldots, u_n)$, obviously $Q$ is non-singular.

Then $x'Qy$ is a non-singular transformation.

It reduces the a.f. to, $x'Ax = (Qy)'A(Qy) = y'(Q'AQ)y = y'\Lambda y$.

Thus the theorem is established.

\[ \text{As } A \text{ be a square matrix of order } n \text{ and } \lambda_1, \ldots, \lambda_n \text{ be the eigen values, then an real m.s. matrix } Q \text{ exists, } \theta \]

$Q'AQ = \text{diag} (\lambda_1, \ldots, \lambda_n) = \Lambda$ is true. Using this result there.

**Corollary:** We have a connection between the nature of the eigenvalues and the nature of the matrix (or a. f.).

If $\lambda_i > 0$ for each $i$, then $x'Ax$ is positive definite (p.d).
If $\lambda_i < 0$ for each $i$, then $x'Ax$ is negative definite (n.d).
If $\lambda_i \geq 0 \forall i$ and $\lambda_i = 0$ for some $i$, then $x'Ax$ is p.s.d.
If $\lambda_i \leq 0 \forall i$ and $\lambda_i = 0$ for some $i$, then $x'Ax$ is n.s.d.
If $\lambda_i > 0$ for some $i \& \lambda_i < 0$ for some $i$, then $x'Ax$ is indefinite.
Result

The necessary and sufficient condition for a quadratic form \( x'Ax \) to be positive definite is the leading principal minors of the matrix \( A \) are all positive, i.e.,

Let \( A \) be a matrix of order \( n \times n \), \( A = (a_{ij})_{n \times n} \), then

\[
\begin{vmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix} > 0, \quad \ldots, \quad \begin{vmatrix}
  a_{11}
\end{vmatrix} > 0.
\]

Proof:

Necessity part (Only if part):

Let \( Q(x) = x'Ax \) be a p.d. a.f. in \( n \) variables \( x_1, x_2, \ldots, x_n \), and let \( m \leq n \) be a natural number, putting \( a_{m+1} = a_{m+2} = \ldots = x_n = 0 \) in the p.d. a.f. \( x'Ax \), we arrive to another p.d. a.f. in \( m \) variables \( x_1, x_2, \ldots, x_m \), the determinant of whose matrix is the leading principal minor of the matrix \( A \).

Since, the determinant of every p.d. a.f. is positive,

\( \therefore \) every leading principal minor of the matrix \( A \) is positive.

Sufficiency part (If part):

Method of induction will be used to prove this part. For a single variable \( x \), the a.f.

\( Q(x) = ax^2 > 0 \) for \( a > 0 \).

Now, suppose that the theorem is true for \( m \) variables.

Consider any a.f. in \( (m+1) \) variables with the corresponding symmetric matrix \( A \) and the leading principal minors of \( A \) is positive.

Let us partition of \( A \) as follows:

\[
A_{m+1 \times m+1} = \begin{pmatrix}
  B_{m \times m} & b_1 \\
  b_1' & k
\end{pmatrix}
\]

the leading principal minors of \( A \) and \( B \) are all positive as the theorem is supposed to be true for \( m \) variables.

Let us now determine \( C = A_{m \times m} \)

\[
\begin{pmatrix}
  p & 0 \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  B & b_1 \\
  b_1' & k
\end{pmatrix}
\begin{pmatrix}
  p & 0 \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  p & b_1 \\
  c & c
\end{pmatrix}
\]

The product matrix equals to

\[
\begin{pmatrix}
  p'b & p'b \\
  c'b + b & c'b + k
\end{pmatrix}
\begin{pmatrix}
  a & 1 \\
  b & p
\end{pmatrix}
\begin{pmatrix}
  p'B & p'b \\
  c'B + b & p'
\end{pmatrix}
\begin{pmatrix}
  p & 0 \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  p & b_1 \\
  c & c
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  p'B & p'B \\
  c'B + b & c'B + c'
\end{pmatrix}
\]
We have $p'BC + p'b = l$
$\Rightarrow p'be = -b'p'by$
$\Rightarrow be = -b'y$
$\Rightarrow c = b'y$.

Under this choice of $c$, we also have
$c'BP + b'p = 0$ and
$c'BC + c'b = b'y(B^{-1})B^{-1}b - b'y(B^{-1})b$
$= b'y(B^{-1})b - b'y(B^{-1})b = 0$.

Product matrix is
\[
\begin{pmatrix}
\text{Im} & 0 \\
0 & b'(c + k)
\end{pmatrix}
\]

Taking determinant in both sides,
$|P'| |A| |P| = |\text{Im} (b'(c + k))$
\[|A| |P|^2 = (b'(c + k))
\]

Now by assumption, $|A| > 0$ and $|P| > 0$.

Let $b'(c + k) > 0$

Get $b'(c + k) = 0$.

Similarly, multiplying and post multiplying both sides by
\[
\begin{pmatrix}
\text{Im} & 0 \\
0 & b'
\end{pmatrix}
\]
\[
\begin{pmatrix}
\text{Im} & 0 \\
0 & b'
\end{pmatrix}Q' AQ = \text{Im} + 1
\]

Thus $Q' AQ = \text{Im}+1$, where $Q = \begin{pmatrix} (p' c')(\text{Im} 2' - 1') \end{pmatrix}$.

Since $A$ is congruent to $\text{Im}+1$, therefore $A$ is p.d., i.e., the corresponding r.f. is p.d. Hence, by induction result follows.
Corollary: The necessary and sufficient condition for a q.f. \( Q(x) = \mathbf{x}' A \mathbf{x} \) to be negative definite is that the leading principal minor of a starting from the first one, alternatively negative and positive, i.e.

\[
a_{11} < 0, \quad |a_{11} a_{12}| > 0, \quad |a_{11} a_{12} a_{13}| < 0, \quad \ldots, \quad (-1)^{n} |A| \geq 0 \text{ according as } n \text{ is even or odd.}
\]

Proof: \( \mathbf{x}' A \mathbf{x} \) is negative definite if and only if \( \mathbf{x}' (-A) \mathbf{x} \) is positive definite. By virtue of the above theorem, \( \mathbf{x}' (-A) \mathbf{x} \) is positive definite if and only if

\[
-a_{11} > 0, \quad \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \end{vmatrix} > 0, \quad \ldots, \\
-a_{31} & -a_{32} & -a_{33}
\]

Thus the necessary and sufficient condition for \( \mathbf{x}' (-A) \mathbf{x} \) to be p.d. or, equivalently, \( \mathbf{x}' A \mathbf{x} \) to be n.d. is the following:

\[
-a_{11} > 0, \quad \text{i.e. } a_{11} < 0, \\
(-1)^{1} |a_{11} a_{12}| > 0, \quad \text{i.e. } |a_{11} a_{12}| > 0, \\
(-1)^{3} |a_{11} a_{12} a_{13}| > 0, \quad \text{i.e. } |a_{11} a_{12} a_{13}| > 0, \\
(-1)^{n} |A| \geq 0 \text{ according as } n \text{ is even or odd.}
\]

Hence the proof.

Ex. 1. Consider the quadratic form: \( 4x_{1}^{2} - 10x_{1}x_{2} + 7x_{2}^{2} \).

Here, \( A = \begin{pmatrix} 4 & -5 \\ -5 & 7 \end{pmatrix} \)

As such, \( a_{11} = 4 > 0 \), \( |A| = 2 > 0 \).

Consequently, the quadratic form is positive definite (p.d.)

Ex. 2. Consider now the quadratic form: \( 4x_{1}^{2} - 10x_{1}x_{2} + 3x_{2}^{2} \).

Here, \( A = \begin{pmatrix} 4 & -5 \\ -5 & 3 \end{pmatrix} \)

As such, \( a_{11} = 4 > 0 \), \( |A| = -13 < 0 \), so that the quadratic form is neither positive definite (p.d.) nor negative definite (n.d.).
\[ \text{If } \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]

Show that the matrix \((C \delta_{ij} + \alpha_i \alpha_j)\) is p.d., \(i, j = 1(1)p\).

**Solt**

Let \(A = (C \delta_{ij} + \alpha_i \alpha_j)\)

\[ = I_n + \alpha \alpha^T \]

Then, \(\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \)

Consider an associated Q.F.

\[ \mu' A \mu = \mu' (I_n + \alpha \alpha^T) \mu \]

\[ = \mu' I_n \mu + \mu' \alpha \alpha' \mu \]

\[ = \mu' \mu + (\alpha' \alpha)' (\alpha' \mu) \]

For \(\mu \neq 0\), \(\mu' \mu = \sum_{i=1}^{n} \mu_i^2 > 0\)

\(\Rightarrow \mu' A \mu > 0\)

Hence, \(\mu' A \mu\) is positive definite.

i.e., \(A = (C \delta_{ij} + \alpha_i \alpha_j)\) is p.d.

**Quadratic Form**

- N.n.d.
- N.p.d.
- p.d.
- b.s.p.d.
- n.d.
- N.s.d.

**Remark:**

\[ \Rightarrow \text{If } \alpha' \alpha \text{ is n.n.d. / p.d. / b.s.d. then } \alpha' (-A) \alpha \text{ is n.p.d. / n.d. / m.s.d.} \]

\[ \Rightarrow \text{If } \alpha' \alpha \text{ is n.n.d. on n.p.d. on Indefinite, then } A \text{ is said to be n.n.d. on n.p.d. on indefinite. If a quadratic form has some definiteness, then the associated matrix has the same definiteness.} \]
Problem: Find the values of \( p \) for which \((1-p)I_n + p11'\) is p.d.? 

Solv.

\[
(1-p)I_n + p11' = \begin{bmatrix}
1 & p & p & \ldots & p \\
p & 1 & p & \ldots & p \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
p & p & p & \ldots & 1
\end{bmatrix}
\]

The \( k \)-th order principal minor is

\[
\Delta_k = \begin{bmatrix}
1 & p & p & \ldots & p \\
p & 1 & p & \ldots & p \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
p & p & p & \ldots & 1
\end{bmatrix} = (1 + k-1p)(1-p)^{k-1}
\]

\((1-p)I_n + p11'\) is p.d. iff \( \Delta_k > 0 \) for all \( k = 1, 2, \ldots, n \).

iff \( \Delta_1 = 1 > 0 \), \( \Delta_2 = (1+p)(1-p) > 0 \),

\[
\Delta_k = (1 + k-1p)(1-p)^{k-1} > 0 \quad \forall \ k = 2, 3, \ldots, n
\]

iff \(-1 < p < 1\), \((1+k-1p) > 0\) for \( k = 2, 3, \ldots, n \)

iff \(-\frac{1}{k-1} < p < 1\), \( k = 2, 3, \ldots, n \)

iff \(-\frac{1}{n-1} < p < 1\).
Problem: Reduce the equation \(3x^2 + 2y^2 + 3z^2 + 2xy + 2yz + 2xz = 1\) into canonical form.

Solution:

\[
A = \begin{pmatrix}
3 & 1 & 1 \\
1 & 3 & 1 \\
1 & 1 & 3
\end{pmatrix}, \quad X = \begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\]

\(X^TAX = 1\).

The characteristic equation of \(A\) is
\[
\begin{vmatrix}
3 - \alpha & 1 & 1 \\
1 & 3 - \alpha & 1 \\
1 & 1 & 3 - \alpha
\end{vmatrix} = 0
\]

\(\alpha_1 = 2, \alpha_2 = 3, \alpha_3 = 0\).

The eigenvalues of \(A\) are 2, 3, 0.

The eigen vector corresponding to the eigen value 2 are \(c\begin{pmatrix}1 \\ 0 \\ 0\end{pmatrix}\), \(c \neq 0\).

The other eigen vectors corresponding to the eigen value 3 are \(c\begin{pmatrix}1 \\ 1 \\ 1\end{pmatrix}\), \(c \neq 0\).

Let \(\alpha_1 = \begin{pmatrix}1 \\ 0 \\ 0\end{pmatrix}, \alpha_2 = \begin{pmatrix}0 \\ 1 \\ 0\end{pmatrix}, \alpha_3 = \begin{pmatrix}0 \\ 0 \\ 1\end{pmatrix}\), then the set \(\alpha_1, \alpha_2, \alpha_3\) is an orthogonal set. The orthonormal set of eigen vectors is
\[
\begin{pmatrix}
\frac{1}{\sqrt{2}} \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
0 \\
\frac{1}{\sqrt{2}} \\
0
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\]

Let \(P = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1
\end{pmatrix}\), then \(P\) is an orthogonal matrix.

Let us apply the orthoogonal transformation \(X' = PX\), so here, \(X' = \begin{pmatrix}X' \\ Y' \\ Z'\end{pmatrix}\).

Then the equation transforms to \((X')^T (P^T A P) X' = 1\).

\(P^T A P = (P^{-1} A P)\) is a diagonal matrix \(D\) which has the same eigen values as those of \(A\).

\(A P = \begin{pmatrix}
3 & 1 & 1 \\
1 & 3 & 1 \\
1 & 1 & 3
\end{pmatrix} \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}}
\end{pmatrix}
\]

\(P = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad P D = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.

So, \(P^{-1} A P = D\).

The equation transforms to \((X')^T D X' = 1\),

i.e. to \(2x'^2 + 3y'^2 + 6z'^2 = 1\).
2. Reduce the equation \(7x^2 - 2xy + 7y^2 - 16x + 16y - 8 = 0\) into canonical form and determine the nature of the conic.

**Solution:**

Let \(A = \begin{pmatrix} 7 & -1 \\ -1 & 7 \end{pmatrix}\), \(b = (-16, 16)\), \(\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}\).

Then the equation takes the form \(\mathbf{x}^T A \mathbf{x} + b^T \mathbf{x} - 8 = 0\).

The eigenvalues of \(A\) are 8, 6.

The eigen vectors corresponding the eigen values 8 and 6 are \(\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\), \(\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\), respectively.

The orthogonal set of eigen vectors is \(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}\).

Let \(P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}\), then \(P\) is an orthogonal matrix.

\(P^T A P = \begin{pmatrix} 8 & 0 \\ 0 & 6 \end{pmatrix}\) is a diagonal matrix which has the same eigenvalues as those of \(A\).

So, \(P^T A P = \begin{pmatrix} 8 & 0 \\ 0 & 6 \end{pmatrix}\).

By the orthogonal transformation \(\mathbf{x} = P \mathbf{x}'\), where \(\mathbf{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix}\),

the equation transforms to \(8x'^2 + 6y'^2 - 16 \sqrt{2} x' - 8 = 0\).

Let us apply the translation \(x'' = x' - \sqrt{2}, y'' = y'\).

The equation transforms to \(8x''^2 + 6y''^2 = 24\).

The canonical form is \(8x''^2 + 6y''^2 = 24\).

The equation represents an ellipse.

**END**
SEQUENCES OF REAL NUMBERS: The word "sequence" is used to convey the idea that the things are arranged in order.

Definition: A sequence of real numbers is a function defined on the set \( \mathbb{N} \) of natural numbers whose range is a subset of the set \( \mathbb{R} \) of real numbers; i.e., if for every \( n \in \mathbb{N} \), \( \exists \) a real number \( a_n \), then the ordered set \( a_1, a_2, \ldots, a_n, \ldots \) is said to define a sequence of real nos.

Remark: \( f: A \to B \) is a mapping or function if for every \( x \in A \), \( \exists \) a unique value \( y \in B \).

Hence, we write \( y = f(x) \) whenever \( x \in A \), \( y \in B \).

\( y = f(x) \) is not a function, it's a relation.

Notation: If \( a_n \) is the \( n \)th term of a sequence, then we write \( a_1, a_2, \ldots, a_n, \ldots \), to describe the sequence.

\( f: \mathbb{N} \to \mathbb{R} \) is a sequence.

\( \{a_n, f(a_n), f(f(a_n)), f(f(f(a_n)))\} \).

The main question one is concerned with here is to decide whether or not the term \( a_n \) tends to a finite quantity when \( n \) increases indefinitely.

Definition: A sequence \( \{a_n\} \) is said to have a limit \( L \in \mathbb{R} \) if, for every \( \epsilon > 0 \), \( \exists \) a natural number \( N(\epsilon) \), \( \forall n \geq N(\epsilon) \), \( |a_n - L| < \epsilon \).

Example: (1) Prove that \( \lim_{n \to \infty} \left( \frac{1}{n} \right) = 0 \).

Solution: Let \( \epsilon > 0 \) be an arbitrary number, then

\[ |a_n - L| < \epsilon \]
\[ \Rightarrow |\frac{1}{n} - 0| < \epsilon \]
\[ \Rightarrow \frac{1}{n} < \epsilon \]
\[ \Rightarrow n > \frac{1}{\epsilon} \]

In particular if \( \epsilon = 0.1 \), then
\[ |\frac{1}{n} - 0| < \epsilon = 0.1 \]
\[ \Rightarrow n > \frac{1}{\epsilon} = 10 \]
\[ \therefore |\frac{1}{n} - 0| < \epsilon = 0.1 \]

whenever \( n \geq 10 \), \( \epsilon = 0.1 \).
If $\epsilon = 0.01$, then
\[
\begin{align*}
|\frac{1}{n} - 0| &< \epsilon = 0.01 \\
\Rightarrow m &> \frac{1}{\epsilon} = 100 \\
\Rightarrow m &> 101
\end{align*}
\]
Hence take, $N(\epsilon) = \left[\frac{1}{\epsilon}\right] + 1$

- Choose a natural no. $N(\epsilon)$ which is $> \frac{1}{\epsilon}$.
  Then $\exists$ a natural no. $N(\epsilon)$ such that $|\frac{1}{n} - 0| < \epsilon \quad \forall \; n \geq N(\epsilon)$

Hence, by definition, \( \lim_{n \to \infty} \left( \frac{1}{n} \right) = 0 \)
\[
|an - l| < \epsilon \quad \text{for} \; m > N(\epsilon)
\]
\[
\Rightarrow l - \epsilon < an < l + \epsilon \quad \text{for} \; m > N(\epsilon)
\]

For $0 < \epsilon' < \epsilon$, then in general,
\[
N(\epsilon') > N(\epsilon)
\]

If $\epsilon$ is small, then $N(\epsilon)$ will be sufficiently large to ensure $|an - l| < \epsilon$ for $m > N(\epsilon)$. And all the members $a_n$, $m > N(\epsilon)$ are in the small interval, $(l - \epsilon, l + \epsilon)$ i.e., then an is very close to $l$.

If a sequence $\{a_n\}$ has a finite limit $l'$, then we say that the sequence $\{a_n\}$ converges to $l'$ on the sequence is convergent.
If a sequence does not converge to a finite limit, then it is said to be divergent. If $\{a_n\}$ converges to $l'$; one writes
\[
\lim_{n \to \infty} (a_n) = l', \text{ or } \lim_{n \to \infty} (a_n) = l.
\]

**Example (2).** Prove that \( \lim_{n \to \infty} \left( \frac{1}{n+1} \right) = 0 \).

**Sol.** Let $\epsilon > 0$ be an arbitrary number,
\[
\begin{align*}
|\frac{1}{n+1} - 0| &< \epsilon \\
\Rightarrow \left| \frac{1}{n+1} - 0 \right| &< \epsilon \\
\Rightarrow \frac{1}{n+1} &< \epsilon \\
\Rightarrow m &> \frac{1}{\epsilon} - 1
\end{align*}
\]

Choose a natural number $N(\epsilon)$ which is $> \frac{1}{\epsilon} - 1$.

Then, $\exists$ a natural no. $N(\epsilon)$ such that $|\frac{1}{n+1} - 0| < \epsilon$ for all $n \geq N(\epsilon)$.

By defn $\lim_{n \to \infty} \left( \frac{1}{n+1} \right) = 0$

\[
\begin{align*}
\Rightarrow \left| \frac{1}{n+1} - 0 \right| &< \frac{1}{n+1} < \frac{1}{n} < \epsilon \\
\Rightarrow m &> \frac{1}{\epsilon}
\end{align*}
\]

Take $N(\epsilon) = \left[\frac{1}{\epsilon}\right] + 1$

Then $\exists$ a natural no. $N(\epsilon)$ such that $|\frac{1}{n+1} - 0| < \epsilon$ for all $n \geq N(\epsilon)$.

By defn, $\lim_{n \to \infty} \left( \frac{1}{n+1} \right) = 0$. 
Example (3). Prove that \( \lim_{n \to \infty} \left( \frac{n}{n+1} \right) = 0 \).

**Solution.** Let \( \varepsilon > 0 \) be an arbitrary number,

\[
| an - l | < \varepsilon
\]

\[
\Rightarrow \left| \frac{1}{n+1} - 0 \right| = \frac{1}{n+1} < \frac{1}{n} < \varepsilon
\]

\[
\Rightarrow n > \frac{1}{\varepsilon}
\]

Take \( N(\varepsilon) = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1 \).

Then \( \exists \) a natural no. \( N(\varepsilon) \), \( \exists \) \( n > N(\varepsilon) \).

\[
\Rightarrow \left| \frac{1}{n+1} - 0 \right| < \varepsilon \quad \text{and} \quad n > N(\varepsilon)
\]

By defn, \( \lim_{n \to \infty} \left( \frac{n}{n+1} \right) = 0 \).

Example (4). Prove that \( \lim_{n \to \infty} \left( \frac{2n^2+1}{n^2+n} \right) = 2 \).

**Solution.** Let \( \varepsilon > 0 \) be an arbitrary number,

\[
| an - l | < \varepsilon
\]

\[
\Rightarrow \left| \frac{2n^2+1}{n^2+n} - 2 \right| < \varepsilon
\]

\[
\Rightarrow \frac{1-2n}{n^2+n} < \varepsilon
\]

\[
\Rightarrow \frac{2n-1}{n^2+n} < \frac{2n}{n^2} \quad < \frac{2n}{n^2} < \varepsilon
\]

\[
\Rightarrow n > \frac{2}{\varepsilon}
\]

Take \( N(\varepsilon) = \left\lceil \frac{2}{\varepsilon} \right\rceil + 1 \).

Then \( \exists \) a natural no. \( N(\varepsilon) \), \( \exists \) \( n > N(\varepsilon) \).

\[
\Rightarrow \left| \frac{2n^2+1}{n^2+n} - 2 \right| < \varepsilon \quad \text{and} \quad n > N(\varepsilon)
\]

By defn, \( \lim_{n \to \infty} \left( \frac{2n^2+1}{n^2+n} \right) = 2 \).

Example (5). Prove that \( \lim_{n \to \infty} \left( \frac{1}{n^p} \right) = 0 \), \( p > 0 \).

**Solution.** Let \( \varepsilon > 0 \) be an arbitrary number,

\[
| an - l | < \varepsilon
\]

\[
\Rightarrow \left| \frac{1}{n^p} - 0 \right| < \varepsilon
\]

\[
\Rightarrow \frac{1}{n^p} < \varepsilon
\]

\[
\Rightarrow n^p > \frac{1}{\varepsilon}
\]

\[
\Rightarrow n > \left( \frac{1}{\varepsilon} \right)^{1/p}
\]

Since \( p > 0 \), take \( N(\varepsilon) = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1 \).

Then \( \exists \) a natural no. \( N(\varepsilon) \), \( \exists \) \( n > N(\varepsilon) \).

\[
\Rightarrow \left| \frac{1}{n^p} - 0 \right| < \varepsilon \quad \text{and} \quad n > N(\varepsilon)
\]
Example (7). Prove that $\lim_{n \to \infty} (\ln n) = 0$ if $|n| < 1$.

\[ \text{Soln.} \quad \text{Let } \varepsilon > 0 \text{ be an arbitrary number. Then } \]
\[ \frac{1}{n^2} < \varepsilon \]
\[ \Rightarrow |\ln n| < \varepsilon \]
\[ \Rightarrow n > \frac{\ln \varepsilon}{|\ln n|} \quad \text{[since } |n| < 1, \ln |n| < 0] \]

Choose a natural no. $N(\varepsilon)$ which is $\geq \frac{\ln \varepsilon}{|\ln n|}$

Then $\exists$ a natural no. $N(\varepsilon)$, $\exists |\ln n - 0| < \varepsilon \forall n > N(\varepsilon)$

$\therefore$ By defn. $\lim_{n \to \infty} (\ln n) = 0$ if $|n| < 1$.

Example (8). Prove that $\lim_{n \to \infty} \left(2 - \frac{1}{2^n}\right) = 2$.

\[ \text{Soln.} \quad \text{Let } \varepsilon > 0 \text{ be an arbitrary no. Then } \]
\[ \left| 2 - \frac{1}{2^n} - 2 \right| < \varepsilon \]
\[ \Rightarrow \left| -\frac{1}{2^n} \right| < \varepsilon \]
\[ \Rightarrow \frac{1}{2^n} < \frac{1}{\varepsilon} \]
\[ \Rightarrow n > \frac{1}{\varepsilon} \]

Take $N(\varepsilon) = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1$

Then $\exists$ a natural no. $N(\varepsilon)$, $\exists \left| 2 - \frac{1}{2^n} - 2 \right| < \varepsilon \forall n > N(\varepsilon)$

$\therefore$ By defn. $\lim_{n \to \infty} \left(2 - \frac{1}{2^n}\right) = 2$.

Example (9). Prove that $\lim_{n \to \infty} \left(2^{1/n}\right) = 1$.

\[ \text{Soln.} \quad \text{Let } \varepsilon > 0 \text{ be an arbitrary no. Then } \]
\[ \left| 2^{1/n} - 1 \right| < \varepsilon \]
\[ \Rightarrow 2^{1/n} < \varepsilon + 1 \]
\[ \Rightarrow \frac{1}{n} \ln 2 < \ln (\varepsilon + 1) \]
\[ \Rightarrow \frac{1}{n} < \frac{\ln (\varepsilon + 1)}{\ln 2} \]
\[ \Rightarrow n > \frac{\ln 2}{\ln (\varepsilon + 1)} \]

Take $N(\varepsilon) = \left\lceil \frac{\ln 2}{\ln (\varepsilon + 1)} \right\rceil$

$\therefore n > N(\varepsilon)$. [P]
Example (10). Prove that \( \lim_{n \to \infty} (3^{-n}) = 0 \).

**Soln.**
\[
\text{Let } \epsilon > 0 \text{ be an arbitrary number.}
\]
\[
\Rightarrow \left| 3^{-n} \right| < \epsilon
\]
\[
\Rightarrow \frac{1}{3^n} < \epsilon
\]
\[
\Rightarrow 3^n > \frac{1}{\epsilon}
\]
\[
\Rightarrow m \ln 3 > \ln \frac{1}{\epsilon}
\]
\[
\Rightarrow m > \frac{\ln \frac{1}{\epsilon}}{\ln 3}
\]

Take, \( N(\epsilon) = \left\lceil \frac{\ln \frac{1}{\epsilon}}{\ln 3} \right\rceil \).

\[
\Rightarrow n > N(\epsilon).
\]

Example (11). Prove that \( \lim_{n \to \infty} \left( \frac{10^6 \cdot n}{n^2 + m} \right) = 0 \).

**Soln.**
\[
\text{Let } \epsilon > 0 \text{ be an arbitrary number.}
\]
\[
\Rightarrow \left| \frac{10^6 \cdot n}{n^2 + m} \right| = \frac{10^6}{n + \frac{m}{n}} < \epsilon
\]
\[
\Rightarrow n + \frac{m}{n} > \frac{10^6}{\epsilon}
\]
\[
\Rightarrow n > \frac{10^6}{\epsilon} - 1.
\]

Take \( N(\epsilon) = \left\lceil \frac{10^6}{\epsilon} - 1 \right\rceil + 1 \).

Then \( \exists \) a natural number \( N(\epsilon) \) such that \( \left| \frac{10^6 \cdot n}{n^2 + m} \right| < \epsilon \) for all \( n > N(\epsilon) \).

Example (12). Prove that \( \lim_{n \to \infty} \left( \frac{\sqrt{n}}{n^2 + m} \right) = 0 \).

**Soln.**
\[
\text{Let } \epsilon > 0 \text{ be an arbitrary number.}
\]
\[
\Rightarrow \left| \frac{\sqrt{n}}{n^2 + m} \right| < \epsilon
\]
\[
\Rightarrow \sqrt{n} < \epsilon
\]
\[
\Rightarrow \frac{1}{\sqrt{n}} < \epsilon
\]
\[
\Rightarrow n > \frac{1}{\epsilon^2}
\]

Take \( N(\epsilon) = \left\lceil \frac{1}{\epsilon^2} \right\rceil + 1 \).

Then \( \exists \) a natural number \( N(\epsilon) \) such that \( \left| \frac{\sqrt{n}}{n^2 + m} \right| < \epsilon \) for all \( n > N(\epsilon) \).

\[
\Rightarrow \lim_{n \to \infty} \left( \frac{\sqrt{n}}{n^2 + m} \right) = 0.
\]
**Example 13.** Prove that \( \lim_{n \to \infty} \left( \frac{m^2 - 1}{2n^2 + 3n} \right) = \frac{1}{2} \)

**Solution.** Let \( \varepsilon > 0 \) be an arbitrary number.

\[
\left| \frac{m^2 - 1}{2n^2 + 3n} - \frac{1}{2} \right| < \frac{3m - 3}{4m} < \frac{3m}{4m} = \frac{3}{4} < \varepsilon
\]

\[\Rightarrow m > \frac{3}{4\varepsilon}\]

Take, \( N(\varepsilon) = \left[ \frac{3}{4\varepsilon} \right] + 1 \)

Then \( \exists \) a natural no. \( N(\varepsilon) \) s.t. \( \left| \frac{m^2 - 1}{2n^2 + 3n} - \frac{1}{2} \right| < \varepsilon \) \( \forall n > N(\varepsilon) \)

\( \therefore \) By defn., \( \lim_{n \to \infty} \left( \frac{m^2 - 1}{2n^2 + 3n} \right) = \frac{1}{2} \)

**Example 14.** Prove that \( \lim_{n \to \infty} \left( (-1)^n \frac{1}{n} \right) = 0 \)

**Solution.** Let \( \varepsilon > 0 \) be an arbitrary no.

\[
\left| (-1)^n \frac{1}{n} - 0 \right| < \varepsilon
\]

\[\Rightarrow n > \frac{1}{\varepsilon} \quad \text{[P]}\]

**Example 15.** Show that \( \lim_{n \to \infty} \left( \frac{\sqrt{n+1} - \sqrt{n}}{n} \right) = 0 \)

\( \lim_{n \to \infty} \frac{\sin n}{n^2} = 0 \quad \text{[3]} \)

**Solution.** Let \( \varepsilon > 0 \) be an arbitrary no.

\[
\left| \frac{\sqrt{n+1} - \sqrt{n}}{n} \right| = \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} < \varepsilon \Rightarrow \frac{1}{\sqrt{n+1}} < \varepsilon
\]

\[\Rightarrow \sqrt{n+1} > \varepsilon
\]

\[\Rightarrow n+1 > \varepsilon^2\]

\[\Rightarrow n > \varepsilon^2 - 1 \quad \text{[P]}\]

If \( \varepsilon > 0 \) be an arbitrary no.

\[
\left| \frac{\sin \frac{\pi}{n}}{n^2} - 0 \right| = \frac{\sin \frac{\pi}{n}}{n^2} \frac{\pi}{\pi} < \frac{\varepsilon}{\pi} < \varepsilon
\]

\[\Rightarrow n > \frac{\varepsilon}{\pi}\]

Take, \( N(\varepsilon) = \left[ \frac{\varepsilon}{\pi} \right] + 1 \)

Then \( \exists \) a natural no. \( N(\varepsilon) \) s.t. \( \left| \frac{\sin \frac{\pi}{n}}{n^2} - 0 \right| < \varepsilon \) \( \forall n > N(\varepsilon) \)

\( \therefore \) \( \lim_{n \to \infty} \frac{\sin \frac{\pi}{n}}{n^2} = 0 \).
REMARK:

1. It is important to realize that the definition of the limit of a sequence is used to verify whether a propose value is a limit of particular sequence or not. But in most of the sequences, we can make a guess about the limit by computing first few terms of the sequence.

2. It is also important to realize that the convergence and divergence of a sequence are the property of the tails of the sequence. In other words, the convergence and divergence depend on the behaviour of the ultimate terms. Hence, the alternation of the finite number of term of a sequence does not affect the convergence or divergence of a sequence, i.e., a convergence sequence does not alter the limit. For e.g.,

\[ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \] and \[ 1000, 100, 10, 1, \ldots, \frac{1}{n}, \ldots \] have the same limit.

**Theorem:** If a sequence is convergent then its limit is unique.

**Proof:** Let \( \{a_n\} \) be an convergent sequence. If possible, let \( l \) and \( l' \) be the two limits of \( \{a_n\} \).

Then for a \( \epsilon > 0 \), \( \exists \) a natural no. \( N_1(\epsilon) \) \( \in \mathbb{N} \) such that \( |a_n - l| < \epsilon \) \( \forall n > N_1(\epsilon) \).

Then also, \( \exists \) a natural no. \( N_2(\epsilon) \) \( \in \mathbb{N} \) such that \( |a_n - l'| < \epsilon \) \( \forall n > N_2(\epsilon) \).

Take \( N(\epsilon) = \max\{N_1(\epsilon), N_2(\epsilon)\} \).

Then if \( n > N(\epsilon) \), we have

\[ |a_n - l| < \epsilon \quad \text{and} \quad |a_n - l'| < \epsilon \]

Now, \( 0 \leq |l' - l| = \left| \left( a_n - l' \right) - \left( a_n - l \right) \right| \leq |a_n - l'| + |a_n - l| < 2\epsilon \)

Hence, for any \( \epsilon > 0 \), we have \( 0 \leq |l' - l| < 2\epsilon \)

Hence, \( l' - l = 0 \implies l = l' \)

**Lemma:** If for every \( \epsilon > 0 \), \( 0 \leq a < \epsilon \) then \( a = 0 \).

**Proof:** If possible, let \( a > 0 \)

Take \( \epsilon = \frac{a}{2} > 0 \), then \( d = 2\epsilon \), i.e., \( a + \epsilon \)

which is the contradiction to the fact that for every \( \epsilon > 0 \), \( 0 \leq a < \epsilon \).

Therefore, \( a = 0 \)
Bounded Sequences: A sequence is said to be bounded if there exist real numbers $m$ and $M$ such that $m \leq a_n \leq M$ for all $n \in \mathbb{N}$.

If a sequence is not bounded, then it is called an unbounded sequence.

Example: The sequence $\{\frac{1}{n}\}$, $\{(-1)^n+\frac{1}{n}\}$ are bounded.

Since $0 < \frac{1}{n} \leq 1$ and $(-1)^n+\frac{1}{n} = 1$ if $n$ is odd

The sequence $\{\frac{1}{n}\}$, $\{(-1)^n+\frac{1}{n}\}$ are unbounded.

Theorem: A convergent sequence is bounded.

Proof: Let $\{a_n\}$ converge to the limit $L$, then for a given $\epsilon > 0$, there exists a natural number $N(\epsilon)$ such that $|a_n - L| < \epsilon$ for all $n > N(\epsilon)$.

Take $M = \max\{|1|+\epsilon, \ldots, |a_n(\epsilon)|-1\}$

Clearly, $M$ is finite.

Then $|a_n| \leq M \forall n \in \mathbb{N}$

$\Rightarrow -M \leq a_n \leq M \forall n \in \mathbb{N}$

Corollary: Unbounded sequences are divergent.

Example: A bounded sequence may not be convergent.

The sequence $\{(-1)^n+1\}$ is bounded but not convergent.

Example: Note that the sequence $\{n\}$, $\{(-1)^n+\frac{1}{n}\}$ are unbounded, hence they are divergent by the above corollary.
Divergent Sequence: The term "convergent sequence" is said to convey for the sequence whose limit is finite and unique. The sequence which is not convergent is called "divergent sequence". There are two possibilities:
- a sequence with no unique limit.
- a sequence with an infinite limit.

Definition:
(a) We say that \( \{a_n\} \) diverges to \( +\infty \) and write \( \lim (a_n) = +\infty \) if for every \( \epsilon > 0 \), \( \exists \) a natural no. \( N(\epsilon) \) \( \ni a_n > \epsilon \) \( \forall n \geq N(\epsilon) \).
(b) We say that \( \{a_n\} \) diverges to \( -\infty \) and write \( \lim (a_n) = -\infty \) if for every \( \epsilon > 0 \), \( \exists \) a natural no. \( N(\epsilon) \) \( \ni a_n < -\epsilon \) \( \forall n \geq N(\epsilon) \).

Properly divergent: If \( \lim (a_n) = +\infty \) or \( \lim (a_n) = -\infty \), then the sequence \( \{a_n\} \) is called properly divergent.

Examples:
(a) \( \lim (m^n) = +\infty \)

Solt: \( \lim \frac{m^n}{n} \)

Let \( \epsilon > 0 \) be an arbitrary no.

Try \( m^n > \epsilon \) \( \Rightarrow n > \frac{\log \epsilon}{\log m} \)

Then \( \epsilon < n \)

Take \( N(\epsilon) = \lceil \frac{\log \epsilon}{\log m} \rceil + 1 \)

Then \( \exists \) a natural no. \( N(\epsilon) \) \( \ni m^n < \epsilon \) \( \forall m > N(\epsilon) \).

(b) \( \lim (\sqrt[n]{m}) = +\infty \)

Solt:

Let \( \epsilon > 0 \) be an arbitrary no.

Then \( \frac{h}{\sqrt[n]{m}} \geq \frac{m}{\sqrt[n]{n-1}} = \epsilon \)

Try \( n > \epsilon \)

Take \( N(\epsilon) = \lceil \frac{\log \epsilon}{\log \sqrt[n]{m}} \rceil + 1 \)

Then \( \exists \) a natural no. \( N(\epsilon) \) \( \ni \frac{m}{\sqrt[n]{n-1}} > \epsilon \) \( \forall n > N(\epsilon) \).

Hence, \( \lim \frac{m}{\sqrt[n]{n-1}} = +\infty \).
\[ \lim (\ln n) = +\infty \text{ if } n > 1. \]

**Solution:** Let \( \varepsilon > 0 \) be an arbitrary number. Then \( \ln n > \varepsilon \) for \( n > 1 \).

\[ \Rightarrow \ln n > \frac{\ln \varepsilon}{\ln m} \]
\[ \Rightarrow n > \frac{\varepsilon}{\ln \varepsilon} \]

Choose a natural number \( N(\varepsilon) \) such that \( \frac{\varepsilon}{\ln \varepsilon} < m \).

Then \( \exists \) a natural number \( n \geq N(\varepsilon) \) such that \( \ln n > \varepsilon \).

By definition, \( \lim (\ln n) = +\infty \) if \( n > 1 \).

**Oscillatory Sequences:**

(a) A bounded sequence which is not convergent is said to be an oscillatory sequence with finite oscillation.

(b) An unbounded sequence which is not properly divergent is said to be an oscillatory sequence with infinite oscillation.

**Example:**

(a) The sequence \( \{a^n\} \) is a properly divergent and diverges to \( +\infty \).

(b) \( \{(-1)^n\} \) is a properly divergent and diverges to \( -\infty \).

(c) \( \{(-1)^n + \frac{1}{n}\} \) is a bounded sequence but not convergent. Hence, it is an oscillatory sequence with finite oscillation.

(d) \( \{(1/n)^n\} \) is unbounded and not properly divergent. It is an oscillatory sequence with infinite oscillation.

**Diagram:**

\[ \text{Sequence} \]

- **Convergent** (with unique finite limit)
  - Properly Divergent
    - Oscillatory
      - Finite
        - Infinite
  - **Oscillatory** with finite oscillation
  - **Unbounded**
    - **Convergent**
  - Properly Divergent
    - Oscillatory with infinite oscillation
Example: Check the convergence or divergence of the sequence $a_n$ for different values of $n$.

**Soln.**

If $|a| < 1$, \( \lim (a^n) = 0 \)

If $n = 1$, then $a_n = 1$, becomes a constant sequence: $1, 1, 1, \ldots$, so $\lim (a^n) = 1$.

If $n = -1$, then $a_n = \frac{1}{n}$, and $\lim (a^n) = \frac{1}{\lim n}$, and

If $n > 1$, \( \lim (a^n) = \infty \)

If $n < -1$, then $n = -\frac{1}{n}$, $n > 1$, and $\lim (a^n) = \frac{1}{\lim n}$ is an oscillatory sequence with infinite oscillation.

**Sandwich Theorem**

**Squeeze Theorem**: If $a_n$, $b_n$, and $c_n$ are three sequences of real numbers such that $a_n \leq b_n \leq c_n$ for $n \geq k$, and $\lim (a_n) = x = \lim (c_n)$, then $\lim (b_n) = x$.

Example: Evaluate the following limits.

(a) \( \lim_{n \to \infty} \left( \sqrt{n+1} - \sqrt{n} \right) = 0 \)

**Soln.**

Let $b_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{2\sqrt{n}}$.

Then $a_n = \frac{2}{\sqrt{n}} < b_n = \frac{1}{\sqrt{n+1} + \sqrt{n}} < c_n = \frac{2}{\sqrt{n}}$ for $n \in \mathbb{N}$.

Note that $\lim(a_n) = 0 = \lim(c_n)$

$\therefore$ By Squeeze Theorem, $\lim (b_n) = 0$.

(b) \( \lim_{n \to \infty} \frac{1}{(n+1)^n} + \frac{1}{(n+2)^n} + \cdots + \frac{1}{(n+n)^n} \)

**Soln.**

Let $b_n = \frac{1}{(n+1)^n} + \frac{1}{(n+2)^n} + \cdots + \frac{1}{(n+n)^n}$.

Note, $\frac{1}{(n+1)^n} < b_n < \frac{1}{n^n}$ for $n \geq 1$.

$\therefore a_n = \frac{1}{(n+1)^n} < b_n < \frac{1}{n^n} = c_n$ for $n \geq 1$.

Note that $\lim(a_n) = 0 = \lim(c_n)$

$\therefore$ By Squeeze Theorem, $\lim (b_n) = 0$.
$$\lim_{n \to \infty} \left( \frac{2^n + 3^n}{n^m} \right)$$

**Solu.**

Note that 

$$3^n \leq b_n = \left( \frac{2^n + 3^n}{n^m} \right) \leq \left( \frac{3^n}{n^m} \right)$$

$$\Rightarrow a_n = 3 \leq b_n \leq 3 \cdot \frac{n^m}{n^m} = c_n.$$ 

Clearly, 

$$\lim_{n \to \infty} (a_n) = 3 = \lim_{n \to \infty} (c_n).$$

By squeeze theorem, 

$$\lim_{n \to \infty} (b_n) = 3.$$

**Examples.**

**Show that**

(a) $$\lim_{n \to \infty} (a^{1/n}) = 1 \quad , \quad a > 0.$$ 

**Solu.**

(a) **Case I :** 

$$a > 1,$$

then 

$$(a^{1/n} - 1) > 0 \quad \forall \quad n \in N.$$ 

Define a new sequence 

$$x_n = (a^{1/n} - 1) > 0 \quad \forall \quad n \in N.$$ 

Hence, 

$$a = (1 + x_n)^n > 1 + n \cdot x_n.$$ 

For $$k > 0,$$ 

$$(1 + x_n)^n > 1 + nx_n \quad , \quad n > 1.$$ 

Now, 

$$0 < x_n \leq \frac{a-1}{n}, \quad n > 1.$$ 

Let $$\epsilon$$ be an arbitrary no., then 

$$|x_n - 0| = x_n \leq \frac{a-1}{n} < \epsilon.$$ 

$$\Rightarrow n > \frac{a-1}{\epsilon}.$$ 

Take 

$$N(\epsilon) = \left\lfloor \frac{a-1}{\epsilon} \right\rfloor + 1.$$ 

Then $$\exists$$ a natural no. $$N(\epsilon):$$ 

$$|x_n - 0| < \epsilon \quad \forall \quad n > N(\epsilon).$$ 

Hence, 

$$\lim_{n \to \infty} (x_n) = 0.$$ 

$$\Rightarrow \lim_{n \to \infty} (a^{1/n} - 1) = 0.$$ 

$$\Rightarrow \lim_{n \to \infty} (a^{1/n}) = 1.$$ 

(b) **Case II :** 

$$a = 1,$$

then the sequence $$\{a^{1/n}\}$$ becomes a constant sequence 

$$1, 1, 1, \ldots \ldots$$

$$\Rightarrow \lim_{n \to \infty} (a^{1/n}) = 1.$$
Case III: \( 0 < a < 1 \).

Let \( b = \frac{1}{a} > 1 \).

Then \( \lim_{n \to \infty} (a^{1/n}) = \lim_{n \to \infty} \left( \frac{1}{b^{1/n}} \right) = \lim_{n \to \infty} \left( \frac{1}{b^{1/n}} \right) = \frac{1}{\lim_{n \to \infty} (b^{1/n})} = \frac{1}{1} \)

Since \( b > 1 \), \( \lim(b^{1/n}) = 1 \) by case I.

Hence, \( \lim(a^{1/n}) = 1 \) for \( a > 0 \).

(b) \( \lim_{n \to \infty} (n^{1/n}) = 1 \).

\[ \text{Soln.} \] Define a new sequence,
\[ x_n = n^{1/n} - 1 \geq 0 \text{ for } n > 1 \]

Now,
\[ n = (1 + x_n)^n \geq \frac{m(n-1)}{2} x_n \]

For \( x > 0 \),
\[ (1 + x)^n = 1 + nx + \frac{x(n-1)}{2} + \cdots + \frac{x^n}{n} \geq \frac{m(n-1)}{2} x_n \text{ for } n \geq 2 \]

\[ 0 \leq x_n \leq \frac{2}{\sqrt[n]{n-1}} \text{ for } n \geq 2 \]

Let \( \varepsilon > 0 \) be an arbitrary number.

Then
\[ |x_n - 0| = x_n \leq \frac{2}{\varepsilon} < \varepsilon \text{ for } n > N(\varepsilon) \text{ for some } N(\varepsilon) \]

\[ n - 1 > \frac{2}{\varepsilon} \]
\[ n > \frac{2}{\varepsilon} + 1 \]

Take
\[ N(\varepsilon) = \left[\frac{2}{\varepsilon} + 1\right] + 1 \]

Then \( \exists \) a natural no., \( N(\varepsilon) \) such that \( |x_n - 0| < \varepsilon \text{ for } n > N(\varepsilon) \).

By definition, \( \lim (x_n) = 0 \).

\[ \Rightarrow \lim (n^{1/n}) = 1 \]

\[ \text{Also, } \lim (a_n) = 0 = \lim (C_n) \]

Note that \( \lim (a_n) = 0 = \lim (C_n) \) by squeeze theorem, \( \lim (x_n) = 0 \).

\( \left( \frac{\sin n}{n} \right) = 0 \)

\[ \frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \text{ for } n \in \mathbb{N} \text{ as } -1 \leq \sin n \leq 1 \]

Hence, we can apply squeeze theorem to infer that
\[ \lim (n \sin n) = 0. \]
Example 8: Show that
(a) \( \lim (n^{1/n^n}) = 1 \)
(b) \( \lim (n^{1/n^n}) = 1 \), \( a > 0 \)
(c) \( \lim (n^{a^n}) = 0 \), \( 0 < a < 1 \).

Hints:
(a) Define \( x_n = n^{1/n} - 1 > 0 \) \( \forall n \in \mathbb{N} \)
\[ x_n = (1 + x_n)^{1/n} \]
\[ 1 > x_n \]
\[ 0 \leq x_n \leq \frac{1}{n} \quad \forall n \in \mathbb{N}. \]
(b) Case I: \( a > 1 \)
Let \( x_n = (a_n)^{1/n} - 1 > 0 \), \( \forall n \in \mathbb{N} \)
\[ a_n = (1 + x_n)^{m} \geq \frac{m(1 + x_n)}{2} \]
\[ 0 \leq x_n \leq \frac{2}{m} \quad m \geq 2. \]
Case II: \( 0 < a < 1 \)
Let \( x_n = (a_n)^{1/n} - 1 \leq \frac{2}{m} \quad m \geq 2. \)

\[ \text{[Alt.] } \lim_{n \to 0} (a_n)^{1/n} = \left( \lim_{n \to 0} a_n \right)^{1/n} = 1, 1, 1. \]
(c) \( 0 < a < 1 \).
\[ a = \frac{1}{1 + b}, \quad b > 0 \]
\[ (a_n^n) = \frac{n}{(1 + b)^n} \leq \frac{n}{n(n-1)^{1/b^2}} \]
\[ \text{[For } b > 0, \ (1 + b)^n \leq n^{1/(1-b^2)} \]
\[ 0 \leq a_n < \frac{2}{(m-1)b^2}, \quad m \geq 2 \]
Let \( e > 0 \) be an arbitrary number,
Then \( |a_n - 0| = a_n \leq \frac{2}{(m-1)b^2} < e(\text{choice of } n) \)
\[ \Rightarrow m - 1 > \frac{2}{b^2e} \]
\[ \Rightarrow m > \frac{2}{b^2e} + 1 \]
\[ \text{(Take } N(e) = \left[ \frac{2}{b^2e} + 1 \right] + 1 \]
\[ \Rightarrow \lim_{n \to \infty} (a_n^n) = 0 \]
Limit Theorem: If \( \lim (a_n) = a \) and \( \lim (b_n) = b \) then
\[
\begin{align*}
\lim (a_n \pm b_n) &= \lim (a_n) \pm \lim (b_n) = a \pm b \\
\lim (a_n \cdot b_n) &= \lim (a_n) \cdot \lim (b_n) = a \cdot b \\
\lim \left( \frac{a_n}{b_n} \right) &= \frac{\lim (a_n)}{\lim (b_n)} = \frac{a}{b}
\end{align*}
\]

provided \( \lim (b_n) = b \neq 0 \) and \( b_n \neq 0, n \in \mathbb{N} \).

Proof of 1) Let \( \varepsilon > 0 \) be an arbitrary number. Then \( \exists \) a natural number \( N_1(\varepsilon) \) such that \( |a_n - a| < \varepsilon \) for all \( n > N_1(\varepsilon) \). Similarly, \( \exists \) a natural number \( N_2(\varepsilon) \) such that \( |b_n - b| < \varepsilon \) for all \( n > N_2(\varepsilon) \).

Let \( N(\varepsilon) = \max \{N_1(\varepsilon), N_2(\varepsilon)\} \).

Then if \( n > N(\varepsilon) \), then \( n > N_1(\varepsilon) \) and \( n > N_2(\varepsilon) \), and also
\[
\begin{align*}
|(a_{n+1} + b_{n+1}) - (a + b)| &= |(a_n - a) + (b_n - b)| < \varepsilon + \varepsilon = 2\varepsilon
\end{align*}
\]

By definition, \( \lim (a_{n+1} + b_{n+1}) = a + b \).

Remark:
1) If \( \{a_n\} = \{\frac{1}{n} \} \) and \( \{b_n\} = \{-\frac{1}{n} \} \), then they are divergent but \( \{a_n + b_n\} = \{0\} \) which converges to 0.

\[
\left[\begin{array}{ll}
\text{odd}, & \text{then } a_n = -1, b_n = +1 \\
\text{even}, & \text{then } a_n = +1, b_n = -1
\end{array}\right]
\]

Hence \( \lim (a_{n+1} + b_{n+1}) \) is a finite number but the sequence \( \{a_n\} \), \( \{b_n\} \) are not convergent.

2) Let \( a_n = \frac{1}{n} \), \( b_n = \frac{1}{n^2} \), then \( \lim (a_n) = 0 = \lim (b_n) \). Using 1)

But \( \frac{a_n}{b_n} = \frac{1}{n} \) converges to 0.

Hence, when \( \lim (a_n) = 0 = \lim (b_n) \), then nothing can be said about the sequence \( \{a_n + b_n\} \), the sequence may converge or may diverge.
Example 4

(a) If \( \lim(an) = 0 \) then prove by using the definition of limit that \( \lim(an^2) = 0 \).

\[
\text{Soln. } \quad \text{Let } \epsilon > 0 \text{ be an arbitrary quantity. Then } \exists \text{ a natural } N(\epsilon) \ni |an - 0| < \epsilon \quad \forall n > N(\epsilon)
\]

\[
\Rightarrow |an| < \epsilon \quad \forall n > N(\epsilon)
\]

\[
\Rightarrow a_n < \epsilon \quad \forall n > N(\epsilon)
\]

\[
\Rightarrow |a_n - 0| < \epsilon \quad \forall n > N(\epsilon)
\]

\[
\Rightarrow \lim_{n \to \infty} |a_n| = 0.
\]

By definition \( \lim(an^2) = 0 \).

(b) Prove that \( \lim(an) = 0 \), \( \lim(|an|) = 0 \), give an e.g. to establish that the convergence of \( |an| \) may not imply the convergence of \( an \).

\[\text{Hint} \Rightarrow \]

\[
\lim(an) = 0
\]

\[
\Rightarrow |an - 0| < \epsilon \quad \forall n > N(\epsilon)
\]

\[
\Rightarrow |an| < \epsilon \quad \forall n > N(\epsilon)
\]

\[
\Rightarrow |an| - 0| < \epsilon \quad \forall n > N(\epsilon)
\]

\[
\Rightarrow \lim |an| = 0.
\]

Ex.

\[\text{Let } f(x) = \{(-1)^n\}
\]

N.T. \( f(x) \) is oscillating, i.e., it diverges.

But \( |an| = 1 \) is constant sequence, which converges.

(c) If \( \lim(an) = 0 \) and \( an > 0 \quad \forall n \in \mathbb{N} \), then show that \( \lim(\sqrt{an}) = 0 \).

Example 5

(a) Prove that if \( \lim(an) = 0 \) and sequence \( \{b_n\} \) is bounded (but not necessarily convergent) then \( \lim(an \cdot b_n) = 0 \).

\[\text{Soln. } \quad \text{Let } \epsilon > 0 \text{ be an arbitrary quantity. Then } \exists \text{ a } N(\epsilon) \ni |an - 0| < \epsilon \quad \forall n > N(\epsilon).
\]

Again, \( \{b_n\} \) is bounded,

\[
\Rightarrow |b_n| < M \quad \forall n \in \mathbb{N}
\]

for some \( M \).

Now,

\[
|an \cdot b_n - 0| = |an||b_n| < \epsilon M \quad \forall n > N(\epsilon)
\]

\[
\Rightarrow \lim(an \cdot b_n) = 0.
\]
(b) Prove that if \( \lim (a_n) = l \) then \( \lim (a^n) = l^n \).

Hint → Let \( \varepsilon > 0 \)

Then \( |a_n - l| < \varepsilon \) \( \forall n > N(\varepsilon) \)

Now, \( |a_n^m - l^m| = |a_n - l| |a_n^{m-1}| \leq \varepsilon \cdot |a_n|^{m-1} \cdot |l|^{m-1} \)

Note that, since \( \{a_n\} \) is convergent, \( \{a_n\} \) is bounded, i.e., \( |a_n| < M \) \( \forall n \in \mathbb{N} \) for some \( M \).

So \( |a_n - l| < \varepsilon \) \( \implies M^{m-1} \cdot |l|^{m-1} < \varepsilon \) \( \forall n > N(\varepsilon) \)

By definition, \( \lim (a_n^m) = l^m \).

Monotonic Sequences:

(a) A sequence \( \{a_n\} \) is said to be monotonic increasing if \( a_{n+1} > a_n \) \( \forall n \in \mathbb{N} \).

(b) A sequence \( \{a_n\} \) is said to be monotonic decreasing if \( a_{n+1} \leq a_n \) \( \forall n \in \mathbb{N} \).

A sequence \( \{a_n\} \) is called monotonic sequence if it is either monotonic increasing or decreasing.

- **Theorem:** A monotonic sequence converges if it is bounded.

- **Proof:** Let \( \varepsilon > 0 \).

  Then, \( M - \varepsilon < a_n \leq M \) \( \forall n > N(\varepsilon) \)

  \[ a_{n+1} \leq \frac{a_n}{M} \leq \frac{M}{M-\varepsilon} \]

  \[ \implies M - \varepsilon < a_{n+1} < M + \varepsilon \] \( \forall n > N(\varepsilon) \)

  \[ |a_{n+1} - M| < \varepsilon \] \( \forall n > N(\varepsilon) \)

  \[ \implies \lim (a_n) = M \]

**Corollary:** A monotonic sequence is only divergent if it is unbounded.

In fact, if \( \{a_n\} \) is monotonic increasing and unbounded, then \( \lim (a_n) = +\infty \).

If \( \{a_n\} \) is monotonic decreasing and unbounded, then \( \lim (a_n) = -\infty \).
**Example 6** For the following sequences examine whether or not they are bounded and whether or not they are monotonic. Also establish the convergence or divergence of the sequences.

(a) \( a_n = \frac{1}{n}, \ n \in \mathbb{N} \)

(b) \( a_1 = 1, a_n = \sqrt{2a_{n-1}}, \ n = 2, 3, \ldots \)

(c) \( a_n = \left(1 + \frac{1}{n}\right)^n, \ n \in \mathbb{N} \)

**Soln**

(a) Note that

\[ a_{n+1} - a_n = \frac{1}{n+1} - \frac{1}{n} = \frac{1}{n(n+1)} < 0 \quad \forall \ n \in \mathbb{N}. \]

\(a_n\) is decreasing, clearly \(0 < \frac{1}{n} \leq 1 \quad \forall \ n \in \mathbb{N}.\)

i.e., \(a_n\) is bounded.

Hence, \(a_n\) converges.

(b) Here, \(a_1 = 1\) and \(a_2 = \sqrt{2a_1} = \sqrt{2}\)

Let us suppose that \(a_k > a_{k-1}\).

Now, \(a_{k+1} = \sqrt{2a_k} > 2a_{k-1} = a_k\)

Hence by mathematical induction, \(a_n > a_1 \quad \forall \ n \in \mathbb{N}.\)

So, \(a_n\) is increasing.

Note that, \(a_1 = 2, a_2 < 2\)

Let us assume that \(a_k < 2\).

Now, \(a_{k+1} = \sqrt{2a_k} < \sqrt{2} < 2\)

By mathematical induction, \(a_n < 2 \quad \forall \ n \in \mathbb{N}\)

i.e., \(a_n\) is bounded.

\(\therefore \) \(a_n\) converges.

(c) Here, \(a_n = \left(1 + \frac{1}{n}\right)^n\)

\[= 1 + \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \cdots\]

\[\leq 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{3!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \cdots\]

\[< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\]

\[< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} < 2 \quad \forall \ n \in \mathbb{N}\]
\[ a_n < 1 + \frac{1}{(1 - \frac{1}{2n})} = 1 + 2\left(1 - \frac{1}{2n}\right) < 3, \forall n \in \mathbb{N}. \]

\[ \Rightarrow a_n < 3, \forall n \in \mathbb{N}. \]

Note that,
\[ a_n = 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{3!} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots + \frac{1}{n!} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots \left(1 - \frac{m-1}{n}\right) \]

and,
\[ a_{n+1} = 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n+1}\right) + \frac{1}{3!} \left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right) + \cdots + \frac{1}{n!} \left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots \left(1 - \frac{m-1}{n+1}\right) \]

\[ + \frac{1}{(n+1)!} \left(1 - \frac{1}{n+1}\right)\cdots \left(1 - \frac{m-1}{n+1}\right) \]

Note that,
\[ n+1 > m \Rightarrow \frac{1}{n+1} < \frac{1}{n}. \]

\[ \Rightarrow 1 - \frac{1}{n+1} > 1 - \frac{1}{n}, \quad 1 - \frac{2}{n+1} > 1 - \frac{2}{n}, \text{ etc.} \]

Hence,
\[ a_{n+1} > a_n, \forall n \in \mathbb{N}. \]

Now,
\[ 2 = a_1 \leq a_n < 3, \forall n \in \mathbb{N}. \]

\[ \text{\(a_n\)} \text{ is monotonic, increasing and bounded, i.e.} \]

\[ \text{\(a_n\)} \text{ converges.} \]
SERIES OF REAL NUMBERS: Consider a given sequence \( \{a_n\} \), then the infinite sum \( \sum_{n=1}^{\infty} a_n \) is called a series.

For a given sequence \( \{a_n\} \), define:

\[
S_1 = a_1, \quad S_2 = a_1 + a_2, \quad S_3 = a_1 + a_2 + a_3, \\
\vdots \\
S_n = a_1 + a_2 + \cdots + a_n \left[ = \sum_{i=1}^{n} a_i \right]
\]

If \( \{a_n\} \) is a given sequence, then we define on infinite series or simply a series \( \sum_{n=1}^{\infty} a_n \), by the sequence \( \{S_n\} \) of partial sums, where \( S_n = \sum_{k=1}^{n} a_k \).

Note that, \( \lim_{n \to \infty} (S_n) = \lim_{n \to \infty} (S_n) = \sum_{k=1}^{\infty} a_k \).

Hence, \( \sum_{n=1}^{\infty} a_n \) converges if \( \{S_n\} \) converges, or diverges if \( \{S_n\} \) diverges.

Definition: If the sequence \( \{S_n\} \) of partial sums is convergent and has the limit 'S', then the series \( \sum_{n=1}^{\infty} a_n \) is called convergent and the number 'S' is the value of the sum of the series.

If \( \{S_n\} \) is divergent then the series \( \sum_{n=1}^{\infty} a_n \) is said to diverge and has no sum or value.

Example 1: Check the convergence or divergence of the following series.

\[
\sum_{n=1}^{\infty} \frac{1}{n(n+1)}
\]

Solution: The \( n \)th partial sum is

\[
S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1}.
\]

Let \( \epsilon > 0 \) be an arbitrary number. Then \( \left| S_n - 1 \right| = \frac{1}{n+1} < \frac{1}{n} < \epsilon \) \( \Rightarrow \exists n > \frac{1}{\epsilon} \)

\[
N(\epsilon) = \left[ \frac{1}{\epsilon} \right] + 1
\]

Then \( \exists \) a natural number \( N(\epsilon) \) \( \Rightarrow \left| S_n - 1 \right| < \epsilon, \forall n > N(\epsilon) \)

\( \Rightarrow \lim_{n \to \infty} (S_n) = 1 \)

\( \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \) converges and has the value 1.
\[ \frac{1}{4n^{n-1}} \]

**Solution:**

\[ S_n = \sum_{k=1}^{n} \frac{1}{4k^{n-1}} \]

\[ = \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} \]

\[ = \frac{1}{2} \sum_{k=1}^{n} \left( \frac{1}{2k-1} - \frac{1}{2k+1} \right) \]

\[ = \frac{1}{2} \left[ 1 - \frac{1}{2n+1} \right] \]

\[ \lim (S_n) = \frac{1}{2} \]

Hence, the series converges and has the sum \( \frac{1}{2} \).

**\( \sum_{n=1}^{\infty} \frac{n^{n-1}}{n} \), \(|n| < 1\).**

**Solution:**

\[ S_n = \sum_{k=1}^{n} n^{k-1} \]

\[ = \frac{1 - n^n}{1 - n} \]

Now, \( \lim_{n \to \infty} (S_n) = \frac{1 - \lim_{n \to \infty} (n^n)}{1 - n} \)

\[ = \frac{1 - 0}{1 - n} = \frac{1}{1 - n} \quad [\text{since } \lim (n^n) = 0, \ |n| < 1] \]

Hence, the series \( \sum_{n=1}^{\infty} n^{n-1} \) converges and has the sum \( \frac{1}{1 - n} \), if \(|n| < 1\).

**\( \sum_{n=1}^{\infty} n^{n} \)**

**Solution:**

\[ S_n = \sum_{k=1}^{n} k^{n-1} \]

\[ = \frac{n(n+1)(2n+1)}{6} \]

\[ \therefore S_n \to \infty \quad \text{as } n \to \infty \]

\[ \Rightarrow S_n \text{ diverges to } \infty \]

\[ \Rightarrow \text{the series } \sum_{n=1}^{\infty} n^{n} \text{ diverges and has no value.} \]
Theorem: \( (n^{th} \text{ term test}) \)

If \( \sum_{n=1}^{\infty} a_n \) converges, then \( \lim_{n \to \infty} (a_n) = 0 \).

\[ S_{n+1} = a_1 + a_2 + \cdots + a_n + a_{n+1} \]
\[ S_n = a_1 + a_2 + \cdots + a_n \]

\[ S_{n+1} - S_n = a_{n+1} \]

So, \( a_n = S_n - S_{n-1} \).

If \( \sum_{n=1}^{\infty} a_n \) converges, then \( \sum_{n=1}^{\infty} S_n \) converges.

Let \( \lim_{n \to \infty} (S_n) = S \)

Then,

\[ \lim_{n \to \infty} (a_n) = \lim_{n \to \infty} (S_n - S_{n-1}) \]
\[ = \lim_{n \to \infty} (S_n) - \lim_{n \to \infty} (S_{n-1}) \]
\[ = S - S \]
\[ = 0 \]

Corollary: If \( \lim_{n \to \infty} (a_n) \neq 0 \), then \( \sum_{n=1}^{\infty} a_n \) diverges.

Example: Let a series be \( \sum \cos \left( \frac{\pi}{n} \right) \). We are going to check its convergence or divergence by calculating its limiting value.

\[ \lim_{n \to \infty} \left( \cos \left( \frac{\pi}{n} \right) \right) = \cos \theta = 1 \neq 0 \]

By using the theorem, we can say that \( \sum \cos \left( \frac{\pi}{n} \right) \) diverges.

Series of Non-negative Items: If \( a_n > 0 \), \( n \in \mathbb{N} \), then the series \( \sum_{n=1}^{\infty} a_n \) is a series of non-negative terms.

Theorem: A series of non-negative terms \( \sum_{n=1}^{\infty} a_n \) converges if the sequence \( S_n \) of partial sums is bounded above.

Proof: Let \( S_n = a_1 + a_2 + \cdots + a_n + a_{n+1} \)
\[ S_n - S_{n-1} = a_n \geq 0 \quad \forall n \]

\[ \Rightarrow \]\( S_n \) is increasing.

Hence, \( S_n \) converges if it is bounded above,
\[ \Rightarrow \sum_{n=1}^{\infty} a_n \] converges iff \( S_n \) is bounded above.
NOTE: A series \( \sum_{n=1}^{\infty} a_n \) of non-negative terms either converges or diverges to \((+\infty)\). If \( \sum_{n=1}^{\infty} a_n \) is not bounded above, then \( \sum_{n=1}^{\infty} a_n \) diverges to \(\infty\). 

**Example: Establish the convergence or divergence of the following series:**

(a) \( \sum_{n=1}^{\infty} \frac{1}{n!} \)

**Sol.:** Hence, \( S_n = \sum_{k=1}^{n} \frac{1}{k!} \)

\[ \therefore S_n - S_{n-1} = \frac{1}{n!} > 0 \quad \forall \ n \geq 2 \]

\[ \Rightarrow \{S_n\} \text{ is increasing.} \]

Now, \( S_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} < 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} \)

\[ \therefore S_n < \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 2 \left(1 - \frac{1}{2^n}\right) < 2 \]

\( \therefore 1 = S_1 \leq S_n \leq 2 \quad \forall \ n \in \mathbb{N} \)

\[ \Rightarrow \{S_n\} \text{ is bounded above.} \]

Hence \( \{S_n\} \) converges.

**Cor. 1:** \( \sum_{n=1}^{\infty} \frac{1}{n!} \) converges.

(b) **For the Power Series**, i.e., \( \sum_{n=1}^{\infty} \frac{1}{n^p} \), \( \text{PER} \).

[**Power series:** \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converges iff \( p > 1 \) and diverges if \( p \leq 1 \)]

**Case I:** \( p > 1 \)

Note that, \( S_n = \sum_{k=1}^{n} \frac{1}{k^p} \), \( \forall \ n \in \mathbb{N} \).

\[ \therefore S_n - S_{n-1} = \frac{1}{n^p} > 0 \quad \forall \ n \in \mathbb{N} \]

\[ \Rightarrow \{S_n\} \text{ is increasing.} \]

Now, \( S_{2n-1} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{(2n-1)^p} \)

\[ = 1 + \left( \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \right) + \left( \frac{1}{5^p} + \frac{1}{6^p} + \cdots \right) + \left( \frac{1}{7^p} + \frac{1}{8^p} + \cdots \right) + \cdots \]

\[ < 1 + 2 \cdot \frac{1}{2^p} + 2 \cdot \frac{1}{4^p} + 2 \cdot \frac{1}{8^p} + \cdots + 2 \cdot \frac{1}{(2n-1)^p} \]

\[ = 1 + \frac{1}{2^p} + \frac{1}{4^p} + \cdots + \frac{1}{(2n-1)^p} \]

\[ = 1 + \frac{1}{2^p-1} + \frac{1}{4^p-1} + \cdots + \frac{1}{(2n-1)^p-1} \]
\[ \sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \left(\frac{1}{3^p} + \frac{1}{4^p}\right) + \left(\frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \frac{1}{8^p}\right) + \cdots \]

As \( m \leq 2^{n-1} \) and \( \sum_{n=1}^{2^n} \) is increasing,
\[
1 = S_1 \leq S_n \leq S_{2^n-1} < \frac{2^p-1}{2^p-1}
\]

\( \Rightarrow \) \( S_n \) is bounded if \( p > 1 \).

Hence, \( \frac{S_n}{n^p} \) converges if \( p > 1 \).

**Case II** \( p \leq 1 \)

Note that \( n^p \leq n \), \( n \in \mathbb{N} \).

\[
\frac{1}{n^p} \geq \frac{1}{n}, \quad n \in \mathbb{N}
\]

\[ \sum_{n=1}^{2^n} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{(2^n)^p} \]

\[
= 1 + \frac{1}{2^p} + \left(\frac{1}{3^p} + \frac{1}{4^p}\right) + \left(\frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \frac{1}{8^p}\right) + \cdots \]

\[ \geq 1 + \frac{1}{2} + \left(\frac{1}{3^p} + \frac{1}{4^p}\right) + \left(\frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \frac{1}{8^p}\right) + \cdots \]

\[ > 1 + \frac{1}{2} + 2^{1-p} + 4^{1-p} + \cdots + 2^{m-1} \cdot \frac{1}{2n} \]

\[ \sum_{n=1}^{2^n} > 1 + \frac{m}{2} \]

For any \( m \in \mathbb{N} \), \( \exists \) a natural no. \( m \in 2^m \leq n \leq 2^{m+1} \)

Hence, \( \sum_{n=1}^{2^n} > 1 + \frac{m}{2} \)

[since \( (S_n) \) is increasing]

\( \Rightarrow \) \( S_n \) is unbounded.

As \( S_n \) is increasing and unbounded \( \frac{S_n}{n^p} \) diverges to \( +\infty \).

Hence, \( \frac{1}{n^p} \) diverges if \( p \leq 1 \).
We consider the following tests for convergence or divergence of series of non-negative terms:

1. **Comparison Test:** If \( \sum a_n \) and \( \sum b_n \) are two series of non-negative terms and \( a_n \leq k \cdot b_n \) for all \( n > m \), then:
   - If \( \sum b_n \) converges, then \( \sum a_n \) converges.
   - If \( \sum b_n \) diverges, then \( \sum a_n \) diverges.

2. Example:
   - Test for convergence or divergence of the following series:
     - \( \sum \frac{1}{n!} \)
     - \( \sum \frac{1}{n^{\sqrt{n}+1}} \)
     - \( \sum \frac{1}{n^{n+1}} \)
     - \( \sum \frac{1}{\sqrt{n+1}} \)
     - \( \sum \frac{1}{\sqrt{n+1+\sqrt{n}}} \)
     - \( \sum \frac{1}{\sqrt{n+1+\sqrt{n}}} \)

   **Proof:**
   - Note that \( 0 < \frac{1}{n!} \leq \frac{1}{2^{n-1}} \) for \( n \in \mathbb{N} \).
   - Note that \( \sum \frac{1}{2^{n-1}} \) is a convergent geometric series.
   - Since \( \sum \frac{1}{2^{n-1}} \) converges by comparison test, \( \sum \frac{1}{n!} \) converges.

   - Note that \( \frac{1}{n^{\sqrt{n}+1}} < \frac{1}{n^{1+\sqrt{n}}} \)
   - Since, \( \sum \frac{1}{n^{1+\sqrt{n}}} \) is divergent.
   - Power series as \( p = \frac{1}{2} < 1 \), by comparison, \( \sum \frac{1}{\sqrt{n+1+\sqrt{n}}} \) diverges.

   - Note that \( \frac{1}{\sqrt{2n}} \leq \frac{1}{\sqrt{n+1+\sqrt{n}}} \) for all \( n > 1000 \).
   - Note that \( 0 \leq \frac{1}{n^{\sqrt{n}}} \leq \frac{1}{n!} \) for all \( n \in \mathbb{N} \).
   - Note that \( \sum \frac{1}{n^{\sqrt{n}}} \) converges.

   By comparison test, \( \sum \frac{1}{\sqrt{n+1+\sqrt{n}}} \) converges.

   - Note that \( 0 \leq \frac{1}{n^{\sqrt{n}+1}} < \frac{1}{n} \) for \( n \in \mathbb{N} \).
   - As \( \sum \frac{1}{n} \) converges by comparison test, \( \sum \frac{1}{n^{\sqrt{n}+1}} \) converges.
(II) Limit Comparison Test: \( Z \) if \( a_n \) and \( b_n \) are known series of positive terms and \( \lim \left( \frac{a_n}{b_n} \right) \neq 0 \) then both \( S \) \( a_n \) and \( S \) \( b_n \) converge or diverge simultaneously.

Example: Test the following series for convergence or divergence

(i) \( \sum_{n=1}^{\infty} \frac{1}{n^2 + n} \), (ii) \( \sum_{n=1}^{\infty} \frac{n \cdot \sin \frac{1}{n}}{n^2 + n} \), (iii) \( \sum_{n=1}^{\infty} a^{n-1} \), (iv) \( \sum_{n=1}^{\infty} \frac{(n+1)^{3}}{n^2} - n \)

Solutions:

(i) Let \( a_n = \frac{1}{n^2 + n} \), \( b_n = \frac{1}{n^2 + n} \), \( n \in N \).

\( \lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) = \lim_{n \to \infty} \left( \frac{n^2 + n}{n^2 + n} \right) = 1 \neq 0 \)

Since \( \sum b_n = \sum \frac{1}{n^2} \) converges by limit comparison test, \( \sum a_n = \sum \frac{1}{n^2 + n} \) converges.

(ii) Let \( a_n = \frac{n \cdot \sin \frac{1}{n}}{n^2 + n} \), \( b_n = \frac{1}{n^2} \), \( n \in N \), \( n \to \infty \).

\( \lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) = \lim_{n \to \infty} \left( \frac{n \cdot \sin \frac{1}{n}}{n^2 + n} \right) = 1 \neq 0 \)

Since \( \sum b_n = \sum \frac{1}{n^2} \) diverges by limit comparison test, \( \sum a_n = \sum \frac{n \cdot \sin \frac{1}{n}}{n^2 + n} \) diverges.

(iii) Note that \( \lim_{n \to \infty} \sin \frac{1}{n} = \lim_{x \to 0} \frac{\sin x}{x} = 1 \neq 0 \)

Since \( \sum \frac{1}{n} \) diverges, \( \sum \frac{n \cdot \sin \frac{1}{n}}{n^2 + n} \) diverges by limit comparison test.

(iv) Let \( a_n = \frac{1}{n} \sin \frac{1}{n} \), \( b_n = \frac{1}{n^2} \), \( n \in N \), \( n \to \infty \).

\( \lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) = \lim_{n \to \infty} \left( \frac{1}{n} \sin \frac{1}{n} \right) = 1 \neq 0 \)

Note that \( \lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) = \lim_{n \to \infty} \left( \frac{1}{n} \sin \frac{1}{n} \right) = 1 \neq 0 \)

Since \( \sum \frac{1}{n} \) diverges, \( \sum \frac{1}{n} \) diverges by limit comparison test.
\[
\frac{a_n}{a_{n+1}} = \frac{(n^3 + 1) - n^3}{(n^3 + 1)^{1/3} + n (n^3 + 1)^{1/3} + n} \cdot \frac{1}{n^2 + n + 1}
\]

Take, \( b_n = \frac{1}{n^2 + n + 1} \)

\[
\lim \left( \frac{a_n}{b_n} \right) = \frac{1}{3} \neq 0
\]

Remark: If \( \lim_{n \to \infty} \frac{a_n}{b_n} = 0 \), then one writes \( a_n \sim b_n \) for large \( n \).

Then \( 2a_n \) and \( 2\frac{b_n}{n} \) have similar nature of convergence or divergence.

Note that \( \int \frac{m^p}{(n+1)^{1/3}} \sim \frac{1}{n^{p-1} (1+1)^{1/3}} \), \( \int \frac{m^p}{(n+1)^{1/3}} \sim \frac{1}{n^{p-1}} \).

\( \sin \frac{n}{n} \sim \frac{1}{n} \); \( \sim \frac{1}{n} \).

\( \frac{m^p}{(n+1)^{1/3}} \sim \frac{1}{n^{p-1}} \).

\( \frac{m^p}{(n+1)^{1/3}} \sim \frac{1}{n^{p-1}} \).

(IV) D'alembert's Ratio Test: \( \int \frac{an+1}{an} \). If \( 2n \) is a series of positive terms and \( \lim \left( \frac{an+1}{an} \right) = 1 \), then \( 2n \) converges if \( l < 1 \) and diverges if \( l > 1 \).

Hint: \( \int l < 1, \) then \( \exists \) \( \epsilon > 0 \) \( \Rightarrow l + \epsilon < 1 \).

By definition of limit

\[
\frac{a_{n+1} - l}{an} < \epsilon \forall n > N(\epsilon)
\]

\[
\Rightarrow \frac{a_{n+1}}{an} < l + \epsilon = r (\epsilon) \forall n > N(\epsilon)
\]

Now, \( \frac{ak}{a_{n+1}} = \frac{ak - 1}{a_{k-1}} \cdots \frac{a_{k+1}}{a_1} \)

\[
\frac{ak}{a_{n+1}} < \left( \frac{ak+1}{a_{k+1}} \right) \cdots \frac{ak+1}{a_1}
\]

Note: \( \int l < 1 \) converges, \( 2n \) converges.

Note: When \( l = 1 \) the test fails to give a conclusion.
(iv) **Cauchy’s Root Test:** If \( 2a_n \) is a series of non-negative terms and \( \lim (a_n^{1/n}) = l \), then \( 2a_n \) converges if \( l < 1 \) and diverges if \( l > 1 \).

**Hints:** Let \( l < 1 \), then \( \exists \ a_n \ e > 0, \ s.t. \ l + e < 1 \).

By definition, \( a_n^{1/n} < l + e = m \ < 1 \) \( \forall \ n > N(e) \)

\[ \Rightarrow a_n = n^m \ \forall \ n > N(e) \]

Since \( 2a_n \), \( m < 1 \) converges,
\( 2a_n \) converges by comparison test.

**Example:** Test the following series for convergence or divergence.

(a) \( \sum \frac{(m!)^n}{(2n)!} \)

(b) \( \sum (n^{1/m} - 1)^n \)

**Solution:**

(a) Let \( a_n = \frac{(m!)^n}{(2n)!} \), \( m \in \mathbb{N} \).

Now, \( \lim_{n \to \infty} \left( \frac{a_{n+1}}{a_n} \right) = \lim_{n \to \infty} \frac{(m!)^{n+1}}{(2n+2)!} \times \frac{(2n)!}{(m!)^n} \)

\[ = \lim_{n \to \infty} \frac{(m+1)!}{(2n+2)(2n+1)!} \]

\[ = \lim_{n \to \infty} \frac{m+1}{2(2n+1)} \]

\[ = \frac{1}{4} \]

By ratio test, \( 2a_n = \sum \frac{(m!)^n}{(2n)!} \) converges.

Here, \( l = \lim \left( \frac{a_{n+1}}{a_n} \right) = \frac{1}{4} < 1 \).

(b) Let \( a_n = (n^{1/m} - 1)^n \), \( m \in \mathbb{N} \).

Here, \( l = \lim \left( a_{n/m} \right) = \lim \left( n^{1/m} - 1 \right) ^{n/m} \)

\[ = \left( \frac{n^{1/m}}{m} - 1 \right) \]

\[ = \frac{1 - 1}{m} = 0 < 1 \]

By root test, \( 2a_n = \sum (n^{1/m} - 1)^n \) converges.

**Remark:**

*4th term test:* If \( 2a_n \) converges, then \( \lim (a_n) = 0 \).

But the converse is not true (i.e., the converse theorem is not necessarily true).

Note that \( 2a_n = \sum 1/n \) diverges and \( 2b_n = \sum 1/n \) converges.

But \( \lim (a_n) = 0 = \lim (b_n) \).
\(\text{In the limit test if } l = 1 \text{ then we cannot draw any conclusion. Let } 2a_n = 2^{-\frac{1}{n}} \text{, } 2b_n = 2^{-\frac{1}{n^2}}\)

\[ l = \lim \left( \frac{a_n}{n!} \right) = 1 \text{ and } l = \lim \left( \frac{b_n}{n!} \right) = 1 \]

In both cases, \(l = 1\) but \(2a_n = 2^{-\frac{1}{n}}\) diverges whereas \(2b_n = 2^{-\frac{1}{n^2}}\) converges.

Again, \(l = \lim \left( \frac{a_n}{n!} \right) + \lim \left( \frac{b_n}{n!} \right) = \lim \left( \frac{a_n}{n!} \right) + \lim \left( \frac{b_n}{n!} \right) = 1\)

\[
\sum_{n=1}^{\infty} \left( \frac{(-1)^{n-1}}{n^p} \right) \text{ converges if } p > 0.
\]

**Series of arbitrary terms, not necessarily non-negative:**

**Alternative Series:** If \(\sum_{n=1}^{\infty} (-1)^{n-1} a_n\) with \(a_n > 0\), is an alternating series such that \(\{a_n\}\) is decreasing and \(\lim(a_n) = 0\), then the series converges. This is known as Leibnitz Test.

**Example:** Show that the series \(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}\) converges if \(p > 0\).

**Sol.:** Here, \(a_n = \frac{1}{n^p}, n \in \mathbb{N}\).

Note that, \(a_{n+1} - a_n = \frac{1}{(n+1)^p} - \frac{1}{n^p} < 0 \forall n \in \mathbb{N}\).

\(a_n\) is decreasing.

Note that, \(\lim(a_n) = \lim \left( \frac{1}{n^p} \right) = 0 \text{ if } p > 0\).

By Leibnitz test, the alternating series \(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}\) converges if \(p > 0\).

**Absolute Convergence:** If the series \(\sum_{n=1}^{\infty} |a_n|\) converges, then we say that the series \(\sum_{n=1}^{\infty} a_n\) converges absolutely.

If a series converges absolutely (i.e., \(\sum_{n=1}^{\infty} |a_n|\) is finite), then it is (ordinary) convergent (i.e., \(\sum_{n=1}^{\infty} a_n\) converges).

Since, \(a_n < |a_n|\). But absolute divergence, not necessarily implies the ordinary divergence.

Absolute Convergence \(\Rightarrow\) Ordinary Convergence.

But absolute divergence \(\not\Rightarrow\) Ordinary divergence.
Test for absolute convergence:

- **Ratio Test**: If $\{a_n\}$ is a series of arbitrary terms, with $a_n \neq 0$, and if $l = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$, then:
  1. $\{a_n\}$ converges absolutely if $l < 1$.
  2. $\{a_n\}$ diverges if $l > 1$.

- **Root Test**: If $\{a_n\}$ is a series of arbitrary terms, with $a_n \neq 0$, and if $l = \lim_{n \to \infty} \sqrt[n]{|a_n|}$, then:
  1. $\{a_n\}$ converges absolutely if $l < 1$.
  2. $\{a_n\}$ diverges if $l > 1$.

**Example**: Determine the convergence or divergence of the series:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n+1}{n^3-4}, \quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n!}$$

**Soln.**

1. Let $a_n = (-1)^{n-1} \frac{2n+1}{n^3-4}$, $b_n = \frac{1}{n^3-4}$, $m \in \mathbb{N}$.

Now, $\lim_{n \to \infty} \left| \frac{a_n}{b_n} \right| = \lim_{n \to \infty} \left( \frac{2n+1}{n^3-4} \times n^3 \right) = 2 \neq 0$

Hence, $\sum |a_n|$ & $\sum b_n$ both converge on divergence simultaneously.

As $\sum b_n = \sum \frac{1}{n^3-4}$ converges, $\sum |a_n|$ also converges $\iff \sum a_n$ converges absolutely $\iff \sum (-1)^{n-1} \frac{2n+1}{n^3-4}$ converges.

2. Here $a_n = (-1)^n \frac{m^2}{n!}$, $m \in \mathbb{N}$.

Note that:

$$l = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1 + \frac{1}{n})^n}{n^2} \right| = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e > 1$$

By ratio test (for absolutely convergence on divergence). The series $\sum a_n$ diverges.
Example: Show that \( \sum \frac{\sin nx}{n^p} \) converge absolutely if \( p > 1 \).

\[ 0 \leq \left| \frac{\sin nx}{n^p} \right| \leq \frac{1}{n^p} \cdot n \in \mathbb{N}. \]

Note, \( \frac{1}{n^p} \) converges if \( p > 1 \).

By comparison test, \( \sum \left| \frac{\sin nx}{n^p} \right| \) converges if \( p > 1 \),

\[ \Rightarrow \sum \frac{\sin nx}{n^p} \text{ converges absolutely if } p > 1. \]

Example: Establish the convergence or divergence of the series \( \sum_{n=1}^{\infty} \frac{x^n}{n^v} \) for different values of \( x \).

\[ \sum_{n=1}^{\infty} \frac{x^n}{n^v} \]

**Solution:** Note that \( l = \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)^v} \cdot \frac{n^v}{x^n} \right| \)

\[ = \lim_{n \to \infty} \left| x \cdot \frac{n^v}{(n+1)^v} \right| \]

\[ = |x| \lim_{n \to \infty} \frac{n^v}{(n+1)^v} \]

By ratio test, if \( |x| < 1 \), the series \( \sum \frac{x^n}{n^v} \) converges (absolutely);

if \( |x| > 1 \), the series \( \sum \frac{x^n}{n^v} \) diverges.

The test fails to give any conclusion if \( |x| = 1 \); when \( |x| = 1 \), then

\[ \sum \frac{x^n}{n^v} = \sum \frac{1}{n^v} \]

i.e., \( \sum \frac{1}{n^v} \) converges (absolutely).

Hence, \( \sum \frac{x^n}{n^v} \) converges (absolutely) if \( |x| \leq 1 \) and diverges if \( |x| > 1 \).

**Conditional Convergence:** A series \( \sum a_n \) is said to converge conditionally if \( \sum |a_n| \) converges but \( \sum |a_n| \) diverges.

\( \sum a_n \) is a series of arbitrary terms.

\[ \sum |a_n| \]

\[ \sum |a_n| \text{ converges } \]

\[ \sum |a_n| \text{ diverges } \]
Example: Test the following series for absolute and conditional convergence:

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}, \text{ } p > 0 \]

Solution:

1. By the Leibnitz test, \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} \) converges if \( p > 1 \) and diverges if \( p \leq 1 \).

Note that, \( \sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} x^{n-1} \) converges if \( p > 1 \) and diverges if \( p \leq 1 \).

2. The series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} \) converges absolutely if \( p > 1 \).

But, for \( 0 < p \leq 1 \), \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} \) diverges but

\[ \sum_{n=1}^{\infty} \frac{1}{n^p} \] converges, i.e., \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} \) converges conditionally.

Note that, \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left( \frac{n+1}{n} \right)^p = |x| \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^p = |x| \)

If \( |x| < 1 \), then \( \sum_{n=1}^{\infty} \frac{a_n}{n^p} \) converges absolutely.

If \( |x| > 1 \), then \( \sum_{n=1}^{\infty} \frac{a_n}{n^p} \) diverges by ratio test.

If \( |x| = 1 \), then the test fails to give a conclusion.

If \( x = 1 \), then \( \sum_{n=1}^{\infty} \frac{a_n}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n^p} \) converges (absolutely) if \( p > 1 \), and diverges if \( p \leq 1 \).

If \( x = -1 \), then \( \sum_{n=1}^{\infty} \frac{a_n}{n^p} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^p} \) converges absolutely if \( p > 1 \) and converges conditionally if \( 0 < p \leq 1 \).
Rearrangement of Series: It is a familiar fact that the sum of a finite number of terms is same, no matter how the terms are arranged. But it's not universally true for an infinite series, as and e.g., consider the series:

\[ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \]

= \$ (say)

If we rearranged the terms of the series \( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \) so that each positive term is followed by two negative terms, then we have the series:

\[ 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots \]

\[ = (1 - \frac{1}{2}) - \frac{1}{4} - (\frac{1}{3} - \frac{1}{6}) - \frac{1}{5} + (\frac{1}{10}) - \frac{1}{12} + \cdots \]

\[ = \frac{1}{2} - \frac{1}{6} + \frac{1}{3} - \frac{1}{12} + \cdots \]

\[ = \frac{5}{2} \]

Hence, a rearrangement of the series \( \sum \frac{(-1)^{n-1}}{n} \) alters the sum of the series. So, we naturally ask "Under what conditions may we arrange the terms of a series without altering its value?"

\- Theorem: If a series \( \sum a_n \) converges absolutely, its sum is unaltered by rearrangement.

\- Example: Consider the series \( \sum \frac{(-1)^{n-1}}{n} \), what is the sum of the following series:

\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \]

\- Solution: Since \( \sum \frac{(-1)^{n-1}}{n} \) converges absolutely, let \( \sum \frac{(-1)^{n-1}}{n} = \$ \).

Hence, any rearrangement of the series has the same value(s).

\- Riemann's Theorem: If a series converges conditionally, its sum can be made to have an arbitrary value by suitable rearrangement. It can also be made divergent or oscillatory.
A. Sequences of Functions: It may happen that the terms of a sequence depend on some variable \( x \), in addition to the index \( n \in \mathbb{N} \) and this is indicated by \( f_n(x) \). Suppose that for each \( n \in \mathbb{N} \), there is a function \( f_n: A \to \mathbb{R} \), then we say that \( \{f_n(x)\} \) is a sequence of functions on \( A \). For example, if \( f_n(x) = x^n \), \( x \in \mathbb{R} \), \( n \in \mathbb{N} \). Then \( \{f_n(x)\} \) is a sequence of functions on \( \mathbb{R} \). For each \( x \in A \), such a sequence becomes a sequence of real numbers, and for some values of \( x \in A \), the sequence may converge, and for other values of \( m \), the sequence may diverge. We assume that \( \{f_n(x)\} \) is convergent for all \( x \in A_0 \) and for each \( x \in A_0 \), we have

\[
\lim_{n \to \infty} f_n(x) \text{ as a finite number.}
\]

Then the \( \lim_{n \to \infty} f_n(x) \) defines a function of \( x \), say \( f(x) \) on \( A_0 \).

The function \( f(x) \) defined on \( A_0 \) by the equation \( f(x) = \lim_{n \to \infty} f_n(x) \) is called the limit function of the sequence \( \{f_n(x)\} \) of functions.

Pointwise Convergence:

- Definition:

\( (i) \) The sequence \( \{f_n(x)\} \) of functions converges pointwise to the function \( f(x) \) on \( A_0 \) if for each \( x \in A_0 \), the sequence \( \{f_n(x)\} \) of real numbers converges to the real number \( f(x) \).

\( (ii) \) The sequence \( \{f_n(x)\} \) of functions converges pointwise to the function \( f(x) \) on \( A_0 \) if for each \( x \in A_0 \), for any \( \epsilon > 0 \), there is a natural no. \( N \in \mathbb{N} \) such that \( |f_n(x) - f(x)| < \epsilon \) for \( n > N(e, x) \).

Note:

Uniform convergence \( \Rightarrow \) pointwise convergence
but not vice versa.

However, Non-pointwise convergence \( \nRightarrow \) non-uniform convergence.
Example:

(a) Show that the sequence \( f_n(x) \) of function converges pointwise to the function \( f(x) = \begin{cases} 0 & 0 < x < 1 \\ 1 & x = 1 \end{cases} \) on the interval \([0, 1]\).

\[ \lim_{n \to \infty} f_n(x) = f(x) \]

For any \( \varepsilon > 0 \), there exists a natural number \( N(\varepsilon) \) such that for all \( n > N(\varepsilon) \),

\[ |f_n(x) - f(x)| < \varepsilon \]

Hence, for each \( x \in [0, 1] \),

\[ f_n(x) \to f(x) \quad \text{as} \quad n \to \infty \]

(b) Show that the sequence \( \frac{\sin (nx + \pi)}{n} \) converges pointwise to \( f(x) = 0 \) on \( \mathbb{R} \).

For each \( x \in \mathbb{R} \),

\[ \left| \frac{\sin (nx + \pi)}{n} - 0 \right| < \varepsilon \]

Take \( N(\varepsilon) = \left[ \frac{1}{\varepsilon} \right] + 1 \)

Hence, for each \( x \in \mathbb{R} \), for any \( \varepsilon > 0 \),

\[ \left| \frac{\sin (nx + \pi)}{n} - 0 \right| < \varepsilon \]

Hence, \( \frac{\sin (nx + \pi)}{n} \) converges pointwise to \( f(x) = 0 \) on \( \mathbb{R} \).
**Remark:**

1. In definition, \( |f_n(x) - f(x)| < \varepsilon \) for \( n > N(\varepsilon, x) \), the natural number \( N(\varepsilon, x) \) is a measure of rate of natural number \( N(\varepsilon, x) \) is a measure of rate of convergence of the sequence at \( x \). The value \( N(\varepsilon, x) \) in general depends on both \( \varepsilon \) and \( x \). In Example 1(a), \( N(\varepsilon, x) \) depends on \( \varepsilon \) and \( x \) both. For this the convergence of the sequence is significantly faster at some points (i.e., it is at other points). However, in Example 1(b), the value \( N(\varepsilon, x) \) does not depend on \( x \); hence the rate of convergence is same for all \( x \), i.e., the rate of convergence uniform on \( x \).

2. The question arises: Is there a natural number \( N(\varepsilon, x) \) for a given \( \varepsilon > 0 \), is there a natural number \( N(\varepsilon) \) and \( \forall x \in A_0 \)?

And: If \( N(\varepsilon, x) \leq N(\varepsilon) \) \( \forall x \in A_0 \), then the natural number \( N(\varepsilon) \) may be chosen instead of \( N(\varepsilon, x) \) and \( m > N(\varepsilon) \).

This question is the same question as following:

"For a given \( \varepsilon > 0 \), is the function \( N(\varepsilon, x) \) bounded on \( A_0 \)?"

**Uniform Convergence:**

Definition: A sequence \( \{f_n(x)\} \) of functions converges uniformly to the function \( f(x) \) on \( A_0 \) if for every \( \varepsilon > 0 \), there is a natural number \( N(\varepsilon) \), which does not depend on \( x \) \( \in A_0 \), \( x \in A_0 \), whenever \( n > N(\varepsilon) \).

**Example:**

Show that the sequence \( \{f_n(x)\} = \{\frac{1}{n}x + n\} \) of functions is uniformly convergent on \([0, k]\), whatever \( k \) may be.

**Solution:** For each \( x \in [0, k] \),

\[
\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{x}{n} + n = 0.
\]

The limit function is \( f(x) = 0 \), \( x \in [0, k] \).

Let \( \varepsilon > 0 \) be an arbitrary quantity.

Then, for each \( x \in [0, k] \)

\[
|f_n(x) - f(x)| < \varepsilon
\]

\[
\Rightarrow \frac{1}{x+n} < \varepsilon \Rightarrow n > \frac{1}{\varepsilon} - x.
\]
Choose a natural number \( N(\varepsilon, k) = \left\lceil \frac{\varepsilon}{k} \right\rceil + 1 \).

Note that \( N(\varepsilon, k) \) decreases as \( k \) increases.

Hence, \( N(\varepsilon, k) \leq \left\lceil \frac{\varepsilon}{k} \right\rceil + 1 \forall \varepsilon \in [0, k] \).

Hence, it is possible to find a natural no. \( N(\varepsilon) = \left\lceil \frac{\varepsilon}{\varepsilon} \right\rceil + 1 \),

\[ |f_n(x) - f(x)| < \varepsilon \forall x \in [0, k] \]

Hence, the sequence \( \{f_n(x)\} \) of functions converge uniformly on \([0, k]\).

**Example 3**: Show that the sequence \( \{f_n(x)\} \) of functions is uniformly convergent on \([0, k]\), \( k < 1 \), but not uniformly on \([0, 1]\).

Said \Rightarrow \quad \quad \text{For each} \ x \in [0, 1], \quad \quad \lim_{n \to \infty} f_n(x) = f(x) = 0 \text{, for } 0 \leq x < 1 \text{, and for } x = 1,

\[ \lim_{n \to \infty} f_n(x) = 1. \]

Then, \( f(x) \) is the limit function.

\[ |f_n(x) - f(x)| = |f_n(x) - 0| < \varepsilon \quad \text{for} \quad 0 \leq x < 1. \]

Choose a natural number \( N(\varepsilon, k) = \left\lceil \frac{\varepsilon}{\ln x} \right\rceil + 1 \).

Note that, \( N(\varepsilon, k) \) increases as \( k \) increases, \( N(\varepsilon, k) \to +\infty \) as \( k \to 1 \) -

\[ \text{i.e., } N(\varepsilon, k) \text{ is not bounded in } [0, 1]. \]

**It is not possible to find a natural number \( N(\varepsilon) \in [0, 1] \)**.

Hence, the convergence is non-uniform on \([0, 1]\), i.e.,

\[ \text{pointwise converges} \]

Now, consider the interval \([0, k], k < 1\)

In the interval \([0, k]\), the greatest of \( N(\varepsilon, k) \) is

\[ N(\varepsilon, k) = \left\lceil \frac{\varepsilon}{\ln x} \right\rceil + 1 = N^*(\varepsilon) \]

\[ \therefore N(\varepsilon, x) \text{ increases with } \varepsilon \in [0, 1] \]

Hence, it is possible to find a natural number \( N^*(\varepsilon) \in [0, k] \), \( k < 1 \).

Hence, the convergence is uniform on \([0, k], k < 1\).
Remark: We see that $f_n(x)$ is non-uniformly convergent on the interval $[0, 1]$ and the non-uniform convergence is due to the inflection at point $x = 1$. Hence $x = 1$ cannot be an included point in the given sequence of functions such a point is known as point of non-uniform convergence. Now, if we eliminate the point of non-uniform convergence from the interval, the sequence of functions is convergent on the new interval. Therefore, $f_n(x)$ is uniformly convergent on the interval $[0, 1], k < 1$.

Example 3: Show that for the following sequence of functions $f_n(x)$ converges pointwise but not uniformly on $[0, 1]$. Also, determine an interval of uniform convergence of the following sequence $f_n(x) = \frac{x^n}{1 + x^n}$.

Solution:
For $x \in [0, 1]$, $\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{x^n}{1 + x^n} = 0$, $0 \leq x < 1$.

For $x = 1$, $\lim_{n \to \infty} f_n(x) = \frac{1}{2}$.
Therefore, the limit function is $f(x) = \begin{cases} 1, & x = 1 \\ \frac{1}{2}, & 0 \leq x < 1 \end{cases}$.

Let $\varepsilon > 0$ be an arbitrary quantity. Then for each $x \in (0, 1)$, $
\begin{align*}
|f_n(x) - f(x)| &< \varepsilon \\
\Rightarrow \frac{1}{1 + x^n} &< \varepsilon \\
\Rightarrow x^n &< \frac{1}{\varepsilon - 1} \\
\Rightarrow n &> \ln \left( \frac{1}{\varepsilon - 1} \right)
\end{align*}$

For $x = 0, 1$, $|f_n(x) - f(x)| = 0 < \varepsilon$ for all $n > 1$.

Choose a natural $n_0$, $N(\varepsilon, x) = \left[ \frac{\ln \left( \frac{1}{\varepsilon - 1} \right)}{\ln n} \right] + 1; N(\varepsilon, x) \to +\infty$ as $x \to 1 -$

Hence, it is not possible to find a natural $n_0, N(\varepsilon, x) \to +\infty$ as $x \to 1 -$

Hence, the convergence of $f_n(x)$ to $f(x)$ is pointwise but not uniformly on $[0, 1]$.
Here, the point $x = 1$ is the point of non-uniform convergence, i.e. non-uniform convergence is due to inclusion of the point $x = 1$. Hence, $x = 1$ cannot be included in any interval of uniform convergence of the given sequence of functions.

Therefore, the interval $[0, k]$, $k < 1$ is an interval of uniform convergence of $\frac{\sin(x)}{x}$ to $f(x)$.

**Example 5** Construct an example of a sequence of functions $\frac{\sin(x)}{x}$ such that it is not uniformly convergent on $[0, 1]$ but uniformly convergent on $[\frac{1}{2}, 1]$.

**Hints** Consider the sequence $f_n(x) = \sin(x) = \left\{ \begin{array}{cl} 1 & 0 < x < 1 \\ 0 & x = 0 \end{array} \right.$

Hence $x = 0$ is the point of non-uniform convergence.

Hence $f_n(x)$ converges non-uniformly on $[0, 1]$ but uniformly on $[\frac{1}{2}, 1]$.

**Example 6** Does $\frac{\sin(x)}{x}$ converges uniformly on $A$ ?

- (a) $f_n(x) = \frac{\sin(x)}{nx + 1}$, $A = [0, 1]$
- (b) $f_n(x) = \frac{\sin(x)}{nx}$, $A = [0, \infty)$

**Solution**

(a) For each $x \in [0, 1]$, 

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{\sin(x)}{nx + 1} = 0$$

The limit function is $f(x) = 0$, $x \in [0, 1]$.

Let $\varepsilon > 0$ be an arbitrary quantity. 

Then, for each $x \in (0, 1]$

$$|f_n(x) - f(x)| = \frac{\sin(x)}{nx + 1} < \frac{x}{nx} = \frac{1}{n} < \varepsilon$$

$$\Rightarrow n > \frac{1}{\varepsilon}$$

Take $N(\varepsilon) = \left[ \frac{1}{\varepsilon} \right] + 1$.

Hence, it is possible to find a natural no. $N(\varepsilon)$ such that if $n > N(\varepsilon)$, $|f_n(x) - f(x)| < \varepsilon$ and $x \in A = [0, 1]$. 

Therefore, $\frac{\sin(x)}{x}$ converges uniformly on $[0, 1]$. 

**Example 7** Does $\int_0^1 f_n(x) dx$ converges uniformly on $[0, 1]$?
(b) For each \( x \in [0, \infty) \),
\[
\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} x e^{-nx} = \lim_{n \to \infty} \frac{x}{e^{nx}} = 0.
\]
Limit function is \( f(x) = 0 \), \( x \in \mathbb{R} \).

Let \( \epsilon > 0 \) be an arbitrary no.

Then, for each \( x \in [0, \infty) \),
\[
|f_n(x) - f(x)| = x e^{-nx} \leq \frac{1}{nx} \epsilon.
\]

\[ g(x) = xe^{-nx} \]
\[
\Rightarrow g'(x) = e^{-nx} + xe^{-nx}(-nx) = e^{-nx} - nx e^{-nx} = (1 - nx)e^{-nx} = \left( \frac{1}{n} - x \right) xe^{-nx} = \begin{cases} > 0 & x < \frac{1}{n} \\ < 0 & x > \frac{1}{n} \end{cases}
\]

Note that \( g(x) \) first increases, then achieves its maximum at \( x = \frac{1}{n} \), finally decreases.

\[ \text{max} g(x) = \frac{1}{n} e^{-\frac{1}{n}} = \frac{1}{ne} \]

\( g(x) \) maximum when \( g'(x) = 0 \), \( \Rightarrow x = \frac{1}{n} \Rightarrow g(x) = \frac{1}{ne} \).

Take \( N(\epsilon) = \left[ \frac{1}{\epsilon} \right] + 1 \), which is independent of \( x \).

Hence, it is possible to find a natural number \( N(\epsilon) \) \( \exists \) if \( n > N(\epsilon) \), \( |f_n(x) - f(x)| < \epsilon \) \( \forall x \in [0, \infty) \).

Hence, the convergence of \( x e^{-nx} \) is uniform on \( [0, \infty) \).

**Remark:** Let \( \{f_n(x)\} \) be a sequence of function \( \exists \lim f_n(x) = f(x) \)
on \( x \in \mathbb{R} \), and let \( M_n = \max_{x \in [a, b]} |f_n(x) - f(x)| \) on any upper boundary of \( f_n(x) - f(x) \).

Then \( f_n(x) \) converges uniformly to \( f(x) \) on \( A \) iff \( M_n \to 0 \) as \( n \to \infty \).

**Example 1:** S.T. \( \int_{0}^{\infty} \frac{1}{1 + n^2 x^2} \) is not uniformly convergent on any interval containing \( 0 \). Suggest an interval of uniform convergence.

**Example 2:** S.T. if \( f_n(x) = x + \frac{1}{n} \), \( n \in \mathbb{R} \), then \( f_n(x) \) converges uniformly on \( R \), but \( f_n(x) \) does not converge uniformly on \( R \).
Hints:
(a) Limit function is \( f(x) = 0 \), \( x \in A \), where \( A \) is an interval containing 0.
\[
|f_n(x) - f(x)| = \frac{m |x|}{1 + n^m x^n} < \frac{m |x|}{n^m x^n} = \frac{1}{n|x|} < \epsilon
\]
\[
\Rightarrow \quad n > \frac{1}{\epsilon |x|}
\]
\[
N(\epsilon, x) = \left[\frac{1}{\epsilon |x|}\right] + 1.
\]
Note that, \( N(\epsilon, x) = \left[\frac{1}{\epsilon |x|}\right] + 1 \to \infty \) as \( x \to 0 \),
i.e. \( N(\epsilon, x) \) is not bounded on any interval containing 0.
Here \( x = 0 \) is the point of non-uniform convergence. Hence, any integral of uniform convergence can't include the point \( x = 0 \).
Therefore,
\[
\begin{align*}
&\{k, \infty\}, \quad k > 0 \\
&\{(\infty, -k]\}, \quad k > 0 \\
&\{(\infty, -k] \cup [k, \infty)\}, \quad k > 0
\end{align*}
\]
are the intervals of uniform convergence.

(b) \( f_n(x) = x + \frac{1}{n} \)
\[
\Rightarrow \lim_{n \to \infty} f_n(x) = f(x)
\]
\[
\Rightarrow \quad \frac{1}{n} < \epsilon
\]
\[
\Rightarrow \quad n > \frac{1}{\epsilon}
\]
Choose a natural number \( N(\epsilon) = \left[\frac{1}{\epsilon}\right] + 1 \)
\[
\Rightarrow \quad N(\epsilon) \text{ is independent on } x.
\]
\( \Rightarrow \) It is uniformly convergent.
\[
\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \left(x + \frac{1}{n}\right) = x^\infty, \quad x \in \mathbb{R}
\]
\[
= f_1(x) = f_2(x)
\]
Let \( \epsilon > 0 \) be an arbitrary real no.
Then for each \( x \in \mathbb{R} \)
\[
\left|f_n(x) - f_1(x)\right| = \left|f_n(x) - f_2(x)\right| = \left|f_n(x) - f_2(x)\right| = \frac{2x}{n^n} + \frac{1}{n^n} \leq \frac{2n|x| + 1}{n^n}
\]
\[
\leq \frac{2n|x| + 1}{n^n}
\]
\[
= \frac{2|x| + 1}{n^n} < \epsilon
\]
\[
\Rightarrow \quad n > \frac{2|x| + 1}{\epsilon}
\]
Take \( N(\epsilon, x) = \left[\frac{2|x| + 1}{\epsilon}\right] + 1 \)
Note that, \( N(\epsilon, x) \to \infty \) as \( x \to \infty \), i.e. \( N(\epsilon, x) \) is not bounded on \( \mathbb{R} \).
Hence, it is not possible to find a natural no. \( N(\varepsilon) \) such that
\[
\forall x \in \mathbb{R}, \quad |f^n(x) - f(x)| < \varepsilon \quad \forall \varepsilon > 0.
\]
\[
|f^n(x)| \text{ does not converge uniformly on } \mathbb{R}.
\]

\subsection*{B.3 SERIES OF FUNCTIONS:}
Consider a sequence \( \{f_n(x)\} \) of functions defined on \( A \), then the series \( \sum_{n=1}^{\infty} f_n(x) \) is called a series of functions.

If the sequence \( \{S_n(x)\} \) of partial sums,
\[
S_n(x) = \sum_{k=1}^{n} f_k(x),
\]
converges pointwise to \( S(x) \), we have
\[
S(x) = \lim_{n \to \infty} S_n(x).
\]

Definition: The series \( \sum_{n=1}^{\infty} f_n(x) \) of functions is said to converge uniformly to the function \( S(x) \) on \( A \) iff the sequence \( \{S_n(x)\} \) of functions converges uniformly to \( S(x) \) on \( A \).

Example: Let \( \sum_{n=1}^{\infty} x^n \) converges uniformly to \( S(x) = \frac{1}{1-x} \) on \([0,1] \), \( 0 < a < 1 \), but the convergence is not uniform on \((-1,1)\).

Solution: The \( n \)th partial sum is
\[
S_n(x) = \frac{1-x^n}{1-x}, \quad x \in (-1,1) \quad \text{and} \quad \lim_{n \to \infty} S_n(x) = \frac{1}{1-x}
\]

Now, \( \lim_{n \to \infty} S_n(x) = \frac{1}{1-x} \), \( |x| < 1 \)

Let \( \varepsilon > 0 \) be arbitrary, then for each \( x \in (-1,1) \)
\[
\left| S_n(x) - S(x) \right| = \left| \frac{1-x^n}{1-x} - \frac{1}{1-x} \right| = \frac{|x^n|}{1-x} < \varepsilon
\]
\[
\Rightarrow m \cdot |x| < \ln \varepsilon \quad \Rightarrow m > \frac{\ln \varepsilon}{|x|} \quad (|x| < 1) \]
\[
\Rightarrow m > \frac{\ln \varepsilon}{1-|x|} \quad (|x| < 1-|x|) \]
\[
\Rightarrow m > \frac{\ln \varepsilon}{1-|x|} \quad (|x| < 1-|x|)
\]

Let \( N(\varepsilon, x) = \left[ \frac{\ln \varepsilon}{1-|x|} \right] + 1 \)

Note that \( N(\varepsilon, x) \to +\infty \) as \( x \to +1^- \) or \( x \to -1^+ \)
i.e., \( N(\varepsilon, x) \) is not bounded on \((-1,1)\).

Hence, the convergence of \( \{S_n(x)\} \) is not uniform on \((-1,1)\).

Consider the interval \((-a, a)\) \( 0 < a < 1 \).

N.T., \( N(\varepsilon, x) \) is an increasing function in \(|x|\), i.e.,
\[
N(\varepsilon, x) \leq N(\varepsilon, a) = N(\varepsilon, a) \quad \text{so},
\]
\[
\Rightarrow \text{The convergence of } \{S_n(x)\} \text{ is uniform on } [-a,a].
\]
Example 2: Show that \( \sum_{n=1}^{\infty} \frac{\alpha}{(n-1)x+1)(nx+1)} \) is uniformly convergent on \([k, \infty), \ k > 0\) but is non-uniform on \([0, \infty)\).

Hint: \[ S_n(x) = \sum_{k=1}^{n} \frac{\alpha}{(k-1)x+1)(kx+1)} \]
\[ = \frac{1}{\alpha} \sum_{k=1}^{n} \left( \frac{1}{k-1} - \frac{1}{k} \right) \]
\[ = 1 - \frac{1}{nx+1} = \frac{nx}{nx+1} \]

Weierstrass M-test: If \( |f_n(x)| \leq M_n, \ \forall x \in A_0\),
for \( n > m \), where \( M_n \) is a non-negative constant independent
of \( x \) and if \( \sum M_n \) converges, then \( \sum f_n(x) \) converges uniformly
as well as absolutely on \( A_0 \).

Hint: \[ \sum_{k=1}^{n} |f_k(x)| \leq \sum_{k=1}^{n} M_k \]
By comparison test \( \sum f_n(x) \) converges absolutely if \( \sum M_n \) converges.

Example 3. c.o.

(a) \( \sum_{n=1}^{\infty} \frac{1}{n^p} \sin nx, \ p > 1 \), is uniformly convergent on \( \mathbb{R} \).
(b) Does the series \( \sum_{n=1}^{\infty} (x e^{-x})^n \) converge uniformly on \([0, \infty)\) ?

Solution:

(a) \( |f_n(x)| = \left| \frac{1}{n^p} \sin nx \right| \leq \frac{1}{n^p} = M_n \)

Note that, \( \sum_{n=1}^{\infty} \frac{1}{n^p} \), \( p > 1 \) converges.

By M. Test the series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \sin nx \) converges uniformly on \( \mathbb{R} \).

(b) Let \( f_n(x) = x e^{-x} \)
Now, \( g_n(x) = (1-x) e^{-x} \)
\[ = \begin{cases} 
> 0 & \text{if } x < 1 \\
< 0 & \text{if } x > 1 
\end{cases} \]
Hence \( g(x) \) first increases, then decreases, achieves its maximum at \( x = 1 \) and finally decreases.

\[ \max |g(x)| = g(1) = e^{-1} \]
Hence, \( |f_n(x)| \leq (e^{-1})^n = M_n \) \( \forall n \)

The series \( \sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} (\frac{1}{2})^n \) is a convergent geometric series
with \( r = \frac{1}{2} < 1 \). By M. Test, \( \sum_{n=1}^{\infty} (x e^{-x})^n \) converges uniformly on \([0, \infty)\).
**Power Series:** This is an important class of series of functions and satisfies properties which are not valid in general.

**Definition:** A series of functions of the form \( \sum_{n=0}^{\infty} a_n (x - x_0)^n \) where \( x_0 \) is a fixed real number and the coefficients \( a_n \) depend on \( x \), is called a power series around \( x_0 \).

If we take \( f(x) = (x - x_0)^n \), then the power series reduces to \( \sum_{n=1}^{\infty} a_n x^n \), a power series around 0.

When, we shall consider only the power series of the form

\[
\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots
\]

**Theorem:** If for a power series \( \sum_{n=0}^{\infty} a_n x^n \) exists, then the power series \( \lim_{n \to \infty} \frac{|a_n x^n|}{|x|^{n+1}} \) converges absolutely if \( |x| < R \) and diverges if \( |x| > R \).

**Proof:**

\[
\lim_{n \to \infty} \left| \frac{a_n x^n}{|x|^{n+1}} \right| = \lim_{n \to \infty} \left| \frac{a_n}{|x|} \right| = \frac{|x|}{R}.
\]

By root test, the series converges absolutely if \( |x| < R \) and diverges if \( |x| > R \).

Thus any power series has an interval \((-R, R)\) within which it converges absolutely and outside which convergence is not possible. The interval \((-R, R)\) is symmetrically located about 0 on the region of convergence on interval of convergence of the power series \( \sum_{n=0}^{\infty} a_n x^n \). The number \( R \) is called the radius of convergence.

**Definition:** Let \( \sum_{n=0}^{\infty} a_n x^n \) be a power series. Then, we define the radius of convergence of \( \sum_{n=0}^{\infty} a_n x^n \) as:

\[
R = \begin{cases} 
0 & \text{if } \lim_{n \to \infty} \left| \frac{a_n}{|x|} \right| = \infty, \\
\frac{1}{\lim_{n \to \infty} \left| \frac{a_n x^n}{|x|^{n+1}} \right|} & \text{if } 0 < \lim_{n \to \infty} \left| \frac{a_n x^n}{|x|^{n+1}} \right| < \infty. 
\end{cases}
\]

provided the limit \( \left| \frac{a_n x^n}{|x|^{n+1}} \right| \) exists.
Example 1. Find the radius of convergence of the following power series:
(i) \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \),
(ii) \( \sum_{n=0}^{\infty} \frac{x^n}{n^n} \),
(iii) \( \sum_{n=0}^{\infty} \frac{x^n}{n^n} \).

Also check the series for convergence or divergence at the boundary points.

Solution:
(i) Comparing the given power series with \( \sum a_n x^n \), we have
\[ a_n = \frac{1}{n!} \Rightarrow |a_n|^{1/n} = \frac{1}{n^{1/n}} \rightarrow 0 \text{ as } n \rightarrow \infty. \]

Hence the radius of convergence is \( R = \frac{1}{\lim |a_n|^{1/n}} = 1 \).

The boundary points are \( |x| = 1 \), i.e., \( x = \pm 1 \).

When \( x = 1 \), the series reduces to \( \sum_{n=0}^{\infty} \frac{1}{n!} \), which converges.

When \( x = -1 \), the series reduces to \( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \) which converges (oscillates).

Hence the power series diverges at both the boundary points.

(ii) Comparing the given power series with \( \sum a_n x^n \), we have
\[ a_n = \frac{1}{n^n} \Rightarrow |a_n|^{1/n} = \frac{1}{n^{1/n}} \rightarrow 0 \text{ as } n \rightarrow \infty. \]

Hence, the radius of convergence is \( R = \frac{1}{\lim |a_n|^{1/n}} = 1 \).

The boundary points are \( |x| = 1 \), i.e., \( x = \pm 1 \).

When \( x = 1 \), the series reduces to \( \sum_{n=0}^{\infty} \frac{1}{n^n} \), which converges.

When \( x = -1 \), the series reduces to \( \sum_{n=0}^{\infty} \frac{(-1)^n}{n^n} \) which converges conditionally.

Hence at \( x = -1 \), the power series \( \sum \frac{x^n}{n^n} \) converges conditionally but at \( x = 1 \), the series diverges.

(iii) Comparing the given power series with \( \sum a_n x^n \), we have
\[ a_n = \frac{1}{n^n} \Rightarrow |a_n|^{1/n} = \frac{1}{n^{1/n}} \rightarrow 0 \text{ as } n \rightarrow \infty. \]

Hence, the radius of convergence is \( R = \frac{1}{\lim |a_n|^{1/n}} = 1 \).

The boundary points are \( |x| = 1 \), i.e., \( x = \pm 1 \).

When \( x = 1 \), the series reduces to \( \sum_{n=0}^{\infty} \frac{1}{n^n} \), which converges absolutely.

When \( x = -1 \), the series reduces to \( \sum_{n=0}^{\infty} \frac{(-1)^n}{n^n} \) which converges absolutely.

Hence, the power series \( \sum_{n=0}^{\infty} \frac{x^n}{n^n} \) converges absolutely at \( x = \pm 1 \).
Example 2. Find the radius of convergence and the interval of convergence of the following series:

(i) \( \sum \frac{(2n+3)^n}{n!} \), (ii) \( \sum (2n+3)^n x^n \), (iii) \( \sum \frac{2^n}{n^2} \), (iv) \( \sum \frac{n!}{x^n} \).

\[ \text{Soln.} \quad \text{(i) } y = (x+3)^n. \]

Then the \( \sum \frac{y^n}{n!} \) power series \( \sum \frac{(x+3)^n}{n!} \) reduces to \( \sum \frac{y^n}{n!} \).

Here, \( a_n = \frac{1}{n} \), \( n \in \mathbb{N} \).

The radius of the convergence is \( R = \frac{1}{\lim |a_n|} = \frac{1}{\lim (n!)} = 1 \).

Hence, the power series \( \sum \frac{y^n}{n!} \) converges absolutely if \( |y| < 1 \), i.e., if \( -1 < (x+3) < 1 \Rightarrow -4 < x < 2 \) and diverges if \( |x+3| > 1 \).

Hence, the interval of convergence of \( \sum \frac{(x+3)^n}{n!} \) is \((-4, -2)\).

\[ \text{(ii) Here } a_n = (2n+3)^n, \quad n \in \mathbb{N}. \]

\[ \lim_{n \to \infty} \frac{a_n}{a_{n-1}} = \lim_{n \to \infty} \frac{(2n+3)^n}{(2n-1)^n} = 3. \]

\[ 3 < (2n+3)^n < (2n-1)^n \quad \Rightarrow \quad \lim_{n \to \infty} (2n+3)^n = 3. \]

By Squeeze theorem, \( \lim_{n \to \infty} (2n+3)^n = 3 \). \[ \therefore \lim_{n \to \infty} \frac{(2n+3)^n}{n!} = 3. \]

\[ \therefore R = \frac{1}{\lim_{n \to \infty} \frac{a_n}{a_{n-1}}} = \frac{1}{3} \quad \text{is the radius of convergence.} \]

Note that, the interval of convergence is \((-\frac{3}{2}, \frac{3}{2})\).

\[ \text{(iii) Here, } a_n = \frac{1}{n!}. \]

The radius of the convergence is \( R = \lim_{n \to \infty} \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \lim_{n \to \infty} \frac{n+1}{n!} = 0 \).

Hence, the interval convergence is \((0, \infty)\), i.e., the entire real line.

Therefore, the power series converges absolutely on the entire real line.

\[ \text{(iv) } \sum \frac{n!}{x^n} \text{, Here } a_n = n!. \]

\[ R = \lim_{n \to \infty} \left( \frac{a_n}{a_{n+1}} \right) = \lim_{n \to \infty} \left( \frac{n!}{(n+1)!} \right) = \lim_{n \to \infty} \frac{1}{n+1} = 0 \quad \text{is the radius of convergence.} \]

Note that, every power series has \( \sum a_n x^n \) has the value \( a_0 \), when \( n = 0 \), i.e., every power series is convergent at \( x = 0 \). Therefore, the power series \( \sum n! x^n \) has no interval of convergence but it converges only at \( x = 0 \).
Theorem: If for a given power series \( \sum a_n x^n \), \( 0 < R = \lim \frac{|a_{n+1}|}{|a_n|} \) exist, then \( \sum a_n x^n \) converges absolutely if \( |x| < R \) and diverges if \( |x| > R \).

Definition 2) Let \( \sum a_n x^n \) be a given power series, then one define the radius of convergence of the power series as:

\[
R = \begin{cases} 
\frac{1}{\lim \frac{|a_{n+1}|}{|a_n|}}, & \text{if } 0 < \lim \frac{|a_{n+1}|}{|a_n|} < \infty \\
\infty, & \text{if } \lim \frac{|a_{n+1}|}{|a_n|} = 0 \\
0, & \text{if } \lim \frac{|a_{n+1}|}{|a_n|} = \infty, 
\end{cases}
\]

\[
= \lim \frac{|a_n|}{|a_{n+1}|}, \text{ provided the limit exists.}
\]

Example: Example 2(iii) & (iv) are the example of this type.

Example 2. Find the radius of convergence and the interval of convergence of the power series:

(i) \( \sum \frac{a_n}{n!} x^n \) 
(ii) \( \sum \frac{x^{2n}}{2^n} \).

Solution:
\( a_n = \frac{n^n}{n!} \)

(i) The radius of convergence is
\[
R = \lim_{n \to \infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n \to \infty} \frac{n^n}{n!} \times \frac{(n+1)!}{(n+1)^{n+1}} = \frac{1}{\lim_{n \to \infty} (\frac{n+1}{n})^n} = \frac{1}{e}
\]

The interval of convergence is \((-R, R) =\left(-\frac{1}{e}, \frac{1}{e}\right)\).

(ii) \( \sum \frac{x^{2n}}{2^n} = \sum \frac{(x^2)^n}{4^n} = \sum \frac{y^n}{4^n} \), where \( y = x^2 \).

Note that the radius of convergence of \( \sum \frac{y^n}{4^n} \) is \( R = \lim \frac{1}{4^n} \sqrt[n]{y} = 1 \)

Here, \( \sum \frac{y^n}{4^n} \) converges absolutely if \( |y| < 4 \).

[Note that, from the series \( \sum \frac{y^n}{4^n} = \sum \frac{u_n}{4^n} \), say, \( \lim_{n \to \infty} u_n \sqrt[n]{4^n} = \lim \frac{u_n}{4^n} \).

If \( y < 4 \), by root test the series \( \sum \frac{y^n}{4^n} \) converges absolutely.]

Therefore, the series \( \sum \frac{x^{2n}}{2^n} \) converges absolutely if \( x^2 < 4 \), i.e., \( -2 < x < 2 \), and diverges if \( |x| > 2 \).

Hence, the radius of convergence of \( \sum \frac{x^{2n}}{2^n} \) is \( R' = 2 \).
Example 4) If the radius of convergence of \( \sum_{n=0}^{\infty} a_n x^n \) is \( R > 0 \), find the radius of convergence of the following series:
\[ \sum_{n=0}^{\infty} a_n x^n / m \quad \text{and} \quad \sum_{n=0}^{\infty} a_n x^n \]

Hint: \( R \) Here, \( R = \frac{1}{\log|\lambda|/m} \) or \( \lim \frac{|a_n|/m}{|a_n + 1|} \), provided the limit exists.

Now, the radius of convergence of \( \sum_{n=0}^{\infty} a_n / m \) is \( R' = \lim \frac{|a_n|/m}{|a_n + 1|} = \lim \frac{|a_n|}{|a_n + 1|} \cdot \lim \frac{1}{|1|/m} = 1 \cdot R = R \).

The radius of convergence of \( \sum_{n=0}^{\infty} a_n x^n \) is \( R' = \lim \frac{|a_n|}{|a_{n+1}|} = \lim \frac{|a_n|}{|a_n + 1|} \cdot \lim \frac{1}{|1|/m} = R, 0 = 0 \).

Example 5)
(a) If \( \lim a_n / n = c \), then show that the radius of convergence of \( \sum_{n=0}^{\infty} a_n x^n \) is \( |c| \).
(b) If \( 0 < p \leq |a_n| \leq q \), \( n \in N \), then show that the radius of convergence of \( \sum_{n=0}^{\infty} a_n x^n \) is \( 1 \).

\[ \text{Soln} \]
(a) Let \( c > 0 \) be an arbitrary quantity, then \( |a_n - c| < c \forall n \in N \).
\[ \Rightarrow \frac{c - 1}{m^2} < \frac{c + 1}{m^2} \quad \text{assuming} \quad |c - 1| < |c + 1| \]
\[ \Rightarrow \frac{c - 1}{m^2} < |a_n^m| / m^2 \quad \text{assuming} \quad |a_n^m| / m^2 \quad \text{let} \quad |a_n^m| / m^2 \quad \text{let} \quad |c - 1| = l_1, |c + 1| = l_2. \]
\[ \Rightarrow \frac{l_1}{m^2} < \frac{l_2}{m^2} \]
\[ \Rightarrow \lim_{n \to \infty} \frac{l_1}{m^2} \leq \lim_{n \to \infty} \frac{l_2}{m^2} \]
\[ \Rightarrow \frac{1}{l_1} \leq \lim \frac{|a_n^m|}{m^2} \leq \frac{1}{l_2} \]
\[ \Rightarrow \lim \frac{|a_n^m|}{m^2} = \frac{1}{l_1} \]
\[ \Rightarrow R = \frac{1}{\lim |a_n^m|} = |l_1| \]

(b) \( p \leq |a_n| \leq q \)
\[ \Rightarrow p^m \leq |a_n^m| \leq q^m \]
\[ \Rightarrow \lim_{n \to \infty} \frac{p^m}{m^2} \leq \lim_{n \to \infty} \frac{|a_n^m|}{m^2} \leq \lim_{n \to \infty} \frac{q^m}{m^2} \]
\[ \Rightarrow 1 \leq \lim_{n \to \infty} \frac{|a_n|}{m^2} \leq 1 \]
\[ \Rightarrow \lim_{n \to \infty} \frac{|a_n|}{m^2} = 1 \]
\[ \Rightarrow R = 1. \]
Taylor’s Polynomial: There are several ways to approximate a given function by polynomials, depending on what use is to be made of the approximation.

Consider a function $f(x) = e^x$.

Note that, $f^{(n)}(x) = e^x$, and $f^{(n)}(0) = 1$, $n \in N$.

The 1st degree polynomial $p_1(x) = 1 + x$ also has $p_1(0) = 1 = f(0)$ and $p_1'(0) = 1 = f'(0)$.

The 2nd degree polynomial $p_2(x) = 1 + \frac{x}{2!} + \frac{x^2}{2^2}$ also has $p_2(0) = f(0)$, $p_2'(0) = f'(0)$, $p_2''(0) = f''(0)$.

Clearly, $p_2(x)$ is a better approximation than $p_1(x)$ around $x = 0$ for $f(x) = e^x$.

The Taylor’s Polynomial generated by a function:

Suppose $f(x)$ has derivatives up to order $n$ at $x = x_0$.

Let us try to find out a polynomial $p_n(x)$ which agrees with $f(x)$ and its first $n$ derivatives at $x = x_0$.

Note that there are $(n+1)$ conditions to be satisfied:

\[ p_n(x_0) = f(x_0) \]
\[ p_n^{(n)}(x_0) = f^{(n)}(x_0), \quad n = 1, 2, \ldots, n. \]

From the $(n+1)$ conditions, we can determine only $(n+1)$ unknown constants; so we try with a polynomial of degree $n$, say,

\[ p_n(x) = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots + a_n (x - x_0)^n. \]

When $x = x_0$, $p_n(x_0) = f(x_0)$, so $a_0 = f(x_0)$.

Note that $p_n^{(n)}(x_0) = f^{(n)}(x_0)$ and $p_n^{(n)}(x_0) = f^{(n)}(x_0)$.

\[ a_n = \lim_{x \to x_0} \frac{f^{(n)}(x_0)}{n!}. \]
Hence, we have \( P_n(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n \).

And this is the one and only polynomial of exact degree \( \leq n \), which agrees with \( f(x) \) and its first \( n \) derivatives at \( x=x_0 \). This polynomial \( P_n(x) \) is called the Taylor's polynomial of degree \( n \) generated by \( f(x) \) at \( x=x_0 \).

It is natural to expect that this polynomial will provide a reasonable approximation to \( f(x) \) for points near \( x_0 \) and to measure the quality of approximation.

It is necessary to have information regarding the remainder:

\[ R_n(x) = f(x) - P_n(x). \]

**Taylor's Theorem**: If \( f(x) \) has continuous derivatives up to order \( (n+1) \) is some neighborhood of \( x=x_0 \), then

\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n + R_n(x) \]

where \( R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-x_0)^{n+1} \)

for some \( c \) between \( x \) and \( x_0 \).

**Example**: Obtain the Taylor's polynomial of degree 6 generated by \( f(x) = \sin(x) \) at \( x=0 \). Also estimate the error in approximating the interval \([0, \frac{\pi}{2}]\).

**Solution**:

- The Taylor's polynomial of degree 6 generated by \( f(x) \) at \( x=0 \) is

\[ P_6(x) = \sum_{n=0}^{6} \frac{f^{(n)}(0)}{n!} x^n. \]

Hence, \( f(x) = \sin(x) \), \( f'(x) = \cos(x) \), \( f''(x) = -\sin(x) \), \( f'''(x) = -\cos(x) \).

Hence, \( P_6(x) = 0 + \frac{1}{1!} x + \frac{0}{2!} x^2 + \frac{-1}{3!} x^3 + \frac{0}{4!} x^4 + \frac{1}{5!} x^5 + \frac{0}{6!} x^6 \)

\[ = x - \frac{x^3}{3!} + \frac{x^5}{5!} \]

- By Taylor's theorem:

\[ f(x) = P_6(x) + R_6(x) \]

where \( R_6(x) = \frac{f^{(7)}(c)}{7!} x^7 \)

Hence, \( R_6(x) = -\frac{\cos(c)}{7!} x^7 \)

and \( |R_6(x)| \leq \frac{1}{7!} \)

For \( x \in \left[0, \frac{\pi}{2}\right] \), \( |R_6(x)| \leq \frac{1}{7!} \)

Hence, the maximum error in approximating \( f(x) = \sin(x) \) by \( P_6(x) \) for \( x \in \left[0, \frac{\pi}{2}\right] \) is \( \frac{1}{7!} \).
Example 2. Obtain the Taylor polynomial $P_2(x)$ of degree 2 generated by $f(x) = \sqrt[3]{1 + x}$ at $x = 0$. Hence show that

$$\left| 3^{1/3} \left(\frac{1}{3} + \frac{x}{3} - \frac{x^2}{9} \right) \right| < \frac{5}{81} \text{ for } x > 0.$$ 

Use this inequality to approximate $3^{1/3}$ correctly.

An estimate on the error

**SOLN.** Let $f(x) = 3^{1/3} x^{1/3}$.

Then the 2nd degree Taylor Polynomial generated by $f(x)$ at $x = 0$ is

$$P_2(x) = \sum_{n=0}^{2} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0) x + \frac{f''(0)}{2!} x^2.$$

Now, $f(0) = 1$, $f'(x) = \frac{1}{3} (1 + x)^{-2/3}$, $f''(x) = -\frac{2}{9} (1 + x)^{-5/3}$.

Therefore, $P_2(x) = 1 + \frac{x}{3} - \frac{x^2}{6}$.

By Taylor's theorem,

$$f(x) = P_2(x) + R_2(x),$$

where $R_2(x) = \frac{f^{(3)}(c)}{3!} (x-0)^3$ for some $c$ between 0 and $x$.

Now, $\left| \frac{2}{9} \left(1 + \frac{x}{3} - \frac{x^2}{6} \right) \right| < \frac{5}{81},$ for $x > 0.$

Note that $\sqrt[3]{0.6} = 3^{1/3} (0.6)^{1/3}$ and $x = 0.6$.

Hence we obtain the approximation of $\sqrt[3]{0.6}$ as

$$P_2(0.6) = 1 + 0.6 - \frac{0.6^2}{9} < 1.16,$$

with a maximum error

$$\left| \frac{2}{9} (0.6)^{2} \right| = \frac{4}{81},$$

Example 3. Obtain the Taylor's polynomial $P_n(x)$ generated by $f(x) = \ln(1 + x)$ at $x = 0$. Approximate $\ln(1.5)$ with an error less than 0.001.

**SOLN.** Let $f(x) = \ln(1 + x)$.

Hence, $f(n)(x) = \frac{(-1)^{n-1} (1 + x)^n}{n}$.

Hence, $P_n(x) = \sum_{k=0}^{n} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n$.

$$= 0 + \sum_{n=1}^{n} \frac{(-1)^{n-1} x^n}{n}.$$
By Taylor's theorem, we have,

\[ f(x) = P_n(x) + R_n(x) \]

\[ \Rightarrow \text{the remainder term is } f(x) - P_n(x) \]

\[ = R_n(x) = \left( \frac{f^{(n+1)}(c)}{(n+1)!} \right) x^{n+1} \]

i.e., \[ R_n(x) = \frac{(-1)^n n \ln(1 + c) \cdot x^{n+1}}{(1 + c)^{n+1} (n+1)} \]

For \[ x > 0 \],

\[ \left| R_n(x) \right| = \frac{x^{n+1}}{(n+1)(1+c)^{n+1}} < \frac{x^{n+1}}{n+1} \]

\[ \left[ \text{For } x > 0, \ 0 < c < x, \ (1+c) > 1 \right] \]

\[ \Rightarrow \frac{1}{1+c} < 1 \Rightarrow \frac{1}{(1+c)^{n+1}} < 1 \]

Here, \( \ln(1.5) = \ln(1+0.5) \) is to be approximate by \( P_n(1/2) \).

the maximum error

\[ \frac{1}{2^{n+1}(n+1)} < 0.001 \]

\[ \Rightarrow 2^{n+1}(n+1) > 1000 \]

\[ \Rightarrow 2^{n+1}(n+1) = \int_{0.896}^{1.89} \]

\[ n = 6 \]

\[ 2^{6+1}(6+1) = 1.896 > 1000 \]

and hence, the smallest value of \( n \) is 7.

Thus, we obtain

\[ P_7(0.5) = \sum_{n=0}^{7} (-1)^n \frac{(0.5)^n}{n!} \]

\[ \ln(1.5) \text{ with an error less than } 10^{-3}. \]

**Example:** Obtain the Taylor's Polynomial \( P_3(x) \) of degree 3 generated by \( \ln x \) at \( x = 1 \). Also estimate the error in approximating \( \ln x \) by \( P_3(x) \) for \( x \in \left[ \frac{1}{2}, \frac{3}{2} \right] \).

**Solution:**

\[ f(x) = \ln x, \ x_0 = 1 \]

\[ P_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(1)}{n!} (x-1)^n \]

\[ f'(x) = \frac{1}{x}, \ f''(x) = -\frac{1}{x^2}, \ f'''(x) = \frac{2}{x^3}, \ f^{(4)}(x) = -\frac{6}{x^4} \]

\[ \therefore P_3(x) = \frac{1}{1!} (x-1) + \frac{-1}{2!} (x-1)^2 + \frac{2}{3!} (x-1)^3 \]

By Taylor's theorem,

\[ f(x) = P_3(x) + R_3(x), \ 	ext{where} \]

\[ R_3(x) = \frac{f^{(4)}(c)}{4!} (x-1)^4 = -\frac{6}{4!} (x-1)^4 = -\frac{(x-1)^4}{24} \]

where \( c \) lies between \( 1 \) and \( x \),

\[ \left| R_3(x) \right| \leq \frac{(x-1)^4}{4! c^4} \leq \frac{1}{24} \]

\[ \frac{1}{4! c^4} < \frac{1}{24} \]

\[ \Rightarrow \frac{1}{c^4} < \frac{1}{24} \]
Hence the maximum error in approximating \( f(x) \) by \( P_n(x) \) for \( x \in [-\frac{1}{2}, \frac{3}{2}] \) is \( \frac{1}{20} \).

**B. TAYLOR’S SERIES**: If a function \( f(x) \) has derivatives of every order in an open interval about \( x_0 \), then we can compute the Taylor’s coefficients \( a_n = f^{(n)}(x_0) \), and with these coefficients we can form the power series:

\[
\sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n
\]

The power series converges to \( f(x) \) for \( |x-x_0| < R \) if the sequence of remainder terms converges to 0. Then we shall write:

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n
\]

and we say that the power series is the Taylor’s series expansion of \( f(x) \) at \( x = x_0 \).

**Example**: Obtain the Taylor series expansions of the following functions at \( x = 0 \).

(i) \( \sin x \), (ii) \( e^x \), (iii) \( \log_2(1+x) \), where \(-1 < x \leq 1\).

**Solution**:

(i) Let \( f(x) = \sin x \).

\[
\sin x = (x) + \frac{1}{2!} (x)^2 - \frac{1}{4!} (x)^4 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.
\]

Taylor’s coefficients are:

\[
a_{2n} = f^{(2n)}(0) = 0
\]

\[
a_{2n+1} = f^{(2n+1)}(0) = (-1)^n
\]

Now, \( \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \) is the Taylor’s series expansion of \( f(x) \) at \( x = 0 \) if the sequence of remainder terms converges to 0.

Now, \( R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \)

and \( |R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \left[ \sum_{n=0}^{\infty} \frac{|x|^n}{n!} \right] = \frac{|x|^{n+1}}{n+1} \leq \frac{1}{1} \).

\[
|\sin x| \leq 1.
\]
Note that
\[ \lim \left[ \frac{|x|^{n+2}}{|n+2|} \right] \]
\[ = |x|, \quad \lim \left( \frac{1}{n+1} \right) = 0 < 1 \]
\[ \Rightarrow \sum \frac{|x|^{n+1}}{|n+1|} \text{ converges,} \]
\[ \Rightarrow \lim_{n \to \infty} \frac{|x|^{n+1}}{|n+1|} = 0 \quad \text{[By n'th term test: } \sum a_n \text{ converges,]} \]
Hence,
\[ 0 \leq |R_n(x)| \leq \frac{|x|^{n+1}}{|n+1|} \to 0 \quad \text{as } n \to \infty \forall x \in \mathbb{R}, \]
\[ \Rightarrow \lim_{n \to \infty} R_n(x) = 0 \]
Therefore, we have the Taylor's series expansion of \( \sin x \) about \( x = 0 \) as:
\[ \sin x = \sum_{n=0}^{\infty} a_n (x-0)^n \]
\[ = \sum_{n=0}^{\infty} a_{2n-1} x^{2n-1} \quad \text{as } a_{2n} = 0 \]
\[ = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} , \quad \forall x \in \mathbb{R} \]
\[ = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} + \ldots \]

(ii) \( f(x) = e^x \)
Here, \( f^{(n)}(x) = e^x \)
and the Taylor's coefficient
\[ a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!} , \quad n \in \mathbb{N} \]
Now, \( \frac{f^{(n)}(0)}{n!} \) is the Taylor's series expansion of
\[ f(x) \text{ at } x = 0 \]
iff the sequence \( \{ R_n(x) \} \) of remainder terms converges to 0.
Here,
\[ R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-c)^{n+1} \]
\[ = \frac{e^{c} x^{n+1}}{(n+1)!}, \quad \text{where } c \text{ lies between } 0 \text{ and } x. \]
Now,
\[ |R_n(x)| \leq \frac{|x|^{n+1}}{|n+1|} \quad \text{as } |c| < |x| \]
\[ \quad \text{i.e., } e^{|c|} < e^{|x|} \]
Note that \[ \lim_{n \to \infty} \left( \frac{e^{x_n}}{n+2} \right) \]
\[ = |x_1| \lim_{n \to \infty} \left( \frac{1}{n+2} \right) = 0 < 1 \]
\[ \Rightarrow \sum_{n=1}^{\infty} \frac{|x_1|}{n+1} \text{ converges} \]
\[ \Rightarrow \lim_{n \to \infty} \frac{|x_1|}{n+1} = 0 \]
\[ \Rightarrow \lim_{n \to \infty} R_n(x) = 0 \]
Therefore, the Taylor's series expansion of \( e^x \) at \( x = 0 \) is
\[ e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \]
\[ = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \quad x \in \mathbb{R}. \]

(iii) Here \( f(x) = \log_e (1+x), \quad -1 < x \leq 1, \)
Now, \( f^{(n)}(x) = \frac{(-1)^n}{(1+x)^{n-1}} \)
\[ a_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^n}{(1+0)^{n-1}} \cdot \frac{1}{n!} = \frac{(-1)^n}{n}, \quad n \in \mathbb{N}. \]
Now, \( \sum_{n=0}^{\infty} \frac{(-1)^n}{n} x^n \) is the Taylor's series expansion of \( f(x) \) at \( x = 0 \) if the sequence \( f(R_n(x)) \) converges to zero.
Here, \( R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} = \frac{(-1)^n}{(1+c)^{n+1}} \cdot \frac{x^{n+1}}{n+1} \)
so \( c \) lies between 0 & \( x \).

Case: \( 0 \leq x \leq 1 \)
Then \( 0 < c < x, \leq 1 \)
\[ |R_n(x)| = \frac{|x^{n+1}|}{n+1} \cdot \frac{1}{(1+c)^{n+1}} < \frac{x^{n+1}}{n+1} \]
\[ \Rightarrow |R_n(x)| < \frac{x^{n+1}}{n+1} \leq \frac{1}{n+1} \rightarrow 0 \text{ as } n \to \infty \text{ for } 0 \leq x \leq 1, \]
\[ \Rightarrow \lim_{n \to \infty} R_n(x) = 0, \quad 0 \leq x \leq 1. \]
Case-II: Cauchy's form of remainder.

\[ R_n = \frac{x^{n+1} (1-\theta)^n \cdot f^{(n+1)}(\theta x)}{n !} \]

\[ = \frac{x^{n+1} (1-\theta)^n}{n} \cdot \frac{(-1)^n n !}{(1+\theta x)^{n+1}} \]

\[ = (-1)^n, x^{n+1} \cdot \left( \frac{1-\theta}{1+\theta x} \right)^n \cdot \frac{1}{1+\theta x} \]

Now, \( 1-\theta < 1 + \theta x \)

\[ \Rightarrow \frac{1-\theta}{1+\theta x} < 1 \]

and \( \left( \frac{1-\theta}{1+\theta x} \right)^n \rightarrow 0 \) as \( n \rightarrow \infty \)

Also note that \( x^{n+1} \rightarrow 0 \) as \( n \rightarrow \infty \).

and \( \left[ \frac{1}{1+\theta x} < \frac{1}{1-1} \right] \), moreover, it is independent of \( n \).

Hence, \( R_n(x) \rightarrow 0 \) as \( n \rightarrow \infty \)

Therefore, the Taylor's series expansion of \( \log_e(1+x) \) at \( x=0 \) is given by

\[ \log_e(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot x^n}{n} \]

\[ = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad -1 < x \leq 1, \]
**GAMMA AND BETA INTEGRALS**

(DEFINITIONS & USES)

Gamma Function: - The Gamma function is defined by Euler to be the definite integral

\[ \Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, dx \quad (n > 0) \]

L.H.S will not exist for \( n \leq 0 \). The integral is known to exist for \( n > 0 \) and the value of the integral is a positive number.

Now, by direct integration, we see that

\[ \Gamma(1) = \int_0^\infty e^{-x} \, dx = [e^{-x}]_0^\infty = 0 - 0 = 1. \]

Integrating \( \Gamma(1) \) by parts, we get the following recurrence relation.

We have

\[ \Gamma(n+1) = \int_0^\infty x^n e^{-x} \, dx \]

\[ = \left[ -x^n e^{-x} \right]_0^\infty + n \int_0^\infty x^{n-1} e^{-x} \, dx \]

\[ = -0 + n \Gamma(n) \]

\[ = n \Gamma(n) \]

So, we have \( \Gamma(n+1) = n \Gamma(n) \) or \( \Gamma(n) = \frac{\Gamma(n+1)}{n} \).

Relation between Factorial and Gamma Notation: -

If \( n \) is a positive integer, then \( \Gamma(n+1) = n \Gamma(n) \)

\[ \therefore \Gamma(n+1) = n(n-1) \Gamma(n-2) \]

\[ = n(n-1) \cdots 1 \Gamma(1) \]

\[ = n(n-1) \cdots 1 \cdot 1 \]

\[ = n! \]

\[ \Gamma(n) = (n-1)! \]

\[ \Gamma(0) = \frac{\Gamma(1)}{0} = \text{undefined} \]

Thus \( \Gamma(n) \) is not defined if \( n = 0 \) or a negative integer.
Example:

S.T. \( \Gamma \left( \frac{2k + 1}{2} \right) = \frac{1 \cdot 3 \cdot 5 \cdots (2k - 1)}{2^k} \sqrt{\pi} \)

Proof:

\[
\Gamma \left( \frac{2k + 1}{2} \right) = \Gamma \left( k + \frac{1}{2} \right) \\
= (k - \frac{1}{2}) \Gamma \left( k - \frac{1}{2} \right) \\
= (k - \frac{1}{2}) (k - \frac{3}{2}) \Gamma \left( k - \frac{3}{2} \right) \\
= \cdots \\
= \frac{(2k - 1)(2k - 3) \cdots 1}{2^k} \Gamma \left( \frac{1}{2} \right) \\
= \frac{(2k - 1)(2k - 3) \cdots 1}{2^k} \sqrt{\pi} \quad \text{[\( \Gamma \left( \frac{1}{2} \right) = \sqrt{\pi} \)]} \\
\]

(Proved)

2) Evaluate: \( \int_0^\infty e^{-x^4} \, dx \)

Soln. → \( I = \int_0^\infty e^{-x^4} \, dx \)

Let, \( y = x^4 \)

\( x = y^{1/4} \)

\( \Rightarrow dx = \frac{1}{4} y^{-3/4} \, dy \)

\( I = \frac{1}{4} \int_0^\infty e^{-y} y^{1/4 - 1} \, dy \)

\( = \frac{1}{4} \Gamma \left( \frac{1}{4} \right) = \Gamma \left( \frac{5}{4} + 1 \right) = \Gamma \left( \frac{5}{4} \right) \quad \text{(ANS)} \)
**Beta Integral:** The Beta integral with parameters \( m \) and \( n \) is denoted by \( B(m, n) \) and defined as

\[
B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx \quad ; \quad m, n > 0
\]

**Result:** \( B(m, n) = B(n, m) \)

**Proof:** We define \( B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx \)

Let \( y = 1-x \)

\[
\frac{dy}{dx} = -1 \quad \Rightarrow \quad dx = -dy
\]

So,

\[
B(m, n) = \int_0^1 (1-y)^{m-1} y^{n-1} \, dy
\]

\[
= \int_0^1 y^{n-1} (1-y)^{m-1} \, dy
\]

\[
= B(n, m)
\]

**Result:** \( B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} \)

**Problem:**

Evaluate: \( \int_0^1 \frac{x^n}{\sqrt{1-x^2}} \, dx \)

**Ans:**

\[
I = \int_0^1 \frac{x^n}{\sqrt{1-x^2}} \, dx
\]

Put, \( x^2 = \frac{y}{2} \), then \( dx = \frac{dy}{2} \)

and \( I = \int_0^{1/2} \frac{y^{(n+1)/2}}{\sqrt{1 - y/2}} \, dy \)

\[
= \frac{1}{2} \int_0^{1/2} y^{(n+1)/2 - 1} (1-y)^{-1/2} \, dy
\]

\[
= B\left(\frac{n+1}{2}, \frac{1}{2}\right)
\]

\[
= \frac{\Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)}
\]

\[
= \frac{1}{2} \pi \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)} \quad \text{[Since, } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\text{]}
\]
2) Evaluate: \[ \int_0^\alpha \frac{dz}{(1+z^2)^n} \]

**Ans:** Let \( \alpha = y' \rightarrow x = y^{1/2} \)

\( I = \int_0^\infty \frac{y^{1/2 - 1}}{(1+y)^n} \, dy \)

\[ = \frac{1}{2} \int_0^\infty y^{-1/2} \, dy \]

\[ = \frac{1}{2} \Gamma \left( \frac{1}{2} \right) \Gamma \left( n - \frac{1}{2} \right) \]

\[ = \frac{1}{2} \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( n - \frac{1}{2} \right)}{\Gamma \left( n \right)} \]

\[ = \frac{1}{2} \sqrt{\pi} \cdot \frac{\Gamma \left( n - \frac{1}{2} \right)}{\Gamma \left( n \right)}. \]

**Show that** \( \Gamma \left( \frac{1}{2} \right) = \sqrt{\pi} \)

**Soll** We have \( B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx \)

\( B \left( \frac{1}{2}, \frac{1}{2} \right) = \int_0^1 x^{1/2 - 1} (1-x)^{1/2 - 1} \, dx \)

\[ \text{or,} \quad \frac{\Gamma \left( \frac{1}{2} \right)^2}{\Gamma (1)} = \int_0^1 \frac{dx}{\sqrt{x(1-x)}} \]

\[ \text{or,} \quad \left[ \Gamma \left( \frac{1}{2} \right) \right]^2 = \int_0^{\pi/2} \frac{dx}{\sqrt{\sin x \cos x}} \]

\[ = \int_0^{\pi/2} \frac{2 \sin \theta \cos \theta \, d\theta}{\sin \theta \cos \theta} \]

\[ = 2 \int_0^{\pi/2} d\theta \]

\[ = \pi \]

\[ \Gamma \left( \frac{1}{2} \right) = \sqrt{\pi} \] \[ \left[ \text{Negative sign ignored since it represents an area} \right] \]
IMPROPER INTEGRAL: We first explain the concept of Riemann Integral.

Riemann Integral: Let \( f(x) \) be a bounded function in the closed interval \([a, b]\),

Define, \( M = \sup_{x \in [a, b]} f(x) \)
\( m = \inf_{x \in [a, b]} f(x) \)

A partition of \([a, b]\) is a set of finitely many points, called the points of subdivision, and denoted by \( p: a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b \)

For the partition \( p \), we get

\[
\begin{align*}
M_i &= \sup_{x \in [x_{i-1}, x_i]} f(x) \\
m_i &= \inf_{x \in [x_{i-1}, x_i]} f(x)
\end{align*}
\]

Clearly, \( m_i \leq m \leq M \leq M_i \), \( \forall i \).

For the partition \( p \), we define the upper sum of the function \( f(x) \) as

\[
U(p, f) = \sum_{i=1}^{n} M_i \Delta x_i,
\]
where \( \Delta x_i = x_i - x_{i-1} \).

Similarly, we define the lower sum as

\[
L(p, f) = \sum_{i=1}^{n} m_i \Delta x_i.
\]

[From the graph, \( U(p, f) \) overestimates the actual area and \( L(p, f) \) underestimates the actual area; i.e., only they are known as upper and lower sums]

Note that,

\[
\begin{align*}
&\\m(b-a) \leq \sum_{i=1}^{n} m_i \Delta x_i \leq \sum_{i=1}^{n} M_i \Delta x_i \leq M(b-a)
\end{align*}
\]

The upper integral of \( f(x) \) on \([a, b]\) is defined by

\[
\int_{a}^{b} f(x) \, dx = \inf_{p} U(p, f)
\]

and the lower integral of \( f(x) \) on \([a, b]\) is defined by

\[
\int_{a}^{b} f(x) \, dx = \sup_{p} L(p, f)
\]

\[
L_i, U_i \quad \text{are the lower and upper sums}
\]

for a partition \( P_i \). \( L_i \) and \( U_i \) may be the actual value of the integral.

Then \( \sup_{P_i} \{ L(P_i, f) \} \) and \( \inf_{P_i} \{ U(P_i, f) \} \) will be very closely to \( I \).

If they are equal, then \( I \) exists, otherwise \( I \) does not exist.
Hence, \( \int_{a}^{b} f(x) \, dx \) and \( \int_{a}^{b} g(x) \, dx \) are finite.

**Definition:** A function \( f(x) \) is said to be Riemann integrable on \([a, b]\) if 
\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{b} g(x) \, dx.
\]

Supremum: The supremum of a set is the least upper bound of the set.

\( A = \{ x \leq x \leq 1 \} \), note that, \( u = 1, 1.1, 1.2, \ldots \), i.e., any number \( u \geq 1 \) is an upper bound. The least upper bound is \( U = 1 \), that is supremum over \( x \), \( A = 1 \).

Clearly, the maximum over \( A \) is \( 1 \).

\( A = \{ x < x < 1 \} \), Any number \( u (>1) \) is an upper bound, any number \( u (>1) \) can't be a upper bound, \( \sup A = 1 \).

Hence, the set \( A \) has no maximum.

Infimum: The greatest lower bound of a set \( A \) is called the infimum of \( A \).

**Example:** Show that every constant function \( f(x) = k \) is Riemann integrable on any interval \([a, b]\): \( M = \inf \leq \inf \leq m \).

Soln. Let \( p : a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b \) be a partition of \([a, b]\):
\[
U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i = k \sum_{i=1}^{n} \Delta x_i = k(b-a) \quad (1)
\]
\[
L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i = k(b-a) \quad (2)
\]

Hence,
\[
\int_{a}^{b} f(x) \, dx = \sup L(P, f) = k(b-a) \quad \text{and}
\]
\[
\int_{a}^{b} f(x) \, dx = \inf U(P, f) = k(b-a)
\]

Hence, \( \int_{a}^{b} f(x) \, dx = \int_{a}^{b} g(x) \, dx \)

i.e., \( f(x) \) is integrable on \([a, b]\).
Example: \( \int f(x) \, dx \) is not integrable on \([a, b] \)

\[ f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} \]

\( \mathbb{Q} \) is the set of rational numbers. Show that \( f(x) \)
is not integrable on \([a, b] \).

Solution:
- Let \( P : a = x_0 < x_1 < \cdots < x_n = b \) be a partition of \([a, b] \).
- \( U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i \)
- \( L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} 0 \cdot \Delta x_i = 0 \)
- Hence, \( \int_{a}^{b} f(x) \, dx = \sup_{P} L(P, f) = 0 \)
- and \( \int_{a}^{b} f(x) \, dx = \inf_{P} U(P, f) = (b - a) \)

Hence, \( f(x) \) is not integrable on \([a, b] \).

Improper Integral: The concept of Riemann integrability of a function on an interval is subject to the following limitations:
- The function is bounded.
- The interval is finite.

Hence, we extend the concept of Riemann integrability to include improper integrals, i.e., the integral \( \int_a^b f(x) \, dx \) which are unbounded at some point.

Different types of Improper Integrals:

First Kind: The integral of the form \( \int_{a}^{\infty} f(x) \, dx \) or \( \int_{-\infty}^{b} f(x) \, dx \),
where \( f(x) \) is bounded on the interval of integration, is an improper integral of the first kind.

For e.g. \( \int_{a}^{\infty} \frac{1}{x+1} \, dx \), \( \int_{-\infty}^{b} \frac{1}{x^{2}+1} \, dx \) are improper integrals of first kind.

Second Kind: When \( a \) and \( b \) are real numbers (i.e., finite quantities) but \( f(x) \) is bounded at some point on the interval \([a, b] \), the integral \( \int_{a}^{b} f(x) \, dx \) is an improper integral of the second kind.

For e.g. \( \int_{-\infty}^{\infty} \frac{1}{x^{2}} \, dx \), \( \int_{-\infty}^{\infty} \frac{1}{x^{2}+1} \, dx \) are improper integrals of second kind.
THIRD KIND: When combination of these two difficulties occur, i.e. the interval of integration is infinite as well as the integrand is unbounded at some point in the interval, then the integral is an improper integral of third kind.

For e.g., \( \int \frac{1}{k} \, dx \) and \( \int \frac{1}{k-1} \, dx \) are improper integrals of third kind.

FIRST KIND:

Definition: Let \( f(x) \) be Riemann integrable on \([a, R]\), for every \( R > a \),

\[
\int_a^R f(x) \, dx = \lim_{R \to a^+} \int_a^R f(x) \, dx.
\]

Similarly, if \( f(x) \) is Riemann integrable on \([s, b]\) for \( s < b \), then

\[
\int_s^b f(x) \, dx = \lim_{s \to b^-} \int_s^b f(x) \, dx.
\]

If the appropriate limit is finite, the integral is said to converge, otherwise it diverges.

Example 1: Does \( \int \frac{1}{k^p} \, dx \) converge?

Solution: The interval of integration is infinite but the integrand is bounded on the interval. Hence \( \int \frac{1}{k^p} \, dx \) is of the first kind improper integral.

Now define gives:

\[
\int \frac{1}{k^p} \, dx = \lim_{R \to \infty} \int_a^R \frac{1}{k^p} \, dx = \lim_{R \to \infty} \left[ \frac{k^{-b+1}}{-b+1} \right]_a^R = \frac{1}{-b+1} \lim_{R \to \infty} \frac{R^{1-b} - a^{1-b}}{b-1}.
\]

If \( b > 1 \), then \( \lim_{R \to \infty} \frac{R^{1-b} - a^{1-b}}{b-1} = 0 \) if \( p > 1 \).

Hence, \( \int \frac{1}{k^p} \, dx \) converges if \( p > 1 \) and diverges if \( p \leq 1 \).

Remark: N.T. \( \int \frac{1}{k^p} \, dx \) converges if \( p > 1 \) and diverges if \( p \leq 1 \), when \( a > 0 \).
Example 2) Does \( \int e^{-ax} \, dx \) converge?

**Soln.** This is an improper integral of the first kind.

By definition, \( \int_{0}^{\infty} e^{-ax} \, dx = \lim_{R \to \infty} \int_{0}^{R} e^{-ax} \, dx \). If \( a \neq 0 \),

\[
\lim_{R \to \infty} \int_{0}^{R} e^{-ax} \, dx = \lim_{R \to \infty} \left[ \frac{-e^{-ax}}{a} \right]_{0}^{R} = \lim_{R \to \infty} \left( \frac{1}{a} - \frac{e^{-aR}}{a} \right)
\]

\[
= \lim_{R \to \infty} \frac{1}{a} - \lim_{R \to \infty} \frac{e^{-aR}}{a} = \frac{1}{a} \text{ if } a > 0,
\]

\[
= \lim_{R \to \infty} \frac{-e^{-aR}}{a} = 0 \text{ if } a < 0.
\]

\[
\Rightarrow \int_{0}^{\infty} e^{-ax} \, dx \text{ converges if } a > 0 \text{ and diverges if } a \leq 0.
\]

**SECOND KIND:**

**Definition:** If \( f(x) \) is Riemann integrable on \([a, R]\) for \( R \in (a, b)\) and \( f(x) \) is unbounded at \( x = b \), then \( \int_{a}^{b} f(x) \, dx = \lim_{R \to b^{-}} \int_{a}^{R} f(x) \, dx \).

If \( f(x) \) is Riemann integrable on \([s, b]\) for \( s \in (a, b)\) and \( f(x) \) is unbounded at \( x = a \), then \( \int_{a}^{b} f(x) \, dx = \lim_{s \to a^{+}} \int_{s}^{b} f(x) \, dx \).

If the appropriate limit is finite, the improper integral is said to converge; otherwise, it diverges.

Again, if \( f(x) \) is unbounded at \( x = a_{0} \), \( a < a_{0} < b \) then we say that \( \int_{a}^{b} f(x) \, dx \) converges if \( \int_{a_{0}}^{b} f(x) \, dx \) and \( \int_{a}^{a_{0}} f(x) \, dx \) both converge.
**Example 1**

Does \( \int_0^1 \frac{x}{x^p} \, dx \) converge?

**Solution**

Here \( f(x) \) is unbounded at \( x = 0 \), \( \forall p > 0 \), but the interval of integration is finite. This is an improper integral of 2nd kind.

Now, \( \int_0^1 \frac{x}{x^p} \, dx = \lim_{s \to 0^+} \int_s^1 \frac{x}{x^p} \, dx \)

If \( p \neq 1 \),

\[
\lim_{s \to 0^+} \int_s^1 \frac{x}{x^p} \, dx = \lim_{s \to 0^+} \left[ \frac{x^{1-p}}{1-p} \right]_s^1 = \frac{1}{1-p} \lim_{s \to 0^+} s^{1-p} = \begin{cases} 1 & \text{if } p < 1 \\ \infty & \text{if } p > 1 \end{cases}
\]

If \( p = 1 \),

\[
\lim_{s \to 0^+} \int_s^1 \frac{x}{x^p} \, dx = \lim_{s \to 0^+} \left[ \ln x \right]_s^1 = \lim_{s \to 0^+} (-\ln s) = +\infty.
\]

Hence, \( \int_0^1 \frac{x}{x^p} \, dx \) converges if \( p < 1 \) and diverges if \( p > 1 \).

**Example 2**

Show that \( \int_a^b \frac{dx}{(b-y)^n} \) converges if \( m < 1 \).

**Solution**

The integrand \( f(y) = \frac{1}{(b-y)^n} \) is unbounded at \( y = b \), if \( m > 0 \). This is an improper integral of 2nd kind.

Hence,

\[
\int_a^b \frac{dy}{(b-y)^n} = \lim_{R \to b^-} \int_a^R \frac{dx}{(b-x)^n} = \lim_{R \to b^-} \left[ \frac{(b-x)^{-n+1}}{-n+1} \right]_a^R
\]

If \( n \neq 1 \),

\[
= \lim_{R \to b^-} \left[ \frac{(b-a)^{1-n}}{1-n} \right]_0^R = \frac{(b-a)^{1-n}}{1-n}
\]

If \( n = 1 \),

\[
\lim_{R \to b^-} \int_a^R \frac{dx}{b-x} = \lim_{R \to b^-} \left[ -\ln (b-x) \right]_a^R = \ln (b-a) - \lim_{R \to b^-} \ln (b-x)
\]

Hence, \( \int_a^b \frac{dy}{(b-y)^n} \) converges if \( m < 1 \).
Example 8: Examine the convergence of the following integrals:

\[ \int_{-1}^{1} \frac{1}{x} \, dx \quad \text{and} \quad \int_{0}^{\infty} \frac{1}{x^2} \, dx \]

**Soln.**

For the integral \( \int_{-1}^{1} \frac{1}{x} \, dx \), the integrand \( f(x) = \frac{1}{x} \) is unbounded at \( x = 0 \in (-1, 1) \).

This is an improper integral of 2nd kind. Write,

\[ \int_{-1}^{1} \frac{1}{x} \, dx = \int_{-1}^{0} \frac{1}{x} \, dx + \int_{0}^{1} \frac{1}{x} \, dx. \]

Both integrals converge if \( \int_{-1}^{0} \frac{1}{x} \, dx \) and \( \int_{0}^{1} \frac{1}{x} \, dx \) both converge.

Now,

\[ \int_{-1}^{0} \frac{1}{x} \, dx = \lim_{R \to 0^-} \int_{-1}^{R} \frac{1}{x} \, dx = \lim_{R \to 0^-} \left[ \ln |x| \right]_{-1}^{R} = \lim_{R \to 0^-} \ln R = -\infty \]

and,

\[ \int_{0}^{1} \frac{1}{x} \, dx = \lim_{S \to 0^+} \int_{S}^{1} \frac{1}{x} \, dx = \lim_{S \to 0^+} \left[ \ln |x| \right]_{S}^{1} = \lim_{S \to 0^+} (-\ln S) = +\infty \]

As \( \int_{-1}^{0} \frac{1}{x} \, dx \) and \( \int_{0}^{1} \frac{1}{x} \, dx \) both diverge, so \( \int_{-1}^{1} \frac{1}{x} \, dx \) diverges.

For the integral \( \int_{0}^{\infty} \frac{1}{x^2} \, dx \),

\[ f(x) = \frac{1}{2x - x^2} = \frac{1}{x(2-x)} \]

is unbounded at \( x = 0 \) and \( x = 2 \) on \([0, 2] \).

Hence,

\[ \int_{0}^{2} \frac{1}{2x - x^2} \, dx = \int_{0}^{1} \frac{du}{x(2-x)} + \int_{1}^{2} \frac{du}{x(2-x)} \]

\[ = \lim_{s \to 0^+} \int_{s}^{1} \frac{du}{x(2-x)} + \lim_{R \to 2^-} \int_{1}^{R} \frac{du}{x(2-x)} \]

\[ = \frac{1}{2} \lim_{s \to 0^+} \left[ \log \frac{x}{2-x} \right]_{s}^{1} + \frac{1}{2} \log \frac{2}{2-s} + \left[ \log \frac{R}{2-R} \right]_{1}^{2} \]

\[ = -\frac{1}{2} \lim_{s \to 0^+} \log \left( \frac{s}{2-s} \right) + \frac{1}{2} \log \left( \frac{R}{2-R} \right) \]

\[ = (-\infty) + (+\infty) = \infty \]
Comparison Test: If $0 \leq f(x) \leq g(x)$ for all $x \in [a, R]$, $a < R < \infty$ and $f(x)$ and $g(x)$ are Riemann integrable on $[a, R]$, then

\begin{align*}
\int_a^R g(x) \, dx & \text{ converges } \Rightarrow \int_a^R f(x) \, dx \text{ converges.} \\
\int_a^R f(x) \, dx & \text{ diverges } \Rightarrow \int_a^R g(x) \, dx \text{ diverges.}
\end{align*}

Example: Does \( \int_0^\infty \frac{1}{x^{n+1/2}} \, dx \) converge?

\text{Sol.} \quad \text{Not that } 0 < \frac{1}{x^{n+1/2}} < \frac{1}{x^n} \quad \forall \ n > 1

Since \( \int_0^\infty \frac{1}{x^n} \, dx \text{ converges,} \) 
Hence \( \int_0^\infty \frac{1}{x^{n+1/2}} \, dx \text{ converges, by Comparison test.} \)

Example 2: Does \( \int_0^\infty \frac{1}{x^{n+1/2}} \, dx \) converge?

\text{Sol.} \quad \text{Notice that } 0 < \frac{1}{x^{n+1/2}} < \frac{1}{x^n} \quad \forall \ n > 1

Now, \( \int_0^\infty \frac{1}{x^n} \, dx \text{ converges, by p-integral.} \) 
By Comparison Test, \( \int_0^\infty \frac{1}{x^{n+1/2}} \, dx \text{ converges.} \)

Example 3: Does \( \int_0^\infty \frac{x}{(1+x^3)^2} \, dx \) converge?

\text{Sol.} \quad 0 < \frac{x}{(1+x^3)^2} < \frac{1}{x^2} \quad \forall \ n > 1

Now, \( \int_0^\infty \frac{1}{x^2} \, dx \text{ diverges, by p-integral.} \) 
By Comparison Test \( \int_0^\infty \frac{x}{(1+x^3)^2} \, dx \text{ also converges.} \)
Limit Comparison Test: If \( f(x) \) and \( g(x) \) are two positive-valued functions for \( x \in [a, b] \) such that \( \lim_{x \to a^+} \frac{f(x)}{g(x)} = L = 0 \) or \( \lim_{x \to b^-} \frac{f(x)}{g(x)} = L \) and \( \lim_{x \to a^+} f(x) = \lim_{x \to b^-} g(x) = 0 \), then the integral \( \int_a^b f(x) \, dx \) and \( \int_a^b g(x) \, dx \) both converge or diverge simultaneously.

Example: Does \( \int_0^\infty \frac{x}{(1+x)^n} \, dx \) converge?

Solution:
Note that,
\[
\int_0^\infty \frac{x}{(1+x)^n} \, dx = \int_0^1 \frac{x}{(1+x)^n} \, dx + \int_1^\infty \frac{x}{(1+x)^n} \, dx.
\]

Here, \( \int_0^1 \frac{x}{(1+x)^n} \, dx \) is a proper integral on Riemann integrals and hence it is finite. To examine the convergence of \( \int_1^\infty \frac{x}{(1+x)^n} \, dx \), let \( f(x) = \frac{x}{(1+x)^n} \) and \( g(x) = \frac{1}{x^n} \), \( x > 1 \).

Note that, \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{x^n} = 0 \).

As, \( \int_1^\infty g(x) \, dx = \int_1^\infty \frac{1}{x^n} \, dx \) is a convergent improper integral, by limit comparison test, \( \int_1^\infty f(x) \, dx \) converges.

Hence, \( \int_0^\infty \frac{x}{(1+x)^n} \, dx \) converges.

Third Kind: Third kind improper integrals are a combination of the difficulties of the first kind and second kind improper integrals. The improper integrals of third kind are first written as a sum of the two integrals, one of the first kind and another of the second kind. If both, the first and second kind improper integrals converge then given the third kind improper integral converges.
Example 1: Show that \( \int_{-\infty}^{\infty} x^p \, dx \) diverges for any value of \( p \).

Solution: The interval of integration is infinite, and the integrand \( f(x) = x^p \) is unbounded at \( x = 0 \) if \( p < 0 \). This is an improper integral of third kind.

Note,
\[
\int_{-\infty}^{\infty} x^p \, dx = \lim_{s \to 0^+} \int_{s}^{R} x^p \, dx + \lim_{R \to \infty} \int_{R}^{\infty} x^p \, dx
\]
\[
= \lim_{s \to 0^+} \left[ \frac{x^{p+1}}{p+1} \right]_{s}^{R} + \lim_{R \to \infty} \left[ \frac{x^{p+1}}{p+1} \right]_{R}^{\infty}, \quad \text{if } p \neq -1
\]
\[
= \lim_{s \to 0^+} \frac{1 - s^{p+1}}{p+1} + \lim_{R \to \infty} \frac{R^{p+1} - 1}{p+1}, \quad \text{if } p + 1 < 0
\]
\[
= \lim_{s \to 0^+} \frac{R^{p+1} - 1}{s^{p+1}}\quad \text{if } p + 1 > 0
\]
\[
= \begin{cases} 
+\infty & \text{if } p \neq -1 \\
\infty & \text{if } p + 1 < 0 \\
0 & \text{if } p + 1 > 0
\end{cases}
\]

Hence, for \( p + 1 < 0 \), \( \lim_{R \to \infty} R^{p+1} = 0 \) but \( \lim_{s \to 0^+} s^{p+1} = \infty \).

Hence, \( \int_{0}^{\infty} x^p \, dx \) diverges.

Hence, \( \int_{0}^{\infty} x^p \, dx \) diverges for all \( p \). 

Hence, \( \int_{0}^{\infty} x^p \, dx \) diverges.

Example 2: Examine the convergence of \( \int_{0}^{\infty} \frac{1}{x^{p+\frac{1}{2}}} \, dx \).

Solution: The interval of integration is infinite, and the integrand \( f(x) = \frac{1}{x^{p+\frac{1}{2}}} \) is unbounded at \( x = 0 \). This is an improper integral of third kind.

Note, \( \int_{0}^{\infty} \frac{1}{x^{p+\frac{1}{2}}} \, dx = \int_{0}^{1} \frac{1}{x^{p+\frac{1}{2}}} \, dx + \int_{1}^{\infty} \frac{1}{x^{p+\frac{1}{2}}} \, dx \)

Note that, \( 0 < \frac{1}{x^{p+\frac{1}{2}}} < \frac{1}{x} \) \( \forall \ x \in (0,1] \)

Hence, \( \int_{0}^{1} \frac{1}{x^{p+\frac{1}{2}}} \, dx \) converges, by comparison test, \( \int_{1}^{\infty} \frac{1}{x^{p+\frac{1}{2}}} \, dx \) converges.

Again, \( 0 < \frac{1}{x^{p+\frac{1}{2}}} < \frac{1}{x^2} \) \( \forall \ x > 1 \)

As \( \int_{1}^{\infty} \frac{1}{x^2} \, dx \) converges, by comparison test, \( \int_{0}^{1} \frac{1}{x^{p+\frac{1}{2}}} \, dx \) converges.
Gramma Integral:

Result: Show that the Gamma integral $\int e^{-x} x^{n-1} dx$ converges iff $n > 0$.

Proof: The interval of integration is infinite and the integrand $e^{-x} x^{n-1}$ is unbounded at $x = 0$ if $n < 1$. This is an improper integral of third kind.

Write, $\int e^{-x} x^{n-1} dx = \int e^{-x} x^{n-1} dx + \int e^{-x} x^{n-1} dx$

**Convergence at $x = 0$ for $n < 1$**.

Let $f(x) = e^{-x} x^{n-1}$

$g(x) = \frac{1}{x^{1-n}}, x \in (0, 1]$

**N.T.** $\lim_{x \to 0^+} \frac{g(x)}{f(x)} = \lim_{x \to 0^+} e^{-x} = 1$

As, $\int g(x) dx = \int_{0}^{1} dx$ converges iff $1 - n < 1$, i.e., $n > 0$.

by limit comparison test $\int f(x) dx = \int e^{-x} x^{n-1} dx$ converges iff $n > 0$.

**Convergence at $x = \infty$**.

**N.T.** $\lim_{n \to \infty} \frac{x^{1-n}}{x^0} = \lim_{n \to \infty} e^{-x} = 0 \forall n$

For large $x$, $e^{-x}$ increases more rapidly than $x^{1-n}$.

Then $\exists b, \in \mathbb{R}$ where $e^{-x} < \frac{1}{x^{1-n}}$ for $x > b$.

$\Rightarrow e^{-x} x^{n-1} < \frac{1}{x^{1-n}}$

Since, $\int_{x}^{\infty} \frac{1}{x} dx$ converges, by comparison test $\int_{x}^{\infty} e^{-x} x^{n-1} dx$ converges $\forall n$.

Hence, $\int e^{-x} x^{n-1} dx = \int_{0}^{b} e^{-x} x^{n-1} dx + \int_{b}^{\infty} e^{-x} x^{n-1} dx$

[Proper Integral]

converges for all $n$.

Hence, $\int e^{-x} x^{n-1} dx$ converges if $n > 0$ and $\int e^{-x} x^{n-1} dx$ converges for all $n$.

$\Rightarrow \int e^{-x} x^{n-1} dx$ converges if $n > 0$.
Beta Integral:

Result: Show that the Beta integral 
\[ \int_0^1 x^{m-1} (1-x)^{n-1} \, dx \]
converges if \( m > 0, n > 0 \).

Proof: If \( m > 1, n > 1 \), then the integral is a proper integral (i.e., finite). The integrand \( x^{m-1} (1-x)^{n-1} \) is unbounded at \( x = 0 \) if \( m < 1 \) and is unbounded at \( x = 1 \) if \( m < 1 \).

Let \( m > 1 \) and \( n > 1 \).

Write
\[
\int_0^1 x^{m-1} (1-x)^{n-1} \, dx = \int_0^{1/2} x^{m-1} (1-x)^{n-1} \, dx + \int_{1/2}^1 x^{m-1} (1-x)^{n-1} \, dx.
\]

Note, we check the convergence of \( \int_{1/2}^1 x^{m-1} (1-x)^{n-1} \, dx \) at \( x = 0 \) and the convergence of \( \int_0^{1/2} x^{m-1} (1-x)^{n-1} \, dx \) at \( x = 1 \).

Convergence at \( x = 0 \):

Let \( f(x) = \frac{(1-x)^{n-1}}{x^{1-m}} \) and \( g(x) = \frac{1}{x^{1-m}} \), \( x \in (0, 1/2) \).

N.T. \( \lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} (1-x)^{n-1} = 1 \).

Since, \( \int_{1/2}^1 g(x) \, dx = \int_{1/2}^1 \frac{1}{x^{1-m}} \, dx \) converges iff \( 1-m < 1 \), i.e., \( m > 0 \).

By comparison test, \( \int_0^{1/2} f(x) \, dx = \int_0^{1/2} x^{m-1} (1-x)^{n-1} \, dx \) converges iff \( m > 0 \).

Convergence at \( x = 1 \):

Here, \( f(x) = x^{m-1} \) and \( g(x) = \frac{1}{(1-x)^{1-n}} \), \( x \in (1/2, 1) \).

\[ \lim_{x \to 1^-} \frac{f(x)}{g(x)} = 1 \neq 0. \]

Since, \( \int_{1/2}^1 g(x) \, dx = \int_{1/2}^1 \frac{1}{(1-x)^{1-n}} \, dx \) converges iff \( 1-n < 1 \), i.e., \( m > 0 \).

By limit comparison test, \( \int_{1/2}^1 f(x) \, dx = \int_{1/2}^1 x^{m-1} (1-x)^{n-1} \, dx \) converges iff \( m > 0 \).

Hence, \( \int_0^1 x^{m-1} (1-x)^{n-1} \, dx \) converges iff \( m > 0 \) and \( n > 0 \).
Remark: \[ \int x^{m-1} (1-x)^{n-1} \, dx = \beta (m, n) \quad \text{if} \quad m > 0, \; n > 0 \]

It can be shown that \[ \beta (m, n) = \frac{\Gamma (m) \Gamma (n)}{\Gamma (m+n)} \]

Further Examples:

- Does \[ \int_0^{\pi/2} \frac{\sin x}{x^p} \, dx \] converge?

- Show that \[ \int_0^{\pi/2} \frac{\sin^m x}{x^n} \, dx \] converges if \( n < m + 1 \).

Solution:

Let \( f(x) = \frac{\sin x}{x^p} \), \( g(x) = \frac{1}{x^{p+1}} \), \( x \in (0, \pi/2] \)

\[ \lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} \frac{\sin x}{x^p} = 1. \]

\[ \int_0^{\pi/2} \frac{1}{x^p} \, dx \text{ converges if } -1 < p < 2. \]

Hence, \[ \int_0^{\pi/2} \frac{\sin^m x}{x^n} \, dx \text{ converges if } p < 2. \]

Absolute Convergence:

- Definition: The improper integral \( \int_a^\infty f(x) \, dx \) (1st kind) or \( \int_0^a f(x) \, dx \) (2nd kind) is said to be absolutely convergent if \( \int_a^\infty |f(x)| \, dx \) or \( \int_0^a |f(x)| \, dx \) converges.

Example: (a) Show that \[ \int_0^{\pi/2} \frac{\sin x}{x^p} \, dx \] converges absolutely if \( p > 1 \).

Solution:

\[ \left| \frac{\sin x}{x^p} \right| \leq \frac{1}{x^p}, \quad x > 1. \]

As \( \int_1^\infty \frac{1}{x^p} \, dx \text{ converges if } p > 1 \), then by comparison test, \( \int_0^{\pi/2} \frac{\sin x}{x^p} \, dx \) converges if \( p > 1 \).
\textbf{Duplication Formula:}

\textbf{Result:}\quad \Gamma(2n) = \frac{2^{2n-1} \cdot \Gamma(n) \cdot \Gamma\left(n + \frac{1}{2}\right)}{\sqrt{\pi}}

\textbf{Proof:}\quad 
\beta(n, m) = \int_0^1 x^{n-1} (1-x)^{m-1} \, dx

\begin{align*}
\beta(n, m) &= \int_0^{1/2} \{m(1-x)^{n-1}\} \, dx \\ &= 2 \int_0^{1/2} \{m(1-x)^{n-1}\} \, dx \\
&= \int_\frac{2a}{2}^\frac{2a}{2} f(x) \, dx = 2 \int_0^\frac{2a}{2} f(x) \, dx, \text{ if } (2a-x) = f(x), \text{ Here, } f(x) = \frac{1}{2}
\end{align*}

\[ \therefore \beta(n, m) = \frac{4 \pm \sqrt{16 - 4 \cdot 4} \cdot 2}{2 \cdot 4} \]

\[ \therefore \quad \beta(n, m) = \frac{1 - \sqrt{1 - \frac{1}{2}}}{2}, \quad \text{as } 0 < \kappa < \frac{1}{2} \]

\begin{align*}
\text{Let } x &= 4 (1 - 2\kappa) \\
\Rightarrow dx &= \frac{dx}{4\sqrt{1 - \frac{1}{2}}} \\
\text{(**) reduces to} \\
&= 2 \int_0^1 \left(\frac{x}{4}\right)^{n-1} (1-2\kappa)^{1/2-1} \, dx \\
&= \frac{1}{2^{2n-1}} \int_0^1 x^{n-1} (1-x)^{1/2-1} \, dx \\
&= \frac{1}{2^{2n-1}} \beta\left(n, \frac{1}{2}\right) \\
\Rightarrow \quad \frac{\Gamma(n) \cdot \Gamma(m)}{\Gamma(2n)} = \frac{1}{2^{2n-1}} \cdot \frac{\Gamma(n) \cdot \Gamma\left(n + \frac{1}{2}\right)}{\Gamma(n+1) \cdot \Gamma\left(n + \frac{1}{2}\right)}
\end{align*}

\[ \therefore \quad \Gamma(2n) = \frac{2^{2n-1} \cdot \Gamma(n) \cdot \Gamma\left(n + \frac{1}{2}\right)}{\sqrt{\pi}} \quad \text{[Proved]} \]

\[ \therefore \quad \square \quad \square \quad \square \]
Definition of limit:

\[ \lim_{x \to a} f(x) = l \text{ if for every } \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow |f(x) - l| < \varepsilon. \]

Theorem: \( \lim_{x \to a} f(x) = l \) if for every sequence \( \{a_n\} \) converging to \( a \), i.e., \( \lim_{n \to \infty} a_n = a \), then \( \lim_{n \to \infty} f(a_n) = l \).

Remark: Two sequences \( \{a_n\} \) and \( \{b_n\} \) are said to be equivalent if \( \lim_{n \to \infty} \frac{a_n}{b_n} = l \neq 0 \).

Similarly, two functions \( f(x) \) and \( g(x) \) are said to be equivalent if for large \( x \), \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = l \neq 0 \).

Examples:

(i) \( \frac{n}{n+1} \sim \frac{1}{1} \)
(ii) \( \sin \frac{1}{n} \sim \frac{1}{n} \)
(iii) \( a_{n+1} = \frac{1}{n+1} \)
(iv) \( \frac{1}{\sqrt{n+1}} \)
(v) \( \frac{n^2 - n}{n+1} \sim \frac{n^2}{n+1} \)
(vi) \( (n+1)^2 - n^2 \sim n^2 \)

Note: \( \lim_{x \to a} f(x) = l \iff \text{for given } \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow |f(x) - l| < \varepsilon. \)

In other words, if \( f(x) \) is a function and \( a \) is a real number, then \( \lim_{x \to a} f(x) = l \) if for every \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that for every \( x \) in the neighborhood of \( a \), \( f(x) \) is in the neighborhood of \( l \).

Examples:

If \( f(x) = \frac{x^2 - a^2}{x-a}, x \neq a \), then show that \( \lim_{x \to a} f(x) = 2a \).

\[ |f(x) - 2a| < \varepsilon \]
\[ \Rightarrow |\frac{x^2 - a^2 - 2a(x-a)}{x-a}| < \varepsilon \]
\[ \Rightarrow |x-a| < \varepsilon \]

Now if we choose a number \( \delta \) such that \( 0 \leq \delta \leq \varepsilon \), then
\[ |f(x) - 2a| < \varepsilon \text{ whenever } 0 < |x-a| < \delta. \]

Thus, \( \lim_{x \to a} f(x) = 2a. \)
Right-hand limit: A real number \( l \) is said to be a limit of a function \( f \) as \( x \) tends to \( a \) from above (from the right) iff for every given \( \varepsilon > 0 \), \( \exists \) a \( \delta > 0 \) such that:
\[
|f(x) - l| < \varepsilon \quad \text{whenever} \quad a < x < a + \delta.
\]
We write
\[
\lim_{x \to a^+} f(x) = l = f(a^+).
\]
Here \( a < x < a + \delta \Rightarrow l - \varepsilon < f(x) < l + \varepsilon \).

Left-hand limit: A real number \( l \) is said to be a limit of a function \( f \) as \( x \) tends to \( a \) from below (from the left) iff for every given \( \varepsilon > 0 \), \( \exists \) a \( \delta > 0 \) such that:
\[
|f(x) - l| < \varepsilon \quad \text{whenever} \quad a - \delta < x < a.
\]
We write
\[
\lim_{x \to a^-} f(x) = l = f(a^-).
\]
Here \( a - \delta < x < a \Rightarrow l - \varepsilon < f(x) < l + \varepsilon \).

Question: A function \( f(x) \) is said to have a limit \( l \) iff both the RHL and LHL exist and are equal to \( l \).

Solution:
By definition of limit,
\[
\lim_{x \to a} f(x) = l \iff \text{for a given } \varepsilon > 0, \exists \text{ a } \delta > 0 \text{ such that}
\]
\[
|f(x) - l| < \varepsilon \quad \text{whenever} \quad a - \delta < x < a + \delta.
\]
Hence the result.

Example: If a function \( f \) is defined as \( f(x) = [1 - x] \varepsilon x \in \mathbb{R} \).
Then \( \lim_{x \to 0} f(x) ≠ \lim_{x \to 0^-} f(x) \).

Solution:
\[
\forall \alpha \in (0, 1) \Rightarrow 0 < x < 1 \Rightarrow 0 < 1 - x < 1 \Rightarrow f(x) = 0 \quad \forall \alpha \in (0, 1)
\]
So for any \( \varepsilon > 0 \) and for any \( \delta > 0 \), we have
\[
|f(x) - 0| < \varepsilon \quad \text{whenever} \quad x \in (0, \delta).
\]
\[
\lim_{x \to 0^+} f(x) = 0.
\]
\[
\forall \alpha \in (-1, 0), \Rightarrow -1 < x < 0 \Rightarrow 1 < 1 - x < 2 \Rightarrow f(x) = 1 \quad \forall \alpha \in (-1, 0)
\]
So for any \( \varepsilon > 0 \) and for \( \delta > 0 \), we have
\[
|f(x) - 1| < \varepsilon \quad \text{whenever} \quad x \in (-\delta, 0)
\]
\[
\lim_{x \to 0^-} f(x) = 1.
\]
\[
\therefore f(0^+) ≠ f(0^-).
\]
Rule I: To evaluate \( \lim_{x \to a} \frac{f(x)}{g(x)} \), divide the numerators and denominator by the highest power of \( x \) involved in \( f(x) \) and \( g(x) \).

Examples:

(i) \( \lim_{x \to a} \frac{\sqrt{x+1} - \sqrt{x}}{x-a} \)

\( \text{Sol.} \quad \lim_{x \to a} \frac{\sqrt{x+1} - \sqrt{x}}{x-a} \)

\( = \lim_{x \to a} \frac{1}{\sqrt{x}} \)

\( = 0. \)

Now, \( \sqrt{x+1} - \sqrt{x} \approx \frac{1}{\sqrt{x}} \)

(ii) \( \lim_{x \to a} \frac{\sqrt{x+\sqrt{2+x}} - \sqrt{x}}{x-a} \)

\( \text{Sol.} \quad \sqrt{x+\sqrt{2+x}} - \sqrt{x} \)

\( = \frac{x+\sqrt{2+x} - x}{\sqrt{x+\sqrt{2+x}} + \sqrt{x}} \)

\( = \frac{\sqrt{1+\frac{1}{\sqrt{x}}}}{\sqrt{1+\frac{1}{\sqrt{x}}}} \)

\( \Rightarrow \lim_{x \to \infty} \frac{\sqrt{1+0}}{\sqrt{1+0}} + 1 = \frac{1}{2} \quad \text{as} \ x \to \infty. \)

(iii) \( \lim_{x \to \infty} \frac{\alpha^3}{\sqrt{x^2 + \sqrt{1+x^4} - \alpha x^2}} \)

\( \text{Sol.} \quad \lim_{x \to \infty} \frac{\alpha^3}{\sqrt{x^2 + \sqrt{1+x^4} + \sqrt{2}}} \)

\( = \lim_{x \to \infty} \frac{\alpha^3}{\sqrt{x^2 + \sqrt{1+x^4} + \alpha \sqrt{2}}} \)

\( = \lim_{x \to \infty} \frac{\alpha^3}{\sqrt{1+\frac{1}{\alpha^4}} + \alpha \sqrt{2}} \)

\( = \lim_{x \to \infty} \frac{1}{\sqrt{1+\frac{1}{\alpha^4}} + \sqrt{2}} \)

\( = \frac{1}{\sqrt{1+0} + \sqrt{2}} \)

\( = \frac{1}{\sqrt{2}}. \)
2. (i) \[ \lim_{x \to \infty} \frac{a x^p + b x^{p-2} + c}{d x^p + a x^{p-2} + b}, \quad p > 0 \]

\[ \text{Sol.} \]
\[ = \lim_{x \to \infty} \frac{a + \frac{b}{x^2} + \frac{c}{x^p}}{d + \frac{a}{x^2} + \frac{b}{x^p}} \quad , \quad p > 0 \]
\[ = \frac{a}{d} \]

(ii) \[ \lim_{x \to \infty} \frac{x^4 \sin \frac{1}{x} + x^2}{1 + |x|^3} \]

\[ \text{Sol.} \]
\[ = \lim_{x \to -\infty} \frac{x \sin \frac{1}{x} + \frac{1}{x}}{x^3 - 1} \quad \left[ \text{Here, } \lim_{x \to -\infty} \alpha \sin \frac{1}{x} = 1 \right] \]
\[ = \frac{1 + 0}{0 - 1} \]
\[ = -1. \]

3. (i) If \[ \lim_{n \to \infty} \left( a_n - \frac{n + n^2}{n+1} \right) = b, \quad \text{then find the values of } a \text{ and } b. \]

\[ \text{Solution:} \]
\[ b = \lim_{n \to \infty} \left( a_n - \frac{n + n^2}{n+1} \right) \]
\[ = \lim_{n \to \infty} \left( \frac{a_n + a_n^2 - 1 - n^2}{1 + n} \right) \]
\[ = \lim_{n \to \infty} \left( \frac{(a-1)n^2 + (a-1)}{(1+n)} \right) \]

If \( a-1 \neq 0 \) then \[ \lim_{n \to \infty} \frac{(a-1)n^2 + (a-1)}{(1+n)} = +\infty \]

but it is given that the limiting value is \( b \) (finite).

\( a-1 = 0 \Rightarrow a = 1 \), \( b = \lim_{n \to \infty} \frac{n-1}{n+1} = 1 \).

(iii) If \[ \lim_{x \to \infty} \left( \sqrt{x^4 + \alpha^2 + 1} - \alpha x^2 - b \right) = 0, \quad \text{then find } a \text{ and } b. \]

\[ \text{Solution:} \]
\[ 0 = \lim_{x \to \infty} \left( \sqrt{x^4 + \alpha^2 + 1} - \alpha x^2 - b \right) \]
\[ = \lim_{x \to \infty} \left( \frac{(\sqrt{x^4 + \alpha^2 + 1} - (\alpha x^2 + b))^2}{\sqrt{x^4 + \alpha^2 + 1} + \alpha x^2 + b} \right) \]
\[ = \lim_{x \to \infty} \left( \frac{(1-\alpha^2)x^4 - (1+2\alpha b)x^2 + (1-b^2)}{\sqrt{x^4 + \alpha^2 + 1} + \alpha x^2 + b} \right) \]

Hence \( 1-\alpha^2 = 0 \Rightarrow \alpha = \pm 1 \).

If \( \alpha = 1 \), then \( -(1+2b) = 0 \)
\[ \Rightarrow b = -1/2 \]

If \( \alpha = -1 \), then \( (1-2b) = 0 \)
\[ \Rightarrow b = 1/2. \]
Rule II: \[ \lim_{x \to 0} \frac{x^n}{e^x} = 0, n > 0. \]

since, \( e^x \) increases more rapidly than \( x^n, n > 0 \), for large \( x \).

Note: \[ \lim_{x \to 0} \frac{e^x}{x^n} = \infty, n > 0. \]

Attention: \[ \lim_{y \to \infty} \frac{\log(y)}{y} = 0; \text{ where } y = e^x. \]

Example: 1.

(i) \[ \lim_{x \to 0} (\log x - x) \]

\[ \log x - x = \log x \left( \frac{x}{x^2} \right) \to -\infty \text{ as } n \to \infty. \]

(ii) \[ \lim_{x \to \infty} \frac{\log x}{x} \]

\[ \log x = \log x \text{ for large } x. \]

\[ \lim_{x \to \infty} \frac{\log x}{x} = \lim_{x \to \infty} \frac{\log x}{x} = 0. \]

Rule III: \[ \lim_{n \to \infty} n^0 = 0, 1 < n < 1. \]

If \( n > 1, \lim_{n \to 0} n = +\infty. \]

Example: 1. \( \alpha \in [0,1] \), then \( f(x) = \lim_{m \to 0} \lim_{n \to 0} \left( 1 + \cos^2 \left( \frac{\pi}{n} \right) \right) \)

is (i) \( 1 \) on 2 according as \( \alpha \) is rational or irrational.

(ii) \( 1 \) on 2 according as \( \alpha \) is irrational or rational.

(iii) \( 2 \) for all \( x \).

(iv) \( 2 \) for all \( x \).

Solution: If \( \alpha \) is rational, \( \alpha = \frac{p}{q} \); \( p \) and \( q \) being integers and \( \ln \alpha = \ln \frac{p}{q} \) is an integer, for sufficiently larger.

\[ \text{Now, } \cos^2 \left( \frac{\pi}{n} \right) = \cos^2 \left( \frac{\pi}{n} \right) = 1. \]

\[ L = \lim_{m \to 0} \lim_{n \to 0} \left( 1 + \cos^2 \left( \frac{\pi}{n} \right) \right) \]

\[ = 1 + 1 \]

\[ = 2. \]

If \( \alpha \) is irrational, \( \ln \alpha \neq \text{ an integer } \).

\[ \Rightarrow \alpha = \cos^2 \left( \frac{\pi}{n} \right) < 1. \]

\[ \Rightarrow \cos^2 \left( \frac{\pi}{n} \right) = \cos^2 \left( \frac{\pi}{n} \right) \to 0 \text{ as } m \to \infty. \]

\[ L = 1 + 0 = 1. \]

And: (ii)
Example 2.
(i) \[ \lim_{n \to \infty} \frac{2^n + 3^n}{3^n - 2^n} \]
Solution:
\[ \lim_{n \to \infty} \frac{2^n + 3^n}{3^n - 2^n} = \lim_{n \to \infty} \frac{\left(\frac{2}{3}\right)^n + 1}{1 - \left(\frac{2}{3}\right)^n} \]
\[ = 3 \left( \frac{0 + 1}{1 + 0} \right) = 3 \]

(ii) \[ \lim_{n \to \infty} \frac{\alpha^{2n} - 1}{\alpha^{2n} + 1} \to \alpha \in \mathbb{R} \]
Solution:
\[ L = \lim_{n \to \infty} \frac{\alpha^{2n} - 1}{\alpha^{2n} + 1}, \alpha \in \mathbb{R} \]
If \[ |\alpha| < 1, \alpha^{2n} \to 0 \text{ as } n \to \infty \]
\[ A_L = \frac{0 - 1}{1 + 0} = -1 \]
If \[ |\alpha| > 1, \text{ then } \alpha^2 > 1, \frac{1}{\alpha^{2n}} \to 0 \text{ as } n \to \infty \]
\[ L = \lim_{n \to \infty} \left( \frac{1 - \frac{1}{\alpha^{2n}}}{1 + \frac{1}{\alpha^{2n}}} \right) = \frac{1 - 0}{1 + 0} = 1 \]
If \[ |\alpha| = 1, \alpha^{2n} = 1 \]
\[ L = \frac{1 - 1}{1 + 1} = 0 \]

(iii) \[ \lim_{n \to \infty} \frac{4^n + 3^n}{4^n - 3^n} \]
Solution:
\[ \lim_{n \to \infty} \frac{4^n + 3^n}{4^n - 3^n} = \lim_{n \to \infty} \frac{1 + \left(\frac{3}{4}\right)^n}{1 - \left(\frac{3}{4}\right)^n} \]
\[ = \frac{1 + 0}{1 - 0} = 1 \]
Rule IV:

(i) $\lim_{\alpha \to 0} \log \left(\frac{1 + \alpha}{\alpha}\right) = 1$

(ii) $\lim_{\alpha \to 0} \log \left(\frac{1 + \frac{1}{\alpha}}{\alpha}\right) = 1$

(iii) $\lim_{\alpha \to 0} \log \left(\frac{1 + \frac{a}{\alpha} + \frac{b}{\alpha^2}}{\alpha}\right) = a$

Proof:

$\ln L = \lim_{\alpha \to 0} \ln \left(1 + \frac{a}{\alpha} + \frac{b}{\alpha^2}\right)$

$= \lim_{\alpha \to 0} \left[ a + \frac{b}{\alpha} + \cdots \right]$

$= a$

Examples:

1. If $\alpha, \beta$ are the roots of $ax^2 + bx + c = 0$, then

$\lim_{\alpha \to \infty} \frac{1 + \alpha \alpha^2 + b \alpha + c}{1 - \alpha}$ is

(i) $a(\alpha - \beta)$

Solution:

$\lim_{\alpha \to \infty} \frac{1 + \alpha \alpha^2 + b \alpha + c}{1 - \alpha}$

$= \lim_{\alpha \to \infty} \frac{1 + \alpha \alpha^2 + b \alpha + c}{\alpha - \alpha}$

$= \lim_{\alpha \to \infty} \left[ 1 + a(\frac{\alpha^2}{\alpha} + \frac{b}{\alpha^2} \alpha + \frac{c}{\alpha}) \right]$

$= \lim_{\alpha \to \infty} \left[ 1 + a(\alpha - \alpha)(\alpha - \beta) \right]$

$= \lim_{\alpha \to \infty} \left[ \frac{1 + (1 + \alpha)}{1 - \alpha} \right]$

$= a(\alpha - \beta)$
2. \(\lim_{x \to 0} \left( \frac{x+e}{x+1} \right)^{x+4}\)

\[\text{Sol.} \quad \lim_{x \to 0} \left( \frac{x+e}{x+1} \right)^{x+4} = \lim_{x \to 0} \left( \frac{1+ \frac{e}{x}}{1+ \frac{1}{x}} \right)^{x+4} \cdot \left( \frac{1+ \frac{e}{x}}{1+ \frac{1}{x}} \right)^{4} = e^{\frac{e}{2}} \cdot \left( 1+0 \right)^{4} = e^{\frac{e}{2}}.\]

(iii) \(\lim_{x \to 0} \left( \frac{x-1}{3x+2} \right)^{\frac{x+1}{3}}\)

\[\text{Sol.} \quad \lim_{x \to 0} \left( \frac{x-1}{3x+2} \right)^{\frac{x+1}{3}} = \lim_{x \to 0} \left( \frac{3- \frac{1}{x}}{3 + \frac{2}{x}} \right)^{\frac{x+1}{3}} = \left( \frac{3- \frac{1}{e}}{3 + \frac{2}{e}} \right)^{\frac{1}{3}} = e^{-\frac{2}{3}}.\]

(iii) \(\lim_{x \to 0} \left( \frac{x+5x^2+3}{x^2+5x+2} \right)^{x}\)

\[\text{Sol.} \quad \lim_{x \to 0} \left( \frac{x+5x^2+3}{x^2+5x+2} \right)^{x} = \lim_{x \to 0} \left( \frac{1 + \frac{5x^2}{x^2}}{1 + \frac{5x}{x^2}} \right)^{x} = \left( \frac{1 + \frac{5x^2}{1+3x^2}}{1+0} \right)^{x} = e^{\frac{e^5}{e^3}} = e^2.\]
Rule V: \( \lim_{x \to 0} \text{using expansion:} \)

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \quad [\text{Use these expansions only when variable in the expansion tends to zero, i.e. } x \to 0]
\]

Example:
1. Find the values of \(a\), \(b\), \(c\) such that

\[
\lim_{x \to 0} \frac{x^3}{a(1 + ax^2) - bx^2 + \ldots} = 1.
\]

Sol.

\[
1 = \lim_{x \to 0} \frac{x^3}{a(1 + ax^2) - bx^2 + \ldots} = \frac{x^3}{a(1 + a - b)} + \frac{\alpha (\frac{a^2}{2!} - \frac{b}{3!})}{x^3} + \ldots
\]

Here \(1 + a - b = 0\) and \(-\frac{a}{2!} + \frac{b}{3!} = 1\)
\(\Rightarrow\) \(a - b = -1\), \(\Rightarrow\) \(b - 3a = 8\)
\(a = -5/2, b = -3/2\).

(b) \(\lim_{x \to 0} \frac{ax e^{-2 \log(x + 1)} + 2e}{x^2} = 2\).

Sol.

\[
2 = \lim_{x \to 0} \frac{ax(1 + \frac{a^2}{2!} + \ldots) - bx + \frac{a^3}{3!} + \ldots}{x^2}
\]

\[
= \lim_{x \to 0} \frac{2 - \frac{a^2}{3!} + \ldots}{x^2}
\]

\(a + \frac{b}{2} - c = 0, a + \frac{b}{2} - c = 0, \frac{a}{2} - \frac{b}{3} + \frac{c}{2} = 2\)
\(\Rightarrow a = 3, b = 12, c = 9\).

(c) \(\lim_{x \to 0} \frac{ax e^{0} + bx e^{-5}}{x^2} = a, \text{ finite} \).

Sol.

\[
c = \lim_{x \to 0} \frac{a \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) + bx \left(1 - \frac{x^2}{3!} + \ldots\right)}{x^2}
\]

\[
= \lim_{x \to 0} \frac{(a - 5) + (-\frac{a}{2}) + \frac{b}{2} - \frac{a}{3} + \frac{b}{3}) x^2 + \ldots}{x^2}
\]

\(a - 5 = 0, a + \frac{b}{2} - b = 0\)
\(\Rightarrow a = 5, b = \frac{5}{2}, c = 12, c = \frac{5}{24}, 5 = c\)
2. (i) \[ \lim_{x \to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5} \]
\[ = \lim_{x \to 0} \frac{\alpha^5}{5!} + \frac{\alpha^3}{360} \]
\[ = \frac{1}{5!} \]
\[ = \frac{1}{120} \]

(ii) \[ \lim_{x \to 0} \left( \sin \frac{10x}{x} \right)^{1/x^2} \]
\[ = \left( \frac{10x}{x} \right)^{1/x^2} \]
\[ = (10)^{1/x^2} \]
\[ \to e^{1/3} \text{ as } x \to 0. \]

(iii) \[ \lim_{x \to a} \frac{\sin x}{\sin x} \]
\[ = \frac{\log_{t_{0x}} - \log_{t_{0a}}}{(x-a)} \]
\[ = \frac{f(x) - f(a)}{x-a} \]
\[ = f'(a) \]
\[ = \frac{d}{da} \left( \log_{t_{0a}} \right) \]
\[ = \frac{1}{t_{0a}} \cdot \sec^2 a \]
\[ = 2 \cos \sec 2a. \]

(iv) \[ \lim_{x \to 0} \left( \sin \frac{1}{x} + \cos \frac{1}{x} \right)^x \]
\[ = \lim_{x \to 0} \left( \frac{1}{x} + 1 \right) x \]
\[ = e. \]
Rule VI: \[ \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \quad \text{[form } \frac{0}{0} \text{ on } \infty \infty \text{]} \]

\[ = \lim_{x \to a} \frac{f'(x)}{g'(x)} \]

Ex. 1. (i) \[ \lim_{x \to \infty} \frac{\alpha^n}{e^x}, n \in \mathbb{N} \]

Sol. \[ \lim_{x \to \infty} \frac{\alpha^n}{e^x} \]

\[ = \lim_{x \to \infty} \frac{\alpha^{n-1}}{e^x} \]

\[ = \lim_{x \to \infty} \frac{\alpha^{n-1}}{e^x} \]

\[ = 0 \]

(ii) \[ \lim_{x \to 1^+} \frac{\int_2^{x+1} dt}{\sin(x-1)} \]

Sol. \[ \lim_{x \to 1^+} \frac{\int_2^{x+1} dt}{\sin(x-1)} \]

\[ = \frac{1}{\cos(1)} \]

\[ = 2\cos(1) \]

(iii) \[ \int_0^\infty \frac{2t}{x-1} dt \]

(iv) \[ \lim_{x \to 0} \frac{\int_0^x e^{t^2}(e^{t^2}-1)^2 dt}{\sqrt{x}} \]

Sol. \[ \lim_{x \to 0} \frac{\int_0^x e^{t^2}(e^{t^2}-1)^2 dt}{\sqrt{x}} \]

\[ = \left( \lim_{x \to 0} e^{x^2} \right) \cdot \lim_{x \to 0} \left( \frac{e^{-x^2}}{x} \right) \]

\[ = 1 \cdot 1 = 1 \]

\[ = \frac{2}{3} \]
**Theorem:** If \( \lim_{x \to a} f(x) \) exists, then it is unique.

**Non-existance of limit:**

Ex. 1. Prove that \( \lim_{x \to 0} \frac{|x|}{x} \) does not exist.

**Solution:**

\[ f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} \]

For given \( \varepsilon > 0 \)

Now \( |f(x)| = 1 \) when \( 0 < x < 1 \) for any \( \delta_1 > 0 \).

\( f(0+0) = 1 \)

Again \( |f(x)| = 1 \) when \( -1 < x < 0 \) for any \( \delta_2 > 0 \).

Hence \( f(0+0) \neq f(0-0) \)

Thus limit does not exist here.

Ex. 2. Show that \( \lim_{x \to 0} \frac{1}{x} \) does not exist.

Sol. Here \( f(x) = \frac{1}{x} \) is defined in \( \mathbb{R} - \{0\} \) but unbounded in every nod of \( f(x) \).

\( \lim_{x \to 0^+} \frac{1}{x} = +\infty \) and \( \lim_{x \to 0^-} \frac{1}{x} = -\infty \).

Ex. 3. \( \sin \frac{1}{x} \) does not exist.

Sol. Do yourself.

**Theorem:** Let \( f, g, \) and \( h \) be functions on \( D \subset \mathbb{R} \), and \( c \in D \). If \( f \) is bounded on \( N(c) \setminus \{c\} \) and \( \lim_{x \to c} g(x) = 0 \), then \( \lim_{x \to c} (f \cdot g)(x) = 0 \).

**Examples:**

1. Prove that \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \).

Sol. Here \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \) is bounded in some deleted neighborhood of 0, Therefore \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \).

2. Prove that \( \lim_{x \to 0} \frac{\sin x}{x} = 0 \).

Sol. \( \lim_{x \to 0} \frac{\sin x}{x} = 0 \) and \( \frac{1}{x} \) is bounded in some deleted neighborhood of 0.

Hence \( \lim_{x \to 0} \frac{\sin x}{x} = 0 \).

**Sandwich Theorem:** Let \( D \subset \mathbb{R} \) and \( f, g, h \) be functions on \( D \subset \mathbb{R} \). Let \( c \in D' \). If \( f(\alpha) \leq g(\alpha) \leq h(\alpha) \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \lim_{x \to c} h(x) = l \), then \( \lim_{x \to c} g(x) = l \).

Ex. S.T. \( \lim_{x \to 0} \cos \frac{x}{2} = 0 \).

Sol. \( -1 \leq \cos \frac{x}{2} \leq 1 \) as \( x > 0 \)

\( -\frac{1}{2} \leq \cos \frac{x}{4} \leq 1 \) as \( x > 0 \)

As \( \frac{1}{x} \rightarrow 1 \pm 0 \), so \( \frac{x}{2} \rightarrow 0 \) as \( x \rightarrow 0 \)
SOLVED EXAMPLES:

1. Examine if \( \lim_{x \to 0} (\sin \frac{x}{2} + \frac{x}{2} \sin \frac{x}{2}) \) exists or not.

**Solution:** We have that \( \lim_{x \to 0} \sin \frac{x}{2} \) does not exist.

We know if \( \lim f(x) \) and \( \lim g(x) \) both exist, then
\[ \lim f(x) + g(x) \] exists.

Hence \( \lim_{x \to 0} \left( \sin \frac{x}{2} + \frac{x}{2} \sin \frac{x}{2} \right) \) does not exist.

2. Show that \( \lim_{x \to 0} \frac{[x]}{x} = 1 \), where \([x]\) denotes the greatest integer contained in \(x\) not greater than \(x\).

**Sol.** We see that \( |[x] - x| < 1 \).

Choose \( \epsilon > 0 \)

Then \( | \frac{[x]}{x} - 1 | = \left| \frac{[x] - x}{x} \right| < \frac{1}{x} < \epsilon \) if \( 2x \leq 1 \).

Hence \( \lim_{x \to 0} \frac{[x]}{x} = 1 \).

3. If \( f(x) = x \), then is \(x\) is a rational
\( = -x\), then \(x\) is irrational

Show that \( \lim_{x \to a} f(x) \) exists only when \(a = 0\).

**Sol.** The function is defined for all real numbers of which \(a\) is a limit point.

For \(a > 0\), let \(x \in \mathbb{Q}\) and \(x \in \mathbb{R} - \mathbb{Q}\) in \((a, a + 2)\).

Then \( | f(x) - f(x) | = |x + a| > 2a \).

Hence by Cauchy's condition the following limit does not exist for \(a > 0\).

Similarly, for \(a < 0\), \( \lim f(x) \) does not exist.

When \(a = 0\), we see that \( | f(x) - 0 | < \epsilon \)
\[ \Rightarrow |x| < \epsilon \text{ whenever } 0 < x - 0 < \epsilon \text{.} \]

Thus the limit exists only when \(a = 0\).

4. If \( \lim_{x \to a} g(x) = b \) and \( \lim_{y \to b} f(y) = f(b) \), then
\[ \lim_{x \to a} f(g(x)) = f(\lim_{y \to b} g(x)) \]

**Sol.** Use Cauchy's criterion.

This above result is known as limit of a function of a function.
**CONTINUITY AND DIFFERENTIABILITY**

**Continuity:** A function \( f(x) \) is continuous at \( x = a \) if for every \( \epsilon > 0 \), \( \exists \delta > 0 \) such that

\[
|f(x) - f(a)| < \epsilon \quad \Rightarrow \quad |x-a| < \delta.
\]

**Alternative:** A function \( f(x) \) is continuous at \( x = a \) if

\[
\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a).
\]

**Note:** The discontinuity is removable by defining a new function \( g(x) \) as

\[
g(x) = \begin{cases} 
    f(x) & \text{if } x \neq a \\
    f(a) & \text{if } x = a
\end{cases}
\]

\( f(x) \) is discontinuous at \( x = a \), this discontinuity is not removable as \( \lim_{x \to a} f(x) \) does not exist.

1. If for two sequences \( x_n \) and \( y_n \) converge to \( a \), but \( \lim_{n \to \infty} f(x_n) \neq f(y_n) \) then \( \lim_{x \to a} f(x) \) does not exist.

**Theorem:** \( \lim_{x \to a} f(x) = L \) if for every sequence \( x_n \) converges to \( a \), i.e.

\[
\lim_{n \to \infty} f(x_n) = L.
\]

**Examples:**

1. Let \( f(x) = \begin{cases} 
    1 & \text{if } x \text{ is rational} \\
    2 & \text{if } x \text{ is irrational}
\end{cases} \)

   Then show that \( f(x) \) is nowhere continuous.

   **Solution:**

   Let \( a \in \mathbb{R} \) is a fixed real number.

   Consider a sequence \( x_n \) of rational numbers converging to \( a \).

   Then \( \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} 1 = 1 \).

   Now consider a sequence \( y_n \) of irrational numbers converging to \( a \).

   Then \( \lim_{n \to \infty} f(y_n) = \lim_{n \to \infty} 2 = 2 \).

   Hence \( \lim_{x \to a} f(x) \) does not exist \( \forall a \in \mathbb{R} \).

   \( \Rightarrow f(x) \) is discontinuous everywhere in \( \mathbb{R} \).
2. Let \( f(x) = \begin{cases} \frac{0}{x^2} & \text{if } x \text{ is rational} \\ \frac{0}{x^2} & \text{if } x \text{ is irrational} \end{cases} \)

then show that \( f(x) \) is continuous at \( x = 0 \).

Solution: Let \( a \in \mathbb{R} \) be a fixed real number.

Then consider a sequence \( x_n \) of rational numbers converging to \( a \).

Then \( \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} 0 = 0 \).

Let \( y_n \) be another sequence of irrational numbers converging to \( a \).

Then \( \lim_{n \to \infty} f(y_n) = \lim_{n \to \infty} y_n^2 = a^2 \).

Now, \( f(x) \) is continuous when \( \lim_{x \to a} f(x_n) = \lim_{n \to \infty} f(y_n) \Rightarrow a^2 = 0 \iff a = 0 \).

Thus, \( f(x) \) is continuous at \( x = 0 \).

More similar problems:

Differentiability: A function \( f(x) \) is differentiable at \( x = a \),

if \( \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \) exists,

iff \( \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \frac{LHD - RHD}{2} \lim_{h \to 0} f(a+h) - f(a) \)

Note that, \( f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \).

Remark: Every differentiable function is continuous but the converse is not true.

Note: By Taylor's series,

\[
\begin{align*}
    f(a+h) &= f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots \\
    f(a-h) &= f(a) - h f'(a) + \frac{h^2}{2!} f''(a) - \cdots \\
\end{align*}
\]

Note that, \( \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \{ f'(a) + \frac{h^2}{2!} f''(a) + \cdots \} = f'(a) \),

\( \lim_{h \to 0} \frac{f(a+h) - f(a-h)}{2h} = \lim_{h \to 0} \{ f'(a) + \frac{h^2}{2!} f''(a) + \cdots \} = f'(a) \),

and \( \lim_{h \to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2} = \lim_{h \to 0} \{ f''(a) + \frac{h^2}{12} f''''(a) + \cdots \} = f''(a) \).
Ex. 1. If \( f(x) = \begin{cases} x^n \sin \frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases} \) show that \( f(x) \) is continuous but not differentiable at \( x = 0 \) if \( 0 < n \leq 1 \).

**Solution:**

\[
|f(x) - f(0)| = |x^n \sin \frac{1}{x}| = |x^n| \sin \frac{1}{x} \leq |x^n| \leq \epsilon \\
\text{whenever } |x - 0| < \epsilon \forall \epsilon > 0 \text{ if } n > 0.
\]

Hence \( n > 0 \), \( f(x) \) is continuous at \( x = 0 \).

For differentiability,

\[
\left| \frac{f(x) - f(0)}{x - 0} \right| = \left| x^{n-1} \sin \frac{1}{x} \right| \leq |x|^{n-1} < \epsilon
\]

\[\text{whenever } |x - 0| < \epsilon^{1/(n-1)} = \delta \text{ if } n > 1.
\]

\[
\therefore \text{for } 0 \leq n \leq 1, f(x) \text{ is not differentiable.}
\]

Ex. 2. \( f(x) = \begin{cases} ax^2 - b & ; |x| < 1 \\ -\frac{1}{|x|} & ; |x| \geq 1 \end{cases} \)

The above function is continuous & differentiable. Find \( a \) and \( b \).

**Solution:**

\[
f(x) = \begin{cases} \frac{1}{x} & ; x \leq -1 \\ ax^2 - b, & -1 < x < 1 \\ -\frac{1}{x} & ; x > 1 \end{cases}
\]

\( f(x) \) is continuous at \( x = \pm 1 \).

Now,

\[
\begin{align*}
\left. f(x) \right|_{x=1^-} &= f(1) = a \cdot 1^2 - b = a - b \\
\left. f(x) \right|_{x=1^+} &= \frac{1}{1} = 1 \\
\end{align*}
\]

\[
\text{and } \left. f(x) \right|_{x=-1^-} = f(-1) = a \cdot (-1)^2 - b = a - b - 1
\]

\[
\therefore a - b = -1.
\]

\[
f'(x) = \begin{cases} -\frac{1}{x^2} & ; x < -1 \\ 2ax, & -1 < x < 1 \\ \frac{1}{x^2}, & x > 1 \end{cases}
\]

\[
\text{L.H.D.} = \left. \frac{d}{dx}f(x) \right|_{x=1^-} = \left. \frac{d}{dx}(ax^2 - b) \right|_{x=1^-} = 2a - 1
\]

\[
\text{R.H.D.} = \left. \frac{d}{dx}f(x) \right|_{x=1^+} = \left. \frac{d}{dx}\left(-\frac{1}{x}\right) \right|_{x=1^+} = 1
\]

\[
\Rightarrow 2a - 1 = 1 \implies a = \frac{1}{2}.
\]

\[
\therefore a - b = -1
\]

\[
\therefore b = \frac{3}{2}.
\]
Ex. (5) - s.t. \( f(x) = 2 |x| \) is differentiable at \( x \neq 0 \) but not twice differentiable at \( x = 0 \).

\[
\begin{align*}
\text{Sol.} & \\
& f(x) = \begin{cases} 2x, & x > 0 \\ -2x, & x < 0 \end{cases}
\end{align*}
\]

\[
\begin{align*}
f'(x) = \begin{cases} 2, & x > 0 \\ -2, & x < 0 \end{cases}
\end{align*}
\]

\[
\begin{align*}
f''(x) = \begin{cases} 0, & x > 0 \\ 2, & x < 0 \end{cases}
\end{align*}
\]

At \( x = 0 \), it is not twice differentiable since \( f''(0) \) does not exist.

Ex. (6) - Show that \( f(x) = |x| \) is continuous but not differentiable at \( x = 0 \).

\[
\begin{align*}
\text{Sol.} & \\
& f(x) = |x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}
\end{align*}
\]

\[
\begin{align*}
l_h(x) = l_{x<0} (x) & = 0, \\
l_r(x) = l_{x>0} (x) & = 0.
\end{align*}
\]

Hence \( f(x) \) is continuous at \( x = 0 \).

Note that \( f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} \)

\[
\begin{align*}
f'(0-) = 1, & \quad f'(0+) = -1.
\end{align*}
\]

LHD \( \neq \) RHD

Hence \( f'(0) \) does not exist.

Ex. (7) - \( \ln|x| \) is not differentiable at \( x = 1 \) but continuous at \( x = 1 \).

\[
\begin{align*}
\text{Sol.} & \\
& f(x) = \ln|x| = \ln|1| \text{ is not differentiable at } x = 0, i.e. \ln x = 0 \Rightarrow x = 1.
\end{align*}
\]

Ex. (8) - \( \sin|x| \) is not differentiable at \( x = 0 \).

\[
\begin{align*}
\text{Sin.} & \\
& \sin x = 0 \Rightarrow x = n\pi, n \in \mathbb{N}.
\end{align*}
\]

Ex. (9) - At which values of \( x \), \( f(x) = |x^2 - 3x + 2| + \cos |x| \) is not-differentiable.

\[
\begin{align*}
\text{Sol.} & \\
& |x^2 - 3x + 2| = |(x-1)(x-2)| \text{ is not differentiable at } x = 1, 2.
\end{align*}
\]

\[
\begin{align*}
& \cos |x| = \begin{cases} \cos x, & x > 0 \\ \cos(-x), & x < 0 \end{cases}.
\end{align*}
\]

\[
\begin{align*}
& \text{At } x = 1, 2; f'(x) \text{ is not differentiable.}
\end{align*}
\]

Ex. (10) - At which values of \( x \), \( f(x) = (x-1)^2 |x+1| + |x-2| \) is not differentiable.

\[
\begin{align*}
\text{Sol.} & \\
& f(x) = \begin{cases} (x-1)^2 |x+1| + |x-2|, & x < 1 \\ (x-1)^2 |x+1| + |x-2|, & x > 1 \end{cases}
\end{align*}
\]

is not differentiable at \( x = 1, 2 \).
Ex. 9. If \( f(x) = a \sin x + b|x| + c|x|^3 \) is differentiable at \( x = 0 \), then \( a = b = 0 \), \( c \in \mathbb{R} \).

Solution:-
\[ f(x) = a \sin x + b|x| + c|x|^3 \]
\( |\sin x| \) is not differentiable at \( x = n \pi \) where \( n \) is an integer.
\( |x| \) is not differentiable at \( x = 0 \)
but \( |x|^3 \) is differentiable at \( x = 0 \).
\( \therefore \ a = b = 0 \), \( c \in \mathbb{R} \).

Ex. 10. \( f(x) = |x|^p \) is differentiable if \( p > 1 \).

Solution:-
\[ f(x) = \begin{cases} x^p & x > 0 \\ (-x)^p & x < 0 \end{cases} \]
\( f'(x) = \begin{cases} px^{p-1} & x > 0 \\ p(-x)^{p-1}(-1) & x < 0 \end{cases} \)
\( f'(0-) = \lim_{x \to 0^-} f'(x) = \lim_{x \to 0^-} x^{p-1} = 0 \) if \( p > 1 \)
\( f'(0+) = 0 \) if \( p > 1 \).

Ex. 11. If the derivative of \( f(x) = \begin{cases} bx^2 + ax + 4 & x > 1 \\ ax^2 + b & x < 1 \end{cases} \) is everywhere continuous then
(i) \( a = 2, b = 3 \)
(ii) \( a = -2, b = -3 \).

Solution:-
\( f(x) = \begin{cases} bx^2 + ax + 4 & x > 1 \\ ax^2 + b & x < 1 \end{cases} \)
\( f'(x) = \begin{cases} 2bx + a & x > 1 \\ 2ax & x < 1 \end{cases} \)
\( f'(1-) = 2b(-1) + a \)
\( f'(1+) = 2b(1) + a \)
\( \Rightarrow 2a = 2b - a \)
\( \Rightarrow b = 3 \)
Rolle's Theorem: \( f(x) \) is

(i) continuous on \([a,b]\)
(ii) differentiable on \((a,b)\)
(iii) \( f(a) = f(b) \)

Then \( \exists \ c \in (a,b) \) such that \( f'(c) = 0 \).

Ex. Show that for the function \( f(x) = \sqrt{x} - x, 0 \leq x \leq 1 \), Rolle's theorem is not applicable but \( \exists \ c \in (0,1) \) \( f'(c) = 0 \).

Sol.

Let \( f'(x) = \lim_{x \to 0} \frac{f(x) - f(0)}{x-0} \) does not exist.

Hence, Rolle's theorem is not applicable.

Now, \( f'(x) = \frac{1}{2\sqrt{x}} - 1 \neq 0 \)
\( \Rightarrow \alpha = \frac{1}{4} \in (0,1) \).

Lagrange's mean value theorem: \( f(x) \) is

(i) continuous on \([a,b]\)
(ii) differentiable on \((a,b)\)

Then \( \exists \ c \in (a,b) \) such that

\( f(b) - f(a) = f'(c)(b-a) \)
\( \Rightarrow f(b) = f(a) + f'(c)(b-a) \).

Ex. If \( |f(x) - f(y)| \leq |x-y|^2 \), \( \forall x,y \), then \( \exists \ c \) \( f(x) \) is constant.

Sol.

\[ \left| \frac{f(x) - f(y)}{x-y} \right| \leq |x-y| \]
\( 0 \leq \left| \frac{f(x) - f(y)}{x-y} \right| \leq |x-y| \)
\( \Rightarrow 0 \leq |f'(c)| \leq 0 \)
\( \Rightarrow f'(c) = 0 \forall y \)
\( \Rightarrow f(x) \) is constant.
Examples of Rolle's Theorem & Lagrange's M.V.T:

1. Show that \( \frac{b-a}{1+b^2} < \frac{\tan^{-1}b - \tan^{-1}a}{b-a} < \frac{b-a}{1+a^2} \).

**Sol.** Let \( f(x) = \tan^{-1}x \).

From Mean-Value Theorem,

\[ \frac{f(b) - f(a)}{b-a} = f'(c) \]

\[ \frac{\tan^{-1}b - \tan^{-1}a}{b-a} = \frac{1}{1+c^2}, \quad a < c < b \]

\[ \Rightarrow \frac{\tan^{-1}b - \tan^{-1}a}{b-a} = \frac{b-a}{1+c^2} < \frac{b-a}{1+a^2} \]

2. Let \( f(x) = \sum_{k=0}^{n} a_k x^k \), where \( a_k \)'s satisfy \( \sum_{k=0}^{n} \frac{a_k}{k+1} = 0 \),

s.t. \( f(x) = 0 \) in the interval \( (0,1) \).

**Sol.** Define \( g(x) = \int_{0}^{x} f(t) \, dt \)

\[ g(x) = \int_{0}^{x} \sum_{k=0}^{n} \frac{a_k}{k+1} t^k \, dt = \sum_{k=0}^{n} \frac{a_k}{k+1} \int_{0}^{x} t^k \, dt = \sum_{k=0}^{n} \frac{a_k}{k+1} \frac{x^{k+1}}{k+1} \]

Apply Rolle's theorem,
Taylor's Theorem:

\[ f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(c) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(c) + \frac{(x-a)^n}{n!}f^{(n)}(c), \quad c \text{ lies between } a \text{ and } x. \]

Ex. (1): If \( f'(a) = f''(a) = \cdots = f^{(n-1)}(a) = 0 \) but \( f^{(n)}(a) > 0 \) and \( f^{(n)}(x) \) is continuous at \( x = a \), then show that \( f(x) \) is local minimum.

Solution: By Taylor's theorem,

\[ f(a) = f(a) + \frac{(x-a)^1}{1!}f'(a) \]
\[ \Rightarrow f(x) - f(a) = \frac{(x-a)}{1!}f'(a) > 0 \\forall a \in (a-e, a+e) \]
\[ \Rightarrow f(a) > f(a) \text{ } \forall \text{ } x \in (a-e, a+e) \]

Ex. (2): Let \( f: E \to \mathbb{R} \) be a function \( \left| f(x) - f(y) \right| \leq |x-y| \text{ } \forall x, y \in E \) \( \Rightarrow \frac{\text{Lebesgue}}{a \leq y \leq b} \text{ } f'(x) \).

Prove that for any function \( f \) satisfying the above condition, also satisfies

\[ \left| \int_a^b f(x) \, dx \right| \leq \frac{1}{2} (b-a)^2, \]

provided \( f \) is integrable on \([a, b] \).

Sol.

\[ \left| \frac{f(x) - f(y)}{x-y} \right| \leq 1 \text{ } \forall x, y \in E \]

\[ \Rightarrow \left| \frac{\text{Lebesgue}}{a \leq y \leq b} \text{ } f'(x) \right| \leq 1 \text{ } \forall y \in E \]

Define \( F(x) = \int_a^x f(t) \, dt \)

\[ = \int_a^b f(x) \, dx = f(b) + (b-a)F'(a) + \frac{(b-a)^2}{2}F''(c) \]
\[ = c + (b-a)F(a) + \frac{(b-a)^2}{2}f(c) \]

\[ \Rightarrow \left| \int_a^b f(x) \, dx - (b-a)F(a) \right| \leq \frac{(b-a)^2}{2} \left| f'(c) \right| \]

\[ \leq \frac{(b-a)^2}{2}. \]
Ex. (1): List \( f(0) = 0 \); show that \( \lim_{h \to 0} \frac{f(h) + f(-h)}{2h^2} = f''(0) \).

Solution:
\[
\begin{align*}
\frac{f(h) + f(-h)}{2h^2} &= \frac{f(h) + f(h) + \frac{h^2}{2!} f''(0) + \frac{h^3}{3!} f'''(c)}{2h^2} \\
&= \frac{h f'(0) + \frac{h^2}{2!} f''(0) + \frac{h^3}{3!} f'''(c)}{2h^2} \\
&= \frac{f'(0) + \frac{h}{2} f''(0) + \frac{h^2}{3} f'''(c)}{2h}
\end{align*}
\]

Similarly,
\[
\frac{f(h) + f(-h)}{2h^2} = -h f'(0) + \frac{h^2}{2!} f''(0) - \frac{h^3}{3!} f'''(c)
\]

\[
\therefore \frac{f(h) + f(-h)}{2h^2} = \frac{h^2 f''(0)}{2h}
\]

\[
\lim_{h \to 0} \frac{f(h) + f(-h)}{2h^2} = f''(0).
\]

Ex. (2): List \( g \) be a function (continuous) with \( g(1) = 1 \) and \( g(x+y) = 5g(x)g(y) \). Find \( g(x) \).

Sol.
\[
\begin{align*}
g(x+y) &= 5g(x)g(y) \\
5g(x+y) &= 5g(x)5g(y) \\
\Rightarrow \log [5g(x+y)] &= \log [5g(x)] + \log [5g(y)] \\
\Rightarrow g(x+y) &= g(x) + g(y) \\
\Rightarrow g(x) &= 5 \cdot 5g(y) \\
\Rightarrow g(x) &= 5^x \\
\Rightarrow g(x) &= 5^{x-1}.
\end{align*}
\]
**Monotonic Function**

A function is monotonically increasing on \( D \) if for every \( x_1 < x_2 \) from \( D \), \( f(x_1) \leq f(x_2) \).

**Theorem:** A differentiable function is increasing on \( D \) if \( f'(x) > 0 \) for \( x \in D \).

**Proof:** By Taylor's theorem,

\[
f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x^*)
\]

Let \( h > 0 \)

Then

\[
f(x+h) > f(x) + hf'(x^*)
\]

\[
f(x+h) > f(x)
\]

\[
f' is increasing.
\]

**Ex. 1:** Find the intervals of monotonicity of the function \( y = 2x^2 - 2 \log|x| \).

**Sol.**

\[
y = 2x^2 - 2 \log|x|, \quad x \neq 0
\]

\[
\frac{dy}{dx} = 4x - \frac{2}{x}
\]

\[
\frac{dy}{dx} > 0 \Rightarrow \frac{2x - 1}{x^2} > 0 \Rightarrow (x+\frac{1}{2})(x-\frac{1}{2}) > 0
\]

\[
x \in (-\infty, 0) \cup (\frac{1}{2}, \infty)
\]

**Ex. 2:** If \( g(x) = f(\cos x) + f(\sin x) \) and \( f''(x) > 0 \), \( 0 \leq x < 1 \), then find the intervals of monotonicity of \( g(x) \).

**Sol.**

\[
g(x) = f(\cos x) + f(\sin x)
\]

From \( g(x) \) to be increasing \( \Rightarrow g'(x) > 0 \)

\[
g'(x) = f'(\cos x)(-\sin x) + f'(\sin x)\cos x
\]

\[
\Rightarrow f'(\cos x) > 0, \quad f'(\sin x) > 0
\]

**Ex. 3:** Let \( f(x) = (x+1)e^{x+1} \), then \( f(0) > 0 \), \( f'(x) = (x+1)e^{x+1} + e^{x+1}
\]

\[
f(0) = 2 > 0
\]

**Sol.**

For \( x > 0, \) \( f'(x) > 0 \), then \( f(x) > f(0) \).

For \( x < 0, \) \( f'(x) < 0 \), then \( f(x) > f(0) \).

Hence \( f(x) > 0 \) for \( x \in \mathbb{R} \).
Ex.(9): Show that \( 1 + x \log \left( x + \sqrt{x^2 + 1} \right) \geq \sqrt{1 + x^2} \) for all \( x > 0 \).

Solution:

\[ f(x) = 1 + x \log \left( x + \sqrt{x^2 + 1} \right) - \sqrt{1 + x^2} \]
\[ f'(x) = \frac{x}{x + \sqrt{x^2 + 1}} \cdot 2x \cdot \log (x + \sqrt{x^2 + 1}) - \frac{1}{2\sqrt{1 + x^2}} \]

\[ \therefore f'(x) = \log \left( x + \sqrt{x^2 + 1} \right) > 0 \text{ if } x > 0 \]

\[ \Rightarrow f'(x) \text{ is increasing for } x > 0 \]

\[ \therefore f(x) > f(0) \]

\[ 1 + x \log \left( x + \sqrt{x^2 + 1} \right) > \sqrt{1 + x^2} \text{ for all } x > 0. \]

Ex.(5): Which one is larger: \( \pi^e \) and \( e^\pi \)?

Solution:

\[ \pi^e \geq e^\pi \]
\[ \Rightarrow \pi^{1/e} \geq e^{1/\pi} \]

Let \( f(x) = x^{1/x} \)
\[ \ln f(x) = \frac{1}{x} \ln x \]

\[ \therefore \frac{1}{x} f'(x) = \frac{1}{x^2} - \frac{1}{x} \ln x = \frac{1 - \ln x}{x^2} \]

\[ \therefore f'(x) > 0 \text{ if } 1 - \ln x > 0 \text{ if } x > e. \]

As \( \pi > e \), \( f(\pi) < f(e) \)

\[ \Rightarrow \pi^{1/\pi} < e^{1/e} \]
\[ \Rightarrow \pi^e < e^\pi. \]

\[ \therefore \pi^e < e^\pi. \]

\[ \therefore \pi^e < e^\pi. \]
FUNCTION OF SEVERAL VARIABLES

Functions \( z = f(x, y) \) be a function defined on \( D \subseteq \mathbb{R}^2 \).

**Bolzano-Weierstrass Theorem in \( \mathbb{R}^2 \):** Every bounded infinite set \( S \subseteq \mathbb{R}^2 \) has at least one limit point in \( \mathbb{R}^2 \).

The limit of a function:

Let \( f(x, y) = L \) means for every \( \epsilon > 0 \), there corresponds a \( \delta > 0 \)
\( (x, y) \to (a, b) \)
such that \( |f(x, y) - L| < \epsilon \) whenever \( (x - a)^2 + (y - b)^2 < \delta^2 \).

i.e. \( |x - a| < \delta \), \( |y - b| < \delta \).

Examples:

1. \( \lim_{x\to 0, y\to 0} \frac{xy}{\sqrt{x^2+y^2}} = 0 \)
2. \( \lim_{(x,y)\to (0,0)} \frac{\alpha^3 - \alpha y^3}{\alpha^2 + y^2} = 0 \)
3. \( \lim_{(x,y)\to (0,0)} \alpha^y \left( \frac{x^2}{x^2 + y^2} \right) = 0 \)
4. \( \lim_{(x,y)\to (0,0)} e^{-\left(\frac{\alpha^2y^2}{\alpha^2 + y^2}\right)} = 1 \)
5. \( \lim_{(x,y)\to (1,2)} \alpha x^2 + 2y = 5 \)
6. \( \lim_{(x,y)\to (0,0)} \frac{x^2 - y^2}{1 + x^2 + y^2} = 0 \)
7. \( \lim_{(x,y)\to (0,0)} \frac{\alpha x^4 + y^4}{\alpha^2 + y^2} = 0 \)

Solution:

Let \( \epsilon > 0 \) be given, then

\[
\left| f(x, y) - L \right| = \left| \frac{xy}{\sqrt{x^2+y^2}} - 0 \right| = \left| \frac{\alpha x \sin \theta}{\sqrt{\alpha^2 \cos^2 \theta + \sin^2 \theta}} \right| = \left| \frac{\alpha}{\sqrt{\alpha^2 + 1}} \right| \leq \frac{\alpha}{\sqrt{\alpha^2 + 1}} \leq \epsilon
\]

whenever \( x^2 + y^2 < 4\epsilon^2 \).

Then by definition, \( \lim_{(x, y)\to (0,0)} f(x, y) = 0 \).

Alternatively:

\[
\left| \frac{xy}{\sqrt{x^2+y^2}} - 0 \right| = \left| 1 \frac{\alpha |x| |y|}{\sqrt{\alpha^2 + y^2}} \right| \leq \frac{\alpha |x| |y|}{\sqrt{\alpha^2 + y^2}} \leq \frac{\alpha \sqrt{x^2 + y^2}}{\sqrt{\alpha^2 + y^2}} \leq \frac{\alpha \sqrt{x^2 + y^2}}{\sqrt{\alpha^2 + y^2}} \leq \epsilon
\]

if \( \alpha^2 + y^2 < \epsilon^2 \).

By definition, \( \lim_{(x, y)\to (0,0)} f(x, y) = 0 \).
(ii) \[ \left| \frac{x^2 + y^3 - 1}{x^2 + y^2 - 0} \right| \leq \frac{1}{r^2} (\cos^3 \theta - \sin^3 \theta) \]
\[ \leq \frac{1}{r^2} (\cos^3 \theta + |\sin^3 \theta|) \]
\[ \leq 2 \varepsilon \sqrt{x^2 + y^2} < \varepsilon \]

whenever \( x^2 + y^2 < \delta^2 = \frac{\varepsilon^2}{4} \).

By definition, \( \lim_{(x,y) \to (0,0)} f(x,y) = 0. \)

(iii) Let us put \( x = r \cos \theta, y = r \sin \theta. \)
\[ x^2 + y^2 = r^2 \quad \text{implies} \quad r \to 0 \quad \text{as} \quad (x,y) \to (0,0) \]
\[ f(x,y) = r^2 \sin \theta \cos \theta \cos \theta \]
Now,
\[ |x| \frac{x^2 - y^2}{x^2 + y^2} | \leq \frac{r^2}{4} \sin \theta \leq \frac{\varepsilon^2}{4} < \varepsilon \]
\[ r^2 < 4 \varepsilon \quad \text{on} \quad x^2 + y^2 < \delta^2 (\leq \varepsilon) \]

By definition, \( \lim_{(x,y) \to (0,0)} f(x,y) = 0. \)

Alternatively: We see \( |x| \leq \sqrt{x^2 + y^2} \) and \( |y| \leq \sqrt{x^2 + y^2} \).

Also, \[ \left| \frac{x^2 - y^2}{x^2 + y^2} \right| \leq 1. \]

2. \[ |f(x,y) - 0| = \left| \frac{x^2 - y^2}{x^2 + y^2} \right| \leq x^2 + y^2 < \varepsilon \]

if \( x^2 + y^2 < \delta^2 (\leq \varepsilon) \).

By definition, \( \lim_{(x,y) \to (0,0)} f(x,y) = 0. \)

(iv) \[ \left| \frac{e^{-x^2 - y^2} - 1}{x^2 + y^2} \right| \geq \frac{\varepsilon^2}{4} \quad \text{if} \quad x^2 + y^2 < \delta^2 (\leq \varepsilon) \]

(v) \[ \lim_{(x,y) \to (0,0)} e^{-x^2 - y^2} = 1. \]

When \( |x - 1| \leq \delta \) and \( |y - 2| \leq \delta \), we get:
\[ -4 \delta + \delta^2 < x^2 + y^2 < 5 \delta^2 + 4 \delta \]
Choosing \( 0 < \varepsilon \leq 1 \), we get \( -5 \delta \leq x^2 + y^2 - 5 < 5 \delta \)

on, \( |x^2 + y^2 - 5| < 5 \delta (\leq \varepsilon) \)
Thus \( |x^2 + y^2 - 5| < \varepsilon \) when \( |x - 1| < \delta \) and \( |y - 2| < \delta \) (\( \leq \varepsilon \))
(vi) \[ \lim_{(x,y) \to (0,0)} \frac{\sqrt{1+\alpha x^2 y^2} - 1}{x^2+y^2} = \left| \frac{\frac{1}{2} \cdot \alpha x^2 y^2}{x^2+y^2} \right| = \frac{\alpha}{8} \left| \sin^2 2\theta \right| \leq \frac{\alpha^2}{8} < \epsilon \]

\[ \lim_{(x,y) \to (0,0)} \frac{\sqrt{1+\alpha x^2 y^2} - 1}{x^2+y^2} = 0 \text{ if } \alpha^2 < 8 \epsilon \text{ on } x^2+y^2 < \delta^2 (= 8 \epsilon) \]

(vii) We put \( x = r \cos \Theta, \ y = r \sin \Theta \)

\[ \frac{x^2+y^2}{1+x^2+y^2} = \frac{r^2 \cos^2 \Theta}{1+r^2} \leq r^2 < \epsilon \text{ if } x^2+y^2 < \delta^2 (= \epsilon) \]

\[ \lim_{(x,y) \to (0,0)} \frac{x^2+y^2}{1+x^2+y^2} = 0. \]

(viii) \[ \frac{x^4+y^4}{x^2+y^2} - 0 = \frac{x^4+y^4}{x^2+y^2} \leq \frac{10^{1.9} + y^{1.9}}{x^2+y^2} \leq 2 (x^2+y^2)^2 < \epsilon \]

\[ \therefore x^2 < x^2+y^2, \ y^2 < (x^2+y^2)^2 \text{ and } y^2 < (x^2+y^2)^2 \]

\[ \text{if } x^2+y^2 < \delta^2 (= \epsilon^2) \]
Non-Existence of Limit — Let \( f(x, y) = \frac{a}{b} \)

Consider a path \( y = \phi(x) \) of approaching to the limit \((a, b)\),

i.e., \( \phi(x) = y \to \) as \( x \to a \).

Then for any path \( y = \phi(x) \) of approaching to \((a, b)\),

\[
\lim_{x \to a} f(x, \phi(x)) = l.
\]

If for two paths (different) \( y = \phi_1(x) \) and \( y = \phi_2(x) \) of

\[
\lim_{x \to a} f(x, \phi_1(x)) \neq \lim_{x \to a} f(x, \phi_2(x)),
\]

then \( \lim_{(x, y) \to (a, b)} f(x, y) \) does not exist.

Examples: — Show that the following limits do not exist:

(i) \( \lim_{(x, y) \to (0, 0)} \frac{a^2 y}{x^2 + y^2} \)

(ii) \( \lim_{(x, y) \to (0, 0)} \frac{a^2 y}{x^4 + y^2} \)

(iii) \( \lim_{(x, y) \to (0, 0)} \frac{a^2 y}{x^4 + y^6} \)

(iv) \( \lim_{(x, y) \to (0, 0)} \frac{a^2 y}{x^4 + y^4} \)

(v) \( \lim_{(x, y) \to (0, 0)} \frac{a^2 y}{x^2 + y^2} \)

(vi) \( \lim_{(x, y) \to (0, 0)} \frac{a^2 y}{x^2 + y^2} \)

In each case we shall show that for two different modes of

approach of \((x, y) \to (0, 0)\) the limits are different and hence the

limit will cease to exist.

Solution: —

(i) Let \( y = mx \)

As \( x \to 0, y = mx \to 0 \)

\[
\lim_{(x, y) \to (0, 0)} \frac{a^2 y}{x^2 + y^2} = \lim_{x \to 0} \frac{a^2 mx}{x^2(1 + m^2)} = \frac{m}{1 + m^2}, \text{ depends on } m.
\]

Hence \( \lim_{(x, y) \to (0, 0)} \frac{a^2 y}{x^2 + y^2} \) does not exist.

(ii) Let \( y = m^2 x \)

\( x \to (0, 0) \)

\[
\lim_{(x, y) \to (0, 0)} \frac{a^2 y}{x^2 + y^2} = \lim_{x \to 0} \frac{m^2 x^4}{x^4(1 + m^2)} = \frac{m}{1 + m^2}, \text{ depends on } m.
\]

Hence the limit does not exist.

(iii) Let \( y = m^3 x \)

\( x \to (0, 0) \)

\[
\lim_{(x, y) \to (0, 0)} \frac{a^2 y}{x^2 + y^6} = \lim_{x \to 0} \frac{m^6 x^6}{x^2(1 + m^6)} = \frac{m}{1 + m^6}, \text{ depends on } m.
\]

Hence the limit does not exist.
(v) Letting \((x, y) \to (0,0)\) along \(y = mx\), we see that the limit depends on \(m\). Hence the limit does not exist.

\[
\lim_{(x, y) \to (0,0)} \frac{x^3 + y^3}{x - y} = \lim_{x \to 0} \frac{x^3 + (x - mx^3)^3}{x^3}
\]

\[
= \lim_{x \to 0} \frac{1 + (1-mx^2)^3}{m}
\]

\[
= \frac{2}{m}, \text{ which has different values for different } m.
\]

A limit does not exist.

(vi) \((x, y) \to (0,0)\) along \(y = 0\) and \(y = x^2\) separately.

\[
\lim_{(x, y) \to (0,0)} f(x, y) = \lim_{x \to 0} f(x, 0) = \lim_{x \to 0} \frac{x}{-x^2} = 0.
\]

Also, \[
\lim_{(x, y) \to (0,0)} f(x, y) = \lim_{x \to 0} f(x, x^2)
\]

\[
= \lim_{x \to 0} \frac{x + x^2}{x^2 - x^2(1 + x)^2}
\]

\[
= \lim_{x \to 0} \frac{(1 + x)(2 + 2x + x^2)}{-(2 + x)}
\]

\[
= -1.
\]

\[
\lim_{(x, y) \to (0,0)} f(x, y) \text{ does not exist.}
\]

(vii) Let \((x, y) \to (0,0)\) along \(y = 0\) and \(y = x^2\) separately.

\[
\lim_{x \to 0} f(x, 0) = 0 \text{ and } \lim_{x \to 0} f(x, x) = 1.
\]

\[
\lim_{x \to 0} f(x, 0) \neq \lim_{x \to 0} f(x, x)
\]

\[
\text{Limit does not exist.}
\]
Iterated or Repeated Limits: If \( f(x,y) \) is defined in some neighbourhood of \((a,b)\), then \( \lim_{y \to b} f(x,y) \), if it exists, is a function of \(x\), say \( \phi(x) \).

If \( \lim_{x \to a} \phi(x) \) exists, then

\[
\lim_{x \to a} f(x,y) = \lim_{y \to b} \left( \lim_{x \to a} f(x,y) \right) = l_1, \text{ say and } l_2 \text{ say}.
\]

That is, \( \lim_{x \to a} \lim_{y \to b} f(x,y) \) is a repeated limit of \( f \) as \( x \to b, y \to a \).

Similarly, the other \( \lim_{y \to b} \lim_{x \to a} f(x,y) \) is also a repeated limit of \( f \) as \( x \to a, y \to b \).

Note: If the simultaneous limits exist then two repeated limits exist and are equal, but the converse is not necessarily true, if the repeated limits are not equal, then simultaneous limit may not exist.

Ex. (1)&: Give an example to show that repeated limits of a function \( f(x,y) \) exist and are equal but the double limit does not exist as \((x,y) \to (a,b)\).

Let \( f(x,y) = \frac{xy}{x^2 + y^2} \).

Find the two repeated limit of \( f(x,y) \) at \((0,0)\). Also comment on the existence of \( \lim_{(x,y) \to (0,0)} f(x,y) \).

Solution:

\[
\lim_{y \to 0} \left( \lim_{x \to 0} f(x,y) \right) = \lim_{y \to 0} \left[ \lim_{x \to 0} \frac{xy}{x^2 + y^2} \right] = \lim_{x \to 0} \left[ \lim_{y \to 0} \frac{xy}{x^2 + y^2} \right] = \lim_{x \to 0} \left[ \lim_{y \to 0} \frac{xy}{x^2 + y^2} \right] = 0.
\]

So, both repeated limits are equal.

Now, let \( y = mx, \) a.s \( x \to 0, mx \to 0 \).

Now, \( \lim_{x \to 0} f(x, mx) = \lim_{x \to 0} \frac{mx^2}{(m^2+1)x^2} = \frac{m}{m^2+1} \), depends on \( m \).

Hence, \( \lim_{(x,y) \to (0,0)} f(x,y) \) does not exist.
Ex. (2): Show that \( \lim_{(x,y) \to (0,0)} f(x,y) = 0 \). But the repeated limits do not exist, where \( f(x,y) \) is defined as

\[
f(x,y) = \begin{cases} \frac{x \sin \frac{y}{x} + y \sin \frac{x}{y}}{x} & \text{when } xy \neq 0 \\ 0 & \text{when } xy = 0. \end{cases}
\]

**Solution:**

\[
\lim_{x \to 0} f(x,y) = \left( \lim_{x \to 0} \frac{x \sin \frac{y}{x} + y \sin \frac{x}{y}}{x} \right)
\]

The limit does not exist.

\[
\lim_{y \to 0} f(x,y) = \left( \lim_{y \to 0} \frac{x \sin \frac{y}{x} + y \sin \frac{x}{y}}{x} \right)
\]

Similarly, \( \lim_{y \to 0} f(x,y) \) does not exist.

Hence \( \lim_{x \to 0} f(x,y) \) and \( \lim_{y \to 0} f(x,y) \) do not exist.

Note, for \( \varepsilon > 0 \), \( |f(x,y) - 0| = \left| \frac{x \sin \frac{y}{x} + y \sin \frac{x}{y}}{x} \right| \leq |x| + |y| < \varepsilon \)

whenever \( |x| < \frac{\varepsilon}{2} \) and \( |y| < \frac{\varepsilon}{2} \).

Let \( \lim_{(x,y) \to (0,0)} f(x,y) = 0 \).

Ex. (3): Show that for \( f(x,y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases} \)

Double limit \( f \) both repeated limits exist at \( (0,0) \).

**Solution:** We have already shown that \( \lim_{(x,y) \to (0,0)} f(x,y) = 0 \).

Now, \( \lim_{x \to 0} \lim_{y \to 0} x y \left( \frac{x^2 - y^2}{x^2 + y^2} \right) = \lim_{y \to 0} \lim_{x \to 0} \left\{ yx \right\} = 0. \)

Also, \( \lim_{y \to 0} \lim_{x \to 0} \left( \frac{x^2 - y^2}{x^2 + y^2} \right)^2 = 0. \)
Ex. (4):— Evaluate the following limits:

(i) \( \lim_{(x,y) \to (0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} \)

\[ L = \lim_{(x,y) \to (0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} \]

\[ = 0 \]

(ii) \( \lim_{(x,y) \to (0,0)} \frac{\sin^{-1}(xy-2)}{\tan^{-1}(3xy-6)} \)

\[ L = \lim_{u \to (xy-2) \to 0} \frac{\sin^{-1}u}{\tan^{-1}3u} \]

(iii) \( \lim_{(x,y) \to (0,0)} \frac{x^2+y^2-1}{\sqrt{x^2+1}-y} \)

\[ L = \lim_{(x,y) \to (0,0)} \frac{x^2+y^2-1}{\sqrt{x^2+1}-y} \]

Ex. (5):— Let \( f(x,y) = \frac{x \sin \frac{\pi}{2} + \frac{x^2-y^2}{x^2+y^2}}{\frac{x^2+y^2}{2}} \).

Show that \( \lim_{x \to 0} \lim_{y \to 0} f(x,y) \) exists, but neither \( \lim_{y \to 0} \lim_{x \to 0} f(x,y) \) nor \( \lim_{x \to 0} \lim_{y \to 0} f(x,y) \) exist.

Solution:— We first show that \( \lim_{(x,y) \to (0,0)} f(x,y) \) does not exist.

\[ = \lim_{(x,y) \to (0,0)} \left( \frac{x \sin \frac{\pi}{2} + \frac{x^2-y^2}{x^2+y^2}}{\frac{x^2+y^2}{2}} \right) \]

\[ = \lim_{(x,y) \to (0,0)} \frac{x \sin \frac{\pi}{2}}{\frac{x^2+y^2}{2}} + \lim_{(x,y) \to (0,0)} \frac{\frac{x^2-y^2}{x^2+y^2}}{\frac{x^2+y^2}{2}} \]
\[
\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x \to 0} \frac{x^2(1 - m^2)}{(1 + m^2)} = 1 - m^2, \text{ depends on } m.
\]

\[
\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2} \text{ does not exist.}
\]

\[
\lim_{(x,y) \to (0,0)} f(x,y) \text{ does not exist.}
\]

\[
\lim_{x \to 0} \lim_{y \to 0} f(x,y) = \lim_{y \to 0} \left( \lim_{x \to 0} \frac{x^2}{x^2 + y^2} \right)
\]

\[
\lim_{x \to 0} \sin \frac{1}{x} \text{ does not exist, so } \lim_{y \to 0} \lim_{x \to 0} f(x,y) \text{ does not exist.}
\]

\[
\lim_{y \to 0} \lim_{x \to 0} \left\{ \sin \frac{1}{x} + \frac{x^2 - y^2}{x^2 + y^2} \right\}
\]

\[
= 0 + \frac{-y^2}{y^2} = -1.
\]
Continuity of \( f(x,y) \):

A function \( f(x,y) \) is continuous at \((a,b)\) if for every \( \varepsilon > 0 \), there corresponds a \( \delta > 0 \) such that \( |f(x,y) - f(a,b)| < \varepsilon \) whenever \((x-a)^2 + (y-b)^2 < \delta^2\).

Note: If \( \lim_{(x,y) \to (a,b)} f(x,y) \) does not exist, the question of continuity of \( f \) at \((a,b)\) does not arise.

Examples:
Examine the continuity of the following functions:

(i) \( f(x,y) = \begin{cases} \frac{1}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} \)

Solution:

\[ |f(x,y) - f(0,0)| = \frac{1}{x^2+y^2} \]

wherever \( x^2+y^2 > \frac{1}{\varepsilon} = \delta^2 \)

Hence the limit does not exist, i.e., \( f(x,y) \) is not continuous at \((0,0)\).

(ii) \( f(x,y) = \begin{cases} \frac{x^3+y^3}{x-y} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} \)

Solution:

Let \((x,y) \to (0,0)\) along \( x = my \), \( y = z - mx \), \( z \to 0 \).

\[ \lim_{x \to 0} \frac{1 + (1-m^2)x^3}{m} = \frac{2}{m} \]

The limit does not exist and hence the function is not continuous at \((0,0)\).

(iii) \( f(x,y) = \begin{cases} \frac{x^4+y^4}{x-y} & \text{when } x \neq y \\ 0 & \text{when } x = y \end{cases} \)

s.t. \( f \) is not continuous at \((0,0)\).

Solution:

\( x - y = mx \)

\[ \lim_{(x,y) \to (0,0)} f(x,y) = \frac{2}{m} \]

\[ \lim_{(x,y) \to (0,0)} f(x,y) \] does not exist. Hence \( f(x,y) \) is not continuous at \((0,0)\).
(iv) \[ f(x, y) = \begin{cases} \frac{1}{2} - \frac{2y}{x^2 - 4xy + y^2} & \text{if } x \neq 2y \\ 0 & \text{if } x = 2y \end{cases} \]

\text{Sol.} \quad \text{Let } u = x - 2y \\
\text{As, } (x, y) \to (0, 0), \ u \to 0. \\
\therefore \lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{u \to 0} \frac{2}{u^2} = +\infty.

\therefore \text{The limit does not exist.}
\therefore f(x, y) \text{ is not continuous at } (0, 0).

(v) \[ f(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \]

\text{Sol.} \quad |f(x, y) - f(0, 0)| = \left| \frac{x^2 - y^2}{x^2 + y^2} \right| = \frac{|x^2 - y^2|}{|x^2 + y^2|} \leq 2|1| 1g |\leq \frac{x^2 + y^2}{2} < \varepsilon

\text{if } x^2 + y^2 < 2 \varepsilon \quad (= \varepsilon^2).

\therefore f(x, y) \text{ is continuous at } (0, 0).

(vi) \[ f(x, y) = \begin{cases} \frac{3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \]

\text{Sol.} \quad \text{We know } x^2 \leq x^2 + y^2 \quad \text{and } x^2 \leq (x^2 + y^2)^{3/2}

\text{Similarly, } y^2 \leq (x^2 + y^2)^{3/2}

\text{Note, } |f(x, y) - f(0, 0)| = \left| \frac{x^2 - y^2}{x^2 + y^2} \right| \leq \frac{2(x^2 + y^2)^{3/2}}{(x^2 + y^2)} \leq 2(x^2 + y^2)^{1/2} < \varepsilon

\text{if } x^2 + y^2 < \varepsilon^2 \quad (= \frac{\varepsilon^2}{4}).

\therefore f(x, y) \text{ is continuous at } (0, 0).

(vii) \[ f(x, y) = \begin{cases} \frac{1}{\sqrt{x^2 + y^2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \]

\text{Sol.} \quad |f(x, y) - f(0, 0)| = \left| \frac{1}{\sqrt{x^2 + y^2}} \right| \leq \frac{x^2 + y^2}{\sqrt{x^2 + y^2}} \leq \sqrt{x^2 + y^2} < \varepsilon

\text{when } |x^2 + y^2| \leq \varepsilon^2 \quad (= \varepsilon^2).

\therefore f(x, y) \text{ is continuous at } (0, 0).
(viii) \[ f(x,y) = \begin{cases} \alpha \sin \frac{1}{x} + y \sin \frac{1}{x} & \text{when } xy \neq 0 \\ 0 & \text{when } xy = 0 \end{cases} \]

S.T. \( f \) is continuous at \( (0,0) \).

Sol.
\[ \left| f(x,y) - f(0,0) \right| = \left| \alpha \sin \frac{1}{x} + y \sin \frac{1}{x} \right| \leq |\alpha| + |y| < \varepsilon \]

if \( |x| < \delta \) and \( |y| < \delta \) \( (\delta = \frac{\varepsilon}{2}) \)

\[ \lim_{(x,y) \to (0,0)} f(x,y) = 0 = f(0,0) \]

\[ \therefore f(x,y) \text{ is continuous at } (0,0). \]

(cix) \[ f(x,y) = \begin{cases} \frac{xy}{\alpha^2 + y^2} & \text{when } \alpha^2 + y^2 \neq 0 \\ 0 & \text{when } \alpha^2 + y^2 = 0 \end{cases} \]

S.T. \( f \) is not continuous at \( (0,0) \).

Sol.
\[ y = mx \]

\[ \lim_{(x,y) \to (0,0)} f(x,y) = \lim_{x \to 0} f(x, mx) = \lim_{x \to 0} \frac{\alpha^2 m}{(1 + m^2) \alpha^2} = \frac{m}{(1 + m^2)}, \text{ depends on } m. \]

\[ \lim_{(x,y) \to (0,0)} f(x,y) \text{ does not exist.} \]

\[ \therefore f(x,y) \text{ is not continuous at } (0,0). \]

(e) \[ f(x,y) = \begin{cases} (px + qy) \sin \frac{1}{y} \sin \frac{1}{y} & , y \neq 0 \\ 0 & , y = 0 \end{cases} \]

Is \( f \) continuous at \( (0,0) \)?

Sol.
\[ f(0,0) = 0. \]

\[ \left| f(x,y) - f(0,0) \right| = \left| (px + qy) \sin \frac{1}{y} \sin \frac{1}{y} - 0 \right| = \left| (px + qy) \right| \sin \frac{1}{y} \sin \frac{1}{y} \]

\[ \leq |px + qy| \leq |p|x + |q| |y| \leq |p| \frac{\varepsilon}{2|p|} + |q| \frac{\varepsilon}{2|q|} \]

when \( |x| < \frac{\varepsilon}{2|p|} = \delta_1 \) and \( |y| < \frac{\varepsilon}{2|q|} = \delta_2 \)

\[ \left| f(x,y) - f(0,0) \right| < \varepsilon \text{ when } |x| < \delta_1 \text{ and } |y| < \delta_2, \text{ where } \delta = \min \{\delta_1, \delta_2\} \]

\[ \therefore f(x,y) \text{ is continuous at } (0,0). \]
Partial Derivatives:

The partial derivatives of a function \( f(x,y) \) at \((a,b)\) with respect to \(x\) and \(y\) are respectively,

\[
\frac{\partial f}{\partial x} \text{ on } f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}, \text{ if it exists.}
\]

\[
\frac{\partial f}{\partial y} \text{ on } f_y = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}, \text{ if it exists.}
\]

Note: If the partial derivatives \( f_x \) and \( f_y \) exist and are bounded in a region \( R \subseteq \mathbb{R}^2 \), then \( f \) is continuous in \( R \).

Remark: Unlike the situation for functions of one variable, the existence of 1st order partial derivatives does not imply the continuity at the point.

Example: 1. If \( f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases} \)

Show that the partial derivatives exist at \((0,0)\) but the function is not continuous there.

Solution: Now, \( f_x(0,0) = \lim_{h \to 0} \frac{f(0+h, 0) - f(0,0)}{h} \)

\[
= \lim_{h \to 0} \frac{0}{h} = 0.
\]

\( f_x(0,0) \) exist and is equal to zero.

Similarly, \( f_y(0,0) = \lim_{k \to 0} \frac{f(0,0+k) - f(0,0)}{k} \)

\[
= \lim_{k \to 0} \frac{0}{k} = 0.
\]

Also, \( \lim_{(x,y) \to (0,0)} \frac{xy}{x^2+y^2} \) does not exist.

Hence \( f(x,y) \) is not continuous at \((0,0)\).
Ex. 2. Show that for the function \( f(x, y) = |x^2 + y^2| \), partial derivatives \( f_x \) and \( f_y \) do not exist at \((0,0)\) but \( f(x, y) \) is continuous at \((0,0)\).

**Sol.**

\[
\lim_{\alpha \to 0} \frac{f(\alpha, 0) - f(0, 0)}{\alpha} = \lim_{\alpha \to 0} \frac{|\alpha|}{\alpha} = \pm 1, \quad \text{which does not exist,}
\]

\[f_x(0,0) \text{ does not exist.}\]

Similarly, \( f_y(0,0) \) does not exist.

Here \( |f(x, y) - f(0, 0)| = |x^2 + y^2| - 0| = |x^2 + y^2| < \epsilon \)

if \( |x| < \frac{\epsilon}{2} \) and \( |y| < \frac{\epsilon}{2} \)

choosing \( \delta = \frac{\epsilon}{2} \), then

\[|f(x, y) - f(0, 0)| < \epsilon \quad \text{whenever} \quad |x| < \delta \quad \text{and} \quad |y| < \delta.
\]

\( f(x, y) \) is continuous at \((0,0)\).

Ex. 3. Let \( f(x, y) = \begin{cases} 
ax^2 \sin \frac{1}{x} + y^2 \sin \frac{1}{y} & \text{when} \quad x \neq 0, y \neq 0 \\
ax^2 \sin \frac{1}{x} & \text{when} \quad y = 0, x \neq 0 \\
y^2 \sin \frac{1}{y} & \text{when} \quad x = 0, y \neq 0 \\
0 & \text{when} \quad x = 0, y = 0
\end{cases} \)

Find \( f_x(0, y) \) and \( f_y(x, 0) \).

**Solution:**

For any fixed \( y \),

\[
\lim_{\alpha \to 0} \frac{f(\alpha, y) - f(0, y)}{\alpha} = \lim_{\alpha \to 0} \frac{ax^2 \sin \frac{1}{x} + y^2 \sin \frac{1}{y} - y^2 \sin \frac{1}{y}}{\alpha} = \lim_{\alpha \to 0} ax^2 \sin \frac{1}{x} = 0.
\]

Also,

For any fixed \( x \),

\[
\lim_{\gamma \to 0} \frac{f(x, \gamma) - f(x, 0)}{\gamma} = \lim_{\gamma \to 0} \frac{ax^2 \sin \frac{1}{x} + y^2 \sin \frac{1}{y} - ax^2 \sin \frac{1}{x}}{\gamma} = \lim_{\gamma \to 0} y^2 \sin \frac{1}{y} = 0,
\]

\( f_x(0, y) = f_y(x, 0) \).
Ex. (4) - For the function \( f(x, y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \)

Examine the existence of partial derivatives \( f_x \) and \( f_y \) at \((0, 0)\).

Solution: Here, \( \lim_{x \to 0} f(x, 0) - f(0, 0) \)

\[
= \lim_{x \to 0} \frac{0}{x^2} = 0
\]

Also, \( \lim_{y \to 0} \frac{f(0, y) - f(0, 0)}{y} = \lim_{y \to 0} \frac{0}{y} = 0 \).

\( f_x(0, 0) = 0 = f_y(0, 0) \).

Ex. (5) - If \( f(x, y) = \begin{cases} \frac{x^2-y^3}{x^2+y^2} & \text{when } x^2+y^2 \neq 0 \\ 0 & \text{when } x^2+y^2 = 0 \end{cases} \)

Show that \( f_x(0, 0) = 1 \) and \( f_y(0, 0) = -1 \).

Solution: Here, \( \lim_{x \to 0} f(x, 0) - f(0, 0) \)

\[
= \lim_{x \to 0} \frac{x^2-0}{x^2} = 1
\]

Also, \( \lim_{y \to 0} \frac{f(0, y) - f(0, 0)}{y} = \lim_{y \to 0} \frac{0-y^3}{0} = -1 \).

\( f_x(0, 0) = 1 \) and \( f_y(0, 0) = -1 \).
Differentiability of a function $f(x,y)$:

**Theorem 1:** If $f(x,y)$ is differentiable at $(a,b)$, partial derivative $f_x(a,b)$ and $f_y(a,b)$ exist.

**Theorem 2:** If $f(x,y)$ is differentiable at $(a,b)$, it is continuous there.

**Theorem 3:** (Sufficient condition for differentiability $f(x,y)$ at $(a,b)$)

Let $(a,b) \in D$, the domain of the function of $f$. If $f_x(a,b)$ exists and $f_y(a,b)$ is continuous at $(a,b)$, then $f(x,y)$ is differentiable at $(a,b)$.

**Examples:**

(1). The function $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if } x^2+y^2 = 0 \end{cases}$ is not differentiable at $(0,0)$.

**Solution:** We see that $f(x,y)$ is not continuous at $(0,0)$.

For, let $(x,y) \to (0,0)$ along $y = mx$.

$$\lim_{(x,y) \to (0,0)} f(x,y) = \lim_{x \to 0} f(x, mx) = \lim_{x \to 0} \frac{x^2 m}{x^2 (1+m^2)} = \frac{m}{1+m^2}$$

which is a function of $m$. Hence $f(x,y)$ is not continuous at $(0,0)$. Therefore $f(x,y)$ is not differentiable at $(0,0)$ because differentiability would imply continuity.

**Examples:**

(2). Show that $f(x,y) = |x+y|$ is not differentiable at $(0,0)$.

**Solution:** We have to show that $f(x,y)$ is not continuous at $(0,0)$.

Here, $\lim_{x \to 0} \frac{f(x+y) - f(0)}{x} = \lim_{x \to 0} \frac{|x|}{x}$ does not exist.

If $f(x,y)$ has no partial derivative, say, $\partial f / \partial x$ at $(0,0)$.

Similarly, $f_y(0,0)$ does not exist.

If $f(x,y)$ is not differentiable at $(0,0)$, then partial derivatives could exist there.

**Ex. (3):** Let $f(x,y) = |x+1| (1+y)$. Examine whether $f(x,y)$ is differentiable at $(0,0)$.

**Solution:**

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h, 0) - f(0,0)}{h} = \lim_{h \to 0} \frac{|h| - 0}{h}$$

does not exist.

And $f_y(0,0) = \lim_{k \to 0} \frac{f(0,0+k) - f(0,0)}{k}$

$$= \lim_{k \to 0} \frac{0-0}{k} = 0$$

Now, we know the necessary condition for the differentiability of a function at a point is the existence of partial derivatives at that point.

So, here $f$ is not differentiable at $(0,0)$. 

---

**Note:** The content of the image is a mathematical text discussing the differentiability of functions and providing examples to illustrate the concepts. The text is clear and readable, with mathematical expressions and theorems presented in a logical flow. The examples and solutions are worked out to demonstrate the application of the theorems.
Ex. (y). Let \( f(x, y) = x \), when \( 1|y| < |x| \)
- \( \infty \), when \( 1|y| \geq |x| \)
Examine whether \( f \) is differentiable at \((0,0)\).

\[ \begin{align*}
\text{Sol.} & \quad (f_x)(0,0) = \lim_{h \to 0} \frac{f(0+h, 0) - f(0, 0)}{h} \\
& = \lim_{h \to 0} \frac{h}{h} = 1 \\
\text{and} \quad (f_y)(0,0) = \lim_{k \to 0} \frac{f(0,0+k) - f(0,0)}{k} \\
& = \lim_{k \to 0} \frac{-k}{k} = -1.
\end{align*} \]

If \( f(x, y) \) is to be differentiable at \((0,0)\), we must have 
\[ f(h, 0) - f(0,0) = h(f_x(0,0)) + k(f_y(0,0)) + o(h, k) \]
where \( |o(h, k)| \to 0 \) as \((h,k) \to (0,0)\).

\[ \begin{align*}
\text{So,} \quad f(h, 0) - f(0,0) = h - k + k\phi - k\psi \\
\text{In particular if} \quad k = -\phi, \text{ we get} \\
-1 = -2k - k\phi - k\psi, \\
\Rightarrow 1 = \phi - \phi \\
\text{As} \quad k \to 0, \quad \text{RHS} \to 0 \text{ but LHS} \to 0. \quad \text{So,} \quad f \text{ is not differentiable at} \quad (0,0).
\end{align*} \]

Ex. (b) - Let \( f(x, y) = xy^2 \), \( y > 0 \)
\[ \begin{align*}
\text{8.v.} \quad (f_y)(0,0) = f_y(x, y), \quad x \geq 0, \quad y \leq 0 \]

\[ \begin{align*}
\text{Sol.} & \quad (f_x)(0,0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} \\
& = \lim_{h \to 0} \frac{h}{h} = 0 \\
\text{and} \quad (f_y)(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} \\
& = \lim_{k \to 0} \frac{k^2}{k} = 0
\end{align*} \]

Therefore, \( \lim_{k \to 0^+} \frac{f_x(0,k) - f_x(0,0)}{k} = \lim_{k \to 0^-} \frac{k^2}{k} = 0 \)

\begin{align*}
& \text{Therefore,} \quad \lim_{k \to 0^+} \frac{f_x(0,k) - f_x(0,0)}{k} = 0 \\
& \text{and} \quad \lim_{k \to 0^-} \frac{f_x(0,k) - f_x(0,0)}{k} = 0 \\
& \text{So,} \quad \frac{f_y(0,0)}{x} = 0.
\end{align*}
Differentiability: Let \( f(x,y) \) be a function defined on \( \mathbb{R}^2 \).

Consider the increment:
\[
\Delta f = f(x+h,y+k) - f(x,y) = Ah + Bk + h \phi(x,h,k) + k \psi(x,h,k),
\]
where \( A = \frac{\partial f}{\partial x}(x,y) \), \( B = \frac{\partial f}{\partial y}(x,y) \) and \( \phi \) and \( \psi \) are functions of \( (x,h,k) \).

Then \( f \) is differentiable at \( (x,y) \).

Then \( \Delta f = f_x \Delta x + f_y \Delta y \) is the differential at \( (x,y) \).

Note:
\[
\lim_{(h,k) \to (0,0)} \left[ \frac{f(x+h,y+k) - f(x,y)}{\sqrt{h^2+k^2}} \right] = \lim_{(h,k) \to (0,0)} \left[ \frac{Ah + Bk + h \phi(x,h,k) + k \psi(x,h,k)}{\sqrt{h^2+k^2}} \right] = 0.
\]

Hence differentiability of \( f(x,y) \) implies the continuity and also the existence of \( f_x \) and \( f_y \). If a function is not continuous, then it is not differentiable.

Ex. (i): Examine the differentiability of:
\[
f(x,y) = \begin{cases} \frac{x^3 - y^3}{x^2 + y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}
\]
at \( (x,y) = (0,0) \).

Solution: The differentiability of \( f \) at \( (0,0) \),
we must have
\[
\Delta f = f(h,k) - f(0,0) = Ah + Bk + h \phi + k \psi
\]
where, \( A = \frac{\partial f}{\partial x}(0,0) \), \( B = \frac{\partial f}{\partial y}(0,0) \) and \( \phi, \psi \to 0 \) as \( (h,k) \to (0,0) \).

Now, \( A = \frac{\partial f}{\partial x}(0,0) = \lim_{k \to 0} \frac{f(h,k) - f(0,0)}{h} = \lim_{h \to 0} \frac{(\frac{h^3}{h^2+0^2}) - 0}{h} = 1 \).

and \( B = \frac{\partial f}{\partial y}(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \to 0} \frac{0 - 0}{0^2+k^2} = -1 \).

Now,
\[
\frac{h^2 - k^2}{h^2+k^2} = \frac{-h(\frac{h^2+k^2}{h^2+k^2}) + k(h^2+k^2)}{h^2+k^2} = h(1) + \frac{-k^2}{h^2+k^2} + k \left( \frac{k^2}{h^2+k^2} \right) = h(1) + k(-1) + \frac{k^2}{h^2+k^2} + k \psi.
\]
but \( \lim \frac{\partial f}{\partial x} = \lim_{(h,k) \to (0,0)} \frac{-k^2}{h^2 + k^2} \) &
\lim \frac{\partial f}{\partial y} = \lim_{(h,k) \to (0,0)} \frac{-2h}{h^2 + k^2} \) do not exist.

Hence, \( f(x,y) \) is not differentiable at \((0,0)\).

Example 2: Show that the function \( f(x,y) = \sqrt{x^2 + y^2} \), \( (x,y) \neq (0,0) \)
is continuous and possesses partial derivatives but not differentiable at \((0,0)\).

Solving, let us first show that \( f \) is continuous at \((0,0)\).

\[
\lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2}} - 0 \leq \frac{x^2 + y^2}{\sqrt{x^2 + y^2}} = \frac{x^2 + y^2}{\sqrt{x^2 + y^2}} < \varepsilon
\]
whenever \( x^2 + y^2 < \delta^2 \) \((\varepsilon, \delta)\).

\( f(x,y) \) is continuous at \((0,0)\).

Note, \( f_x (0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0 \).

Similarly, \( f_y (0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = 0 \).

Note, one can show that \( f(x,y) \) is not differentiable at \((0,0)\).

If possible let, \( f(x,y) \) is differentiable at \((0,0)\).

\[
f(x,y) = f(0,0) = A(x^2 + y^2) + \varepsilon_1 + \varepsilon_2
\]
where, \( \varepsilon_1, \varepsilon_2 \to 0 \) as \( x, y \to 0 \).

\[
f(h,k) - f(0,0) = Ah^2 + 2h + k + \varepsilon_1 + \varepsilon_2 \quad \text{[Limit A=0, \varepsilon=0]}
\]

\[
\frac{h^2}{h^2 + k^2} \Rightarrow \varepsilon = \frac{hk}{h^2 + k^2}
\]
does not tend to a unique limit as \((h,k) \to (0,0)\).

In fact, \( \frac{h^2}{h^2 + k^2} \) along \( k=mh \), we see that

\[
\lim_{(h,k) \to (0,0)} \frac{h^2}{h^2 + k^2} = \lim_{h \to 0} \frac{mh^2}{h^2 + m^2 h^2} = \lim_{h \to 0} \frac{m^2 h^2}{h^2 (1+m^2)} = \frac{m^2}{1+m^2}
\]
depends on \( m \).

Therefore, \( \lim_{(h,k) \to (0,0)} \varepsilon \) does not exist. Hence, \( f(x,y) \) is not differentiable at \((0,0)\).
Ex. (3): If \( f(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases} \),

Show that \( f(x, y) \) is differentiable at \((0, 0)\).

Solution: Partial derivatives \( f_x(0, 0) \) and \( f_y(0, 0) \) exist.

\[
\lim_{(x, y) \to (0, 0)} \frac{f(x+h, y) - f(x, y)}{h} = \lim_{(x, y) \to (0, 0)} \frac{f(x, k) - f(x, 0)}{k}
\]

\[
= \lim_{h \to 0} \frac{0 - 0}{h} = 0
\]

\[
f_x(0, 0) = 0,
\]

Similarly, \( f_y(0, 0) = 0 \).

Now, \( f(0+h, 0+k) - f(0, 0) = f(h, k) - f(0, 0) = f(h, k) = \frac{hk(h^2-k^2)}{h^2+k^2} \).

If we write \( \frac{hk(h^2-k^2)}{h^2+k^2} = 0 \), \( h > 0 \), \( h \to \infty \), \( k \to 0 \), then \( h^2+k^2 = h \sqrt{h^2+k^2} \).

\[
\therefore \ |h(1) - 0| = \left| \frac{hk(h^2-k^2)}{h^2+k^2} \right| \leq \frac{\sqrt{h^2+k^2}}{\sqrt{h^2+k^2}} \leq \sqrt{\sqrt{h^2+k^2}} \leq \sqrt{\eta^2} = \eta,
\]

\[
\therefore \ \lim_{(h,k) \to (0,0)} f(h,k) = 0
\]

\( f(x, y) \) is differentiable at \((0, 0)\).
**Liebnitz's Rule**

**Theorem:** If \( f \) is a continuous function on \([a,b]\) and \( u(x) \) and \( v(x) \) are differentiable functions on \( a \) whose values lie in \([a,b]\) then

\[
\frac{d}{dx} \left\{ \int_{u(x)}^{v(x)} f(t) \, dt \right\} = f(v(x)) \frac{dv(x)}{dx} - f(u(x)) \frac{du(x)}{dx}
\]

**Proof:**

Let \( \frac{d}{dx} F(x) = f(x) \)

\[
\int_{u(x)}^{v(x)} f(t) \, dt = F(v(x)) - F(u(x))
\]

\[
\frac{d}{dx} \left\{ \int_{u(x)}^{v(x)} f(t) \, dt \right\} = \frac{d}{dx} \left( F(v(x)) - F(u(x)) \right)
\]

\[
\frac{d}{dx} \left( \int_{u(x)}^{v(x)} f(t) \, dt \right) = v'(x) \frac{d}{dx} F(u(x)) - u'(x) \frac{d}{dx} F(v(x))
\]

\[
\frac{d}{dx} \left( \int_{u(x)}^{v(x)} f(t) \, dt \right) = f(u(x)) \frac{du(x)}{dx} - f(v(x)) \frac{dv(x)}{dx}
\]

**Example 1.** If \( y = \int_{2}^{x} \frac{1}{\log t} \, dt \) (where \( x > 0 \)), then find \( \frac{dy}{dx} \).

**Solution:**

\[
y = \int_{2}^{x} \frac{1}{\log t} \, dt
\]

\[
\frac{dy}{dx} = \frac{1}{\log x^2} - \frac{1}{\log 2}
\]

\[
= \frac{2x^2 - 2x}{2x \log x} = x(x-1)(\log x)^{-1}
\]

**Example 2.** If \( \int_{\sin x}^{1} t^2 f(t) \, dt = 1 - \sin x \), where \( x \in (0, \frac{\pi}{2}) \), then find the value of \( f \left( \frac{1}{\sqrt{3}} \right) \).

**Solution:**

\[
\int_{\sin x}^{1} t^2 f(t) \, dt = 1 - \sin x
\]

Differentiating both sides w.r.t. \( x \), we get

\[
1^2 f(1) - \sin^2 x f'(\sin x) \cos x = -\cos x
\]

\[
f'(\sin x) = \cos x \sec^2 x = \frac{1}{\sin^2 x}
\]

\[
\Rightarrow f'(1) = \frac{1}{2^2}
\]

\[
\Rightarrow f \left( \frac{1}{\sqrt{3}} \right) = 3
\]
Ex. (3): Evaluate \( \lim_{x \to 0} \left( \int_0^x e^{2z} \, dz \right)^2 \).

**Sol.**

Since \( e^{2x} > 0 \), \( e^{2x} > 0 \) in \([0, x]\), where \( x > 0 \).

\[
\int_0^x e^{2z} \, dz = \left[ \frac{1}{2} e^{2z} \right]_0^x = \frac{1}{2} e^{2x} - \frac{1}{2} = \frac{1}{2} e^{2x}
\]

So, \( \left( \int_0^x e^{2z} \, dz \right)^2 = \left( \frac{1}{2} e^{2x} \right)^2 = \frac{1}{4} e^{4x} \).

\[
L = \lim_{x \to 0} \frac{\left( \int_0^x e^{2z} \, dz \right)^2}{\int_0^x e^{2z} \, dz} = \lim_{x \to 0} \frac{\left( \frac{1}{2} e^{2x} \right)^2}{\frac{1}{2} e^{2x}} = \lim_{x \to 0} \frac{1}{2} e^{2x} = 0.
\]

Using L'Hôpital's Rule,

\[
L = \lim_{x \to 0} \frac{2e^{2x} \int_0^x e^{2z} \, dz}{2e^{2x}} = 2 \lim_{x \to 0} \int_0^x e^{2z} \, dz = 2 \lim_{x \to 0} \left[ \frac{1}{2} e^{2z} \right]_0^x = 2 \lim_{x \to 0} \left( \frac{1}{2} e^{2x} - \frac{1}{2} \right) = 2 \lim_{x \to 0} \frac{1}{2} e^{2x} = 0.
\]

Ex. (4):

Let \( f: (0, \infty) \to (0, \infty) \) be a differentiable function satisfying \( x \int_0^x (1-t)f(t) \, dt = \int_0^x f(t) \, dt \) for all \( x \in \mathbb{R}^+ \) and \( f(1) = 1 \).

Determine \( f(x) \).

**Sol.**

\[
\alpha \int_0^x (1-t)f(t) \, dt = \int_0^x f(t) \, dt
\]

Differentiating both sides with respect to \( x \), we get,

\[
\alpha \int_0^x (1-t)f(t) \, dt = \int_0^x f(t) \, dt
\]

Differentiating both sides with respect to \( x \), we again get,

\[
\alpha^2 f(x) + 2\alpha f(x) = (1-x) f(x)
\]

\[
\Rightarrow \frac{f(x)}{x} = \frac{1-2x}{x^2}
\]

\[
\Rightarrow \int \frac{f(x)}{x} \, dx = \int \frac{1-2x}{x^2} \, dx
\]

\[
\Rightarrow \log \left( \frac{f(x)}{e} \right) = -\frac{1}{x} - 3\log x
\]

Given \( f(1) = 1 \) \( \Rightarrow \log \left( \frac{1}{e} \right) = -1 \) \( \Rightarrow e = e \)

\[
\Rightarrow \log \left( \frac{f(x) \cdot x^3}{e} \right) = -\frac{1}{x} \Rightarrow f(x) = \frac{x^3}{e^{\frac{1}{x}}}
\]
Theorem: If the functions \( \phi(x) \) and \( \psi(x) \) are defined on \([a, b]\) and differentiable functions at a point \( x \in (a, b) \) and \( f(x,t) \) is continuous then,

\[
\frac{d}{dx} \left[ \int_a^b \frac{\psi(x)}{\phi(x)} \int_a^t f(x,t) \, dt \right] = \int_a^b \frac{d\psi(x)}{dx} \frac{1}{\phi(x)} f(x,t) \, dt + \left\{ \frac{d\phi(x)}{dx} \int_a^t f(x,t) \, dt \right\} \phi(x) - \left\{ \frac{d\phi(x)}{dx} \right\} \frac{d\psi(x)}{dx} \int_a^t f(x,t) \, dt.
\]

Corollary: If \( f \) is continuous on \( R = [a, b] \), then,

\[
\int_a^b \int_a^b f(z, y) \, dy \, dz = \int_a^b \int_a^b f(z, y) \, dz \, dy.
\]

i.e., the two repeated (iterated) integrals are equal.

Worked Examples:

1. If \( f(x) = \int_a^x t^3 e^t \, dt \), then find \( f'(x) \).

Solution:
\[
\frac{d}{dx} f(x) = \frac{d}{dx} \int_a^x t^3 e^t \, dt = \left[ t^3 e^t \right]_a^x = x^3 e^x - a^3 e^a.
\]

2. If \( \phi(x) = \int_a^x \sin(x^2) \, dt \), then \( \phi'(4) = ? \)

Solution:
\[
\phi'(x) = \int_a^x \sin(x^2) \, dt = \frac{1}{2} \sin x^2 + \frac{1}{2} \sin \frac{1}{x^2}.
\]
\[
\phi'(4) = \frac{1}{2} \sin 4^2 + \frac{1}{2} \sin \frac{1}{4^2} = \frac{3}{2} \sin 4.
\]

3. If \( f(x) = \int_a^x \sin\sqrt{t} \, dt \), then \( f'(x) = ? \)

Solution:
\[
f'(x) = \int_a^x \sin\sqrt{t} \, dt = 4x^3 \sin(x^2) - 2x \sin x^2.
\]

4. If \( f(t) = \int_a^t \frac{dx}{1 + x^2} \), then \( f'(t) = ? \)

Solution:
\[
f(t) = \left[ \arctan x \right]_a^t = \arctan t + \arctan a = \frac{\pi}{2} - \arctan a.
\]
\[
f'(t) = 2 \cdot \frac{1}{1 + t^2},
\]
\[
f'(1) = 1.
\]
5. If \( f(x) = \frac{1}{x^{2/3}} \int_{1}^{x} \left( 4t^{2} - 2F(t) \right) \, dt \), then \( F'(4) \) equals?

Solution (Sol.):
\[ F'(x) = \frac{1}{x^{2/3}} \left( 4x^{2} - 2F(x) \right) = \frac{2}{x^{2/3}} \int_{1}^{x} \left( 4t^{2} - 2F(t) \right) \, dt \]
\[ \Rightarrow F'(4) = \frac{1}{16} \left[ 64 - 2F(4) \right] = 0 \]
\[ \Rightarrow F'(4) = \frac{32}{3} \]

6. If \( \int_{0}^{\pi/2} f'(\sin x) \, dx = A \int_{0}^{\pi/2} f'(\sin x) \, dx \), then \( A = ? \)

Solution (Sol.):
\[ I = \int_{0}^{\pi/2} f'(\sin x) \, dx \]
\[ 2\pi I = 2 \pi \int_{0}^{\pi/2} f'(\sin x) \, dx \]
\[ A = \pi \]

7. The points of intersection of \( F_{1}(x) = \int_{2}^{x} (2t - 5) \, dt \) and \( F_{2}(x) = \int_{2}^{x} t \, dt \) are:

(A) \( \left( \frac{6}{5}, \frac{36}{25} \right) \), (B) \( \left( \frac{3}{5}, \frac{4}{5} \right) \), (C) \( \left( \frac{1}{5}, \frac{1}{5} \right) \), (D) \( \left( \frac{1}{5}, \frac{1}{25} \right) \)

Solution (Sol.):
(A) \( y_{1} = \int_{2}^{x} (2t - 5) \, dt \), \( y_{2} = \int_{2}^{x} t \, dt \)

Now point of intersection means those points at which
\[ y_{1} = y_{2} \]
\[ \Rightarrow x^{2} - 5x + c \]
and \( y_{2} = x^{2} \)

Now, solving, we have \( x = \frac{6}{5}, x^{2} = \frac{36}{25} \).

8. If \( f(x) = \int_{x^{2}}^{x^{2}+1} e^{-t^{2}} \, dt \) then find the interval in which the function \( f(x) \) is increasing.

Solution (Sol.):
\[ f(x) \text{ is said to be increasing if } f'(x) > 0, \]
\[ f'(x) = e^{-(x^{4}+1)^{2}} - e^{-(x^{2})^{2}} \cdot 2x = 2xe^{-x^{4}+2x^{2}}(1 - e^{-x^{2}}) \]
\[ \Rightarrow f'(x) > 0 \Rightarrow x \in (-\infty, 0). \]
9) If the value of the definite integral \( \int_{a}^{b} \frac{x}{x^3 + 16} \, dx \) lies in the interval \([a, b]\), then find the smallest integral.

**Solution:**

The function \( f(x) = \frac{x}{x^3 + 16} \) is an increasing function.

So, \( \min f(x) = f(0) = 0 \) and \( \max f(x) = f(1) = \frac{1}{17} \).

Therefore, by the law \( m(b-a) \leq \int_{a}^{b} f(x) \, dx \leq M(b-a) \)

\[ 0 \leq \int_{a}^{b} \frac{x}{x^3 + 16} \, dx \leq \frac{1}{17}. \]

10) Find the points of extremum of the following integral:

\[ \int_{0}^{x} \frac{t^2 - 5t + 4}{2 + e^t} \, dt. \]

**Solution:**

Let \( F(x) = \int_{0}^{x} \frac{t^2 - 5t + 4}{2 + e^t} \, dt \)

\[ F'(x) = \frac{x^2 - 5x + 4}{2 + e^x} \cdot 2x \]  
(By Leibniz's rule)

From \( F'(x) = 0 \), we get \( x = 0 \) or \( x = \frac{5 \pm \sqrt{25-16}}{2} = 2 \, \text{or} \, 1 \).

Hence \( x = 0, 1, 2 \).

11) Let \( f, g: \mathbb{R} \to \mathbb{R} \) be continuous functions, then the value of the integral \( \int_{-\pi/2}^{\pi/2} [f(x) + f(-x)] g(x) \, dx = ? \)

**Solution:**

Let \( h(x) = f(x) + f(-x) \)

\[ h(x) = -h(-x) \]

\( h(x) \) is an odd function,

\[ \int_{-\pi/2}^{\pi/2} h(x) \, dx = 0. \]

12) Let \( f(x) \) be a function satisfying \( f'(x) = f(x) \) with \( f(0) = 1 \) and \( g(x) \) be the function satisfying \( f(x) + g(x) = e^x \), then the value of the integral \( \int_{0}^{ln5} f(x) g(x) \, dx \) is \( ? \)

**Solution:**

\[ f'(x) = f(x) \]

\[ \frac{f(x)}{f(x)} = 1 \]

\[ \log f(x) = x + \log c \]  
(since \( f(0) = 1 \), \( c = 1 \)).

\[ f(x) = c e^x \]

\[ f(x) = e^x \]  
\( g(x) = x^2 - e^x \).

\[ \int_{0}^{\ln 5} f(x) g(x) \, dx = \int_{0}^{\ln 5} e^x (x^2 - e^x) \, dx \]
(13) Let the function \( f(x) \) be defined as \( f(x) = \int_0^x \sqrt{2 - t^2} \, dt \) then find the no. of real roots of the equation \( x^2 - f''(x) = 0 \).

**Sol.**

\[
\int \sqrt{2 - x^2} \, dx = \frac{1}{2} \left( x \sqrt{2 - x^2} - 2 \arcsin \frac{x}{\sqrt{2}} \right) + C
\]

\( x^2 - \frac{1}{2} \left( x \sqrt{2 - x^2} - 2 \arcsin \frac{x}{\sqrt{2}} \right) = 0 \)

\( \Rightarrow \frac{1}{2} \left( x \sqrt{2 - x^2} - 2 \arcsin \frac{x}{\sqrt{2}} \right) = 0 \)

\( \Rightarrow x^2 = \frac{4}{\pi} \)

\( \Rightarrow x = \pm \frac{2}{\sqrt{\pi}} \). Hence there are 2 roots.

(14) Let \( \frac{d}{dx} F(x) = \left( \frac{e^{\sin x}}{x} \right), \ x > 0 \), if \( \int_{1}^{4} \frac{e^{\sin x^2}}{x} \, dx = F(x) - F(0) \) then find one of the possible value of \( F(4) \).

**Sol.**

\[
\frac{d}{dx} F(x) = \frac{e^{\sin x}}{x}
\]

\[
\int_{1}^{4} \frac{e^{\sin x^2}}{x} \, dx = \int \frac{3x^2}{x^2} \, e^{\sin x^2} \, dx - (x = 1)
\]

Put \( x^2 = t \) \( \Rightarrow 2x \, dx = dt \)

\[
F(t) = \int_{1}^{4} \frac{e^{\sin t}}{t} \, dt = \int_{1}^{4} F(t) \, dt = F(4) - F(1)
\]

\( \therefore k = 64 \).

(15) Let \( g \) and \( f \) be continuous function on \( R \) so \( f(x) = \int_{0}^{x} g(t) \, dt \) and \( g(x) = \int_{0}^{x} f(t) \, dt + 1 \). Prove that \( (f(x))^2 + (g(x))^2 = 1 \).

**Sol.**

Applying leibniz rule on \( f(x) \) and \( g(x) \) we have

\[
f'(x) = \frac{d}{dx} \left( \int_{0}^{x} g(t) \, dt \right) = g(x)
\]

\[
g'(x) = \frac{d}{dx} \left[ \int_{0}^{x} f(t) \, dt + 1 \right] = -f(x)
\]

\[
f''(x) = g'(x) = -f(x)
\]

\[
g''(x) = -f'(x) = -f(x)
\]

\[
f'''(x) f'(x) + f''(x) f'(x) = -f(x) f'(x) - f'(x) f'(x) = 0
\]

Integrating \( [f'(x)]^2 + [g'(x)]^2 = 1 \) \( \Rightarrow [f(x)]^2 + [g(x)]^2 = 1 \).
Extrema of Functions of several variables:

If \( D \subseteq \mathbb{R}^n \) be the domain of a function \( f \). We shall denote a point in \( D \) as \( \mathbf{x} \) and the value of the function at \( \mathbf{x} \) by \( f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n) \).

Definition: A function \( f \) has a local (or relative) maximum or minimum at \( \mathbf{x}_0 \) if there is some neighbourhood \( N_\delta(\mathbf{x}_0) = \{ \mathbf{x} : |\mathbf{x} - \mathbf{x}_0| < \delta \} \) such that

\[
\forall \mathbf{x} \in N_\delta(\mathbf{x}_0), \quad f(\mathbf{x}) \leq f(\mathbf{x}_0) \quad \text{or} \quad f(\mathbf{x}) \geq f(\mathbf{x}_0).
\]

A necessary condition for an extremum of a differentiable function \( f \) at \( \mathbf{x}_0 \) is

\[
\left( f_{x_1}(\mathbf{x}_0), f_{x_2}(\mathbf{x}_0), \ldots, f_{x_n}(\mathbf{x}_0) \right) = 0.
\]

OR

\[
\nabla f(\mathbf{x}_0) = 0 \quad \text{or} \quad f_{x_i}(\mathbf{x}_0) = 0 \quad \forall \ i = 1(1)n,
\]

The partial derivative of \( f \) with respect to \( x_i \) is given by

\[
f_{x_i}(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x}_1, \ldots, x_i + h, \ldots, x_n) - f(\mathbf{x}_1, \ldots, x_i, \ldots, x_n)}{h}.
\]

Notation: \( \frac{\partial f(\mathbf{x})}{\partial x_i} \), etc.

A Functions of two variables:

Theorem 1: If the 2nd order partial derivatives of \( f \) are continuous at each point in \( D \) is an open domain in \( \mathbb{R}^2 \), and \( (x_0, y_0) \in D \) such that \( \nabla f(x_0, y_0) = 0 \), i.e., \( f_{x}(x_0, y_0) = f_{y}(x_0, y_0) \), then

\[
f(f(x_0, y_0)) \text{ is a local maximum if and only if } D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \text{ is negative definite.}
\]

Hence, \( f_{xx}(x_0, y_0) < 0 \) and \( f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 > 0 \).
ii) \( f(x_0, y_0) \) is a local minimum if \( D \) is positive definite

i.e. \( f_{xx}(x_0, y_0) > 0 \) and

\[ f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 > 0 \]

iii) \( f(x_0, y_0) \) is neither a maximum nor a minimum if

\( D \) is indefinite, i.e.

\[ f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 < 0. \]

\[ \star \text{Example: } \Rightarrow 1. \text{ Find all the local maximum and minimum} \]

\[ f(x, y) = 2x^2 - xy + 2y^2 - 20x. \]

\[ f_x(x, y) = 4x - y - 20 \]

\[ f_y(x, y) = -x + 4y \]

For points of extremum,

\[ f_x(x, y) = 0 \Rightarrow 4x - y = 20 \Rightarrow \{ x = \frac{16}{3} \} \]

\[ f_y(x, y) = 0 \Rightarrow -x + 4y = \frac{16}{3} \Rightarrow \{ y = \frac{4}{3} \} \]

Now,

\[ f_{xx}(x, y) = 4 \quad \text{and} \quad f_{xy}(x, y) = -1 \]

Now,

\[ D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ -1 & 4 \end{vmatrix} = 16 - 1 = 15 > 0 \]

Hence, \( D \) is positive definite.

Therefore, \( f \left( \frac{16}{3}, \frac{4}{3} \right) \) is the minimum value.
Example 2. Give \( n \) points \((x_i, y_i)\), where \( x_i \)'s are distinct, find \( a \) and \( b \) such that the function \( f(a, b) = \sum_{i=1}^{n} (ax_i + b - y_i)^2 \) is minimum.

Solution. Hence, \( f_a = \sum_{i=1}^{n} 2(ax_i + b - y_i) x_i \)

\[ f_b = \sum_{i=1}^{n} 2(ax_i + b - y_i) \]

\[ f_{aa} = 2 \sum x_i \]

\[ f_{bb} = 2 \text{ and } f_{ab} = 2 \sum x_i \]

For points of extreme: \( f_a = 0 \) \( \Rightarrow \) \( \sum (x_i x - \bar{x})(y_i - \bar{y}) = \lambda \sum (x_i - \bar{x}) \)

\( f_b = 0 \) \( \Rightarrow \) \( b = \frac{\lambda}{n} - \lambda \bar{x} = \lambda \bar{y} \)

Note, \( D = \begin{bmatrix} f_{aa} & f_{ab} \\ f_{ab} & f_{bb} \end{bmatrix} = 2 \begin{bmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & n \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \)

Hence, \( f_{aa}(\lambda, \bar{y}) = 2 \sum x_i^2 > 0 \)

and \( |D| = 4 \sum x_i^2 - (\sum x_i)^2 \bar{y} = 4x_i \sum x_i = 2 \sum x_i \bar{y} > 0 \)

Hence, \( f(a, b) \) is the minimum value of \( f(a, b) \).

Example 3. Show that \( f(x, y) = y^2 + x^2 + ax \) has a minimum at \((0, 0)\).

Solution. For points of extreme:

\( 0 = f_x = 2xy + 4x^3 \)

\( 0 = f_y = 2y + a \)

\( \Rightarrow \) \( x = 0 = y \)

Again, \( f_{xx} = 2y = 12x^2 \)

\( f_{yy} = 2 = 2 \)

Note that, \( D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \)

\( |D| = f_{xx}(0, 0) f_{yy}(0, 0) - f_{xy}(0, 0)^2 = 0 \)

Hence, it is a doubtful case and requires further investigation.
Note that, 
\[ f(x, y) = \left( y + \frac{2y \cdot \frac{x}{2} - \frac{x^4}{4}}{2} \right) + \frac{3x^4}{4} \]

\[ = \left( y + \frac{3x^4}{4} \right) + \frac{3x^4}{4} \geq 0 \]

Now, \( f(x, y) = 0 \) if \( y + \frac{3x^4}{4} = 0 \) if \( x = 0 \) y = 0,

Hence, \( f(x, y) \) has a minimum value at \( (0, 0) \).

**Example 1:** Show that the function, \( f(x, y) = 2x^4 - 3xy + y^3 \) has neither a maximum nor a minimum at \( (0, 0) \).

**Solution:**
\[ \frac{\partial f}{\partial x} = 8x^3 - 3y \]
\[ \frac{\partial f}{\partial y} = 3x^4 + 3y^2 \]
\[ \frac{\partial^2 f}{\partial x^2} = 24x^2 - 6y \]
\[ \frac{\partial^2 f}{\partial y^2} = 6y \]
\[ \frac{\partial^2 f}{\partial x \partial y} = 2y \]

Now, \( f_x(0, 0) = 0 = f_y(0, 0) \),

Also, \( f_{xx}(0, 0) f_{yy}(0, 0) - f_{xy}(0, 0)^2 = 0 \)

Hence, it is a doubtful case and requires further investigation.

Now, \( f(x, y) - f(0, 0) \)
\[ = 2x^4 - 3xy + y^3 \]
\[ = (y - x^4) (y - 2x) \]

Thus, \( S < 0 \) as \( x < y < 2x \)
\[ \text{as } y > 2x, y < x \text{ and } y > 2x \]

In every neighbourhood of \( (0, 0) \), \( f(x, y) \) has neither a maximum nor a minimum, i.e. \( f(x, y) \) has a saddle point at \( (0, 0) \).
**Ex. 5** Find all local maxima and minima of

\[ f(x, y) = xy e^{-x}. \]

**Soln.**

\[ f(x, y) = xy e^{-x}. \]

\[ f_x = y e^{-x} - xye^{-x}, \]

\[ f_y = xe^{-x}. \]

For points of extremum, \( f_x = 0, f_y = 0 \)

\[ \Rightarrow y e^{-x} - xye^{-x} = 0 \]

\[ \Rightarrow y = 0 \] on \( x = 1, \) \( \Rightarrow x = 0 \)

\[ \Rightarrow (x, y) = (0, 0) \]

Now, \( f_{xx} = -ye^{-x}(1-x) - ye^{-x} \]

\[ = -ye^{-x}(2-x) \]

\[ f_{yy} = 0, \quad f_{xy} = e^{-x} - xe^{-x} = e^{-x}(1-x). \]

Now, at \((0, 0), f_{xx} f_{yy} - f_{xy}^2 = 0 - 1^2 = -1 \]

i.e., \[ D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} \]

is indefinite.

Hence \( f(0, 0) \) is neither a maxima nor a minima, and \( f(x, y) \) has no maximum or no minimum at \((0, 0).\)
Theorem: If the function $f$ has 2nd order partial derivatives that are continuous at every point in $D$, where $D$ is an open region and $x_0 \in D$, such that $\nabla f(x) = 0$, i.e., $f_x(x) = 0, f_y(x) = 0$, then

i) $f(x_0)$ is a local maximum if the matrix of 2nd order partial derivatives at $x_0$, i.e.,
$$D = \left| \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right|_{x_0}$$

is negative definite.

ii) $f(x_0)$ is a local minimum if $D$ is positive definite.

iii) $f(x_0)$ is neither a maximum nor a minimum if $D$ is indefinite.

Example: Find all the local maximum and minimum of
$$f(x, y, z) = x^2 + 2y^2 + z^2 - 2xy + yz.$$

Solution:
$$f_x = 2x - 2y$$
$$f_y = 4y - 2x + z$$
$$f_z = 2z + y$$

Now, for points of extremum,
$$0 = f_x = 2x - 2y$$
$$0 = f_y = 4y - 2x + z$$
$$0 = f_z = 2z + y$$

$$\Rightarrow \begin{pmatrix} 2 & -2 & 0 \\ -2 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

As, $\begin{vmatrix} 2 & -2 & 0 \\ -2 & 4 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 6 \neq 0$

It is non-singular, hence inverse exists.

Now, $f_{xx} = 2$, $f_{yy} = 4$, $f_{xy} = 2$, $f_{xz} = 1$, $f_{zx} = 0, f_{zz} = 2$. 

Solve...
Note that, 
\[
\begin{bmatrix}
\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\
\frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\
\frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2}
\end{bmatrix}
= \begin{bmatrix}
2 & -2 & 0 \\
-2 & 4 & 1 \\
0 & 1 & 2
\end{bmatrix}
\]
Here, \(2 > 0\), \(\begin{vmatrix} 2 & -2 \\ -2 & 4 \end{vmatrix} = 4 > 0\)
\(|D| = 6 > 0 \Rightarrow \text{D is p.d. matrix.}\)
\(\Rightarrow f(0,0,0)\) is the minimum value.

\(\star \text{Ex. 2.}\) Show that the function has a minimum at \((1,1,1)\).

\(f(x, y, z) = x^4 + y^4 + z^4 - 4xyz\)

**Solu.**
Here,
\(f_x = 4x^3 - 4y^2, \quad f_y = 4y^3 - 4z^2, \quad f_z = 4z^3 - 4xy\)

For points of extremum,
\(f_x = 0 \Rightarrow x^3 = y^2, \quad f_y = 0 \Rightarrow y^3 = 2z^2, \quad f_z = 0 \Rightarrow z^3 = xy\)
\(\Rightarrow (x^3)^2 \cdot (2z^2) = x^6 \cdot 2z^2 \Rightarrow (xyz)^2 = (xy)^2 = 1 \Rightarrow 2xyz = 0, \text{ or } 1\).

Now, \(x^4 = y^4 = z^4 = 2xyz = 2 \Rightarrow (x, y, z) = (0, 0, 0), (1, 1, 1)\).

Hence, \((1,1,1)\) is a point of extremum.
At the point \((1,1,1)\), \(f_x = f_y = f_z = 0\).

Now, \(f_{xx} = 12x^2, \quad f_{xy} = -4z, \quad f_{xz} = -4y, \quad f_{yy} = 12y, \quad f_{yz} = -4x, \quad f_{zz} = 12z\).

Now,
\[
D = \begin{bmatrix}
\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\
\frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\
\frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2}
\end{bmatrix}
= \begin{bmatrix}
12 & -4 & -4 \\
-4 & 12 & -4 \\
-4 & -4 & 12
\end{bmatrix}
\]
Here, $D > 0$, $\begin{vmatrix} 12 & 4 \\ -4 & 12 \end{vmatrix} > 0$, $|D| > 0$,

i.e., $D$ is p.d.

Hence, $f(1,1,1)$ is a minimum of $f(x,y,z)$.

Method of Lagrange's Multipliers:

Suppose, we wish to minimize or maximize $Z = f(x,y)$ where $(x,y)$ is constrained to satisfy $g(x,y) = 0$.

Assuming that these function have continuous derivatives, we can visualize $g(x,y) = 0$ as a curve along which the level curve of $f(x,y) = z$ (constant).

Intuitively, if we move the level curve in the direction of increasing $Z$, the largest or smallest $Z$ occurs at a point where the level curve touches $g(x,y) = 0$.

The quadrants of $f'$ and $g'$ should be in the same or opposite direction. Then

$$(\lambda = \text{constant}) \Rightarrow \nabla f + \lambda \nabla g = 0$$

$\nabla f = (f_x, f_y)$

$$f_x = -\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial x}$$

$$f_y = -\frac{\partial f}{\partial y} = -\frac{\partial g}{\partial y}$$

By problem, at point of tangency, $-f_x = f_y = 0$.

Hence, to find the maximum or minimum of $f(x,y)$ subject to $g(x,y) = 0$, find all the solution of the equation,

$$\nabla f + \lambda \nabla g = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}$$

Local maxima and minima will be among the solutions. If the curve $g(x,y) = 0$ is closed and bounded, then the absolute maxima and minima of $f(x,y)$ exist and are among these solutions. (Existence)
Example 11. Find all the (local) maximum and minimum of
\[ f(x, y) = g - (x-1)^2 - y^2 \] on \( x^2 + y^2 = 4 \).

Solution: Here, \( z = f(x, y) \) and \( g(x, y) = x^2 + y^2 - 4 \). To maximize or minimize \( z = f(x, y) \), subject to \( g(x, y) = 0 \).

Graphical Method:
Clearly, the level curve \( g(x, y) = 0 \) touches the curve \( g(x, y) = 0 \) at two points \((\pm 2, 0)\).

As the curve is closed and bounded, the absolute maximum and minimum are at \((\pm 2, 0)\).

Note that,
\[ f(2, 0) = 9 - (2 - 1)^2 - 0^2 = 8 \]
and \[ f(-2, 0) = 9 - (-2 - 1)^2 - 0^2 = 0 \].

Hence, \( f(2, 0) = 8 \) and \( f(-2, 0) = 0 \) are the maximum and minimum values of \( f(x, y) \).

Alternate Method:
Define, \( F(x, y) = f(x, y) + 2g(x, y) \)
\[ = 9 - (x - 1)^2 - y^2 + 2(x^2 + y^2 - 4) \]

For points of extremum,
\[
\begin{align*}
\frac{\partial F}{\partial x} &= -2(x-1) + 2x = 0 \\
\frac{\partial F}{\partial y} &= -2y + 2y = 0 \\
\text{and } g(x, y) &= 0
\end{align*}
\]

\[ \Rightarrow \begin{cases} (1-\lambda)x = 1 \\ (1-\lambda)y = 0 \\ \text{and } x^2 + y^2 = 4 \end{cases} \]
From \( \lambda 
eq 1 \), then \( \epsilon \Rightarrow y = 0 \), from \( \Omega \Rightarrow x = \frac{1}{1-\lambda} \).

\[ \left( \frac{1}{1-\lambda} \right)^2 + 0^2 = 4 \Rightarrow \lambda = \frac{3}{2}, \frac{1}{2}. \]

\( \Rightarrow x = \pm 2, \quad (x, y) = (\pm 2, 0) \)

Since \( f(x, y) = 0 \), i.e., \( x^2 + y^2 = 4 \) is a closed bounded curve, the absolute maximum and minimum of \( f(x, y) \) exist and are one among the solutions.

Note that, \( f(2, 0) = 8 \) and \( f(-2, 0) = 0 \)

Clearly, \( f(2, 0) = 8 \) and \( f(-2, 0) = 0 \) are the absolute maximum and minimum.

Ex. 2. Show that the volume of the greatest rectangular parallelepiped that can be inscribed in the ellipsoid

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \]

is \( \frac{abc}{3\sqrt{3}} \).

Solu. \(
\text{The volume of the parallelepiped,}
\]

\( V = xyz \)

To maximize \( \int xydez \) subject to

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow \text{To max.} \ln(V) = \ln(xydez) \]

\( \text{subject to} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \).

Define, \( F(x, y, z) = \ln(xydez) + \lambda \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right) \)

where \( \lambda \) is the Lagrange’s multiplier.

For points of extremum,

\[ \frac{DF}{dx} = \frac{DF}{dy} = \frac{DF}{dz} = 0 \quad \text{cotic} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \]

Hence,

\[ x = -\frac{a}{2\lambda}, \] \( y = -\frac{b^2}{2\lambda}, \]

and \( \lambda = 3, \frac{1}{2} \)

\[ \Rightarrow \lambda = \frac{a^2}{2}, \Rightarrow \lambda = \frac{a^2}{3}, \frac{b^2}{3}, \frac{c^2}{3} \]

Hence, \( \lambda = \frac{a^2}{3}, \frac{b^2}{3}, \frac{c^2}{3} \)

\( \Rightarrow (x, y, z) = \left( \frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}} \right) \) as \( (x, y, z) > 0 \)
As \( \frac{a}{at} + \frac{y}{bt} + \frac{z}{ct} = 1 \) is closed bounded curve, then absolute maximum or minimum exist and is at the solution.

At \( (x, y, z) = \left( \frac{a}{2}, \frac{b}{2}, \frac{c}{\sqrt{2}} \right) \), then \( V = \frac{abc}{4\sqrt{2}} \).

At \( (x, y, z) = \left( \frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}} \right) \), then \( V = \frac{abc}{3\sqrt{3}} \).

Hence, the maximum volume is \( \frac{abc}{3\sqrt{3}} \).

**Ex. 3.** A rectangular box, open at the top, is to have a volume 32 cu. ft. What must be the dimensions so that the total surface area is minimum?

**Ans:** 48 \((4, 4, 2)\)

**Soli:** Volume = \( V = xyz \), and total surface area

\[ = xy + 2(y^2 + 2x) \]

To maximize or minimize \( S \), subject to \( V = 32 = 2y^2 \).

Let, \( f(x, y) = 2x + 2y + 2z + 2 \) \( g(x, y) = 2xyz - 32\).

**Geometry and definition of Lagrange's multipliers:**

Solve the following equations simultaneously,

\[ \sum_{i=1}^{k} \lambda_i g_i = 0 \text{ and } \sum_{i=1}^{k} \lambda_i g_i y_i = 0 \text{ for } i = 1 \text{ to } k \]

The numbers \( \lambda_1, \lambda_2, \ldots, \lambda_k \) are called Lagrange's multipliers.
Ex. 4. The temperature \( T \) at any point is given by \( T = 200 (x^2 y + z^2) \). Find the highest and lowest temperature on the surface of sphere \( x^2 + y^2 + z^2 = 1 \).

**Solution:**
To maximize or minimize \( T = 200 (x^2 y + z^2) \) subject to \( x^2 + y^2 + z^2 = 1 \).

Define \( F = \ln (200 (x^2 y + z^2)) + x (x^2 + y^2 + z^2 - 1) \).

For points of extremum,
\[
0 = \frac{\partial F}{\partial x} = \frac{1}{x^2 y + z^2} + 2y \Rightarrow x = \frac{1}{2y}.
\]
\[
0 = \frac{\partial F}{\partial y} = \ln (200 (x^2 y + z^2)) + 2x y + x \Rightarrow y = \frac{1}{2x}.
\]
\[
0 = \frac{\partial F}{\partial z} = \frac{1}{x^2 y + z^2} + 2y \Rightarrow z = \frac{1}{2x}.
\]

Next, \( x^2 + y^2 + z^2 = 1 \)
\[
\Rightarrow \frac{1}{2y} - \frac{1}{2y} - \frac{1}{2} = 1
\]
\[
\Rightarrow y = z = 2.
\]
\[
(x, y, z) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})
\]
\[
(x, y, z) = (\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2})
\]

The curve \( x^2 + y^2 + z^2 = 1 \) is closed and bounded, the absolute maximum and minimum of \( f(x, y, z) \) exists and are among the solutions.
For points \((x, y, z)\) for which \( x y = \frac{1}{2} \),
\[
f(x, y, z) = 200 \times \frac{1}{2} \times \frac{1}{2} = 25,
\]
For points \((x, y, z)\) for which \( x y = -\frac{1}{2} \),
\[
f(x, y, z) = 200 \times (-\frac{1}{2}) \times \frac{1}{2} = -25,
\]
Hence the highest temperature is 25 and the lowest temperature is -25.
Ex. 5. If \( ax^2 + by^2 = ab \), then the maximum and minimum value of \( (x^2 + y^2 + 1)^2 \) will be the values of \( F \) given by

\[
\Delta \ = \ a(\theta - a)(\theta - b) = ab. \quad \text{e.u.} \quad (5)
\]

Solve \( F(x, y) = (x^2 + xy + y^2) + 2(ax^2 + by^2 - ab) \)

For points of extrema,

\[
\frac{\delta F}{\delta x} = 2x + y + 2ax = 0 \quad (1)
\]
\[
\frac{\delta F}{\delta y} = x + 2y + 2by = 0 \quad (2)
\]

and \( ax^2 + by^2 = ab \),

Now, \( 0 \times x + 0 \times y \Rightarrow 2(x^2 + xy + y^2) + 2(\Delta) = 0 \) for which \( x^2 + y^2 = 0 \), are the max. or min. values

\[
\Rightarrow 2(\Delta) = 0
\]

\[
\Delta = -\frac{ab}{2}
\]

Note that, \( (x, y) = (0, 0) \) is not a solution of \( ax^2 + by^2 = ab \)

Hence, \( 2x(1 + 2\Delta) = 0 \)

\[
2 + 2y(1 + \Delta) = 0 \quad [A^2 = 2, \Delta \neq 0]
\]

has a non-trivial solution.

\[
\Rightarrow 4(1 + 2\Delta)(1 + \Delta) - 1 = 0
\]

\[
\Rightarrow 4 \left( 1 - \frac{\theta}{b} \right) \left( 1 - \frac{\theta}{a} \right) = 1
\]

\[
\Rightarrow 4(b - \theta)(a - \theta) = ab
\]

\[
\Rightarrow 4(0 - \theta)(\theta - b) = ab.
\]
Remark:— In general, to maximize or minimize

\[ \alpha = f(x_1, x_2, \ldots, x_n) \]

subject to the constraints

\[ g_i(x_1, x_2, \ldots, x_n) = 0, \quad i = 1(1)k, \]

solve the following equations simultaneously,

\[ \nabla \left\{ f + \sum_{i=1}^{k} \lambda_i g_i \right\} = 0 \quad \text{and} \]

\[ g_i(x_1, x_2, \ldots, x_n) = 0, \quad i = 1(1)k. \]

The numbers \( \lambda_1, \lambda_2, \ldots, \lambda_k \) are called the

Lagrange's Multipliers. The method for finding extrema

of a function subject to some constraints, is
called the method of Lagrange's Multipliers.

Ex. 6. If \( xy^2 = k^3 \), show that \( (x+a)(y+b)(z+c) \) is
minimum when \( \frac{a}{x} = \frac{b}{y} = \frac{c}{z} = \frac{k}{\sqrt[3]{abc}}. \)

Soln. \(
\text{Let } F = \log s (x+a)(y+b)(z+c) + \lambda(xy^2 - k^3)
\)

for points of extrema,

\[ 0 = \frac{\partial F}{\partial x} = \frac{1}{x+a} + 2y + \frac{\lambda}{x+a} + 3k \Rightarrow \frac{\lambda}{x+a} = \frac{-3k}{1+3k}. \]

\[ 0 = \frac{\partial F}{\partial y} = \frac{1}{y+b} + 2x + \frac{\lambda}{y+b} + 3k \Rightarrow \frac{\lambda}{y+b} = \frac{-3k}{1+3k}. \]

\[ 0 = \frac{\partial F}{\partial z} = \frac{1}{z+c} + 2y + \frac{\lambda}{z+c} + 3k \Rightarrow \frac{\lambda}{z+c} = \frac{-3k}{1+3k}. \]

\[ kxy^2 = k^3 \]

\[ \frac{(abc) \cdot (-3k^3)^3}{(1+3k^3)^3} = k^3 \]

\[ \Rightarrow \frac{-3k^3}{1+3k^3} = \frac{k}{(abc)^{\frac{1}{3}}} \]

\[ \Rightarrow \frac{-3k^3}{1+3k^3} = \frac{k}{(abc)^{\frac{1}{3}}} \]

\[ \Rightarrow \frac{-3k^3}{1+3k^3} = \frac{k}{(abc)^{\frac{1}{3}}} \]

is a point of maximum on

minimum. (continued...)
For $(x, y, z) = (a u, b u, c u)$, where $u = \frac{k}{a b c}$

\[(x + a)(y + b)(z + c) = (a b c) \cdot (u + 1) \left(\frac{u}{2} + 1\right)(2u + 1)\]

\[\geq (a b c) \cdot (u + 1)^3 \geq (a b c) \left(\frac{u}{2} + 1\right)^3 \geq (x + a)(y + b)(z + c)\]

where $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{k}{(a b c) v^3}$

\[\text{Ex. 7. Given } n \text{ positive numbers } c_1, c_2, \ldots, c_n \text{ find the maximum value of } \sum_{i=1}^{n} c_i x_i, \text{ where the variables } x_i \text{'s are restricted so that } \sum_{i=1}^{n} x_i = 1.\]

\[\text{Solu} \Rightarrow \text{To maximize } f(x) = \sum_{i=1}^{n} c_i x_i \text{ subject to } g(x) = \sum_{i=1}^{n} x_i = 1.\]

Define, \( F(x) = f(x) + \lambda g(x) \)
\[= \sum_{i=1}^{n} c_i x_i + \lambda \left(\sum_{i=1}^{n} x_i - 1\right) \]

for points of extrema,
\[n \sum_{i=1}^{n} c_i x_i = 0 \text{ for } i = 1(n), \text{ and } \sum_{i=1}^{n} x_i = 1.\]

\[\Rightarrow \sum_{i=1}^{n} c_i + 2 \lambda x_i = 0 \text{ for } i = 1(n) \text{ and } \sum_{i=1}^{n} x_i = 1.\]

\[\Rightarrow x_i = -\frac{c_i}{2\lambda} \]

and \[\frac{1}{4\lambda} = \sum_{i=1}^{n} c_i x_i = \frac{1}{4\lambda} \left(\sum_{i=1}^{n} c_i^2\right) \]

\[\Rightarrow 2\lambda = \pm \sqrt{\sum_{i=1}^{n} c_i x_i} \]

\[\Rightarrow x_i = \pm \sqrt{\sum_{i=1}^{n} c_i x_i} \text{ for } i = 1(n) \text{.}\]

Since \( \sum_{i=1}^{n} x_i = 1 \) is a closed and bounded curve, the absolute maximum and minimum are among the solutions.
For, $\alpha_i = -\frac{c_i}{\sqrt{\sum_{i=1}^{n} c_i^2}}$, for $\sum_{i=1}^{n} c_i \alpha_i = -\sqrt{\sum_{i=1}^{n} c_i^2}$.

For, $\alpha_i = \frac{c_i}{\sqrt{\sum_{i=1}^{n} c_i^2}}$, $\sum_{i=1}^{n} c_i \alpha_i = +\sqrt{\sum_{i=1}^{n} c_i^2}$

Hence, $\sqrt{\frac{n}{\sum_{i=1}^{n} c_i^2}}$ is the maximum.

Remark: By Cauchy-Schwarz inequality,
$$\left(\sum_{i=1}^{n} c_i \alpha_i\right)^2 \leq \left(\sum_{i=1}^{n} c_i^2\right)\left(\sum_{i=1}^{n} \alpha_i^2\right) = \sum_{i=1}^{n} c_i^2$$

$$\Rightarrow -\sqrt{\sum_{i=1}^{n} c_i^2} \leq \sum_{i=1}^{n} c_i \alpha_i \leq \sqrt{\sum_{i=1}^{n} c_i^2}$$

Ex. 8. Maximize $f(x_1, x_2) = x_1 x_2$ subject to $x_1 + x_1 x_2 = 12$.

Sol.: Let $f(x_1, x_2) = x_1 x_2 + \lambda (x_1 + x_1 x_2 - 12)$

$$0 = \frac{\partial f}{\partial x_1} = 2x_1 x_2 + \lambda (2x_1 + x_2) \quad (1)$$

$$0 = \frac{\partial f}{\partial x_2} = x_1 + 2x_2 \quad (2)$$

and $x_1 + x_1 x_2 = 12$.

$$\Rightarrow x_1 (x_1 + \lambda) = 0$$

$$\Rightarrow x_1 = -\lambda \text{ as } x_1 = 0 \text{ is not a soln. of } x_1 + x_1 x_2 = 12.$$  

$$\Rightarrow -2x_2 = 2\lambda \Rightarrow x_2 = -2\lambda$$

Now, $x_1 + x_1 x_2 = 12$

$$\Rightarrow x_1^2 + 2x_1 x_2 = 12 \Rightarrow x_1 = 4$$

$$\Rightarrow x_2 = 2 \Rightarrow -2\lambda = 2 \Rightarrow \lambda = 2$$

$(x_1, x_2) = (-2, 4)$ or $(2, 4)$

Hence, the maximum value of $x_1 x_2$ is 16 and the minimum value is -16.
\[ \text{INTEGRALS} \rightarrow \]

A. Double Integrals: We assume that the real-valued function \( f(x, y) \) is defined and bounded on the rectangle \( R = [a, b] \times [c, d] \).

A partition of \( R \) is constructed by passing lines through parallel to \( y \)-axis: \( x = x_i \) and parallel to \( x \)-axis: \( y = y_j \) where,
\[
a = x_0 < x_1 < \ldots < x_n = b \quad \text{and} \quad c = y_0 < y_1 < \ldots < y_m = d.
\]

We let, \( M = \sup_{(x, y) \in R} f(x, y) \)
and \( m = \inf_{(x, y) \in R} f(x, y) \).

For the partition \( P = \{R_{ij}\} \),
\[
R_{ij} = f(x, y) : x_{i-1} < x < x_i \quad \text{and} \quad y_{j-1} < y < y_j.
\]

We define, \( M_{ij} = \sup_{(x, y) \in R_{ij}} f(x, y) \)
and \( m_{ij} = \inf_{(x, y) \in R_{ij}} f(x, y) \).

The upper sum and the lower sum are defined as below,
\[
U(P, f) = \sum_{i=1}^{n} \sum_{j=1}^{m} M_{ij} A(R_{ij})
\]
and \( L(P, f) = \sum_{i=1}^{n} \sum_{j=1}^{m} m_{ij} A(R_{ij}) \).

We define the upper double integral of \( f \) over \( R \) to be
\[
\iint f = \inf_{P} U(P, f)
\]

Similarly, we define the lower double integral of \( f \) over \( R \) by
\[
\iint f = \sup_{P} L(P, f).
\]

[For any \( P \), \( L(P, f) \leq \text{actual volume} \leq U(P, f) \)]
Calculation of Double integral over a closed region:

If \( f(x,y) \) is integrable on a closed region \( D = \{(x,y) : a \leq x \leq b, \phi(x) \leq y \leq \psi(x)\} \) and if \( f(x,y) \) dy exists for each \( x \in [a,b] \), then

\[
\int \int_D f(x,y) \, dx \, dy = \int_a^b \left( \int_{\phi(x)}^{\psi(x)} f(x,y) \, dy \right) \, dx
\]

Define, \( f^*(x,y) = \begin{cases} f(x,y), & \text{if} \ (x,y) \in D \\ 0, & \text{if} \ (x,y) \in \mathbb{R} - D \end{cases} \)

Now, \( \int \int_D f(x,y) \, dx \, dy \) can be evaluated by Riemann integral. Note that,

\[
\int \int_D f(x,y) \, dx \, dy = \int \int_D f^*(x,y) \, dx \, dy + \int \int_{\mathbb{R} - D} f^*(x,y) \, dx \, dy
\]

Evaluate the integral.
Evaluate: \( \int y \, dy \), where \( D \) is the region bounded by \( y = 2 \) and \( y = 4x - x^2 \).

**Soln.**

For points of intersection:

\[ x = 4x - x^2 \]
\[ x^2 - 3x = 0 \]
\[ x(x - 3) = 0 \]
\[ x = 0, 3 \]

The domain of the integration is:

\[ D = \{(x, y) : 0 \leq x \leq 3, 4x - x^2 \leq y \leq 0\} \]

Hence,

\[ \int y \, dy = \int \left( \int y \, dy \right) \, dx \]

\[ = \int_{0}^{3} \left( \frac{4x - x^2}{2} \right) \, dx \]

\[ = \int_{0}^{3} \left[ \frac{-1}{2} \right] x \left( 4x - x^2 \right) \, dx \]

\[ = \frac{1}{2} \int_{0}^{3} x^4 - 8x^3 + 15x^2 \, dx \]

\[ = \frac{1}{2} \left[ \frac{x^5}{5} \right]_0^3 - 4 \left[ \frac{x^4}{4} \right]_0^3 \]

\[ + \frac{15}{2} \left[ \frac{x^3}{3} \right]_0^3 \]

\[ = 10.8 \]
Ex. 2. Evaluate $\iint_D (x^2 + y^2) \, dxdy$ where $D$ is the region bounded by $y = x+1$ and $y = x-1$.

**Solu.**

1. For points of intersection:
   \[ x+1 = x-1 \]
   \[ \Rightarrow 2x = 0 \]
   \[ \Rightarrow x = 0 \]

2. The domain of the integral is $D = \{(x, y) : -1 \leq x \leq 2, x-1 \leq y \leq x+1\}$

Hence,

$$
\iint_D (x^2 + y^2) \, dxdy = \int_{x=0}^{2} \int_{y=x-1}^{x+1} (x^2 + y^2) \, dy \, dx
$$

$$
= \int_{x=0}^{2} \left[ x^2 y + \frac{y^3}{3} \right]_{x-1}^{x+1} \, dx
$$

$$
= \int_{x=0}^{2} \left[ x^2 (x+1) + \frac{(x+1)^3}{3} - \left( x^2 - 1 \right) - \frac{x^3}{3} \right] \, dx
$$

$$
= \frac{117}{14}
$$
**Ex. 3.** Evaluate \( \iint_D e^{x-y} \, dx \, dy \)

where \( D \) is the region bounded by \( y = x, y = x - 1, y = 0 \) and \( y = 1 \).

**Solution:**

\[ D = \left\{ (x, y) : 0 \leq y \leq 1, y \leq x \leq y + 1 \right\} \]

Now,

\[
\iint_D e^{x-y} \, dx \, dy = \int_0^1 \left( \int_0^{y+1} e^{x-y} \, dx \right) e^{-y} \, dy
\]

\[
= \int_0^1 \left[ \frac{1}{2} e^{y+1} \right]_{x=0}^{x=y+1} e^{-y} \, dy
\]

\[
= \int_0^1 \left( e^{y+1} - e^y \right) e^{-y} \, dy
\]

\[
= \int_0^1 e^{y+1} - 1 \, dy
\]

\[
= \frac{1}{2} \left[ \left( \frac{e^{y+1}}{2} \right) - y \right]_0^1
\]

\[
= \frac{1}{2} \left[ \frac{e^3 - e}{2} - 1 \right]
\]

\[
= \frac{e^3 - e - 2}{4}
\]
Ex. 4. Evaluate \( \iiint_D e^{y/x} \, dx \, dy \), where \( D \) is the region bounded by \( y = x^2 \), \( y = 0 \), \( x = 1 \).

\[ \text{Soln.} \]

The domain of integration is \( \{(x, y) : 0 < x < 1, 0 < y < x^2\} \).

Hence, evaluating the double integral by repeated integral,

\[ \iiint_D e^{y/x} \, dx \, dy = \int_0^1 \left( \int_0^{x^2} e^{y/x} \, dy \right) \, dx \]

\[ = \int_0^1 \left[ \frac{e^{y/x}}{1/x} \right]_0^{x^2} \, dx \]

\[ = \int_0^1 2x(e-1) \, dx \]

\[ = (e-1) \left[ \frac{x^2}{2} \right]_0^1 \]

\[ = \frac{1}{2} (e-1). \]

Ex. 5. Evaluate \( \iiint_R f(x, y) \, dx \, dy \), where \( R = [0, 1] \times [0, 1] \)

and \( f(x, y) = \int_0^{\min(x+y, x-y)} (x+y) \, dz \), \( x < y < 2x \).

\[ \text{Soln.} \]

Here,

\[ D = \{(x, y) : 0 < y < 1, 0 < x < \min(y, 2y)\}. \]

Hence,

\[ \iiint_R f(x, y) \, dx \, dy = \int_0^1 \left( \int_0^{\min(x+y, x-y)} (x+y) \, dz \right) \, dy \]

\[ = \int_0^1 \left[ \frac{(x+y)^2}{2} + yx \right]_{x-y}^{x+y} \, dy \]
\[
\begin{align*}
\int \left\{ \left( \frac{y}{4} \right)^{1/2} + 2y^{5/2} \left( 1 - \frac{1}{4} \right) \right\} dy & = \left[ \frac{y^{3/2}}{3/2} + \frac{2y^{7/2}}{7} \left( 1 - \frac{1}{4} \right) \right]_0^1 \\
& = \frac{1}{8} + \frac{2(3^2-1)}{5\sqrt{2}} \\
& = \frac{21-8\sqrt{2}}{40}.
\end{align*}
\]

\[\star \text{ Ex. E. Evaluate } \iiint_D \frac{x}{y} \, dx \, dy \, z, \text{ where } D \text{ is bounded by } x = 2, y = x, xy = 1.\]

\[\text{Solv.} \Rightarrow \]

Hence,

\[D = \{(x,y) : 1 < x < 2, \frac{1}{2} < y < x\}.\]

\[\text{Note:} \iiint_D \frac{x}{y} \, dx \, dy = -2.\]

\[\text{Now,} \iiint_D \frac{x}{y} \, dx \, dy = \int_{1/2}^1 \left[ \int_{1/2}^x \frac{1}{y} \, dy \right] \, dx = \int_{1/2}^1 \left[ \frac{1}{2} - \frac{1}{y} \right] \, dx = \left[ \frac{x^2}{4} - \frac{x}{2} \right]_{1/2}^1 = 4 - \frac{1}{4} - 2 + \frac{1}{2} = \frac{9}{4}.\]
Change of variables in a Double Integral:

Jacobian:

Consider the double integral

\[ \iint_{D} f(x, y) \, dx \, dy \]

over a bounded region \( D \) in the \( xy \)-plane.

Consider the transformation:

\[ u = \phi(x, y) \]
\[ v = \psi(x, y) \]

If the transformation represents a continuous one-to-one mapping of the closed region \( D \) in the \( xy \)-plane on a region \( D^* \) in the \( uv \)-plane and if \( \alpha = \phi_1(u, v) \) and \( \beta = \phi_2(u, v) \) have the continuous partial derivatives with Jacobian (also called the local magnification factor):

\[ J = \begin{vmatrix} \frac{\partial \alpha}{\partial u} & \frac{\partial \alpha}{\partial v} \\ \frac{\partial \beta}{\partial u} & \frac{\partial \beta}{\partial v} \end{vmatrix} \neq 0 \]

Then,

\[ \iint_{D} f(x, y) \, dx \, dy = \iint_{D^*} f(\phi_1(u, v), \phi_2(u, v)) |J| \, du \, dv \]

The tangent vectors are

\[ (\frac{\partial \phi_1}{\partial u}, \frac{\partial \phi_1}{\partial v}) \, du \text{ and } (\frac{\partial \phi_2}{\partial u}, \frac{\partial \phi_2}{\partial v}) \, dv \]

Area \( (\epsilon_{ij}) \) is the area of the parallelogram formed by the tangent vectors:

\[ \epsilon_{ij} = \begin{vmatrix} \frac{\partial \phi_1}{\partial u} & \frac{\partial \phi_1}{\partial v} & 0 \\ \frac{\partial \phi_2}{\partial u} & \frac{\partial \phi_2}{\partial v} & 0 \\ 0 & 0 & 1 \end{vmatrix} \, du \, dv \]

This is the significance of the Jacobian.
Example 1. Evaluate: \( \iint_D e^{x+y} \, dx \, dy \), where \( D \) is the region bounded by \( x=0, y=0, x+y=1 \).

Solution: The domain of the integration is \( D = \{(x, y): 0 \leq x, y \leq 1, x+y \leq 1\} \).

Let \( u = \frac{x+y}{2}, v = x+y \).

Then \( y = uv, x = v(1-u) \).

As \( (x,y) \in D \),
\( 0 \leq x \leq 1, 0 \leq y \leq 1 \) and \( x+y \leq 1 \).

\( \Rightarrow 0 \leq uv \leq 1, 0 \leq u(1-u) \leq 1, v \leq 1 \).

\( \Rightarrow 0 \leq u \leq 1, 0 \leq v \leq 1 \).

The new domain is \( D^* = \{(u,v): 0 \leq u, v \leq 1\} \).

The Jacobian is \( J = \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right| = -u, 1-u \).

Hence, \( \iint_D e^{x+y} \, dx \, dy \)

\( = \iint_{D^*} e^{u+v} \, du \, dv \)

\( = \int_{0}^{1} \left[ \int_{0}^{1} e^{u+v} \, dv \right] \, du \)

\( = \int_{0}^{1} \left[ \int_{0}^{1} e^{u+v} \, dv \right] \, du \)

\( = \left( \int_{0}^{1} e^{u} \, du \right) \left( \int_{0}^{1} e^{v} \, dv \right) \)

\( = \frac{1}{2} (e-1). \)
Ex. 2. Evaluate: \[ \iint_D \sin \left( \frac{x-y}{x+y} \right) \, dx \, dy \], where \( D \) is the region bounded by \( x = 0, y = 0, x + y = 1 \).

S oln. \rightarrow

Let, \( u = x - y, v = x + y \)

\[ x = \frac{u+v}{2}, \quad y = \frac{v-u}{2} \]

The transformed domain is,\[ D^* = \{ (u, v) : 0 \leq u \leq 1, -v \leq u \leq v \} \]

Jacobian is,\[ J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2} \]

Hence, \[ \iint_D \sin \left( \frac{x-y}{x+y} \right) \, dx \, dy = \iint_{D^*} \sin \left( \frac{u}{v} \right) \frac{1}{2} \, du \, dv \]

\[ = \frac{1}{2} \int_0^1 \left\{ \int_{-v}^v \sin \left( \frac{u}{v} \right) \, du \right\} \, dv \]

\[ = \frac{1}{2} \int_0^1 \left[ -v \cos \left( \frac{u}{v} \right) \right]_{u=0}^{u=v} \, dv \]

\[ = \frac{1}{2} \int_0^1 \left[ -v \cos 1 - (-v \cos v) \right] \, dv \]

\[ = \frac{1}{2} \int_0^1 (-v) \left[ \cos 1 - \cos v \right] \, dv \]

\[ = 0. \]
Example 8. Evaluate \( \int \int_D (y - x) \, dxdy \) where \( D \) is the region bounded by the lines \( y = x + 3 \), \( y = x - 1 \), \( 3y + x = 5 \) and \( y = x + 1 \).

Solution:

Let \( u = y - x \), \( v = 3y + x \)

The new domain is \( D^* = \{(u, v) : -1 \leq u \leq 3, 5 \leq v \leq 7\} \).

Hence, \[ J = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{9} \cdot \frac{1}{9} = -\frac{1}{4} \]

Hence, \[ \int \int_D (y - x) \, dxdy = \int_{-1}^{3} \int_{5}^{7} (u - \frac{1}{4}u) \, dudv = \frac{1}{4} \int_{-1}^{3} \left[ \frac{3u^2}{2} \right]_{0}^{7} \, du = \frac{1}{4} \int_{-1}^{3} \left( \frac{81}{2} - \frac{1}{2} \right) \, du = \frac{8}{2} \cdot \frac{1}{4} \left[ 9 - 5 \right] = \frac{24}{9} \cdot \frac{1}{4} \left( \frac{m}{(m+1)(n)} \right)^7 \approx \left( \frac{m+n+1}{7} \right)^7 \]
**Ex. 4**: Evaluate \( \iiint_D x^{l-1} y^{m-1} (1-x-y)^{p-1} \, dx \, dy \), where \( D \) is the region bounded by \( x=0, y=0 \) and \( x+y=1 \).

**Solution**

Hence, \( D = \{(x, y) : 0 \leq x, y \leq 1, x+y \leq 1 \} \)

Let, \( x+y = u, \ x = uv \)

\( u = xvu, y = u(1-v) \)

As, \((x, y) \in D\)

\( 0 \leq u \leq 1, \ 0 \leq v \leq 1 \)

The transformed domain is

\[ D^* = \{ (u, v) : 0 \leq u, v \leq 1 \} \]

\[ |J| = \text{mod.} \begin{vmatrix} u & v \\ v & -u \end{vmatrix} = 1-u^2 = v \]

\[ \iint_D x^{l-1} y^{m-1} (1-x-y)^{p-1} \, dx \, dy \]

\[ = \iiint_{D^*} (uv)^{l-1} \{ u(1-v) \}^{m-1} (1-u)^{p-1} \, u \, dv \, du \]

\[ = \int_0^1 u^{l+m-1} (1-u)^{p-1} \, du \int_0^{1-u} u^{m-1} (1-u)^{p-1} \, du \]

\[ = B(l+m, p) \cdot B(l, m) \]

\[ = \frac{\Gamma(l+m) \Gamma(p) \Gamma(l) \Gamma(m)}{\Gamma(l+m+p) \Gamma(l+m)} \]

\[ = \frac{\Gamma(l) \Gamma(m) \Gamma(p)}{\Gamma(l+m+p)} \]

[This is a Dirichlet integral]
Ex. 5. Show that: \( \beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} \), \( m > 0, n > 0 \)

**Solution:**

\[
\Gamma(m) \cdot \Gamma(n) = \left( \int_0^\infty e^{-x} x^{m-1} dx \right) \left( \int_0^\infty e^{-y} y^{n-1} dy \right) = \int_0^\infty \int_0^\infty e^{-(x+y)} x^{m-1} y^{n-1} dx \, dy
\]

where, \( D = \{(x, y) : 0 < x, y < \infty\} \)

Let \( u = x+y \), \( v = \frac{x}{x+y} \), \( \alpha = u \), \( \beta = \frac{1}{1-v} \), \( v > 0 \), \( 0 < u < \infty \).

As \( (x, y) \in D \), \( 0 < y < \infty \), \( 0 < u < \infty \), \( 0 < u(1-v) < \infty \) \( \Rightarrow 0 < \frac{x}{x+y} < 1 \), \( \Rightarrow 0 < u < 1 \)

The transformed domain is \( D^* = \{(u, v) : 0 < u < \infty, 0 < v < 1\} \)

and \( \det J = \det \begin{vmatrix} u & 1-u \\ v & 1-v \end{vmatrix} = 1-w^2 = w \)

Hence, \( \Gamma(m) \Gamma(n) = \int \int e^{-u} (uv)^{m-1} \{u(1-u)\}^{n-1} \, dudv \)

\( D^* \)

\[
= \left( \int_0^\infty \int_0^1 e^{-u} u^{m+n-1} \, du \right) \left( \int_0^\infty \int_0^1 (1-v)^{n-1} \, dv \right)
\]

\[
= \Gamma(m+n) \cdot \beta(m, n)
\]

Hence, \( \beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} \)
Polar transformation in $R^2$:

Let $x = r \cos \theta$, $y = r \sin \theta$

Here, $r = \sqrt{x^2 + y^2}$ is the distance of $P(x, y)$ from the origin.

and, $\theta$ is the angle which $OP$ makes with the direction of $x$-axis.

As $(x, y) \in R^2$, $0 < r < \infty$, $0 < \theta < 2\pi$.

The transformed plane of $R^2$ under polar transformation is

$D^* = \{(r, \theta) : 0 < r < \infty, 0 < \theta < 2\pi\}$

Hence, Jacobian is

$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r (\cos \theta \sin \theta + \sin \theta \cos \theta) = r \cos \theta$

$|J| = r \cos \theta$

$\star$ Ex. 11 Evaluate: $\iiint_D x \, \, dy \, \, dx$, where $D$ is the region bounded by the semi-circle $y = \sqrt{1-x^2}$ and $x$-axis.

SOLN: Let $x = r \cos \theta$, $y = r \sin \theta$.

As $(x, y) \in D$, $0 < r < \infty$, and $0 < \theta < \pi$.

Hence, the transformed domain

$D^* = \{(r, \theta) : 0 < r < 1, 0 < \theta < \pi\}$

and $|J| = r \cos \theta$.

$J = \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} \int_{0}^{2\pi} r \cos \theta \, \, dr \, \, d\theta \, \, dy$
Hence,

\[
\iint_{D} e^{x+y} \, dx \, dy
\]

\[
= \iint_{D^*} e^{r \cos \theta + r \sin \theta} \, r \, dr \, d\theta
\]

\[
= \int_{0}^{\pi} \left( \int_{1}^{2} r e^{r \cos \theta + r \sin \theta} \, dr \right) \, d\theta
\]

\[
= \left[ \frac{e^r}{2} \right]_{1}^{2} \left[ \theta \right]_{0}^{\pi}
\]

\[
= \pi (e - 1).
\]

Ex. 2. Evaluate \(\int \int_{D} \sqrt{x^2 + y^2} \, dx \, dy\), where \(D\) is the region bounded by the circles \(x^2 + y^2 = 1\) and \(x^2 + y^2 = 9\).

\[\text{Sol.} \Rightarrow\]

\(D = \{(x, y) : 1 \leq x^2 + y^2 \leq 9\}\)

Let, \(x = r \cos \theta, y = r \sin \theta, 0 < \theta < 2\pi\).

As, \((x, y) \in D, 1 \leq r < 2; 0 < \theta < 2\pi,\)

The transformed domain is \(D^* = \{(r, \theta) : 1 \leq r < 2, 0 < \theta < 2\pi, \}\)

Hence, \(\left| J \right| = 1\).

Hence,

\[
\int \int_{D} \sqrt{x^2 + y^2} \, dx \, dy
\]

\[
= \int_{0}^{\pi} \int_{1}^{2} r \, e^{r \cos \theta + r \sin \theta} \, dr \, d\theta
\]

\[
= \int_{0}^{\pi} \left( \int_{1}^{2} r e^{r \cos \theta + r \sin \theta} \, dr \right) \, d\theta
\]

\[
= \left[ \frac{e^r}{2} \right]_{1}^{2} \left[ \theta \right]_{0}^{\pi}
\]

\[
= \pi (e^{2} - 1).
\]
Ex. 3. Evaluate: \( \iint e^{-(x^2+y^2)} \, dx \, dy \) using Polar transformation.

Hence, evaluate \( \int e^{-x^2} \, dx \).

**SOLN:**
\[
\iint e^{-(x^2+y^2)} \, dx \, dy = \iint e^{-(x^2+y^2)} \, r \, dr \, d\theta
\]

Let, \( x = r \cos \theta \), \( y = r \sin \theta \).

As, \( (x, y) \in D \), \( 0 < r < \infty, \ 0 < \theta < \pi/2 \)

The transformed domain is
\[
D^* = \{(r, \theta) : 0 < r < \infty, 0 < \theta < \pi/2\}
\]

and \( |J| = r \).

Hence, \( \iint e^{-(x^2+y^2)} \, dx \, dy = \iint e^{-r^2} \, r \, dr \, d\theta \)

\[
= \left( \int_0^\infty e^{-r^2} \, r \, dr \right) \left( \int_0^{\pi/2} d\theta \right)
\]

\[
= \left( \frac{1}{2} \cdot \frac{\pi}{2} \right) = \frac{\pi}{4}
\]

Hence, \( \iint e^{-(x^2+y^2)} \, dx \, dy = \pi/4 \).

\[
\Rightarrow \left( \int_0^\infty e^{-x^2} \, dx \right) \left( \int_0^\infty e^{-y^2} \, dy \right) = \pi/4
\]

\[
\int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}
\]

**Remark:**
\[
\int_0^\infty e^{-x^2} \, dx = \frac{1}{2} \cdot \int_0^\infty e^{-t} \, dt = \frac{1}{2} \Gamma \left( 1/2 \right)
\]

\[
\Rightarrow \Gamma \left( \frac{1}{2} \right) = \sqrt{\pi}
\]
TRIPLE INTEGRAL (Theorem):

If the transformation \( U = \phi_1(x, y, z), \) \( V = \phi_2(x, y, z), \) and \( W = \phi_3(x, y, z) \) represents a continuous one-to-one transformation of the closed region \( D \) in \( xyz \)-plane on a region \( E \) in \( uvw \)-plane \( \phi \), and if the functions \( \phi_1, \phi_2, \phi_3 \) and if the functions \( \phi_1, \phi_2, \phi_3 \) have continuous partial derivatives then the Jacobian is

\[
J = \frac{F(x, y, z)}{F(u, v, w)}
\]

\[
= \frac{d(\text{old variable})}{d(\text{new variable})};
\]

and

\[
\iiint_D f(x, y, z) \, dx \, dy \, dz = \iiint_E f(\phi_1^*, \phi_2^*, \phi_3^*) \, du \, dv \, dw,
\]

where

\[
x = \phi_1^* (u, v, w),
\]

\[
y = \phi_2^* (u, v, w)
\]

\[
z = \phi_3^* (u, v, w),
\]

\[
\left( u, v, w \right).
\]
B. **Multiple Integrals**:

i. **Polar transformation in \( \mathbb{R}^3 \)**:

Polar transformation of \((x, y, z) \in \mathbb{R}^3\) is

\[
\begin{align*}
  x &= r \cos \theta, \\
  y &= r \sin \theta \cos \phi, \\
  z &= r \sin \theta \sin \phi
\end{align*}
\]

**Note that**, in \( yz \)-plane

\[
0N^2 + NM^2 = OM^2
\]

\[
\Rightarrow r \sin \theta \left( \cos^2 \phi + \sin^2 \phi \right) = \sin \theta
\]

Again, \( OM^2 + MP^2 = OP^2 \)

\[
\Rightarrow r \sin \theta + r \cos \phi = h
\]

Clearly, \( r = \sqrt{x^2 + y^2 + z^2} \) is the distance between \( O \) and \( P \).

\( \theta_1 \) is the angle which \( OP \) makes with the positive direction of \( x \)-axis, and

\( \theta_2 \) is the angle which \( OM \) (the projection of \( OP \) on \( yz \)-plane) makes with the positive direction of \( y \)-axis.

From figure, \( 0 < r < \infty \), \( 0 \leq \theta_1 < \pi \), \( 0 \leq \theta_2 < 2\pi \).

The Jacobian of the transformation is

\[
J = \frac{\partial (x, y, z)}{\partial (r, \theta_1, \theta_2)} = \frac{\partial (x, y, z)}{\partial (r, \theta_1, \theta_2)}
\]

\[
= \begin{vmatrix}
  \cos \theta_1 & -r \sin \theta_1 & 0 \\
  \sin \theta_1 \cos \theta_2 & r \cos \theta_1 \cos \theta_2 & -r \sin \theta_1 \sin \theta_2 \\
  \sin \theta_1 \sin \theta_2 & r \cos \theta_1 \sin \theta_2 & r \sin \theta_1 \cos \theta_2 \\
\end{vmatrix}
\]

\[
= r^2 \sin \theta_1 \left( \cos \theta_1 \cos \theta_2 \cos \theta_2 \right)
\]

\[
= \begin{vmatrix}
  1 & -r \tan \theta_1 & 0 \\
  1 & \cot \theta_1 & -r \tan \theta_2 \\
  1 & \cot \theta_1 & \cot \theta_2 \\
\end{vmatrix}
\]
\[
\begin{align*}
&= \sin^2 \Theta_1 \cos \Theta_1 \cos \Theta_2 \sin \Theta_2 \\
&\quad \left| \begin{array}{c}
1 - \tan^2 \Theta_1 \\
\tan^2 \Theta_1 \\
\sin \Theta_1 \cos \Theta_1 \\
\sin \Theta_1 \cos \Theta_1 \\
\sin \Theta_2 \cos \Theta_2 \\
\sin \Theta_2 \cos \Theta_2
\end{array} \right|
\frac{\tan^2 \Theta_1}{\sin \Theta_1}
\begin{array}{c}
1 - \tan^2 \Theta_1 \\
\tan^2 \Theta_1 \\
\cos \Theta_1 \\
\cos \Theta_1 \\
\cos \Theta_2 \\
\cos \Theta_2
\end{array}

&= \sin \Theta_1 \cos \Theta_1 \cos \Theta_2 \sin \Theta_2 \\
&\quad \left| \begin{array}{c}
R_3' = R_3 - R_1, \\
R_2' = R_2 - R_1
\end{array} \right|
\frac{\tan^2 \Theta_1}{\sin \Theta_1}
\begin{array}{c}
1 - \tan^2 \Theta_1 \\
\tan^2 \Theta_1 \\
\cos \Theta_1 \\
\cos \Theta_1 \\
\cos \Theta_2 \\
\cos \Theta_2
\end{array}

&= \frac{\tan^2 \Theta_1}{\sin \Theta_1}
\begin{array}{c}
1 - \tan^2 \Theta_1 \\
\tan^2 \Theta_1 \\
\cos \Theta_1 \\
\cos \Theta_1 \\
\cos \Theta_2 \\
\cos \Theta_2
\end{array}
\sin \Theta_1 \cos \Theta_1 \sin \Theta_2 \cos \Theta_2
\end{align*}
\]

\textbf{Ex.1.} Evaluate: \( \iiint d\chi d\Theta d\phi \) in the sphere \( x^2 + y^2 + z^2 = 1 \).
Also evaluate \( \iiint x \cdot d\chi d\Theta d\phi \).

\textbf{Solution:}
Let, \( x = n \cos \Theta_1 \),
\( y = n \sin \Theta_1 \cos \Theta_2 \),
\( z = n \sin \Theta_1 \sin \Theta_2 \).
As, \( x^2 + y^2 + z^2 \leq 1 \),
\( 0 < n \leq 1 \), \( 0 < \Theta_1 < \pi \), \( 0 < \Theta_2 \leq 2\pi \).
Hence, \( \iiint = n\sin \Theta_1 \).

Hence, \( \iiint d\chi d\Theta d\phi = \int_0^1 \int_0^\pi \int_0^{2\pi} \sin \Theta_1 \, d\phi \, d\Theta \, dn \).

\[
\begin{align*}
\iiint d\chi d\Theta d\phi &= \int_0^1 \int_0^\pi \int_0^{2\pi} \sin \Theta_1 \, d\phi \, d\Theta \, dn \\
&= \int_0^1 \int_0^\pi \sin \Theta_1 \, d\Theta \, dn \\
&= \frac{1}{3} \left[ - \cos \Theta_1 \right]_0^{2\pi} \\
&= \frac{4\pi}{3}.
\end{align*}
\]
Ex. 2. Evaluate: \[ \iiint_D \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}} \, dx \, dy \, dz \]

where \( D \) is the region bounded by \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \).

Solv.: Hence, \( D = \{ (x, y, z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \} \)

Let, \( \frac{x}{a} = \rho \cos \vartheta_1, \)

\( \frac{y}{b} = \rho \sin \vartheta_1 \cos \vartheta_2, \)

\( \frac{z}{c} = \rho \sin \vartheta_1 \sin \vartheta_2. \)

As \( (x, y, z) \in D, \)

0 < \rho < 1, \quad 0 < \vartheta_1 < \pi, \quad 0 < \vartheta_2 < 2\pi.

Hence the Jacobian of the transformation \( (x, y, z) \rightarrow (\rho, \vartheta_1, \vartheta_2) \) is \( J = \frac{\partial (x, y, z)}{\partial (\rho, \vartheta_1, \vartheta_2)} \)

\( = abc. \rho. \sin \theta_1. \)

Hence, \( \iiint_D \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}} \, dx \, dy \, dz \)

\( = \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{1} (abc. \rho^2 \sin \vartheta_1) \, d\rho \, d\vartheta_1 \, d\vartheta_2 \)

\( = (abc) \left( \int_{0}^{\pi} \sin \vartheta_1 \, d\vartheta_1 \right) \left( \int_{0}^{2\pi} d\vartheta_2 \right) \left( \int_{0}^{1} \rho^2 \, d\rho \right) \)

\( = (abc) \cdot \left( \int_{0}^{\pi} \sin \vartheta_1 \, d\vartheta_1 \right) \left( \int_{0}^{2\pi} d\vartheta_2 \right) \left( \int_{0}^{1} \rho^2 \, d\rho \right) \)

\( = \frac{2\pi (abc) \cdot \int_{2/2}^{3/2} (1 - \zeta)^{3/2} - 1 \, d\zeta}{\zeta} \), where \( \zeta = \frac{\rho}{a} \)

\( = \frac{2\pi (abc) \cdot B(3/2, 3/2)}{\Gamma(3/2) \Gamma(3/2)} \cdot \frac{1}{2} \)

\( = \frac{\pi (abc) \cdot B^2(3/2)}{2} \)

\( = \frac{\pi (abc) \cdot \pi(3)}{2} \)

\( = \frac{\Pi (abc)}{2} \)
Extrema of Functions of single variable:

Let $Y = f(x)$ be defined on $D$, domain.

**Definition:** (a) A function $y = f(x)$ is maximum at $a \in D$ if $f(x) \leq f(a)$, for all $x \in D$. A function $y = f(x)$ is minimum at $a \in D$ if $f(x) \geq f(a)$, for all $x \in D$.

Here $f(a)$ is the global or absolute maximum on minimum at $y = f(x)$ on $D$.

**Definition:** (b) A function $y = f(x)$ has a local or relative maximum or minimum at $x_0$ if there exists a neighborhood $N_{\delta}(x_0) = \{ x : |x - x_0| < \delta \}$ of $x_0$ such that $f(x) \leq f(x_0)$ for $x \in N_{\delta}(x_0)$.

Then, $f(x) \geq f(x_0)$ for $x \in N_{\delta}(x_0)$

Consider a graph of a function $y = f(x)$:

From the graph, $f(a)$ is the smallest or global minimum of $f(x)$ on $D$ and $f(x_0)$ is the largest value of $f(x)$ on $D$.

Also, note that $f(x_0)$, $f(x_*)$ are the local minimum and $f(x_1)$, $f(x_3)$ are the local minimum of $f(x)$.

Here a local maximum $f(x_3)$ is larger than a local maximum $f(x_0)$. 
A necessary condition for an extremum of a differentiable function \( f(x) \) at \( x_0 \) is \( f'(x_0) = 0 \). From graph, at the points of local maximum or minimum, the tangents of the \( y = f(x) \) is parallel to \( x \)-axis. Then at the points of local extremum we must have \( f'(x) = 0 \).

**Tests for extremum:**

(i) **First derivative test:** A function \( y = f(x) \) has a local minimum at \( x = x_0 \) if \( f'(x) < 0 \), if \( x \in (x_0 - s, x_0) \) and \( f'(x) > 0 \), if \( x \in (x_0, x_0 + s) \) for some \( s > 0 \).

\[\begin{array}{c}
\text{The function is decreasing here.} \\
\text{The function is increasing here.}
\end{array}\]

(ii) A function \( y = f(x) \) has a local maximum at \( x = x_0 \) if \( f'(x) > 0 \), if \( x \in (x_0 - s, x_0) \) and \( f'(x) < 0 \), if \( x \in (x_0, x_0 + s) \) for some \( s > 0 \).
Second derivative test:

(a) A function \( y = f(x) \) has a local minimum at \( x = x_0 \)
if \( f'(x_0) = 0 \) and \( f''(x_0) > 0 \).

(b) A function \( y = f(x) \) has a local maximum at \( x = x_0 \)
if \( f'(x_0) = 0 \) and \( f''(x_0) < 0 \).

Example:

1. Find the max. or min. value of \( y = f(x) \) over \( D : - \)
   \( y = |x|, \ x \in \mathbb{R} = D \).
   \[ y = \begin{cases} x, & \text{if } x > 0 \\ -x, & \text{if } x < 0 \end{cases} \]
   \[ \frac{dy}{dx} = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases} \]
   
   N.T. when \( x < 0 \), \( \frac{dy}{dx} < 0 \) and when \( x > 0 \), \( \frac{dy}{dx} > 0 \).
   
   Hence by first derivative test, \( y = |x| \) has minimum at \( x = 0 \).

2. Find the max. or min. value of \( y = 3 - 2(x-2)^{2/3}, x \in \mathbb{R} = D \).

   \[ y = 3 - 2(x-2)^{2/3} \]
   \[ \frac{dy}{dx} = -\frac{4}{3} (x-2)^{-1/3} \]

   Note that, \( \frac{dy}{dx} \) does not exist at \( x = 2 \); Hence, \( x = 2 \) may
   be a point of extremum, when \( x \in (2-d, 2) \) then
   \[ \frac{dy}{dx} > 0 \]
   and when \( x \in (2, 2+d) \), \( \frac{dy}{dx} < 0 \).

   By 1st derivative test, \( y \) is maximum at \( x = 2 \).
3. Find the extremum of the function

\[ f(x) = 5 - 3(x-2)^{2/3} \]

Solution:

\[ f(x) = 5 - 3(x-2)^{2/3} \]
\[ f'(x) = -2 \cdot \left( \frac{2}{3} \right) (x-2)^{-1/3} \]
\[ = -2 \cdot \frac{2}{3 \sqrt[3]{x-2}} \]

\[ f''(x) \] does not exist,

\[ \Rightarrow x = 2 \] is a stationary point.

Note, \( f'(2) = \begin{cases} \frac{2}{3} & \text{if } x < 2 \\ \frac{2}{3} & \text{if } x > 2 \end{cases} \)

\[ \Rightarrow f(x) \] has a unique local maximum at \( x = 2 \) and \( f(2) \) is the absolute maximum or largest value.

A necessary condition: \( f(x) \) has a local maximum or minimum at \( x = a \), then \( f'(a) \) does not exist or \( f''(a) = 0 \).

Question:

\[ \int \max \{ 2-x, 1+x \} \, dx \]

\[ \text{is (i) } 4 \quad \text{(ii) } \frac{9}{2} \quad \text{(iii) } 2 \quad \text{(iv) None.} \]

Solution:

\[ (\text{ii}) \text{ :- Ans.} \]

\[ \int \max \{ 2-x, 1+x \} \, dx = \int \left( 2-x \right) \frac{1}{x} \, dx + \frac{9}{2} \int \left( 1+x \right) \, dx - \int x \, dx \]

\[ = \int 2 \, dx + \int 2x \, dx - \int x \, dx \]

\[ = 2x + \frac{x^2}{2} - \frac{x^2}{2} \]
Theorem: (1st Derivative Test):
If $f'(x) < 0$ for $a - \delta < x < a$ and $f'(x) > 0$ for $a < x < a + \delta$ for some $\delta > 0$, then $f(a)$ is local minimum.

Proof: Lagrange mean-value theorem states that if $f(x)$ is continuous on $[a, b]$ and differentiable in $(a, b)$, then
\[
\frac{f(b) - f(a)}{b - a} = f'(c)
\]
where $c \in (a, b)$.

For $x \in (a - \delta, a)$
\[
\frac{f(x) - f(a)}{x - a} = f'(c_x) < 0
\]
\[
\Rightarrow f(x) - f(a) = (x - a) f'(c_x) < 0
\]

For $x \in (a, a + \delta)$
\[
\frac{f(x) - f(a)}{x - a} = f'(c_x) > 0
\]
\[
\Rightarrow f(x) - f(a) = (x - a) f'(c_x)
\]

Hence for all $x \in (a - \delta, a + \delta)$, $f(x) > f(a)$
\[
\Rightarrow f(a) \text{ is a local minimum.}
\]
Theorem: - (2nd derivative test) : If \( f''(a) = 0 \) and \( f''(a) < 0 \) then \( f(a) \) is the local maximum of \( f(x) \).

Proof: - By Rolle's Theorem,
\[
 f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(c) \quad \text{for some } c \text{ lies between } a \text{ and } x.
\]
Here, \( f'(a) = 0 \) and \( f''(a) < 0 \).

Then,
\[
 f(x) - f(a) = \frac{(x-a)^2}{2!} f''(c) < 0
\]
assuming \( f(x) \) is continuous at \( x = a \).

For \( x \in (a-\delta, a+\delta) \)
\[
 f(x) \leq f(a) \quad \forall x \in (a-\delta, a+\delta)
\]
\[
 f(x) \text{ is local maximum.}
\]

Ques: - Evaluate the double integral \( \int \int [x+y] \, dx \, dy \)

Solution:
\[
 \int \int [x+y] \, dx \, dy
\]
\[
 = \int_{0}^{2} \int_{0}^{2} [x+y] \, dx \, dy + \int_{0}^{2} \int_{1}^{2} [x+y] \, dx \, dy + \int_{0}^{2} \int_{2}^{3} [x+y] \, dx \, dy
\]
\[
0 < (x+y) \leq 1 \quad 1 < (x+y) \leq 2 \quad 2 < (x+y) \leq 3
\]
\[
= 0 + \int_{0}^{2} \int_{0}^{2} \, dx \, dy + 2 \int_{0}^{2} \int_{1}^{2} \, dx \, dy + 3 \int_{0}^{2} \int_{2}^{3} \, dx \, dy
\]
\[
= 6 \int_{0}^{2} \int_{0}^{2} \, dx \, dy
\]
\[
= 24.
\]
 Question

1. Evaluate the double integral
\[ \int \int \sin(z+y) \, dz \, dy \quad \text{over} \quad R = \{ 0 \leq z \leq \frac{\pi}{2}, \quad 0 \leq y \leq \frac{\pi}{2} \} \]

\[ \text{Ans:} \quad - \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin(z+y) \, dz \, dy \]
\[ = - \int_0^{\frac{\pi}{2}} \left[ \sin(z+y) \right]_0^{\pi/2} \, dy \]
\[ = - \int_0^{\frac{\pi}{2}} \left( \sin\left( \frac{\pi}{2} + y \right) - \sin y \right) \, dy \]
\[ = - \int_0^{\frac{\pi}{2}} \left( \cos y - \sin y \right) \, dy \]
\[ = \int_0^{\frac{\pi}{2}} \left( \cos y + \sin y \right) \, dy = \left. \left( \sin y - \cos y \right) \right|_0^{\frac{\pi}{2}} = 1 - 1 = 0 \]

2. Evaluate \( \int \int \cos(z+y) \, dz \, dy \)
DEFINITE INTEGRALS

Ex. Evaluate the following definite integrals:

(i) \[ \int_{-1}^{2} [x] \, dx \]

Sol. \[ [x] = \begin{cases} -2, & -2 \leq x < -1 \\ -1, & -1 \leq x < 0 \\ 0, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \end{cases} \]

\[ I = \int_{-1}^{2} [x] \, dx = \int_{-1}^{0} (-1) \, dx + \int_{0}^{1} (0) \, dx + \int_{1}^{2} (1) \, dx \]

\[ = [-x]_{-1}^{0} + 0 + [x]_{1}^{2} \]

\[ = 0. \]

(ii) \[ \int_{0}^{\infty} e^{x-[x]} \, dx \]

Sol. \[ \int_{0}^{\infty} e^{x-[x]} \, dx = \sum_{k=1}^{\infty} \int_{k-1}^{k} e^{x-[x]} \, dx \]

\[ = \sum_{k=1}^{\infty} \left( \int_{k-1}^{k} e^{x-k} \, dx \right) \]

\[ = \sum_{k=1}^{\infty} \left( \int_{0}^{1} e^{y} \, dy \right) \]

\[ = 150 \left( e - 1 \right) \]

Area of the shaded region = \( \frac{1}{2} \times 7 \times 7 + \frac{1}{2} \times 1 \times 1 \)

All the integrals:

\[ \int_{-4}^{4} |x-3| \, dx = 2 \]

\[ = -\int_{-4}^{0} (x-3) \, dx + \int_{0}^{4} (x-3) \, dx \]

\[ = \frac{1}{2} x^2 \bigg|_{-4}^{0} + \frac{1}{2} x^2 \bigg|_{0}^{4} \]

\[ = 25 \]

\[ \int_{-4}^{4} |3-2x| \, dx = 2 \]

\[ = \int_{-4}^{0} (2x-3) \, dx + \int_{0}^{4} (3-2x) \, dx \]

\[ = \frac{1}{2} x^2 \bigg|_{-4}^{0} + \frac{1}{2} x^2 \bigg|_{0}^{4} \]

\[ = 25 \]
(iv) \[ \int [x^2] \, dx \]

Solu.
\[ [x^2] = \begin{cases} 0 & , 0 \leq x^2 < 1 \Rightarrow 0 \leq x < 1 \\ 1 & , 1 \leq x^2 < 2 \Rightarrow 1 \leq x < \sqrt{2} \\ \frac{1}{2} & , 2 \leq x^2 < 3 \Rightarrow \frac{\sqrt{2}}{2} \leq x < \sqrt{3} \end{cases} \]

\[ \int [x^2] \, dx = \int_0^1 1 \, dx + \int_1^{\sqrt{2}} 1 \, dx + \int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{2} \, dx \]

\[ = (\sqrt{2} - 1) + 2(1.5 - \sqrt{2}) \]

\[ = (2 - \sqrt{2}) \cdot \]

(v) If \( \int e^t \frac{dt}{t + 1} = a \), then \( \int \frac{e^{t-b-1}}{t-b-1} \, dt \) is equal to \( (b-a) \cdot e^b \).

(a) \( a^e - b \)  
(b) \( -ae^b \)  
(c) \( -be^{-a} \)  
(d) \( ae^b \).

Solution:

\[ a = \int_0^1 \frac{e^t}{t + 1} \, dt \]

\[ \int_{b-1}^b \frac{e^{-t}}{t - b - 1} \, dt = \int_1^0 \frac{e^{u-b}}{-(u+1)} (-du) ; u = b - t \]

\[ = -e^{-b} \int_0^1 \frac{e^u}{u+1} \, du \]

\[ = -e^{-b} \cdot a \]

Ans. \( a(b) - ae^{-b} \).
Ex. (1): Find the difference between the greatest and least value of \( f(x) = \int_0^x (t^2 + t + 1) \, dt \) on \([2, 3]\).

**Solution:**

Let \( x_1, x_2, \ldots, x_k \) be the stationary points and one solution of \( f'(x) = 0 \). Then the absolute maximum value

\[
\max \left\{ f(a), f(b), f(x_1), \ldots, f(x_k), f(b) \right\}
\]

Here \( f'(x) = x^2 + x + 1 \)

\[
= (x + \frac{1}{2})^2 + \frac{3}{4} > 0 \quad \Rightarrow f(x) \text{ is increasing.}
\]

\[
\Rightarrow \text{The required difference is } = f(3) - f(2)
\]

\[
= \int_2^3 (t^2 + t + 1) \, dt
\]

\[
= \left[ \frac{t^3}{3} + \frac{t^2}{2} + t \right]_2^3
\]

\[
= 8 + \frac{2}{3} = \frac{16}{3}
\]