MATHMATICS BOOK FOR 10+2 ENTRANCES

BOOKS FOR B.SC MATHEMATICS/STATISTICS ENTRANCE EXAMS:

1. Challenge & Thrill of Pre-College Mathematics
2. Mathematics Olympiad by Rajeev Manocha
3. An excursion in Mathematics by Modak
4. Mathematical Circles by Fomin
5. Play with Graphs by Arihant
6. Maths Wit Volume 1 & 2 by S.Chatterjee
7. Plane Trigonometry by SL Loney
8. Coordinate Geometry by SL Loney
9. Essential Calculus early transcendentals by James Stewart
10. Test of Mathematics at 10+2 level by East-West Press
Infinite series

\[ f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^r}{r!} f^{(r)}(0) + \ldots \]

\[ f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots + \frac{(x-a)^r}{r!} f^{(r)}(a) + \ldots \]

\[ f(a + x) = f(a) + xf'(a) + \frac{x^2}{2!} f''(a) + \ldots + \frac{x^r}{r!} f^{(r)}(a) + \ldots \]

\[ e^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots, \quad \text{all } x \]

\[ \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1} \frac{x^r}{r} + \ldots, \quad -1 < x \leq 1 \]

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^{r+1} \frac{x^r}{(2r+1)!} + \ldots, \quad \text{all } x \]

\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots, \quad \text{all } x \]

\[ \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^{r+1} \frac{x^{2r+1}}{(2r+1)!} + \ldots, \quad -1 < x \leq 1 \]

\[ \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2r+1}}{(2r+1)!} + \ldots, \quad \text{all } x \]

\[ \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2r}}{(2r)!} + \ldots, \quad \text{all } x \]

\[ \tanh x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2r+1}}{(2r+1)} + \ldots, \quad -1 < x < 1 \]

Hyperbolic functions

\[ \cosh^2 x - \sinh^2 x = 1, \quad \sinh 2x = 2 \sinh x \cosh x, \quad \cosh 2x = \cosh^2 x + \sinh^2 x \]

Complex Numbers

\[ \{ r(\cos \theta + j \sin \theta) \}^n = r^n(\cos n\theta + j \sin n\theta) \]

\[ e^{j\theta} = \cos \theta + j \sin \theta \]

The roots of \( z^n = 1 \) are given by \( z = \exp\left( \frac{2\pi k j}{n} \right) \) for \( k = 0, 1, 2, \ldots, n-1 \)

Finite series

\[ \sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1) \quad \sum_{r=1}^{n} r^3 = \frac{1}{4} n^2(n+1)^2 \]
Perpendicular distance of a point from a line and a plane

Line: \( (x_1, y_1) \) from \( ax + by + c = 0 \):
\[
\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}
\]

Plane: \( (\alpha, \beta, \gamma) \) from \( n_1x + n_2y + n_3z + d = 0 \):
\[
\frac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}
\]

Vector product
\[
a \times b = |a||b| \sin \theta \ n = \begin{vmatrix} i & a_1 & b_1 \\ j & a_2 & b_2 \\ k & a_3 & b_3 \end{vmatrix} = \begin{pmatrix} a_1b_3 - a_2b_2 \\ a_2b_1 - a_3b_2 \\ a_3b_1 - a_1b_3 \end{pmatrix}
\]

Conics

<table>
<thead>
<tr>
<th>Ellipse</th>
<th>Parabola</th>
<th>Hyperbola</th>
<th>Rectangular hyperbola</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard form</td>
<td>( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 )</td>
<td>( y^2 = 4ax )</td>
<td>( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 )</td>
</tr>
<tr>
<td>Parametric form</td>
<td>((acos\theta, bsin\theta))</td>
<td>((at^2, 2at))</td>
<td>((usec\theta, bttan\theta))</td>
</tr>
<tr>
<td>Eccentricity</td>
<td>( e &lt; 1 )</td>
<td>( e = 1 )</td>
<td>( e &gt; 1 )</td>
</tr>
<tr>
<td>Foci</td>
<td>((\pm ae, 0))</td>
<td>((a, 0))</td>
<td>((\pm ae, 0))</td>
</tr>
<tr>
<td>Directrices</td>
<td>( x = \pm \frac{a}{e} )</td>
<td>( x = -a )</td>
<td>( x = \pm \frac{a}{e} )</td>
</tr>
<tr>
<td>Asymptotes</td>
<td>none</td>
<td>none</td>
<td>( \frac{x}{a} = \pm \frac{y}{b} )</td>
</tr>
</tbody>
</table>

Any of these conics can be expressed in polar coordinates (with the focus as the origin) as: 
\[
\frac{r}{\sqrt{1 + e \cos \theta}} = \frac{l}{r}
\]
where \( l \) is the length of the semi-latus rectum.

Mensuration

Sphere: Surface area = \( 4\pi r^2 \)

Cone: Curved surface area = \( \pi r \times \text{slant height} \)
### Differentiation \( f(x) \)

<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative ( f'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tan kx )</td>
<td>( k \sec^2 kx )</td>
</tr>
<tr>
<td>( \sec x )</td>
<td>( \sec x \tan x )</td>
</tr>
<tr>
<td>( \cot x )</td>
<td>( -\csc^2 x )</td>
</tr>
<tr>
<td>( \cosec x )</td>
<td>( -\cosec x \cot x )</td>
</tr>
<tr>
<td>( \arcsin x )</td>
<td>( \frac{1}{\sqrt{1-x^2}} )</td>
</tr>
<tr>
<td>( \arccos x )</td>
<td>( -\frac{1}{\sqrt{1-x^2}} )</td>
</tr>
<tr>
<td>( \arctan x )</td>
<td>( \frac{1}{1+x^2} )</td>
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<td>( \sinh x )</td>
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<td>( \tanh x )</td>
<td>( \sech^2 x )</td>
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<tr>
<td>( \arcsinh x )</td>
<td>( \frac{1}{\sqrt{1+x^2}} )</td>
</tr>
<tr>
<td>( \arccosh x )</td>
<td>( \frac{1}{\sqrt{x^2-1}} )</td>
</tr>
<tr>
<td>( \artanh x )</td>
<td>( \frac{1}{1-x^2} )</td>
</tr>
</tbody>
</table>

### Integration \( f(x) \)

<table>
<thead>
<tr>
<th>Function</th>
<th>Integral ( \int f(x) , dx ) (+ a constant)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sec^2 kx )</td>
<td>( \frac{1}{2} ) ( \tan kx )</td>
</tr>
<tr>
<td>( \tan x )</td>
<td>( \ln</td>
</tr>
<tr>
<td>( \cot x )</td>
<td>( \ln</td>
</tr>
<tr>
<td>( \cosec x )</td>
<td>( -\ln</td>
</tr>
<tr>
<td>( \arcsin x )</td>
<td>( \frac{1}{2} ) ( \ln \frac{x-a}{x+a} )</td>
</tr>
<tr>
<td>( \arccos x )</td>
<td>( \frac{1}{2} ) ( \arccos \frac{x}{a} ), (</td>
</tr>
<tr>
<td>( \arctan x )</td>
<td>( \frac{1}{a} ) ( \arctan \frac{x}{a} )</td>
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<tr>
<td>( \sinh x )</td>
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<tr>
<td>( \cosh x )</td>
<td>( \sinh x )</td>
</tr>
<tr>
<td>( \tanh x )</td>
<td>( \frac{1}{a} ) ( \tanh \frac{x}{a} )</td>
</tr>
<tr>
<td>( \arcsinh x )</td>
<td>( \frac{1}{a} ) ( \arcsinh \frac{x}{a} ) or ( \ln (x + \sqrt{x^2 + a^2}) ),</td>
</tr>
<tr>
<td>( \arccosh x )</td>
<td>( \frac{1}{a} ) ( \arccosh \frac{x}{a} ) or ( \ln (x + \sqrt{x^2 - a^2}) ), (x &gt; a ), (a &gt; 0 )</td>
</tr>
</tbody>
</table>

### Quotient rule

\[ y = \frac{u}{v}, \quad \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \]

### Trapezium rule

\[ \int_a^b \frac{1}{2} \left( y_0 + y_n \right) + 2 \left( y_1 + y_2 + \ldots + y_{n-1} \right), \text{ where } h = \frac{b-a}{n} \]

### Integration by parts

\[ \int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx \]

### Area of a sector

\[ A = \frac{1}{2} \int r^2 \, d\theta \] (polar coordinates)

\[ A = \frac{1}{2} \int (xy - yx) \, dt \] (parametric form)

### Arc length

\[ s = \int \sqrt{\left( \frac{dy}{dx} \right)^2 + 1} \, dx \] (parametric form)

\[ s = \int \sqrt{\left( \frac{dr}{d\theta} \right)^2 + 1} \, d\theta \] (polar coordinates)

### Surface area of revolution

\[ S_x = \int y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

\[ S_y = \int x \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

### Curvature

\[ \kappa = \frac{\partial y}{\partial s} = \frac{x \ddot{y} - \dot{x} \dot{y}}{(x^2 + y^2)^{3/2}} = \frac{\dot{y}^2}{(1 + \left( \frac{\dot{y}}{\dot{x}} \right)^2)^{3/2}} \]

### Radius of curvature

\[ \rho = \frac{1}{\kappa}, \quad \text{Centre of curvature } c = r + \rho \hat{n} \]

### L'Hôpital's rule

If \( f(a) = g(a) = 0 \) and \( g'(a) \neq 0 \) then

\[ \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \]

### Multi-variable calculus

\[ \nabla g = \left( \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right) \]

For \( w = g(x, y, z) \), \( \delta w = \frac{\partial w}{\partial x} \, \delta x + \frac{\partial w}{\partial y} \, \delta y + \frac{\partial w}{\partial z} \, \delta z \]
**Induction**

If an positive integer equation is satisfied by \( n = 1 \), \( n = k \) where \( k \) is any natural number and if this is satisfied by \( n = k+1 \) given for \( n = k \) this is true then the equation is satisfied by any positive integer. This is called law of induction.

**Problem 1: Prove that** \( 1^3 + 2^3 + 3^3 + ..... + n^3 = \{n(n+1)/2\}^2 \)

**Solution 1:** Let us check for \( n = 1 \).

Now, putting \( n = 1 \) in RHS we get, \( \{1(1 + 1)/2\}^3 = 1 = \text{LHS} \).
So, let us consider that this is true for \( n = k \).

That is, we have, \( 1^3 + 2^3 + 3^3 + ..... + k^3 = \{k(k+1)/2\}^2 \)
Now, we will see what happens with \( n = k + 1 \).
LHS = \( 1^3 + 2^3 + 3^3 + ..... + k^3 + (k + 1)^3 \)
\[= \{k(k+!)/2\}^2 + (k+1)^3 \]
\[= \{(k+1)/2\}^2(k^2 + 4k + 4) \]
\[= \{(k+1)(k+2)/2\}^2 \]
= RHS for \( n = k + 1 \).
So, this is true for \( n = 1 \), \( k \) and \( k + 1 \) given this is true for \( n = k \).
So, this will hold for any positive integer \( n \) according to the law of induction.
Number Theory

Congruency: \( a \equiv b \pmod{c} \)
It reads a congruent to b modulus c.
It means a, b, c are integers and a – b is divisible by c or if a is divided by c then the remainder is b.

Example: \( 7 \equiv 2 \pmod{5} \)

Properties:
(a) If \( a \equiv b \pmod{c} \) then \( a^n \equiv b^n \pmod{c} \)
Proof: \( a^n - b^n = (a - b)(a^{n-1} + ba^{n-2} + ... + b^{n-2}a + b^{n-1}) \)
Now, a – b is divisible by c.
\( \Rightarrow a^n - b^n \) is divisible by c.
So, \( a^n \equiv b^n \pmod{c} \)

(b) If \( a \equiv b \pmod{c} \) then \( ad \equiv bd \pmod{c} \).
Proof: \( ad - bd = d(a -b) (a - b) \) is divisible by c
So, (a – b)d is also divisible by c.
\( \Rightarrow ad \equiv bd \pmod{c} \)

(c) If \( a \equiv b \pmod{c} \) then \( a + d \equiv b + d \pmod{c} \).

Proof: \( (a + d) - (b + d) = (a - b) \) divisible by c.
\( \Rightarrow a + d \equiv b + d \pmod{c} \)

(d) If \( a \equiv b \pmod{c} \) then \( a - d \equiv b - d \pmod{c} \).

Proof: \( (a - d) - (b - d) = (a - b) \) divisible by c.
So, \( a - d \equiv b - d \pmod{c} \)

(e) If \( a \equiv b \pmod{c} \) and \( a \equiv b \pmod{d} \) then \( a \equiv b \pmod{cd} \)
if $\gcd(c, d) = 1$.

Proof: $a - b$ is divisible by $c$ and $d$ and $\gcd(c, d) = 1$.

$\Rightarrow a - b$ is divisible by $cd$.

$\Rightarrow a \equiv b \pmod{cd}$

(f) If $a \equiv b \pmod{c}$ then $a/d \equiv b/d \pmod{c}$ where $\gcd(c, d) = 1$.

Proof: $(a/d) - (b/d) = (a - b)/d$

Now, $c$ divides $a - b$ so $c$ divides $(a - b)/d$ as $c$ and $d$ are relatively prime.

$\Rightarrow a/d \equiv b/d \pmod{c}$

Corollary: If $a \equiv b \pmod{c}$ and $\gcd(c, d) = g$ then $a/d \equiv b/d \pmod{c/g}$

Proof: $(a/d) - (b/d) = (a - b)/d = (a - b)/gd_1$ where $d = gd_1$

Now, $\gcd(d_1, c) = 1$ Now,

$c$ divides $(a - b)$

$\Rightarrow c/g$ divides $(a - b)/g$.

$\Rightarrow a/d \equiv b/d \pmod{c/g}$

Example: Find the remainder when $(n^2 + 1)/2$ is divided by 4, where $n$ is odd.

Solution: Now, as $n$ is odd so $n \equiv 1$ or $3 \pmod{4}$

$\Rightarrow n \equiv \pm 1 \pmod{4}$

$\Rightarrow n^2 \equiv (\pm 1)^2 \pmod{4}$

$\Rightarrow n^2 \equiv 1 \pmod{4}$

$\Rightarrow n^2 + 1 \equiv 1 + 1 \pmod{4}$

$\Rightarrow n^2 + 1 \equiv 2 \pmod{4}$

$\Rightarrow (n^2 + 1)/2 \equiv 2/2 \pmod{4}$

$\Rightarrow (n^2 + 1)/2 \equiv 1 \pmod{4}$

This is wrong as we have divided by 2 and $\gcd(2, 4) = 2$ and we have not changed the modulus accordingly.

This is a common mistake done by lots of students.

Right procedure is,

$n \equiv \pm 1, \pm 3 \pmod{8}$ (We are starting from modulus 8 as we have to divide by 2 and the modulus will become $8/2 = 4$ and we will get the result)

$\Rightarrow n^2 \equiv 1, 9 \pmod{8}$
\[ n^2 \equiv 1 \pmod{8} \quad \text{(As } 9 \equiv 1 \pmod{8} \text{)} \]
\[ n^2 + 1 \equiv 2 \pmod{8} \]
\[ (n^2 + 1)/2 \equiv 2/2 \pmod{8/2} \quad \text{(As } \gcd(8, 2) = 2 \text{)} \]
\[ (n^2 + 1)/2 \equiv 1 \pmod{4} \]

Note that though the answer was correct but the procedure was wrong.

**Tips to solve number theory problems:**

1. (Any odd number) \[2\] \[\equiv 1 \pmod{4}\]

Proof: Let \( n \) is odd.
Then \( n \equiv \pm 1 \pmod{4} \)
\[ \Rightarrow n^2 \equiv (\pm 1)^2 \pmod{4} \]
\[ n^2 \equiv 1 \pmod{4} \]

Application:

**Problem 1: Prove that sum of squares of 8n+4 consecutive positive integers cannot be perfect square.**

Solution: There are \( 8n + 4 \) consecutive integers.
- There are \( 4n + 2 \) odd integers and \( 4n + 2 \) even integers.
Now, (any even integer) \[2\] \[\equiv 0 \pmod{4}\] (As there must be a factor at least 2 in even integer and when it is squared then \( 2^2 = 4 \) is at least a factor of square of an even number).

So, sum of all even integer square \( \equiv 0 \pmod{4} \)

Now, there are \( 4n + 2 \) odd integers.

If they are summed up it will give an even integer as number of odd terms.

So, let us consider the equation, \( a_1^2 + a_2^2 + \ldots + a_{4n+2}^2 \) are odd \( i = 1, 2, \ldots, 4n+2 \) and \( b \) is even.

Now, dividing the equation by 4 we get,
\[ 1 + 1 + \ldots \ (4n + 2) \text{ times} \equiv 0 \pmod{4} \quad \text{(As } b \text{ is even)} \]
\[ \Rightarrow 4n + 2 \equiv 0 \pmod{4} \]
\[ 2 \equiv 0 \pmod{4} \quad \text{(As } 4n \equiv 0 \pmod{4} \text{)} \]

Which is impossible.
So, sum of squares of \( 8n + 4 \) consecutive positive integers cannot be perfect square.
Problem 2: Prove that sum of squares of $4n+3$ number of odd integers cannot be perfect square where $n \geq 0$.

Solution 2: Let us take the equation, $a_1^2 + a_2^2 + \ldots + a_{4n+3}^2 = b^2$ where $a_i$’s are odd $i = 1, 2, \ldots, 4n+3$

Now, $b^2$ is sum of odd number $(4n + 3)$ of odd numbers ($a_i^2$)

$\Rightarrow$ $b^2$ is odd.

Now, dividing the equation by 4 we get,

$1 + 1 + \ldots (4n+3 \text{ times}) \equiv 1 \pmod{4}$

$\Rightarrow 4n + 3 \equiv 1 \pmod{4}$

$\Rightarrow 3 \equiv 1 \pmod{4}$ (As $4n \equiv 0 \pmod{4}$)

Which is impossible.

2. (Any odd integer)$^2 \equiv 1 \pmod{8}$

Proof: Let, $n$ is any odd integer.

Now, $n \equiv 1, 3, 5, 7 \pmod{8}$

$\Rightarrow n^2 \equiv 1, 9, 25, 49 \pmod{8}$

$\Rightarrow n^2 \equiv 1 \pmod{8}$ (As, $9, 25, 49$ all $\equiv 1 \pmod{8}$)

3. (Any odd integer)$^4 \equiv 1 \pmod{16}$

Proof: Let $n$ is any odd integer.

Now, $n^2 \equiv 1 \pmod{8}$

$\Rightarrow n^2 \equiv 1, 9 \pmod{16}$

$\Rightarrow (n^2)^2 \equiv 1, 81 \pmod{16}$

$\Rightarrow n^4 \equiv 1 \pmod{16}$ (As $81 \equiv 1 \pmod{16}$)

4. If $m$ is any odd integer not divisible by 3 then $m^2 \equiv 1 \pmod{24}$.

Proof: As $m$ is not divisible by 3, $m \equiv \pm 1 \pmod{3}$

$\Rightarrow m^2 \equiv 1 \pmod{3}$

And from above we have, $m^2 \equiv 1 \pmod{8}$
As \(\gcd(3, 8) = 1\) so we have, \(m^2 \equiv 1 \pmod{3 \times 8}\)

**Problem 1: Prove that sum of squares of \(24n + 17\) primes where all the primes > 3 cannot be a square number.**

Solution 1: This problem cannot be solved by dividing by 4 or 8.
This problem needs to be solved by dividing by 24.
As all the primes > 3 so they don’t have a factor of 3.
Consider the equation, \(a_1^2 + a_2^2 + \ldots + a_{24n+17}^2 = b^2\) where \(a_i\)'s are odd primes and \(b\) is odd.

Dividing the equation by 24 we get,

\[
1 + 1 + \ldots (24n+17 \text{ times}) \equiv 1 \text{ or } 9 \pmod{24} \quad (\text{If } b \text{ is not divisible by } 3 \text{ then } 1 \text{ if divisible then } 9)
\]
\[
\Rightarrow 24n + 17 \equiv 1 \text{ or } 9 \pmod{24}
\]
\[
\Rightarrow 17 \equiv 1 \text{ or } 9 \pmod{24}
\]

Which is impossible.

5. If any \(n\) consecutive positive integers are divided by \(n\) then remainders are 0, 1, 2, 3, ..., \(n-1\) where the loop remains same but starting and ending rotates.

Proof: Self-explanatory.

**Problem 1: Prove that sum of squares of \(n\) consecutive positive integers is divisible by \(n\) where \(n\) is odd and \(n\) is not divisible by 3.**

Solution: Now, we have seen that the remainders are 0, 1, 2, ..., \(n-1\) in order but the starting point may be different. So, if we take sum then we are good to go to start with 1.

So, sum of square of remainders = \(1^2 + 2^2 + 3^2 + \ldots + (n-1)^2\)
\[
= (n-1)n(2n-1)/6
\]

Now, \(n\) is odd so \(n\) is not divisible by 2 and also \(n\) is not divisible by 3.

\[
\Rightarrow 6 \text{ doesn’t divide } n.
\]
\[
\Rightarrow (n - 1)n(2n - 1)/6 \text{ is divisible by } n.
\]
Theorems:

1. Fermat’s Little Theorem: If $p$ is a prime then $a^{(p - 1)} \equiv 1 \pmod{p}$ where $a$ and $p$ are relatively prime.

Proof: Let us consider the integers $a, 2a, 3a, ..., (p - 1)a$.

None of these divisible by $p$. Also no two of these are congruent modulo $p$, because if $ra \equiv sa \pmod{p}$ for some integers $r, s$ such that $1 \leq r < s \leq p - 1$, then cancelling $a$ (since $a$ is prime to $p$) we must have $r \equiv s \pmod{p}$, a contradiction.

This means the integers $a, 2a, 3a, ..., (p - 1)a$ are congruent to $1, 2, 3, ..., p - 1$ modulo $p$, taken in some order.

Therefore, $a \times 2a \times 3a \times ... \times (p - 1)a \equiv 1 \times 2 \times 3 \times ... \times (p - 1) \pmod{p}$

$\Rightarrow a^{(p - 1)} \times (p - 1)! \equiv (p - 1)! \pmod{p}$

$\Rightarrow a^{(p - 1)} \equiv 1 \pmod{p}$ (since $\gcd(p, (p - 1)!) = 1$).

Problem 1: There are $p$ number of leaves kept in a circular form where $p$ is prime. The leaves are numbered $L_1, L_2, L_3, ..., L_p$ in anti-clockwise direction. A frog first jumps to any leaf say, $L_n$ leaf. Then it jumps to leaf $L_{2n}$, then to leaf $L_{4n}$, then $L_{8n}$, ... Prove that after $p^{th}$ jump the frog will reach the first leaf where he started i.e. $L_n$.

Solution 1:
First understand that the frog will stay in the leaf $L_i$ after $j^{th}$ jump if $t_j \equiv i \pmod{p}$.

Let us take an example to understand this.

Let there are 7 leaves.

Let the frog first jumps to $L_3$.

The frog will then jump to $L_6$, then $L_{12}$. Now $L_{12}$ is $12 \equiv 5 \pmod{7}$.

So the frog will be staying in $L_5$ after $3^{rd}$ jump.

Now, let us get back to the general case as depicted in the problem.

Let, the frog first jumps to $L_n$.

At $2^{nd}$ jump the term is $n \times 2$. At
3rd jump the term $n \times 2^2$.
...
...
At $p^{th}$ jump the term is $n \times 2^{p-1}$.
Now, we need to find the remainder of $n \times 2^{p-1}$ divided by $p$.
Now, as per Fermat’s little theorem, $2^{p-1} \equiv 1 \pmod{p}$
\[ \Rightarrow n \times 2^{p-1} \equiv n \pmod{p} \]
So, the frog will reach $L_n$ after $p^{th}$ jump.

**Corollary:** If $p$ and $p+2$ are twin prime i.e. both $p$ and $p+2$ are primes then $p \times 2^p + 1 \equiv 0 \pmod{p+2}$

**Proof:** Now, $p \equiv -2 \pmod{p+2}$

\[ \Rightarrow p \times 2^p \equiv -2^{(p+1)} \pmod{p+2} \]
\[ \Rightarrow p \times 2^p \equiv -1 \pmod{p+2} \] (By Fermat’s Little theorem, $2^{(p+1)} \equiv 1 \pmod{p+2}$)
\[ \Rightarrow p \times 2^p + 1 \equiv 0 \pmod{p+2} \]

**Problem:** Find the remainder when $41 \times 2^{41} + 1$ is divided by $43$.

**Solution:** Now, 41 and 43 are both prime, putting $p = 41$ in the above expression we get,

$41 \times 2^{41} + 1 \equiv 0 \pmod{43}$

2. If $p$ is a prime then $\binom{p-1}{r} \equiv (-1)^r \pmod{p}$

**Proof:** Now, $\binom{p-1}{r} = (p-1)/(p-r)!(p-r+1)...(p-2)/r$!

Now, $p - 1 \equiv -1 \pmod{p}$
$p - 2 \equiv -2 \pmod{p}$
$p - 3 \equiv -3 \pmod{p}$
p - r ≡ -r (mod p)

(p-1)(p-2)...(p-r) ≡ (-1)(-2)(-3)...(-r) (mod p)
(p-1)(p-2)...(p-r) ≡ (-1)^r \times r! (mod p)
(p-1)(p-2)...(p-r)/r! ≡ (-1)^r (mod p)

(we can divide because p is prime and r! doesn’t contain any factor of p)

⇒ (p-1)C_r ≡ (-1)^r (mod p)

3. Wilson’s Theorem: If p is a prime then (p - 1)! + 1 ≡ 0 (mod p).

4. Euler’s Theorem : \(a^{\Phi(n)} \equiv 1 \pmod{n}\) where n is any natural number and \(\Phi(n)\) is Euler’s totient function and \(\gcd(a, n) = 1\) i.e. a and n are relatively prime.

**Factor, GCD:** Factors of a positive integer is defined as the positive integers less than the integer and which divides the integer.

For example, if \(p_1, p_2, p_3, p_4 < n\) and divides n then \(p_1, p_2, p_3, p_4\) are called factors of n. For example, the factors of 12 are 1, 2, 3, 4, 6, 12.

Note that when a factor of a positive integer occurs then another factor by default occurs to produce result \(p_1 \times p_2 = n\). So every number has even Number of factors but what if \(p_1 \times p_1 = n\) then \(p_1\) comes twice and it is counted as 1 factor. In that case the numbers have odd number of factors. In the above example we see that n is a square number and \(n = p_1^2\) i.e., only the square numbers have odd number of factors.
Problem 1: There are 1000 doors named $D_1, D_2, D_3, \ldots, D_{1000}$. There are 1000 persons named $P_1, P_2, P_3, \ldots, P_{1000}$. At first all the doors are closed. $P_1$ goes and opens all the doors. Then $P_2$ goes and closes even numbered doors i.e., $D_2, D_4, D_6, \ldots, D_{1000}$ and leaves the odd numbered doors i.e. $D_1, D_3, D_5, \ldots, D_{999}$ as it is. Then $P_3$ goes and changes the state of the doors (If open then closes and if closed then opens) which are multiple of 3 i.e., the doors $D_3, D_6, \ldots$. Then $P_4$ goes and changes the state of the doors which are multiple of 4. Similarly in this way 1000 persons goes and changes the states of the doors accordingly. At the end how many doors will be open and how many doors will be closed?

Solution 1: Let us take an example.

Let us see how many times and which persons are operating on $D_{28}$.

The persons who are operating on $D_{28}$ are $P_1, P_2, P_4, P_7, P_{14}, P_{28}$. Now, $1, 2, 4, 7, 14, 28$ are factors of 28.

So, any door $D_i$ is getting operated by the persons $P_j$ where $j$’s are the factors of $i$.

In this case we are not interested to find the factors of every number up to 1000. Rather we are interested in which doors are getting operated odd number of times and which doors are getting operated even number of times i.e. which numbers have odd number of factors and which numbers have even number of factors. If odd number of factors then odd number of operation and the door will be open and if even number of factors then even number of operation and the door will be closed.

We have seen that every number has even number of factors except the square numbers which have odd number of factors.

So, $D_1, D_4, D_9, D_{16}, \ldots, D_{961}$ these 31 doors will stay open and rest of the doors will stay closed.

**Number of factors of an integer**: Let, $n = p_1^{r_1} \times p_2^{s_2}$

Then number of factors of $n = (r_1 + 1)(r_2 + 1)$

Proof: Now, $p_1, p_1^2, p_1^3, \ldots, p_1^r$ divides $n$. So there are $r$ factors.

Similarly, $p_2, p_2^2, p_2^3, \ldots, p_2^s$ divides $n$.

So there are $s$ factors.
Now, \( p_1 \times p_2, p_1 \times p_2^2, \ldots, p_1 \times p_2^s \) divides \( n \).

So, for \( p_1 \) number of factors occurring with the set of \( p_2 \) is \( s \).

Similarly, for \( p_1^2 \) number of factors occurring with the set of \( p_2 \) is \( s \).

And \( s \) number of factors occurring for every \( p_1^i \) with the set of \( p_2 \). So, number of factors in this way = \( r \times s \).

And 1 divides \( n \).

So, number of factors of \( n = r + s + r \times s + 1 = (r + 1)(s + 1) \)

This can be extended or generalized for any \( n = (p_1^{a_1})(p_2^{a_2})\ldots(p_m^{a_m}) \) then, Number of factors of \( n = (a_1 + 1)(a_2 + 1)\ldots(a_m + 1) \)

**Example: Find the number of factors of 12.**

Solution: \( 12 = 2^2 \times 3 \)

So, number of factors of 12 = \( (2 + 1)(1 + 1) = 6 \)

Those are = 1, 2, 3, 4, 6, 12.

**Sum of the factors:** Let, \( n = p_1^r \times p_2^s \)

Then sum of the factors of \( n = \{(p_1^{r+1} - 1)/(p_1 - 1)\} \times \{(p_2^{s+1} - 1)/(p_2 - 1)\} \)

Proof: Now, sum of the factors of \( n = 1 + (p_1 + p_1^2 + \ldots + p_1^r) + (p_2 + p_2^2 + \ldots + p_2^s) + \ldots \}

= \{(p_1^{r+1} - 1)/(p_1 - 1)\} \times \{(p_2^{s+1} - 1)/(p_2 - 1)\}
This can be extended to general case as, If

\[ n = (p_1^{a_1})(p_2^{a_2}) \ldots (p_m^{a_m}) \]

Then, sum of factors of \( n \) = \[
\left\{ \frac{p_1^{(a_1+1)} - 1}{p_1 - 1} \right\} \cdot \frac{p_2^{(a_2+1)} - 1}{p_2 - 1} \ldots \left\{ \frac{p_m^{(a_m+1)} - 1}{p_m - 1} \right\} 
\]

**Example:** Find the sum of factors of 12.

**Solution:**

12 = \( 2^2 \times 3 \)

Sum of factors of 12 = \[
\left\{ \frac{2^{(2+1)} - 1}{2 - 1} \right\} \cdot \left\{ \frac{3^{(1+1)} - 1}{3 - 1} \right\} = (8 - 1)(9 - 1)/2 = 7 \times 8/2 = 7 \times 4 = 28.
\]

Check, sum of factors of 12 = 1 + 2 + 3 + 4 + 6 + 12 = 28.

**Theorem:** If \( p, q \) are odd then either of \( (p - q)/2 \) and \( (p + q)/2 \) is odd and another is even.

**Proof:** Let, \( p + q = 2^s \times m_1 \) where \( m_1 \) is odd. And \( p - q = 2^t \times m_2 \) where \( m_2 \) is odd. Adding the two equations we get, \( 2p = 2^s \times m_1 + 2^t \times m_2 \)

\[ \Rightarrow p = 2^{s-1} \times m_1 + 2^{t-1} \times m_2 \]

Now, \( p \) is odd.

\[ \Rightarrow \]

Either \( s = 1, t > 1 \) or \( t = 1 \) and \( s > 1 \).

Either of \( (p - q)/2 \) and \( (p + q)/2 \) is odd and another is even.

**Problem 1:** There are two square boards with \( (p - q)/2 \) and \( (p + q)/2 \) number of rows or columns where \( p, q \) are odd. The boards have \( \{(p - q)/2\}^2 \) and \( \{(p + q)/2\}^2 \) number of small squares respectively. The small squares are numbered 2, 3, 4, ..., row-wise. Define a function \( f : \mathbb{N} \rightarrow \mathbb{N} + \{0\} \) such that \( f(m) = m\{(m - 1)! - 1\} \). At first all the small squares are coloured black. One paints white and counts 1, then black counts 2, then white again counts 3, then black again counts 4, ..., till he reaches counting \( f(m) \) for \( m^{th} \) box. Prove that out of the two boards exactly one board will turn into a CHESS board.

**Solution 1:**

Now, \( f(m) = m\{(m - 1)! - 1\} = m! - m \)

Now, \( m! \) is always even. So \( m! - m \) is even if \( m \) is even and odd if \( m \) is odd.

Now, consecutive integers i.e. 2, 3, 4, ... comes in alternate even and odd.

For \( m = 2 \), \( f(2) = 0 = \text{even} \).
For \( m = 3 \), \( f(3) = 3 \) = odd;
For \( m = 4 \), \( f(4) = 20 \) = even.

...

So, if we alternately paint white and black and count to \( f(m) \) for every \( m \) then we will have white, black, white, black, ... this colours in the small squares of the board row-wise.

Now, if the board has even number of rows then if it starts with black then it will end the row with white and then black again in the second row first column small square. So, all the first column small squares of the board will be black then second column white, then black,... So it will not turn into a CHESS board.

Now, if the board has odd number of rows then if it starts with black in the first row first column then it will end up with black. So, second row first column square will be black. And then continues white, black,... So, it will turn into a CHESS board.

Now, out of \( (p - q)/2 \) and \( (p + q)/2 \) one is odd and another is even. The even will not turn into CHESS board and the odd will turn into CHESS board.

\[ \Rightarrow \] Out of the two boards exactly one will turn into a CHESS board.

**Definition of GCD:** GCD (greatest common divisor) of two positive integers is defined as the greatest common divisor of both the integers.

For example, if \( n_1 = p_1^r \times p_2^s \) and \( n_2 = p_1^t \times p_3^u \) where \( t < r \).

Then GCD \((n_1, n_2) = p_1^t \)

For example, \( 12 = 2^2 \times 3 \) and \( 28 = 2^2 \times 7 \)

Then GCD \((12, 28) = 2^2 = 4 \).

**Relatively prime:**
If there is no factor between \( n_1 \) and \( n_2 \) (\( n_1, n_2 \) may be prime or composite) then \( n_1 \) and \( n_2 \) are said to be relatively prime to each other.

Another definition, if GCD \((n_1, n_2) = 1 \) then \( n_1 \) and \( n_2 \) are called relatively prime.
For example, 12 and 35 are relatively prime though they both are composite number.

**Euler’s totient/Phi function:** Euler’s totient function denoted by $\Phi(n)$ of a positive integer $n$ is defined as number of positive integers that are less than $n$ and relatively prime to $n$.

For example, $\Phi(12) = 4$ (These are 1, 5, 7, 11)

Let, $n = p^k$ where $p$ is prime.

Now, $p, 2p, 3p, ..., p^k$ are not relatively prime to $n$.

Clearly, the number of such integers = $p^{k-1}$

So, number of positive integers that are less than $n$ and relatively prime to $n = p^k - p^{k-1} = p^k(1 - 1/p)$

Now, $\Phi(mn) = \Phi(m)\Phi(n)$ where, m and n are relatively prime.

**Problem 1:** There are 1000 doors named D1, D2, D3, ...., D1000. There are 1000 persons named P1, P2, P3, ......, P1000. At first, all the doors are closed. Now, P1 goes and opens all the doors. Then P2 goes and closes D3, D5,...., D999 i.e. the numbers that are relatively prime and leaves other doors as it is. Then P3 goes and reverses Di (i.e. if D i is open then closes and if it is closed then opens) where gcd of (i, 3) = 1 and i > 3. Note that gcd(2, 3) = 1 but P3 will not reverse D2 because 2 < 3. In this way Pt goes and changes the doors Dj where gcd(j, t) = 1 and j > t. In this way P1000 goes and leaves all the doors unchanged because 1000 is the greatest number here. At the end which door(s) will be open and which door(s) will be closed?

**Solution 1:** Let us take an example.

Let us see which persons are operating on D12 and how many times D12 is getting operated.

Clearly, P1, P5, P7, P11 are operating on D12.

Now, 1, 5, 7, 11 are relatively prime to 12 and less than 12.

So, any door Di is getting operated by Pj where j’s are less than i and relatively prime to i.

This is the definition of Euler’s totient function.
Now, inspect the expression and it is easy to verify that $\Phi(n)$ is even except for $\Phi(1) = \Phi(2) = 1$.

So, every door will be operated even number of times except $D_1$ and $D_2$.

So all the doors will be closed except $D_1$ and $D_2$ which will stay opened.

**Problem 2:** There is an nxn board with $n^2$ small squares where $n$ is odd. All are painted white. They are numbered 3, 4, 5, 6, ..., $n^2 + 2$. There are $n^2+1$ brushes numbered $B(2), B(3), \ldots, B(n^2), B(n^2+1), B(n^2+2)$. First $B(2)$ will be picked up and it will paint black on the i numbered square if $\text{gcd}(i, 2) \neq 1$ i.e. it will paint all the even numbered squares black. Now, $B(3)$ will be picked up and it will change the colour (if it is painted white then black and if it is painted black then white) of j numbered square if $\text{gcd}(j, 3) \neq 1$ and $j > 3$. In this way $B(t)$ will be picked up and it will change the colour of k numbered square if $\text{gcd}(t, k) \neq 1$ and $k > t$. For example $\text{gcd}(6, 12) = 6$ but $B(12)$ will not change the colour of 6 numbered square as $6 < 12$. Prove that after the painting of $B(n^2+2)$ brush i.e. all the brushes the board will turn into a CHESS board.

**Solution 2:** Let us take an example.

Let us see which brushes are painting on 12th square and how many times 12th square is getting painted.

12th square is getting painted by 2, 3, 4, 6, 8, 9, 10, 12.

The set is nothing but $12 - \Phi(12)$.

So, any square i is getting painted by $i - \Phi(i)$.

Now, $\Phi(i)$ is always even as there is no 1 or 2 numbered square.

$\Rightarrow i - \Phi(i)$ is even if i is even and odd if i is odd.

Now, if odd number of operation then square will be painted black and if even number of operation then square will be painted white.

So, consecutive squares (odd, even, odd, even, ..) will be black, white, black, white, ...

Now, $n$ is odd.

So, first row will start with black and end with black. So, second row will start with white and end with white.

Similarly goes the rest of the rows.

So, the board is turning into a CHESS board.
**Tips:** $a^4 + 4b^4 = (a^2)^2 + (2b^2)^2 = (a^2 + 2b^2)^2 - 2 \times a^2 \times 2b^2$

$= (a^2 + 2b^2)^2 - (2ab)^2 = (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2)$

So, we see that if we get an expression of the form $a^4 + 4b^4$ then it cannot be prime as it can be factored.

**Problem:** Prove that $n^4 + 4^n$ cannot be prime where $n$ is natural number.

Solution: Clearly if $n$ is even then it is even and cannot be prime.

So, $n$ needs to be odd.

Note that if $n$ is odd then $n - 1$ is even.

Now, $n^4 + 4^n = n^4 + 4 \times 4^{n-1}$

$= a^4 + 4b^4$ form.

So cannot be prime as it can be factored.
Inequality

First we will discuss about AM (Arithmetic mean), GM (Geometric mean) and HM (Harmonic mean).

AM of any n positive real number is defined as \( \frac{x_1 + x_2 + \ldots + x_n}{n} \).

GM of any n positive real number is defined as \( (x_1x_2\ldots x_n)^{1/n} \).

HM of any n positive real number is defined as \( \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}} \).

**Theorem: AM ≥ GM ≥ HM**

Proof: Let, a and b are two positive real numbers.

Then, \((a - b)^2 \geq 0\)

\( (a + b)^2 - 4ab \geq 0 \)

\( (a + b)^2 \geq 4ab \)

\( \left\{\frac{(a + b)}{2}\right\}^2 \geq ab \)

\( \frac{(a + b)}{2} \geq \sqrt{ab} \)

\( \frac{a + b}{2} \geq \sqrt{ab} \)

This is proved for two positive real numbers. It can be extended to any number of positive real numbers.

Now, we have, \( \frac{a + b}{2} \geq \sqrt{ab} \)

\( 1/\sqrt{ab} \geq 2/(a + b) \)

\( \sqrt{ab}/ab \geq 2/(a + b) \)

\( \sqrt{ab} \geq 2ab/(a + b) \)

\( \sqrt{ab} \geq 2/\{(1/a) + 1/b\} \)

\( GM \geq HM \)

This is proved for two positive real numbers. It can be extended to any number of positive real numbers.

Equality holds when \( a = b \).

Note that “positive” is written in *Italic* to emphasize on the word that whenever you will be using AM ≥ GM ≥ HM then all the real numbers must be positive.
Problem 1: Prove that \(a^3 + b^3 + c^3 \geq 3abc\) where \(a, b, c\) are positive real numbers.

Solution 1: Applying AM \(\geq\) GM on \(a^3, b^3\) and \(c^3\) we get,

\[
\frac{(a^3 + b^3 + c^3)}{3} \geq (a^3 b^3 c^3)^{1/3}
\]

\[
\Rightarrow \frac{(a^3 + b^3 + c^3)}{3} \geq abc
\]

\[
\Rightarrow a^3 + b^3 + c^3 \geq 3abc.
\]

Weighted AM, GM, HM: If \(x_1, x_2, \ldots, x_n\) are \(n\) real numbers with weights \(w_1, w_2, \ldots, w_n\) then weighted AM is defined as,

Weighted AM = \((x_1w_1 + x_2w_2 + \ldots + x_nw_n)/(w_1 + w_2 + \ldots + w_n)\)

Weighted GM = \(((x_1w_1)(x_2w_2)\ldots(x_nw_n))^{1/(w_1 + w_2 + \ldots + w_n)}\)

Weighted HM = \((w_1 + w_2 + \ldots + w_n)/((w_1/x_1) + (w_2/x_2) + \ldots + (w_n/x_n))\)

Also, Weighted AM \(\geq\) Weighted GM \(\geq\) Weighted HM but remember the word positive whenever applying this.
**Permutation and Combination**

Fundamental theorem: If a work can be done in m ways and another work can be done in n ways then the two works can be done simultaneously in mn ways.

Note: Emphasize on the word simultaneously because most of the students get confused where to apply multiplication and where to apply addition. When both the works need to be done to complete a set of work then apply multiplication and if the works are disjoint then apply addition.

**Permutation:** There are n things and we need to take r things at a time and we need to arrange it with respect to order then the total number of ways is \( ^nP_r \) where \( ^nP_r = n!/(n – r)! \).

For example there are 3 tuples (1, 2, 3)

6 permutations are possible = (1, 2, 3); (1, 3, 2); (2, 1, 3); (2, 3, 1); (3, 1, 2) and (3, 2, 1)

Now, we will check by the formula.

Here number of permutations = \(^3P_3 = 3!/(3 – 3)! = 3\times2/1 = 6.\)

Proof: We can take 1\(^{st}\) thing in n ways, 2\(^{nd}\) thing in (n – 1) ways, ......, r\(^{th}\) thing in \{n – (r – 1)\} = (n – r + 1) ways.

By fundamental theorem,

total number of ways = n(n – 1)(n – 2)....(n – r + 1)

= n(n – 1)(n – 2)....(n – r + 1)(n – r)(n – r – 1)....2\times1/{(n – r)(n – r – 1)..... \times 2\times1}\)

(Multiplying numerator and denominator by (n – r)(n – r – 1).... \times 2\times1)

= n!/(n – r)!

So, \(^nP_r = n!/(n – r)! \)

Proved.
**Combination:** If there are \( n \) things and we need to select \( r \) things at a time (order is not important) then total number of ways of doing this = \( ^nC_r = \frac{n!}{(n - r)! \times r!} \).

For example, there are 5 numbers (1, 2, 3, 4, 5). We need to select 3 at a time.

Total number of ways = (1, 2, 3); (1, 2, 4); (1, 2, 5); (1, 3, 4); (1, 3, 5); (1, 4, 5); (2, 3, 4); (2, 3, 5); (2, 4, 5); (3, 4, 5) i.e. 10 number of ways.

Now, we will check by formula.

Here number of combinations = \( ^5C_3 = \frac{5!}{(5 - 3)! \times 3!} = \frac{5 \times 4 \times 3!}{3! \times 2} = 10 \).

Note that here (1, 2, 3) is equivalent to (1, 3, 2) etc. as order is not important in combination but this is important in permutation.

**Proof:** Now, the order is not important.

Hence the number of ways \( r \) things can permutate among themselves is \( r! \).

Therefore, \( r! \times ^nC_r = ^nP_r = \frac{n!}{(n - r)!} \)

\( \Rightarrow \ ^nC_r = \frac{n!}/\{(n - r)! \times r!\} \) Proved.

**Number of non-negative solution:**

**Problem:** \( a_1 + a_2 + .... + a_r = n \) where \( a_1, a_2, ..., a_n, n \) are all integers.

Prove that the number of non-negative solution of the equation is \( ^{n+r-1}C_{r-1} \) (or, \( \binom{n+r-1}{r-1} \))

We will prove it by induction.

Clearly, this is true for \( r = 1 \).
Let, this is true for \( r = k \) i.e. number of non-negative solution when there are \( k \) variables in the LHS is \( ^{(n + k - 1)}C_{(k - 1)} \).

Now, number of non-negative solution for \( r = k + 1 \) i.e. when an extra variable gets added in LHS is \( \sum_{i=0}^{n} \binom{n-i+k-1}{k-1} \)

Now, we have to prove that, \( \sum_{i=0}^{n} \binom{n-i+k-1}{k-1} = \binom{n+k}{k} \).

Now, we will prove this by another induction. For, \( n = 1 \),

LHS = \( ^kC_{(k - 1)} + (k - 1)C_{(k - 1)} = k + 1 \)

RHS = \( (k + 1)C_k = k + 1 \).
So, this is true for \( n = 1 \).

Let, this is true for \( n = p \) i.e. we have, \( \sum_{i=0}^{p} \binom{p-i+k-1}{k-1} = \binom{p+k}{k} \).

For, \( n = p + 1 \), LHS = \( \sum_{i=0}^{p+1} \binom{p+1-i+k-1}{k-1} \)
\[ = (p + k)C_{(k - 1)} + \sum_{i=0}^{p} \binom{p+i+k-1}{k-1} \]
\[ = (p + k)C_{(k - 1)} + (p + k)C_{k} \]
\[ = (p + k)!/{((p + 1)!(k - 1)!)} + (p + k)!/{p!xk!} \]
\[ = [(p + k)!/{((p + 1)!xk!}])k + p + 1) \]
\[ = (p + k + 1)!/{((p + 1)!xk!}] \]
\[ = \binom{p+1+k}{k} \]
\[ = \text{RHS for } n = p + 1. \]

**Number of positive solutions:**

**Problem:** \( a_1 + a_2 + \ldots + a_r = n \) where \( a_1, a_2, \ldots, a_r, n \) are all positive integers. Prove that number of positive solutions of this equation is \( n-1 \text{C}_{r-1} \).

**Solution:** We will prove this by induction.

Clearly this is true for \( r = 1 \).

Let this is true for \( r = k \) i.e. if there are \( k \) number of variables in LHS then number of solutions of the equation is \( n-1 \text{C}_{k-1} \).

Now, for \( r = k + 1 \) i.e. if an extra variable gets added then number of solutions of the equation is \( \sum_{i=1}^{n-k} \binom{n-i-1}{k-1} \)
We have to prove that, \( \sum_{i=1}^{n-k} \binom{n-i-1}{k-1} = n-1 \text{C}_{k} \). We will prove this by another induction.

Clearly this is true for \( n = 1 \).

Let this is true for \( n = p \) i.e. \( \sum_{i=1}^{p-k} \binom{p-i-1}{k-1} = p-1 \text{C}_{k} \).

Now, for \( n = p + 1 \), LHS = \( \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} = p-1 \text{C}_{k-1} + p-1 \text{C}_{k} \) (from above)
\[ = p \text{C}_{k} = \text{RHS for } n = p + 1. \]
Proved.
Problem: Find number of terms in the expansion of \((x + y + z + w)^n\).

Solution:

Now, there are 4 variables and any term consists of the 4 terms such that,
\[x_1 + y_1 + z_1 + w_1 = n\] where \(x_1, y_1, z_1, w_1\) are powers of \(x, y, z, w\) in any term.

Now, \(x_1, y_1, z_1, w_1\) runs from 0 to \(n\).

So, we need to find number of non-negative solution of this equation and we are done with number of terms of this equation.

From previous article we know, number of non-negative solution of this equation = \(n+4-1C_{4-1} = n+3C_3\).

In general if there are \(r\) variables then number of terms = \(n+r-1C_{r-1}\).

Problem: Find number of terms which are independent of \(x\) in the expansion of \((x + y + z + w)^n\).

Solution:

Now, \(x_1 = 0\). So, we need to find number of non-negative solution of the equation, \(y_1 + z_1 + w_1 = n\) and then we need to subtract this from \(n+3C_3\) and we are done with the number of terms which are independent of \(x\).

It is, \(n+3-1C_{3-1} = n+2C_2\).

So, number of terms which are independent of \(x\) = \(n+3C_3 - n+2C_2\).

In general if there are \(r\) variables then number of terms excluding one variable = \(n+r-1C_{r-1} - n+r-2C_{r-2}\).

Similarly, we can find number of terms independent of \(x\) and \(y\) and so on.

Problem: Prove that number of ways of distributing \(n\) identical things among \(r\) members where every member gets at least 1 thing is \(n-1C_{r-1}\).

Solution: Let, first member gets \(x_1\) things, second member gets \(x_2\) things and so on i.e. \(t^{th}\) member gets \(x_t\) things.

So, \(x_1 + x_2 + ... + x_r = n\)
Now, we need to find number of positive solutions of this equation because x_1, x_2, ..., x_n > 0.

From above it is n^{-1}C_{r-1}.

Problem: In an arrangement of m H’s and n T’s, an uninterrupted sequence of one kind of symbol is called a run. (For example, the arrangement HHHTHTTTTH of 6 H’s and 4T’s opens with an H-run of length 3, followed successively by a T-run of length 1, an H-run of length 2, a T-run of length 3 and, finally an H-run if length 1.)

Find the number of arrangements of m H’s and n T’s in which there are exactly k H-runs.

Solution: Now, m H’s can be put in k places with k+1 holes (spaces) between them in m^{-1}C_{k-1} ways.

Now, k – 1 spaces between the H’s must be filled up by at least one T.

So, number of ways is n^{-1}C_{k-2}.

So, in this case number of ways = m^{-1}C_{k-1} \times n^{-1}C_{k-2}.

Now, if k spaces (i.e. one space from either side first or last) can be filled by n T’s where in every space at least one T is there in n^{-1}C_{k-2} ways.

So, total number of ways in this case = 2 \times m^{-1}C_{k-1} \times n^{-1}C_{k-1}.

Now, if k+1 spaces (i.e. including first and last space) can be filled up by n T’s where in every space at least one T is there in n^{-1}C_{k} ways.

So, in this case total number of ways = m^{-1}C_{k-1} \times n^{-1}C_{k}.

So, total number of ways = m^{-1}C_{k-1}(n^{-1}C_{k-2} + 2 \times n^{-1}C_{k-1} + n^{-1}C_{k})

= m^{-1}C_{k-1}(n^{-1}C_{k-2} + n^{-1}C_{k-1} + n^{-1}C_{k-1} + n^{-1}C_{k})

= m^{-1}C_{k-1}(nC_{k-1} + nC_k) = n^{-1}C_{k-1} \times n+1C_{k}

Problem: Show that number of ways in which four distinct integers can be chosen from 1, 2, ..., n (n ≥ 7) such that no two are consecutive is equal to n^{-3}C_{4}.

Solution:
We choose 4 integers as shown in figure by circle.
So, there are maximum 5 spaces between them shown in figure by boxes.
Now, let us say, 2, 3, 4 spaces i.e. boxes are to be filled by other \( n - 4 \) integers (4 integers already chosen for 4 circles).

Number of ways = \( n-5 \binom{2}{} \) (As number of ways is \( n-1 \binom{1}{r-1} \) for at least one to be there)

Similarly, for 1, 2, 3, 4, boxes and 2, 3, 4, 5 boxes to be filled by other \( n-4 \) integers number of ways = \( 2 \times (n-5 \binom{3}{} ) \).

For 5 boxes to be filled by other \( n-4 \) integers number of ways = \( n-5 \binom{4}{} \).

Total number of ways = \( n-5 \binom{2}{} + 2 \times (n-5 \binom{3}{} ) + (n-5 \binom{4}{} ) = n-4 \binom{3}{} + n-4 \binom{4}{} = n-3 \binom{4}{} . \)

**Problem:** Prove that number of ways of distributing \( n \) identical things to \( r \) members (no condition) is \( n+r-1 \binom{r-1}{} \).

**Solution:** Let first member gets \( x_1 \) things, second member gets \( x_2 \) things and so on i.e. \( t^{th} \) member gets \( x_t \) things.

We have, \( x_1 + x_2 + \ldots + x_r = n \)

We need to find number of non-negative solutions of this equation.

From above it is \( n+r-1 \binom{r-1}{} \).

Proved.

**Problem:** Find the number of all possible ordered \( k \)-tuples of non-negative integers \((n_1, n_2, \ldots, n_k)\) such that \( \sum_{i=1}^{k} n_i = 100 \).

**Solution:** Clearly, it needs number of non-negative solution of the equation, \( n_1 + n_2 + \ldots + n_k = 100 \).

It is \( n+k-1 \binom{k-1}{} \).
Problem: Show that the number of all possible ordered 4-tuples of non-negative integers \((n_1, n_2, n_3, n_4)\) such that \(n_1 + n_2 + n_3 + n_4 \leq 100\) is \(\binom{104}{4}\).

Solution: Clearly, required number = \(3C_3 + 4C_3 + 5C_3 + \ldots + 103C_3\) = \(\binom{104}{4}\).

Problem: How many 6-letter words can be formed using the letters A, B and C so that each letter appears at least once in the word?

Solution: Let \(x_1\) number of A, \(x_2\) number of B and \(x_3\) number of C are chosen where \(x_1, x_2, x_3 > 0\)

Now, \(x_1 + x_2 + x_3 = 6\).

Number of positive solution of this equation is \(6^{-1}C_{3-1} = 5C_2 = 10\).

So, combinations are as follows,

- 4 A, 1 B, 1 C, number of words = \(6!/4! = 30\)
- 3 A, 2 B, 1 C, number of words = \(6!(3!×2!) = 60\)
- 3 A, 1 B, 2 C, number of words = \(6!(3!×2!) = 60\)
- 2 A, 1 B, 3 C, number of words = \(6!(2!×3!) = 60\)
- 2 A, 2 B, 2 C, number of words = \(6!(2!×2!×2!) = 90\)
- 2 A, 3 B, 1 C, number of words = \(6!(2!×3!) = 60\)
- 1 A, 1 B, 4 C, number of words = \(6!/4! = 30\)
- 1 A, 2 B, 3 C, number of words = \(6!(2!×3!) = 60\)
- 1 A, 3 B, 2 C, number of words = \(6!(3!×2!) = 60\)
- 1 A, 4 B, 1 C, number of words = \(6!/4! = 30\)

So, total number of words = 30 + 60 + 60 + 60 + 90 + 60 + 30 + 60 + 60 + 60 + 30 = 540.
**Problem:** All the permutations of the letters a, b, c, d, e are written down and arranged in alphabetical order as in a dictionary. Thus the arrangement abcde is in the first position and abced is in the second position. What is the position of the arrangement debac?

**Solution:**

Now, first fix a at first place. Number of arrangements = 4!

Now, fix b at first place. Number of arrangements = 4!

Now, fix c at first place. Number of arrangements = 4!

Now comes d the first letter of the required arrangement.

Now fix d at first position & a at second position.
Number of arrangement = 3!

Fix b at second place. Number of arrangement = 3!

Fix c at second place. Number of arrangement = 3!

Now, comes e at second place and we have de.

Now, fix a at third place. Number of arrangement = 2!

Now comes b which is required and we have deb.

Then comes a and then c.

So, debac comes after (4! + 4! + 4! + 3! + 3! + 3! + 2!) = 92 arrangement.

So, it will take 92 + 1 = 93rd position.

**Problem:** x red balls, y black balls and z white balls are to be arranged in a row. Suppose that any two balls of the same color are indistinguishable. Given that x + y + z = 30, show that the number of possible arrangements is the largest for x = y = z = 10.

**Solution:**

Clearly, number of possible arrangement is \((x + y + z)! / (x! \times y! \times z!)\)

= \(30! / (x! \times y! \times z!)\)

Now, it will be largest when \(x! \times y! \times z!\) = minimum.

Let us say, \(x = 12\) and \(y = 8\)
Now, \(12!\times 8! = 12\times 11\times 10!\times 10!/(10\times 9) = (12\times 11/10\times 9)\times (10!)^2\)

\[\Rightarrow 12!\times 8!/(10!)^2 = (12\times 11)/(10\times 9) > 1\]

\[\Rightarrow (10!)^2 < 12\times 8!\]

It will be least when \(x = y = z = 10\).

**Problem:** Find number of arrangements of the letters of the word MISSISSIPPI.

**Solution:**

Number of letters = 10

Number of I’s = 4, number of S’s = 3, number of P’s = 2

Therefore, total number of words that can be formed from the letters of the word is \(10!/(4!\times 3!\times 2!)\).

**Problem:** Find the number of words (meaningful or non-meaningful) that can be formed from the letters of the word MOTHER.

**Solution:**

Number of letters = 6.

All are distinct.

Hence total number of words = 6!

**Problem:** Show that the number of ways one can choose a set of distinct positive integers, each smaller than or equal to 50, such that their sum is odd, is \(2^{49}\).

**Solution:** The sum is odd.

We need to select odd number of integers.

Now, we can select 1 integer from 50 integers in \(50C_1\) ways.

We can select 3 integers from 50 integers in \(50C_3\) ways.

...  

We can select 49 integers from 50 integers in \(50C_{49}\) ways.
So, number of ways = $50C_1 + 50C_3 + \ldots + 50C_{49}$.

Now, $50C_0 + 50C_1 + \ldots + 50C_{49} + 50C_{50} = 2^{50}$

Now, $50C_0 - 50C_1 + \ldots - 50C_{49} + 50C_{50} = 0$

Subtracting the above two equations we get,

$2(50C_1 + 50C_3 + \ldots + 50C_{49}) = 2^{50}$

$\Rightarrow 50C_1 + 50C_3 + \ldots + 50C_{49} = 2^{49}$.

**Number of ways of distributing n distinct things to r persons (r < n) so that every person gets at least one thing.**

Total number of ways = $r^n$

Now, let $A_i$ denotes that the $i^{th}$ person doesn’t get a gift and $B$ denotes that every person gets at least one gift.

Therefore, $r^n = |B| + |A_1UA_2UA_3U\ldotsUA_r|$

Now, $|A_1UA_2U\ldotsUA_r|$

$= \sum A_i - \sum A_i\cap A_j + \sum A_i\cap A_j\cap A_k - \ldots + (-1)^{r-1}[A_1\cap A_2\cap\ldots\cap A_r]$

Now, $|A_i| = (r - 1)^n$, $|A_i\cap A_j| = (r - 2)^n$, $\ldots$, $A_1\cap A_2\cap\ldots\cap A_r = (r - r)^n = 0^n$

So, $|A_1UA_2U\ldotsUA_r| = rC_1(r - 1)^n - rC_2(r - 2)^n + rC_3(r - 3)^n - \ldots + (-1)^{r-1}rC_r0^n$

Therefore, $|B| = r^n - [rC_1(r - 1)^n - rC_2(r - 2)^n + rC_3(r - 3)^n - \ldots + (-1)^{r-1}rC_r0^n]$

$= r^n - rC_1(r - 1)^n + rC_2(r - 2)^n - rC_3(r - 3)^n + \ldots + (-1)^{r}rC_r0^n$
Polynomial

Let, \( P(x) \) be a polynomial of degree \( d \), then \( P(x) \) can be written as,
\[
P(x) = a_1x^d + a_2x^{d-1} + a_3x^{d-2} + \ldots + a_{d-1}x^2 + a_dx + a_{d+1}
\]

**Remainder theorem:** Consider a polynomial of degree \( d > 1 \). \( P(x) \) gives the remainder \( P(a) \) if \( P(x) \) is divided by \( x - a \).

Proof: Let \( P(x) = (x - a)Q(x) + R \) where \( R \) is constant as the divider is linear so at most degree of \( R \) is 0 i.e. free of \( x \) or constant.

\( Q(x) \) is the quotient and \( R \) is remainder.

Putting \( x = a \) in the above expression we get,
\[
P(a) = (a - a)Q(a) + R
\]
\[\therefore R = P(a)\]

**Problem 1:** Find the remainder when \( P(x) = x^2 + x + 1 \) is divided by \( x + 1 \).

Solution 1: From above we have remainder = \( R = P(-1) \)
So, \( R = P(-1) = (-1)^2 + (-1) + 1 = 1 \).

Consider a polynomial \( P(x) \) of degree \( d > 1 \). We will now find the remainder if there is any repeated root in divider. So, we will find the remainder when \( P(x) \) is divided by \( (x - a)^2 \).

Let, \( P(x) = (x - a)^2Q(x) + R(x) \) (Note that this time \( R(x) \) is not constant and have degree 1 as divider is quadratic)

Let, \( R(x) = Ax + B \).

\( Q(x) \) is the quotient.

Now, putting the value of \( R(x) \) in the above equation the equation becomes,
\[
P(x) = (x - a)^2Q(x) + Ax + B
\]
Now, putting \( x = a \) in the above equation we get,
P(a) = (a - a)^2Q(a) + Aa + B

Aa + B = P(a)

Now, differentiating the above equation w.r.t. x we get,

P''(x) = 2(x - a)Q(x) + (x - a)^2Q''(x) + A

Putting x = a in the above equation we get,

P''(a) = 2(a - a)Q(a) + (a - a)^2Q''(a) + A

A = P''(a)

B = P(a) - Aa = P(a) - a×P''(a)

R(x) = Ax + B = P''(a)x + P(a) - a*P''(a) = P''(a)(x - a) + P(a)

**Problem 1:** Consider a polynomial \( P(x) \) of degree \( d > 2 \). Let \( R(x) \) be the remainder when \( P(x) \) is divided by \((x - 1)^2\). \( P'(1) = P(1) = 1 \). Find \( R(x) \).

**Solution 1:** From the above result we have,

\[ R(x) = P''(1)(x - 1) + P(1) = 1*(x - 1) + 1 = x. \]

Consider a polynomial \( P(x) \) of degree \( d > 1 \). Now we will find the remainder when the divider is quadratic and have two distinct roots. Let us find the remainder when \( P(x) \) is divided by \((x - a)(x - b)\).

Let, \( P(x) = (x - a)(x - b)Q(x) + R(x) \)

\( Q(x) \) = quotient and \( R(x) \) = remainder = \( Ax + B \).

Putting value of \( R(x) \) in the above equation we get,

\[ P(x) = (x - a)(x - b)Q(x) + Ax + B \]

Putting \( x = a \) in the above equation we get,

\[ P(a) = (a - a)(a - b)Q(a) + Aa + B \]

\[ Aa + B = P(a) \quad ....(i) \]

Now, putting \( x = b \) in the above equation we get,

\[ P(b) = (b - a)(b - b)Q(b) + Ab + B \]
Ab + B = P(b) ......(ii)

Now, from (i) and (ii) we get, Aa + B - Ab - B = P(a) - P(b)

A = \{P(a) - P(b)\}/(a - b)
B = \{aP(b) - bP(a)\}/(a - b)

R(x) = Ax + B = \{P(a) - P(b)\}x/(a - b) + \{aP(b) - bP(a)\}/(a - b)

Consider a polynomial of degree d > 1. Now, we will work with quotient. Let Q(x) be the quotient when P(x) is divided by (x - a). Then we will have the relation Q(a) = P''(a).

Proof: Let, P(x) = (x - a)Q(x) + R where R is remainder and note that R is constant as divider is linear i.e. of degree 1.

Differentiating the above equation w.r.t. x we get,

P''(x) = Q(x) + (x - a)Q''(x) + R''

Note that R'' = 0 as R is constant.

So, we have, P''(x) = Q(x) + (x - a)Q''(x)

Putting x = a in the above equation we get,

P''(a) = Q(a) + (a - a)Q''(a)

Q(a) = P''(a)

Problem 1: Consider a polynomial P(x) = x^3 + 3x^2 + 2x + 1. Q(x) is the quotient when P(x) is divided by x - 1. Find the value of Q(1).

Solution 1: From the above we have the result, Q(1) = P''(1)

Given P(x) = x^3 + 3x^2 + 3x + 1
P''(x) = 3x^2 + 6x + 3
P''(1) = 3x1^2 + 6x1 + 3
Q(1) = P''(1) = 12.

Problem 2: Consider a polynomial P(x) of degree d > 1. Q(x) = 4x + 3 is the quotient when P(x) is divided by x - 7. Find the slope of P(x) at x = 7 or put in other way find P'(7).

Solution 2: From above we have, P''(7) = Q(7) = 4*7 + 3 = 31.
Consider a polynomial $P(x)$ of degree $d > 1$. Now, we will see relation between quotient i.e. $Q(x)$ and $P(x)$ when there is repeated root in the divider. Let, $Q(x)$ is the quotient when $P(x)$ is divided by $(x - a)^2$. Then we will have the relation $Q(a) = \frac{P''(a)}{2}$.

**Proof:** Let, $P(x) = (x - a)^2 Q(x) + R(x)$ (Note that $R(x)$ is linear here and so $R^{'''}(x) = 0$)

Differentiating w.r.t. $x$ we get,

$$P''(x) = 2(x - a)Q(x) + (x - a)^2 Q''(x) + R''(x)$$

Differentiating again w.r.t. $x$ we get,

$$P^{'''}(x) = 2Q(x) + 2(x - a)Q''(x) + 2(x - a)^2 Q'''(x) + R^{'''}(x)$$

Putting $x = a$ in the above expression we get,

$$P^{'''}(a) = 2Q(a) + 2(a - a)Q''(a) + 2(a - a)^2 Q'''(a) + 0$$

$\therefore Q(a) = \frac{P^{'''}(a)}{2}$.

**Problem 1:** Let $Q(x) = 3x^2 + 2$ is the quotient when $P(x)$ is divided by $(x - 1)^2$. Find the value of $P^{'''}(1)$.

**Solution 1:** From the above result we have,

$$P^{'''}(1) = 2Q(1) = 2(3*1^2 + 2) = 10.$$  

**Problem 2:** Let $P(x)$ be a polynomial of degree $d > 2$. $Q(x)$ is the quotient when $P(x)$ is divided by $(x - 2)^2$. $Q(2) = 4$. Find $P^{'''}(2)$.

**Solution 2:** From the above result we have,

$$P^{'''}(2) = 2Q(2) = 2*4 = 8.$$  

**Tips to solve problems:**

1. The remainder of $P(x)$ divided by $x + a$ can be found by putting $x = -a$ i.e. $P(-a)$ will give the remainder when $P(x)$ is divided by $x + a$.
2. If there is a root between $(a, b)$ then $P(a)$ and $P(b)$ will be of opposite sign.
3. If $P(x)$ is strictly increasing or decreasing then $P(x)$ have at most one real
root. P(x) can be proved increasing if P''(x) > 0 and decreasing if P''(x) < 0.

4. To find number of real roots in P(x) draw the graph of LHS and RHS and count the number of intersection points and that is the answer.

5. If there is any mention of sum of coefficients then think of P(1) and vice versa.

6. If there is any repeated root think of derivative.

7. If there is any involvement of quotient Q(x) then write the equation P(x) = D(x)Q(x) + R(x) where D(x) is the divider and think of derivative.

8. If there is any mention of becoming the polynomial prime think of P(0) i.e. the constant term.

9. R(x) has at most degree d - 1 where d is the degree of divider.

10. Complex roots come in pair. If a polynomial is of degree d which is odd then the polynomial must have at least one real root.

11. If there is any question/mention of multiplicity of a root then do derivative for m + 1 times where m is at most multiplicity of the root and show that P^{(m+1)}(x) doesn't have the root.

Problem 1: Consider the polynomial P(x) = 30x^7 - 35x^6 + 42x^5 + 210x^3 - 1470. Prove that P(x) = 0 have only one real root and the root lies between (1, 2).

Solution 1: Now, P(x) = 30x^7 - 35x^6 + 42x^5 + 210x^3 - 1470

P''(x) = 210x^6 - 210x^5 + 210x^4 + 630x^2

P''(x) = 210x^4(x^2 - x + 1) + 630x^2

P''(x) = 210x^4{(x - \frac{1}{2})^2 + \frac{3}{4}} + 630x^2

Which is always greater than 0.

P(x) is increasing.

P(x) has at most one real root.

Now, P(1) = 30\times1^7 - 35\times1^6 + 42\times1^5 + 210\times1^3 - 1470 < 0

And, P(2) = 30\times2^7 - 35\times2^6 + 42\times2^5 + 210\times2^3 - 1470 > 0

There is a root between (1, 2) & this root is the only real root of P(x) = 0.

Problem 2: Prove that x = \cos x has only one real root.

Solution 2: Now, drawing the graph of LHS and RHS i.e., y = x and y = \cos x.
Clearly, there is one point of intersection. There is one real root.

**Problem 3:** P(x) and Q(x) are two polynomials such that the sum of the coefficient is same for both. Prove that the remainders when P(x) and Q(x) are divided by x - 1 are same.

**Solution 3:** Let, $P(x) = a_1x^d + a_2x^{d-1} + \ldots + a_d + a_{d+1}$

Putting $x = 1$ we get,

$P(1) = a_1 + a_2 + \ldots + a_d + a_{d+1} =$ sum of the coefficients.

Now, from remainder theorem (also tips number 1) we have the remainder $P(x)$ gives when $P(x)$ is divided by $(x - 1)$ is $P(1) =$ sum of coefficients.

Similar thing goes for $Q(x)$ i.e., $Q(1) =$ sum of coefficients.

Now, it is given that sum of coefficients of $P(x) =$ sum of coefficients of $Q(x)$

$P(1) = Q(1)$.

**Problem 4:** P(x) is a polynomial of degree $d > 1$ with integer coefficients. P(1) is divisible by 3. All the coefficients are placed side by side in any order to make a positive integer. For example if there is 2 coefficients 20 and 9 then the numbers formed are 209 or 920. Prove that the number thus generated is divisible by 3.

**Solution 4:** Now, in the previous example we have seen that $P(1) =$ sum of coefficients of $P(x)$.

Now, the sum is divided by 3.

As per the rule of divisibility by 3 says a number is divisible by 3 if sum of the digits is divisible by 3.
The sum of the digits of the coefficients of $P(x)$ is divisible by 3.
Now, whatever be the order of placing the digits of the coefficients of $P(x)$ the sum of their digits must be same.

Thus the number generated with whatever be the order of placing the digits is divisible by 3.

**Problem 5:** $P(x)$ and $Q(x)$ be two polynomials with integer coefficients. Suppose the sum of the coefficients of both the polynomials are equal to $S$. It is given that $(P(x))^3 - (Q(x))^3 = P(x^3) - Q(x^3)$ for all $x$.

1. Show that $P(x) - Q(x) = (x - 1)^aR(x); a \geq 1$
2. Show that $S^2 = 3^{a-1}$

**Solution 5:** This question deals with sum of the coefficients are same means $P(1) = Q(1)$ and repeated roots at $x = 1$ i.e. we have to go for differentiation and prove that $(a - 1)^{th}$ time derivative have the root at $x = 1$. Thus we follow the below steps to solve the problem.

Let, $F(x) = P(x) - Q(x)$

$P(1) = Q(1) = \text{sum of coefficients} = S.$ (As per tips 5)

$F(1) = P(1) - Q(1) = S - S = 0$

We will show that $F(x)$ have repeated roots at $x = 1$. For this we need to show that $F'(1), F''(2), \ldots$ have roots $(x - 1)$ (As per tips 6)

We have $(P(x))^3 - (Q(x))^3 = P(x^3) - Q(x^3)$

$(P(x) - Q(x))^3 - 3P(x)Q(x)(P(x) - Q(x)) = F(x^3)$

$(F(x))^3 - 3P(x)Q(x)F(x) = F(x^3)$

Differentiating w.r.t. $x$ we get,

$3(F(x))^2F(x) - 3(P''(x)Q(x)F(x) + P(x)Q''(x)F(x) + P(x)Q(x)F''(x))$

$= 3x^2F''(x^3)$ ......(A)

Putting $x = 1$ we get,

$3(F(1))^2F''(1) - 3(P''(1)Q(1)F(1) + P(1)Q''(1)F(1) + P(1)Q(1)F''(1)) = 3F''(1)$

$3*0^2*F''(1) - 3(P''(1)Q(1)*0 + P(1)Q''(1)*0 + S*S*F''(1)) = 3F''(1)$

(As $F(1) = 0)$

$3S^2F''(1) = 3F''(1)$

Now, either $S^2 = 1$ or $F''(1) = 0$
We consider $F''(1) = 0$

Now, again differentiating equation (A) w.r.t. $x$ we get,

$$6F(x)F''(x) + 3F(x)F'''(x) = 3(P''(x)Q(x)F(x) + P'(x)Q''(x)F(x) + P'(x)Q'(x)F'(x) + P''(x)Q''(x)F''(x) + P(x)Q'''(x)F''(x) + P(x)Q''(x)F''(x) + P(x)Q'(x)F'(x) + P'(x)Q(x)F''(x) + P'(x)Q'(x)F''(x) + P(x)Q(x)F''(x)) = 6xF''(x^3) + 9x^4F''''(x^3)$$

Putting $x = 1$ and $F(1) = F''(1) = 0$ we get,

$$3S^2F''(x) = 9\times 1^4 F''(x)$$

$$S^2F''(x) = 3F''(x)$$

Now, either $S^2 = 3$ or $F''(x) = 0$.

**Problem 6: Prove that there is no non-constant polynomial $P(x)$ with integer coefficients such that $P(n)$ is prime number for all positive integers $n$.**

Solution 6: Let, there is a polynomial $P(x)$ with such property.

Let, $P(x) = a_1x^m + a_2x^{m-1} + ... + a_{m+1}$.

Now, $P(a_{m+1}) = a_1a_{m+1}^m + a_2a_{m+1}^{m-1} + .... + a_{m+1}$

$= a_{m+1}(a_1a_{m+1}^{m-1} + a_2a_{m-1}^{m-2} + ... + 1)$ which is clearly not a prime.

(Putting $x = a_{m+1} = P(0)$ as per tips 8)

Now, if there is no constant term $a_{m+1}$ then it is divisible by all numbers.

- Our assumption was wrong.
- There is no non-constant polynomial with such property.

**Problem 7: Show that the equation $x(x - 1)(x - 2).....(x - 2009) = c$ has real roots of multiplicity at most 2.**

Solution 7: (tips number 11)

As per the tips we need to show $P'''(x)$ doesn’t have the root which $P(x)$ and $P''(x)$ has.

We have, $x(x - 1)(x - 2).....(x - 2009) = c$

Differentiating w.r.t. $x$ we get,

$$(x - 1)(x - 2)....(x - 2009) + x(x - 2)(x - 3)....(x - 2009) + x(x - 1)(x - 2)....(x - 2009) + x(x - 1)(x -...$$
3)...(x – 2009) + ..... + x(x – 1)(x – 2)....(x – 2008) = 0

- c/x + c/(x – 1) + c/(x – 2) + .... + c/(x – 2009) = 0

(Putting value from the given equation)

- c{1/x + 1/(x – 1) + 1/(x – 2) + .... + 1/(x – 2009)} = 0

Now, differentiating again w.r.t. x we get,

-c × {1/x^2 + 1/(x – 1)^2 + 1/(x – 2)^2 + .... + 1/(x – 2009)^2} = 0

This equation cannot hold true as sum of squares of real numbers equal to 0 but they are always greater than 0.

The given polynomial cannot have real roots of multiplicity more than 2.

**Problem 8: Prove that the polynomial P(x) = x^3 + x – 2 have at least one real root.**

Solution 8: (Tips number 10)

Let all the roots of P(x) is complex.

Complex roots come in pair.

There needs to be 4 roots of P(x)

But P(x) have at most 3 roots as the degree of the polynomial is 3.

\[\Rightarrow\] Our assumption was wrong. It may have at most 2 complex root.

There is at least one real root of P(x).

**Solved examples:**

1. Consider a polynomial P(x) of degree d > 1. Q(x) is the quotient when P(x) is divided by x - a. Prove that P'(a) = Limiting value of Q(x) as x -> a.

Solution:

We can write, P(x) = (x – a)Q(x) + R(x) where R(x) is remainder when P(x) is divided by (x – a).

\[\Rightarrow\] R(x) is constant.
Now, \( P''(a) = \frac{P(x) - P(a)}{(x - a)} \) as \( x \to a \)

- \( P''(a) = \{(x - a)Q(x) + R(x) - R(a)/(x - a) \) as \( x \to a \) (\( P(a) = R(a) \))
- \( P''(a) = Q(x) \) as \( x \to a \) (as \( R''(a) = 0 \) because \( R(x) \) is constant)
- \( P''(a) = Q(x) \) as \( x \to a \) (as \( R''(a) = 0 \) because \( R(x) \) is constant)

2. Consider a polynomial \( P(x) \) of degree \( d > 2 \). Let \( R(x) \) be the remainder when \( P(x) \) is divided by \( (x - 1)^2 \). \( P'(1) = P(1) = 1 \). Find \( R(x) \).

Solution: We can write, \( P(x) = (x - 1)^2 Q(x) + R(x) \) where \( Q(x) \) is the quotient when \( P(x) \) is divided by \( (x - 1)^2 \).

- \( P''(x) = 2(x - 1)Q(x) + (x - 1)^2 Q''(x) + R''(x) \)
- \( P''(1) = R''(1) = 1 \)

And, \( P(1) = R(1) = 1 \)

Now, \( R(x) \) is remainder when \( P(x) \) is divided by \( (x - 1)^2 \)

- \( R(x) \) is linear.

Say, \( R(x) = ax + b \)

Now, \( R''(x) = a \)
\( R''(1) = a = 1 \)

Now, \( R(1) = a + b = 1 \)

- \( b = 0 \). So, \( R(x) = x \).

3. Let \( P(x) \) be a polynomial of degree \( d > 2 \). \( Q(x) \) is the quotient when \( P(x) \) is divided by \( (x - 2)^2 \). \( Q(2) = 4 \). Find \( P''(2) \).

Solution: We can write, \( P(x) = (x - 2)^2 Q(x) + R(x) \) where \( R(x) \) is remainder when \( P(x) \) is divided by \( (x - 2)^2 \).

So, \( R(x) \) is linear.

Now, \( P''(x) = 2(x - 2)Q(x) + (x - 2)^2 Q''(x) + R''(x) \)
\( P''(x) = 2Q(x) + 2(x - 2)Q''(x) + 2(x - 2)Q''(x) + (x - 2)^2 Q''(x) + R''(x) \)
\( P''(x) = 2Q(2) + R''(2) \)

Now, \( R''(2) = 0 \) as \( R(x) \) is linear.
\( P''(2) = 2Q(2) = 2 \times 4 = 8 \).
4. \( P(x) \) and \( Q(x) \) are two polynomials such that the sum of the coefficient is same for both. Prove that the remainders when \( P(x) \) and \( Q(x) \) are divided by \( x - 1 \) are same.

Solution: Sum of the coefficients of \( P(x) \) and \( Q(x) \) are same.

\[ P(1) = Q(1). \]

Now, as per Remainder theorem \( P(1) \) and \( Q(1) \) are remainders when \( P(x) \) and \( Q(x) \) are divided by \( (x - 1) \) and they are clearly same.

5. \( P(x) \) is a polynomial of degree \( d > 1 \) with integer coefficients. \( P(1) \) is divisible by 3. All the coefficients are placed side by side in any order to make a positive integer. For example if there is 2 coefficients 20 and 9 then the numbers formed are 209 or 920. Prove that the number thus generated is divisible by 3.

Solution: \( P(1) \) is divisible by 3.

\[ \Rightarrow \text{Sum of the coefficients is divisible by 3.} \]

\[ \Rightarrow \text{If we place the coefficients side by side then the number formed will be divisible by 3 as the sum of the digits is divisible by 3 as per } P(1) \text{ is divisible by 3.} \]

6. Consider a polynomial \( P(x) \) of degree \( d > 2 \). \( Q(x) \) is the quotient when \( P(x) \) is divided by \( D(x) \). \( D(x) \) is quadratic and \( x = a \) is a root. \( P''(a) = b \) and \( Q(a) = b/2 \). Prove that \( D(x) \) has repeated root at \( x = a \).

Solution: Let, \( D(x) = (x - a)(x - d) \)

We can write, \( P(x) = D(x)Q(x) + R(x) \) where \( R(x) \) is remainder when \( P(x) \) is divided by \( D(x) \). \( R(x) \) is linear as \( D(x) \) is quadratic.

\[ P(x) = (x - a)(x - d)Q(x) + R(x) \]
\[ P''(x) = (x - d)Q(x) + (x - a)Q(x) + (x - a)(x - d)Q''(x) + R''(x) \]
\[ P'''(x) = Q(x) + (x - d)Q'''(x) + Q(x) + (x - a)Q''(x) + (x - d)Q''(x) + (x - a)Q'(x) + (x - c)(x - d)Q''(x) + R'''(x) \]
\[ P''(a) = 2Q(a) + 2(a - d)Q''(a) + R''(a) \]
\[ b = 2(b/2) + 2(a - d)Q''(a) \]
\[ (a - d)Q''(a) = 0 \text{ as } R(x) \text{ is linear} \]
\[ d = a \]
\[ D(x) \text{ has repeated root at } x = a. \]
7. Consider two polynomials \( P(x) \) and \( Q(x) \) of degree \( d > 0 \) with integer coefficients. \( P(0) = Q(0) \). Prove that there exists an integer \( n \) which divides both \( P(n) \) and \( Q(n) \).

Solution: \( P(0) = Q(0) \)

- The constant term of \( P(x) \) and \( Q(x) \) are equal. Let,

\[
P(x) = a_1x^p + a_2x^{p-1} + \ldots + a_p x + n
\]

Let, \( Q(x) = b_1x^q + b_2x^{q-1} + \ldots + b_q x + n \)

Clearly, \( n \) divides both \( P(n) \) and \( Q(n) \).

8. Let \( P(x) \) be a polynomial of degree \( 3d - 1 \) where \( d > 0 \).

Let \( P^{(i)}(0) = 3 \times i! \) where \( P^{(i)}(x) \) is \( i \)-th derivative of \( P(x) \) w.r.t. \( x \). Prove that \( P(1) \) is divisible by 9.

Solution:

\[
P(x) = a_1x^n + a_2x^{n-1} + \ldots + a_{n-1}x^2 + anx + an+1 P(0)
\]

\[
= an+1 = 3
\]

\[
P''(0) = an = 3
\]

\[
P''(0) = (2!)an-1 = 3 \times 2!
\]

\[
\Rightarrow an-1 = 3.
\]

Similarly, \( an-2 = an-3 = \ldots = a_1 = 3 \).

Now, \( P(1) = a_1 + a_2 + \ldots + an+1 = a_1 + a_2 + \ldots + a_{3d} \) as \( n = 3d - 1 \).

\[
\Rightarrow P(1) = 3 + 3 + \ldots. 3d \text{ times.} = 3 \times 3d = 9d
\]

\[
\Rightarrow P(1) \text{ is divisible by 9.}
\]

9. Consider a polynomial \( P(x) \) of degree \( d > 1 \). Given \( P(0) = 25 \). All the roots of \( P(x) \) are distinct positive integers. \( P^{(d)}(0) = d! \). Find the value of \( P^{(d-1)}(0)/(d-1)! \) where \( P^{(m)}(x) \) is \( m \)-th derivative of \( P(x) \) w.r.t. \( x \).

Solution:

Let, \( P(x) = a_1x^d + a_2x^{d-1} + \ldots + a_{d-1}x^2 + adx + 25 \) (As \( P(0) = 25 \))

Now, \( P''(0) = ad = 1 \)

\[
P''(0) = (2!)ad-1 = 2!
\]

\[
\Rightarrow ad-1 = 1
\]

Similarly, \( ad-2 = ad-3 = ad-4 = \ldots = a_2 = a_1 = 1 \).
Now, \(P^{(d-1)}(0) = (d - 1)! \times a_2\)
\[\Rightarrow \frac{P^{(d-1)}(0)}{(d - 1)!} = a_2 = 1\]

10. Let \(P(x)\) and \(Q(x)\) be two polynomials of degree \(d_1\) and \(d_2\) respectively where \(d_1\) and \(d_2\) are both odd. Prove that the sum of the squares of the number of real roots of \(P(x)\) and \(Q(x)\) cannot be equal to \(a^n\) where \(a\) and \(n\) are positive integers, \(n > 1\).

Solution: Now, if \(P(x)\) have complex roots then they will come in pair (complex + conjugate)
So, number of real roots of \(P(x)\) must be odd as degree = \(d_1 =\) odd.
Similarly, number of real roots of \(Q(x)\) must be odd.
Let, number of real roots of \(P(x)\) and \(Q(x)\) are \(u\) and \(v\) respectively.
Now, let, \(u^2 + v^2 = a^n\)
As \(u\) and \(v\) are both odd, \(a\) is even.
Now, dividing the equation by 4 we get,
\[1 + 1 \equiv 0 \pmod{4}\] as \(n > 1\)
\[\Rightarrow 2 \equiv 0 \pmod{4}\]
Which is impossible

Sum of squares of the number of real roots of \(P(x)\) and \(Q(x)\) cannot be equal to \(a^n\) where \(a\) and \(n\) are positive integers, \(n > 1\).

10. Let \(a, b, c\) be three distinct integers, and let \(P\) be a polynomial with integer coefficients. Show that in this case the conditions \(P(a) = b, P(b) = c, P(c) = a\) cannot be satisfied simultaneously.

Solution:
Suppose the conditions are satisfied. We derive a contradiction.
\(P(x) - b = (x - a)P_1(x) \) .................. (1)
\(P(x) - c = (x - b)P_2(x) \) .................. (2)
\(P(x) - a = (x - c)P_3(x) \) .................. (3)

Among the numbers \(a, b, c\), we choose the pair with maximal absolute difference.
Suppose this is $|a - c|$. Then we have

$$|a - b| < |a - c|\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldot...
Symmetric Polynomials:

A polynomial \( f(x, y) \) is symmetric, if \( f(x, y) = f(y, x) \) for all \( x, y \).

(a) The elementary symmetric polynomials in \( x \) and \( y \)
\[ \sigma_1 = x + y \quad \text{and} \quad \sigma_2 = xy \]

(b) The power sums, \( S_i = x^i + y^i \quad i = 0, 1, 2, \ldots \)

A polynomial symmetric in \( x, y \) can be represented as a polynomial in \( \sigma_1, \sigma_2 \).

\[ S_n = x^n + y^n = (x + y)(x^{n-1} + y^{n-1}) - xy(x^{n-2} + y^{n-2}) = \sigma_1 s_{n-1} - \sigma_2 s_{n-2} \]

We have the recursion \( s_0 = 2, s_1 = \sigma_1 \) and \( s_n = \sigma_1 s_{n-1} - \sigma_2 s_{n-2}, n \geq 2 \).

**Problem 1:** Solve the system \( x^5 + y^5 = 33; \quad x + y = 3 \).

Solution: We set \( \sigma_1 = x + y \) and \( \sigma_2 = xy \).

Then the system becomes, \( \sigma_1^5 - 5\sigma_1^3\sigma_2 + 5\sigma_1\sigma_2^2 = 33, \sigma_1 = 3 \).

Substituting \( \sigma_1 = 3 \) in the first equation we get, \( \sigma_2^2 - 9\sigma_2 + 14 = 0 \) with two solutions, \( \sigma_2 = 2 \) and \( \sigma_2 = 7 \). Now we must solve \( x + y = 3, xy = 2; \) and \( x + y = 3, xy = 7 \) resulting in \((2, 1); (1, 2);\)

\((x_3, y_3) = (3/2 + i\sqrt{19}/2, 3/2 - i\sqrt{19}/2); (x_4, y_4) = (y_3, x_3)\).

**Problem 2:** Find the real solutions of the equation \( 4\sqrt{(97 - x)} + 4\sqrt{x} = 5 \).

Solution: We set, \( 4\sqrt{x} = y \) and \( 4\sqrt{(97 - x)} = z \)

We get, \( y^4 + z^4 = x + 97 - x = 97 \) and \( y + z = 5 \).

Setting \( \sigma_1 = y + z \) and \( \sigma_2 = yz \), we get system of equations \( \sigma_1 = 5 \) and \( \sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2 = 97 \) resulting in \( \sigma_2^2 - 50\sigma_2 + 264 = 0 \) with solutions, \( \sigma_2 = 6, \sigma_2 = 44 \).

We must solve the system \( y + z = 5, yz = 6 \) with solutions \((2, 3); (3, 2)\).

Now, \( x_1 = 16, x_2 = 81 \).

The solutions, \( y + z = 5, yz = 44 \) gives complex values.
How to solve the questions like “how many real solutions does this equation have?” and you have a four-degree equation.

1. First check if you can factorize using Vanishing method with 1, 2, \(-1, -2\), maximum verify by 3.
2. Then check the degree of the equation. If it is odd then it has at least one real solution. If it is even then it may have no real solution at all because complex roots come in pair.
3. Then use Descartes” sign rule to evaluate if there is any positive or negative real roots. Descartes” sign rule says: check number of sign changes of the coefficients from higher degree to lower degree of the polynomial and that says number of maximum possible positive roots of the equation. If it has 4 sign changes then it may have 4 or 2 or 0 number of positive roots i.e. it comes down by an even number 2. Check of negative roots is my same method but of the polynomial \(P(-x)\). So, put \(x = -x\) and then find number of negative roots of the equation. If there is no then all roots are complex, otherwise it may have real roots.
4. Check whether the polynomial is increasing or decreasing for some value of \(x\). For example, it is a fourth degree equation and we have evaluated that it may have 4 positive roots. And you see the polynomial is increasing for \(x > 0\). Implies the polynomial doesn’t meet the x-axis after \(x > 0\). Therefore, all the roots it has negative but from Descartes” sign rule we have zero negative roots. Implies all the roots of the equation are complex.
5. Take any complex root of the equation as \(a + ib\), then \(a - ib\) is also a root of the equation. Now, do \(P(a + ib) - P(a - ib) = 0\) and check whether \(b = 0\) for sure. If it comes out to be \(b = 0\), then it has no imaginary roots as the imaginary part of the root is zero.

**Problem: The number of real roots of** \(x^5 + 2x^3 + x^2 + 2 = 0\)

**Solution:** First check \(x = -1\) is a solution. So, we will first factorize it by vanishing method.

Now, \(x^5 + 2x^3 + x^2 + 2\)

\[
= x^5 + x^4 - x^4 - x^3 + 3x^3 + 3x^2 - 2x^2 - 2x + 2x + 2 \\
= x^4(x + 1) - x^3(x + 1) + 3x^2(x + 1) - 2x(x + 1) + 2(x + 1) \\
= (x + 1)(x^4 - x^3 + 3x^2 - 2x + 2)
\]

It is a fifth degree equation and we have evaluated one real root \(x = -1\).
Now, we have a four degree equation, \( x^4 - x^3 + 3x^2 - 2x + 2 = 0 \)

Number of sign change = 4. Therefore, it may have 4, 2 or 0 positive roots.
And it has 0 negative roots.

\[ P(x) = x^4 - x^3 + 3x^2 - 2x + 2 \]

\[ P(0) = 2, \ P(1) = 3, \ P(2) = 18 \]
and we are seeing that it is increasing with (+)-ve value of \( x \).

So, \[ P''(x) = 4x^3 - 3x^2 +3x - 2 \]

\[ = x(4x^2 - 4x + 1) + (x^2 + 2x - 2) \]

\[ = x(2x - 1)^2 + (x - 1)(x + 2) + x > 0 \text{ for } x > 2 \]

\( \circ \) \( P(x) \) is increasing for \( x > 2 \).

\( \circ \) \( P(x) \) may have negative real roots but from Descartes’ sign rule it has no negative roots.

\( \circ \) All the roots of the four degree equation are complex.

\( \circ \) The equation has only one real root and that is \( x = -1 \).
Elements of Combinatorial Analysis

**RULE – I**: If there are two groups $G_1$ & $G_2$;

$G_1 = \{a_1, a_2, \ldots, a_n\}$ consisting of n elements and $G_2 = \{b_1, b_2, \ldots, b_m\}$ consisting of m elements then the no. of pairs $(a_i, b_j)$ formed by taking one element $a_i$ from $G_1$ and $b_j$ from $G_2$ is $n \times m$.

If there are k groups $G_1, G_2, \ldots, G_k$, such that

\[
G_1 = \{a_1, a_2, \ldots, a_{n_1}\} \\
G_2 = \{b_1, b_2, \ldots, b_{n_2}\} \\
\vdots \\
G_k = \{t_1, t_2, \ldots, t_{n_k}\}
\]

Then the number ordered k-tuples $(a_{i_1}, b_{i_2}, \ldots, t_{i_k})$ formed by taking one element from each group is $n_1 \times n_2 \times \ldots \times n_k$.

**Example**: ‘Placing balls into the cells’ amounts to choose one cell for each ball. Let there are r balls and n cells. For the 1st ball, we can choose any one of the n cells. Similarly, for each of the balls, we have n choices, assuming the capacity of each cell is infinite or we can place more than one ball in each cell. Hence the r balls can be placed in the n cells in $n^r$ ways.

**Applications**:

1. A die is rolled r times. Find the probability that –
   i) No ace turns up.
   ii) No ace turns up.

**Solution**:

i) The experiment of throwing a die r times has $6 \times 6 \times 6 \ldots r \text{ times} = 6^r$ possible outcomes.

Assume that all possible cases are equally likely. The no. of cases favorable to the event (A), ‘no ace turns up’ is $5^r$.

By Classical Definition, $P[A] = \frac{N(A)}{N} = \frac{5^r}{6^r}$.

ii) $P[\text{an ace turns up}] = 1 - P[\text{no ace turns up}] = 1 - \frac{5^r}{6^r}$.

**Remark**: The all possible outcomes of ‘r’ throw of a die correspond to the placing r balls into $n = 6$ cells.
RULE—II:

Ordered Samples: Consider a population of $n$ elements $a_1, a_2, \ldots, a_n$ any order arrangement $a_{j1}, a_{j2}, \ldots, a_{jr}$ of $r$ elements is called an ordered sample of size $r$, drawn from the population. Two procedure are possible –

i) Sampling with replacement: Here an element is selected from the population and the selected element is returned to the population before the next selection is made. Each selection is made from the entire population, so that the same element can be drawn more than ones.

ii) Sampling without replacement: Here an element once chosen is removed from the population, so that the sample becomes an arrangement without repetition.

- For a population with $n$ elements and a prescribed sample size $r$, there are $n^r$ different ordered samples with replacement and $n(n-1)\ldots(n-r+1) = n_p^r$ or $(n)_r$ different ordered samples without replacement.

Remark:

1. $n_p^r = n(n-1)\ldots(n-r+1)$ is defined if $n \in N$ and $r$ is a non-negative integers. But $(n)_r = n(n-1)\ldots(n-r+1)$ is defined if $n \in R$ and $r$ is non-negative integer. In the same way if $n \in \mathbb{R}$ then

$$n_c^r = \binom{n}{r} = \frac{n(n-1)\ldots(n-r+1)}{r!}$$

Example: 1) A random sample of size ‘$r$’ with replacement is taken from a population of $n$ elements. Find the probability that in the sample no element appear twice.

Solution: There are $n^r$ sample in all. As the samples are drawn randomly, all samples are equally likely. The no. of the samples in which in which no element appears twice is the no. of samples drawn without replacement.

Favorable sample is $= n(n-1)\ldots(n-r+1) = (n)_r$

Hence, the probability is $\frac{(n)_r}{n^r}$

Example: 2) If $n$ balls are randomly placed into $n$ cells, what is the probability that each cell will be occupied.

Solution: $P(A) = \frac{n!}{n^n}$. 

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SOLVED EXAMPLES:

1. Find the probability that among five randomly selected digits, all digits are different.

Ans:

\[ P(A) = \frac{10^5}{10^5} \]

2. In a city seven accidents occur each week in a particular week there occurs one accidents per day. Is it surprising?

Ans:

\[ P(A) = \frac{7!}{7^7} \]

3. An elevator (lift) stands with 7 passengers and stops at 10th floor. What is the probability that no two passengers leave at the same floor?

Solution:

\[ N = 10,10 ....10 \ (7 \ times) \]
\[ = 10^7 \]
\[ N(A) = 10.9.8.7.6.5.4 \]
\[ = (10)_7 \]
\[ P(A) = \frac{(10)_7}{10^7} \]

4. What is the probability that r individuals have different birthdays? Also show that the probability is approximately equal to \( e^{-r(r-1)/730} \). How many people are required to make the prob. of distinct birthdays less than ½ ?

Solution:

\[ p = \frac{(365)_r}{365^r} = \frac{365 \cdot 364 \ldots (365 - r + 1)}{365 \cdot 365 \cdot 365 \ldots \ldots \ldots 365} \]
\[ = 1 \left( 1 - \frac{1}{365} \right) \left( 1 - \frac{2}{365} \right) \ldots \left( 1 - \frac{r - 1}{365} \right) \]
\[ \therefore \ln p = \sum_{k=1}^{r-1} \ln \left( 1 - \frac{k}{365} \right) \]

For, \( 0 < x < 1, \ln(1 - x) \approx -x. \)
\[ \ln p \approx \sum_{k=1}^{r-1} \left( -\frac{k}{365} \right) = -\frac{1}{365} \sum_{k=1}^{r-1} k \]

\[ = -\frac{r(r-1)}{2(365)} = -\frac{r(r-1)}{730} \]

\[ \therefore p = e^{-\frac{r(r-1)}{730}} \]

For \( p = \frac{1}{2}, \ln p = -\ln 2 = -0.693, \)

\[ \therefore \frac{r(r-1)}{730} = 0.693 \]

\[ \therefore r^2 - r - 506 = 0 \]

\[ \Rightarrow (r - 23)(r + 22) = 0 \]

\[ \Rightarrow r = 23 \]

\[ \therefore \text{More than 23 people are required.} \]

5. Six dice are thrown. What’s the prob. that every possible number will appear.

Hints: \( p = \frac{6!}{6^6} = 0.0154 \)

6. There are four children in a family. Find the prob. that
(a) At least two of them have the same birthday?
(b) Only the oldest and the youngest have the same birthday?

Hints: (a)

\[ p_1 = 1 - \left( \frac{(365)_4}{365^4} \right) = 1 - p \{ \text{them have different birthdays} \} \]

(b)

\[ p_2 = \frac{365 \times 365 \times 363}{365^4} = \frac{(365)_3}{365^4} \]
7. The number 1, 2, ..., n are arranging in a random order. Find the probability that digits (a) 1, 2, (b) 1, 2, 3 appears as neighbours in the order named.

Hints: consider (1, 2) as a single digit then there are (n−1) entities which can be arranged in (n−1)! ways.

(a) 
\[
\text{Required prob. is } = \frac{(n-1)!}{n!} = \frac{1}{n}
\]

(b) 
\[
\text{Required prob. is } = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}
\]

8. (i) In sampling with replacement find the prob. that a fixed element be included at least once.
(ii) In sampling without replacement find the prob. that a fixed element of a population of n elements to be included in a random sample of size r.

Hints:

(i) 
\[
P_1 = 1 - P \text{ [ the fixed element is not included in the sample WOR]}
\]
\[
= 1 - \frac{(n - 1)^r}{n^r}
\]

(ii) 
\[
P_2 = 1 - P \text{ [ a fixed element is not included in the sample WR]}
\]
\[
= 1 - \frac{(n - 1)_r}{(n)_r}  \left(1 - \frac{n - r}{n}\right) = \frac{r}{n}
\]

9. There is 3 volume dictionary among 30 books is arranged in a shelf in random way. Find the prob. of 3 volume standing in an increasing order from left to right? (The vols. are not necessary side by side).

Solution: The order of the 3 vols. doesn’t depend on the arrangement of the remaining books. Here 3 vols. can be arranged in 3! ways of which only one case \(V_1, V_2, V_3\) is favorable. Hence prob. is \(1/3!\).

10. Two fair dice are thrown 10 times. Find the prob. that the first 3 throws result in a sum of 7 and the last 7 throws in a sum of 8.

Solution: \(\Omega_k = \{(i,j): i, j = 1(1)6, k = 1(1)10, be the sample space of the kth throw of a pair of dice, the sample space of the experiment is \)
\[
\Omega = \Omega_1 \times \Omega_2 \times \Omega_3 \times ... \times \Omega_{10}.
\]
\[
N = n(\Omega) = n(\Omega_1 \times \Omega_2 \times \Omega_3 \times ... \times \Omega_{10}) = 36^{10}
\]
Let, \(A = \{(i,j): i + j = 7, i,j = 1(1)6\}, the event of getting a sum of 7 in a throw of a pair of dice.
And \(B = \{(i,j): i + j = 8, i,j = 1(1)6\}, the event of getting a sum of 8 in a throw of a pair of dice.
Our event is \(A \times A \times A \times B \times ... B\)
Favorable cases are \( \{(3, 4), (2, 5), (1, 6), (2, 6), (3, 5), (4, 4)\} \)

\[
N(A) = \{n(A)\}^3\{n(B)\}^7 = 6^3 \times 5^7 \ldots \}
\]

\[
\therefore \text{Required Probability} = \frac{6^3 \times 5^7}{36^{10}}
\]

11.
(i) If \( n \) men, among whom A and B, stand in a row. What’s the prob. that there will be exactly \( r \) men between A and B?
(ii) If they stand in a ring instead of in a row, show that the prob. is independent of ‘\( r \)’.

[In the circular arrangement, consider only that they are leading from A to B in the +ve direction.]

Solution:

(i) \( n \) persons can be arranged among themselves in \( n! \) ways. Since, the persons are randomly, all possible cases are equally likely. For the favorable cases if A occupies a position to the left of B, then A may choose any of the positions:

1st, 2nd, \ldots, \((n-r-1)\)th from the left, with \( r \) persons between A and B. The remaining \((n-2)\) persons can stand in \((n-2)\) places in \((n-2)!\) Ways. Similar thing for B on the left of A.

Hence, no. of favorable cases, \( N(A) = 2(n - r - 1)(n - 2)! \)

Required probability

\[
= \frac{2(n - r - 1)(n - 2)!}{n!} = \frac{2(n - r - 1)}{n(n - 1)}
\]

(ii) If they form a ring, then the no. of possible arrangement is \((n-1)!\) which is obtained by keeping the place for any person fixed and arranging the remaining \((n-1)\) persons.

For the favorable cases, we fixed the places for A and B, with \( r \) individuals between them and then remaining \((n-2)\) persons can be arranged in \((n-2)!\) ways.

\[
\text{Required probability} = \frac{(n - 2)!}{(n - 1)!} = \frac{1}{n - 1}, \text{it is independent of } r.
\]
RULE–III:

Subpopulations and Groups: Consider a subpopulation of size ‘r’ from a given population of size ‘n’, let the no. of the groups of size r be x.

Now the r elements in a group can be arranged in r! ways. Hence x.r! ordered samples of size r.

\[ x \cdot r! = (n)_r \]

So, \[ x = \binom{n}{r} \]

Application:

1. Each of the 50 states has two senator. Find the prop. of the event that in a committee of 50 senators chosen randomly –
   (a) A given state is represented.
   (b) All states are represented.

Solution: We can choose a group of 50 senators in \( \binom{100}{50} \) ways & since 50 senators are chosen randomly 50 all possible outcomes are equally likely.

(a) There are 100 senators and 98 not from the given state.

Required probability = \( P[ \text{the given state is not represented}] \)

\[ = 1 - \frac{98}{100} \]

(b) All states will be represented if one senators from each state is selected. A committee of 50 with one senator from 50 states can be selected in \( \frac{2 \times 2 \times \ldots \times 2}{50 \text{ times}} \) ways.

\[ \text{Required probability} = \frac{2^{50}}{\binom{100}{50}} \]

2. If n balls are placed at random in n cells, find the probability that exactly one cell remains empty.

Solution: \( N = n^r \)

Since k balls can be chosen in \( \binom{r}{k} \) ways which are to be placed in the specified cells and the remaining \( (r-k) \) balls can be placed in the remaining \( (n-1) \) cells in \( (n - 1)^{r-k} \) ways.

\[ \text{Required prob.} = \frac{\binom{r}{k} (n-1)^{r-k}}{(n)^r} = \binom{r}{k} \left( \frac{1}{n} \right)^k \left( 1 - \frac{1}{n} \right)^{r-k} \]
3. If n balls are placed at a random order in n cells, find the prob. that exactly one cell remains empty.

Solution: \( N = n^n \)

For the favorable cases, the empty cell can be chosen in \( n \) ways and the two balls to be kept in the same cell can be chosen in \( \binom{n}{2} \) ways.

Consider the two balls as a single ball or entity, then \((n-1)\) entities can be arranged in \((n-1)\) cells in \((n-1)!\) ways.

So, the required prob. \( = \frac{n\binom{n}{2}(n-1)!}{n^n} \)

4. A closet contains \( n \) pairs of shoes. If \( 2r \) shoes chosen at random \((2r < n)\). What is the prob. that there will be:

   (a) No complete pair
   (b) Exactly one complete pair
   (c) Exactly two complete pair among them.

Solution: (a)

\[
\text{required prob.} = \binom{n}{2r} \frac{2^{2r}}{\binom{2n}{2r}}
\]

(b)

\[
\text{Required prob.} = \binom{n}{1} \binom{n-1}{2r-2} \frac{2^{2r-2}}{\binom{2n}{2r}}
\]

(c)

\[
\text{Req. prob.} = \binom{n}{2} \binom{n-2}{2r-4} \frac{2^{2r-4}}{\binom{2n}{2r}}
\]

5. A car is parked among \( N \) cars in a row, not at either end. On the return the car owner finds that exactly \( r \) of the \( N \) places are still occupied. What’s the prob. that both neighbouring places are empty?

Solution: Required Prob. \( = \binom{N-3}{r-1} \frac{(N-1)!}{(r-1)!} \)
**RULE – IV:**

The no. of ways in which a population of n elements can be divided into K–ordered parts of which 1\(^{st}\) contains \(r_1\), 2\(^{nd}\) contains \(r_2\) elements and so on is

\[
\frac{n!}{r_1! \cdot r_2! \cdot \ldots \cdot r_k!}, \text{ where } \sum_{i=1}^{k} r_i = n
\]

**Application:**

1. In a bridge table, calculate the prob. that
   (a) Each of the 4 players has an ace
   (b) One of the player receives all 13 spades.

**Solution:**

(a) In a bridge table 52 cards are partitioned into four equal groups and the no. of different hands is

\[
\binom{52}{13} \binom{39}{13} \binom{26}{13} \binom{13}{13}
\]

For the favorable cases, 4 aces can be arranged in 4! ways and each arrangement represents one possibility of given one ace to each player and the remaining 48 cards can be distributed equally among the 4 players in

\[
\binom{48}{12} \binom{36}{12} \binom{24}{12} \binom{12}{12}
\]

ways

\[
\text{Required prob.} = \frac{4! \left( \binom{48}{12} \binom{36}{12} \binom{24}{12} \binom{12}{12} \right)}{\binom{52}{13} \binom{39}{13} \binom{26}{13} \binom{13}{13}}
\]

\[
= \frac{4! \cdot 48!}{12!^4} \cdot \frac{52!}{\binom{13}{13}^4}
\]

(b)

\[
\text{Required prob.} = \frac{4! \cdot 39!}{52! \binom{13}{13}^3}
\]
2. In a bridge hand of cards consists of 13 cards drawn at random WOR from a deck of 52 cards.

Find the prob. that a hand of cards will contain
(a) $\gamma_1$ clubs, $\gamma_2$ spades, $\gamma_3$ diamonds
(b) $\gamma$ aces
(c) $\gamma_1$ aces and $\gamma_2$ kings.

Solution:
(a) 
\[
\text{prob.} = \frac{\binom{13}{\gamma_1} \binom{13}{\gamma_2} \binom{13}{\gamma_3} \binom{13}{13-\gamma_1-\gamma_2-\gamma_3}}{\binom{52}{13}}
\]

(b) 
\[
\text{Prob.} = \frac{\binom{4}{\gamma} \binom{48}{13-\gamma}}{\binom{52}{13}}
\]

(c) 
\[
\text{Prob.} = \frac{\binom{4}{\gamma_1} \binom{4}{\gamma_2} \binom{44}{4-\gamma_1-\gamma_2}}{\binom{52}{3}}
\]

3. 4 cards are drawn at random from a full deck of 52 cards. What’s the prob. that
(i) They are of different denominations?
(ii) They are of different suits?
(iii) Both?

Solution:
(i) In a deck of cards there are 13 denominations and 4 suits.

For favorable cases select a group of 4 denominations from 13 and then choose one card from each of the 4 denomination.

So, no. of favorable cases = \(\binom{13}{4} \binom{4}{1}^4\).

\[
\text{Required prob.} = \frac{\binom{13}{4} \binom{4}{1}^4}{\binom{52}{4}}
\]
(ii)  
\[
\text{required prob.} = \frac{\binom{13}{4}}{\binom{52}{4}}
\]

(iii) For favorable cases, selecting 4 denominations from 13 and then taking one card from the 1st denomination in 4 ways from the 4 suits. Then taking 2nd from the 2nd denomination in 3 ways & so on.

\[
\text{Required probability} = \frac{\binom{13}{4} \times 4!}{\binom{52}{4}}
\]

4. From a deck of 52 cards are drawn successively until an ace appears. What is the prob. that the 1st ace will appear?
   (a) At the nth draw,
   (b) After the nth draw.

Solution:

(a) For the favourable cases, at the nth draw an ace can occur in 4 ways and the first (n−1) cards are to be taken from 48 non-ace cards which can be done in \(\binom{48}{n-1}\) ways.

\[
\therefore \text{Required prob.} = \frac{4 \times \binom{48}{n-1}}{\binom{52}{n}}
\]

(b) For the favorable cases, 1st n cards contain no ace.

\[
\therefore \text{Req prob.} = \frac{\binom{48}{n}}{\binom{52}{n}}
\]

5. (Spread of Rumours) In a town of \((n+1)\) inhabitants, a person tells a rumour to a second person, who in turn, repeats it to a third person, etc. At each step the receipt of the rumour is chosen at random from n people available.

(i) Find the prob. that the rumour will be told r times without
   (a) Returning to the originator.
   (b) Being repeated to any person.

(ii) Do the same problem when at each step the rumour is told by one person to a gathering of \(N\) randomly chosen individuals.

Solution:

(i) Since any person can tell the rumour to any one of the \(n\) available persons in \(n\) ways, total possible cases = \(n^r\).
(a) The originator can tell the rumour to anyone of the remaining \( n \) persons in \( n \) ways & each of the 
\((r-1)\) receipts of the rumour can tell to anyone of the remaining \((n-1)\) persons without returning 
to the originator in \((n-1)\) ways.

(b)

\[
\text{Req. prob.} = \frac{n(n-1)^{r-1}}{n^r} \\
\text{Req. prob.} = \frac{(n)_r}{n^r}
\]

(ii)

(a)

\[
P_a = \frac{\left(\frac{n}{N}\right)\left(\frac{n-1}{N}\right)^{r-1}}{\left\{\binom{n}{N}\right\}^r} \\
= \left\{\frac{(n-1)^{r-1}}{\left(\frac{n}{N}\right)^r}\right\} = \left(1 - \frac{N}{n}\right)^{r-1}
\]

(b)

\[
P_b = \frac{\left(\frac{n}{N}\right)\left(\frac{n-N}{N}\right)\left(\frac{n-2N}{N}\right) \ldots \left(\frac{n-(r-1)N}{N}\right)}{\left\{\binom{n}{N}\right\}^r} \\
= \frac{n!}{\left(\frac{n^r}{\left(\frac{n}{N}\right)^r}\right)} \\
= \frac{(n)_{rN}}{\left(\frac{n}{N}\right)^r}
\]

6. 5 cards are taken at random from a full deck. Find the probability that
(a) They are different denominations?
(b) 2 are of same denominations?
(c) One pair of of one denomination & other pair of a different denomination and one odd?
(d) There are of one denomination & two scattered?
(e) 2 are of one denomination and 3 of another?
(f) 4 are of one denomination and 1 of another?
Solution: (a)

\[
P(a) = \frac{\binom{13}{5} \binom{4}{1}^5}{\binom{52}{5}}
\]

(b)

\[
P(b) = \frac{\binom{13}{1} \binom{4}{2} \binom{12}{3} \binom{4}{1}^3}{\binom{52}{5}}
\]

(c)

\[
P(c) = \frac{\binom{13}{2} \binom{4}{2}^2 \binom{11}{1} \binom{4}{1}}{\binom{52}{5}}
\]

(d)

\[
P(d) = \frac{\binom{13}{1} \binom{4}{3} \binom{12}{2} \binom{4}{1}^2}{\binom{52}{5}}
\]

(e)

\[
P(e) = \frac{\binom{13}{2} \binom{4}{2} \binom{13}{3} \binom{4}{3}}{\binom{52}{5}}
\]

(f)

\[
P(f) = \frac{\binom{13}{1} \binom{4}{4} \binom{12}{1} \binom{4}{1}}{\binom{52}{5}}
\]
RULE-V:

- **Occupancy Problem**: In many situations it is necessary to treat the balls indistinguishable, e.g., in statistical studies of the distribution of accidents among week days, here one is interested only in the number of occurrences and not in the individual involved.

Such an example is completely described by its occupancy numbers $r_1, r_2, \ldots, r_n$; where, $r_k$ denotes the number of balls in the $k$th cell.

Here we are interested in number of possible distribution, i.e., the number of different $n$-tuples $(r_1, r_2, \ldots, r_n)$ such that $r_1 + r_2 + \cdots + r_n = r$ ($r_i \geq 0$).

- **Theorem 1**: The number of different distributions of ‘$r$’ indistinguishable balls in $n$ cells, i.e., the number of different solution of the above fact is
  \[
  \binom{n + r - 1}{n - 1}.
  \]

- **Theorem 2**: The number of different distribution of ‘$r$’ indistinguishable balls in the $n$ cells in which no cell remains empty is
  \[
  \binom{r - 1}{n - 1}.
  \]

**Ex**: $r$ distinguishable balls are distributed into $n$ cells and all possible distributions are equally likely. Find the prob. that exactly $m$ cells remain empty.

**Solution**: The $m$ cells which are to be kept empty can be chosen from $n$ cells in $\binom{n}{m}$ ways and $r$ indistinguishable balls can be distributed in the remaining $(n-m)$ cells so that no cell remain empty is in
\[
\binom{r - 1}{n - m - 1} \text{ ways}.
\]

No. of favorable cases = $\binom{n}{m} \binom{r - 1}{n - m - 1}$

\[\therefore \text{Required prob.} = \frac{\binom{n}{m} \binom{r - 1}{n - m - 1}}{\binom{n + r - 1}{r}}\]
Application:

1. Show that \( r \) indistinguishable balls can be distributed in \( n \) cells i.e., the no. of different solution \((r_1, r_2, ..., r_n)\) such that \( r_1 + r_2 + \cdots + r_n = r \) is \( \binom{n + r - 1}{r} \), where \( r_i \geq 0 \).

Solution: Denoting the choices of \( r_1 \), i.e., 0, 1, ..., \( r \) in the indices, we get the factors \((x^0 + x^1 + \cdots + x^r)\).

The no. of different solutions \((r_1, r_2, ..., r_n)\) of

\[
\sum_{i=1}^{n} r_i = r, \text{ where } r_i \geq 0
\]

= The coefficient of \( x^r \) in

\[
\frac{(x^0 + x^1 + \cdots + x^r) \cdots (x^0 + x^1 + \cdots + x^r)}{n \text{ times}}
\]

= The coefficient of \( x^r \) in

\[
\left(\frac{1 - x^{r+1}}{1 - x}\right)^n
\]

= The coefficient of \( x^r \) in the expression \((1 - x^{r+1})^n (1 - x)^{-n}\)

= The coefficient of \( x^r \) in

\[
\left\{1 - nx^{r+1} + \binom{n}{2} x^{2r+2} + \cdots\right\}\left\{1 + nx + \binom{n}{2} x^2 + \cdots + \binom{n}{r} x^r + \cdots\right\}
\]

\[
= \binom{n + r - 1}{r}.
\]

2. Show that the no. of different distributions of \( r \) indistinguishable balls in \( n \) cells where no cell remains empty is \( \binom{n - 1}{r - 1} \).

Hints:

The coefficients of \( x^r \) in \((x + x^2 + \cdots + x^r)^n\)

= The coefficient of \( x^r \) in \( x^n \left(\frac{1-x^r}{1-x}\right)^n \)

= The coefficient of \( x^{n-r} \) in \((1 - x^r)^n (1 - x)^{-n}\)

\[
= \binom{n + r - n - 1}{r - n} = \binom{r - 1}{n - 1}.
\]
STUDY MATERIALS ON BASIC ALGEBRA

TOPIC : COMPLEX NUMBERS (DE’ MOIVRE’S THEOREM)

A complex number $z$ is an ordered pair of real numbers $(a,b)$: $a$ is called Real part of $z$, denoted by $\text{Re} \ z$ and $b$ is called imaginary part of $z$, denoted by $\text{Im} \ z$. If $\text{Re} \ z=0$, then $z$ is called purely imaginary; if $\text{Im} \ z =0$, then $z$ is called real. On the set $C$ of all complex numbers, the relation of equality and the operations of addition and multiplication are defined as follows:

$$(a,b)=(c,d) \iff a=b \text{ and } c=d, \ (a,b)+(c,d)=(a+c,b+d), \ (a,b). (c,d)= (ac-bd,ad+bc)$$

The set $C$ of all complex numbers under the operations of addition and multiplication as defined above satisfies following properties:

- For $z_1, z_2, z_3 \in C$, (1) $(z_1+z_2)+z_3=z_1+(z_2+z_3)$ (associativity), (2) $z_1+(0,0)=z_1$, (3) for $z=(a,b) \in C$, there exists $-z=(-a,-b) \in C$ such that $(-z)+z=z+(z)=(0,0)$, (4) $z_1+z_2=z_2+z_1$.
- For $z_1, z_2, z_3 \in C$, (1) $(z_1. z_2).z_3=z_1.(z_2. z_3)$ (associativity), (2) $z_1.(1,0)=z_1$, (3) for $z=(a,b) \in C, z \neq (0,0)$, there exists $\frac{1}{z} \in C$ such that $z. \frac{1}{z}=\frac{1}{z}. z=1$, (4) $z_1.z_2=z_2.z_1$.
- For $z_1, z_2, z_3 \in C$, $z_1.(z_2+z_3)=(z_1.z_2)+(z_1.z_3)$.

**Few Observations**

1. Denoting the complex number $(0,1)$ by $i$ and identifying a real complex number $(a,0)$ with the real number $a$, we see $z=(a,b)=(a,0)+(0,b)=(a,0)+(0,1)(b,0)$ can be written as $z=a+ib$.
2. For two real numbers $a, b$, $a^2+b^2=0$ implies $a=0= b$; same conclusion need not follow for two complex numbers, for example, $1^2+i^2=0$ but $1=(1,0)\neq (0,0) =0$ and $i=(0,1) \neq (0,0)$ ($\equiv$ denotes identification of a real complex number with the corresponding real number).
3. For two complex numbers $z_1, z_2$, $z_1z_2=0$ implies $z_1=0$ or $z_2=0$.
4. $i^2=(0,1)(0,1)=(-1,0) =-1$.
5. Just as real numbers are represented as points on a line, complex numbers can be represented as points on a plane: $z=(a,b) \leftrightarrow P: (a,b)$. The line containing points representing the real complex numbers $(a,0)$, a real, is called the real axis and the line containing points representing purely imaginary complex numbers $(0,b)$ $\equiv ib$ is called the imaginary axis.
The plane on which the representation is made is called Gaussian Plane or Argand Plane.

**Definition 1.1** Let \( z=(a,b) = a+ib \). The conjugate of \( z \), denoted by \( \bar{z} \), is \( a-ib \).

Geometrically, the point (representing) \( \bar{z} \) is the reflection of the point (representing) \( z \) in the real axis. The conjugate operation satisfies the following properties:

1. \( \bar{\bar{z}} = z \),
2. \( \bar{z_1 + z_2} = \bar{z_1} + \bar{z_2} \),
3. \( \bar{z_1z_2} = \bar{z_1}\bar{z_2} \),
4. \( \frac{\bar{z_1}}{\bar{z_2}} = \frac{\bar{z_1}}{\bar{z_2}} \),
5. \( z + \bar{z} = 2 \text{Re}(z) \)
6. \( z - \bar{z} = 2i \text{Im}(z) \)

**Definition 1.2** Let \( z=(a,b) = a+ib \). The modulus of \( z \), written as \( |z| \), is defined as \( \sqrt{a^2 + b^2} \).

Geometrically, \( |z| \) represents the distance of the point representing \( z \) from the origin (representing complex number \( (0,0) = 0+i0 \)). More generally, \( |z_1 - z_2| \) represents the distance between the points \( z_1 \) and \( z_2 \). The modulus operation satisfies the following properties:

1. \( |z_1 + z_2| \leq |z_1| + |z_2| \),
2. \( |z_1z_2| = |z_1|\cdot |z_2| \),
3. \( \frac{|z_1|}{|z_2|} = \left| \frac{z_1}{z_2} \right| \),
4. \( ||z_1| - |z_2|| \leq |z_1 - z_2| \)

**GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS: THE ARGAND PLANE**

Let \( z=a+ib \) be a complex number. In the Argand plane, \( z \) is represented by the point whose Cartesian co-ordinates is \( (a,b) \) referred to two perpendicular lines as axes, the first co-ordinate axis is called the real axis and the second the imaginary axis. Taking the origin as the pole and the real axis as the initial line, let \( (r,\theta) \) be the polar co-ordinates of the point \( (a,b) \). Then \( a=r \cos \theta \), \( b=r \sin \theta \). Also \( r=\sqrt{a^2 + b^2} = |z| \). Thus \( z=a+ib=|z|(\cos \theta +isin \theta) \): this is called modulus-amplitude form of \( z \). For a given \( z \neq 0 \), there exist infinitely many values of \( \theta \) differing from one another by an integral multiple of \( 2\pi \): the collection of all such values of \( \theta \) for a given \( z \neq 0 \) is denoted by \( \text{Arg} \ z \) or \( \text{Amp} \ z \). The principal value of \( \text{Arg} \ z \), denoted by \( \text{arg} \ z \) or \( \text{amp} \ z \), is defined to be the angle \( \theta \) from the collection \( \text{Arg} \ z \) that satisfies the condition \( -\pi < \theta \leq \pi \). Thus \( \text{Arg} \ z = \{ \text{arg} \ z + 2n\pi: n \text{ an integer} \} \). \( \text{arg} \ z \) satisfies following properties:

1. \( \text{arg}(z_1z_2) = \text{arg}z_1 + \text{arg}z_2 + 2k\pi \), where \( k \) is a suitable integer from the set \{-1, 0, 1\}
such that $-\pi < \arg z_1 + \arg z_2 + 2k\pi \leq \pi$, (2) $\arg \left( \frac{z_1}{z_2} \right) = \arg z_1 - \arg z_2 + 2k\pi$, where $k$ is a suitable integer from the set $\{-1, 0, 1\}$ such that $-\pi < \arg z_1 - \arg z_2 + 2k\pi \leq \pi$.

**Note** An argument of a complex number $z=a+ib$ is to be determined from the relations $\cos \theta = a/|z|$, $\sin \theta = b/|z|$ simultaneously and not from the single relation $\tan \theta = b/a$.

**Example 1.1** Find $\arg z$ where $z=1+i \tan \frac{3\pi}{5}$.

Let $1+i\tan\frac{3\pi}{5}=r(\cos \theta + i \sin \theta)$. Then $r^2 = \sec^2\frac{3\pi}{5}$. Thus $r = -\sec\frac{3\pi}{5} > 0$. Thus $\cos \theta = -\cos \frac{3\pi}{5}$, $\sin \theta = -\sin \frac{3\pi}{5}$. Hence $\theta = \pi + \frac{3\pi}{5}$. Since $\theta > \pi$, $\arg z = \theta - 2\pi = -\frac{2\pi}{5}$.

**Theorem 1.1** (De Moivre’s Theorem) If $n$ is an integer and $\theta$ is any real number, then $$(\cos \theta + i \sin \theta)^n = \cos n \theta + i \sin n \theta.$$ If $n = \frac{p}{q}$, $q$ natural, $p$ integer, $|p|$ and $q$ are relatively prime, $\theta$ is any real number, then $$(\cos \theta + i \sin \theta)^n$$ has $q$ number of values, one of which is $\cos n \theta + i \sin n \theta$.

**Proof:** Case 1: Let $n$ be a positive integer.

Result holds for $n=1$: $$(\cos \theta + i \sin \theta)^1 = \cos 1 \theta + i \sin 1 \theta.$$ Assume result holds for some positive integer $k$: $$(\cos \theta + i \sin \theta)^k = \cos k \theta + i \sin k \theta.$$ Then $$(\cos \theta + i \sin \theta)^{k+1} = (\cos \theta + i \sin \theta)^k(\cos \theta + i \sin \theta) = (\cos k \theta + i \sin k \theta)(\cos \theta + i \sin \theta) = \cos(k+1) \theta + i \sin(k+1) \theta.$$ Hence result holds by mathematical induction.

Case 2: Let $n$ be a negative integer, say, $n=-m$, $m$ natural.

$$(\cos \theta + i \sin \theta)^n = (\cos \theta + i \sin \theta)^{-m} = \frac{1}{(\cos \theta + i \sin \theta)^m} = \frac{1}{\cos m \theta + i \sin m \theta} \quad \text{(by case 1)}$$

$$= \cos m \theta - i \sin m \theta = \cos(-m) \theta + i \sin(-m) \theta = \cos n \theta + i \sin n \theta.$$  

Case 3: $n=0$: proof obvious.

Case 4 Let $n=\frac{p}{q}$, $q$ natural, $p$ integer, $|p|$ and $q$ are relatively prime.

Let $$(\cos \theta + i \sin \theta)^{\frac{p}{q}} = \cos \varphi + i \sin \varphi.$$ Then $$(\cos \theta + i \sin \theta)^{\frac{p}{q}} = (\cos \varphi + i \sin \varphi)^q.$$ Thus $p \theta + i \sin p \theta = \cos q \varphi + i \sin q \varphi$. Thus $q \varphi = 2k\pi + p \theta$, that is, $$\varphi = \frac{2k\pi + p \theta}{q}.$$ Hence $$(\cos \theta + i \sin \theta)^{\frac{p}{q}} = \cos \left( \frac{2k\pi + p \theta}{q} \right) + i \sin \left( \frac{2k\pi + p \theta}{q} \right),$$ where $k=0,1,\ldots,q-1$ are the distinct $q$ values.
Some Applications of De’ Moivre’s Theorem

(1) Expansion of $\cos n \theta$, $\sin n \theta$ and $\tan n \theta$ where $n$ is natural and $\theta$ is real.

$\cos n \theta+i \sin n \theta=(\cos \theta+is\sin \theta)^n=\cos^n \theta+i^n \sin^n \theta=(\cos^n \theta-\frac{n!}{2!} \cos^{n-2} \theta i^2 \sin^2 \theta+\ldots+i^n \sin^n \theta)+i(\frac{n!}{3!} \cos^{n-3} \theta \sin^3 \theta+\ldots).$ Equating real and imaginary parts, $\cos n \theta=\cos^n \theta-\frac{n!}{2!} \cos^{n-2} \theta \sin^2 \theta+\ldots$ and $\sin n \theta=\frac{n!}{3!} \cos^{n-3} \theta \sin^3 \theta+\ldots\ldots$.

(2) Expansion of $\cos^n \theta$ and $\sin^n \theta$ in a series of multiples of $\theta$ where $n$ is natural and $\theta$ is real.

Let $x=\cos \theta+i \sin \theta$. Then $x^n=\cos n \theta+i \sin n \theta$, $x^n=(x+\frac{1}{x})^n$.

$=(x^\frac{1}{n}+\frac{1}{x^n})+\frac{n!}{1!} (x^{n-2}+\frac{1}{x^{n-2}})+\ldots=2 \cos \theta+\frac{n!}{1!} (2 \cos(n-2) \theta)+\ldots$.

Similarly, expansion of $\sin^n \theta$ in terms of multiple angle can be derived.

(3) Finding $n$ th roots of unity

To find $z$ satisfying $z^n=1=\cos(2k\pi)+i \sin(2k\pi)$, where $k$ is an integer.

Thus $z=[\cos(2k\pi)+i \sin(2k\pi)]^{1/n}=\cos\left(\frac{2k\pi}{n}\right)+i \sin\left(\frac{2k\pi}{n}\right)$, $k=0,1,\ldots,n-1$;

replacing $k$ by any integer gives rise to a complex number in the set $A=\{\cos\left(\frac{2k\pi}{n}\right)+i \sin\left(\frac{2k\pi}{n}\right)\mid k=0,1,\ldots,n-1\}$. Thus $A$ is the set of all $n$ th roots of unity.

Example 1.2 Solve $x^6+x^5+x^4+x^3+x^2+x+1=0$.

We have the identity $x^6+x^5+x^4+x^3+x^2+x+1=\frac{x^7-1}{x-1}$. Roots of $x^7-1=0$ are $\cos\left(\frac{2k\pi}{7}\right)+i \sin\left(\frac{2k\pi}{7}\right)$, $k=0,1,\ldots,6$. Putting $k=0$, we obtain root of $x-1=0$. Thus the roots of given equation are $\cos\left(\frac{2k\pi}{7}\right)+i \sin\left(\frac{2k\pi}{7}\right)$, $k=1,\ldots,6$. &
Example 1.3 Prove that the sum of 99th powers of all the roots of $x^7-1=0$ is zero.

The roots of $x^7-1=0$ are $\{1, \alpha, \alpha^2, \ldots, \alpha^6\}$, where $\alpha=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$. Thus sum of 99th powers of the roots is $1+\alpha^{99}+(\alpha^2)^{99}+\ldots+(\alpha^6)^{99} = 1+\alpha^{99}+(\alpha^{99})^2 + \ldots+(\alpha^{99})^6 = \frac{1-\alpha^{99.7}}{1-\alpha^{99}} = 0$, since $\alpha^{99.7}=1$ and $\alpha^{99} \neq 1$.

Example 1.4 If the amplitude of the complex number is $\frac{\pi}{4}$, show that $z$ lies on a circle in the Argand plane.

Let $z=x+iy$. Then $\frac{z-i}{z+1} = \frac{x^2+x+y^2-y}{(x+1)^2+y^2} + i \frac{y-x-1}{(x+1)^2+y^2}$. By given condition, $\frac{y-x-1}{x^2+x+y^2-y} = 1$. On simplification, $(x+1)^2+(y-1)^2 = 1$. Hence $z$ lies on the circle centred at $(-1,1)$ and radius 1.

Example 1.5 If $A, B, C$ represent complex numbers $z_1, z_2, z_3$ in the Argand plane and $z_1+z_2+z_3=0$ and $|z_1|=|z_2|=|z_3|$, prove that $ABC$ is an equilateral triangle.

$z_1+z_2=-z_3$. Hence $|z_1+z_2|^2 = |z_3|^2$, that is, $|z_1|^2+|z_2|^2+2z_1z_2=-|z_3|^2$. By given condition, $|z_1||z_2|\cos \theta = |z_1|^2$, where $\theta$ is the angle between $z_1$ and $z_2$. Thus $\cos \theta = \frac{1}{2}$, that is, $\theta=120^0$. Hence the corresponding angle of the triangle $ABC$ is $60^0$. Similarly other angles are $60^0$.

Example 1.6 Let $z$ and $z_1$ be two complex numbers satisfying $z=\frac{1+z_1}{1-z_1}$ and $|z_1|=1$. Prove that $z$ lies on the imaginary axis.

$z=\frac{z-1}{z+1}$. By given condition, $1 = \left| \frac{z-1}{z+1} \right| = \left| \frac{z-1}{|z+1|} \right|$. If $z=x+iy$, $x=0$. Hence.

Example 1.7 Complex numbers $z_1, z_2, z_3$ satisfy the relation $z_1^2+z_2^2+z_3^2-2z_1z_2-2z_2z_3-z_3z_1=0$ iff $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$.

$0=z_1^2+z_2^2+z_3^2-2z_1z_2-2z_2z_3-z_3z_1=(z_1+wz_2+w^2z_3)(z_1+w^2z_2+wz_3)$, where $w$ stands for an imaginary cube roots of unity. If $z_1+wz_2+w^2z_3=0$, then $(z_1-z_2)=-w^2(z_3-z_2)$; hence $|z_1 - z_2| = |w^2||z_2 - z_3| = |z_2 - z_3|$. Similarly other part.
Conversely, if $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$, then $z_1, z_2, z_3$ represent vertices of an equilateral triangle. Then $z_2 - z_1 = (z_3 - z_1)(\cos 60^0 + i\sin 60^0)$, $z_1 - z_2 = (z_3 - z_2)(\cos 60^0 + i\sin 60^0)$; by dividing respective sides, we get the result.

**Example 1.8** Prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$, for two complex numbers $z_1, z_2$.

$$|z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = |z_1|^2 + |z_2|^2 + 2z_1 z_2;$$ similarly $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2z_1 z_2$; Adding we get the result.

**Example 1.9** If $\cos \alpha + \cos \beta + \cos \gamma = \sin \alpha + \sin \beta + \sin \gamma = 0$, then prove that (1) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3(\cos \alpha + \beta + \gamma)$, (2) $\sum \cos^2 \alpha = \sum \sin^2 \alpha = 3/2$.

Let $x = \cos \alpha + i \sin \alpha$, $y = \cos \beta + i \sin \beta$, $z = \cos \gamma + i \sin \gamma$. Then $x + y + z = 0$. Also $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$; hence $xy + yz + zx = 0$. Thus $x^2 + y^2 + z^2 = 0$. By De’ Moivre’s Theorem, $(\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma) = 3[\cos(\alpha + \beta + \gamma) + isin(\alpha + \beta + \gamma)]$. Equating, we get result.

Let $x = \cos \alpha + i \sin \alpha$, $y = \cos \beta + i \sin \beta$, $z = \cos \gamma + i \sin \gamma$. Then $x + y + z = 0$. Also $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$; hence $xy + yz + zx = 0$. Thus $x^2 + y^2 + z^2 = 0$. By De’ Moivre’s Theorem, $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$. Hence $\sum \cos^2 \alpha = 3/2$. Using $\sin^2 \alpha = 1 - \cos^2 \alpha$, we get other part.

**Example 1.10** Find the roots of $z^n = (z+1)^n$, where $n$ is a positive integer, and show that the points which represent them in the Argand plane are collinear.

Let $w = \frac{z+1}{z}$. Then $z = \frac{1}{w-1}$. Now $z^n = (z+1)^n$ implies $w^n = 1$. Thus, $w = \cos \frac{2k\pi}{n} + i\sin \frac{2k\pi}{n}, k = 0, \ldots, n-1$.

So $z = \frac{1}{\cos \frac{2k\pi}{n} + i\sin \frac{2k\pi}{n}}, k = 1, \ldots, n-1$

$$= \frac{1}{2} - \frac{i}{2} \cot \frac{k\pi}{n}.$$ Thus all points $z$ satisfying $z^n = (z+1)^n$ lie on the line $x = \frac{1}{2}$. 
TOPIC: THEORY OF EQUATIONS

An expression of the form \( a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n \), where \( a_0, a_1, \ldots, a_n \) are real or complex constants, \( n \) is a nonnegative integer and \( x \) is a variable (over real or complex numbers) is a polynomial in \( x \). If \( a_0 \neq 0 \), the polynomial is of degree \( n \) and \( a_0x^n \) is the leading term of the polynomial. A non-zero constant \( a_0 \) is a polynomial of degree 0 while a polynomial in which the coefficients of each term is zero is said to be a zero polynomial and no degree is assigned to a zero polynomial.

**Equality:** two polynomials \( a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n \) and \( b_0x^n + b_1x^{n-1} + \ldots + b_{n-1}x + b_n \) are equal iff \( a_0 = b_0, a_1 = b_1, \ldots, a_n = b_n \).

**Addition:** Let \( f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n \), \( g(x) = b_0x^n + b_1x^{n-1} + \ldots + b_{n-1}x + b_n \). the sum of the polynomials \( f(x) \) and \( g(x) \) is given by

\[
\begin{align*}
f(x) + g(x) &= a_0x^n + \ldots + a_{n-m}x^{m+1} + (a_{n-m} + b_0)x^m + \ldots + (a_n + b_m), \quad \text{if } m < n \\
&= (a_0 + b_0)x^n + \ldots + (a_n + b_n), \quad \text{if } m = n \\
&= b_0x^m + \ldots + b_{m-n}x^{n+1} + (b_{m-n} + a_0)x^n + \ldots + (b_m + a_n), \quad \text{if } m > n.
\end{align*}
\]

**Multiplication:** Let \( f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n \), \( g(x) = b_0x^n + b_1x^{n-1} + \ldots + b_{n-1}x + b_n \). the product of the polynomials \( f(x) \) and \( g(x) \) is given by

\[
f(x)g(x) = c_0x^{m+n} + c_1x^{m+n-1} + \ldots + c_{m+n}; \quad \text{where } c_i = a_0b_i + a_1b_{i-1} + \ldots + a_ib_0. \quad c_0 = a_0b_0 \neq 0; \quad \text{hence degree of } f(x)g(x) \text{ is } m+n.
\]

**Division Algorithm:** Let \( f(x) \) and \( g(x) \) be two polynomials of degree \( n \) and \( m \) respectively and \( n \geq m \). Then there exist two uniquely determined polynomials \( q(x) \) and \( r(x) \) satisfying \( f(x) = g(x)q(x) + r(x) \), where the degree of \( q(x) \) is \( n-m \) and \( r(x) \) is either a zero polynomial or the degree of \( r(x) \) is less than \( m \). In particular, if degree of \( g(x) \) is 1, then \( r(x) \) is a constant, identically zero or non-zero.

**Theorem 1.1 (Remainder Theorem)** If a polynomial \( f(x) \) is divided by \( x-a \), then the remainder is \( f(a) \).

»Let \( q(x) \) be the quotient and \( r \) (constant) be the remainder when \( f \) is divided by \( x-a \). then \( f(x) = (x-a)q(x) + r \) is an identity. Thus \( f(a) = r \).
Theorem 1.2 (Factor Theorem) If \( f \) is a polynomial, then \( x-a \) is a factor of \( f \) iff \( f(a)=0 \).

By Remainder theorem, \( f(a) \) is the remainder when \( f \) is divided by \( x-a \); hence, if \( f(a)=0 \), then \( x-a \) is a factor of \( f \). Conversely, if \( x-a \) is a factor of \( f \), then \( f(x)=(x-a)g(x) \) and hence \( f(a)=0 \).

Example 1.1 Find the remainder when \( f(x)=4x^5+3x^3+6x^2+5 \) is divided by \( 2x+1 \).

The remainder on dividing \( f(x) \) by \( x-(\frac{1}{2})=x+\frac{1}{2} \) is \( f(-\frac{1}{2})=6 \). If \( q(x) \) be the quotient, then \( f(x)=q(x)(x+\frac{1}{2})+6=\frac{q(x)}{2}(2x+1)+6 \). Hence 6 is the remainder when \( f \) is divided by \( 2x+1 \).

Theorem 1.3 (Fundamental Theorem of Classical Algebra)

Every polynomial equation of degree \( \geq 1 \) has a root, real or complex.

Corollary A polynomial equation of degree \( n \) has exactly \( n \) roots, multiplicity of each root being taken into account.

Corollary If a polynomial \( f(x) \) of degree \( n \) vanishes for more than \( n \) distinct values of \( x \), then \( f(x) = 0 \) for all values of \( x \).

Polynomial equations with Real Coefficients

Theorem 1.4 If \( a+ib \) is a root of multiplicity \( r \) of the polynomial equation \( f(x)=0 \) with real coefficients, then \( a-ib \) is a root of multiplicity \( r \) of \( f(x)=0 \).

Note: \( 1+i \) is a root of \( x^2-(1+i)x=0 \) but not so is \( 1-i \).

Example 1.2 Prove that the roots of \( \frac{1}{x-1} + \frac{2}{x-2} + \frac{3}{x-3} = \frac{1}{x} \) are all real.

The given equation is \( \frac{1}{x-1} + \frac{4}{x-2} + \frac{9}{x-3} = -5 \) (*). Let \( a+ib \) be a root of the polynomial equation (*) with real coefficients. Then \( a-ib \) is also a root of (*). Thus
\[
\frac{1}{(a-1)+ib} + \frac{4}{(a-2)+ib} + \frac{9}{(a-3)+ib} = -5 \quad \text{and} \quad \frac{1}{(a-1)-ib} + \frac{4}{(a-2)-ib} + \frac{9}{(a-3)-ib} = -5.
\]
Subtracting, \(-2ib[\frac{1}{(a-1)^2+b^2} + \frac{4}{(a-2)^2+b^2} + \frac{9}{(a-3)^2+b^2}]=0\) which gives \( b=0 \). Hence all roots of given equation must be real.
Example 1.3 Solve the equation \( f(x) = x^4 + x^2 - 2x + 6 = 0 \), given that \( 1+i \) is a root.

Since \( f(x) = 0 \) is a polynomial equation with real coefficients, \( 1-i \) is also a root of \( f(x) = 0 \). By factor theorem, \((x-1-i)(x-1+i) = x^2 - 2x + 2\) is a factor of \( f(x) \). By division, \( f(x) = (x^2 - 2x + 2)(x^2 + 2x + 3) \). Roots of \( x^2 + 2x + 3 = 0 \) are \( -1 \pm \sqrt{2}i \). Hence the roots of \( f(x) = 0 \) are \( 1 \pm i, -1 \pm \sqrt{2}i \).

Theorem 1.5 If \( a + \sqrt{b} \) is a root of multiplicity \( r \) of the polynomial equation \( f(x) = 0 \) with rational coefficients, then \( a - \sqrt{b} \) is a root of multiplicity \( r \) of \( f(x) = 0 \) where \( a, b \) are rational and \( b \) is not a perfect square of a rational number.

Since every polynomial with real coefficients is a continuous function from \( \mathbb{R} \) to \( \mathbb{R} \), we have

Theorem 1.6 (Intermediate Value Property) Let \( f(x) \) be a polynomial with real coefficients and \( a, b \) are distinct real numbers such that \( f(a) \) and \( f(b) \) are of opposite signs. Then \( f(x) = 0 \) has an odd number of roots between \( a \) and \( b \). If \( f(a) \) and \( f(b) \) are of same sign, then there is an even number of roots of \( f(x) = 0 \) between \( a \) and \( b \).

Example 1.4 Show that for all real values of \( a \), the equation \((x+3)(x+1)(x-4)+a(x+2)(x-1)(x-3) = 0\) has all its roots real and simple.

Let \( f(x) = (x+3)(x+1)(x-4)+a(x+2)(x-1)(x-3) \). Then \( \lim_{x \to -\infty} f(x) = \infty \), \( f(-2) < 0 \), \( f(1) > 0 \), \( f(3) < 0 \), \( \lim_{x \to \infty} f(x) = \infty \). Thus each of the intervals \((-\infty, -2), (-2, 1), (1, 3), (3, \infty)\) contains a real root of \( f(x) = 0 \). Since the equation is of degree 4, all its roots are real and simple.

Theorem 1.7 (Rolle’s Theorem) Let \( f(x) \) be a polynomial with real coefficients. Between two distinct real roots of \( f(x) = 0 \), there is at least one real root of \( f^{(1)}(x) = 0 \).

Note:

1. Between two consecutive real roots of \( f^{(1)}(x) = 0 \), there is at most one real root of \( f(x) = 0 \).
2. If all the roots of \( f(x) = 0 \) be real and distinct, then all the roots of \( f^{(1)}(x) = 0 \) are also real and distinct.
Example 1.5 Show that the equation \( f(x)=(x-a)^3+(x-b)^3+(x-c)^3+(x-d)^3=0 \), where \( a, b, c, d \) are not all equal, has only one real root.

> Since \( f(x)=0 \) is a cubic polynomial equation with real coefficients, \( f(x)=0 \) has either one or three real roots. If \( \alpha \) be a real multiple root of \( f(x)=0 \) with multiplicity 3, then \( \alpha \) is also a real root of \( f^{(1)}(x)=3[(x-a)^2+(x-b)^2+(x-c)^2+(x-d)^2]=0 \), and hence \( \alpha=a=b=c=d \) (since \( \alpha, a, b, c, d \) are real), contradiction. If \( f(x)=0 \) has two distinct real roots, then in between should lie a real root of \( f^{(1)}(x)=0 \), contradiction since not all of \( a, b, c, d \) are equal. Hence \( f(x)=0 \) has only one real root.

Example 1.6 Find the range of values of \( k \) for which the equation \( f(x)=x^4+4x^3-12x+k=0 \) has four real and unequal roots.

> Roots of \( f^{(1)}(x)=0 \) are \(-3, -1, 1\). Since all the roots of \( f(x)=0 \) are to be real and distinct, they will be separated by the roots of \( f^{(1)}(x)=0 \). Now \( \lim_{x\to-\infty} f(x)=\infty \), \( f(-3)=-9+k \), \( f(-1)=7+k \), \( f(1)=-9+k \). \( \lim_{x\to\infty} f(x)=-\infty \). Since \( f(-3)<0 \), \( f(-1)>0 \) and \( f(1)<0 \), \( -7<k<9 \).

Example 1.7 If \( c_1, c_2, \ldots, c_n \) be the roots of \( x^n+nax+b=0 \), prove that \( (c_1-c_2)(c_1-c_3)\ldots(c_1-c_n)=n(c_1^{n-1}+a) \).

> By factor theorem, \( x^n+nax+b=(x-c_1)(x-c_2)\ldots(x-c_n) \). Differentiating w.r.t. \( x \), \( n(x^{n-1}+a)=(x-c_2)\ldots(x-c_n)+(x-c_1)(x-c_3)\ldots(x-c_n)\ldots+(x-c_2)(x-c_3)\ldots(x-c_n) \).

Replacing \( x \) by \( c_1 \) in this identity, we obtain the result.

Example 1.8 If \( a \) is a double root of \( f(x)=x^n+p_1x^{n-1}+\ldots+p_n=0 \), prove that \( a \) is also a root of \( p_1x^{n-1}+2p_2x^{n-2}+\ldots+np_n=0 \).

> Since \( a \) is a double root of \( f(x)=0 \), both \( f(a)=0 \) and \( f^{(1)}(a)=0 \) hold. Thus \( a^n+p_1a^{n-1}+\ldots+p_n=0 \) (1) and \( na^{n-1}+(n-1)p_1a^{n-2}+\ldots+p_{n-1}=0 \) (2). Multiplying both side of (1) by \( n \) and both side of (2) by \( a \) and subtracting, we get \( p_1a^{n-1}+2p_2a^{n-2}+\ldots+np_n=0 \). Hence the result.

Example 1.9 Prove that the equation \( f(x)=1+x+x^2/2!+\ldots+x^n/n!=0 \) cannot have a multiple root.

> If \( a \) is a multiple root of \( f(x)=0 \), then \( 1+a+a^2/2!+\ldots+a^n/n!=0 \) and \( 1+a+a^2/2!+\ldots+a^{n-1}/(n-1)!=0 \). It thus follows that \( a^n/n!=0 \), so that \( a=0 \); but 0 is not a root of given equation.

Hence no multiple root.
Descartes’ Rule of signs

**Theorem 1.8** The number of positive roots of an equation \( f(x)=0 \) with real coefficients does not exceed the number of variations of signs in the sequence of the coefficients of \( f(x) \) and if less, it is less by an even number.

The number of negative roots of an equation \( f(x)=0 \) with real coefficients does not exceed the number of variations of signs in the sequence of the coefficients of \( f(-x) \) and if less, it is less by an even number.

**Example 1.10** If \( f(x)=2x^3+7x^2-2x-3 \), express \( f(x-1) \) as a polynomial in \( x \). Apply Descartes’ rule of signs to both the equations \( f(x)=0 \) and \( f(-x)=0 \) to determine the exact number of positive and negative roots of \( f(x)=0 \).

Let \( g(x)=f(x-1)=2x^3+x^2-10x+4 \). By Descartes’ Rule, \( g(x)=0 \) has exactly one negative root, say, \( c \). Thus \( g(c)=f(c-1)=0 \); hence \( c-1(<0) \) is a negative root of \( f(x)=0 \). Since there are 2 variations of signs in the sequence of coefficients of \( f(-x) \) and since \( c-1 \) is a negative root of \( f(x)=0 \), \( f(x)=0 \) has two negative roots. Also, \( f(x)=0 \) has exactly one positive root, by Descartes’ rule.

**Relations between roots and coefficients**

Let \( c_1, \ldots, c_n \) be the roots of the equation \( a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=0 \). By factor theorem,

\[
a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=a_0(x-c_1)(x-c_2)\ldots(x-c_n).
\]

Equating coefficients of like powers of \( x \), \( a_1=a_0(-\sum c_1) \), \( a_2=a_0\sum c_1 c_2, \ldots, a_n=a_0 (-1)^n c_1 c_2 \ldots c_n \). Hence

\[
\sum c_1=\frac{a_1}{a_0}, \quad \sum c_1 c_2=\frac{a_2}{a_0}, \ldots, \quad c_1 c_2 \ldots c_n=(-1)^n a_n \frac{a_n}{a_0}.
\]

**Example 1.11** Solve the equation \( 2x^3-x^2-18x+9=0 \) if two of the roots are equal in magnitude but opposite in signs.

Let the roots be \( -a, a, b \). Using relations between roots and coefficients, \( b=(-a)+a+b=\frac{1}{2} \) and \( -a^2b=\frac{9}{2} \). Hence \( a^2=9 \), that is, \( a=\pm 3 \). Hence the roots are \( 3, -3, \frac{1}{2} \).
SETS AND FUNCTIONS

A set is a collection of objects having the property that given any abstract (the thought of getting 100% marks at the term-end examination) or concrete (student having a particular Roll No. of semester II mathematics general) object, we can say without any ambiguity whether that object belongs to the collection (collection of all thoughts that came to one’s mind on a particular day or the collection of all students of this class) or not. For example, the collection of ‘good’ students of semester II will not be a set unless the criteria of ‘goodness’ is made explicit! The objects of which a set $A$ is constituted of are called elements of the set $A$. If $x$ is an element of a set $A$, we write $x \in A$; otherwise $x \notin A$.

If every element of a set $X$ is an element of set $Y$, $X$ is a subset of $Y$, written as $X \subseteq Y$. $X$ is a proper subset of $Y$ if $X \subseteq Y$ and $Y \not\subseteq X$, written as $X \subset Y$. For two sets $X = Y$ iff (if and only if, bi-implication) $X \subseteq Y$ and $Y \subseteq X$. A set having no element is called null set, denoted by $\emptyset$.

Example 1.1 $a \neq \{a\}$ (a letter inside envelope is different from a letter without envelope!), $\{a\} \in \{a, \{a\}\}$, $\{a\} \not\subseteq \{a, \{a\}\}$, $\emptyset \subseteq A$(the premise $x \in \emptyset$ of the implication $x \in \emptyset \Rightarrow x \in A$ is false and so the implication holds vacuously ), $A \subseteq A$, for every set $A$.

Set Operations: formation of new sets

Let $X$ and $Y$ be two sets. Union of $X$ and $Y$, denoted by $X \cup Y$, is the set \{a| a \in X or a \in Y or both\}. Intersection of $X$ and $Y$, denoted by $X \cap Y$, is the set \{a| a \in X and a \in Y\}. The set difference of $X$ and $Y$, denoted by $X - Y$, is the set \{a| a \in X and a \not\in Y\}. The set difference $U - X$ is called complement of the set $X$, denoted by $X'$, where $U$ is the universal set. The symmetric set difference of $X$ and $Y$, denoted by $X \Delta Y$, is the set $(X - Y) \cup (Y - X)$. For any set $X$, the power set of $X$, $P(X)$, is the set of all subsets of $X$. Two sets $X$ and $Y$ are disjoint iff $X \cap Y = \emptyset$. The Cartesian product of $X$ and $Y$, denoted by $X \times Y$, is defined as the set \{(x,y)| x \in X, y \in Y\} [ (x,y) is called an ordered pair. Two ordered pairs (x,y) and
(u,v) are equal, written (x,y) = (u,v), iff \( x = u \) and \( y = v \). If we take \( X = \{1,2\} \) and \( Y = \{3\} \), then \( X \times Y = \{(1,3),(2,3)\} \neq \{(3,1),(3,2)\} = Y \times X \). Thus Cartesian product between two distinct sets are not necessarily commutative (Is \( \emptyset \times \{1\} \neq \{1\} \times \emptyset \)?).

**Laws governing set operations**

For sets \( X, Y, Z, \)

- Idempotent laws: \( X \cup X = X, \ X \cap X = X \)
- Commutative laws: \( X \cup Y = Y \cup X, \ X \cap Y = Y \cap X \)
- Associative laws: \( (X \cup Y) \cup Z = X \cup (Y \cup Z), \ (X \cap Y) \cap Z = X \cap (Y \cap Z) \)
- Distributive laws: \( X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z), \ X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z) \)
- Absorptive laws: \( X \cap (X \cap Y) = X, \ X \cup (X \cap Y) = X \)
- De’ Morgan’s laws: \( X -(Y \cup Z) = (X - Y) \cap (X - Z), \ X -(Y \cap Z) = (X - Y) \cup (X - Z) \)

**Example 1.1** Let \( A, B, C \) be three sets such that \( A \cap B = A \cap C \) and \( A \cup B = A \cup C \), then prove \( B = C \).

\[
B = B \cup (A \cap B) = B \cup (A \cap C) = (B \cup A) \cap (B \cup C) \text{ (distributivity of } \cup \text{ over } \cap \) = (C \cup A) \cap (B \cup C) = C \cup (A \cap B) = C \cup (A \cap C) = C.
\]

**Example 1.2** \( A \Delta C = B \Delta C \) implies \( A = B \): prove or disprove.

**NOTE:** Proving will involve consideration of arbitrary sets \( A, B, C \) satisfying the given condition, whereas disproving consists of giving counter-examples of three particular sets \( A, B, C \) that satisfies the hypothesis \( A \Delta C = B \Delta C \) but for which the conclusion \( A = B \) is false.

**NOTATION:** \( N, Z, Q, R, C \) will denote set of all positive integers, integers, rational numbers ,real numbers and the complex numbers respectively.
A function from a set A to a set B, denoted by f: A→B, is a correspondence between elements of A and B having the properties:

✓ For every x∈A, the corresponding element f(x)∈B. f(x) is called the image of x under the correspondence f and x is called a pre-image of f(x). A is called domain and B is called the codomain of the correspondence. Note that we differentiate between f, the correspondence, and f(x), the image of x under f.

✓ For a fixed x∈A, f(x)∈B is unique. For two different elements x and y of A, images f(x) and f(y) may be same or may be different.

In brief, a function is a correspondence under which

➢ both existence and uniqueness of image of all elements of the domain is guaranteed but

➢ neither the existence nor the uniqueness of pre-image of some element of codomain is guaranteed.

Example 1.7 Prove that f(A∩B) ⊆f(A)∩f(B) ; give a counterexample to establish that the reverse inclusion may not hold.

» y∈ f(A∩B)⇒y = f(x), x∈ A∩B⇒ y = f(x), x∈ A and x∈B⇒y∈f(A)and y∈f(B) ⇒y∈f(A)∩f(B). Hence f(A∩B) ⊆f(A)∩f(B). Consider the counter example: f: R→R,f(x) = x², A = {2}, B = {-2}.

Example 1.3 Let f: R→R, f(x) = 3x²-5. f(x) = 70 implies x = ±5. Thus f⁻¹{70} = {-5, 5}. Hence f[f⁻¹{70}] = {f(-5), f(-5)} = {70}. Also, f⁻¹({-11}) = ∅ [x∈ f⁻¹({-11})⇒3x²-5=-11⇒x²=-2].

A function under which uniqueness of pre-image is guaranteed is called an injective function. A function under which existence of pre-image is guaranteed is called a surjective function. Put in a different language, f: A→B is injective iff a₁, a₂∈A, f(a₁) =
f(a_2) implies a_1 = a_2. f is **surjective** iff codomain and range coincide. A function which is both injective and surjective is called **bijective**.

**NOTE:** The injectivity, surjectivity and bijectivity depends very much on the domain and codomain sets and may well change with the variation of those sets even if expression of the function remains unaltered e.g. f:Z→Z, f(x) = x^2 is not injective though g:N→Z, g(x) = x^2 is injective.

**Example1.4** f: R→R, f(x) = x^2 – 3x+4. f(x_1) = f(x_2) implies (x_1-x_2)(x_1+x_2-3) = 0. Thus f(1) = f(2) though 1≠2; hence f is not injective [Note: for establishing non-injectivity, it is sufficient to consider particular values of x]. Let y∈R and x∈f^1{y}. Then y = f(x) = x^2 – 3x+4. We get a quadratic equation x^2 – 3x+(4-y) = 0 whose roots, considered as a quadratic in x, give pre-image(s) of y. But the quadratic will have real roots if the discriminant 4y-7≥0, that is, only when y≥7/4. Thus, for example, f^1{1} =∅. Hence f is not surjective.

If f:A→B and g:B→C, we can define a function g_0f:A→C, called the composition of f and g, by (g_0f)(a) = g(f(a)), a∈A.

**Example1.5** f:Z→Z and g: Z→Z by f(n) = (-1)^n and g(n) = 2n. Then g_0f: Z→Z, (g_0f)(n)=g((-1)^n) = 2(-1)^n and (f_0g)(n) = (-1)^2n. Thus g_0f ≠ f_0g. Commutativity of composition of functions need not hold.
LINEAR ALGEBRA

MATRICES

Definition: A rectangular array of mn elements $a_{ij}$ into m rows and n columns, where the elements $a_{ij}$ belong to a field F, is called a matrix of order mxn over F. It is denoted by $[a_{ij}]_{mxn}$ or by \[
\begin{bmatrix}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 \vdots & \vdots & \ddots & \vdots \\
 a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\] or by
\[
\begin{pmatrix}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 \vdots & \vdots & \ddots & \vdots \\
 a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]. F is called field of scalars. In particular, if F be the field R of real numbers, a matrix over R is said to be a real matrix. The element $a_{ij}$ appearing in the i th row and j th column of the matrix is said to be ij th element. If m=1, the matrix is said to be a row matrix and if n = 1, it is called a column matrix. If each element of a matrix is 0, it is called a null matrix and denoted by $O_{mxn}$. If m=n, matrix is called a square matrix. Two matrices $[a_{ij}]_{mxn}$ and $[b_{ij}]_{pxq}$ are equal iff m = p, n = q and $a_{ij} = b_{ij}$ for each i and j. A square matrix whose elements on the principal diagonal are all equal to 1 and all the elements off the main diagonal are 0 is called identity matrix and is denoted by $I_n$. If $A = [a_{ij}]_{mxn}$, then transpose of $A$, denoted by $A^T$, is defined as $A^T = [b_{ij}]_{nxm}$, where $b_{ij} = a_{ji}$, for each i and j. A square matrix is a diagonal matrix if all the elements not lying on the main diagonal are zero.

OPERATION ON MATRICES

Equality of matrices $[a_{ij}]_{mxn} = [b_{ij}]_{pxq}$ iff m=p, n=q and $a_{ij} = b_{ij}$, for each i,j.

Multiplication by a scalar for a scalar c, \( c[a_{ij}]_{mxn} = [ca_{ij}]_{mxn} \)

Addition two matrices $[a_{ij}]_{mxn}$ , $[b_{ij}]_{pxq}$ are conformable for addition iff m = p and n = q and in that case
\[
[a_{ij}]_{mxn} + [b_{ij}]_{mxn} = [a_{ij}+b_{ij}]_{mxn}
\]
Multiplication two matrices $[a_{ij}]_{mxn}$, $[b_{ij}]_{pxq}$ are conformable for multiplication iff $n = p$. In that case

$$[a_{ij}]_{mxn} [b_{ij}]_{nxq} = [c_{ij}]_{mxq}, c_{ij} = \sum_{s=1}^{n} a_{is} b_{sj}. $$

**ALGEBRA OF MATRICES**

1. Matrix addition is commutative and associative.
2. Matrix multiplication is NOT commutative.
3. Matrix multiplication is associative. Let $A = [a_{ij}]_{mxn}$, $B = [b_{ij}]_{nxp}$, $C = [c_{ij}]_{pxq}$. Then $AB = [d_{ij}]_{mxp}$, where $d_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. Thus $(AB)C = [e_{ij}]_{mxq}$ where $e_{ij} = \sum_{s=1}^{p} d_{is} b_{sj} = \sum_{s=1}^{p} \sum_{k=1}^{n} a_{ik} b_{ks} c_{sj}$. Again, $BC = [f_{ij}]_{mxq}$, where $f_{ij} = \sum_{t=1}^{p} b_{lt} c_{tj}$. Thus $A(BC) = [g_{ij}]_{mxq}$ where $g_{ij} = \sum_{u=1}^{n} a_{iu} f_{uj} = \sum_{u=1}^{n} \sum_{s=1}^{p} a_{iu} b_{ut} c_{tj} = e_{ij}$, for all $i,j$. Hence $A(BC) = (AB)C$.
4. Matrix multiplication is distributive over addition.
5. $(A^T)^T = A$
6. $(A+B)^T = A^T + B^T$.
7. $(AB)^T = B^T A^T$ (supposing $A,B$ are conformable for product). Let $A = [a_{ij}]_{mxn}$, $B = [b_{ij}]_{nxp}$. $AB = [c_{ij}]_{mxp}$, where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. So $(AB)^T = [d_{ij}]_{pxm}$, $d_{ij} = \sum_{k=1}^{n} a_{jk} b_{ki}$. $B^T = [e_{ij}]_{pxn}$, $A^T = [f_{ij}]_{nxm}$ where $e_{ij} = b_{ji}$, $f_{ij} = a_{ij}$. Hence $B^T A^T = [g_{ij}]_{pxm}$, where $g_{ij} = \sum_{s=1}^{n} e_{is} f_{sj} = \sum_{s=1}^{n} \sum_{t=1}^{p} a_{js} b_{st} = d_{ij}$.

**Symmetric and skew-symmetric matrix**

A square matrix $A$ is symmetric iff $A = A^T$. A square matrix $A$ is skew-symmetric iff $A = -A^T$.

**Results involving symmetric and skew-symmetric matrices**

1. If $A$ and $B$ are symmetric matrices of the same order, then $A+B$ is symmetric.
(2) If $A$ and $B$ are symmetric matrices of the same order, then $AB$ is symmetric iff $AB = BA$.

If $AB$ is symmetric, then $AB = (AB)^T = B^T A^T = BA$. If $AB = BA$, then $AB = BA = B^T A^T = (AB)^T$, so that $AB$ is symmetric.

(3) $AA^T$ and $A^T A$ are both symmetric.

(4) A real or complex square matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrix.

Let $A$ be a square matrix. Then $A$ can be expressed as $A = \frac{1}{2} (A + A^T) + \frac{1}{2} (A - A^T)$, where $\frac{1}{2} (A + A^T)$ is symmetric and $\frac{1}{2} (A - A^T)$ is skew-symmetric. Uniqueness can be proved.

**DETERMINANT**

Let $M$ stand for the set of all square matrices over $\mathbb{R}$. We define a function $\det : M \rightarrow \mathbb{R}$ inductively as follows:

Step 1 $\det X = a_{11} = (-1)^{1+1} a_{11}$, if $X = [a_{11}]_{1 \times 1}$. $\det X = (-1)^{1+1} a_{11} det(a_{22}) + (-1)^{1+2} a_{21} det(a_{12})$, if $X = [a_{ij}]_{2 \times 2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Step 2 let us assume the definition is valid for a square matrix of order $n$: thus for $X = (a_{ij})_{n \times n}$, $\det X = \sum_{r=1}^{n} (-1)^{1+r} a_{1r} \det X_{1r}$, where $X_{1r}$ is a matrix of order $n-1$, obtained from $X$ by deleting the first row and $r$ th column. We now consider $X = (a_{ij})_{(n+1) \times (n+1)}$. We now define $\det X = \sum_{r=1}^{n+1} (-1)^{1+r} a_{1r} \det X_{1r}$, where $X_{1r}$ is a matrix of order $n$, obtained from $X$ by deleting the first row and $r$ th column. By assumption, $\det X_{1r}$ can be evaluated for all $r$. so the definition is valid for a square matrix of order $(n+1)$. 
Hence by induction, for a matrix $X = (a_{ij})_{n\times n}$, we define $\det X = \sum_{r=1}^{n}(-1)^{1+r}a_{1r}\det X_{1r}$, where $X_{1r}$ is a matrix of order $n-1$, obtained from $X$ by deleting the first row and $r$th column.

Following this definition, we can define $\det X$, for any $X \in M$.

Example 2.1 for the matrix $X = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $\det X = (-1)^{1+1}a_{11}X_{11}+(-1)^{1+2}a_{12}X_{12}+(-1)^{1+3}a_{13}X_{13}$, where $X_{1r}$ is the determinant of the matrix obtained by deleting the first row and $r$th column of $A$. Now $X_{11} = (-1)^{2+2}a_{22}\det(a_{33})+(-1)^{2+3}a_{23}\det(a_{32}) = a_{22}a_{33}-a_{23}a_{32}$.

Similarly value for $X_{12}$ and $X_{13}$ can be found and finally value of $\det X$ can be calculated.

Properties of determinants

1. $\det X = \det X^T$, where $X = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. By actual calculation, we can verify the result.

   NOTE: By virtue of this property, a statement obtained from an established result by interchanging the words ‘row’ and ‘column’ thoroughly will be established.

2. Let $A$ be a matrix and $B$ is obtained from $A$ by interchanging any two rows (columns) of a matrix $A$, then $\det A = -\det B$.

3. If $A$ be a matrix containing two identical rows (or columns), then $\det A = 0$. Result follows from (2).

4. If elements of any row of a determinant is expressed as sum of two elements, then the determinant can be expressed as a sum of two determinants.
Example: $\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + p & a_{22} + q & a_{23} + r \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} +$

$\det \begin{bmatrix} p & q & r \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, which can be verified using definition and earlier properties.

(5) If elements of any row of a determinant is multiplied by a constant, then the determinant is multiplied by the same constant.

Example: $\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ ca_{21} & ca_{22} & ca_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = c \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, which can be verified using definition and earlier properties.

(6) In an $n \times n$ matrix $A$ if a scalar multiple of one row(column) be added to another row(column), then $\det A$ remains unaltered.

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + ca_{31} & a_{22} + ca_{32} & a_{23} + ca_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} +$$

$\det \begin{bmatrix} ca_{31} & ca_{32} & ca_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, [by property (4)]

$$= \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$+ c \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, [by property(5)]

$$= \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + c.0 \ , \ [by \ property(3)]

(7) In an $n \times n$ matrix $A$, if one row(column) be expressed as a linear combination of the remaining rows(columns), then $\det A = 0$. 

9
Example: Prove without expanding the determinant \[ |1 \ a \ b^2 + c^2 + bc| = 0 \]

\[ |1 \ b \ c^2 + a^2 + ca| \]
\[ |1 \ c \ a^2 + b^2 + ab| \]

= \[ |1 \ a \ -a^2| + |1 \ b \ -b^2| + |1 \ c \ -c^2| \]

= \[ |1 \ a \ a^2| + |1 \ b \ b^2| + |1 \ c \ c^2| \]

= \[ |1 \ a | + |1 \ b | + |1 \ c | \] = 0.

Cofactors and Minors:

Let \( A = [a_{ij}]_{m \times n} \). \( M_{ij} \), minor of the element \( a_{ij} \), is the determinant of the matrix obtained by deleting \( i \) th row and \( j \) th column of the matrix \( A \). \( A_{ij} \), cofactor of the element \( a_{ij} \), is defined as \((-1)^{i+j}M_{ij}\).

Theorem: For a matrix \( A = [a_{ij}]_{3 \times 3} \), \( a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = \det A \), if \( i = k \) and \( 0 \), if \( i \neq k \).

\[ a_{11}A_{11} + a_{12}A_{12} + A_{13} = a_{11}(-1)^{1+1}\det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} + a_{12}(-1)^{1+2}\det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}(-1)^{1+3}\det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \det A, \text{ by definition. Also}, \]

\[ \]
\[ a_{11} A_{21} + a_{12} A_{22} + a_{13} A_{23} = a_{11}(-1)^{2+1} \det \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} + a_{12}(-1)^{2+2} \det \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}(-1)^{2+3} \det \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix} \]

= \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{(by definition of determinant)} = 0 \text{ (by property (3))}.

Similarly other statement can be proved.

**Multiplication of determinants**

**Theorem:** If \( A \) and \( B \) are two square matrices of the same order, then

\[ \det(AB) = \det A \cdot \det B = \det A^T \cdot \det B \]

**Example:** prove

\[ \begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ca - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix} = (a^3 + b^3 + c^3 - 3abc)^2 \]

we have \[ \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \]

\[ = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} -a & -b & -c \\ c & a & b \\ b & c & a \end{vmatrix} = \begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ca - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix} \]

**Example:** Prove that

\[ \begin{vmatrix} 0 & (a - b)^2 & (a - c)^2 \\ (b - c)^2 & 0 & (b - c)^2 \\ (c - a)^2 & (c - b)^2 & 0 \end{vmatrix} = 2(a-b)^2(b-c)(c-a)^2. \]

we have \[ \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a). \]

Now \[ \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \]
Definition: If $A=(a_{ij})$ be a square matrix and $A_{ij}$ be the cofactor of $a_{ij}$ in det $A$, then det($A_{ij}$) is the adjoint of det $A$.

Theorem: If $A=(a_{ij})_{3x3}$, then det($A_{ij}$)=[det ($a_{ij}$)]$^2$, if det ($a_{ij}$)$\neq$0.

\[
\begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix}
=\begin{vmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{vmatrix}
=\begin{vmatrix}
a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} & a_{11}A_{12} + a_{12}A_{22} + a_{13}A_{32} & 0 \\
0 & \det A & 0 \\
0 & 0 & \det A
\end{vmatrix}
=(\det A)^3. \text{ Hence } \det(A_{ij})=[\det (a_{ij})]^2, \text{ if } \det (a_{ij})\neq 0.

Example: Prove \[
\begin{vmatrix}
bc - a^2 & ca - b^2 & ab - c^2 \\
ca - b^2 & ab - c^2 & bc - a^2 \\
ab - c^2 & bc - a^2 & ca - b^2
\end{vmatrix}=(a^3+b^3+c^3-3abc)^2
\]

\[
\begin{vmatrix}
a & b & c \\
b & c & a \\
c & a & b
\end{vmatrix} = -(a^3+b^3+c^3-3abc).
\]
Now \[
\begin{vmatrix}
bc - a^2 & ca - b^2 & ab - c^2 \\
ca - b^2 & ab - c^2 & bc - a^2 \\
ab - c^2 & bc - a^2 & ca - b^2
\end{vmatrix}
\]
\[
= \text{adj} \begin{vmatrix}
a & b & c \\
b & c & a \\
c & a & b
\end{vmatrix} = \begin{vmatrix}
a & b & c \\
b & c & a \\
c & a & b
\end{vmatrix}^2 = \text{RHS}
\]

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Adjoint of a square matrix

Definition Let $A = (a_{ij})$ be a square matrix. Let $A_{ij}$ be the cofactor of $a_{ij}$ in $\det A$. The adjoint of $A$, denoted by $\text{Adj} A$, is defined as $(A_{ij})^T$.

Theorem: Let $A$ be a square matrix of order $n$. then $A \cdot (\text{Adj} A) = (\text{Adj} A) \cdot A = |A| I_n$.

The $i,j$ th element of $A \cdot (\text{Adj} A)$ is $a_{i1}A_{1j} + a_{i2}A_{2j} + \ldots + a_{in}A_{nj}$ which equals 0, if $i \neq j$ and equals $|A|$, if $i = j$. Hence $A \cdot (\text{Adj} A) = \begin{bmatrix} |A| & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |A| \end{bmatrix} = |A| I_n$. Similarly other part.

Definition: A square matrix is singular if $|A| = 0$ and is non-singular if $|A| \neq 0$.

Definition: A square matrix of order $n$ is invertible if there exists a matrix $B$ such that $AB = BA = I_n$. $B$ is called an inverse to $A$. If $C$ be an inverse to $A$ also, then $AC = CA = I_n$. Using associativity of product of matrices, it is easy to verify that $B = C$. So inverse of a square matrix, if it exists, is unique. Note also that since $AB$ and $BA$ both are to be defined, $A$ must be a square matrix.

Theorem: An $n \times n$ matrix $A$ is invertible iff it is non-singular.

Necessity let $A_{n \times n}$ be invertible. Then there exists $B_{n \times n}$ such that $AB = BA = I_n$. then $|A||B| = |AB| = |I_n| = 1$ so that $|A| \neq 0$.

Sufficiency Let $|A| \neq 0$. we know that $A(\text{Adj} A) = (\text{Adj} A)A = |A| I_n$. hence $A \left[ \frac{1}{|A|} (\text{Adj} A) \right] = \left[ \frac{1}{|A|} (\text{Adj} A) \right] A = I_n$, proving the result.
Theorem: If \( A, B \) be invertible matrices of the same order, then \( AB \) is invertible and \((AB)^{-1} = B^{-1}A^{-1}\).

\[ |AB| = |A||B| \neq 0, \text{ hence } AB \text{ is invertible. Using associativity,} \]
\[ (AB)(B^{-1}A^{-1}) = I_n = (B^{-1}A^{-1})(AB). \]

Theorem: If \( A \) be invertible, then \( A^{-1} \) is also invertible and \((A^{-1})^{-1} = A\).

\[ \text{From } AA^{-1} = A^{-1}A = I_n, \text{ it follows from the definition of inverse that } (A^{-1})^{-1} = A. \]

Theorem: If \( A \) be an invertible matrix, then \( A^T \) is invertible and \((A^T)^{-1} = (A^{-1})^T\).

\[ A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n = (AA^{-1})^T = (A^{-1})^T A^T. \text{ Hence } (A^T)^{-1} = (A^{-1})^T. \]

Orthogonal matrix

Definition: A square matrix \( A \) of order \( n \) is orthogonal iff \( AA^T = I_n \).

Theorem: If \( A \) is orthogonal, then \( A \) is non-singular and \( |A| = \pm 1 \).

\[ |A|^2 = |A||A^T| = |AA^T| = |I_n| = 1 \text{ implies } |A| = \pm 1 \neq 0. \]

Theorem: If \( A \) be an \( nxn \) orthogonal matrix, then \( A^TA = I_n \).

\[ AA^T = I_n \Rightarrow A^T( AA^T ) = A^TI_n \Rightarrow (A^TA)^T = A^T = A^T(A^TA-I_n)A^T = O \Rightarrow (A^TA-I_n)[A^T(A^T)^{-1}] = O \]
\[ (A \text{ is orthogonal} \Rightarrow A \text{ is non-singular} \Rightarrow A^T \text{ is non-singular} \Rightarrow A^T \text{ is invertible}) \Rightarrow A^TA = I_n. \]

Theorem: If \( A \) and \( B \) are orthogonal matrices of the same order, then \( AB \) is orthogonal.

\[ (AB)(AB)^T = (AB)(B^TA^T) = A( BB^T)A^T = (AI_n)A^T = AA^T = I_n. \]

Theorem: If \( A \) is orthogonal, \( A^{-1} \) is orthogonal.

\[ (A^{-1})(A^{-1})^T = (A^{-1})(A^T)^{-1} = (A^TA)^{-1} = I_n^{-1} = I_n. \]

Note: If \( A \) be an orthogonal matrix, \( A^T = A^{-1} \).
Rank of a matrix

Definition Let A be a non-zero matrix of order mxn. Rank of A is defined to be the greatest positive integer r such that the determinant of the matrix formed by elements of A lying at the intersection of some r rows and some r columns is nonzero. Rank of null matrix is defined to be zero.

Note: (1) $0 < \text{rank } A \leq \min\{m,n\}$, for a non-zero matrix A.

(2) for a square matrix A of order n, rankA < n or =n according as A is singular or non-singular.

(3) $\text{Rank } A = \text{Rank } A^T$.

Elementary row operations

An elementary operation on a matrix A over a field F is an operation of the following three types:

- Interchange of two rows(columns) of A
- Multiplication of a row (or column) by a non-zero scalar $c \in F$
- Addition of a scalar multiple of one row (or column) to another row (or column)

When applied to rows, elementary operations are called elementary row operations.

Notation: interchange of $i$ th and $j$ th row will be denoted by $R_{ij}$. Multiplication of $i$ th row by $c$ will be denoted by $cR_i$. Addition of $c$ times the $j$ th row to the $i$ th row is denoted by $R_i + cR_j$.

Definition: An m xn matrix B is row equivalent to a m xn matrix A over the same field F iff B can be obtained from A by a finite number of successive elementary row operations.
Note Since inverse of an elementary row operation is again an elementary row operation, if B is row equivalent to A, then A is also row equivalent to B.

**Definition:** An m x n matrix is row reduced iff

- The first non-zero element in a non-zero row is 1 and
- each column containing the leading 1 of some row has all other elements zero.

**Example:**

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 2 \\
1 & 0 & \cdots & 0 & 0
\end{bmatrix}
\]

**Definition:** An m x n matrix A is row reduced echelon matrix iff

- A is row reduced
- Each zero row comes below each non-zero row and
- If first r rows are non-zero rows of A and if the leading element of row I occurs in column k_i, then k_1<k_2<…<k_r.

**Example:**

\[
\begin{bmatrix}
0 & 1 & 3 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

**Algorithm which row-reduces a matrix to echelon form**

**Step-1:** Suppose that j_1 column is the first column with a nonzero entry. Interchange the rows so that this nonzero entry appears in the first row, that is, so that a_{1j_1}≠0

**Step-2:** For each i>1, apply the operation \(R_i→a_{1j_1}R_1 + a_{1j_1}R_i.\)

Repeat steps 1 and 2 with the submatrix formed by all the rows excluding the first.

**Theorem:** For a given matrix A, a row-reduced echelon matrix B equivalent to A can be found by elementary row operations.
Theorem: If a matrix $A$ is equivalent to a row-reduced echelon matrix having $r$ non-zero rows, then $\text{Rank } A = r$.

Example: Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

» First Method: $\text{rank } A \geq 1$, since $\text{det} [1] = 1 \neq 0$. Though $\text{det} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 0$, $\text{det} \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \neq 0$.

Hence $\text{rank } A \geq 2$. Since $\text{det } A = 0$, rank $A$ is not equal to 3. hence rank $A = 2$.

Second Method: $A \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 6 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow{R_3 + \frac{1}{6}R_3} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{6}R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Hence rank $A = 2$.

**SYSTEM OF LINEAR EQUATIONS**

A system of $m$ linear equations in $n$ unknowns $x_1, x_2, \ldots, x_n$ is of the form

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m,$$ where $a_{ij}$’s and $b_i$’s are given elements of a field.
An ordered n-tuple \((c_1, c_2, \ldots, c_n)\) is a solution of the system if each equation of the system is satisfied by \(x_1 = c_1, \ldots, x_n = c_n\). A system of equations is consistent if it has a solution; otherwise it is inconsistent.

Matrix representation: Let \(A = (a_{ij})_{mxn}\), \(X = (x_j)_{nx1}\), and \(B = (b_i)_{mx1}\). Then the system can be written as \(AX = B\). The matrix \[
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
\vdots & \vdots & & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{bmatrix}
\] is called the augmented matrix, denoted by \((A, B)\).

Definition: Two systems \(AX = B\) and \(CX = D\) are equivalent systems if the augmented matrices \((A, B)\) and \((C, D)\) are row equivalent.

Theorem: If \(AX = B\) and \(CX = D\) are equivalent systems and if \((e_1, e_2, \ldots, e_n)\) be a solution of \(AX = B\), then \((e_1, e_2, \ldots, e_n)\) is also a solution of \(CX = D\). If one of two equivalent systems is inconsistent, then the other is also so.

Ex: Solve, if possible, the system \(x_1 + 2x_2 - x_3 = 10\), \(-x_1 + x_2 + 2x_3 = 4\), \(2x_1 + x_2 - 3x_3 = 2\).

\[
\begin{bmatrix}
1 & 2 & -1 & 10 \\
-1 & 1 & 2 & 4 \\
2 & 1 & -3 & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & -1 & 10 \\
0 & 3 & 1 & 14 \\
2 & 1 & -3 & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & -1 & 10 \\
0 & 3 & 1 & 14 \\
0 & 0 & 0 & -4
\end{bmatrix}
\]

Thus the given system is equivalent to \(x_1 + 2x_2 - x_3 = 10\), \(3x_2 + x_3 = 14\), \(0 = -4\), which is inconsistent. Thus the given system is inconsistent.

Theorem: A necessary and sufficient condition that a given system of linear equations \(AX = B\) is consistent is that \(\text{rank } A = \text{rank}(A, B)\).
Solution of a system of linear equations having same number of variables as that of equations in which coefficient matrix is nonsingular

METHOD 1: Cramer’s rule

Let

\[ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \]
\[ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \]
\[ \vdots \]
\[ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \]

be a system of \( n \) linear equations in \( n \) unknowns where \( \det A = \det(a_{ij})_{n \times n} \neq 0 \). Then there exists a unique solution of the system given by \( x_1 = \frac{\det A_1}{\det A}, \ldots, x_n = \frac{\det A_n}{\det A} \), where \( A_i \) is the \( n \times n \) matrix obtained from \( A \) by replacing its \( i \) th column by the column \([b_1 \ b_2 \ldots b_n]^T\), \( i = 1, 2, \ldots, n \).

\[ x_1 \det A = \det \begin{bmatrix} x_1 a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ x_1 a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \det \begin{bmatrix} x_1 a_{11} + x_2 a_{12} + \cdots + x_n a_{1n} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ x_1 a_{n1} + x_2 a_{n2} + \cdots + x_n a_{nn} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \det \begin{bmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \det A_1. \] Similarly others.
Example: let us consider the system \( x+2y-3z = 1, \ 2x - y+z = 4, \ x+3y = 5. \)

Determinant of the coefficient matrix = 
\[
\begin{vmatrix}
1 & 2 & -3 \\
2 & -1 & 1 \\
1 & 3 & 0
\end{vmatrix}
\]

\( = -22 \neq 0. \) By Cramer’s rule,

\[
x = \begin{vmatrix} 1 & 2 & -3 \\ 4 & -1 & 1 \\ 5 & 3 & 0 \end{vmatrix} = -2, \quad y = \begin{vmatrix} 1 & 1 & -3 \\ 2 & 4 & 1 \\ 1 & 5 & 0 \end{vmatrix} = 1, \quad z = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 4 \\ 1 & 3 & 5 \end{vmatrix} = 1.
\]

METHOD 2: Matrix Inversion method

Let \( A = (a_{ij})_{n \times n}, \ X = [x_1, \ldots, x_n]^T, \ B = [b_1, b_2, \ldots, b_n]^T. \) Then the above system of linear equations can be written as \( AX = B, \)

where \( \det A \neq 0. \) Thus \( A^{-1} \) exists and \( X = A^{-1}B. \)

Example 2.12 \( 3x+y = 2, \ 2y+3z = 1, \ x+2z = 3. \)

Let \( A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}, \ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \ B = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}. \ \det A \neq 0. \)

\[
A^{-1} = \frac{1}{\det A} AdjA = \begin{bmatrix}
\frac{4}{15} & -\frac{2}{15} & \frac{3}{15} \\
\frac{3}{15} & \frac{6}{15} & -\frac{9}{15} \\
\frac{2}{15} & \frac{1}{15} & \frac{6}{15}
\end{bmatrix}. \ X = A^{-1}B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}. \ Thus the solution is
\]

\( x = 1, y = -1, z = 1. \)
EIGEN VECTOR AND EIGEN VALUE CORRESPONDING TO A SQUARE MATRIX

Let A be an nxn matrix over a field F. A non-zero vector \( \mathbf{x} \in \mathbb{F}^n \) is an eigen vector or a characteristic vector of A if there exists a scalar \( a \in \mathbb{F} \) such that \( A\mathbf{x} = a\mathbf{x} \) holds. Thus \((A-cI_n)x = 0\) holds. This is a homogeneous system of n equations in n unknowns. If \( \det(A-cI_n) \neq 0 \), then by Cramer’s rule, \( x = 0 \) will be the only solution. Since we are interested in non-zero solution, \( \det(A-cI_n) = 0 \) equation is called the characteristic equation of A.

A root of the above equation equation in a, is called an eigen value of A.

Example: let \( A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \). The characteristic equation is \( \begin{vmatrix} 1-a & 3 \\ 4 & 5-a \end{vmatrix} = 0 \), or

\[ a^2 - 6a - 7 = 0 \text{, thus eigen values are -1,7. The eigenvector } \begin{bmatrix} x \\ y \end{bmatrix} \text{ corresponding to the eigenvalue -1 is given by } \begin{bmatrix} 1 \\ 4 \\ 5 \\ y \end{bmatrix} = 1 \begin{bmatrix} x \\ y \end{bmatrix}. \text{ Thus } 2x + 3y = 0, 4x + 6y = 0. \text{ Thus } x = -3y/2. \text{ hence the eigenvector corresponding to -1 is } \begin{bmatrix} x \\ y \end{bmatrix} = k \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \text{ where } k \neq 0. \text{ Similarly eigenvector corresponding to eigenvalue 7 can be found.}

Theorem: The eigen values of a diagonal matrix are its diagonal elements.

Theorem: If \( c \) is an eigen value of a nonsingular matrix A, then \( c^{-1} \) is an eigen value of \( A^{-1} \).

Theorem: If A and P be both nxn matrices and P be non-singular, then A and \( P^{-1}AP \) have the same eigen values.

Theorem: To an eigen vector of A, there corresponds a unique eigen value of A.

» if possible, let there be two distinct eigen values \( c_1 \) and \( c_2 \) of A corresponding to an eigen vector \( \mathbf{x} \). Thus \( A\mathbf{x} = c_1\mathbf{x} = c_2\mathbf{x} \). hence \((c_1-c_2)\mathbf{x} = 0\); but this is a contradiction since \( a_1 \neq a_2 \) and \( \mathbf{x} \) is non-zero vector.
Theorem: (Cayley Hamilton theorem)

Every square matrix satisfies its own characteristic equation.

Ex: let $A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$. Verify that $A$ satisfies its characteristic equation. Hence find $A^{-1}$.

» The characteristic equation is $x^2 - 7x + 7 = 0$. Now $A^2 - 7A + 7I = 0$ can be verified by actual calculation. Hence Cayley Hamilton theorem is verified. Hence $A^{-1} = -\frac{1}{7} (A - 7I_2) = I_2$. Thus $A^{-1} = \frac{1}{7} \begin{bmatrix} 5 & -1 \\ -3 & 2 \end{bmatrix}$.

REAL QUADRATIC FORM

An expression of the form $\sum_{i,j} a_{ij}x_i x_j$ (i,j = 1,2,...,n) where $a_{ij}$ are real and $a_{ij} = a_{ji}$, is said to be a real quadratic form in n variables $x_1,x_2,...,x_n$. The matrix notation for the quadratic form is $x^T A x$, where $x = [x_1 \ x_2 \ ... \ x_n]^T$, $A = (a_{ij})_{n\times n}$. $A$ is a real symmetric matrix since $a_{ij} = a_{ji}$ for all i,j. $A$ is called the matrix associated with the quadratic form.

Example 2.27 $x_1x_2-x_2x_3$ is a real quadratic form in three variables $x_1,x_2,x_3$. The associated matrix is

$$
\begin{bmatrix}
0 & 1/2 & 0 \\
1/2 & 0 & -1/2 \\
0 & -1/2 & 0
\end{bmatrix}.
$$
Definition: A real quadratic form $Q = x^T A x$ is

(1) Positive definite if $Q > 0$ for all $x \neq 0$
(2) Positive semi definite if $Q \geq 0$ for all $x \neq 0$
(3) Negative definite if $Q < 0$ for all $x \neq 0$
(4) Negative semidefinite if $Q \leq 0$ for all $x \neq 0$
(5) Indefinite if $Q \geq 0$ for some $x \neq 0$ and $Q \leq 0$ for some other $x \neq 0$

Example: consider the quadratic form $Q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 4x_3^2 + 2x_1x_2 - 4x_2x_3 - 2x_3^2 = (x_1 + x_2 - x_3)^2 + (x_2 - x_3)^2 + 2x_3^2 \geq 0$ and $Q = 0$ only when $x_1 + x_2 - x_3 = x_2 - x_3 = x_3 = 0$, that is, when $x_1 = x_2 = x_3 = 0$. Thus $Q$ is positive definite.

For a real quadratic form $Q = X^T A X$ where $A$ is real symmetric matrix of rank $r (\leq n)$, there exists a non-singular matrix $P$ such that $P^T A P$ becomes a diagonal matrix $\begin{bmatrix} I_m & & \\ & -I_{r-m} & \\ & & 0 \end{bmatrix}$ of rank $r$, where $0 \leq m \leq r$. Thus by a suitable transformation $X = P Y$, where $P$ is nonsingular, the real quadratic form $Q$ transforms to $y_1^2 + \ldots + y_m^2 - y_{m+1}^2 - \ldots - y_r^2$ where $0 \leq m \leq r \leq n$. This is called normal form of $Q$.

Theorem: A real quadratic form of rank $r$ and index $m$ is

(1) Positive definite, if $r = n, m = r$
(2) Positive semidefinite, if $r < n, m = r$
(3) Negative definite, if $r = n, m = 0$
(4) Negative semidefinite, if $r < n, m = 0$
(5) Indefinite, if $r \leq n, 0 < m < r$
Example: Reduce the quadratic form $5x^2+y^2+14z^2-4yz-10zx$ to its normal form and show that it is positive definite.

The associated symmetric matrix is $A = \begin{bmatrix} 5 & 0 & -5 \\ 0 & 1 & -2 \\ -5 & -2 & 14 \end{bmatrix}$. Let us apply congruent operations on $A$ to reduce it to the normal form.

\[
A \xrightarrow{R_3+R_1} \begin{bmatrix} 5 & 0 & -5 \\ 0 & 1 & -2 \\ -2 & 9 \end{bmatrix} \xrightarrow{c_3+c_1} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 9 \end{bmatrix} \xrightarrow{R_3+2R_2} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow{c_3+2c_2} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}
\]

\[
\frac{1}{\sqrt{5}}R_1, \frac{1}{\sqrt{5}}R_3 \rightarrow \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} \xrightarrow{\frac{1}{\sqrt{5}}c_1, \frac{1}{\sqrt{5}}c_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

The normal form is $x^2+y^2+z^2$. The rank of the quadratic form is 3 and its signature is 3. Thus the quadratic form is positive definite.
We will be discussing two types of co-ordinate system, namely, rectangular Cartesian co-ordinate system and polar co-ordinate system.

**Rectangular Cartesian co-ordinate system**

We take two directed lines $X'OX$ and $Y'OY$ in a plane at right angles to each other, intersecting at $O$ and with their positive directions from $O$ to $X$ and $O$ to $Y$ respectively. We call $O$ the origin, $X'OX$ the $x$-axis and $Y'OY$ the $y$-axis.

Next we choose a unit for measuring distances along the $x$ and $y$ axes, $O$ being the origin for both. Let $P$ be an arbitrary point in the plane. Draw perpendiculars from $P$ on the lines $X'/OX$ and $Y'/OY$, intersecting them at $M$ and $N$ respectively. The distances of the point $P$ along the $x$ and $y$ axes, that is $OM = x$ and $MP = y$ are called respectively the $x$ co-ordinate and $y$ co-ordinate of $P$, and $x$ and $y$ are together called the rectangular Cartesian co-ordinates of $P$, $x$ being called the abscissa and $y$ the ordinate of $P$. If $x$ and $y$ are the abscissa and the ordinate of a point, then the point is denoted by $(x,y)$. The position of a point $P$ in the plane is completely determined by its rectangular Cartesian co-ordinates. Each point of the plane will have a completely determined
pair of co-ordinates \( x,y \) in the system. Conversely, for any two real numbers \( x \) and \( y \), there will exist a completely determined point \( P \) in the plane, whose abscissa is \( x \) and the ordinate is \( y \) in the co-ordinate system. The \( x \) and \( y \) axes together are called the co-ordinate axes.

**Half-planes and Quadrants**

It is customary to take the \( x \)-axis of a rectangular Cartesian co-ordinate system as horizontal and the direction to the right as its positive direction; the \( y \) axis is taken as vertical and the upward direction as positive. A straight line divides a plane into two parts each of which is called a half-plane. Thus the \( y \) axis divides the plane into two half-planes; the half-plane containing the positive half of the \( x \)-axis is called the right half-plane while the other is called the left half-plane. Similarly, the \( x \)-axis divides the plane into two half-planes; the half-plane containing the positive half of the \( y \)-axis is called the upper half-plane while the other is called the lower half-plane. The two co-ordinate axes divide the plane into four parts, each of which is called a quadrant. They are numbered first, second, third and fourth in the following manner:

The first quadrant is both in the right and the upper half planes.

The second quadrant is both in the left and the upper half planes.

The third quadrant is both in the left and the lower half planes.

The fourth quadrant is both in the right and the lower half planes.

**Projection of a directed segment on a directed line**

Let \( \overline{PQ} \) be an arbitrary directed segment and \( g \) be a directed line. From \( P \) and \( Q \) draw
perpendiculars on g to meet it at A and B respectively. Then the value of the segment $\overline{AB}$ of g is called the projection of the segment $\overline{PQ}$ on g.

Some results

1. If $P(x_1,y_1)$ and $Q(x_2,y_2)$ are any two points in a plane, then the projection of $\overline{PQ}$ on the x and y axes are $x_2 - x_1$ and $y_2 - y_1$ respectively.
2. Let P and Q be two distinct points of a plane and g be the straight line determined by them. Let g be made a directed line by assigning a positive direction on it. Now choose a third point R on g different from P and Q. Consider the number $k$ determined by the relation $k = \frac{PR}{RQ}$. Then $k$ is called the ratio in which R divides the directed line segment $\overline{PQ}$.
3. If R is a point of the segment $\overline{PQ}$, then $k$ is positive, while if it lies on g but outside $\overline{PQ}$, then $k$ is negative.
4. If $P(x_1,y_1)$ and $Q(x_2,y_2)$ are any two points in a plane and a point $P(x,y)$ divides the segment $\overline{PQ}$ in the ratio $k$, then $x = \frac{x_1 + kx_2}{1 + k}$ and $y = \frac{y_1 + ky_2}{1 + k}$, $1 + k \neq 0$.

Concept of locus and its equation

Let P be a variable point which obeys certain geometric conditions for its different positions. Then the set of points corresponding to its different positions is said to form the locus of the point.

For example let a variable point P obey the geometric condition that its distance
from the origin is always one. Then the set of points corresponding to its different positions form the circle whose centre is the origin and radius is unity. Hence the locus of P is a circle.

Every locus has a defining geometric condition. This condition can be expressed in terms of the co-ordinates of every point on the locus. Thus the geometric condition for the locus considered above can be stated as \( \sqrt{x^2 + y^2} = 1 \), where \((x,y)\) are the rectangular Cartesian co-ordinates of a point on the locus. This relation is an equation because it is not valid identically i.e., not valid for any pair of numbers \(x, y\). The relation is satisfied by the co-ordinates of every point on the locus (unit circle) and is not satisfied by any point not on the locus. Thus the geometric condition defining a locus leads to an equation involving the co-ordinates of any point on it. Such an equation is called the equation of the locus which is satisfied by the co-ordinates of any point on it and is not satisfied by the co-ordinates of any point that is not on the locus.

**Transformation of Rectangular Cartesian co-ordinates**

In analytic geometry the position of a geometric entity is determined with the help of a co-ordinate system. Sometimes the replacement of a co-ordinate system by another is found to be more convenient for treatment of different problems in geometry. Hence, if in a certain problem two co-ordinate systems are to be used and the co-ordinates of a point with respect to one co-ordinate system be known, it becomes necessary to find the co-ordinates of the point with respect to the other co-ordinate system. This is done by means of co-ordinate transformation formulas corresponding to the required change in the co-ordinate system.

We shall establish the formulas corresponding to the change in the co-ordinate system of the following types:

1. when the origin is changed but the direction of the axes and the scale is not changed.
2. When both the axes are turned through the same angle in the same sense, but the origin and the scale are not changed.
(3) When the change of co-ordinate system consists of a combination of a change of type (1) and a change of type (2) taken in either order. In cases (1), (2) and (3) we say that the formulas are respectively those for
(i) transformation by translation of axes or simply translation
(ii) transformation by rotation of axes or simply rotation and
(iii) transformation by a combination of a translation and rotation of axes taken in either order.

**Formula for translation of axes**

Let O and O' be the old and new origins respectively, OX, OY be the old coordinate axes and O'X' and O'Y' be the new co-ordinate axes. Denote the co-ordinates of O' with respect to the old axes by (a, b). Let (x, y) and (x', y') be the co-ordinates of any point P with respect to the old and new co-ordinate systems. Denote the projection of O' on OX by Q and the projection of P on OX and O'X' by R and S respectively. Then QR = O'S = x'. Also since OQ = a and OR = x. As OR = OQ + QR, therefore x = a + x' = x' + a . Similarly, y = y' + b.
Hence the required transformation formulas are
x = x' + a, y = y' + b.
The above formula can also be written as
x' = x - a, y' = y - b

**Formula for rotation of axes**

Let OX, OY be the old coordinate axes and O'X', and O'Y' be the new co-ordinate axes and let α be the angle through which the old axes are rotated to coincide with the new axes. Let (x, y) and (x', y') be the co-ordinates of an arbitrary point P with respect to the old and new co-ordinate systems.
Let the projections of $P$ on $OX$ and $OY$ be $Q$ and $R$ and on $OX'$ and $OY'$ be $S$ and $T$ respectively. Then $OQ = x, PQ = OR = y, OS = x'$ and $PS = OT = y'$.

From the diagram it is clear that $\theta = \theta' + \alpha$. Also, $x = r \cos \theta, y = r \sin \theta, x' = r \cos \theta', y' = r \sin \theta'$, where $OP = r$.

Hence $x = r \cos (\theta' + \alpha) = r (\cos \theta' \cos \alpha - \sin \theta' \sin \alpha) = x' \cos \alpha - y' \sin \alpha$.

Similarly, $y = r \sin (\theta' + \alpha) = r (\sin \theta' \cos \alpha + \cos \theta' \sin \alpha) = y' \cos \alpha + x' \sin \alpha$.

Hence the required transformation formulas are $x = x' \cos \alpha - y' \sin \alpha$ and $y = y' \cos \alpha + x' \sin \alpha$.

Since the new co-ordinate system is obtained by a rotation of the old co-ordinate system through an angle $\alpha$, the old system will be obtained by a rotation of the new system through an angle $-\alpha$. Hence interchanging the old and the new co-ordinates in the above formula, and replacing $\alpha$ by $-\alpha$ we get $x' = x \cos \alpha + y \sin \alpha$ and $y' = -x \sin \alpha + y \cos \alpha$.

**Formula for a combination of a translation and rotation of axes in either order**

Let the axes $OX$ and $OY$ be first translated to the point $O'$ whose co-ordinates with respect to $OX$ and $OY$ are $(a, b)$ and let $O'X'$ and $O'Y'$ be the positions of $OX$ and $OY$, after being shifted by translation.

Let the axes $O'X'$ and $O'Y'$ be then rotated through an angle $\alpha$ and let $O''X'$, $O''Y'$ be $O''$.

The positions of $O'X'$ and $O'Y'$ after rotation.

Let $P$ be an arbitrary point and $(x, y), (x', y')$ be its co-ordinates with respect to the old and new co-ordinate systems. We now consider the co-ordinate system whose origin is $O'$ and whose axes are $O'X'$ and $O'Y'$ have the same direction as that of
OX and OY. Denote the co-ordinates of P with respect to this system by 
\((x^/, y^/\).\)
Then \(x = x^/ + a, y = y^/ + b\)
Also \(x^/ = x^/ \cos \alpha - y^/ \sin \alpha\) and \(y^/ = y^/ \cos \alpha + x^/ \sin \alpha\)
Combining, \(x = x^/ \cos \alpha - y^/ \sin \alpha + a\) and \(y = y^/ \cos \alpha + x^/ \sin \alpha + b\)
which are the required formula.
From these two results we can solve for \(x^/\) and \(y^/\) and get
\[x^/ = (x - a) \cos \alpha + (y - b) \sin \alpha\] and
\[y^/ = -(x - a) \sin \alpha + (y - b) \cos \alpha.\]
If the rotation and translation are performed in the reverse order, the above formula still remains valid.

The transformation formulas deduced above can also be represented in matrix form. We give the matrix form of the above formulas below.

Let \(X\) denote the column matrix \((x, y)\) and \(X^/\) denote the column matrix \((x^/, y^/)\) and 
\(B\) denote the column matrix \((a, b)\).
Then the formula \(x = x^/ + a, y = y^/ + b\) can be expressed as \(X = AX^/ + B\)
where \(A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \)
\(x = x^/ \cos \alpha - y^/ \sin \alpha, y = y^/ \cos \alpha + x^/ \sin \alpha\) can be expressed as \(X = TX^/\)
where \(T = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}\) and
\(x = x^/ \cos \alpha - y^/ \sin \alpha + a\) and \(y = y^/ \cos \alpha + x^/ \sin \alpha + b\) can be expressed as 
\(X = TX^/ + B,\)
From the above three formulas we can deduce
\[X^/ = A^{-1}X - A^{-1}B,\]
\[X^/ = T^{-1}X,\]
\[X^/ = T^{-1}X - T^{-1}B\text{ where }A^{-1}\text{ and }T^{-1}\text{ denote the inverse of }A\text{ and }T\text{ respectively. It is obvious that the matrices }A\text{ and }T\text{ are orthogonal and the determinant of each of them is 1. For this reason each of the transformations – translation, rotation and a combination of rotation and translation is called an orthogonal transformation. Therefore the formula for an orthogonal transformation in general can be}
written as

\[ x = px' - qy' + r , \ y = qx' + py' + s . \ p^2 + q^2 = 1. \]

If \( p = 1, q = 0 \) then it is a translation and
if \( r = 0, s = 0 \) then it is a rotation.

**Note:** The degree of an equation in \( x, y \) is an invariant under orthogonal transformation of co-ordinate axes.

**Some invariants**

1. If by an orthogonal transformation of co-ordinate axes the co-ordinates of two points \( P(x_1, y_1), Q(x_2, y_2) \) becomes \( (x'_1, y'_1), (x'_2, y'_2) \) respectively, then

\[
(x'_1 - x'_2)^2 + (y'_1 - y'_2)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2
\]

2. If by an orthogonal transformation of co-ordinate axes the co-ordinates of three points \( P(x_1, y_1), Q(x_2, y_2), R(x_3, y_3) \) becomes \( (x'_1, y'_1), (x'_2, y'_2) \),

\[
\begin{pmatrix}
  x'_1 & y'_1 & 1 \\
  x'_2 & y'_2 & 1 \\
  x'_3 & y'_3 & 1 \\
\end{pmatrix}
= \begin{pmatrix}
  x_1 & y_1 & 1 \\
  x_2 & y_2 & 1 \\
  x_3 & y_3 & 1 \\
\end{pmatrix}
\]

3. If by a transformation of rotation of the co-ordinate axes the expression \( ax^2 + 2hxy + by^2 \) is changed to \( a'x'^2 + 2h'x'y' + b'y'^2 \), then

\[
a + b = a' + b', \ ab - h^2 = a'b' - h'^2.
\]

All the above results can be verified using the transformation formulas. We give below the proof of the result 3 using matrices.

Let \( D = \begin{pmatrix} a & h \\ h & b \end{pmatrix} \) and \( D' = \begin{pmatrix} a' & h' \\ h' & b' \end{pmatrix} \).

We consider a rotation of co-ordinate axes given by \( X = TX' \) where \( X = \begin{pmatrix} x \\ y \end{pmatrix} \). Then

\[
ax^2 + 2hxy + by^2 = (x \ y) \begin{pmatrix} a & h \\ h & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = X^*DX, \] where \( X^* \) is the transpose of \( X \).

Putting \( X = TX' \) we get

\[
X^*DX = (TX')^*D(TX') = X'^*(T^*DT)X' = X'^*D/X' \] where \( D' = T^*DT \)

Since \( T \) is orthogonal, \( T^* = T^{-1} \), so \( T^*DT = T^{-1}DT \). Hence \( T^*DT \) is similar to \( T \).
Since similar matrices have the same trace, so, \( \text{trace } T^*DT = \text{trace } D \)
or, \( \text{trace } D' = \text{trace } D \)
i.e. \( a' + b' = a + b \)
Again \( \det D' = \det T^*DT = \det T^* \det D \det T = 1 \cdot \det D \cdot 1 = \det D \)
as \( T \) is orthogonal, \( \det T = \det T^* = 1 \)
Hence \( a' b' - h'^2 = ab - h^2 \).

General equation of second degree

Second order curve

The curve represented by a general equation of the second degree in \( x \) and \( y \) is called a second order curve.
The general equation of the second degree in \( x \) and \( y \) is usually written in the form
\[ ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \]
The above equation can also be written as
\[ x(ax + hy + g) + y(hx + by + f) + (gx + fy + c) = 0. \]
This form will often be found to be useful.

Centre of a second order curve

A centre of a second order curve is a point defined as follows:
If there is a point \( S \) such that where \( P \) is a point on the curve, another point \( P' (\neq P) \) lying on the line \( SP \) and satisfying \( |SP| = |SP'| \), is also a point on the curve, then \( S \) is called a centre of the curve.
Consider the equation \( 3x^2 - 4xy - 2y^2 + 3x - 12y - 7 = 0 \).
Shifting origin to \( O' \left( -\frac{3}{2}, -\frac{3}{2} \right) \) by the translation \( x = x' - \frac{3}{2}, y = y' - \frac{3}{2} \), the above equation takes the form
\[ 12x'^2 - 16x'y' - 8y'^2 - 1 = 0. \]
It is easy to verify that if \( (x_1, y_1) \) is a point on the curve represented by the transformed equation, and then \( (-x_1, -y_1) \) is also a point on the curve. Hence \( O' \) is a centre of the curve and so \( \left( -\frac{3}{2}, -\frac{3}{2} \right) \) is a centre of the original curve.
We state below an important result without proof.

A point \((x_1, y_1)\) is a centre of the curve represented by 
\[ ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \]
if and only if the co-ordinates \((x_1, y_1)\) satisfy the equations 
\[ +hy + g = 0, hx + by + f = 0. \]

**Central second order curve**

A second order curve having a single centre is called a central second order curve. From the previous result, the curve represented by 
\[ ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \]
is a central second order curve if the system of equations 
\[ ax + hy + g = 0, hx + by + f = 0 \]
has a unique solution. The condition for such a situation is \(ab - h^2 \neq 0\) and when this condition is fulfilled, the centre is 
\[ \left( \frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2} \right). \]

**Canonical form of the equation of a second order curve**

Given an equation of a second order curve in the general form, it is required
(i) to remove the term in \(xy\) from the equation,
(ii) to reduce the number of terms of degree one in \(x, y\) to a minimum,
(iii) to remove the constant term also, if possible, by transformation of co-ordinates.
If an equation of a second order curve fulfils these conditions, then it is called its canonical equation. The form of the canonical equation is called its canonical form or normal form.

**Reduction of the equation of a central second order curve to its canonical form**

Let us consider a central second order curve given by the equation
\[ ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \]  
(1)

Then \(ab - h^2 \neq 0\).

Denoting the centre of the curve by \((x_0, y_0)\) and applying the translation \(x = x' + x_0, y = y' + y_0\), the equation becomes
\[ ax'^2 + 2hxy + by'^2 + D = 0 \]  
(2)
Next we apply the rotation \( x' = Xc\cos \alpha - Y\sin \alpha, \ y' = X\sin \alpha + Y\cos \alpha. \)
Then (2) takes the form
\[
\begin{align*}
\alpha(X\cos \alpha - Y\sin \alpha)^2 + 2h(X\cos \alpha - Y\sin \alpha)(X\sin \alpha + Y\cos \alpha) + b(X\sin \alpha + Y\cos \alpha)^2 + D = 0
\end{align*}
\]
or,
\[
\begin{align*}
X^2(\alpha\cos^2 \alpha + 2h\sin \alpha \cos \alpha + b\sin^2 \alpha) + 2XY(\alpha\cos^2 \alpha - h\sin^2 \alpha - \alpha \sin \alpha \cos \alpha + b\sin \alpha \cos \alpha) + Y^2(\alpha \sin^2 \alpha + b\sin^2 \alpha - 2h\sin \alpha \cos \alpha) + D = 0
\end{align*}
\]
We choose \( \alpha \) such that \( \alpha \cos^2 \alpha - h\sin^2 \alpha - \alpha \sin \alpha \cos \alpha + b\sin \alpha \cos \alpha = 0. \)
Then the above equation reduces to
\[
\begin{align*}
X^2(\alpha\cos^2 \alpha + 2h\sin \alpha \cos \alpha + b\sin^2 \alpha) + Y^2(\alpha \sin^2 \alpha + b\sin^2 \alpha - 2h\sin \alpha \cos \alpha) + D = 0
\end{align*}
\]
or,
\[
\begin{align*}
a'/X^2 + b'/Y^2 + D = 0 \tag{3}
\end{align*}
\]
where \( ab - h^2 = a'/b' - h^2 \) (invariants of the transformation)
or, \( ab - h^2 = a'/b' 
eq 0 \neq h^2 \) \( \ [\because \ h = 0] \).
Hence \( a' \neq 0, b' \neq 0. \)
So (3) is the required canonical form of the curve where \( a' \neq 0, b' \neq 0. \)

**Types of central second order curve**

Equation (3) is the canonical form of a central second order curve \( 1 \)
Here \( \delta = ab - h^2 \neq 0. \)
There are two types of curve according as \( \delta > 0 \) or \( \delta < 0. \) The curve is said to be elliptic or hyperbolic according as \( \delta > 0 \) or \( \delta < 0. \)

Case I First we consider an elliptic type of curve \( a'/X^2 + b'/Y^2 + D = 0 \)
where \( \delta > 0. \)
Then \( ab - h^2 = a'/b' > 0. \)
Hence \( a' \) and \( b' \) are of the same sign.
Without loss of generality we take both \( a' \) and \( b' \) to be positive.
Now we consider the following three cases:
(i) $D > 0$ , (ii) $D < 0$ , (iii) $D = 0$

The equation can be written as $\frac{a/x^2}{-D} + \frac{b/y^2}{-D} = 1$ .

Case (i) Here $D > 0$ , therefore $\frac{a}{-D}$ , $\frac{b}{-D}$ are both negative.

So we may write them as $-\frac{1}{A^2}$ and $-\frac{1}{B^2}$ respectively.

Hence (2) assumes the form $\frac{x^2}{A^2} + \frac{y^2}{B^2} = -1$.

In this case the equation represents no geometric locus, sometimes also called an imaginary ellipse.

Case (ii) Here $D < 0$ , therefore $\frac{a}{-D}$ , $\frac{b}{-D}$ are both positive.

So we may write them as $\frac{1}{A^2}$ and $\frac{1}{B^2}$ respectively.

Hence (2) assumes the form $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$.

In this case the equation represents an ellipse.

Case (iii) Here (2) takes the form $a/x^2 + b/y^2 = 0$ .

There is only one point on this locus, namely the point (0,0) . The curve is called a point ellipse.

Case II Now we consider a hyperbolic type of curve $a/ X^2 + b/Y^2 + D = 0$ where $\delta < 0$ .

Then $ab - h^2 = a/b/ < 0$.

Hence $a/ and $b/ are of opposite signs.

Without loss of generality we take $a/ to be positive and $b/ to be negative.

Again we consider the three cases:
(i) $D > 0$ , (ii) $D < 0$ , (iii) $D = 0$

Case (i) Here $D > 0$ , then $\frac{a/}{-D}$ is negative, while $\frac{b/}{-D}$ is positive.

So we can write them as $-\frac{1}{A^2}$ and $\frac{1}{B^2}$ respectively.

The equation then takes the form $-\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$ which is a hyperbola.
Case (ii) Here $D < 0$, therefore \( \frac{a'}{-D} \) is positive but \( \frac{b'}{-D} \) is negative.
So we may write them as \( \frac{1}{A^2} \) and \( -\frac{1}{B^2} \) respectively.
Hence (2) assumes the form \( \frac{X^2}{A^2} - \frac{Y^2}{B^2} = 1 \) which is again a hyperbola.
Case (iii) Here (2) takes the form \( a'X^2 + b'Y^2 = 0 \)
or, \( Y^2 + \frac{a'}{b'}X^2 = 0 \).
Since \( a' \) and \( b' \) are of opposite signs, we write \( \frac{a'}{b'} \) as \( -\lambda^2 (\lambda \neq 0) \).
Hence the equation takes the form \( Y^2 - \lambda^2 X^2 = 0 \) which represents two intersecting straight lines.

Reduction of the equation of a parabolic type of a curve to its canonical form

Let us consider a central second order curve given by the equation
\[
ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad (1)
\]
where \( \delta = ab - h^2 = 0 \)
Then the curve is said to be of parabolic type.
We apply the rotation \( x = x'\cos\alpha - y'\sin\alpha \), \( y = x'\sin\alpha + y'\cos\alpha \)
Then equation (1) is changed into
\[
a(x'\cos\alpha - y'\sin\alpha)^2 + 2h(x'\cos\alpha - y'\sin\alpha)(x'\sin\alpha + y'\cos\alpha) +
b(x'\sin\alpha + y'\cos\alpha)^2 + 2g(x'\cos\alpha - y'\sin\alpha) + 2f(x'\sin\alpha + y'\cos\alpha) +
c = 0
\]
or, \(x'^2(acos^2\alpha + 2hsinacos\alpha + bsin^2\alpha) + 2x'/y'(hcos^2\alpha - hsin^2\alpha +
(b - a)sinacos\alpha) + y'^2(asin^2\alpha + bcos^2\alpha - 2hsinacos\alpha) + 2x'(gcos\alpha +
f\sin\alpha) + 2y'(f\cos\alpha = g\sin\alpha) + c = 0 \)
or, \(a'x'^2 + 2h/x'y' + b'y'^2 + 2g/x' + 2f/y' + c = 0 \quad (2)
We now choose \( \alpha \) such that \( h(cos^2\alpha - sin^2\alpha) + (b - a)sinacos\alpha = 0 \quad (3)\)
Then the equation (2) takes the form
\[
a'x'^2 + b'y'^2 + 2g/x' + 2f/y' + c = 0 \quad (4)
\]
where \( a'/b' - h'^2 = ab - h^2 \) (by invariants of an orthogonal transformation)
or, \( a'/b' = ab - h^2 = 0 \quad (since \ h' = 0 ) \quad (5)\)
Since the equation is of degree 2, therefore \( a' = 0 \) and \( b' = 0 \) cannot hold simultaneously. Hence it follows that either \( a' = 0 \) or \( b' = 0 \).

So we consider the following cases:

(i) \( a' = 0 \) and \( b' \neq 0 \)

(ii) \( a' \neq 0 \) and \( b' = 0 \)

**Case (i)** In this case equation (4) can be written as

\[
b'/y'^2 + 2g'/x' + 2f'/y' + c = 0
\]  
(6)

Supposing \( g' \neq 0 \), (6) can be expressed as

\[
\left(y' + \frac{f'}{b'}\right)^2 + \frac{2g'}{b'}\left(x' + \frac{b'c - f'2}{2g'\cdot b'}\right) = 0
\]  
(7)

Applying the transformation

\[X = x' + \frac{b'c - f'2}{2g'\cdot b'}\quad \text{and} \quad Y = y' + \frac{f'}{b'},\]

equation (7) takes the form \( Y^2 = \lambda X \)

where \( \lambda = -\frac{2g'}{b'} \neq 0 \), which represents a parabola.

Next suppose \( g' = 0 \). Then (6) becomes

\[
b'/y'^2 + 2f'/y' + c = 0
\]

or,

\[
\left(y' + \frac{f'}{b'}\right)^2 + \frac{c}{b'} - \frac{f'2}{b'^2} = 0
\]

or,

\[
\left(y' + \frac{f'}{b'}\right)^2 = \frac{f'2 - b'c}{b'^2}
\]

(9)

Applying the transformation

\[X = x'\quad \text{and} \quad Y = y' + \frac{f'}{b'},\]

equation (9) becomes \( Y^2 = k \)

(10)

where \( k = \frac{f'2 - b'c}{b'^2} \).

The equation (10) represents a pair of parallel straight lines, no locus or a pair of coincident straight lines according as \( k > 0, < 0 \) or \( k = 0 \).

**Case (ii)** Here (4) becomes

\[
a'/x'^2 + 2g'/x' + 2f'/y' + c = 0
\]  
(11)

It can be shown in a similar manner as in case (i) that when \( f' \neq 0 \), equation (11) represents a parabola, and when \( f' = 0 \), it represents a pair of parallel straight lines, no locus or a pair of coincident straight lines according as \( g'^2 - a'/c \) is >, < or = 0.
Note: When $a' = 0, b' \neq 0, g' \neq 0$, then $\Delta' = -b'g'/2 \neq 0$, hence $\Delta \neq 0$ (refer to example 5 under invariants).
Similarly for the other cases the conditions involving $\Delta$ and $\delta$ can be obtained.

A complete classification of second order curves according to different possible values of $\Delta$ and $\delta$ is given below in a tabular form.

<table>
<thead>
<tr>
<th>$\delta = ab - h^2$</th>
<th>$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta &gt; 0$</td>
<td>$\Delta &lt; 0$</td>
<td>Ellipse</td>
</tr>
<tr>
<td>$\delta &gt; 0$</td>
<td>$\Delta &gt; 0$</td>
<td>No geometric locus/Imaginary ellipse</td>
</tr>
<tr>
<td>$\delta &gt; 0$</td>
<td>$\Delta = 0$</td>
<td>Point ellipse</td>
</tr>
<tr>
<td>$\delta &lt; 0$</td>
<td>$\Delta &lt; 0$</td>
<td>Hyperbola</td>
</tr>
<tr>
<td>$\delta &lt; 0$</td>
<td>$\Delta &gt; 0$</td>
<td>Hyperbola</td>
</tr>
<tr>
<td>$\delta &lt; 0$</td>
<td>$\Delta = 0$</td>
<td>A pair of intersecting straight lines</td>
</tr>
<tr>
<td>$\delta = 0$</td>
<td>$\Delta \neq 0$</td>
<td>Parabola</td>
</tr>
<tr>
<td>$\delta = 0$</td>
<td>$\Delta = 0$</td>
<td>a pair of parallel straight lines, no locus or a pair of coincident straight lines according as $\lambda &gt;, &lt;$ or $= 0$ or $k &gt; 0, &lt; 0$ or $= 0$</td>
</tr>
</tbody>
</table>
Rank of a second order curve

For a second order curve \( ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \), the rank of the matrix
\[
\begin{pmatrix}
    a & h & g \\
h & b & f \\
g & f & c
\end{pmatrix}
\]
will be called the rank of the curve.

A second order curve is said to be non-singular, singular or degenerate according as its rank is 3, 2 or 1. From the table above we see that an ellipse, a hyperbola and a parabola are non-singular, a point ellipse, a pair of intersecting straight lines and a pair of parallel straight lines are singular, while a pair of coincident straight lines is degenerate.

Matrix approach for reduction of the equation of a second order curve to its canonical form

Consider the vector space \( R^3 \) and two ordered bases \( B = \{\alpha_1, \alpha_2, \alpha_3\} \) and \( B'/B = \{\alpha'_1, \alpha'_2, \alpha'_3\} \). Let \( \xi \) be an arbitrary vector in \( R^3 \) with co-ordinates \((x_1, x_2, x_3)\) relative to the basis \( B \) and \((x'_1, x'_2, x'_3)\) relative to the basis \( B'/B \).

Then \( \xi = x_1 \alpha_1 + x_2 \alpha_2 + x_3 \alpha_3 = x'_1 \alpha'_1 + x'_2 \alpha'_2 + x'_3 \alpha'_3 \)

Let
\[
\begin{align*}
    \alpha'_1 &= a_{11} \alpha_1 + a_{21} \alpha_2 + a_{31} \alpha_3 \\
    \alpha'_2 &= a_{12} \alpha_1 + a_{22} \alpha_2 + a_{32} \alpha_3 \\
    \alpha'_3 &= a_{13} \alpha_1 + a_{23} \alpha_2 + a_{33} \alpha_3
\end{align*}
\]
Then \( \xi = x'_1 \alpha'_1 + x'_2 \alpha'_2 + x'_3 \alpha'_3 \)
\[
= x'_1 (a_{11} \alpha_1 + a_{21} \alpha_2 + a_{31} \alpha_3) \\
+ x'_2 (a_{12} \alpha_1 + a_{22} \alpha_2 + a_{32} \alpha_3) \\
+ x'_3 (a_{13} \alpha_1 + a_{23} \alpha_2 + a_{33} \alpha_3)
\]
Since \( \{ \alpha_1, \alpha_2, \alpha_3 \} \) is a linearly independent set, therefore

\[
\begin{align*}
    x_1 &= a_{11}x'_1 + a_{12}x'_2 + a_{13}x'_3 \\
    x_2 &= a_{21}x'_1 + a_{22}x'_2 + a_{23}x'_3 \\
    x_3 &= a_{31}x'_1 + a_{32}x'_2 + a_{33}x'_3
\end{align*}
\]

So we can write \( X = AX' \) where \( X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, A = (a_{ij})_{3,3}, X' = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} \).

Therefore \( X = AX' \) gives a transformation of co-ordinates from the basis \( B \) to the basis \( B' \). \( A \) is obviously a non-singular matrix and hence the transformation is a non-singular transformation. This transformation is said to be orthogonal if \( A \) is an orthogonal matrix.

In Euclidean geometry the quadratic forms appear in connection with conics, quadric surfaces etc.

In the plane, the equation \( ax^2 + 2hxy + by^2 = k \) represents a conic. The left hand side expression is a quadratic form in two variables \( x,y \).

In 3-space, the equation \( ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = k \) represents a quadric surface. The left hand side expression is a quadratic form in three variables \( x,y,z \). The matrix notation of the equation \( ax^2 + 2hxy + by^2 = k \) is \( X^\top AX = kl_1 \) where \( X = \begin{pmatrix} x \\ y \end{pmatrix} \) and \( A = \begin{pmatrix} a & h \\ h & b \end{pmatrix} \).

We want to find a suitable non-singular transformation \( X = PX' \) which will reduce the equation to its simplest form in the new variables \( X' \). Since \( A \) is a real symmetric matrix, there exists a non-singular \( P \) such that \( P^\top AP \) is a diagonal matrix. For such a matrix \( P \), the transformed equation will be free from the terms containing \( xy \).

The matrix notation of the equation \( ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = k \) is \( X^\top AX = kl_1 \) where \( X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \) and \( A = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} \). In a similar manner, \( P \) can be obtained.
For example, consider the equation \( x^2 + 6xy - 7y^2 = 1 \).

This equation can be written as \( X^t AX = I_1 \), where \( A = \begin{pmatrix} 1 & 3 \\ 3 & -7 \end{pmatrix} \), \( X' = \begin{pmatrix} x' \\ y' \end{pmatrix} \).

The matrix \( P = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \) is such that \( P^t AP = \begin{pmatrix} 1 & 0 \\ 0 & -16 \end{pmatrix} \).

The transformation \( X = PX' \) where \( X' = \begin{pmatrix} x' \\ y' \end{pmatrix} \), reduces the equation to \( X'^t(P^tAP)X = 1 \), or, \( x'^2 - 16y'^2 = 1 \).

From this we can at once conclude that the equation represents a hyperbola. But \( P \) is not an orthogonal matrix. Therefore the transformation is not an orthogonal transformation. We know that in an orthogonal transformation, distance between any two points remains invariant. Therefore the distance between any two points for a non-orthogonal transformation may not remain the same. Hence the nature of the curve represented by the equation also may not remain the same under the transformation.

Since the purpose of using a transformation is to determine the nature of the curve represented by an equation, we must have an orthogonal transformation. So the matrix \( P \) should be an orthogonal matrix. As \( A \) is a symmetric matrix, from the properties of eigenvalues and eigenvectors we know that there exists an orthogonal matrix \( P \) such that \( P^{-1}AP \) is diagonal. The diagonal elements of \( P^{-1}AP \) are the eigenvalues of \( A \) and the column vectors of \( P \) are n orthonormal eigenvectors of \( A \) (order of the matrix \( A \) being \( n \)).

We refer back to the example considered above. The eigenvalues of
\( A = \begin{pmatrix} 1 & 3 \\ 3 & -7 \end{pmatrix} \) are 2, -8.

The eigen vectors corresponding to 2 are \( c \begin{pmatrix} 3 \\ 1 \end{pmatrix} \), where \( c \neq 0 \).

The eigen vectors corresponding to -8 are \( d \begin{pmatrix} -1 \\ 3 \end{pmatrix} \), where \( d \neq 0 \).

The eigen vectors are normalized by taking \( c = \frac{1}{\sqrt{10}} \), \( d = \frac{1}{\sqrt{10}} \).

Let \( P = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \).

Then \( P^{-1}AP = P^tAP = \frac{1}{10} \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 3 & -7 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -8 \end{pmatrix} \).
By the transformation $X = PX'$, the equation $x^2 + 6xy - 7y^2 = 1$ transforms to $2x'^2 - 8y'^2 = 1$.

The equation represents a hyperbola whose principal axes are $x', y'$ axes which are mutually orthogonal.

**Reduction to canonical form (matrix method)**

(A) In two dimensions

The general equation of second degree in $x, y$ is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

In matrix notation, $X^tAX + BX + cI_1 = 0$.

where $A = \begin{pmatrix} a & h \\ h & b \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $B = \begin{pmatrix} 2g & 2f \end{pmatrix}$.

Since $A$ is a real symmetric matrix, there exists an orthogonal matrix $P$ such that $P^tAP$ is a diagonal matrix $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

where $\lambda_1, \lambda_2$ are the eigenvalues of $A$.

Therefore by the orthogonal transformation $X = PX'$

where $X' = \begin{pmatrix} x' \\ y' \end{pmatrix}$,

the equation transforms to $X'^tDX' + BX' + cI_1 = 0$ or, $\lambda_1 x'^2 + \lambda_2 y'^2 + 2g_1x' + 2f_1y' + c = 0$ (A)

We consider the following cases:

Case I  Rank of A = 2.

In this case $\lambda_1, \lambda_2$ are both non-zero.

Transferring origin to the point $\left( -\frac{g_1}{\lambda_1}, -\frac{f_1}{\lambda_2} \right)$ the equation (A) takes the form $\lambda_1 x^2 + \lambda_2 y^2 + d = 0$.

If $d = 0$, then it represents a pair of straight line if $\lambda_1 \lambda_2 < 0$,
a point ellipse if $\lambda_1 \lambda_2 > 0$, and $\lambda_1 \neq \lambda_2$,
a point circle if $\lambda_1 = \lambda_2$. 

If $d \neq 0$, the equation represents
an ellipse if $\lambda_1\lambda_2 > 0$, $\lambda_1 d < 0$;
a conic without real trace if $\lambda_1\lambda_2 > 0$, $\lambda_1 d > 0$;
a hyperbola if $\lambda_1\lambda_2 < 0$.

Case II  \hspace{1em} \text{Rank A} = 1.
In this case one of $\lambda_1, \lambda_2$ is zero. Let $\lambda_2 = 0$.

Transferring origin to the point $\left(\frac{-g_1}{\lambda_1}, 0\right)$ the equation (A) takes the form
$\lambda_1x^2 + 2f_1y + d = 0$. \hspace{1em} (B)
If $f_1 \neq 0$, transferring origin to the point $\left(0, \frac{-d}{2f_1}\right)$ the equation (B) takes the form
$\lambda_1x^2 + 2f_1y = 0$ which is a parabola.
If $f_1 = 0$, the equation represents
a pair of coincident straight lines if $d = 0$,
a pair of parallel straight lines if $\lambda_1 d < 0$,
a pair of imaginary lines if $\lambda_1 d > 0$.

(B) \textbf{In three dimensions}

The general equation of second degree in $x, y, z$ is
$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx + 2ux + 2vy + 2wz + d = 0$. In
matrix notation, $X^tAX + BX + dl = 0$, where

$$A = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = (2u \hspace{1em} 2v \hspace{1em} 2w).$$

Since $A$ is a real symmetric matrix, there exists an orthogonal matrix $P$ such that

$P^tAP$ is a diagonal matrix

$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$, where $\lambda_1, \lambda_2, \lambda_3$ are the eigen
values of $A$.
Therefore by the orthogonal transformation $X = PX'$
where \( X' = \left( \begin{array}{c} x' \\ y' \\ z' \end{array} \right) \),

the equation transforms to \( X'^{t}DX' + BPX' + cI_1 = 0 \)
or, \( \lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2 + 2u_1 x' + 2v_1 y' + 2w_1 z' + d = 0 \). \hspace{1cm} (A)

The following cases come up for consideration.

Case I  \hspace{0.5cm} \text{Rank of } A = 3.

In this case \( \lambda_1, \lambda_2, \lambda_3 \) are all non-zero.

Transferring origin to the point \( \left( \frac{-u_1}{\lambda_1}, \frac{-v_1}{\lambda_2}, \frac{-w_1}{\lambda_3} \right) \), the equation \( (A) \) takes the form

\( \lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d_1 = 0 \).

If \( d_1 = 0 \), the equation represents a cone.

If \( d_1 \neq 0 \), the equation represents a central quadric.

Case II  \hspace{0.5cm} \text{Rank of } A = 2.

In this case one of \( \lambda_1, \lambda_2, \lambda_3 \) is zero. \( \lambda_3 = 0 \).

Transferring origin to the point \( \left( \frac{-u_1}{\lambda_1}, \frac{-v_1}{\lambda_2}, 0 \right) \), the equation \( (A) \) takes the form

\( \lambda_1 x^2 + \lambda_2 y^2 + 2wz + d_1 = 0 \).

If \( w_1 \neq 0 \), transferring origin to the point \( (0,0,\frac{d_1}{2w_1}) \) the equation takes the form

\( \lambda_1 x^2 + \lambda_2 y^2 + 2w_1 z = 0 \). This is a paraboloid.

If \( w_1 = 0 \), the equation represents a pair of planes (real or imaginary) if \( d_1 = 0 \); a hyperbolic or elliptic cylinder if \( d_1 \neq 0 \).

Case III  \hspace{0.5cm} \text{Rank of } A = 1.

In this case two of \( \lambda_1, \lambda_2, \lambda_3 \) are zeroes. \( \lambda_2 = \lambda_3 = 0 \).

Transferring origin to the point \( \left( \frac{-u_1}{\lambda_1}, 0,0 \right) \), the equation \( (A) \) takes the form

\( \lambda_1 x^2 + 2v_1 y + 2w_1 z + d_1 = 0 \). \hspace{1cm} (B)

If at least one of \( v_1, w_1 \) is not zero, say \( v_1 \neq 0 \), transferring origin to the point \( \left( 0, \frac{d_1}{2v_1}, 0 \right) \), the equation \( (B) \) takes the form

\( \lambda_1 x^2 + 2v_1 y + 2w_1 z = 0 \).

By the orthogonal transformation \( X = PX' \) where
This is a parabolic cylinder.
If $v_1 = w_1 = 0$, the equation represents
a pair of coincident planes if $d_1 = 0$,
a pair of parallel planes if $d_1 \neq 0$.

**Pair of straight lines**

The nature of a general equation of second degree is determined by reducing it to
its canonical form. Here we want to discuss the conditions for the equation to
represent a pair of straight lines only.
From the previous discussion, we already know that the general equation of second
degree represent a pair of straight lines if and only if $
abla = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ and

$$= ab - h^2 \leq 0 .$$

The general equation of second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

(i) a pair of intersecting straight lines if and only if $\Delta= 0$ and $\delta < 0$ , the point of
intersection given by $(bg=hf, af=gh)$$

(ii) a pair of parallel or coincident straight lines or no locus if and only if $\Delta= 0$
and $\delta = 0$ .

From the above we can also conclude that the equation $ax^2 + 2hxy + by^2 = 0$
represents a pair of straight lines through origin if and only if $\delta = ab - h^2 \leq 0$ .

**Equation of the pair of lines joining the origin to the points in which the line**

$$lx + my = n$$

**intersects the conic**

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Let the line $lx + my = n$ meet the curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$
at P and Q. We are to find the equation of the pair of straight lines OP and OQ. Making the equation (2) homogeneous of degree two in x and y with the help of (1), we get,

\[ ax^2 + 2hxy + by^2 + (2gx + 2fy) \left( \frac{lx+my}{n} \right) + c \left( \frac{lx+my}{n} \right)^2 = 0 \]  

(3)

The equation (3) is homogeneous of degree two in \( x \) and \( y \), so it represents a pair of straight lines through the origin. Also, (3) is satisfied by co-ordinates of P and Q as the co-ordinates of P and Q satisfy both the equations (1) and (2) and hence (3). Hence (3) is the required equation of the pair of straight lines OP and OQ.

**Angle between two intersecting lines given by** \( ax^2 + 2hxy + by^2 = 0 \)  

(1)

Since (1) represents a pair of intersecting lines, we have \( h^2 - ab > 0 \).  

(2)

Let the angle between the lines represented by (1) be \( \theta \). Since (1) represents a pair of intersecting lines, the expression \( ax^2 + 2hxy + by^2 \) may be transformed into the canonical form \( Y^2 - \lambda^2 X^2 = 0 \) by a suitable rotation of co-ordinate axes.

Hence by property of invariants we have,

\[ a + b = 1 - \lambda^2 \]  

(3)

and

\[ ab - h^2 = -\lambda^2 - 0 = -\lambda^2 \]  

(4)

Since angle remains invariant under orthogonal transformations, considering the angle between the lines \( Y^2 - \lambda^2 X^2 = 0 \) we get

\[ \tan \theta = \frac{\lambda - (-\lambda)}{1 + \lambda(-\lambda)} \quad \therefore \tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2} \]

\[ = \frac{2\lambda}{1 - \lambda^2} = \frac{2\sqrt{h^2 - ab}}{a+b} \quad \text{using (3) and (4). Note that} \ h^2 - ab > 0 \text{ by (2)} \]

Hence the required angle is given by

\[ \tan \theta = \frac{2\sqrt{h^2 - ab}}{a+b} \]  

(5)

When \( a + b = 0 \), from (3) we get = ±1. Hence the equation \( Y^2 - \lambda^2 X^2 = 0 \) takes the form \( Y^2 - X^2 = 0 \).

(6)

But the lines (6) are at right angles to each other. Hence when \( +b = 0 \), the lines (1) are at right angles to each other.
Equation of the bisectors of the angles between two intersecting lines given by

$$ax^2 + 2hxy + by^2 = 0$$  \hspace{1cm} (1)

Let \( x^2 + 2hxy + by^2 = a(x + py)(x + qy) \).

Comparing coefficients from both sides we get

$$a(p + q) = 2h, \quad apq = b$$  \hspace{1cm} (2)

Let \( Q(x_1, y_1) \) be a point on any bisector. Then the lengths of the perpendiculars from \( Q \) on the lines \( (x + py) = 0 \) and \( (x + qy) = 0 \) must be equal.

Hence

$$\frac{(x_1 + py_1)^2}{1+p^2} = \frac{(x_1 + qy_1)^2}{1+q^2},$$

or, \((1 + q^2)(x_1^2 + 2px_1y_1 + p^2y_1^2) = (1 + p^2)(x_1^2 + 2qx_1y_1 + q^2y_1^2)\).

or, \((p^2 - q^2)(x_1^2 - y_1^2) = 2x_1y_1[p(1 + q^2) - q(1 + p^2)]\).

or, \((p^2 - q^2)(x_1^2 - y_1^2) = 2x_1y_1(p - q)(1 - pq)\)  \hspace{1cm} (3)

Since (1) represents a pair of intersecting straight lines, \( h^2 - ab > 0 \)  \hspace{1cm} (4)

From (2), \((p - q)^2 = (p + q)^2 - 4pq = \frac{4h^2}{a^2} - \frac{4b}{a} = \frac{4(h^2 - ab)}{a^2} > 0 \) by (4)

Hence from (3) we get, \((p + q)(x_1^2 - y_1^2) = 2x_1y_1(1 - pq)\)

or, \(\frac{(x_1^2 - y_1^2)}{a-b} = \frac{x_1y_1}{h}\).  \hspace{1cm} (5)

Thus \((x_1, y_1)\) lies on \(\frac{x^2 - y^2}{a-b} = \frac{xy}{h}\)

Equation (5) can be written as \(hx^2 - (a - b)xy - hy^2 = 0\) which is a homogeneous equation of second degree in \(x, y\) for which

\[\left(\frac{b-a}{2}\right)^2 + h^2 > 0.\]

Hence (5) represents a pair of straight lines through the origin and is the required equation of the bisectors.

Since the sum of the coefficients of \(x\) and \(y\) in (5) is zero, it follows from the previous discussion that the lines represented by (5) are at right angles to each other.

Some results

1. If the equation \(ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0\) represents a pair of straight lines, then \(ax^2 + 2hxy + by^2 = 0\) represents a pair of straight lines through the origin parallel to the first pair.

2. If \(\theta\) be the angle between the straight lines represented by the equation \(ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0\), then it will have the same value
as the angle between the two lines represented by \( x^2 + 2hxy + by^2 = 0 \).

Thus \( \frac{2 \sqrt{h^2 - ab}}{a + b} \).

3. If the two straight lines represented by the equation
\( ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \) are at right angles, then the two lines represented by \( ax^2 + 2hxy + by^2 = 0 \) are also at right angles and for this the condition is \( a + b = 0 \).

4. Let the two straight lines represented by the equation
\( ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \) (1)
be parallel. Let the two lines represented by it be \( lx + my + n = 0 \) and \( l'/x + m'/y + n'/ = 0 \). Then (1) can be written as
\( (lx + my + n)(l'/x + m'/y + n') = 0 \).
\( \therefore \ l^2 = a, m^2 = b, lm = h, l(n + n') = 2g, m(n + n') = 2f, nn' = c. \)

Hence \( h^2 = l^2m^2 = ab \) giving \( \frac{a}{h} = \frac{h}{b} \).

Again \( 2bg = bl(n + n') = lm^2(n + n') \)
and \( 2hf = hm(n + n') = lm^2(n + n'). \)

Therefore \( 2bg = 2hf \), or, \( bg = hf \), whence \( \frac{h}{b} = \frac{g}{f} \).

Thus, for parallelism of two straight lines represented by the equation
\( ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \) the condition is
\( \frac{a}{h} = \frac{h}{b} = \frac{g}{f}. \)

**Second order curves and conics**

By conics we usually mean a parabola, ellipse or a hyperbola whose equations in rectangular Cartesian co-ordinates \( x, y \) are equations of the second degree in \( x \) and \( y \). Also, from the earlier discussion we see that a general equation of the second degree in \( x \) and \( y \) may represent a parabola, an ellipse, a hyperbola, a pair of intersecting , parallel or coincident straight lines, a point only or may not represent any geometric object at all. If we now define a second order curve to be a conic, then according to this definition of a conic, there are nine distinct algebraic classes of conics and not just three that we get from the focus-directrix definition.

From now on, by a conic we shall mean a second order curve. Thus a circle and a pair of straight lines are also conics according to this definition.
Note: Although every general equation of first degree in \( x \) represents a straight line, every general equation of second degree in \( x \) and \( y \) does not represent a conic, for it may not represent any geometric object at all.

**Some properties of conics**

(i) Parametric equation of a parabola \( y^2 = 4ax \) is \( x = at^2, y = 2at, t \) being a variable parameter.

(ii) Parametric equation of an ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) is \( x = acost, y = bsint, t \) being a variable parameter. The circle described on the major axis of an ellipse as a diameter is called the auxiliary circle of the ellipse.

From the parametric equation of an ellipse, we can easily see that the coordinate of \( P \) is \((acost, bsint)\) and that of \( Q \) is \((acost, asint)\). Clearly, from the above diagram, \( \angle NOQ = t \) called the eccentric angle of an ellipse.

Also, it follows immediately that \( \frac{PN}{QN} = \frac{b}{a} \).

Therefore the ordinates of the corresponding points are in constant ratio.

(iii) Parametric equation of a hyperbola \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \) is \( x = asect, y = btant, t \) being a variable parameter.

\( AA' \) and \( BB' \) are the conjugate and the transverse axes.

The equation of the conjugate hyperbola is \( \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \).

If the transverse axis and the conjugate axis of the hyperbola are equal...
then the hyperbola is called rectangular hyperbola whose equation is \( x^2 - y^2 = a^2 \) whose eccentricity is \( \sqrt{2} \). The equation of the rectangular hyperbola conjugate to \( x^2 - y^2 = a^2 \) is \( x^2 - y^2 = -a^2 \).

**Tangent to a non-singular conic**

Let \( C \) be a non-singular conic with equation
\[
ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \tag{1}
\]
[a conic is non-singular if the rank of the matrix \( \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} \) is 3]

Let \( P(x_1, y_1), Q(x_2, y_2) \) be two distinct points on \( C \). The line through \( P \) and \( Q \) is called a secant of \( C \). We now consider \( P \) as fixed and \( Q \) is allowed to approach \( P \) along the conic. Then as \( Q \) approaches \( P \), the secant will in general, approach a limiting position represented by the line \( PT \). The line \( PT \) is defined to be the tangent to \( C \) at \( P \) and the point \( P \) is called the point of contact of the tangent.

Let \( y = mx + d \) be a secant of \( C \). Then by putting \( mx + d \) for \( y \) in (1) we get the equation
\[
ax^2 + 2hx(mx + d) + b(mx + d)^2 + 2gx + 2f(mx + d) + c = 0
\]
or, \( x^2(bm^2 + 2hm + a) + 2x(bmd + hd + fm + g) + (bd^2 + 2fd + c) = 0 \) \( \tag{3} \)

If the secant \( C \) becomes the tangent to the conic (1) at \( P \), then the two points of intersection of (1) and (2) coincides with \( P \). In that case, the roots of (3) are equal. Hence the equality of the roots of (3) may be taken as a condition for the line (2) to be a tangent to the conic (1).

Instead of considering a quadratic equation in \( x \), the condition can also be obtained by considering a quadratic equation in \( y \).
Equation of the tangent to the non-singular conic

\[ ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \] at a point \((x_1, y_1)\) on it

Let the equation of the tangent at \((x_1, y_1)\) be \( (y - y_1) = m(x - x_1) \) \( (1) \)

Putting \( y = y_1 + m(x - x_1) \) in the equation of the conic we get

\[ ax^2 + 2hx[m(x - x_1) + y_1] + b[m(x - x_1) + y_1]^2 + 2gx + 2f[m(x - x_1) + y_1] + c = 0 \]

or,

\[ x^2(bm^2 + 2hm + a) + 2x(-m^2bx_1 + mby_1 - mhx_1 + mf + hy_1 + g) + (m^2bx_1^2 - 2mby_1x_1 - 2mfx_1 + by_1^2 + 2fy_1 + c) = 0 \] \( (2) \)

Since the line (1) is a tangent to the conic, the two roots of (2) must be equal and each of the roots will be \( x_1 \).

Hence, \( 2x_1 = \frac{-2(-m^2bx_1 + mby_1 - mhx_1 + mf + hy_1 + g)}{bm^2 + 2hm + a} \)

Simplifying, \( m = \frac{-(ax_1 + hy_1 + g)}{hx_1 + by_1 + f} \).

Substituting the value of \( m \) in (1), we get

\[ (y - y_1)(hx_1 + by_1 + f) + (x - x_1)(ax_1 + hy_1 + g) = 0 \]

or, \( axx_1 + byy_1 + h(x_1y + xy_1) - ax_1^2 - by_1^2 - 2hx_1y_1 - g x_1 - fy_1 + gx + fy = 0 \)

or, \( axx_1 + byy_1 + h(x_1y + xy_1) + 2gx_1 + 2fy_1 + c = 0 \)

[Since \( ax_1^2 + by_1^2 + 2hx_1y_1 + 2gx_1 + 2fy_1 + c = 0 \)]

or, \( axx_1 + h(x_1y + xy_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0 \) \( (3) \)

which is the required equation.

Particular cases of the conic \( ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \) \( (4) \)

(i) Putting \( a = 0, h = 0, b = g = -2a, f = 0 \) and \( c = 0 \) in (4) we get

\[ y^2 - 4ax = 0 \] which represents a parabola. In this case (3) takes the form

\[ yy_1 - 2a(x + x_1) = 0. \]

Hence the equation of the tangent to a parabola \( y^2 = 4ax \) at the point \((x_1, y_1)\) is \( y_1 = 2a(x + x_1) \).

Condition that the straight line \( y = mx + c \) is a tangent to the parabola \( y^2 = 4ax \) can be obtained by solving the two equations and using the
condition that the two solutions are the same, which is \( \frac{a}{m^2} \). The point of contact in this case is \( \left( \frac{a}{m^2}, \frac{2a}{m} \right) \).

Similarly it can be shown that

(ii) the equation of the tangent to an ellipse \( \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \) at the point \( (x_1, y_1) \) is \( \frac{xx_1}{A^2} + \frac{yy_1}{B^2} = 1 \).

Condition that the straight line \( y = mx + c \) is a tangent to the ellipse \( \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \) is \( c^2 = a^2m^2 + b^2 \) and the points of contact are

\[
\left( -\frac{a^2m}{\sqrt{a^2m^2 + b^2}}, \frac{b^2}{\sqrt{a^2m^2 + b^2}} \right), \left( \frac{a^2m}{\sqrt{a^2m^2 + b^2}}, -\frac{b^2}{\sqrt{a^2m^2 + b^2}} \right).
\]

(iii) the equation of the tangent to a hyperbola \( \frac{x^2}{A^2} - \frac{y^2}{B^2} = 1 \) at the point \( (x_1, y_1) \) is \( \frac{xx_1}{A^2} - \frac{yy_1}{B^2} = 1 \).

Condition that the straight line \( y = mx + c \) is a tangent to the above hyperbola is \( c^2 = a^2m^2 - b^2 \) and the points of contact are

\[
\left( -\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \frac{b^2}{\sqrt{a^2m^2 - b^2}} \right), \left( \frac{a^2m}{\sqrt{a^2m^2 - b^2}}, -\frac{b^2}{\sqrt{a^2m^2 - b^2}} \right).
\]

From the above discussion we see that through every point on a non-singular conic there passes a unique tangent to the conic. Now we will try to find an answer to whether there passes a tangent to a non-singular conic through every point which is not on the conic.

Outside and inside points of an ellipse

If \( ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \) represents an ellipse, then a point \( P(x_1, y_1) \) is said to be an outside and inside point of it according as

\[
ax_1^2 + by_1^2 + 2hx_1y_1 + 2gx_1 + 2fy_1 + c > 0 \text{ or } < 0.
\]

Outside and inside points of an ellipse are sometimes called external and internal points.

Let us consider the conic \( 8x^2 - 12xy + 17y^2 + 16x - 12y + 3 = 0 \)  \( \text{(1)} \).

This is an ellipse as in this case \( ab - h^2 = 8.17 - 6^2 = 136 - 36 = 100 > 0 \).

This ellipse has \( (-1, 0) \) as its centre. (Verify). It can easily be verified using the
above definition that \((-1,0)\) is an inside point of this ellipse.

The equation of a line through \((-1,0)\) can be written as \(y = m(x + 1)\) \hspace{1cm} (2)

The \(x\) co-ordinates of the points of intersection of (2) and (1) are given by

\[
8x^2 - 12xm(x + 1) + 17y[m(x + 1)]^2 + 16x - 12m(x + 1) + 3 = 0
\]

or,

\[
x^2(17m^2 - 12m + 8) + 2x(17m^2 - 12m + 8) + (17m^2 - 12m + 3) = 0 \hspace{1cm} (3)
\]

If (2) is a tangent to (1) then there must exist a real value of \(m\) such that the two roots of (3) are equal, the condition for which is

\[
4(17m^2 - 12m + 8)^2 = 4(17m^2 - 12m + 8)(17m^2 - 12m + 3)
\]

or,

\[
(17m^2 - 12m + 8)(17m^2 - 12m + 8 - 17m^2 + 12m - 3) = 0
\]

or,

\[
17m^2 - 12m + 8 = 0
\]

(4)

But there is no real value of \(m\) satisfying (4). Hence (2) cannot be a tangent to (1).

This is an example to show that there may not exist a tangent to a non-singular conic through every point which is not on the conic.

In fact, there may exist more than one tangent from an external point.

Again consider the equation (1).

The point \((0,0)\) is an outside point of the ellipse (1).

Equation of a line through \((0,0)\) is of the form \(y = mx\) \hspace{1cm} (5)

The \(x\) co-ordinate of the point of intersections of (1) and (5) are given by

\[
8x^2 - 12mx^2 + 17m^2x^2 + 16x - 12mx + 3 = 0
\]

or,

\[
x^2(17m^2 - 12m + 8) + 4x(4 - 3m) + 3 = 0 \hspace{1cm} (6)
\]

The equation (5) will be a tangent to (1) if the roots of (6) are equal, the condition for which is

\[
16(4 - 3m)^2 = 12(17m^2 - 12m + 8)
\]

i.e.,

\[
15m^2 + 60m - 40 = 0
\]

or,

\[
3m^2 + 12m - 8 = 0 \hspace{1cm} (7)
\]

The two values of \(m\) given by (7) are real. Hence there are two tangents to the ellipse (1) through the point \((0,0)\).

Now we generalize the above observations.
If there exist a tangent to the conic \( ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \) through a given point \( A(x_1, y_1) \) not lying on it, to find the equation of these tangents.

Let \( g \) be a line through \( A \) and let \( P \) be a point of its intersection with the conic. Take a point \( B \) on \( g \) which is not on the conic. Denote the co-ordinates of \( P \) and \( B \) by \( (\bar{x}, \bar{y}) \) and \( (x, y) \) respectively. Let \( P \) divide the segment \( \overline{AB} \) in the ratio \( k \).

Then
\[
\bar{x} = \frac{x_{1} + kx}{1 + k}, \quad \bar{y} = \frac{y_{1} + ky}{1 + k}, \quad (1 + k \neq 0).
\]

Since \( (\bar{x}, \bar{y}) \) lies on the conic we get
\[
a\bar{x}^2 + 2h\bar{x}y + b\bar{y}^2 + 2g\bar{x} + 2f\bar{y} + c = 0
\]
or,
\[
a\left(\frac{x_{1} + kx}{1 + k}\right)^2 + 2h\left(\frac{x_{1} + kx}{1 + k}\right)\left(\frac{y_{1} + ky}{1 + k}\right) + b\left(\frac{y_{1} + ky}{1 + k}\right)^2 + 2g\left(\frac{x_{1} + kx}{1 + k}\right) + 2f\left(\frac{y_{1} + ky}{1 + k}\right) + c = 0
\]

Arranging the above equation as a quadratic in \( k \) we get,
\[
k^2(ax^2 + 2hxy + by^2 + 2gx + 2fy + c) + 2k[ax_{1} + h(x_{1}y + xy_{1}) + by_{1} + g(x + x_{1}) + f(y + y_{1}) + c] + ax_{1}^2 + by_{1}^2 + 2hx_{1}y_{1} + 2gx_{1} + 2fy_{1} + c = 0 \tag{1}
\]
If the line \( g \) be a tangent to the conic, then the two roots of the equation \( (1) \) must be equal. Hence we have
\[
[ax_{1} + h(x_{1}y + xy_{1}) + by_{1} + g(x + x_{1}) + f(y + y_{1}) + c]^2 = 4(ax^2 + 2hxy + by^2 + 2gx + 2fy + c)(ax^2 + 2hxy + by^2 + 2gx + 2fy + c) \tag{2}
\]
Thus the relation \( (2) \) is satisfied by the co-ordinates \( (x, y) \) of any point on a tangent to the conic through \( (x_{1}, y_{1}) \). For any point \( (x, y) \), not on the tangent through \( (x_{1}, y_{1}) \), the relation \( (2) \) is not satisfied by the co-ordinates of \( Q \). Hence \( (2) \) is the equation of any tangent to the conic through \( (x_{1}, y_{1}) \). Since \( (2) \) is of second degree it follows that there are two tangents to the conic through \( (x_{1}, y_{1}) \). The equation \( (2) \) therefore is the equation of these two tangents.

Note: Writing \( S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c \),
\[
S_1 = ax_{1}^2 + by_{1}^2 + 2hx_{1}y_{1} + 2gx_{1} + 2fy_{1} + c \quad \text{and}
\]
\[
T = ax_{1} + h(x_{1}y + xy_{1}) + by_{1} + g(x + x_{1}) + f(y + y_{1}) + c,
\]
the equation \( (2) \) can be expressed as
\[
SS_1 = T^2.
\]
Particular cases

(i) For the parabola \( y^2 = 4ax \), the equation of the tangents to the curve from a two tangent point \((x_1, y_1)\) is

\[
(y^2 - 4ax)(y_1^2 - 4ax_1) = [yy_1 - 2a(x + x_1)]^2
\]

(iv) For the ellipse \( \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \), the equation of the tangents to the curve from a two tangent point \((x_1, y_1)\) is

\[
\left( \frac{x^2}{A^2} + \frac{y^2}{B^2} - 1 \right) \left( \frac{x_1^2}{A^2} + \frac{y_1^2}{B^2} - 1 \right) = \left( \frac{x_1}{A^2} + \frac{y_1}{B^2} - 1 \right)^2
\]

(ii) For the hyperbola \( \frac{x^2}{A^2} - \frac{y^2}{B^2} = 1 \), the equation of the tangents to the curve from a two tangent point \((x_1, y_1)\) is

\[
\left( \frac{x^2}{A^2} - \frac{y^2}{B^2} - 1 \right) \left( \frac{x_1^2}{A^2} - \frac{y_1^2}{B^2} - 1 \right) = \left( \frac{x_1}{A^2} - \frac{y_1}{B^2} - 1 \right)^2
\]

Equation of the chord of contact of the tangents to the conic

\[ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0\] from a point \((x_1, y_1)\)

Let \((x_2, y_2), (x_3, y_3)\) be the co-ordinates of the points of contact of the tangents to the conic from \((x_1, y_1)\). The equations of the tangents to the conic at \((x_2, y_2)\) and \((x_3, y_3)\) are

\[
ax_2 + h(x_2y + xy_2) + by_2 + g(x + x_2) + f(y + y_2) + c = 0 \quad (1)
\]

and

\[
ax_3 + h(x_3y + xy_3) + by_3 + g(x + x_3) + f(y + y_3) + c = 0 \quad (2)
\]

Since \((x_1, y_1)\) is a point on (1) and (2), we have

\[
ax_1x + h(x_2y_1 + x_1y_2) + by_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0 \quad (3)
\]

and

\[
ax_1x + h(x_3y_1 + x_1y_3) + by_1y_3 + g(x_1 + x_3) + f(y_1 + y_3) + c = 0 \quad (4)
\]

From (3) and (4) it follows that \((x_2, y_2)\) and \((x_3, y_3)\) lie on the line

\[
ax_1x + h(xy_1 + x_1y) + by_1y + g(x_1 + x) + f(y_1 + y) + c = 0
\]

which is the chord of contact.
**Equation of the chord of the conic**

\[ ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \]

which is bisected at the point \((x_1, y_1)\)

Let \((x_2, y_2), (x_3, y_3)\) be the co-ordinates of the extremities of the chord AB which is bisected at the point \((x_1, y_1)\).

Let us consider the equation

\[
a(x - x_2)(x - x_3) + h((x - x_2)(y - y_3) + (x - x_3)(y - y_2)) + b(y - y_2)(y - y_3) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c \quad (1)
\]

The locus represented by (1) passes through the points \((x_2, y_2), (x_3, y_3)\), because its equation is satisfied by these co-ordinates.

The equation (1) can be expressed as

\[
x\{a(x_2 + x_3) + h(y_2 + y_3) + 2g\} + \{h(x_2 + x_3) + b(y_2 + y_3) + 2f\} = ax_2x_3 + h(x_2y_3 + x_3y_2) + by_2y_3 - c \quad (2)
\]

Since (2) is a linear equation in \(x\) and \(y\), the locus represented by it is a straight line. Thus the equation (1) or (2) is the equation of the chord AB. Since \((x_1, y_1)\) are the co-ordinates of the middle point of the chord AB, so \(2x_1 = x_2 + x_3\) and \(2y_1 = y_2 + y_3\).

Denoting the slope of (2) by \(m\), we get

\[
m = -\frac{a(x_2 + x_3) + h(y_2 + y_3) + 2g}{h(x_2 + x_3) + b(y_2 + y_3) + 2f} = -\frac{2ax_1 + 2hy_1 + 2g}{2hx_1 + 2by_1 + 2f} = \frac{-ax_1 + hy_1 + g}{hx_1 + by_1 + f}
\]

Now the equation of AB can be written as

\[
y - y_1 = m(x - x_1) = -\frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f}(x - x_1)
\]

Therefore the required equation is

\[
(y - y_1)(hx_1 + by_1 + f) + (x - x_1)(ax_1 + hy_1 + g) = 0
\]

Particular cases

(i) When the equation of the conic is \(y^2 = 4ax\), the equation of the chord bisected at the point \((x_1, y_1)\) is

\[
yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1
\]

(ii) When the equation of the conic is \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\), the equation of the chord bisected at the point \((x_1, y_1)\) is

\[
\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1.
\]
(iii) When the equation of the conic is \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \), the equation of the chord bisected at the point \((x_1, y_1)\) is

\[
\frac{x x_1}{a^2} - \frac{y y_1}{b^2} = 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1.
\]

**Equation of the normal to the conic**

\( ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \) at a point \((x_1, y_1)\) on it

Equation of the tangent at \((x_1, y_1)\) is

\[
x(ax_1 + hy_1 + g) + y(hx_1 + by_1 + f) + gx_1 + fy_1 + c = 0.
\]

Equation of a line perpendicular to it is

\[
x(hx_1 + by_1 + f) - y(ax_1 + hy_1 + g) + D = 0.
\]

If this perpendicular line passes through the point \((x_1, y_1)\), then we get

\[
D = -x_1(ax_1 + hy_1 + g) + y_1(hx_1 + by_1 + f).
\]

Using this value of \(D\), the equation of the normal at \((x_1, y_1)\) is

\[
x(hx_1 + by_1 + f) - y(ax_1 + hy_1 + g) - x_1(ax_1 + hy_1 + g) + y_1(hx_1 + by_1 + f) = 0.
\]

On simplification \(\frac{x-x_1}{ax_1+hy_1+g} = \frac{y-y_1}{hx_1+by_1+f}\) is the required equation of the normal.

**Particular cases**

(i) When the equation of the conic is \( y^2 = 4ax \), the equation of the normal at the point \((x_1, y_1)\) is

\[
\frac{x-x_1}{-2a} = \frac{y-y_1}{y_1}
\]

(ii) When the equation of the conic is \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) the equation of the normal at the point \((x_1, y_1)\) is

\[
\frac{x-x_1}{a^2} = \frac{y-y_1}{b^2}
\]

(iii) When the equation of the conic is \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \), the equation of the normal at the point \((x_1, y_1)\) is

\[
\frac{x-x_1}{a^2} = \frac{y-y_1}{-b^2}
\]
**Polar co-ordinate system in a plane**

Let OA be a half-line emanating from a fixed point O and P be an arbitrary point of the plane. Denote the distance of P from O by \( r = |OP| \) and the measure of the angle \( \angle AOP \) by \( \theta \). Then the numbers \( r \) and \( \theta \) are together defined to be the polar co-ordinates of P.

Here \( r \) is called the radius vector and \( \theta \) the vectorial angle of P. O is called the pole and the half-line OA is called the polar axis or the initial line. If the radius vector and the vectorial angle of a point P are \( r \) and \( \theta \) respectively, then its polar coordinates is denoted by \( (r, \theta) \). The polar coordinate of O is \( (0, \theta) \).

**Relation between Cartesian and polar Co-ordinates**

Let the pole and polar axis of a polar co-ordinate system coincide respectively with the origin and positive x-axis of a rectangular Cartesian system. Let \( (x, y) \) and \( (r, \theta) \) be the Cartesian and polar co-ordinates of a point P. Then we get:

\[
\cos \theta = \frac{x}{r}, \sin \theta = \frac{y}{r}
\]

or,

\[
x = r \cos \theta, \quad y = r \sin \theta
\]

From (1),

\[
r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}.
\]

(1) and (2) give formula for getting Cartesian co-ordinates from polar and polar co-ordinates from Cartesian respectively.

**Polar equation of a straight line**

Consider a rectangular Cartesian co-ordinate system whose positive x-axis coincides with the polar axis of a polar co-ordinate system. We consider a straight line whose equation in this Cartesian system is \( ax + by + c = 0 \) (1) where \( a \) and \( b \) are not simultaneously zero.

Putting \( x = r \cos \theta, y = r \sin \theta \) in (1) we get:

\[
ar \cos \theta + br \sin \theta + c = 0,
\]
or, $\frac{-c}{r} = acos \theta + bsin \theta$,

or, $\frac{A}{r} = acos \theta + bsin \theta$, where $A = -c$,

which is the polar equation of a straight line.

Thus the polar equation of a straight line is of the form $\frac{A}{r} = acos \theta + bsin \theta$, $A,a,b$ are constants, $a$ and $b$ are not simultaneously zero.

Corollary

1. The polar equation of a straight line parallel to the straight line $\frac{A}{r} = acos \theta + bsin \theta$ can be written as $\frac{B}{r} = acos \theta + bsin \theta$, where $B$ is a constant different from $A$.

2. The polar equation of a straight line perpendicular to the straight line $\frac{A}{r} = acos \theta + bsin \theta$ can be written as $\frac{B}{r} = acos(\frac{\pi}{2} + \theta) + bsin(\frac{\pi}{2} + \theta)$, where $B$ is a constant different from $A$.

Equation of a line passing through $(r_1, \theta_1), (r_2, \theta_2)$

The general form of the polar equation of a straight line is

$$acos \theta + bsin \theta - \frac{1}{r} = 0$$

(1)

If it passes through $(r_1, \theta_1), (r_2, \theta_2)$, then

$$acos \theta_1 + bsin \theta_1 - \frac{1}{r_1} = 0$$

(2)

and

$$acos \theta_2 + bsin \theta_2 - \frac{1}{r_2} = 0$$

(3)

Eliminating $a$ and $b$ from (1), (2) and (3) the required equation is

$$\begin{vmatrix}
    cos \theta & sin \theta & \frac{1}{r} \\
    cos \theta_1 & sin \theta_1 & \frac{1}{r_1} \\
    cos \theta_2 & sin \theta_2 & \frac{1}{r_2}
\end{vmatrix} = 0 ,$$

or, $\frac{l}{r} sin(\theta_1 - \theta_2) + \frac{l}{r_1} sin(\theta_2 - \theta) + \frac{l}{r_2} sin(\theta - \theta_1) = 0$.
Polar equation of a circle

Let C be the centre and d be the radius of a circle. Take O as the pole and OX as the polar axis of a polar coordinate system. Denote the co-ordinates of C by $(\rho, \alpha)$. Let P be any point on the circle with $(r, \theta)$ as its polar co-ordinates. Then $OC = \rho$, $OP = r$, $\angle XOP = \theta$, $\angle XOC = \alpha$, $CP = d$.

Now, $CP^2 = CO^2 + OP^2 - 2 \cdot OC \cdot OP \cdot \cos COP$, so, $d^2 = \rho^2 + r^2 - 2r\rho \cos(\theta - \alpha)$ which is the required equation of the circle whose centre is $(\rho, \alpha)$ and radius is d.

Particular cases

(i) when the polar axis touches the circle
In this case $\rho \sin \alpha$. Hence the equation of the circle becomes
$r^2 - 2r \rho \cos(\theta - \alpha) + \rho^2(1 - \sin^2 \alpha) = 0$

or, $r^2 - 2r \rho \cos(\theta - \alpha) + \rho^2 \cos^2 \alpha = 0$

(ii) when the pole is on the circle
In this case $OC = d = \rho$. Hence the equation of the circle becomes
$r^2 - 2rd \cos(\theta - \alpha) = 0$

or, $r = 2d \cos(\theta - \alpha)$

(iii) when the pole is on the circle and its centre is on the polar axis
Here $p = d$, $\alpha = 0$.
Hence the equation of the circle is
$r = 2d \cos \alpha$

(iv) when the pole is at the centre of the circle
In this case $\rho = 0, r = d$. Hence the equation of the circle is $r^2 = d^2$, i.e., $r = d$. 
**Definition of a conic**

Let \( g \) be a fixed line in a plane and \( S \) be a fixed point of it not lying on \( g \). If \( P \) be a variable point of the plane such that the ratio of its distance from \( S \) to its distance from \( g \) is always a constant, then the locus of \( P \) is called a conic section or simply a conic.

The fixed point \( S \) is called the focus, the fixed line \( g \) is called the directrix and the constant is called the eccentricity of the conic.

The above definition is called the focus-directrix definition of a conic.

The eccentricity of a conic is a positive constant. We shall denote it by \( e \). Conics are divided into the following classes according to the value of \( e \).

A conic is called a parabola if \( e = 1 \), an ellipse if \( e < 1 \), a hyperbola if \( e > 1 \).

**Polar equation of a conic**

Let \( C \) denote an ellipse, a parabola or one branch of a hyperbola.

Denote by \( S \) and \( g \) the focus of \( C \) and the corresponding directrix, in case (iii) \( S \) and \( g \) denoting the focus and the directrix nearest to the branch under consideration.

We choose a polar co-ordinate system such that \( S \) is the pole and the polar axis is along the axis of \( C \) directed from \( S \) and away from the directrix \( g \). Let \((r, \theta)\) be the co-ordinates of any point \( P \) on \( C \) in the chosen co-ordinate system. Let the perpendicular from \( P \) on \( g \) meet it at \( R \). Denote by \( e \) the eccentricity of \( C \).

Then \[ \frac{|SP|}{|PR|} = e . \] (1)

But \(|SP| = r \).

Hence \[ \frac{r}{|PR|} = e . \] (2)

Let \( N \) be the foot of the perpendicular from \( P \) on the polar axis and \( M \) be its intersection with \( g \). Then

\[ |PR| = |MN| = |MS| + |SN| = |MS| + r \cos \theta \] (3)
Let $Q$ be the point on $C$ such that the line $SQ$ is perpendicular to the polar axis. Denote $|MS|$ by $l$ and the foot of the perpendicular from $Q$ on $g$ by $T$.

Then $\frac{|SQ|}{|TQ|} = e$. So $|TQ| = \frac{|SQ|}{e} = \frac{l}{e}$.

Now $|TQ| = |MS|$. Hence $|MS| = \frac{l}{e}$.

Therefore from (3) we get $|PR| = \frac{l}{e} + r\cos\theta$

or, $\frac{r}{e} = \frac{l}{e} + r\cos\theta$ [by (2)]

or, $r\left(\frac{1}{e} - \cos\theta\right) = \frac{l}{e}$ whence $r = \frac{l}{1 - e\cos\theta}$

or, $\frac{l}{r} = 1 - e\cos\theta$ (4)

Hence (4) is the equation of $C$, where $l$ is its semi latus rectum.

Note 1: If the polar axis is taken along the axis of $C$ directed from $S$ towards the directrix $g$, then it can be shown that the equation of $C$ is (5)

Note 2: If (4) is the equation of the right branch of a hyperbola, then the equation of its left branch is $\frac{l}{r} = -(1 + e\cos\theta)$.

**Equation of a chord joining $\theta = \alpha - \beta$ and $\theta = \alpha + \beta$**

Let $\frac{l}{r} = 1 - e\cos\theta$ (1) be the equation of the conic and $\frac{l}{r} = a\cos\theta + b\sin\theta$ (2) be that of the chord.

For the common points of (1) and (2), $1 - e\cos\theta = a\cos\theta + b\sin\theta$. At the common points $\theta = \alpha - \beta$ and $\theta = \alpha + \beta$,

$1 - e\cos(\alpha - \beta) = a\cos(\alpha - \beta) + b\sin(\alpha - \beta)$ and

$1 - e\cos(\alpha + \beta) = a\cos(\alpha + \beta) + b\sin(\alpha + \beta)$

From the above two equations,

$(a + e)\cos(\alpha - \beta) + b\sin(\alpha - \beta) - 1 = 0$ and

$(a + e)\cos(\alpha + \beta) + b\sin(\alpha + \beta) - 1 = 0$

By cross multiplication,

\[ \frac{a + e}{-\sin(\alpha - \beta) + \sin(\alpha + \beta)} = \frac{b}{-\cos(\alpha + \beta) + \cos(\alpha - \beta)} = \frac{1}{\cos(\alpha - \beta)\sin(\alpha + \beta) - \sin(\alpha - \beta)\cos(\alpha + \beta)} \]

or, $\frac{a + e}{2\cos\alpha\sin\beta} = \frac{b}{2\sin\alpha\sin\beta} = \frac{1}{2\sin\beta\cos\beta}$ which gives $a = \frac{\cos\alpha}{\cos\beta} - e$, $b = \frac{\sin\alpha}{\cos\beta}$. 
Therefore the equation of the cord is
\[
\frac{l}{r} = \left(\frac{\cos\alpha}{\cos\beta} - e\right)\cos \theta + \frac{\sin\alpha}{\cos\beta}\sin\theta
\]
or, \[
\frac{l}{r} = -ecos \theta + sec\beta \cos(\theta - \alpha)
\]

**Equation of the tangent at \( \theta = \alpha \)**

The two points corresponding to \( \theta = \alpha - \beta \) and \( \theta = \alpha + \beta \) will coincide if \( \beta = 0 \). In this case the chord will be the tangent at \( \alpha \). Hence the equation of the tangent at \( \alpha \) is \[
\frac{l}{r} = -ecos \theta + \cos(\theta - \alpha)
\]

**Equation of the normal at \( \theta = \alpha \)**

Since the normal is perpendicular to the tangent, let the equation of the normal at \( \alpha \) be \[
\frac{k}{r} = -ecos\left(\frac{\pi}{2} + \theta\right) + \cos\left(\frac{\pi}{2} + \theta - \alpha\right) \quad \text{or} \quad \frac{k}{r} = esin \theta - \sin(\theta - \alpha)
\]

It passes through the point of contact \( \left(\frac{l}{1-ecosa}, \alpha\right) \). Therefore \[
\frac{k(1-ecosa)}{l} = e \sin\alpha, \quad \text{or} \quad k = \frac{el \sin \alpha}{1-ecosa}
\]
Putting this value of \( k \), the equation of the normal is \[
\frac{el \sin \alpha}{r(1-ecosa)} = esin \theta - \sin(\theta - \alpha)
\]

Note: If the equation of the conic be \( \frac{1}{r} = 1 + ecos\theta \), then the equation of the tangent and the normal at \( \alpha \) respectively are
\[
\frac{l}{r} = ecos \theta + \cos(\theta - \alpha) \quad \text{and} \quad \frac{el \sin \alpha}{r(1+ecosa)} = esin \theta + \sin(\theta - \alpha)
\]
LIMIT FORMULAE

1. \( \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1} \).
2. \( \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{x}{\sin x} = \lim_{x \to 0} \frac{\sin^{-1} x}{x} = \lim_{x \to 0} \frac{x}{\sin^{-1} x} = 1 \).
3. \( \lim_{x \to \infty} \frac{\sin x}{x} = 0 \).
4. \( \lim_{x \to 0} \frac{(1+x)^{n-1}}{x} = n \).
5. \( \lim_{x \to 0} \frac{e^x - 1}{x} = 1 \).
6. \( \lim_{x \to 0} \frac{\log_e(1+x)}{x} = 1 \).
7. \( \lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a \), \((a > 0)\).
8. \( \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e \), \(\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e \).
9. \( \lim_{x \to \infty} \frac{\log x}{x} = \lim_{x \to \infty} \frac{\log|x|}{x} = 0 \).
10. \( \lim_{x \to \infty} \frac{x^n}{e^x} = 0 \), \((n > 0)\), \(\lim_{x \to \infty} \frac{e^x}{x^n} = \infty \), \((n > 0)\), \(\lim_{x \to \infty} \frac{x^n}{n!} = 0 \).
11. \( \lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1 \).
12. \( \lim_{n \to \infty} \left(1 + \frac{\mu}{n}\right)^n = e^\mu = \lim_{n \to \infty} (1 + \mu n)^{\frac{1}{n}} \).
13. \( \lim_{x \to -\infty} e^x = 0 \), \(\lim_{x \to \infty} e^{-x} = 0 \).
14. \( \lim_{x \to \infty} x^n = \begin{cases} 0 & \text{if } |x| < 1 \\ \pm 1 & \text{if } |x| = 1 \\ \pm \infty & \text{if } |x| > 1 \end{cases} \), \(\lim_{n \to \infty} x^n = 0 \), \((-1 < x < 1)\).
15. \(\lim_{x \to 0^+} \frac{1}{x} = \infty \), \(\lim_{x \to 0^-} \frac{1}{x} = -\infty \), \(\text{so } \lim_{x \to 0} \frac{1}{x} \text{ does not exist.}\)
16. \(\lim_{x \to 0} \frac{1}{x^2} = \infty \text{ but } \lim_{x \to \infty} \frac{1}{x^2} = 0 \).
17. \(\lim_{x \to \infty} x^{\frac{1}{x}} = 1 \).
18. \(\lim_{x \to a} |x| = |a| \).
19. \(\lim_{x \to \infty} x \sin \frac{1}{x} = 1 \).
20. \(\lim_{x \to 0} \sin \frac{1}{x} \text{ does not exist.}\)
21. \[ \lim_{x \to 0} x \sin \frac{1}{x} = \lim_{x \to 0} x^2 \sin \frac{1}{x} = \lim_{x \to 0} x \sin \frac{1}{x} = \lim_{x \to 0} x \cos \frac{1}{x} = \lim_{x \to 0} x \cos \frac{1}{x} = \]
\[ \lim_{x \to 0} \sqrt{x} \sin \frac{1}{x} = 0 . \]
22. \[ \lim_{x \to a} e^{f(x)} = e^{\lim_{x \to a} f(x)} \]
23. \[ \lim_{x \to a} (f \cdot g)(x) = f(\lim_{x \to a} g(x)) . \]
24. \[ \lim_{x \to a^-} [x] = a - 1 , \lim_{x \to a^+} [x] = a , \quad a \in \mathbb{Z} \text{ (set of integers)} . \]
25. \[ \lim_{x \to \infty} \frac{ae^x + be^{-x}}{ce^x + de^{-x}} = \frac{a}{c} \quad (c \neq 0) , \quad \lim_{x \to -\infty} \frac{ae^x + be^{-x}}{ce^x + de^{-x}} = \frac{b}{d} \quad (d \neq 0) . \]

**SOME IMPORTANT FORMULAE TO REMEMBER**

1. \((\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta . \]
2. \(e^{i\theta} = \cos \theta + i \sin \theta , \quad e^{-i\theta} = \cos \theta - i \sin \theta . \]
3. If \( \lim_{x \to a} f(x) \) exists but \( \lim_{x \to a} g(x) \) does not exist then 
   \( \lim_{x \to a} [f(x) \pm g(x)] \) does not exist .
4. If \( \lim_{x \to a} [f(x)g(x)] \ exists \ then \ lim_{x \to a} f(x) \ and \ lim_{x \to a} g(x) \ may \ not \ exist . \) [ e.g. \( f(x) = x , \ g(x) = 1/x \) ]
5. Product and Ratio of two odd function is even .

**SOME USEFUL RESULTS ON CONTINUOUS FUNCTION**

If \( f \) and \( g \) are continuous at \( x = a \) then

1. \( f \pm g \) is continuous at \( x = a \)
2. \( f \cdot g \) is continuous at \( x = a \)
3. \( f / g \) is continuous at \( x = a \) , where \( g(a) \neq 0 \)
4. \( f[g(x)] \) is continuous at \( x = a \)
5. Every Polynomial function is continuous at every point of the 
   real line .
6. Every Rational function is continuous at every point where its 
   denominator is different from zero .
SOME USEFUL RESULTS ON RELATION AND MAPPING

1. Let \( n(A) = m \), \( n(B) = n \); then the total number of relations from \( A \) to \( B \) is \( 2^{mn} \).
2. Let \( f : A \rightarrow B \) be a function where \( n(A) = m \), \( n(B) = n \), then total number of functions is \( n^m \).
3. If \( f : A \rightarrow B \) be a function where \( n(A) = m \), \( n(B) = n \), then total number of injection is
   \[
   \begin{cases} 
   n^m, & n \geq m \\
   0, & n < m 
   \end{cases}
   \]
4. If \( f : A \rightarrow B \) be a function where \( n(A) = n \), \( n(B) = n \), then total number of bijection is \( n! \).
5. Let \( f : A \rightarrow B \) be a function where \( n(A) = m \), \( n(B) = n \), then total number of onto functions is
   \[
   \sum_{r=1}^{n} (-1)^{n-r} \binom{n}{r} r^m; \quad 1 \leq n \leq m.
   \]

SOME USEFUL RESULTS ON APPLIED CALCULUS

1. \( B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \), where \( m > 0, n > 0 \) is called Beta Integral.
2. \( \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \), where \( n > 0 \) is called Gamma Integral.
3. \( \Gamma(n+1) = n\Gamma(n) = n! \), \( \Gamma(1) = 1 \), \( \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \).
4. \( B(m,n) = B(n,m) \), where \( B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \).
5. \( \Gamma(n)(1-m) = \pi \csc(m\pi) \), \( 0 < m < 1 \).
6. \( \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \).
7. \( \int_0^{\pi/2} \sin^p \theta \cos^q \theta d\theta = \frac{1}{2} \cdot B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \), where \( p > -1, q > -1 \).
8. \( \int_0^\infty \frac{\sin bx}{x} dx = \pm \frac{\pi}{2} \) according as \( b > 0, b < 0 \).
9. \( \int_0^{\pi/2} \sin^p \theta d\theta = \int_0^{\pi/2} \cos^q \theta d\theta = \frac{\pi}{2} \cdot \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)} \) (\( p > -1 \)).
10. \( \int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n} \), \( k > 0 \).
SERIES FORMULAE

1. \((x + a)^n = \sum_{k=0}^{n} \binom{n}{k} x^k a^{n-k}\).

2. \((1 + x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \ldots\)

3. \(e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\), \(-\infty < x < \infty\)

4. \(a^x = 1 + \frac{x \log a}{1!} + \frac{(x \log a)^2}{2!} + \frac{(x \log a)^3}{3!} + \ldots\)

5. \((1 - t)^{-r} = \sum_{x=0}^{\infty} \left(\frac{x+r-1}{x}\right) t^x\) for \(|t| < 1\).

6. \(\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\) [Use these kinds of expansions only when the variable in the expansion tends to 0]

7. \(\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\)

8. \(\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\)

9. \(\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\)

10. \(\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\)

11. \((1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots\)

12. \((1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots\)

13. \((1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\)

14. \((1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\)

15. \((1 + x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots\)

16. \((1 - x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots\)
1 Functions

In this Chapter we will cover various aspects of functions. We will look at the definition of a function, the domain and range of a function, what we mean by specifying the domain of a function and absolute value function.

1.1 What is a function?

1.1.1 Definition of a function

A function \( f \) from a set of elements \( X \) to a set of elements \( Y \) is a rule that assigns to each element \( x \) in \( X \) exactly one element \( y \) in \( Y \).

One way to demonstrate the meaning of this definition is by using arrow diagrams.

\[
\begin{align*}
\text{Set } X & \quad \text{Set } Y \\
1 & \quad 5 \\
2 & \quad 3 \\
3 & \quad 2 \\
4 & \\
\end{align*}
\]

\( f : X \to Y \) is a function. Every element in \( X \) has associated with it exactly one element of \( Y \).

\[
\begin{align*}
\text{Set } X & \quad \text{Set } Y \\
1 & \quad 5 \\
2 & \quad 3 \\
3 & \quad 2 \\
4 & \\
g & \quad 6 \\
1 & \quad 3 \\
2 & \quad 3 \\
4 & \\
\end{align*}
\]

\( g : X \to Y \) is not a function. The element 1 in set \( X \) is assigned two elements, 5 and 6 in set \( Y \).

A function can also be described as a set of ordered pairs \((x, y)\) such that for any \( x \)-value in the set, there is only one \( y \)-value. This means that there cannot be any repeated \( x \)-values with different \( y \)-values.

The examples above can be described by the following sets of ordered pairs.

\[
\begin{align*}
\text{F} & = \{(1,5),(3,3),(2,3),(4,2)\} \text{ is a function.} \\
\text{G} & = \{(1,5),(4,2),(2,3),(3,3),(1,6)\} \text{ is not a function.}
\end{align*}
\]

The definition we have given is a general one. While in the examples we have used numbers as elements of \( X \) and \( Y \), there is no reason why this must be so. However, in these notes we will only consider functions where \( X \) and \( Y \) are subsets of the real numbers.

In this setting, we often describe a function using the rule, \( y = f(x) \), and create a graph of that function by plotting the ordered pairs \((x, f(x))\) on the Cartesian Plane. This graphical representation allows us to use a test to decide whether or not we have the graph of a function: The Vertical Line Test.
1.1.2 The Vertical Line Test

The Vertical Line Test states that if it is not possible to draw a vertical line through a graph so that it cuts the graph in more than one point, then the graph is a function.

This is the graph of a function. All possible vertical lines will cut this graph only once.

This is not the graph of a function. The vertical line we have drawn cuts the graph twice.

1.1.3 Domain of a function

For a function $f : X \rightarrow Y$ the domain of $f$ is the set $X$.

This also corresponds to the set of $x$-values when we describe a function as a set of ordered pairs $(x, y)$.

If only the rule $y = f(x)$ is given, then the domain is taken to be the set of all real $x$ for which the function is defined. For example, $y = \sqrt{x}$ has domain; all real $x \geq 0$. This is sometimes referred to as the natural domain of the function.

1.1.4 Range of a function

For a function $f : X \rightarrow Y$ the range of $f$ is the set of $y$-values such that $y = f(x)$ for some $x$ in $X$.

This corresponds to the set of $y$-values when we describe a function as a set of ordered pairs $(x, y)$. The function $y = \sqrt{x}$ has range; all real $y \geq 0$.

Example

a. State the domain and range of $y = \sqrt{x + 4}$.

b. Sketch, showing significant features, the graph of $y = \sqrt{x + 4}$. 
Solution

a. The domain of \( y = \sqrt{x + 4} \) is all real \( x \geq -4 \). We know that square root functions are only defined for positive numbers so we require that \( x + 4 \geq 0 \), ie \( x \geq -4 \). We also know that the square root functions are always positive so the range of \( y = \sqrt{x + 4} \) is all real \( y \geq 0 \).

b.

The graph of \( y = \sqrt{x + 4} \).

Example

a. State the equation of the parabola sketched below, which has vertex \((3, -3)\).

b. Find the domain and range of this function.

Solution

a. The equation of the parabola is \( y = \frac{x^2 - 6x}{3} \).

b. The domain of this parabola is all real \( x \). The range is all real \( y \geq -3 \).

Example

Sketch \( x^2 + y^2 = 16 \) and explain why it is not the graph of a function.

Solution

\( x^2 + y^2 = 16 \) is not a function as it fails the vertical line test. For example, when \( x = 0 \) \( y = \pm 4 \).
The graph of $x^2 + y^2 = 16$.

Example

Sketch the graph of $f(x) = 3x - x^2$ and find

a. the domain and range

b. $f(q)$

c. $f(x^2)$

d. $\frac{f(2+h)-f(2)}{h}, h \neq 0$.

Solution

The graph of $f(x) = 3x - x^2$.

a. The domain is all real $x$. The range is all real $y$ where $y \leq 2.25$.

b. $f(q) = 3q - q^2$
c. \( f(x^2) = 3(x^2) - (x^2)^2 = 3x^2 - x^4 \)

d. 
\[
\frac{f(2 + h) - f(2)}{h} = \frac{(3(2 + h) - (2 + h)^2) - (3(2) - (2)^2)}{h} \\
= \frac{6 + 3h - (h^2 + 4h + 4) - 2}{h} \\
= \frac{-h^2 - h}{h} \\
= -h - 1
\]

Example

Sketch the graph of the function \( f(x) = (x - 1)^2 + 1 \) and show that \( f(p) = f(2 - p) \).
Illustrate this result on your graph by choosing one value of \( p \).

Solution

\[
f(2 - p) = ((2 - p) - 1)^2 + 1 \\
= (1 - p)^2 + 1 \\
= (p - 1)^2 + 1 \\
= f(p)
\]
The sketch illustrates the relationship $f(p) = f(2 - p)$ for $p = -1$. If $p = -1$ then $2 - p = 2 - (-1) = 3$, and $f(-1) = f(3)$.

1.2 Specifying or restricting the domain of a function

We sometimes give the rule $y = f(x)$ along with the domain of definition. This domain may not necessarily be the natural domain. For example, if we have the function

\[ y = x^2 \text{ for } 0 \leq x \leq 2 \]

then the domain is given as $0 \leq x \leq 2$. The natural domain has been restricted to the subinterval $0 \leq x \leq 2$.

Consequently, the range of this function is all real $y$ where $0 \leq y \leq 4$. We can best illustrate this by sketching the graph.

The graph of $y = x^2$ for $0 \leq x \leq 2$. 
1.3 The absolute value function

Before we define the absolute value function we will review the definition of the absolute value of a number.

The Absolute value of a number $x$ is written $|x|$ and is defined as

$$|x| = x \text{ if } x \geq 0 \quad \text{or} \quad |x| = -x \text{ if } x < 0.$$ 

That is, $|4| = 4$ since 4 is positive, but $|-2| = 2$ since $-2$ is negative.

We can also think of $|x|$ geometrically as the distance of $x$ from 0 on the number line.

$$\begin{align*}
\text{Distance from } 0 &\quad \text{Distance from } 4 \\
\text{Distance from } -2 &\quad \text{Distance from } 4
\end{align*}$$

More generally, $|x - a|$ can be thought of as the distance of $x$ from $a$ on the number line.

$$\begin{align*}
\text{Distance from } a &\quad \text{Distance from } x \\
|a - x| &\quad |x - a|
\end{align*}$$

Note that $|a - x| = |x - a|$.

The absolute value function is written as $y = |x|$.

We define this function as

$$y = \begin{cases} 
+x & \text{if } x \geq 0 \\
-x & \text{if } x < 0
\end{cases}$$

From this definition we can graph the function by taking each part separately. The graph of $y = |x|$ is given below.

The graph of $y = |x|$.
Example

Sketch the graph of $y = |x - 2|$.

Solution

For $y = |x - 2|$ we have

$$y = \begin{cases} 
  +(x-2) & \text{when } x - 2 \geq 0 \ \text{or} \ x \geq 2 \\
  -(x-2) & \text{when } x - 2 < 0 \ \text{or} \ x < 2 
\end{cases}$$

That is,

$$y = \begin{cases} 
  x - 2 & \text{for } x \geq 2 \\
  -x + 2 & \text{for } x < 2 
\end{cases}$$

Hence we can draw the graph in two parts.

We could have sketched this graph by first of all sketching the graph of $y = x - 2$ and then reflecting the negative part in the $x$-axis. We will use this fact to sketch graphs of this type in Chapter 2.

1.4 Exercises

1. a. State the domain and range of $f(x) = \sqrt{9-x^2}$.
   
   b. Sketch the graph of $y = \sqrt{9-x^2}$.

2. Given $\psi(x) = x^2 + 5$, find, in simplest form, $\frac{\psi(x+h) - \psi(x)}{h}$ \quad \text{for } h \neq 0.$

3. Sketch the following functions stating the domain and range of each:
   
   a. $y = \sqrt{x-1}$
b. \( y = |2x| \)

c. \( y = \frac{1}{x-4} \)

d. \( y = |2x| - 1. \)

4. a. Find the perpendicular distance from \((0, 0)\) to the line \(x + y + k = 0\)

b. If the line \(x + y + k = 0\) cuts the circle \(x^2 + y^2 = 4\) in two distinct points, find the restrictions on \(k\).

5. Sketch the following, showing their important features.

a. \( y = \left(\frac{1}{2}\right)^x \)

b. \( y^2 = x^2 \).

6. Explain the meanings of function, domain and range. Discuss whether or not \(y^2 = x^3\) is a function.

7. Sketch the following relations, showing all intercepts and features. State which ones are functions giving their domain and range.

a. \( y = -\sqrt{4 - x^2} \)

b. \( |x| - |y| = 0 \)

c. \( y = x^3 \)

d. \( y = \frac{x}{|x|}; x \neq 0 \)

e. \( |y| = x \).

8. If \(A(x) = x^2 + 2 + \frac{1}{x^2}, x \neq 0\), prove that \(A(p) = A(\frac{1}{p})\) for all \(p \neq 0\).

9. Write down the values of \(x\) which are not in the domain of the following functions:

a. \( f(x) = \sqrt{x^2 - 4x} \)

b. \( g(x) = \frac{x}{x^2 - 1} \)

10. If \(\phi(x) = \log\left(\frac{x}{x-1}\right)\), find in simplest form:

a. \( \phi(3) + \phi(4) + \phi(5) \)

b. \( \phi(3) + \phi(4) + \phi(5) + \cdots + \phi(n) \)

11. a. If \(y = x^2 + 2x\) and \(x = (z - 2)^2\), find \(y\) when \(z = 3\).

b. Given \(L(x) = 2x + 1\) and \(M(x) = x^2 - x\), find

i. \( L(M(x)) \)

ii. \( M(L(x)) \)
12. Using the sketches, find the value(s) of the constants in the given equations:

\[ y = ax^2 - b \]

\[ y = \frac{a}{bx^2 + 1} \]

13. a. Define \(|a|\), the absolute value of \(a\), where \(a\) is real.

b. Sketch the relation \(|x| + |y| = 1\).

14. Given that \(S(n) = \frac{n}{2n+1}\), find an expression for \(S(n - 1)\).

Hence show that \(S(n) - S(n - 1) = \frac{1}{(2n-1)(2n+1)}\).
2  More about functions

In this Chapter we will look at the effects of stretching, shifting and reflecting the basic functions, \( y = x^2, y = x^3, y = \frac{1}{x}, y = |x|, y = a^x, x^2 + y^2 = r^2 \). We will introduce the concepts of even and odd functions, increasing and decreasing functions and will solve equations using graphs.

2.1  Modifying functions by shifting

2.1.1  Vertical shift

We can draw the graph of \( y = f(x) + k \) from the graph of \( y = f(x) \) as the addition of the constant \( k \) produces a vertical shift. That is, adding a constant to a function moves the graph up \( k \) units if \( k > 0 \) or down \( k \) units if \( k < 0 \). For example, we can sketch the function \( y = x^2 - 3 \) from our knowledge of \( y = x^2 \) by shifting the graph of \( y = x^2 \) down by 3 units. That is, if \( f(x) = x^2 \) then \( f(x) - 3 = x^2 - 3 \).

We can also write \( y = f(x) - 3 \) as \( y + 3 = f(x) \), so replacing \( y \) by \( y + 3 \) in \( y = f(x) \) also shifts the graph down by 3 units.

2.1.2  Horizontal shift

We can draw the graph of \( y = f(x - a) \) if we know the graph of \( y = f(x) \) as placing the constant \( a \) inside the brackets produces a horizontal shift. If we replace \( x \) by \( x - a \) inside the function then the graph will shift to the left by \( a \) units if \( a < 0 \) and to the right by \( a \) units if \( a > 0 \).
For example we can sketch the graph of $y = \frac{1}{x-2}$ from our knowledge of $y = \frac{1}{x}$ by shifting this graph to the right by 2 units. That is, if $f(x) = \frac{1}{x}$ then $f(x - 2) = \frac{1}{x-2}$.

Note that the function $y = \frac{1}{x-2}$ is not defined at $x = 2$. The point (1, 1) has been shifted to (1, 3).

### 2.2 Modifying functions by stretching

We can sketch the graph of a function $y = bf(x)$ ($b > 0$) if we know the graph of $y = f(x)$ as multiplying by the constant $b$ will have the effect of stretching the graph in the $y$-direction by a factor of $b$. That is, multiplying $f(x)$ by $b$ will change all of the $y$-values proportionally.

For example, we can sketch $y = 2x^2$ from our knowledge of $y = x^2$ as follows:

The graph of $y = x^2$.  

The graph of $y = 2x^2$. Note, all the $y$-values have been multiplied by 2, but the $x$-values are unchanged.

We can sketch the graph of $y = \frac{1}{2}x^2$ from our knowledge of $y = x^2$ as follows:
The graph of $y = x^2$.

The graph of $y = \frac{1}{2}x^2$. Note, all the $y$-values have been multiplied by $\frac{1}{2}$, but the $x$-values are unchanged.

2.3 Modifying functions by reflections

2.3.1 Reflection in the $x$-axis

We can sketch the function $y = -f(x)$ if we know the graph of $y = f(x)$, as a minus sign in front of $f(x)$ has the effect of reflecting the whole graph in the $x$-axis. (Think of the $x$-axis as a mirror.) For example, we can sketch $y = -|x|$ from our knowledge of $y = |x|$.

The graph of $y = |x|$.

The graph of $y = -|x|$. It is the reflection of $y = |x|$ in the $x$-axis.

2.3.2 Reflection in the $y$-axis

We can sketch the graph of $y = f(-x)$ if we know the graph of $y = f(x)$ as the graph of $y = f(-x)$ is the reflection of $y = f(x)$ in the $y$-axis.
For example, we can sketch \( y = 3^{-x} \) from our knowledge of \( y = 3^x \).

The graph of \( y = 3^x \).

The graph of \( y = 3^{-x} \). It is the reflection of \( y = 3^x \) in the \( y \)-axis.

### 2.4 Other effects

We can sketch the graph of \( y = |f(x)| \) if we know the graph of \( y = f(x) \) as the effect of the absolute value is to reflect all of the negative values of \( f(x) \) in the \( x \)-axis. For example, we can sketch the graph of \( y = |x^2 - 3| \) from our knowledge of the graph of \( y = x^2 - 3 \).

The graph of \( y = x^2 - 3 \).

The graph of \( y = |x^2 - 3| \). The negative values of \( y = x^2 - 3 \) have been reflected in the \( x \)-axis.

### 2.5 Combining effects

We can use all the above techniques to graph more complex functions. For example, we can sketch the graph of \( y = 2 - (x+1)^2 \) from the graph of \( y = x^2 \) provided we can analyse the combined effects of the modifications. Replacing \( x \) by \( x + 1 \) (or \( x - (-1) \)) moves the
The effect of the $-$ sign in front of the brackets turns the graph upside down. The effect of adding 2 moves the graph up 2 units. We can illustrate these effects in the following diagrams.

Similarly, we can sketch the graph of $(x-h)^2 + (y-k)^2 = r^2$ from the graph of $x^2 + y^2 = r^2$. Replacing $x$ by $x - h$ shifts the graph sideways $h$ units. Replacing $y$ by $y - k$ shifts the graph up or down $k$ units. (We remarked before that $y = f(x) + k$ could be written as $y - k = f(x)$.)

For example, we can use the graph of the circle of radius 3, $x^2 + y^2 = 9$, to sketch the graph of $(x - 2)^2 + (y + 4)^2 = 9$. 
The graph of $x^2 + y^2 = 9$.

This is a circle centre $(0, 0)$, radius 3.

The graph of $(x - 2)^2 + (y + 4)^2 = 9$.

This is a circle centre $(2, -4)$, radius 3.

Replacing $x$ by $x - 2$ has the effect of shifting the graph of $x^2 + y^2 = 9$ two units to the right. Replacing $y$ by $y + 4$ shifts it down 4 units.

2.6 Graphing by addition of ordinates

We can sketch the graph of functions such as $y = |x| + |x - 2|$ by drawing the graphs of both $y = |x|$ and $y = |x - 2|$ on the same axes then adding the corresponding $y$-values.
The graph of $y = |x| + |x - 2|$. 

At each point of $x$ the $y$-values of $y = |x|$ and $y = |x - 2|$ have been added. This allows us to sketch the graph of $y = |x| + |x - 2|$. 

This technique for sketching graphs is very useful for sketching the graph of the sum of two trigonometric functions.

### 2.7 Using graphs to solve equations

We can solve equations of the form $f(x) = k$ by sketching $y = f(x)$ and the horizontal line $y = k$ on the same axes. The solution to the equation $f(x) = k$ is found by determining the $x$-values of any points of intersection of the two graphs.
For example, to solve $|x - 3| = 2$ we sketch $y = |x - 3|$ and $y = 2$ on the same axes.

The $x$-values of the points of intersection are 1 and 5. Therefore $|x - 3| = 2$ when $x = 1$ or $x = 5$.

Example

The graph of $y = f(x)$ is sketched below.

For what values of $k$ does the equation $f(x) = k$ have
1. 1 solution
2. 2 solutions
3. 3 solutions?

Solution

If we draw a horizontal line $y = k$ across the graph $y = f(x)$, it will intersect once when $k > 0$ or $k < -4$, twice when $k = 0$ or $k = -4$ and three times when $-4 < k < 0$. Therefore the equation $f(x) = k$ will have
1. 1 solution if $k > 0$ or $k < -4$

2. 2 solutions if $k = 0$ or $k = -4$

3. 3 solutions if $-4 < k < 0$.

### 2.8 Exercises

1. Sketch the following:
   a. $y = x^2$  
   b. $y = \frac{1}{3}x^2$  
   c. $y = -x^2$  
   d. $y = (x + 1)^2$

2. Sketch the following:
   a. $y = \frac{1}{x}$  
   b. $y = \frac{1}{x-2}$  
   c. $y = -\frac{2}{x}$  
   d. $y = \frac{1}{x+1} + 2$

3. Sketch the following:
   a. $y = x^3$  
   b. $y = |x^3 - 2|$  
   c. $y = 3 - (x - 1)^3$

4. Sketch the following:
   a. $y = |x|$  
   b. $y = 2|x - 2|$  
   c. $y = 4 - |x|$  

5. Sketch the following:
   a. $x^2 + y^2 = 16$  
   b. $x^2 + (y + 2)^2 = 16$  
   c. $(x - 1)^2 + (y - 3)^2 = 16$

6. Sketch the following:
   a. $y = \sqrt{9 - x^2}$  
   b. $y = \sqrt{9 - (x - 1)^2}$  
   c. $y = \sqrt{9 - x^2} - 3$

7. Show that $\frac{x - 1}{x - 2} = \frac{1}{x - 2} + 1$.

Hence sketch the graph of $y = \frac{x - 1}{x - 2}$.

8. Sketch $y = \frac{x + 1}{x - 1}$.

9. Graph the following relations in the given interval:
   a. $y = |x| + x + 1$ for $-2 \leq x \leq 2$ [Hint: Sketch by adding ordinates]
   b. $y = |x| + |x - 1|$ for $-2 \leq x \leq 3$
   c. $y = 2^x + 2^{-x}$ for $-2 \leq x \leq 2$
   d. $|x - y| = 1$ for $-1 \leq x \leq 3$.

10. Sketch the function $f(x) = |x^2 - 1| - 1$. 

11. Given \( y = f(x) \) as sketched below, sketch
   a. \( y = 2f(x) \)
   b. \( y = -f(x) \)
   c. \( y = f(-x) \)
   d. \( y = f(x) + 4 \)
   e. \( y = f(x - 3) \)
   f. \( y = f(x + 1) - 2 \)
   g. \( y = 3 - 2f(x - 3) \)
   h. \( y = |f(x)| \)

12. By sketching graphs solve the following equations:
   a. \( |2x| = 4 \)
   b. \( \frac{1}{x-2} = -1 \)
   c. \( x^3 = x^2 \)
   d. \( x^2 = \frac{1}{x} \)

13. Solve \( |x - 2| = 3 \).
   a. algebraically
   b. geometrically.

14. The parabolas \( y = (x - 1)^2 \) and \( y = (x - 3)^2 \) intersect at a point \( P \). Find the coordinates of \( P \).

15. Sketch the circle \( x^2 + y^2 - 2x - 14y + 25 = 0 \). [Hint: Complete the squares.] Find the values of \( k \), so that the line \( y = k \) intersects the circle in two distinct points.

16. Solve \( \frac{4}{5-x} = 1 \), using a graph.

17. Find all real numbers \( x \) for which \( |x - 2| = |x + 2| \).

18. Given that \( Q(p) = p^2 - p \), find possible values of \( n \) if \( Q(n) = 2 \).

19. Solve \( |x - 4| = 2x \).
   a. algebraically
   b. geometrically.
2.9 Even and odd functions

Definition:

A function, \( y = f(x) \), is even if \( f(x) = f(-x) \) for all \( x \) in the domain of \( f \).

Geometrically, an even function is symmetrical about the \( y \)-axis (it has line symmetry).
The function \( f(x) = x^2 \) is an even function as \( f(-x) = (-x)^2 = x^2 = f(x) \) for all values of \( x \). We illustrate this on the following graph.

![The graph of \( y = x^2 \).](image)

Definition:

A function, \( y = f(x) \), is odd if \( f(-x) = -f(x) \) for all \( x \) in the domain of \( f \).

Geometrically, an odd function is symmetrical about the origin (it has rotational symmetry).
The function \( f(x) = x \) is an odd function as \( f(-x) = -x = -f(x) \) for all values of \( x \). This is illustrated on the following graph.

![The graph of \( y = x \).](image)
Example

Decide whether the following functions are even, odd or neither.

1. \( f(x) = 3x^2 - 4 \)
2. \( g(x) = \frac{1}{2x} \)
3. \( f(x) = x^3 - x^2 \).

Solution

1. \[
   f(-x) = 3(-x)^2 - 4 = 3x^2 - 4 = f(x)
   \]
   The function \( f(x) = 3x^2 - 4 \) is even.

2. \[
   g(-x) = \frac{1}{2(-x)} = \frac{1}{-2x} = -\frac{1}{2x} = -g(x)
   \]
   Therefore, the function \( g \) is odd.

3. \[
   f(-x) = (-x)^3 - (-x)^2 = -x^3 - x^2
   \]
   This function is neither even (since \(-x^3 - x^2 \neq x^3 - x^2\)) nor odd (since \(-x^3 - x^2 \neq -(x^3 - x^2)\)).

Example

Sketched below is part of the graph of \( y = f(x) \).

![Graph of f(x)](image)

Complete the graph if \( y = f(x) \) is

1. odd
2. even.
2.10 Increasing and decreasing functions

Here we will introduce the concepts of increasing and decreasing functions. In Chapter 5 we will relate these concepts to the derivative of a function.

Definition:

A function is increasing on an interval $I$, if for all $a$ and $b$ in $I$ such that $a < b$, $f(a) < f(b)$.

The function $y = 2^x$ is an example of a function that is increasing over its domain. The function $y = x^2$ is increasing for all real $x > 0$. 
The graph of $y = 2^x$. This function is increasing for all real $x$.

Notice that when a function is increasing it has a positive slope.

Definition:

A graph is decreasing on an interval $I$, if for all $a$ and $b$ in $I$ such that $a < b$, $f(a) > f(b)$.

The function $y = 2^{-x}$ is decreasing over its domain. The function $y = x^2$ is decreasing on the interval $x < 0$.

Notice that if a function is decreasing then it has negative slope.

### 2.11 Exercises

1. Given the graph below of $y = f(x)$:
   
   a. State the domain and range.
   
   b. Where is the graph
i increasing?
ii decreasing?

c. if \( k \) is a constant, find the values of \( k \) such that \( f(x) = k \) has
i no solutions
ii 1 solution
iii 2 solutions
iv 3 solutions
v 4 solutions.

d. Is \( y = f(x) \) even, odd or neither?

2. Complete the following functions if they are defined to be (a) even (b) odd.

3. Determine whether the following functions are odd, even or neither.

a. \( y = x^4 + 2 \)  

b. \( y = \sqrt{4 - x^2} \)  

c. \( y = 2^x \)  

d. \( y = x^3 + 3x \)  

e. \( y = \frac{x}{x^2} \)  

f. \( y = \frac{1}{x^2 - 4} \)  

g. \( y = \frac{1}{x^2 + 4} \)  

h. \( y = \frac{x}{x^3 + 3} \)  

i. \( y = 2^x + 2^{-x} \)  

j. \( y = |x - 1| + |x + 1| \)  

4. Given \( y = f(x) \) is even and \( y = g(x) \) is odd, prove

a. if \( h(x) = f(x) \cdot g(x) \) then \( h(x) \) is odd

b. if \( h(x) = (g(x))^2 \) then \( h(x) \) is even
c. if \( h(x) = \frac{f(x)}{g(x)} \), \( g(x) \neq 0 \), then \( h(x) \) is odd

d. if \( h(x) = f(x) \cdot (g(x))^2 \) then \( h(x) \) is even.

5. Consider the set of all odd functions which are defined at \( x = 0 \). Can you prove that for every odd function in this set \( f(0) = 0 \)? If not, give a counter-example.
3 Piecewise functions and solving inequalities

In this Chapter we will discuss functions that are defined piecewise (sometimes called piecemeal functions) and look at solving inequalities using both algebraic and graphical techniques.

3.1 Piecewise functions

3.1.1 Restricting the domain

In Chapter 1 we saw how functions could be defined on a subinterval of their natural domain. This is frequently called restricting the domain of the function. In this Chapter we will extend this idea to define functions piecewise.

Sketch the graph of $y = 1 - x^2$ for $x \geq 0$.

![Graph of $y = 1 - x^2$ for $x \geq 0$.]

Sketch the graph of $y = 1 - x$ for $x < 0$.

![Graph of $y = 1 - x$ for $x < 0$.]
We can now put these pieces together to define a function of the form

\[
f(x) = \begin{cases} 
  1 - x^2 & \text{for } x \geq 0 \\
  1 - x & \text{for } x < 0
\end{cases}
\]

We say that this function is defined *piecewise*. First note that it *is* a function; each value of \(x\) in the domain is assigned exactly one value of \(y\). This is easy to see if we graph the function and use the vertical line test. We graph this function by graphing each piece of it in turn.

![Graph of piecewise function](image)

The graph shows that \(f\) defined in this way is a function. The two pieces of \(y = f(x)\) meet so \(f\) is a continuous function.

The absolute value function

\[
f(x) = \begin{cases} 
  x & \text{for } x \geq 0 \\
  -x & \text{for } x < 0
\end{cases}
\]

is another example of a piecewise function.

**Example**

Sketch the function

\[
f(x) = \begin{cases} 
  x^2 + 1 & \text{for } x \geq 0 \\
  2 & \text{for } x < 0
\end{cases}
\]
Solution

This function is not continuous at $x = 0$ as the two branches of the graph do not meet.

Notice that we have put an open square (or circle) around the point $(0, 2)$ and a solid square (or circle) around the point $(0, 1)$. This is to make it absolutely clear that $f(0) = 1$ and not 2. When defining a function piecewise, we must be extremely careful to assign to each $x$ exactly one value of $y$.

3.2 Exercises

1. For the function

\[ f(x) = \begin{cases} 
1 - x^2 & \text{for } x \geq 0 \\
1 - x & \text{for } x < 0 
\end{cases} \]

evaluate

a. $2f(-1) + f(2)$

b. $f(a^2)$

2. For the function given in 1, solve $f(x) = 2$.

3. Below is the graph of $y = g(x)$. Write down the rules which define $g(x)$ given that its pieces are hyperbolic, circular and linear.
4. a. Sketch the graph of \( y = f(x) \) if
\[
f(x) = \begin{cases} \frac{-\sqrt{4 - x^2}}{x^2 - 4} & \text{for } -2 \leq x \leq 0 \\ x^2 - 4 & \text{for } x > 0 \end{cases}
\]

b. State the range of \( f \).

c. Solve
   i. \( f(x) = 0 \)
   ii. \( f(x) = -3 \).

d. Find \( k \) if \( f(x) = k \) has
   i. 0
   ii. 1
   iii. 2 solutions.

5. Sketch the graph of \( y = f(x) \) if
\[
f(x) = \begin{cases} 1 - |x - 1| & \text{for } x \geq 0 \\ |x + 1| & \text{for } x < 0 \end{cases}
\]

6. Sketch the graph of \( y = g(x) \) if
\[
g(x) = \begin{cases} \frac{2}{x+2} & \text{for } x < -1 \\ 2 & \text{for } -1 \leq x < 1 \\ 2^x & \text{for } x \geq 1 \end{cases}
\]
7. McMaths burgers are to modernise their logo as shown below.

Write down a piecewise function that represents this function using (a) 4 (b) 3 (c) 2 pieces (i.e. rules that define the function).

8. a. The following piecewise function is of the form

\[ f(x) = \begin{cases} 
  ax^2 + b & \text{for } 0 < x \leq 2 \\
  cx + d & \text{for } x > 2 
\end{cases} \]

Determine the values of \( a, b, c \) and \( d \).

b. Complete the graph so that \( f(x) \) is an odd function defined for all real \( x, x \neq 0 \).

c. Write down the equations that now define \( f(x), x \neq 0 \).
3.3 Inequalities

We can solve inequalities using both algebraic and graphical methods. Sometimes it is easier to use an algebraic method and sometimes a graphical one. For the following examples we will use both, as this allows us to make the connections between the algebra and the graphs.

<table>
<thead>
<tr>
<th>Algebraic method</th>
<th>Graphical method</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Solve $3 - 2x \geq 1$.</td>
<td></td>
</tr>
<tr>
<td>This is a (2 Unit) linear inequality.</td>
<td></td>
</tr>
<tr>
<td>Remember to reverse the inequality sign when multiplying or dividing by a negative number.</td>
<td></td>
</tr>
<tr>
<td>$3 - 2x \geq 1$</td>
<td></td>
</tr>
<tr>
<td>$-2x \geq -2$</td>
<td></td>
</tr>
<tr>
<td>$x \leq 1$</td>
<td></td>
</tr>
<tr>
<td>When is the line $y = 3 - 2x$ above or on the horizontal line $y = 1$? From the graph, we see that this is true for $x \leq 1$.</td>
<td></td>
</tr>
</tbody>
</table>

<p>| 2. Solve $x^2 - 4x + 3 &lt; 0$. |
| This is a (2 Unit) quadratic inequality. Factorise and use a number line. |
| $x^2 - 4x + 3 &lt; 0$ |
| $(x - 3)(x - 1) &lt; 0$ |
| The critical values are 1 and 3, which divide the number line into three intervals. We take points in each interval to determine the sign of the inequality; eg use $x = 0$, $x = 2$ and $x = 4$ as test values. |</p>
<table>
<thead>
<tr>
<th>positive</th>
<th>negative</th>
<th>positive</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>$0$</td>
<td>$1$</td>
</tr>
<tr>
<td>$2$</td>
<td>$3$</td>
<td>$4$</td>
</tr>
<tr>
<td>When does the parabola have negative $y$-values? OR When is the parabola under the $x$-axis? From the graph, we see that this happens when $1 &lt; x &lt; 3$.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Thus, the solution is $1 < x < 3$. 
3. Solve $\frac{1}{x - 4} \leq 1$.
This is a 3 Unit inequality. There is a variable in the denominator. Remember that a denominator can never be zero, so in this case $x \neq 4$.
First multiply by the square of the denominator
\[
\begin{align*}
x - 4 &\leq (x - 4)^2, x \neq 4 \\
x - 4 &\leq x^2 - 8x + 16 \\
0 &\leq x^2 - 9x + 20 \\
0 &\leq (x - 4)(x - 5)
\end{align*}
\]
Mark the critical values on the number line and test $x = 0$, $x = 4.5$ and $x = 6$.

\[
\begin{array}{c|c|c}
\text{positive} & \text{neg} & \text{positive} \\
0 & 1 & 2 \\
3 & 4 & 5 \\
6 & & \\
\end{array}
\]

Therefore, $x < 4$ or $x \geq 5$.

4. Solve $x - 3 < \frac{10}{x}$.
Consider $x - 3 = \frac{10}{x}$, $x \neq 0$.
Multiply by $x$ we get
\[
\begin{align*}
x^2 - 3x &= 10 \\
x^2 - 3x - 10 &= 0 \\
(x - 5)(x + 2) &= 0
\end{align*}
\]
Therefore, the critical values are $-2$, $0$ and $5$ which divide the number line into four intervals. We can use $x = -3$, $x = -1$, $x = 1$ and $x = 6$ as test values in the inequality. The points $x = -3$ and $x = 1$ satisfy the inequality, so the solution is $x < -2$ or $0 < x < 5$.

(Notice that we had to include $0$ as one of our critical values.)

Let $y = \frac{1}{x - 4}$.

$y = \frac{1}{x - 4}$ is not defined for $x = 4$. It is a hyperbola with vertical asymptote at $x = 4$. To solve our inequality we need to find the values of $x$ for which the hyperbola lies on or under the line $y = 1$. $(5, 1)$ is the point of intersection. So, from the graph we see that $\frac{1}{x - 4} \leq 1$ when $x < 4$ or $x \geq 5$.

Sketch $y = x - 3$ and then $y = \frac{10}{x}$. Note that second of these functions is not defined for $x = 0$.

For what values of $x$ does the line lie under the hyperbola? From the graph, we see that this happens when $x < -2$ or $0 < x < 5$. 
Example

Sketch the graph of \( y = |2x - 6| \).

Hence, where possible,

\[ a. \text{ Solve} \]
\[ \begin{align*}
&i \quad |2x - 6| = 2x \\
&ii \quad |2x - 6| > 2x \\
&iii \quad |2x - 6| = x + 3 \\
&iv \quad |2x - 6| < x + 3 \\
v \quad |2x - 6| = x - 3
\]

\[ b. \text{ Determine the values of } k \text{ for which } |2x - 6| = x + k \text{ has exactly two solutions.} \]

Solution

\[ f(x) = |2x - 6| = \begin{cases} 
2x - 6 & \text{for } x \geq 3 \\
-(2x - 6) & \text{for } x < 3
\end{cases} \]

\[ a. \]
\[ i \quad \text{Mark in the graph of } y = 2x. \text{ It is parallel to one arm of the absolute value graph. It has one point of intersection with } y = |2x - 6| = -2x + 6 \text{ (} x < 3 \text{) at } x = 1.5. \]
\[ ii \quad \text{When is the absolute value graph above the line } y = 2x? \text{ From the graph, when } x < 1.5. \]
To solve \(|2x-6|=x+3\), take \(|2x-6|=2x-6=x+3\) when \(x \geq 3\). This gives us the solution \(x = 9\). Then take \(|2x-6|=-2x+6=x+3\) when \(x < 3\) which gives us the solution \(x = 1\).

iv. When is the absolute value graph below the line \(y = x+3\)?

From the graph, \(1 < x < 9\).

v. \(y = x - 3\) intersects the absolute value graph at \(x = 3\) only.

b. \(k\) represents the \(y\)-intercept of the line \(y = x + k\). When \(k = -3\), there is one point of intersection. (See (a) (v) above). For \(k > -3\), lines of the form \(y = x + k\) will have two points of intersection. Hence \(|2x-6|=x+k\) will have two solutions for \(k > -3\).

### 3.4 Exercises

1. Solve
   a. \(x^2 \leq 4x\)
   b. \(\frac{4p}{p+3} \leq 1\)
   c. \(\frac{7}{9-2x} > -1\)

2. a. Sketch the graph of \(y = 4x(x-3)\).
   b. Hence solve \(4x(x-3) \leq 0\).

3. a. Find the points of intersection of the graphs \(y = 5-x\) and \(y = \frac{4}{x}\).
   b. On the same set of axes, sketch the graphs of \(y = 5-x\) and \(y = \frac{4}{x}\).
   c. Using part (ii), or otherwise, write down all the values of \(x\) for which

\[
5-x > \frac{4}{x}
\]

4. a. Sketch the graph of \(y = 2^x\).
   b. Solve \(2^x < \frac{1}{2}\).
   c. Suppose \(0 < a < b\) and consider the points \(A(a, 2^a)\) and \(B(b, 2^b)\) on the graph of \(y = 2^x\). Find the coordinates of the midpoint \(M\) of the segment \(AB\).
   Explain why

\[
\frac{2^a + 2^b}{2} > 2^{\frac{a+b}{2}}
\]

5. a. Sketch the graphs of \(y = x\) and \(y = |x-5|\) on the same diagram.
   b. Solve \(|x-5| > x\).
   c. For what values of \(m\) does \(mx = |x-5|\) have exactly
      i. two solutions
      ii. no solutions

6. Solve \(5x^2 - 6x - 3 \leq |8x|\).
4 Polynomials

Many of the functions we have been using so far have been polynomials. In this Chapter we will study them in more detail.

Definition

A real polynomial, \( P(x) \), of degree \( n \) is an expression of the form

\[
P(x) = p_n x^n + p_{n-1} x^{n-1} + \cdots + p_2 x^2 + p_1 x + p_0
\]

where \( p_n \neq 0 \), \( p_0, p_1, \ldots, p_n \) are real and \( n \) is an integer \( \geq 0 \).

All polynomials are defined for all real \( x \) and are continuous functions.

We are familiar with the quadratic polynomial, \( Q(x) = ax^2 + bx + c \) where \( a \neq 0 \). This polynomial has degree 2.

The function \( f(x) = \sqrt{x} + x \) is not a polynomial as it has a power which is not an integer \( \geq 0 \) and so does not satisfy the definition.

4.1 Graphs of polynomials and their zeros

4.1.1 Behaviour of polynomials when \( |x| \) is large

One piece of information that can be a great help when sketching a polynomial is the way it behaves for values of \( x \) when \( |x| \) is large. That is, values of \( x \) which are large in magnitude.

The term of the polynomial with the highest power of \( x \) is called the leading or dominant term. For example, in the polynomial \( P(x) = x^6 - 3x^4 - 1 \), the term \( x^6 \) is the dominant term.

When \( |x| \) is large, the dominant term determines how the graph behaves as it is so much larger in magnitude than all the other terms.

How the graph behaves for \( |x| \) large depends on the power and coefficient of the dominant term.

There are four possibilities which we summarise in the following diagrams:

1. Dominant term with even power and positive coefficient, eg \( y = x^2 \).
2. Dominant term with even power and negative coefficient, eg \( Q(x) = -x^2 \).
3. Dominant term with odd power and positive coefficient, eg $y = x^3$.
4. Dominant term with odd power and negative coefficient, eg $Q(x) = -x^3$.

This gives us a good start to graphing polynomials. All we need do now is work out what happens in the middle. In Chapter 5 we will use calculus methods to do this. Here we will use our knowledge of the roots of polynomials to help complete the picture.

### 4.1.2 Polynomial equations and their roots

If, for a polynomial $P(x)$, $P(k) = 0$ then we can say
1. $x = k$ is a root of the equation $P(x) = 0$.
2. $x = k$ is a zero of $P(x)$.
3. $k$ is an $x$-intercept of the graph of $P(x)$.

### 4.1.3 Zeros of the quadratic polynomial

The quadratic polynomial equation $Q(x) = ax^2 + bx + c = 0$ has two roots that may be:
1. real (rational or irrational) and distinct,
2. real (rational or irrational) and equal,
3. complex (not real).

We will illustrate all of these cases with examples, and will show the relationship between the nature and number of zeros of $Q(x)$ and the $x$-intercepts (if any) on the graph.

1. Let $Q(x) = x^2 - 4x + 3$.
   We find the zeros of $Q(x)$ by solving the equation $Q(x) = 0$.
   
   
   $x^2 - 4x + 3 = 0$
   $(x - 1)(x - 3) = 0$
   
   Therefore $x = 1$ or $3$.

   The roots are rational (hence real) and distinct.
2. Let $Q(x) = x^2 - 4x - 3$.
Solving the equation $Q(x) = 0$ we get,
\[ x^2 - 4x - 3 = 0 \]
\[ x = \frac{4 \pm \sqrt{16 + 12}}{2} \]
Therefore $x = 2 \pm \sqrt{7}$.
The roots are irrational (hence real) and distinct.

3. Let $Q(x) = x^2 - 4x + 4$.
Solving the equation $Q(x) = 0$ we get,
\[ x^2 - 4x + 4 = 0 \]
\[ (x - 2)^2 = 0 \]
Therefore $x = 2$.
The roots are rational (hence real) and equal. $Q(x) = 0$ has a repeated or double root at $x = 2$.

Notice that the graph turns at the double root $x = 2$.

4. Let $Q(x) = x^2 - 4x + 5$.
Solving the equation $Q(x) = 0$ we get,
\[ x^2 - 4x + 5 = 0 \]
\[ x = \frac{4 \pm \sqrt{16 - 20}}{2} \]
Therefore $x = 2 \pm \sqrt{-4}$.
There are no real roots. In this case the roots are complex.

Notice that the graph does not intersect the $x$-axis. That is $Q(x) > 0$ for all real $x$. Therefore $Q$ is positive definite.
We have given above four examples of quadratic polynomials to illustrate the relationship between the zeros of the polynomials and their graphs.

In particular we saw that:

i. if the quadratic polynomial has two real distinct zeros, then the graph of the polynomial cuts the $x$-axis at two distinct points;

ii. if the quadratic polynomial has a real double (or repeated) zero, then the graph sits on the $x$-axis;

iii. if the quadratic polynomial has no real zeros, then the graph does not intersect the $x$-axis at all.

So far, we have only considered quadratic polynomials where the coefficient of the $x^2$ term is positive which gives us a graph which is **concave up**. If we consider polynomials $Q(x) = ax^2 + bx + c$ where $a < 0$ then we will have a graph which is **concave down**.

For example, the graph of $Q(x) = -(x^2 - 4x + 4)$ is the reflection in the $x$-axis of the graph of $Q(x) = x^2 - 4x + 4$. (See Chapter 2.)

\[ \text{The graph of } Q(x) = x^2 - 4x + 4. \]
\[ \text{The graph of } Q(x) = -(x^2 - 4x + 4). \]

### 4.1.4 Zeros of cubic polynomials

A real cubic polynomial has an equation of the form

\[ P(x) = ax^3 + bx^2 + cx + d \]

where $a \neq 0$, $a$, $b$, $c$ and $d$ are real. It has 3 zeros which may be:

i. 3 real distinct zeros;

ii. 3 real zeros, all of which are equal (3 equal zeros);

iii. 3 real zeros, 2 of which are equal;

iv. 1 real zero and 2 complex zeros.

We will illustrate these cases with the following examples:
1. Let $Q(x) = 3x^3 - 3x$.
   Solving the equation $Q(x) = 0$ we get:
   \[ 3x^3 - 3x = 0 \]
   \[ 3x(x - 1)(x + 1) = 0 \]
   Therefore $x = -1$ or $0$ or $1$.
   The roots are real (in fact rational) and distinct.

2. Let $Q(x) = x^3$.
   Solving $Q(x) = 0$ we get that $x^3 = 0$.
   We can write this as $(x - 0)^3 = 0$.
   So, this equation has three equal real roots at $x = 0$.

3. Let $Q(x) = x^3 - x^2$.
   Solving the equation $Q(x) = 0$ we get,
   \[ x^3 - x^2 = 0 \]
   \[ x^2(x - 1) = 0 \]
   Therefore $x = 0$ or $1$.
   The roots are real with a double root at $x = 0$ and a single root at $x = 1$.

4. Let $Q(x) = x^3 + x$.
   Solving the equation $Q(x) = 0$ we get,
   \[ x^3 + x = 0 \]
   \[ x(x^2 + 1) = 0 \]
   Therefore $x = 0$.
   There is one real root at $x = 0$.
   $x^2 + 1 = 0$ does not have any real solutions.

The graph intersects the $x$-axis once only.
Again, in the above examples we have looked only at cubic polynomials where the coefficient of the $x^3$ term is positive. If we consider the polynomial $P(x) = -x^3$ then the graph of this polynomial is the reflection of the graph of $P(x) = x^3$ in the $x$-axis.

The graph of $Q(x) = x^3$.

The graph of $Q(x) = -x^3$.

4.2 Polynomials of higher degree

We will write down a few rules that we can use when we have a polynomial of degree $\geq 3$. If $P(x)$ is a real polynomial of degree $n$ then:

1. $P(x) = 0$ has at most $n$ real roots;

2. if $P(x) = 0$ has a repeated root with an even power then the graph of $P(x)$ turns at this repeated root;

3. if $P(x) = 0$ has a repeated root with an odd power then the graph of $P(x)$ has a horizontal point of inflection at this repeated root.

For example, 1. tells us that if we have a quartic polynomial equation $f(x) = 0$. Then we know that $f(x) = 0$ has $\leq 4$ real roots.

We can illustrate 2. by the sketching $f(x) = x(x - 2)^2(x + 1)$. Notice how the graph sits on the $x$-axis at $x = 2$.

The graph of $f(x) = x(x + 1)(x - 2)^2$. 
We illustrate 3. by sketching the graph of \( f(x) = x(x - 2)^3 \). Notice the horizontal point of inflection at \( x = 2 \).

The graph of \( f(x) = x(x - 2)^3 \).

### 4.3 Exercises

1. Sketch the graphs of the following polynomials if \( y = P(x) \) is:
   - a. \( x(x + 1)(x - 3) \)
   - b. \( x(x + 1)(3 - x) \)
   - c. \( (x + 1)^2(x - 3) \)
   - d. \( (x + 1)(x^2 - 4x + 5) \)

2. The graphs of the following quartic polynomials are sketched below. Match the graph with the polynomial.
   - a. \( y = x^4 \)
   - b. \( y = x^4 - 1 \)
   - c. \( y = x^4 + 1 \)
   - d. \( y = 1 - x^4 \)
   - e. \( y = (x - 1)^4 \)
   - f. \( y = (x + 1)^4 \)
3. Sketch the graphs of the following quartic polynomials if \( y = C(x) \) is:
   a. \( x(x - 1)(x + 2)(x + 3) \)
   b. \( x(x - 1)(x + 2)(3 - x) \)
   c. \( x^2(x - 1)(x - 3) \)
   d. \( (x + 1)^2(x - 3)^2 \)
   e. \( (x + 1)^3(x - 3) \)
   f. \( (x + 1)^3(3 - x) \)
   g. \( x(x + 1)(x^2 - 4x + 5) \)
   h. \( x^2(x^2 - 4x + 5) \).

4. By sketching the appropriate polynomial, solve:
   a. \( x^2 - 4x - 12 < 0 \)
   b. \( (x + 2)(x - 3)(5 - x) > 0 \)
   c. \( (x + 2)^2(5 - x) > 0 \)
   d. \( (x + 2)^3(5 - x) \geq 0 \).

5. For what values of \( k \) will \( P(x) \geq 0 \) for all real \( x \) if \( P(x) = x^2 - 4x - 12 + k \)?

6. The diagrams show the graph of \( y = P(x) \) where \( P(x) = a(x - b)(x - c)^d \).
   In each case determine possible values for \( a, b, c \) and \( d \).

   a. 
   b. 
   c. 
   d. 
   e. 
   f. 

7. The graph of the polynomial \( y = f(x) \) is given below. It has a local maximum and minimum as marked. Use the graph to answer the following questions.
   a. State the roots of \( f(x) = 0 \).
   b. What is the value of the repeated root.
   c. For what values of \( k \) does the equation \( f(x) = k \) have exactly 3 solutions.
d. Solve the inequality $f(x) < 0$.

e. What is the least possible degree of $f(x)$?

f. State the value of the constant of $f(x)$.

g. For what values of $k$ is $f(x) + k \geq 0$ for all real $x$.

The graph of the polynomial $y = f(x)$

4.4 Factorising polynomials

So far for the most part, we have looked at polynomials which were already factorised. In this section we will look at methods which will help us factorise polynomials with degree $> 2$.

4.4.1 Dividing polynomials

Suppose we have two polynomials $P(x)$ and $A(x)$, with the degree of $P(x) \geq$ the degree of $A(x)$, and $P(x)$ is divided by $A(x)$. Then

$$\frac{P(x)}{A(x)} = Q(x) + \frac{R(x)}{A(x)},$$

where $Q(x)$ is a polynomial called the quotient and $R(x)$ is a polynomial called the remainder, with the degree of $R(x) <$ degree of $A(x)$.

We can rewrite this as

$$P(x) = A(x) \cdot Q(x) + R(x).$$

For example: If $P(x) = 2x^3 + 4x + 3$ and $A(x) = x - 2$, then $P(x)$ can be divided by $A(x)$ as follows:

$$x - 2 | \begin{array}{c} 2x^3 + 4x + 12 \\ 2x^3 - 4x^2 \\ 4x^2 + 4x - 3 \\ 4x^2 - 8x \\ 12x - 3 \\ 12x - 24 \\ 21 \end{array}$$
The quotient is $2x^2 + 4x + 12$ and the remainder is 21. We have
\[
\frac{2x^3 + 4x + 3}{x - 2} = 2x^2 + 4x + 12 + \frac{21}{x - 2}.
\]
This can be written as
\[
2x^3 + 4x - 3 = (x - 2)(2x^2 + 4x + 12) + 21.
\]
Note that the degree of the ”polynomial” 21 is 0.

### 4.4.2 The Remainder Theorem

If the polynomial $f(x)$ is divided by $(x - a)$ then the remainder is $f(a)$.

**Proof:**

Following the above, we can write
\[
f(x) = A(x) \cdot Q(x) + R(x),
\]
where $A(x) = (x - a)$. Since the degree of $A(x)$ is 1, the degree of $R(x)$ is zero. That is, $R(x) = r$ where $r$ is a constant.

\[
f(x) = (x - a)Q(x) + r \quad \text{where } r \text{ is a constant.}
\]
\[
f(a) = 0 \cdot Q(a) + r
\]
\[
= r
\]

So, if $f(x)$ is divided by $(x - a)$ then the remainder is $f(a)$.

**Example**

Find the remainder when $P(x) = 3x^4 - x^3 + 30x - 1$ is divided by $a$. $x + 1$, $b$. $2x - 1$.

**Solution**

**a.** Using the Remainder Theorem:

\[
\text{Remainder} = P(-1)
\]
\[
= 3 - (-1) - 30 - 1
\]
\[
= -27
\]

**b.**

\[
\text{Remainder} = P\left(\frac{1}{2}\right)
\]
\[
= 3\left(\frac{1}{2}\right)^4 - \left(\frac{1}{2}\right)^3 + 30\left(\frac{1}{2}\right) - 1
\]
\[
= \frac{3}{16} - \frac{1}{8} + 15 - 1
\]
\[
= 14\frac{1}{16}
\]
Example

When the polynomial \( f(x) \) is divided by \( x^2 - 4 \), the remainder is \( 5x + 6 \). What is the remainder when \( f(x) \) is divided by \( (x - 2) \)?

Solution
Write \( f(x) = (x^2 - 4) \cdot q(x) + (5x + 6) \). Then

\[
\text{Remainder} = f(2) = 0 \cdot q(2) + 16 = 16
\]

A consequence of the Remainder Theorem is the Factor Theorem which we state below.

4.4.3 The Factor Theorem

If \( x = a \) is a zero of \( f(x) \), that is \( f(a) = 0 \), then \( (x - a) \) is a factor of \( f(x) \) and \( f(x) \) may be written as

\[
f(x) = (x - a)q(x)
\]

for some polynomial \( q(x) \).

Also, if \( (x - a) \) and \( (x - b) \) are factors of \( f(x) \) then \( (x - a)(x - b) \) is a factor of \( f(x) \) and

\[
f(x) = (x - a)(x - b) \cdot Q(x)
\]

for some polynomial \( Q(x) \).

Another useful fact about zeros of polynomials is given below for a polynomial of degree 3.

If a (real) polynomial

\[
P(x) = ax^3 + bx^2 + cx + d,
\]

where \( a \neq 0, a, b, c \) and \( d \) are real, has exactly 3 real zeros \( \alpha, \beta \) and \( \gamma \), then

\[
P(x) = a(x - \alpha)(x - \beta)(x - \gamma) \quad (1)
\]

Furthermore, by expanding the right hand side of (1) and equating coefficients we get:

i

\[
\alpha + \beta + \gamma = -\frac{b}{a};
\]

ii

\[
\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a};
\]

iii

\[
\alpha\beta\gamma = -\frac{d}{a}.
\]
This result can be extended for polynomials of degree \( n \). We will give the partial result for \( n = 4 \).

If

\[
P(x) = ax^4 + bx^3 + cx^2 + dx + e
\]

is a polynomial of degree 4 with real coefficients, and \( P(x) \) has four real zeros \( \alpha, \beta, \gamma \) and \( \delta \), then

\[
P(x) = a(x - \alpha)(x - \beta)(x - \gamma)(x - \delta)
\]

and expanding and equating as above gives

\[
\alpha \beta \gamma \delta = \frac{e}{a}.
\]

If \( a = 1 \) and the equation \( P(x) = 0 \) has a root which is an integer, then that integer must be a factor of the constant term. This gives us a place to start when looking for factors of a polynomial. That is, we look at all the factors of the constant term to see which ones (if any) are roots of the equation \( P(x) = 0 \).

Example

Let \( f(x) = 4x^3 - 8x^2 - x + 2 \)

a. Factorise \( f(x) \).

b. Sketch the graph of \( y = f(x) \).

c. Solve \( f(x) \geq 0 \).

Solution

a. Consider the factors of the constant term, 2. We check to see if \( \pm 1 \) and \( \pm 2 \) are solutions of the equation \( f(x) = 0 \) by substitution. Since \( f(2) = 0 \), we know that \( x - 2 \) is a factor of \( f(x) \). We use long division to determine the quotient.

\[
x - 2 \left[ \frac{4x^2 - 1}{4x^3 - 8x^2 - x + 2} \right]
\]

\[
= \frac{4x^2 - 1}{4x^3 - 8x^2 - x + 2}
\]

So,

\[
f(x) = (x - 2)(4x^2 - 1)
\]

\[
= (x - 2)(2x - 1)(2x + 1)
\]
b. 

The graph of \( f(x) = 4x^3 - 8x^2 - x + 2 \).

c. \( f(x) \geq 0 \) when \(-\frac{1}{2} \leq x \leq \frac{1}{2}\) or \( x \geq 2 \).

**Example**

Show that \((x - 2)\) and \((x - 3)\) are factors of \( P(x) = x^3 - 19x + 30 \), and hence solve \( x^3 - 19x + 30 = 0 \).

**Solution**

\( P(2) = 8 - 38 + 30 = 0 \) and \( P(3) = 27 - 57 + 30 = 0 \) so \((x - 2)\) and \((x - 3)\) are both factors of \( P(x) \) and \((x - 2)(x - 3) = x^2 - 5x + 6\) is also a factor of \( P(x) \). Long division of \( P(x) \) by \( x^2 - 5x + 6 \) gives a quotient of \((x + 5)\).

So,

\[
P(x) = x^3 - 19x + 30 = (x - 2)(x - 3)(x + 5).
\]

Solving \( P(x) = 0 \) we get \((x - 2)(x - 3)(x + 5) = 0\).

That is, \( x = 2 \) or \( x = 3 \) or \( x = -5 \).

Instead of using long division we could have used the facts that

i. the polynomial cannot have more than three real zeros;

ii. the product of the zeros must be equal to \(-30\).

Let \( \alpha \) be the unknown root.

Then \( 2 \cdot 3 \cdot \alpha = -30 \), so that \( \alpha = -5 \). Therefore the solution of \( P(x) = x^3 - 19x + 30 = 0 \) is \( x = 2 \) or \( x = 3 \) or \( x = -5 \).
4.5 Exercises

1. When the polynomial $P(x)$ is divided by $(x - a)(x - b)$ the quotient is $Q(x)$ and the remainder is $R(x)$.
   a. Explain why $R(x)$ is of the form $mx + c$ where $m$ and $c$ are constants.
   b. When a polynomial is divided by $(x - 2)$ and $(x - 3)$, the remainders are 4 and 9 respectively. Find the remainder when the polynomial is divided by $x^2 - 5x + 6$.
   c. When $P(x)$ is divided by $(x - a)$ the remainder is $a^2$. Also, $P(b) = b^2$. Find $R(x)$ when $P(x)$ is divided by $(x - a)(x - b)$.

2. a. Divide the polynomial $f(x) = 2x^4 + 13x^3 + 18x^2 + x - 4$ by $g(x) = x^2 + 5x + 2$. Hence write $f(x) = g(x)q(x) + r(x)$ where $q(x)$ and $r(x)$ are polynomials.
   b. Show that $f(x)$ and $g(x)$ have no common zeros. (Hint: Assume that $\alpha$ is a common zero and show by contradiction that $\alpha$ does not exist.)

3. For the following polynomials, 
   i. factorise
   ii. solve $P(x) = 0$
   iii. sketch the graph of $y = P(x)$.
   a. $P(x) = x^3 - x^2 - 10x - 8$
   b. $P(x) = x^3 - x^2 - 16x - 20$
   c. $P(x) = x^3 + 4x^2 - 8$
   d. $P(x) = x^3 - x^2 + x - 6$
   e. $P(x) = 2x^3 - 3x^2 - 11x + 6$
5 Solutions to exercises

1.4 Solutions

1. a. The domain of \( f(x) = \sqrt{9 - x^2} \) is all real \( x \) where \(-3 \leq x \leq 3\). The range is all real \( y \) such that \( 0 \leq y \leq 3 \).

b. \[ \]

The graph of \( f(x) = \sqrt{9 - x^2} \).

2. \[
\frac{\psi(x + h) - \psi(x)}{h} = \frac{(x + h)^2 + 5 - (x^2 + 5)}{h} = \frac{x^2 + 2xh + h^2 + 5 - x^2 - 5}{h} = \frac{h^2 + 2xh}{h} = h + 2x
\]

3. a. \[
\]

The graph of \( y = \sqrt{x - 1} \). The domain is all real \( x \geq 1 \) and the range is all real \( y \geq 0 \).

b. \[
\]

The graph of \( y = |2x| \). Its domain is all real \( x \) and range all real \( y \geq 0 \).
c.

The graph of \( y = \frac{1}{x-4} \). The domain is all real \( x \neq 4 \) and the range is all real \( y \neq 0 \).

d.

The graph of \( y = |2x| - 1 \). The domain is all real \( x \), and the range is all real \( y \geq -1 \).

4. a. The perpendicular distance \( d \) from \((0, 0)\) to \( x + y + k = 0 \) is \( d = \frac{|k|}{\sqrt{2}} \).

   b. For the line \( x + y + k = 0 \) to cut the circle in two distinct points \( d < 2 \), ie \( |k| < 2\sqrt{2} \) or \( -2\sqrt{2} < k < 2\sqrt{2} \).

5. a.

   The graph of \( y = \left(\frac{1}{2}\right)^x \).

   b.

   The graph of \( y^2 = x^2 \).

6. \( y^2 = x^3 \) is not a function.
7. a. The graph of \( y = -\sqrt{4 - x^2} \). This is a function with the domain: all real \( x \) such that \(-2 \leq x \leq 2\) and range: all real \( y \) such that \(-2 \leq y \leq 0\).

b. The graph of \( |x| - |y| = 0 \). This is not the graph of a function.

c. The graph of \( y = x^3 \). This is a function with the domain: all real \( x \) and range: all real \( y \).

d. The graph of \( y = \frac{x}{|x|} \). This is the graph of a function which is not defined at \( x = 0 \). Its domain is all real \( x \neq 0 \), and range is \( y = \pm 1 \).

e. The graph of \( |y| = x \). This is not the graph of a function.
8. 
\[ A\left(\frac{1}{p}\right) = \left(\frac{1}{p}\right)^2 + 2 + \frac{1}{\left(\frac{1}{p}\right)^2} \]
\[ = \frac{1}{p^2} + 2 + \frac{1}{p^2} \]
\[ = \frac{1}{p^2} + 2 + p^2 \]
\[ = A(p) \]

9. a. The values of \(x\) in the interval \(0 < x < 4\) are not in the domain of the function.
   b. \(x = 1\) and \(x = -1\) are not in the domain of the function.

10. a. \(\phi(3) + \phi(4) + \phi(5) = \log(2.5)\)
    b. \(\phi(3) + \phi(4) + \phi(5) + \cdots + \phi(n) = \log\left(\frac{n}{2}\right)\)

11. a. \(y = 3\) when \(z = 3\).
    b. i  \(L(M(x)) = 2(x^2 - x) + 1\)
        ii \(M(L(x)) = 4x^2 + 2x\)

12. a. \(a = 2, b = 2\) so the equations is \(y = 2x^2 - 2\).
    b. \(a = 5, b = 1\) so the equation is \(y = \frac{5}{x^2+1}\).

13. b. 

The graph of \(|x| + |y| = 1\).

14. \(S(n - 1) = \frac{n - 1}{2n - 1}\)
   
   Hence 
   \[ S(n) - S(n - 1) = \frac{n}{2n + 1} - \frac{n - 1}{2n - 1} \]
   \[ = \frac{n(2n - 1) - (2n + 1)(n - 1)}{(2n - 1)(2n + 1)} \]
   \[ = \frac{2n^2 - n - (2n^2 - n - 1)}{(2n - 1)(2n + 1)} \]
   \[ = \frac{1}{(2n - 1)(2n + 1)} \]
2.8 Solutions

1. a. The graph of $y = x^2$.

b. The graph of $y = \frac{x^2}{3}$.

c. The graph of $y = -x^2$.

d. The graph of $y = (x + 1)^2$.

2. a. The graph of $y = \frac{1}{x}$.

b. The graph of $y = \frac{1}{x-2}$.
3. a. The graph of $y = x^3$.

b. The graph of $y = |x^3 - 2|$.

c. The graph of $y = 3 - (x - 1)^3$.

4. a. The graph of $y = |x|$.

b. The graph of $y = 2|x - 2|$.
c.

The graph of $y = 4 - |x|$.

5.

a. The graph of $x^2 + y^2 = 16$.

b. The graph of $x^2 + (y + 2)^2 = 16$.

c. The graph of $(x - 1)^2 + (y - 3)^2 = 16$. 

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6. a. The graph of $y = \sqrt{9 - x^2}$.

b. The graph of $y = \sqrt{9 - (x - 1)^2}$.

c. The graph of $y = \sqrt{9 - x^2} - 3$.

7. \[
\frac{1}{x - 2} + 1 = \frac{1 + (x - 2)}{x - 2} = \frac{x - 1}{x - 2}
\]

The graph of $y = \frac{x - 1}{x - 2}$.

8. The graph of $y = \frac{x + 1}{x - 1}$.
9. a. The graph of \( y = |x| + x + 1 \) for \(-2 \leq x \leq 2\).

c. The graph of \( y = 2^x + 2^{-x} \) for \(-2 \leq x \leq 2\).

9. b. The graph of \( y = |x| + |x - 1| \) for \(-2 \leq x \leq 3\).

d. The graph of \( |x - y| = 1 \) for \(-1 \leq x \leq 3\).

10. The graph of \( f(x) = |x^2 - 1| - 1 \).
11. a. The graph of \( y = 2f(x) \).

b. The graph of \( y = -f(x) \).

c. The graph of \( y = f(-x) \).

d. The graph of \( y = f(x) + 4 \).

e. The graph of \( y = f(x - 3) \).

f. The graph of \( y = f(x + 1) - 2 \).
The graph of $y = 3 - 2f(x - 3)$.

The graph of $y = |f(x)|$.

12. a. $x = -2$ and $x = 2$ are solutions of the equation $|2x| = 4$.

b. $x = 1$ is a solution of $\frac{1}{x-2} = -1$.

c. $x = 0$ and $x = 1$ are solutions of the equation $x^3 = x^2$.

d. $x = 1$ is a solution of $x^2 = \frac{1}{x}$.

13. a. For $x \geq 2$, $|x - 2| = x - 2 = 3$. Therefore $x = 5$ is a solution of the inequality. (Note that $x = 5$ is indeed $\geq 2$.)

For $x < 2$, $|x - 2| = -(x - 2) = -x + 2 = 3$. Therefore $x = -1$ is a solution. (Note that $x = -1$ is $< 2$.)
b.

The points of intersection are \((-1, 3)\) and \((5, 3)\).
Therefore the solutions of \(|x - 2| = 3\) are \(x = -1\) and \(x = 5\).

14. The parabolas intersect at \((2, 1)\).

15. \(y = k\) intersects the circle at two distinct points when \(2 < k < 12\).

16.

The point of intersection is \((1, 1)\). Therefore the solution of \(\frac{4}{5-x} = 1\) is \(x = 1\).
17. The point of intersection is (0, 2). Therefore the solution of $|x - 2| = |x + 2|$ is $x = 0$.

18. $n = -1$ or $n = 2$.

19. a. For $x \geq 4$, $|x - 4| = x - 4 = 2x$ when $x = -4$, but this does not satisfy the condition of $x \geq 4$ so is not a solution.
   For $x < 4$, $|x - 4| = -x + 4 = 2x$ when $x = \frac{4}{3}$. $x = \frac{4}{3}$ is $< 4$ so is a solution.
   Therefore, $x = \frac{4}{3}$ is a solution of $|x - 4| = 2x$.

b. The graph of $y = |x - 4|$ and $y = 2x$ intersect at the point $(\frac{4}{3}, \frac{8}{3})$. So the solution of $|x - 4| = 2x$ is $x = \frac{4}{3}$.

2.11 Solutions

1. a. The domain is all real $x$, and the range is all real $y \geq -2$.

   b. i $-2 < x < 0$ or $x > 2$
      ii $x < -2$ or $0 < x < 2$

   c. i $k < -2$
      ii There is no value of $k$ for which $f(x) = k$ has exactly one solution.
      iii $k = 2$ or $k > 0$
iv \quad k = 0

v \quad -2 < k < 0
d. \quad y = f(x) \text{ is even}

2. a. \hspace{1cm} b.
\begin{align*}
\text{y} & = f(x) \text{ is even.} \\
\text{y} & = f(x) \text{ is odd.}
\end{align*}

\begin{align*}
\text{y} & = g(x) \text{ is even.} \\
\text{y} & = g(x) \text{ is odd.}
\end{align*}

3. a. even \quad b. even \quad c. neither \quad d. odd \quad e. odd
\quad f. even \quad g. even \quad h. neither \quad i. even \quad j. even

4. a.
\begin{align*}
h(-x) & = f(-x) \cdot g(-x) \\
& = f(x) \cdot -g(x) \\
& = -f(x) \cdot g(x) \\
& = -h(x)
\end{align*}

Therefore \( h \) is odd.

b.
\begin{align*}
h(-x) & = (g(-x))^2 \\
& = (-g(x))^2 \\
& = (g(x))^2 \\
& = h(x)
\end{align*}

Therefore \( h \) is even.
c. 
\[ h(-x) = \frac{f(-x)}{g(-x)} \]
\[ = \frac{f(x)}{-g(x)} \]
\[ = -\frac{f(x)}{g(x)} \]
\[ = -h(x) \]

Therefore \( h \) is odd.

d. 
\[ h(-x) = f(-x) \cdot (g(-x))^2 \]
\[ = f(x) \cdot (-g(x))^2 \]
\[ = f(x) \cdot (g(x))^2 \]
\[ = h(x) \]

Therefore \( h \) is even.

5. If \( f \) is defined at \( x = 0 \)
\[ f(0) = f(-0) \quad \text{ (since } 0 = -0) \]
\[ = -f(0) \quad \text{ (since } f \text{ is odd)} \]
\[ 2f(0) = 0 \quad \text{ (adding } f(0) \text{ to both sides)} \]

Therefore \( f(0) = 0 \).

3.2 Solutions

1. \( a. \) \( 2f(-1) + f(2) = 2(1 - (-1)) + (1 - (2)^2) = 4 + (-3) = 1. \)
\( b. \) \( f(a^2) = 1 - (a^2)^2 = 1 - a^4 \) since \( a^2 \geq 0. \)

2. You can see from the graph below that there is one solution to \( f(x) = 2 \), and that this solution is at \( x = -1. \)
3. \[ g(x) = \begin{cases} \frac{1}{x+1} & \text{for } x < 1 \\ \sqrt{1-x^2} & \text{for } -1 \leq x \leq 1 \\ -1 & \text{for } x > 1 \end{cases} \]

4. a. The domain of \( f \) is all real \( x \geq -2 \).

\[ \text{b. The range of } f \text{ is all real } y > -4. \]

\[ \text{c. i } f(x) = 0 \text{ when } x = -2 \text{ or } x = 2. \]

\[ \text{ii } f(x) = -3 \text{ when } x = 1. \]

\[ \text{d. i } f(x) = k \text{ has no solutions when } k \leq -4. \]

\[ \text{ii } f(x) = k \text{ has 1 solution when } -4 < k < -2 \text{ or } k > 0. \]

\[ \text{iii } f(x) = k \text{ has 2 solutions when } -2 \leq k \leq 0. \]

5. Note that \( f(0) = 0. \)

6. The domain of \( g \) is all real \( x, x \neq -2. \)
The range of \( g \) is all real \( y < 0 \) or \( y \geq 2 \).

7. Note that there may be more than one correct solution.
   a. Defining \( f \) as

   \[
   f(x) = \begin{cases} 
   x + 6 & \text{for } x \leq -3 \\
   -x & \text{for } -3 < x < 0 \\
   x & \text{for } 0 \leq x \leq 3 \\
   -x + 6 & \text{for } x > 3
   \end{cases}
   \]

   gives a function describing the McMaths burgers' logo using 4 pieces.

   b. Defining \( f \) as

   \[
   f(x) = \begin{cases} 
   x + 6 & \text{for } x \leq -3 \\
   |x| & \text{for } -3 < x < 3 \\
   -x + 6 & \text{for } x \geq 3
   \end{cases}
   \]

   gives a function describing the McMaths burgers' logo using 3 pieces.

   c. Defining \( f \) as

   \[
   f(x) = \begin{cases} 
   3 - |x + 3| & \text{for } x \leq 0 \\
   3 - |x - 3| & \text{for } x > 0
   \end{cases}
   \]

   gives a function describing the McMaths burgers' logo using 2 pieces.

8. a. Here \( a = 1, b = -4, c = 2 \) and \( d = -4 \). So,

   \[
   f(x) = \begin{cases} 
   x^2 - 4 & \text{for } 0 < x \leq 2 \\
   2x - 4 & \text{for } x > 2
   \end{cases}
   \]

   b. Defining \( f \) to be an odd function for all real \( x, x \neq 0 \), we get
c. We can define $f$ as follows

$$f(x) = \begin{cases} 
2x + 4 & \text{for } x < -2 \\
4 - x^2 & \text{for } -2 \leq x < 0 \\
x^2 - 4 & \text{for } 0 < x \leq 2 \\
2x - 4 & \text{for } x > 2 
\end{cases}$$

### 3.4 Solutions

1. a. $0 \leq x \leq 4$
   
   b. $-3 < p \leq 1$
   
   c. $x < -4$ or $-3 < x < 3$ or $x > 4$

2. a. The graph of $y = 4x(x - 3)$ is given below

b. From the graph we see that $4x(x - 3) \leq 0$ when $0 \leq x \leq 3$. 
3.  
   a. The graphs \( y = 5 - x \) and \( y = \frac{4}{x} \) intersect at the points (1, 4) and (4, 1).
   
   b. The graphs of \( y = 5 - x \) and \( y = \frac{4}{x} \).

   ![Graph of y = 5 - x and y = 4/x](image)

   c. The inequality is satisfied for \( x < 0 \) or \( 1 < x < 4 \).

4.  
   a. The graph of \( y = 2^x \).

   ![Graph of y = 2^x](image)

   b. \( 2^x < \frac{1}{2} \) when \( x < -1 \).

   c. The midpoint M of the segment AB has coordinates \( \left( \frac{a+b}{2}, \frac{2^a + 2^b}{2} \right) \).

   Since the function \( y = 2^x \) is concave up, the \( y \)-coordinate of M is greater than \( f\left( \frac{a+b}{2} \right) \). So,

   \[
   \frac{2^a + 2^b}{2} > 2^{\frac{a+b}{2}}
   \]
5.

a. 

b. $|x - 5| > x$ for all $x < 2.5$.

c. i $mx = |x - 5|$ has exactly two solutions when $0 < m < 1$.
   ii $mx = |x - 5|$ has no solutions when $-1 < m < 0$.

6. $-1 \leq x \leq 3$

4.3 Solutions

1. a. 

The graph of $P(x) = x(x + 1)(x - 3)$.

b. 

The graph of $P(x) = x(x + 1)(3 - x)$.

c. 

The graph of $P(x) = (x + 1)^2(x - 3)$.

d. 

The graph of $P(x) = (x + 1)(x^2 - 4x + 5)$.

2. a. iv. b. v. c. i. d. iii. e. ii. f. vi.
3. a. The graph of $P(x) = x(x-1)(x+2)(x+3)$.

b. The graph of $P(x) = x(x-1)(x+2)(3-x)$.

c. The graph of $P(x) = x^2(x-1)(x-3)$.

d. The graph of $P(x) = (x+1)^2(x-3)^2$.

e. The graph of $P(x) = (x+1)^3(x-3)$.

f. The graph of $P(x) = (x+1)^3(3-x)$.

g. The graph of $P(x) = x(x+1)(x^2-4x+5)$.

h. The graph of $P(x) = x^2(x^2-4x+5)$. 

4. a. 

\[ x^2 - 4x - 12 < 0 \text{ when } -2 < x < 6. \]

b. 

\[ (x + 2)(x - 3)(5 - x) > 0 \text{ when } x < -2 \text{ or } 3 < x < 5. \]

c. 

\[ (x + 2)^2(5 - x) > 0 \text{ when } x < 5. \]

5. \[ x^2 - 4x - 12 + k \geq 0 \text{ for all real } x \text{ when } k = 16. \]

6. a. \[ P(x) = x(x - 4) \]
    b. \[ P(x) = -x(x - 4) \]
    c. \[ P(x) = x^2(x - 4) \]
    d. \[ P(x) = \frac{x(x-4)}{3} \]
    e. \[ P(x) = -x(x - 4)^2 \]
    f. \[ P(x) = \frac{(x+4)(x-4)^2}{8} \]

7. a. The roots of \( f(x) = 0 \) are \( x = -2, x = 0 \) and \( x = 2. \)
b. $x = 2$ is the repeated root.

c. The equation $f(x) = k$ has exactly 3 solutions when $k = 0$ or $k = 3.23$.

d. $f(x) < 0$ when $-2 < x < 0$.

e. The least possible degree of the polynomial $f(x)$ is 4.

f. Since $f(0) = 0$, the constant in the polynomial is 0.

g. $f(x) + k \geq 0$ for all real $x$ when $k \geq 9.91$.

4.5 Solutions

1. a. Since $A(x) = (x - a)(x - b)$ is a polynomial of degree 2, the remainder $R(x)$ must be a polynomial of degree $\leq 1$. That is, $R(x) = mx + c$ where $m$ and $c$ are constants. Note that if $m = 0$ the remainder is a constant.

b. Let $P(x) = (x^2 - 5x + 6)Q(x) + (mx + c) = (x - 2)(x - 3)Q(x) + (mx + c)$.

Then
$$P(2) = (0)(-1)Q(2) + (2m + c) = 2m + c = 4$$

and
$$P(3) = (1)(0)Q(3) + (3m + c) = 3m + c = 9$$

Solving simultaneously we get that $m = 5$ and $c = -6$. So, the remainder is $R(x) = 5x - 6$.

c. Let $P(x) = (x - a)(x - b)Q(x) + (mx + c)$.

Then
$$P(a) = (0)(a - b)Q(a) + (ma + c) = am + c = a^2$$

and
$$P(b) = (b - a)(0)Q(b) + (mb + c) = bm + c = b^2$$

Solving simultaneously we get that $m = a + b$ and $c = -ab$ provided $a \neq b$.

So, $R(x) = (a + b)x - ab$.

2. a. 
$$2x^4 + 13x^3 + 18x^2 + x - 4 = (x^2 + 5x + 2)(2x^2 + 3x - 1) - 2$$

b. Let $\alpha$ be a common zero of $f(x)$ and $g(x)$. That is, $f(\alpha) = 0$ and $g(\alpha) = 0$. 
Then since $f(x) = g(x)q(x) + r(x)$ we have
\[ f(\alpha) = g(\alpha)q(\alpha) + r(\alpha) \]
\[ = (0)q(\alpha) + r(\alpha) \quad \text{since } g(\alpha) = 0 \]
\[ = r(\alpha) \quad \text{since } f(\alpha) = 0 \]
But, from part b. $r(x) = -2$ for all values of $x$, so we have a contradiction.
Therefore, $f(x)$ and $g(x)$ do not have a common zero.
This is an example of a proof by contradiction.

3. a. i $P(x) = x^3 - x^2 - 10x - 8 = (x + 1)(x + 2)(x - 4)$
ii $x = -1$, $x = -2$ and $x = 4$ are solutions of $P(x) = 0$.
iii
\[ \text{The graph of } P(x) = x^3 - x^2 - 10x - 8. \]

b. i $P(x) = x^3 - x^2 - 16x - 20 = (x + 2)^2(x - 5)$.
ii $x = -2$ and $x = 5$ are solutions of $P(x) = 0$. $x = -2$ is a double root.
iii
\[ \text{The graph of } P(x) = x^3 - x^2 - 16x - 20. \]

c. i $P(x) = x^3 + 4x^2 - 8 = (x+2)(x^2+2x-4) = (x+2)(x-(-1+\sqrt{5}))(x-(-1-\sqrt{5}))$
ii $x = -2$, $x = -1 + \sqrt{5}$ and $x = -1 - \sqrt{5}$ are solutions of $P(x) = 0$. 
The graph of $P(x) = x^3 + 4x^2 - 8$.
The zeros are $x = -2, x = -1 + \sqrt{5}$ and $x = -1 - \sqrt{5}$.

d. i $P(x) = x^3 - x^2 + x - 6 = (x - 2)(x^2 + x + 3)$. $x^2 + x + 3 = 0$ has no real solutions.
ii $x = 2$ is the only real solution of $P(x) = 0$.

iii

The graph of $P(x) = x^3 - x^2 + x - 6$.
There is only one real zero at $x = 2$.

e. i $P(x) = 2x^3 - 3x^2 - 11 + 6 = (x + 2)(x - 3)(2x - 1)$.
ii $x = -2, x = \frac{1}{2}$ and $x = 3$ are solutions of $P(x) = 0$.

iii

The graph of $P(x) = 2x^3 - 3x^2 - 11 + 6$. 
DIFFERENTIAL CALCULUS

CHAPTER I: REAL NUMBER SYSTEM

Definition 1.1 By Real Number System we mean a non-empty set R, two binary operations addition and multiplication (denoted by + and . respectively) and an order relation \( \leq \) defined on R which satisfy the following axioms I, II and III:

I. **Algebraic Properties:**
   For all \( a,b,c \) in R,
   (1) \( a+(b+c)=(a+b)+c, \ a.(b.c)=(a.b).c \) (associativity),
   (2) \( a+b=b+a, a.b=b.a \) (commutativity)
   (3) there exists 0,1 in R, 0 \( \neq \) 1, such that \( a+0=a, a.1=a \) (existence of identity for addition and multiplication)
   (4) for all \( a \) in R, there is \(-a\) in R such that \( a+(-a)=0\); for all \( a \neq 0 \) in R, there exists \(-1\) in R such that \( a.(-1)=1 \) (existence of inverse under addition and multiplication)
   (5) \( a.(b+c)=a.b+a.c \) (distributivity of . over +)

A system that has more than one element and satisfies these five axioms is called a FIELD. The basic algebraic properties of R can be proved solely on the basis of these field properties. (Field properties will be discussed in detail in a later semester.)

II. **ORDER PROPERTIES:**
   (1) for \( a,b \) in R, either \( a \leq b \) or \( b \leq a \)
   (2) if \( a \leq b \) and \( b \leq a \), then \( a=b \)
   (3) if \( a \leq b \) and \( b \leq c \), then \( a \leq c \)
   (4) if \( a \leq b \) then \( a+c \leq b+c \)
   (5) if \( a \leq b \) and \( 0 \leq c \), then \( ac \leq bc \)

A field satisfying above five properties is called an ordered field. Most of the algebraic and order properties of R can be established for any ordered field (like Q).

III. The Completeness Axiom:

Differentiation of Real Numbers and Rational Numbers
Let \( \varphi \neq S \subseteq R \). If a real number \( M \) satisfies \( s \leq M \) for all \( s \in S \), then \( M \) is called an upper bound of \( S \) and \( S \) is called bounded above.

If a real number \( M \) satisfies (1) \( M \) is an upper bound of \( S \) and (2) no real number less than \( M \) is an upper bound of \( S \), then \( M \) is the LEAST UPPER BOUND of \( S \) or SUPREMUM of \( S \), written as \( \text{l.u.b. } S \) or \( \sup S \).

**Ex:** \( \sup \{ \frac{1}{n} : n \in N \} = 0 \), \( \sup \{ 2n : n \in N \} \) does not exist, \( \sup \{ 1, 2, 3 \} = \max \{ 1, 2, 3 \} \).

The Completeness Axiom for \( R \) states that:

Every nonempty subset \( S \) of \( R \) that is bounded above has a least upper bound in \( R \).

That is, if a nonempty subset of real numbers has an upper bound, then it has a smallest upper bound in \( R \).

**Definition:** A number is called an algebraic number if it satisfies a polynomial equation \( a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 \) where the coefficients \( a_0, a_1, \ldots, a_n \), are integers, \( a_n \neq 0 \), and \( n \geq 1 \). A real number which is not algebraic (like \( \pi, e \)) is called a transcendental number.

**Note:** Rational numbers are algebraic numbers since a rational number \( x = \frac{m}{n} \), \( m, n \) are integers and \( n \neq 0 \) satisfies the equation \( nx - m = 0 \).

**Theorem:** Suppose that \( a_0, a_1, \ldots, a_n \) are integers and \( r \) is a rational number satisfying the polynomial equation \( a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 \) where \( a_0, a_n \neq 0 \), and \( n \geq 1 \). Write \( r = \frac{p}{q} \), where \( p, q \) are integers having no common factors and \( q \neq 0 \). Then \( q \) divides \( a_n \) and \( p \) divides \( a_0 \).

**Ex:** \( \sqrt{2} \) cannot represent a rational number.

»By theorem above, the only rational numbers that could possibly be solutions of \( x^2 - 2 = 0 \) are \( \pm 1 \) and \( \pm 2 \). But none of the four numbers \( \pm 1 \) and \( \pm 2 \) are solutions of the equation. Since \( \sqrt{2} \) represents a solution of \( x^2 - 2 = 0 \), \( \sqrt{2} \) cannot represent a rational number.

**Note:** In \( \mathbb{Z} \), division of an integer \( p \) by an integer \( q \) is defined iff there exists unique integer \( r \) such that \( p = qr \) holds.
For \( p \neq 0 \), division of \( p \) by 0 is undefined because there does not exist any integer \( r \) such that \( p = 0 \cdot r \) holds. Division of 0 by 0 is undefined because for any integer \( r \), \( 0 = 0 \cdot r \) holds and uniqueness of \( r \) is violated.

The absolute value function and the greatest integer function

For \( a \in R \), absolute value of \( a \), denoted as \( |a| \), is defined to be the distance between (the points representing) 0 and \( a \):

\[
|a| = a, \text{ if } a \geq 0, \\
= -a, \text{ if } < 0.
\]

The absolute value function satisfies the following properties:

1. \( |a| = |-a| \),
2. \( |a + b| \leq |a| + |b| \),
3. \( |ab| = |a||b| \),
4. \( \left| \frac{a}{b} \right| = \frac{|a|}{|b|} (if \ b \neq 0) \),
5. \( |a - b| \leq |a| - |b| \)

Note: \( |a - b| \) gives distance between the (points representing) real numbers \( a \) and \( b \). Hence the statement \( b \) lies between \( a + 1 \) and \( a - 1 \) can be equivalently put as \( |a - b| < 1 \).

The greatest integer function is defined on the set \( R \) of real numbers as follows:

for real number \( a \) satisfying \( m \leq a < m + 1 \), \( (m \text{ an integer}), [a] = m \).

Thus, for example, \( [a] = -1 \), for \(-1 \leq a < 0\),

\[
= 0, \text{ for } 0 \leq a < 1
\]

\[
= 1, \text{ for } 1 \leq a < 2
\]

**CHAPTER 2: SEQUENCES OF REAL NUMBERS**
Definition: A sequence (in $R$) is a function $a: N \rightarrow R$, $a(N)$ is called the $n$th term of the sequence and denoted by $a_n$. The sequence $a$ is often denoted by \{${a_1, a_2, \ldots, a_n}$\} or, more compactly, by $(a_n)$. We must distinguish between a sequence $(a_n)$ and its range set $\{a_n\}$: the range set of $(1, 1/2, 1/3, \ldots)$ is \{1, 1/2, 1/3, \ldots\}; that of $(1, 1, 1, \ldots)$ is \{1\}. We shall use the terms ‘bounded’, ‘unbounded’, ‘bounded above’ and ‘bounded below’ for a sequence iff its range set (as a subset of $R$) has the corresponding property. $(1/n)$ is bounded, $(n^2)$ is bounded below but not above, $(-n^2)$ is bounded above but not below, $((-1)^nn)$ is neither bounded above nor bounded below.

Note: A fraction increases when either the numerator is increased and/or the denominator is decreased.

Ex: Verify whether the following sequences are bounded above and/or bounded below:

1. $(\frac{2n+3}{3n+4})$
2. $\left((-1)^n \frac{3n-1}{n}\right)$
3. $(a_n)$ where $a_n = \frac{\sqrt{n} + 1 - \sqrt{n}}{\sqrt{n^2 + 1}}$
4. $(\frac{\sqrt{n} + 1}{\sqrt{n^2 + n}})$

Solution: (1) $\frac{2n+3}{3n+4} = \frac{2}{3} \left(\frac{3n+4+\frac{1}{2}}{3n+4}\right) = \frac{2}{3} + \frac{1}{3(3n+4)} \leq \frac{2}{3} + \frac{1}{3} = 1$, for all natural number $n$. Thus $0 < \frac{2n+3}{3n+4} < 1$, for all natural $n$; hence $\left(\frac{2n+3}{3n+4}\right)$ is bounded.

(2) $0 < \left|(-1)^n \frac{3n-1}{n}\right| \leq 3 - \frac{1}{n} < 3$ for all natural $n$: hence $\left((-1)^n \frac{3n-1}{n}\right)$ is bounded.

(3) $0 < a_n = \frac{1}{(n+1)^3 + (n+1)^2 n^3 + n^2} < \frac{1}{3}$ for all $n$ (justify!): hence $(a_n)$ is bounded.

(4) $0 < \frac{n}{\sqrt{n^2 + n}} < b_n < \frac{n}{\sqrt{n^2 + 1}} \leq \frac{n}{n} = 1$ for all natural $n$: hence $(b_n)$ is bounded.

Definition: A sequence $(a_n)$ is convergent iff there exists $a \in R$ such that for every $\epsilon > 0$, there exists natural number ($\epsilon$), in general depending on $\epsilon$, so that $n \geq m$ implies $|a_n - a| < \epsilon$. In this case, we say $(a_n)$ converges to $a$ as $n \rightarrow \infty$ or $a$ is a limit of $(a_n)$ and write $\lim_{n \rightarrow \infty}(a_n) = a$. A sequence that is not convergent is called divergent.
Ex: Consider the sequence \( a = (a_n) \), where \( a_n = b \) for all \( n \). Then 
\[ \lim (b, b, \ldots, b) = b, \] since for any \( \epsilon > 0, |a_n - b| = 0 < \epsilon \) for all \( n \geq 1 \).

Note that every constant sequence \((b, b, \ldots)\) is convergent to \( b \).

Ex: Consider the sequence \((a_n)\), where \( a_n = (-1)^n, n \) natural.

\[ \lim (a_n) \neq 1 \] since the open interval \((0.5, 1.5)\) containing 1 does not contain all the infinite number of terms of the sequence with odd suffix. \[ \lim (a_n) \neq -1 \] since the open interval \((-1.5, -0.5)\) containing -1 does not contain all the infinite number of terms of the sequence with even suffix.

If \( a \neq 1, -1 \), let \( \epsilon = \min\{|a - 1|, |a + 1|\} \). Then \((a - \epsilon, a + \epsilon)\) does not contain any term of the sequence. Hence \((a_n)\) diverges.

Ex: If \( a_n = \frac{1}{n} \) for all natural, then 
\[ \lim_{n \to \infty} (a_n) = 0: \] for any \( \epsilon > 0, |a_n - 0| < \epsilon \iff n > \frac{1}{\epsilon} \iff n \geq \left[ \frac{1}{\epsilon} \right] + 1 = m \]

Note: Observe the direction of implication sign carefully.

Note: If \( \lim_{n \to \infty} (a_n) = a \), then the inequality \(|a_n| - |a|\) \leq |a_n - a|, \( n \) natural, shows that \( \lim_{n \to \infty} |a_n| = |a| \). Converse may not hold as can be seen from the counter example \(((-1)^n)\).

Theorem: A convergent sequence is bounded.

\» Let \( \lim (a_n) = a \). Corresponding to \( \epsilon = 1 \), there exists positive integer \( m \) such that \(|a_n - a| < 1\), that is \( a - 1 < a_n < a + 1 \) for all \( n \geq m \). Let \( m = \min\{a_1, a_2, ..., a_{m-1}, a - 1\} \) and 
\[ M = \max\{a_1, a_2, ..., a_{m-1}, a + 1\} \]. Then \( m \leq a_n \leq M \) for all natural \( n \). Hence \((a_n)\) is bounded.

Theorem: (Limit Theorem for Sequences) Let \( \lim_{n \to \infty} (a_n) = a \) and 
\[ \lim_{n \to \infty} (b_n) = b. \]
Then \( \lim_{n \to \infty} (a_n + b_n) = a + b \), \( \lim_{n \to \infty} (r a_n) = r a \), \( \lim_{n \to \infty} (a_n b_n) = ab \), if \( a \neq 0 \), then \( \exists \ m \in N \) such that \( a_n \neq 0 \) for all \( n \geq m \) and \( \lim_{n \to \infty} \left( \frac{1}{a_n} \right) = \frac{1}{a} \).

**Note:** Combined applications of different parts ensures that \( \lim (a_n - b_n) = a - b \) and \( \lim (a_n/b_n) = a/b \) if \( b \neq 0 \).

**Note:** Note that it is possible that \( \lim (a_n) = a \) and \( \lim (b_n) = b \), \( a_n < b_n \), for all natural \( n \) and yet \( a = b \). Consider the sequences \( (a_n), (b_n) \) where \( a_n = 0 \) and \( b_n = 1/n \) for all \( n \).

**Theorem:** If \( (x_n) \) and \( (y_n) \) are convergent sequences of real numbers and if \( x_n < y_n \) for all \( n \geq m \) \((m \ \text{fixed} \ \text{natural} \ \text{number}) \), then \( \lim (x_n) \leq \lim (y_n) \).

**Theorem:** (Sandwich Theorem) Let \( (a_n), (b_n), (c_n) \) be sequences and \( c \in R \) be such that \( a_n \leq b_n \leq c_n \), for all \( n \geq m \) and \( \lim (a_n) = \lim (c_n) = c \). Then \( (b_n) = c \).

**Ex:** Let \( |a| < 1 \). Then \( \lim (a_n) = 0 \).

\( » \) If \( a = 0 \), the result is obvious. Let \( a \neq 0 \). Let \( \frac{1}{|a|} = 1 + h \). Then \( h > 0 \) and \( \frac{1}{|a|^n} = (1 + h)^n \geq 1 + nh > nh \) for all natural \( n \). Hence \( 0 < |a|^n < \frac{1}{nh} \). Since \( \lim (\frac{1}{nh}) = 0 \), by Sandwich Thorem, \( \lim (|a|^n) = 0 \). Hence \( \lim (a_n) = 0 \).

**Ex:** Let \( a > 0 \). Then \( \lim (a^{\frac{1}{n}}) = 1 \).

\( » \) If \( a = 1 \), nothing remains to prove.

Let \( a > 1 \). Then \( a^{\frac{1}{n}} > 1 \) for all natural \( n \).

Let \( b_n = a^{\frac{1}{n}} - 1 \) for natural \( n \). then \( (1 + b_n)^n = 1 + nb_n + \cdots + (b_n)^n > nb_n \),

so that \( 0 < b_n < a/n \) for all \( n \).

By Sandwich Theorem, \( \lim (b_n) = 0 \), so that \( \lim (a^{\frac{1}{n}}) = 1 \).
If \( 0 < a < 1 \), let \( b = 1/a \) so that \( \lim \left( \frac{1}{a^{1/n}} \right) = \lim \left( b^{1/n} \right) = 1 \), since \( b > 1 \) and hence \( \lim (a^{1/n}) = 1 \).

**Ex:** \( \lim (n^{1/n}) = 1 \).

Since \((a^{1/n}) \geq 1\) for all \( n \), we can write \( n^{1/n} = 1 + k_n \) where \( k_n \geq 0 \) for all \( n \).

Hence \( n = (1 + k_n)^n \geq \frac{n(n-1)}{2} (k_n)^2 + 1 \).

Thus \( 0 \leq k_n \leq \sqrt{\frac{2}{n}} \). Thus \( \lim (k_n) = 0 \) and so \( \lim (n^{1/n}) = 1 \).

**Example 2.11** Let \( b_n = \frac{n}{n^2+1} + \cdots + \frac{n}{n^2+n} \), for natural \( n \). Clearly \( a_n \leq b_n \leq c_n \) for all natural \( n \), where \( b_n = \sum_{k=1}^{n} \frac{n}{n^2+n} = \frac{n^2}{n^2+n} \) and \( c_n = \sum_{k=1}^{n} \frac{n}{n^2+1} = \frac{n^2}{n^2+1} \). Since \( \lim (a_n) = \lim (c_n) = 1 \), so \( \lim (b_n) = 1 \).

**Ex:** Use Squeeze Theorem to verify whether following sequences converge: \( \left( n^{1/n^2} \right), \left( (n!)^{1/n^2} \right) \).

\( 1 \leq n^{1/n^2} \leq n^{1/n}, 1 \leq (n!)^{1/n^2} \leq (n^n)^{1/n^2} = n^{1/n} \) for all natural \( n \).

**Note:** Every convergent sequence is bounded. The converse may not hold: consider \(((−1)^n)\). We now consider a class of sequences for which convergence is equivalent to boundedness.

**Definition:** A sequence \( a: N \to R \) or \((a_n)\) is monotonically increasing iff \( a \) is monotonically increasing function, that is, \( a_n \leq a_{n+1} \) for all natural \( n \). A sequence \( a: N \to R \) or \((a_n)\) is monotonically decreasing iff \( a \) is monotonically decreasing function, that is, \( a_n \geq a_{n+1} \) for all natural \( n \). A sequence is monotonic iff it is either monotonically increasing or monotonically decreasing.

**Theorem:** A monotonically increasing sequence is convergent iff it is bounded above.
Ex: Consider \((a_n)\), where \(a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\) for natural \(n\). \((a_n)\) is monotonically increasing. Also \(a_n \leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 1 + 2 \left(1 - \frac{1}{2^n}\right) < 3\) for all natural \(n\). Thus \((a_n)\) is convergent.

\[
\text{Ex: Consider } (a_n), \text{ where } a_n = \left(1 + \frac{1}{n}\right)^n.
\]

Now, \(a_n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \frac{1}{n^2} + \cdots + \frac{n(n-1)(n-2)\cdots 1}{n!} \frac{1}{n^n}
\]
\[= 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{n!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right)
\]
\[< 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}
\]
\[< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3.
\]

Thus \(2 < a_n < 3\), for all \(n\).

Hence \((a_n)\) is bounded.

Also \(a_{n+1} = 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n+1}\right) + \cdots + \frac{1}{n!} \left(1 - \frac{1}{n+1}\right) \left(1 - \frac{2}{n+1}\right) \cdots \left(1 - \frac{n-1}{n+1}\right) + \frac{1}{(n+1)!} \left(1 - \frac{1}{n+1}\right) \left(1 - \frac{2}{n+1}\right) \cdots \left(1 - \frac{n-1}{n+1}\right) \left(1 - \frac{n}{n+1}\right).
\]

Thus \(a_{n+1}\) is sum of \((n + 2)\) summands whereas an is that of \((n + 1)\) summands and each summand (from beginning) of \(a_{n+1}\) is greater than or equal to the corresponding summand of \(a_n\). Hence \((a_n)\) is monotone increasing. Thus \((a_n)\) is convergent. If we denote \(\lim(a_n)\) by \(e\), then \(2 \leq e \leq 3\) since \(2 \leq a_n \leq 3\) for all \(n\).

Ex: Consider the sequence \((a_n)\), where \(a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \) for all \(n\). Clearly \((a_n)\) is monotonically increasing. For each \(n\), \(a_{2^n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \cdots + \left(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\right) \geq 1 + \frac{1}{2} + \frac{2}{4} + \cdots + \frac{2^{n-1}}{2^n} = 1 + \frac{n}{2}\). Hence there is no \(M\) such that \(a_n < M\), for all \(n\). Thus \((a_n)\) is unbounded above and hence is not convergent.
Ex: Consider the sequence \((a_n)\), where \(a_n = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\). Clearly \((a_n)\) is monotonically increasing. Also, for each \(n\), \(a_n = 1 + \frac{1}{1.2} + \cdots + \frac{1}{(n-1)n} = 1 + \left(1 - \frac{1}{2}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 2 - \frac{1}{n} < 2\). So \((a_n)\) is bounded above. Hence \((a_n)\) is convergent.

Ex: Consider the sequence \((a_n)\), where \(a_n = 1 + \frac{1}{2^p} + \cdots + \frac{1}{n^p}\) for each \(n\). Clearly \((a_n)\) is monotonically increasing. We have seen in two examples above that \((a_n)\) is divergent if \(p = 1\) and convergent if \(p = 2\). This implies that \((a_n)\) is divergent if \(p \leq 1\) and convergent if \(p \geq 2\), because for each \(n\), \(0 < \frac{1}{n} \leq \frac{1}{n^p}\) if \(p \leq 1\) while \(0 < \frac{1}{n^p} \leq \frac{1}{n^2}\) if \(p \geq 2\).

PROPERLY DIVERGENT SEQUENCES

Definition: Let \((a_n)\) be a sequence of real numbers. \((a_n)\) tends to \(\infty\) as \(n\) tends to \(\infty\), written as \(\lim_{n \to \infty}(a_n) = \infty\) iff for every \(M > 0\), there exists natural number \(m\) such that \(n \geq m\) implies \(a_n > M\). Similarly, a sequence \((b_n)\) of real numbers tends to \(-\infty\) as \(n\) tends to \(\infty\), written as \(\lim_{n \to \infty}(a_n) = -\infty\) iff for every \(M > 0\), there exists natural number \(m\) such that \(n \geq m\) implies \(a_n < -M\).

CAUCHY’S GENERAL PRINCIPLE OF CONVERGENCE

Theorem: A sequence \((a_n)\) of real numbers is convergent iff for every \(\varepsilon > 0\), there exists positive integer \(m\) such that for \(p, q \geq m\), \(p, q\) natural number, \(|a_p - a_q| < \varepsilon\) holds.

Ex: Verify that the sequence \((a_n)\) where \(a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}\), \(n\) natural, does not converge.

»If \((a_n)\) converges, then corresponding to \(\varepsilon = \frac{1}{2}\), there exists \(m\) such that \(|a_p - a_q| < \frac{1}{2}\) for all \(p, q \geq m\). In particular, \(|a_{2m} - a_m| < \frac{1}{2}\). But \(|a_{2m} - a_m| = \frac{1}{m+1} + \cdots + \frac{1}{2m} > \frac{m}{2m} = \frac{1}{2}\), contradiction.


CHAPTER 3: INFINITE SERIES OF REAL NUMBER

Definition: An infinite series, or, for short, a series of real numbers is an expression of the form $a_1 + a_2 + \cdots + a_n + \cdots$ or, in more compact notation $\sum_{k=1}^{\infty} a_k$, where $a_n$ is a real number, for all $n$. The sequence $(A_n)$, where $A_n = a_1 + a_2 + \cdots + a_n$, is called the sequence of partial sums corresponding to the series $\sum_{k=1}^{\infty} a_k$.

Definition: We say $\sum_{k=1}^{\infty} a_k$ is convergent iff $\lim(A_n)$ exists as a real number, that is, iff the sequence of partial sums of the series is convergent. If $\lim_{n\to\infty}(A_n) = A$, then $A$ is the sum of the series $\sum_{k=1}^{\infty} a_k$ and we write $A = \sum_{k=1}^{\infty} a_k$. An infinite series $\sum_{k=1}^{\infty} a_k$ that is not convergent, is divergent. In particular, we say the series $\sum_{k=1}^{\infty} a_k$ diverges to $\infty$ or to $-\infty$ according as $\lim(A_n) = \infty$ or $\lim(A_n) = -\infty$.

Note: The convergence of a series is not affected by changing finite number of its terms, although its sum may change by doing so. If $\sum_{k=1}^{\infty} a_k$ and $b_1 + b_2 + \cdots + b_m + \sum_{k=m+1}^{\infty} a_k$ be two series obtained by changing first $m$ number of terms, and if $(A_n)$ and $(B_n)$ be the corresponding sequences of partial sums, then for $n > m$, $A_n = B_n + \sum_{k=1}^{m} a_k - \sum_{k=1}^{m} b_k$. Thus $(A_n)$ converges iff $(B_n)$ converges and if they converge to $A$ and $B$ respectively, $A = B + \sum_{k=1}^{m} a_k - \sum_{k=1}^{m} b_k$.

Ex: (Geometric Series) For the series $\sum a^k$ (a real),

$A_n = a_1 + a_2 + \cdots + a_{n-1} = \frac{1-a^n}{1-a}$, if $a \neq 1$ and

$A_n = n$, if $a = 1$.

If $-1 < a < 1$, $\lim(A_n) = \frac{1}{1-a}$. Hence the series $\sum a^k$ converges for $-1 < a < 1$.

If $a \geq 1$, then $A_n \geq n + 1$ for all $n$ (since $a, a^2, \ldots, a^{n-1}$ are $\geq 1$); hence $(A_n)$ and thus $\sum a^k$ diverges to $\infty$. 


If \( a = -1 \), \( A_{2n} = 1 \) and \( A_{2n+1} = 0 \) for all \( n \) and hence \( \sum a^k \) is divergent.

Finally, if \( a < -1 \), then \( \lim(A_{2n}) = \infty \) and \( \lim(A_{2n+1}) = -\infty \); hence \( \sum a^k \) is divergent to \(-\infty\).

**Ex:** (Exponential Series) We have seen earlier that the sequence \((A_n)\) given by
\[
A_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}
\]
is convergent to a real number \( e, 2 \leq e \leq 3 \). Thus
\[
e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots
\]

**Ex:** (Harmonic series and its variants) As seen in example 2.17, \( 1 + \frac{1}{2} + \frac{1}{3} + \cdots \) is divergent for \( p \leq 1 \) and convergent for \( p > 1 \). It can be shown that the sequence of partial sums and hence the series \( \sum (-1)^k \frac{1}{k} \) converges.

Note: Since convergence of a series is defined in terms of convergence of its sequence of partial sums, many results about convergence of a series follows from the corresponding results on sequences.

- The sequence of partial sums of a convergent series is bounded.
- Let \( \sum a_k = A, \sum b_k = B \). Then \( \sum (a_k + b_k) = A + B, \sum (ra_k) = rA \). If \( a_k \leq b_k \), for all \( k \), then \( A \leq B \).
- (Sandwich Theorem) If \((a_k), (b_k), (c_k)\) are sequences of real numbers such that \( a_k \leq b_k \leq c_k \) for each \( k \), and further if \( \sum a_k = A = \sum c_k \), then \( \sum b_k = A \).

**Theorem:** If \( \sum a_k \) is convergent, then \( \lim(a_k) = 0 \). In other words, if \( \lim(a_k) \) does not exist or is not equal to zero, then \( \sum a_k \) is divergent. Converse may not hold: consider \( \sum \frac{1}{k^p} \).

**Ex:** \( \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{kp} \) is divergent if \( p \leq 0 \) since \( \left| (-1)^{k-1} \frac{1}{kp} \right| \geq 1 \) for all \( k \).

**TESTS FOR CONVERGENCE:** series of non-negative terms

**Theorem:** (Comparison Test) Let \((a_k), (b_k)\) be sequences of non-negative real numbers such that \( a_k \leq b_k \) for all \( k \geq m, m \) natural. If \( \sum b_k \) is convergent, then \( \sum a_k \) is convergent and \( \sum a_k \leq \sum b_k \). If \( \sum a_k \) diverges, then \( \sum b_k \) is divergent.
Ex: $\sum \frac{2^{k+k}}{3^{k+k}}$ is convergent since $\frac{2^{k+k}}{3^{k+k}} \leq 2 \left(\frac{2}{3}\right)^k$ for all natural $k$ and $\sum \left(\frac{2}{3}\right)^k$ is convergent.

Ex: $\sum \frac{1}{(1+k^2+k^4)^{\frac{1}{5}}}$ is convergent since $\frac{1}{(1+k^2+k^4)^{\frac{1}{5}}} \leq \frac{1}{k^\frac{1}{5}}$ for all natural $k$ and $\sum \frac{1}{k^\frac{1}{5}}$ is convergent.

**Theorem:** (Limit Form of Comparison Test) Let $(a_k), (b_k)$ be sequences of positive real numbers such that $\lim_{n \to \infty} \left(\frac{a_k}{b_k}\right) = L (\neq 0)$. Then $\sum a_k$ is convergent iff $\sum b_k$ is convergent. If $\lim_{n \to \infty} \left(\frac{a_k}{b_k}\right) = 0$ and if $\sum b_k$ is convergent, then $\sum a_k$ is convergent. If $\lim_{n \to \infty} \left(\frac{a_k}{b_k}\right) = \infty$ and if $\sum b_k$ is divergent, then $\sum a_k$ is divergent.

Ex: The series $\sum \frac{2^{k+k}}{3^{k-k}}$ is convergent by Comparison Test since $\sum \left(\frac{2}{3}\right)^k$ is convergent and $\lim_{n \to \infty} \left(\frac{a_k}{b_k}\right) = 1(\neq 0)$.

Ex: $\sum \sin \left(\frac{1}{n}\right)$ diverges by comparison test since $\sum \frac{1}{n}$ is divergent and $\lim_{n \to \infty} \frac{\sin \left(\frac{1}{n}\right)}{\frac{1}{n}} = 1(\neq 0)$.

**Theorem:** (D’Alembert’s Ratio Test) Let $\sum a_k$ be a series of positive real numbers and let $\lim_{k \to \infty} \left(\frac{a_{k+1}}{a_k}\right) = x$. Then $\sum a_k$ converges if $x < 1$ and $\sum a_k$ diverges if $x > 1$.

**Theorem:** (Cauchy’s Root Test) Let $\sum a_k$ be a series of positive real numbers and let $\lim_{k \to \infty} \left(\sqrt[k]{a_k}\right) = x$. Then $\sum a_k$ converges if $x < 1$ and $\sum a_k$ diverges if $x > 1$.

**Theorem:** (Raabe’s Test) Let $\sum a_k$ be a series of positive real numbers and let $\lim_{k \to \infty} k \left(\frac{a_k}{a_{k+1}} - 1\right) = x$. Then $\sum a_k$ converges if $x > 1$ and $\sum a_k$ diverges if $x < 1$. 

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Ex: Use comparison Test to verify whether the following series converge:

\[ (1) \sum \frac{n+1}{n^3}, \ (2) \sum \frac{1}{\sqrt{n(n+1)}}, \ (3) \sum \frac{\sqrt{n}}{(n+1)^2} \]

Compare the series respectively with (1) \( \sum \frac{1}{n^2} \), (2) \( \sum \frac{1}{n} \) and (3) \( \sum \frac{1}{n^{3/2}} \)

Note: While testing convergence of a series \( \sum a_n \) by comparison test, the series is constructed by considering highest power of \( n \) present in numerator and denominator separately.

ALTERNATING SERIES: LEIBNITZ TEST

Definition: A series of the form \( \sum (-1)^n a_n \), where \( a_n > 0 \) for all positive integer \( n \), is an alternating series.

Theorem: (Leibnitz Test) An alternating series \( \sum (-1)^n a_n \) is convergent if the sequence \( (a_n) \) of positive terms is monotonically decreasing and \( \lim_{n \to \infty} (a_n) = 0 \).

Ex: \( \sum (-1)^n \frac{1}{2n+3} \) converges by Leibnitz Test.

CHAPTER 4

LIMIT OF A REAL VALUED FUNCTION OF A REAL VARIABLE

Definition: Let \( f: R \to R \) and \( c \in R \).

A real number \( L \) is the LEFT (hand) LIMIT of \( f \) at \( c \), written as \( \lim_{x \to c^-} f = L \), iff for every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( c - \delta < x < c \) implies \( |f(x) - L| < \epsilon \).
A real number $L$ is the RIGHT( hand) LIMIT of $f$ at $c$, written as $\lim_{x \to c^+} f = L$, iff for every $\epsilon > 0$, there exists $\delta > 0$ such that $c < x < c + \delta$ implies $|f(x) - L| < \epsilon$.

A real number $L$ is limit of $f$ at $c$, written as $\lim_{x \to c^-} f = L$, iff

(1) $\lim_{x \to c^-} f$ exists and
(2) $\lim_{x \to c^+} f$ exists and
(3) $\lim_{x \to c^-} f = L$.

**Ex:** Prove that $\lim_{x \to 2} 5x = 10$.

»Let $\epsilon > 0$ be given. $|5x - 10| < \epsilon \iff |x - 2| < \frac{\epsilon}{5} = \delta \iff 0 < |x - 2| < \delta, \delta > 0$. Hence the result.

Note: Observe the direction of implication sign carefully.

**Ex:** $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$

»Let $\epsilon > 0$ be given. $\left| x^2 \sin \frac{1}{x} - 0 \right| < \epsilon \iff |x^2| \left| \sin \frac{1}{x} \right| \leq |x^2| < \epsilon \iff |x| < \sqrt{\epsilon} = \delta \iff 0 < |x - 0| < \delta$.

**Ex:** Let $f: R \to R, f(x) = [x]$.

$\lim_{x \to 0^+} f = 0$ since for any $\epsilon > 0$, there exists $\delta = 1/2 > 0$ such that $0 < x < 0 + 1/2$ implies $|f(x) - L| = |0 - 0| < \epsilon$. Similarly, $\lim_{x \to 0^-} f = -1$. Hence $\lim_{x \to 0} f$ does not exist.

**Ex:** Let $g: R \to R, g(x) = e^{\frac{1}{x}}, x \neq 0, g(0) = 0$.

As $x \to 0^+$, $\frac{1}{x} \to \infty$ and so $e^{\frac{1}{x}} \to \infty$.

As $x \to 0^-$, $-\frac{1}{x} \to \infty$ and so $e^{\frac{1}{x}} = \frac{1}{e^{-\frac{1}{x}}} \to 0$.

So $\lim_{x \to 0^+} g$ does not exist while $\lim_{x \to 0^-} g = 0$.

**Theorem:** (Sequential criteria of limits) Let $f: R \to R$ and $c$ be a real number. Then the following are equivalent:
(1) \( \lim_{x \to c} f = L \),

(2) for every sequence \((x_n)\) of real numbers that converges to \(c\) such that \(x_n \neq c\) for all \(n\), the sequence \((f(x_n))\) converges to \(L\).

**Theorem:** (Divergence criteria) Let \(f: R \to R\) and \(c\) be a real number.

\[ \lim_{x \to c} f \neq L \text{ iff there exists a sequence } (x_n) \text{ of real numbers that converges to } c \text{ such that } x_n \neq c \text{ for all } n, \lim(x_n) = c \text{ but } \lim(f(x_n)) \neq L \text{ (either } \lim(f(x_n)) \text{ does not exist or exists but not equal to } L). \]

Alternatively, \( \lim_{x \to c} f \) does not exist iff there exist sequences \((x_n)\) and \((y_n)\) of real numbers, \(x_n, y_n \neq c\), for all \(n\), \(\lim(x_n) = \lim(y_n) = c\), but \(\lim(f(x_n)) \neq \lim(f(x_n))\).

**Ex:** \( \lim_{x \to 0} \sin \left( \frac{1}{x} \right) \) does not exist in \(R\).

» The sequences \((x_n)\) and \((y_n)\) both converge to 0 where \(x_n = \frac{1}{n \pi}\) and \(y_n = \frac{1}{(2n+1)\pi/2}\) but \(\sin \left( \frac{1}{x_n} \right) = (\sin n\pi)\) tends to 0 whereas \(\sin (y_n) = (\sin(2n + 1)\pi/2)\) converges to 1.

**Theorem:** (Limit Theorem of Functions) Let \(f, g: R \to R\) and \(b \in R\). If \(\lim_{x \to c} f = L\) and \(\lim_{x \to c} g = M\), then \(\lim_{x \to c} (f + g) = L + M\), \(\lim_{x \to c} (f - g) = L - M\), \(\lim_{x \to c} (fg) = LM\), \(\lim_{x \to c} (bf) = bL\). If \(h: R \to R\), \(h(x) \neq 0\) for all real \(x\) belonging to \((c - \delta, c + \delta)\), for some \(\delta > 0\), and if \(\lim_{x \to c} h = H \neq 0\), then \(\lim \left( \frac{f}{h} \right) = \frac{L}{H}\).

**Note:** If \(p\) be a polynomial, then \(\lim_{x \to c} p = p(c)\) for any real \(c\).

**Theorem:** (Squeeze Theorem) Let \(f, g, h: R \to R\) and \(c \in R\). If \(f(x) \leq g(x) \leq h(x)\) for all real \(x, x \neq c\), and if \(\lim_{x \to c} f = L = \lim_{x \to c} h\), then \(\lim_{x \to c} g = L\).

**Some Extensions of Limit Concept**

**Definition:** Let \(f: R \to R\) and \(c \in R\). \(f\) tends to \(\infty\) as \(x \to c\), written as \(\lim_{x \to c} f = \infty\), iff for every real \(M\), there exists \(\delta > 0\) such that for all \(x\) satisfying \(0 < |x - c| < \delta\), \(f(x) > M\) holds.
Definition: Let $f : R \to R$ and $c \in R$. $f$ tends to $-\infty$ as $x \to c$, written as $\lim_{x \to c} f = -\infty$, iff for every real $M$, there exists $\delta > 0$ such that for all $x$ satisfying $0 < |x - c| < \delta$, $f(x) < M$ holds.

**Ex:** $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

$(\Rightarrow)$ Let $M > 0$. $\frac{1}{x^2} > M \iff x^2 < \frac{1}{M} \iff -\frac{1}{\sqrt{M}} < x < \frac{1}{\sqrt{M}} \iff 0 < |x - 0| < \frac{1}{\sqrt{M}} = \delta$.

Hence the result.

Definition: Let $f : R \to R$. A real number $L$ is a limit of $f$ as $x \to \infty$, written as $\lim_{x \to \infty} f = L$, iff for any given $\epsilon > 0$, there exists $K$ such that $x > K$ implies $|f(x) - L| < \epsilon$.

Definition: Let $f : R \to R$. $f$ tends to $\infty$ as $x$ tends to $\infty$, written as $\lim_{x \to \infty} f = \infty$, iff given any real $M$, there exists $K$ such that $x > K$ implies $f(x) > M$.

**Example 4.8** Evaluate the following limits or show that they do not exist:

1. $\lim_{x \to 1^+} \frac{x}{x-1}$,
2. $\lim_{x \to 0^+} \frac{x+2}{\sqrt{x}}$,
3. $\lim_{x \to \infty} \frac{\sqrt{x^2 - 5}}{\sqrt{x^2} + 3} \ (x > 0)$,
4. $\lim_{x \to \infty} \frac{\sqrt{x - x}}{\sqrt{x} + x} \ (x > 0)$

$(1)$ Let $M > 1$. $\frac{x}{x-1} > M \iff x < \frac{M}{M-1} \iff 0 < x - 1 < \frac{1}{M-1} = \delta (> 0)$.

Hence $\lim_{x \to 1^+} \frac{x}{x-1} = \infty$.

$(2)$ Let $M > 0$. $\frac{x+2}{\sqrt{x}} > M \iff \frac{2}{\sqrt{x}} > M \iff \sqrt{x} < \frac{2}{M} \iff 0 < x - 0 < \frac{4}{M^2} = \delta$.

Hence $\lim_{x \to 0^+} \frac{x+2}{\sqrt{x}} = \infty$.

$(3)$ Let $\epsilon > 0$. $\frac{\sqrt{x^2 - 5}}{\sqrt{x^2} + 3} > M \iff |\frac{\sqrt{x^2 - 5}}{\sqrt{x^2} + 3} - 1| < \epsilon \iff \frac{8}{\sqrt{x^2} + 3} < \epsilon \iff \frac{8}{\sqrt{x}} < \epsilon \iff x > \frac{64}{\epsilon^2} = M$.

Hence $\lim_{x \to \infty} \frac{\sqrt{x^2 - 5}}{\sqrt{x^2} + 3} = 1$. 


Continuous Functions

Definition: Let $f: R \to R$ and let $c$ be a real number. $f$ is continuous at $c$ iff $\lim_{x \to c^+} f$, $\lim_{x \to c^-} f$ and $f(c)$ exists and $\lim_{x \to c^+} f = \lim_{x \to c^-} f = f(c)$. If $f$ is continuous at every real $c$, $f$ is continuous on $R$.

Theorem: Let $f, g: R \to R$ be continuous at $c \in R$ and $k$ be a real constant. Then the functions $f + g$, $f - g$, $f \cdot g$, $kf$ are continuous at $c$, where the functions are defined as follows:

$$(f + g)(x) = f(x) + g(x),$$
$$(f - g)(x) = f(x) - g(x),$$
$$(f \cdot g)(x) = f(x)g(x),$$
$$(kf)(x) = kf(x) \text{ for } x \in R.$$

The function $\frac{f}{g}$ defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ is continuous at $c$ if $g(x) \neq 0$ for all $x$ in $(c - \delta, c + \delta)$ for some $\delta > 0$.

Ex: Every constant function $f: R \to R$, $f(x) = c$ ($c$ real constant) is continuous on $R$.

Ex: Every polynomial function $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ is continuous on $R$.

Definition: $f: R \to R$ has removable discontinuity at $c$ iff $\lim_{x \to c^+} f = \lim_{x \to c^-} f = \lim_{x \to c} f$ exists but either $f(c)$ does not exist or exists but different from the limiting value $\lim_{x \to c} f$.

$f: R \to R$ has discontinuity of first kind at $c$ iff both $\lim_{x \to c^+} f$ and $\lim_{x \to c^-} f$ exist but $\lim_{x \to c^+} f \neq \lim_{x \to c^-} f$.

$f: R \to R$ has discontinuity of second kind at $c$ iff at least one of $\lim_{x \to c^+} f$ and $\lim_{x \to c^-} f$ does not exist.

Ex: Check the continuity of the following functions at the indicated points:

$$(1) \quad f(x) = \lfloor x \rfloor + \lceil -x \rceil \text{ at } x = 0, \quad (2) \quad f(x) = \frac{x}{1+e^{1/x}} \text{ at } x = 0,$$
(3) \( f(x) = 2 - x, 1 \leq x \leq 2; = x - \frac{x^2}{2}, x > 2 \) at \( x = 2 \),

(4) \( f(x) = [x] \) at 0,

(5) \( f(x) = \frac{1}{x}, x \neq 0; f(0) = 0 \) at 0.

\[ \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-1 + 0) = -1 = \lim_{x \to 0^+} (0 + (-1)) = \lim_{x \to 0^+} f(x) \neq 0 = f(0). \]

Hence \( f \) has removable discontinuity at \( 0 \).

\[ x \to 0^- \Rightarrow 1/x \to -\infty \Rightarrow e^{1/x} \to 0 \Rightarrow 1 + e^{1/x} \to 1 \Rightarrow x/(1 + e^{1/x}) \to 0, \]
\[ x \to 0^+ \Rightarrow 1/x \to \infty \Rightarrow e^{1/x} \to \infty \Rightarrow x/(1 + e^{1/x}) \to 0. \]

Thus \( \lim_{x \to 0^+} f = \lim_{x \to 0^-} f = f(0) \). Hence \( f \) is continuous at 0.

\[ \lim_{x \to 2^+} f = \lim_{x \to 2^+} \left( x - \frac{x^2}{2} \right) = 0; \lim_{x \to 2^-} f = \lim_{x \to 2^-} (2 - x) = 0 \]
\[ f(2) = 0. \] Hence \( f \) is continuous at \( x = 2 \).

\[ \lim_{x \to 0^+} f(x) = 0 \neq -1 = \lim_{x \to 0^-} f(x) \); hence \( f \) has discontinuity of first kind at 0.

\[ \lim_{x \to 0+} f \] does not exist as real number; hence \( f \) has discontinuity of second kind at 0.

**Theorem:** If \( f: R \to R \) be continuous at \( c \) and \( g: R \to R \) be continuous at \( f(c) \),
then \( g \circ f \), defined by \((g \circ f)(x) = g(f(x))\) for real \( x \), is continuous at \( c \).

**Ex:** \( f: R \to R, \ f(x) = \sin x \) and \( g: R \to R, \ g(x) = x^2 \) are continuous on \( R \); hence
\[ f \circ g: R \to R, \ (f \circ g)(x) = \sin(x^2) \] is continuous on \( R \).

**Theorem:** If \( f \) is continuous at \( c \) and \( f(c) \neq 0 \), then there exists \( \delta > 0 \) such that \( f(x) \) has the same sign as \( f(c) \) for all \( x \) in \((c - \delta, c + \delta)\).

**Theorem:** (Bolano’s Intermediate Value Property) Let \( f: R \to R \) be continuous and \( a, b \) be real numbers, \( a < b \), such that \( f(a) \neq f(b) \). Let \( k \) be a real number between \( f(a) \) and \( f(b) \). Then there exists \( c \) in \([a, b]\) such that \( f(c) = k \).

**Note:** Continuity is sufficient but not necessary for the conclusion in the above theorem to hold: for example, let us define
\[ f: \mathbb{R} \to \mathbb{R}, f(x) = x, \text{if } x \leq 1; \]
\[ = x - 1, \text{if } 1 < x, \text{ satisfies IVP but is discontinuous at 1.} \]

Corollary: If \( f(a) \) and \( f(b) \) are of opposite signs, then there exists \( c \) in \((a,b)\) such that \( f(c)=0 \).

Ex: Prove that \( \frac{5}{x-1} + \frac{7}{x-2} + \frac{16}{x-3} = 0 \) has a solution between 1 and 2.

\( \text{»Let } f(x) = 5(x-2)(x-3) + 7(x-1)(x-3) + 16(x-1)(x-2). f, \text{ being a polynomial function, is continuous at any real number and hence on } [1,2]. \text{ Also } f(1) > 0 \text{ and } f(2) < 0. \text{ Thus there exists } c \text{ in } (1,2) \text{ such that } f(c) = 0, \text{ that is } \frac{5}{c-1} + \frac{7}{c-2} + \frac{16}{c-3} = 0 \text{ has a solution in } (1,2). \)

Ex: Show that the equation \( \cos x = x \) has solution in \( (0, \frac{\pi}{2}) \).

\( \text{»Let } f(x) = \cos x - x, x \in [0, \frac{\pi}{2}]. f(0) > 0 \text{ and } f(\frac{\pi}{2}) < 0; \text{ also } f \text{ is continuous on } [0, \frac{\pi}{2}]. \text{ Hence there exists } c \text{ in } (0, \frac{\pi}{2}) \text{ such that } f(c) = 0; \text{ } c \text{ is a solution of the given equation.} \)

Theorem: Let \( f: \mathbb{R} \to \mathbb{R} \) be continuous and \( a, b \) be real numbers, \( a < b \). Then there exist \( c, d \) in \([a, b]\) such that \( f(c) = \max\{f(x): x \in [a, b]\} \) and \( f(d) = \min\{f(x): x \in [a, b]\} \).

Note: The result may not hold if we consider a discontinuous function or a non-closed interval.

Let \( f: \mathbb{R} \to \mathbb{R}, f(x) = \frac{1}{x} \text{ for } x \neq 0; \)
\[ = 1 \text{ at } x = 0. \]

So \( f \) is discontinuous at 0. There does not exist \( c \) in \([-1,1]\) such that \( f(c) = \max\{f(x): x \in [-1,1]\} \text{ gives } (0,1) \to \mathbb{R}, f(x) = x \text{ does not have either maximum or minimum on } (0,1). \)
CHAPTER 5

DERIVATIVE OF A REAL VALUED FUNCTION OF A REAL VARIABLE

Definition: Let $f: [a, b] \to R$, $a < c < b$. The left (hand) derivative of $f$ at $c$, denoted by $f^{(1)}(c-)$, is equal to $\lim_{x \to c^-} \frac{f(x) - f(c)}{x-c}$, provided the limit exists. The right (hand) derivative of $f$ at $c$, denoted by $f^{(1)}(c+)$, is equal to $\lim_{x \to c^+} \frac{f(x) - f(c)}{x-c}$, provided the limit exists.

Ex: Let $f: R \to R$, $f(x) = c$, $c$ real constant. $f$ is derivable at $a$ iff $f^{(1)}(a)$ exists and in that case $f^{(1)}(5) = \lim_{x \to 5} \frac{f(x) - f(5)}{x-5} = \lim_{x \to 5} \frac{c-c}{x-5} = 0$. In general, derivative of a constant function defined on an interval at any point of its domain of definition is zero.

Ex: Let $f: [0,2] \to R$, $f(x) = x^2 + x, 0 \leq x < 1$;

- $2$, at $x = 1$;
- $2x^3 - x + 1$, for $1 < x \leq 2$.

$\Rightarrow f^{(1)}(1+) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1^-} \frac{(2x^3 - x+1) - 2}{x-1} = 5$ while

$f^{(1)}(1-) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1^-} \frac{x^2 + x - 2}{x-1} = 3$. Since $5 = f^{(1)}(1+) \neq f^{(1)}(1-) = 3$, $f$ is not derivable at 1. Since $f^{(1)}(2-) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x-2} = \lim_{x \to 2^-} \frac{2x^2 - x - 14}{x-2} = \lim_{x \to 2^-} (2x^2 + 4x + 7) = 23$, so $f$ is differentiable at 2 and

$f^{(1)}(2) = 23$.

Theorem: If $f: [a, b] \to R$ be differentiable at $c$, $a < c < b$, then $f$ is continuous at $c$.

$\Rightarrow \lim_{x \to c} \{f(x) - f(c)\} = \lim_{x \to c} \left[ \frac{f(x) - f(c)}{x-c} \right] (x - c) = \lim_{x \to c} \frac{f(x) - f(c)}{x-c} \lim_{x \to c} (x - c) = f^{(1)}(c) = 0$. Hence $\lim_{x \to c} f(x) = f(c)$: thus $f$ is continuous at $c$. 

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Note: Converse may not hold: Consider \( f: [-1,1] \rightarrow \mathbb{R}, f(x) = |x| \). \( f \) is continuous at 0 but not differentiable at 0.

Ex: Let \( f(x + y) = f(x) + f(y) \) for all real \( x, y \). Let \( f(5) = 2 \) and \( f^{(1)}(0) = 3 \); prove that \( f^{(1)}(5) = 6. \)

\[
x + 2 = f(5) = f(5 + 0) = f(5) + f(0); \text{ hence } f(0) = 0. \text{ Now } f^{(1)}(5) = \lim_{h \rightarrow 0} \frac{f(5 + h) - f(5)}{h} = \lim_{h \rightarrow 0} \frac{f(5) + f(h) - f(5)}{h} = \frac{f(h) - f(0)}{h} = f^{(1)}(0) = 3.
\]

Definition: Let \( f: [a, b] \rightarrow \mathbb{R} \). \( f \) is (monotonically) increasing on \([a, b] \) iff \( x_1, x_2 \in [a, b], x_1 < x_2 \) imply \( f(x_1) \leq f(x_2) \). \( f \) is (monotonically) decreasing on \([a, b] \) iff \( x_1, x_2 \in [a, b], x_1 < x_2 \) imply \( f(x_1) \geq f(x_2) \). \( f \) is monotonic iff \( f \) is either increasing on \([a, b] \) or decreasing on \([a, b] \). \( f \) is strictly increasing iff \( x_1, x_2 \in [a, b], x_1 < x_2 \) imply \( f(x_1) < f(x_2) \). Similarly strictly decreasing function is defined.

Theorem: Let \( f: [a, b] \rightarrow \mathbb{R} \) be a differentiable function. Then

(1) \( f' \) is nonnegative throughout \([a, b] \) iff \( f \) is monotonically increasing on \([a, b] \),

(2) \( f' \) is positive throughout \([a, b] \) implies \( f \) is strictly increasing on \([a, b] \).

Ex: Prove that \( f(x) = \frac{a \sin x + b \cos x}{c \sin x + d \cos x} \) (\( a, b, c, d \) are constants) is either monotonically increasing on \( R \) or monotonically decreasing on \( R \).

\[
\text{Hence } f^{(1)}(x) = \frac{ad - bc}{(e \sin x + d \cos x)^2} \geq 0 \text{ for all } x \text{ if } ad - bc \geq 0 \text{ and } \leq 0 \text{ for all } x \text{ if } ad - bc \leq 0.
\]
CHAPTER 6
SUCCESSIONAL DIFFERENTIATION

Definition: Let \( f: [a, b] \to R \) be differentiable on \([a, b]\). Then \( f^{(1)}: [a, b] \to R \) is a function. This function \( f^{(1)} \) may again be differentiable at every point of \([a, b]\), the function \((f^{(1)})^{(1)}\) is denoted by \( f^{(2)} \) and called second order derivative of \( f \), \( f^{(2)} \), in its turn, may have derivative at every point of \([a, b]\), which is denoted by \( f^{(3)} \), called third order derivative of \( f \). The process may be continued.

\textbf{Ex:} Let \( y = x^m, x > 0, m \) rational. Show that \( y^{(n)} = m(m - 1) \ldots (m - n + 1)x^{m-n} \), for natural \( n \).

\( y^{(1)} = mx^{m-1} \). Result holds for \( n = 1 \). Let \( y^{(n)} = m(m - 1) \ldots (m - n + 1)x^{m-n} \), for some natural \( n \). Then \( y^{(n+1)} = (y^{(n)})^{(1)} = m(m - 1) \ldots (m - n + 1)(m - n)x^{m-n-1} \). Thus if the result holds for \( n \), it holds for \( n + 1 \). Thus, by mathematical induction, the result holds for all natural \( n \).

\textbf{Ex:} Let \( y = \frac{1}{ax+b} \). Then \( y^{(n)} = \frac{(-1)^nn!a^n}{(ax+b)^{n+1}} \).

\textbf{Ex:} Let \( y = \ln x \). Then \( y^{(1)} = x^{-1} \). Hence \( y^{(n)} = (y^{(1)})^{(n-1)} = (x^{-1})^{(n-1)} = \frac{(-1)^{n-1}(n-1)!}{x^n} \).

\textbf{Ex:} Let \( y = \sin x \). To prove: \( y^{(n)} = \sin\left(\frac{nm\pi}{2} + x\right) \) (\( n \) natural).

\( y^{(1)} = \cos x = \sin\left(\frac{\pi}{2} + x\right) \). Result holds for \( n = 1 \). Let \( y^{(m)} = \sin\left(\frac{m\pi}{2} + x\right) \) for some natural \( m \). Then \( y^{(m+1)} = \cos\left(\frac{m\pi}{2} + x\right) = \sin\left(\frac{\pi}{2} + \frac{m\pi}{2} + x\right) = \sin\left(\frac{(m+1)\pi}{2} + x\right) \). Hence the result holds by induction.

\textbf{Ex:} Let \( y = e^{ax}\sin bx \).

Then \( y(1) = e^{ax}(a \sin bx + b \cos bx) \).

Let \( a = r \cos \theta, \quad b = r \sin \theta, -\pi < \theta \leq \pi \).
Then \( r = \sqrt{a^2 + b^2} \) and \( \theta \) satisfies \( \cos \theta = a/r \) and \( \sin \theta = b/r, -\pi < \theta \leq \pi \).

Thus \( y^{(1)} = e^{ax}\sin(bx + \theta) \).

Hence \( y^{(n)} = e^{ax}r^n \sin(bx + n \theta) = e^{ax}(a^2 + b^2)^{n/2} \sin(bx + n \theta) \).

**Ex:** Let \( y = \frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} \). Equating coefficients of like powers of \( x \) in the identity \( 1 = A(x+2) + B(x-2) \), we get \( A = \frac{1}{4} \) and \( B = -\frac{1}{4} \). Hence

\[
y = \frac{1}{4} \left( \frac{1}{x-2} - \frac{1}{x+2} \right).
\]
Thus \( y^{(n)} = \frac{(-1)^n n!}{4} \left[ \frac{1}{(x-2)^{n+1}} - \frac{1}{(x+2)^{n+1}} \right] \).

**Ex:** Let \( y = \frac{x}{x+1} \). Prove \( y^{(5)}(0) = 5! \).

Thus \( y = 1 - \frac{1}{x+1} \). Thus \( y^{(5)} = \frac{(-1)^{55}}{(x+1)^6} \). Hence \( y^{(5)}(0) = 5! \).

**Theorem:** (Leibnitz’s Theorem) Let \( f, g \) be two functions possessing \( n \) th order derivatives, then \( f \cdot g \) is differentiable \( n \) times and

\[
(f \cdot g)^{(n)} = \binom{n}{0} f^{(n)} g + \binom{n}{1} f^{(n-1)} g^{(1)} + \ldots + \binom{n}{r} f^{(n-r)} g^{(r)} + \ldots + \binom{n}{n} f^{(1)} g^{(n)}.
\]

**Proof:** \((f \cdot g)^{(1)} = f^{(1)} g + f g^{(1)} = \binom{1}{0} f^{(1)} g + \binom{1}{1} f g^{(1)} \). Thus the result holds for \( n = 1 \).

Let \((f \cdot g)^{(m)} = \binom{m}{0} f^{(m)} g + \binom{m}{1} f^{(m-1)} g^{(1)} + \ldots + \binom{m}{r} f^{(m-r)} g^{(r)} + \ldots + \binom{m}{m} f^{(1)} g^{(m)} \).

Then \((f \cdot g)^{(m+1)} = ((f \cdot g)^{(m)})^{(1)} \)

\[
= \left[ \binom{m}{0} f^{(m)} g + \binom{m}{1} f^{(m-1)} g^{(1)} + \ldots + \binom{m}{r} f^{(m-r)} g^{(r)} + \ldots + \binom{m}{m} f^{(1)} g^{(m)} \right]^{(1)}
\]

\[
= \left[ \binom{m}{0} f^{(m+1)} g + \binom{m}{0} f^{(m)} g^{(1)} \right] + \ldots + \left[ \binom{m}{r} f^{(m-r+1)} g^{(r)} + \binom{m}{r} f^{(m-r)} g^{(r+1)} \right] + \ldots + \left[ \binom{m}{m} f^{(1)} g^{(m)} + \binom{m}{m} f g^{(m+1)} \right]
\]

\[
= \binom{m+1}{0} f^{(m+1)} g + \binom{m+1}{1} f^{(m)} g^{(1)} + \ldots + \binom{m+1}{r} f^{(m-r+1)} g^{(r)} + \ldots + \binom{m+1}{m+1} f g^{(m+1)}.
\]
since \( (r_{r-1})^m + (m)_{r} = (m+1)_{r} \) holds. Thus result holds by mathematical induction.

**Ex:** Let \( y = \sin(m \sin^{-1}x) \).

Prove that

1. \( (1 - x^2)y^{(2)} - xy^{(1)} + m^2 y = 0, \)
2. \( (1 - x^2)y^{(n+2)} - (2n + 1)xy^{(n+1)} - (n^2 - m^2)y^{(n)} = 0. \)

\( y^{(1)} = \cos(m \sin^{-1}x) \cdot \frac{m}{\sqrt{1-x^2}}. \)

Squaring and cross multiplying, \( (1 - x^2)(y^{(1)})^2 = m^2 \cos^2(m \sin^{-1}x) = m^2(1 - y^2) \) (from given expression).

Hence \( 2y^{(1)}(1 - x^2) - 2x(y^{(1)})^2 = m^2(-2y)y^{(1)}. \)

Since for the given expression \( y^{(1)} \) is not identically zero and the relation is to hold for all \( x \), cancelling \( 2y^{(1)} \) from both sides, we obtain \( (1 - x^2)y^{(2)} - xy^{(1)} + m^2 y = 0. \)

Next applying Leibnitz’s Theorem (remembering that \( n \)th order derivative of sum and difference of a finite number of functions is sum or difference of their \( n \)th order derivatives), we get

\[
[(1 - x^2)(y^{(2)})^{(n)} + \left(\begin{array}{c} n \\ 1 \end{array}\right)(y^{(2)})^{(n-1)}(-2x) + \left(\begin{array}{c} n \\ 2 \end{array}\right)(y^{(2)})^{(n-2)}(-2)] - [x(y^{(1)})^{(n)} + \left(\begin{array}{c} n \\ 1 \end{array}\right)(y^{(1)})^{(n-1)} + m^2y^{(1)} = 0; \text{ simplifying we obtain the result.}
\]

**CHAPTER 7**

**MEAN VALUE THEOREMS**

**Theorem:** (Rolle’s Theorem) If \( f : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\) and differentiable on \((a, b)\) and if \( f(a) = f(b) \), then there is \( c \in (a, b) \) such that \( f'(c) = 0. \)

**Ex:** If \( f(x) = x^3 + px + q \) for \( x \in \mathbb{R} \), where \( p, q \in \mathbb{R} \) and \( p > 0 \), then \( f \) has a unique real root. To see this, note that if \( f \) had more than one real root, then there would be \( a, b \in \mathbb{R} \) with \( a < b \) and \( f(a) = f(b) = 0. \) Hence, by Rolle’s Theorem, there would be \( c \in (a, b) \) such that \( f'(c) = 0. \) But \( f'(x) = 3x^2 + p \) is not zero.
for any $x \in R$ since $p > 0$. On the other hand, $\lim_{x \to -\infty} f = -\infty$ and $\lim_{x \to \infty} f = \infty$. Thus $f$ takes on negative as well as positive values. Hence $f(c) = 0$ for some $c \in R$, since $f$ has the IVP on $R$. Thus $f$ has a unique real root.

**Ex:** Consider $f:[0,1] \to R$ defined by $f(x)=x$, $x \in [0,1)$ and $f(1)=0$. Then $f$ is differentiable on $(0,1)$ and $f(0)=f(1)=0$ but $f$ is not continuous on $[0,1]$. Note that $f'(c)=1 \neq 0$ for $0<c<1$.

**Ex:** Consider $f:[-1,1] \to R$ defined by $f(x)=|x|$. Then $f$ is continuous on $[-1,1]$ and $f(-1)=f(1)=1$. $f'(x)=1$ for $x>0$, $=-1$, for $x<0$ and $f'(0)$ does not exist. There does not exist $c$, $-1<c<1$, such that $f'(0)=0$.

**Ex:** Consider $f:[0,1] \to R$ defined by $f(x)=x$ for $x \in [0,1)$. Then $f$ is continuous on $[0,1]$ and differentiable on $(0,1)$ but $f'(c)=1 \neq 0$ for every $c \in (0,1)$. Note that $f(0) \neq f(1)$.

**Theorem:** (Lagrange’s Mean Value Theorem) If $f:[a,b] \to R$ is continuous on $[a,b]$ and differentiable on $(a,b)$, then there exists $c \in (a,b)$ such that $f(b)-f(a)=(b-a)f'(c)$.

**Proof:** Consider $F:[a,b] \to R$ defined by $F(x)=f(x)-f(a)-s(x-a)$, where $s = \frac{f(b)-f(a)}{b-a}$. Then $F(a)=0$ and by our choice of constant $s$, $F(b)=0$. So Rolle’s Theorem applies to $F$ and, as a result, there exists $c \in (a,b)$ such that $F'(c)=0$, that is, $f'(c)=s$, as desired.

Note: If we write $b=a+h$, then the conclusion of MVT may be stated as follows: $f(a+h)=f(a)+hf'(a+\theta h)$ for some $\theta \in (0,1)$.

**Corollary:** (Mean Value Inequality) If $f:[a,b] \to R$ is continuous on $[a,b]$ and differentiable on $(a,b)$, and if $m,M \in R$ are such that $m \leq f'(x) \leq M$ for all $x \in (a,b)$, then $m(b-a) \leq f(b)-f(a) \leq M(b-a)$.

**Corollary:** Let $I$ be an interval containing more than one point and $f: I \to R$ be any function. Then $f$ is a constant function on $I$ if and only if $f/exists$ and is identically zero on $I$. 

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Proof: If \( f \) is a constant function on \( I \), then it is obvious that \( f'/I \) exists on \( I \) and \( f'(x) = 0 \) for all \( x \in I \).

Conversely, if \( f'/I \) exists and vanishes identically on \( I \), then for any \( x_1, x_2 \in I \) with \( x_1 < x_2 \), we have \([x_1, x_2] \subseteq I \) and applying MVT to \( f \) on \([x_1, x_2] \), we obtain \([f(x_2) - f(x_1)] = (x_2 - x_1)f'(c) \) (for some \( c \in (x_1, x_2) \subseteq I \)) = 0 and hence \( f(x_1) = f(x_2) \).

Note: The MVT or the Mean Value Inequality may be used to approximate a differentiable function around a point. For example, if \( m \) is natural and \( f(x) = \sqrt{x} \) for \( x \in [m, m + 1] \), then \( \sqrt{m + 1} - \sqrt{m} = f(m + 1) - f(m) = f'(c) = \frac{1}{2 \sqrt{c}} \) for some \( c \in (m, m + 1) \). Hence \( \frac{1}{2 \sqrt{m}} < \sqrt{m + 1} - \sqrt{m} < \frac{1}{2 \sqrt{m}}. \) For example, by putting \( m = 1, \) \( \frac{4}{3} = 1 + \frac{1}{3} < 1 + \frac{1}{2 \sqrt{2}} < \sqrt{2} < 1 + \frac{1}{2} = \frac{3}{2} \).

Corollary: Let \( I \) be an interval containing more than one point and \( f : I \rightarrow R \) be a differentiable function. Then (1) \( f' \) is nonnegative throughout \( I \) iff \( f \) is monotonically increasing on \( I \), (2) \( f' \) is positive throughout \( I \) implies \( f \) is strictly increasing.

Proof: Let \( x_1, x_2 \in I \) with \( x_1 < x_2 \). Then \([x_1, x_2] \subseteq I \) and we can apply MVT to the restriction of \( f \) to \([x_1, x_2] \) to obtain \([f(x_2) - f(x_1)] = (x_2 - x_1)f'(c) \) for some \( c \in (x_1, x_2) \). Thus \( f(x_2) \geq f(x_1) \) iff \( f'(c) \geq 0 \).

Note: \( f : R \rightarrow R, f(x) = x^3 \) is strictly increasing on \([-1, 1]\) but \( f'(0) = 0 \).

Theorem: (Cauchy’s MVT) Let \( f, g : [a, b] \rightarrow R \) are continuous on \([a, b]\) and differentiable on \((a, b)\), then there is \( c \in (a, b) \) such that
\[
g^{(1)}(c)(f(b) - f(a)) = f^{(1)}(c)(g(b) - g(a))
\]

Proof: Consider the function \( F : [a, b] \rightarrow R \) defined by \( F(x) = f(x) - f(a) - s[g(x) - g(a)] \), where
\[
s = \frac{f(b) - f(a)}{g(b) - g(a)}.
\]

Theorem: (L’Hospital’s Rule for \( \frac{0}{0} \) Indeterminate Form) Let \( c \in R \) and \( f, g : (c - r, c) \rightarrow R \) be differentiable function such that \( \lim_{x \to c^-} f = 0 \) and \( \lim_{x \to c^-} g = 0 \).
Suppose \( g^{(1)}(x) \neq 0 \) for all \( x \in (c - r, c) \) and \( \lim_{x \to c^-} \frac{f^{(1)}(x)}{g^{(1)}(x)} = L \). Then
\[
\lim_{x \to c^-} \frac{f(x)}{g(x)} = L.
\]

Here \( L \) can be a real number or \( \infty \) or \(-\infty\).

Note: L’Hospital’s Rule for \( \frac{0}{0} \) Indeterminate Form is also valid for right (hand) limits. The statement is similar to that given above. Combining the two cases follows L’Hospital’s Rule for (two-sided) limits of \( \frac{0}{0} \) Indeterminate Form, which we may state as follows:

**Theorem:** Let \( c \in R \) and \( D = (c - r, c) \cup (c, c + r) \) for some \( r > 0 \). Let \( f, g: D \to R \) be differentiable functions such that \( \lim_{x \to c^-} f = 0 \) and \( \lim_{x \to c^-} g = 0 \). Suppose \( g^{(1)}(x) \neq 0 \) for all \( x \in D \) and \( \lim_{x \to c^-} \frac{f^{(1)}(x)}{g^{(1)}(x)} = L \). Then \( \lim_{x \to c^-} \frac{f(x)}{g(x)} = L \).

Here \( L \) can be a real number or \( \infty \) or \(-\infty\).

L’Hospital’s Rule for \( \frac{0}{0} \) Indeterminate Form are also valid if instead of considering limits as \( x \to c \), where \( c \) is a real number, we consider limits as \( x \to \infty \) or as \( x \to -\infty \). For example, a statement for limits as \( x \to -\infty \) would be as follows:

**Theorem:** Let \( a \) be a real number and \( f, g: (-\infty, a) \to R \) be differentiable functions such that \( \lim_{x \to -\infty} f = 0 \) and \( \lim_{x \to -\infty} g = 0 \). Suppose \( g^{(1)}(x) \neq 0 \) for all \( x \) in \((-\infty, a)\) and \( \lim_{x \to -\infty} \frac{f^{(1)}(x)}{g^{(1)}(x)} = L \). Then \( \lim_{x \to -\infty} \frac{f(x)}{g(x)} = L \). Here \( L \) can be a real number or \( \infty \) or \(-\infty\).

**Theorem:** (L’Hospital’s Rule for \( \frac{\infty}{\infty} \) Indeterminate Form) Let \( I \) be an interval \([a, c)\) where \( a \) is real and either \( c \) is real with \( a < c \) or \( c = \infty \). Let \( f, g: I \to R \) be differentiable functions such that \( \lim_{x \to c^-} |g(x)| = \infty \). Suppose \( g^{(1)}(x) \neq 0 \) for all \( x \in I \) and \( \lim_{x \to c^-} \frac{f^{(1)}(x)}{g^{(1)}(x)} = L \). Then \( \lim_{x \to c^-} \frac{f(x)}{g(x)} = L \). Here \( L \) can be a real number or \( \infty \) or \(-\infty\).
Theorem: (Taylor’s Theorem) Let \( n \) be a nonnegative integer and \( f: [a, b] \rightarrow \mathbb{R} \) be such that \( f^{(1)}, f^{(2)}, \ldots, f^{(n)} \) exist on \([a, b]\) and further \( f^{(n)} \) is continuous on \([a, b]\) and differentiable on \((a, b)\). Then there exists \( c \in (a, b) \) such that

\[
f(b) = f(a) + (b - a)f^{(1)}(a) + \cdots + (b - a)^n \frac{f^{(n)}(a)}{n!} + (b - a)^{n+1} \frac{f^{(n+1)}(c)}{(n+1)!}.
\]

Proof: For \( x \in [a, b] \), let \( P(x) = f(a) + f^{(1)}(a)(x - a) + \frac{f^{(2)}(a)}{2!} (x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!} (x - a)^n\). Consider \( F: [a, b] \rightarrow \mathbb{R} \) defined by \( F(x) = f(x) - P(x) - s(x - a)^{n+1} \), where \( s = \frac{f(b) - f(a)}{(b - a)^{n+1}} \). Then \( F(a) = 0 \) and our choice of \( s \) is such that \( F(b) = 0 \). So Rolle’s Theorem is applicable to \( F \) on \([a, b]\) and, as a result, there is \( c_1 \in (a, b) \) such that \( f'(c_1) = 0 \). Next, \( f^{(1)}(a) = P^{(1)}(a) \) and so \( f^{(1)}(a) = 0 \) as well. Now Rolle’s Theorem applies to the restriction of \( F^{(1)} \) to \([a, c_1]\), and so there is \( c_2 \in (a, c_1) \) such that \( F^{(2)}(c_2) = 0 \). Further, if \( n > 1 \), then \( F^{(2)}(a) = 0 \) and so there exists \( c_3 \in (a, c_2) \) such that \( F^{(3)}(c_3) = 0 \). Continuing this way, we see that there is \( c = c_{n+1} \in (a, c_n) \) such that \( F^{(n+1)}(c) = 0 \). Now \( P^{(n+1)} \) is identically zero, since \( P \) is a polynomial of degree \( n \). In particular, \( P^{(n+1)}(c) = 0 \). Hence \( f^{(n+1)}(c) = s(n + 1)! \), which, in turn, yields desired result.

Note: (1) Note that the Lagrange’s MVT corresponds to the case \( n = 0 \) of Taylor’s Theorem.

(2) In statement of Taylor’s Theorem, the point \( a \) was the left end point of the interval on which \( f \) was defined. There is an analogous version for right endpoint: if \( f \) is as in the statement of Taylor’s Theorem, then there exists \( c \in (a, b) \) such that

\[
f(a) = f(b) + f^{(1)}(b)(a - b) + \cdots + \frac{f^{(n)}(b)}{n!} (a - b)^n + \frac{f^{(n+1)}(c)}{(n+1)!} (a - b)^{n+1}.
\]

Similarly, it can be proved that if \( I \) is any interval, \( a \in I \), and \( f: I \rightarrow \mathbb{R} \) is such that \( f^{(1)}, f^{(2)}, \ldots, f^{(n)} \) exist on \( I \) and \( f^{(n+1)} \) exists at every interior point of \( I \), then for any \( x \in I, x \neq a \), there is \( c \) between \( a \) and \( x \) such that

\[
f(x) = f(a) + f^{(1)}(a)(x - a) + \cdots + \frac{f^{(n)}(a)}{n!} (x - a)^n + \frac{f^{(n+1)}(c)}{(n+1)!} (x - a)^{n+1}.
\]
The last expression is sometimes referred to as the Taylor formula for \( f \) around \( a \). The polynomial given by \( P_n(x) = f(a) + f'(a)(x - a) + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n \) is called the \( n \)th Taylor polynomial of \( f \) around \( a \). The difference \( R_n(x) = f - P_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1} \) is called Lagrange form of remainder after \( n \) terms.

(3) Usually, the \( n^{th} \) Taylor polynomial of \( f \) around \( a \) provides a progressively better approximation to \( f \) around \( a \) as \( n \) increases. For example, let us revisit the estimate of \( \sqrt{2} \) obtained from Lagrange’s MVT and see what happens when we use Taylor’s Theorem.

Thus let \( m \) be natural and \( f: [m, m+1] \to \mathbb{R}, f(x) = \sqrt{x} \). Applying Taylor formula for \( f \) around \( m \), with \( n = 1 \), we have \( f(x) = f(m) + f'(m)(x - m) + \frac{f''(c)}{2!}(x - m)^2 \) for some \( c \) between \( m \) and \( x \).

In particular, for \( x = m + 1 \), we get \( \sqrt{m + 1} = \sqrt{m + \frac{1}{2\sqrt{m}}} - \frac{1}{8c\sqrt{c}} \) for some \( c \in (m, m + 1) \).

For example, by putting \( m = 1 \), we obtain \( 1 + \frac{1}{2} - \frac{1}{8} < \sqrt{2} < 1 + \frac{1}{2} - \frac{1}{16\sqrt{2}} \) and hence

\[
\frac{11}{8} < \sqrt{2} < 1 + \frac{1}{2} - \frac{1}{16\sqrt{2}} = \frac{35}{24},
\]

where in the last inequality we have used the estimate \( \sqrt{2} < 1.5 \). The resulting bounds 11/8 and 35/24 are, in fact, better than the bounds 4/3 and 35/24 obtained using MVT.

**Taylor’s Infinite Series:** Let \( f \) be a function possessing derivatives of all orders in \([a, a + h]\) for some \( h > 0 \). The \( n \)th Taylor Polynomial around \( a \) for different positive integers \( n \) are given by

\[
f(a + h) = f(a) + hf'(a)(a + c_1h) = f(a) + R_1,
\]

\[
f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a + c_2h) = f(a) + hf'(a) + R_2,
\]
f(a + h) = f(a) + hf^{(1)}(a) + \frac{h^2}{2!} f^{(2)}(a) + \frac{h^3}{3!} f^{(3)}(a + c_3 h) = f(a) + \ h f^{(1)}(a) + \frac{h^2}{2!} f^{(2)}(a) + R_3, \quad \ldots \ldots \ldots \ldots, \\

f(a + h) = f(a) + hf^{(1)}(a) + \frac{h^2}{2!} f^{(2)}(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a + c_n h) = f(a) + hf^{(1)}(a) + \frac{h^2}{2!} f^{(2)}(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n, \ldots \quad \text{where} \quad c_1, c_2, \ldots, c_n, \ldots \text{are in general different real numbers lying strictly between 0 and 1.}

Let us denote \( f(a) + hf^{(1)}(a) + \frac{h^2}{2!} f^{(2)}(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) \) by \( S_n \). Thus \( f(a + h) = S_n + R_n \), where \( (S_n) \) is the sequence of partial sums corresponding to the infinite series of real numbers

\[ f(a) + hf^{(1)}(a) + \frac{h^2}{2!} f^{(2)}(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \ldots. \quad \text{If} \quad \text{Lim}(R_n) = 0, \quad \text{then} \quad \text{Lim}(S_n) = f(a + h) \text{ and we can write} \]

\[ f(a + h) = f(a) + hf^{(1)}(a) + \frac{h^2}{2!} f^{(2)}(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \ldots \]

Note: Let \( f \) be a function possessing derivatives of all orders in \([0, h]\) for some \( h > 0 \) and let \( \text{lim} \left( \frac{h^n}{n!} f^{(n)}(c_n h) \right) = 0. \) Then the Taylor’s infinite series corresponding to \( f \) about 0 is:

\[ f(h) = f(0) + hf^{(1)}(0) + \frac{h^2}{2!} f^{(2)}(0) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) + \cdots \]

is called Maclaurin’s infinite series.

\[ \text{Ex:} \ \text{(The Exponential Series)} \]

\( f: R \to R, f(t) = e^t, \) possesses derivatives of every order in \([0, x]\) for an arbitrarily fixed \( x \) and \( R_n = \frac{x^n}{n!} f^{(n)}(c_n x) \to 0 \text{ as } n \to \infty \) since \( \text{lim} \left( \frac{x^n}{n!} \right) = 0 \) and \( e^{c_n x} \) is bounded for all positive integer \( n \) (since \( 0 < c_n < 1 \) implies \( 1 < e^{c_n x} < e^x \)). Thus \( e^x \) can be expanded in Maclaurin’s infinite series and

\[ e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots. \]
Ex: (The sine and cosine series) Let \( f(x) = \sin x \), then \( f^{(n)}(x) = \sin \left( \frac{n\pi}{2} + x \right) \) for all natural number \( n \). Thus \( f \) possesses derivatives of all orders in \([0, x]\). Also \( R_n = \frac{x^n}{n!} \sin \left( \frac{n\pi}{2} + c_n x \right) \to 0 \) as \( n \to \infty \) since \( \lim \left( \frac{x^n}{n!} \right) = 0 \) and \( \left| \sin \left( \frac{n\pi}{2} + c_n x \right) \right| \leq 1 \) for all \( n \). Thus \( \sin x \) can be expanded in Maclaurin’s infinite series \( \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \).

CHAPTER 8
FUNCTIONS OF SEVERAL REAL VARIABLES

Let \( f: \mathbb{R}^2 \to \mathbb{R} \) be a real-valued function of two independent real variables: we shall often write \( z = f(x, y) \) where \( x, y \) are independent real variables and \( z \) is the dependent real variable. Just as we represent function of a single real variable by a planar curve in two-dimensional plane, similarly a real-valued function of two independent real variables is represented by a surface in the three-dimensional space.

Definition: Let \( f: \mathbb{R}^2 \to \mathbb{R} \) and \((a, b) \in \mathbb{R}^2\). A real number \( L \) is limit of \( f \) as \((x, y) \to (a, b)\), written as \( \lim_{(x,y)\to(a,b)} f = L \), if and only if for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \) implies \( |f(x, y) - L| < \varepsilon \).

Ex: \( \lim_{(x,y)\to(0,0)} xy \frac{x^2-y^2}{x^2+y^2} = 0 \).

Let \( \varepsilon > 0 \) be given.
\[
|xy \frac{x^2-y^2}{x^2+y^2} - 0| < \varepsilon \iff |x||y| \left| \frac{x^2-y^2}{x^2+y^2} \right| < \varepsilon \iff |x||y| < \varepsilon \ (\text{since } \left| \frac{x^2-y^2}{x^2+y^2} \right| \leq 1 \text{ for real } x, y)
\]
\[
\iff |x| < \sqrt{\varepsilon}, |y| < \sqrt{\varepsilon} \iff x^2 + y^2 < \varepsilon^2 (\text{since } x^2 \geq 0, y^2 \geq 0) \iff 0 < \sqrt{(x-0)^2 + (y-0)^2} < \varepsilon = \delta \ (\text{say}). \text{ Hence the result.}
Definition: Let $f: \mathbb{R}^2 \to \mathbb{R}$ and $(a, b) \in \mathbb{R}^2$. $f$ is continuous at $(a, b)$ iff
\[ \lim_{(x,y) \to (a,b)} f = f(a, b). \]

Note: $f$ is continuous at $(a, b)$ iff for any two sequences $(x_n) \to a$ and $(y_n) \to b$, the sequence $f(x_n, y_n)$ should converge to $f(a, b)$.

Ex: Let $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \frac{xy}{x^2+y^2}$ for $(x, y) \neq (0,0)$ and $f(0,0) = 0$.

$\lim_{(x,y) \to (0,0)} f$ does not exist since for sequences $\left(\frac{1}{n}\right)$ and $\left(\frac{1}{n}\right)$ (both of which tend to 0 as $n \to \infty$), $f\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{1}{2} \to \frac{1}{2}$ as $n \to \infty$; whereas for the sequences $\left(\frac{1}{n}\right)$ and $\left(-\frac{1}{n}\right)$ (both of which tend to 0 as $n \to \infty$), $f\left(\frac{1}{n}, -\frac{1}{n}\right) = -\frac{1}{2} \to -\frac{1}{2}$ as $n \to \infty$. Hence $f$ is not continuous at $(0,0)$.

Definition: Let $f: \mathbb{R}^2 \to \mathbb{R}$ and $(a, b) \in \mathbb{R}^2$. The partial derivative of $f$ with respect to first independent variable $x$ at $(a, b)$, denoted by $f_x(a, b)$, is defined by

$\lim_{h \to 0} \frac{f(a+h,b)-f(a,b)}{h}$, provided the limit exists. Similarly the partial derivative of $f$ with respect to second independent variable $y$ at $(a, b)$, denoted by $f_y(a, b)$, is defined by

$\lim_{h \to 0} \frac{f(a,b+h)-f(a,b)}{h}$, provided the limit exists.

Ex: Find $f_x(2,1)$ and $f_y(2,1)$ for $(x, y) = \frac{x+y-1}{x+y+1}$, if they exist.

$\Rightarrow f_x(2,1) = \lim_{h \to 0} \frac{1}{2(2+h)} = \frac{1}{8}$.

Definition: The partial derivatives $f_x(x, y)$ and $f_y(x, y)$ of the function $f(x, y)$ are, in turn, functions of $x$ and $y$. Thus, if the associated limits exist, we may define partial derivatives of higher order for $f$ as follows:

$f_{xx}(a, b) = \lim_{h \to 0} \frac{f_x(a+h,b)-f_x(a,b)}{h}$,

$f_{xy}(a, b) = \lim_{h \to 0} \frac{f_y(a+h,b)-f_y(a,b)}{h}$, $f_{yx}(a, b) = \lim_{h \to 0} \frac{f_x(a,b+h)-f_x(a,b)}{h}$.
\( f_{yy}(a, b) = \lim_{h \to 0} \frac{f_y(a, b + h) - f_y(a, b)}{h} \).

Here \( f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}, f_{xx} = \frac{\partial^2 f}{\partial x^2}, f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, f_{yx} = \frac{\partial^2 f}{\partial y \partial x}, f_{yy} = \frac{\partial^2 f}{\partial y^2} \).

**Ex:** If \( u = \ln(x^3 + y^3 + z^3 - 3xyz) \), prove that

\[ (1) \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}, \quad (2) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -3/(x + y + z)^2 \]

\( \Rightarrow x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z) \).

Thus \( u = \ln(x + y + z) + \ln(x + \omega y + \omega^2 z) + \ln(x + \omega^2 y + \omega z) \), where \( \omega \) is an imaginary cube roots of unity. Now

\[
\frac{\partial u}{\partial x} = \frac{1}{x+y+z} + \frac{1}{x+\omega y+\omega^2 z} + \frac{1}{x+\omega^2 y+\omega z},
\]

\[
\frac{\partial u}{\partial y} = \frac{1}{x+y+z} + \frac{\omega}{x+\omega y+\omega^2 z} + \frac{\omega}{x+\omega^2 y+\omega z} - \frac{1}{x+y+z} + \frac{\omega^2}{x+\omega y+\omega^2 z} + \frac{\omega}{x+\omega^2 y+\omega z}.
\]

Thus \( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z} \), since \( 1 + \omega + \omega^2 = 0 \). Similarly the other part can be proved.

**Homogeneous functions : Euler’s Theorem**

**Definition:** \( f : R^2 \to R \) is homogeneous of degree \( n \) iff \((tx, ty) = t^n f(x, y) \) for all \( x, y \) and for every positive \( t \).

\( f(x, y) = \frac{xy}{x^2+y^2} \) is homogeneous of degree 0; \( f(x, y) = xy + x \) is not homogeneous.

**Theorem:** (Euler’s Theorem for two independent variables) If \( f \) is a homogeneous function of degree \( n \) having continuous partial derivatives, then \( x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf \).

**Ex:** If \( u = tan^{-1} \frac{x^3+y^3}{x-y} \), prove that

\[ (1) \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u, \]
(2) \[ x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (1 - 4\sin 2u)\sin 2u. \]

» \( u \) is not homogeneous but \( v = \tan u = \frac{x^3 + y^3}{x - y} \) is homogeneous of degree 2 and partial derivatives \( \frac{\partial v}{\partial x} \) and \( \frac{\partial v}{\partial y} \) exist and are continuous at all points except at \((0,0)\); hence we can apply Euler’s Theorem on \( v \) at all points other than \((0,0)\). We get \( x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = 2v \); or, \( x \frac{\partial v}{\partial u} \frac{\partial u}{\partial x} + y \frac{\partial v}{\partial u} \frac{\partial u}{\partial y} = 2\sin u \cos u \), or, \( x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2 \tan u}{\sec^2 u} = \sin 2u. \)

Differentiating partially w.r.t. \( x \) and \( y \) respectively, \( \frac{\partial u}{\partial x} + x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = 2\cos 2u \frac{\partial u}{\partial x}, \ x \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y \frac{\partial^2 u}{\partial y^2} = 2\cos 2u \frac{\partial u}{\partial y} \). Multiplying the equations by \( x \) and \( y \) respectively and adding,

\[ x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (2 \cos 2u - 1) \left( x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = \sin 2u(2 \cos 2u - 1) = (1 - 4\sin 2u)\sin 2u. \]

Ex: If \( u = xf \left( \frac{y}{x} \right) + g \left( \frac{y}{x} \right) \) and \( f, g \) have continuous partial derivatives, prove that \( x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0. \)

» \( xf \left( \frac{y}{x} \right) \) and \( g \left( \frac{y}{x} \right) \) are homogeneous functions of degree 1 and 0 respectively and the functions have continuous partial derivatives; hence \( \frac{\partial [xf \left( \frac{y}{x} \right)]}{\partial x} + \frac{\partial [xf \left( \frac{y}{x} \right)]}{\partial y} = xf \left( \frac{y}{x} \right) \)

and \( \frac{\partial [g \left( \frac{y}{x} \right)]}{\partial x} + \frac{\partial [g \left( \frac{y}{x} \right)]}{\partial y} = 0. \) From first relation, \( f \left( \frac{y}{x} \right) + xf^{(1)} \left( \frac{y}{x} \right) \left( -y/x^2 \right) +

xf^{(1)} \left( \frac{y}{x} \right) \left( -1/x \right) = xf \left( \frac{y}{x} \right), g^{(1)} \left( \frac{y}{x} \right) \left( -y/x \right) + g \left( \frac{y}{x} \right) \left( 1/x \right) = 0. \) Differentiating the relations w.r.t \( x \) and \( y \) respectively and multiplying by \( x \) and \( y \) and adding, we obtain required relation.
Definition: Let \( f: \mathbb{R}^2 \to \mathbb{R} \) possess partial derivatives \( f_x, f_y \) at \((a, b)\). \( f \) is differentiable at \((a, b)\) iff for all \( h \) and \( k \), \( f(a + h, b + k) - f(a, b) = f_x(a, b)h + f_y(a, b)k + A(h,k)h + B(h,k)k \), where \( \lim_{(h,k)\to(0,0)} A(h,k) = 0 = \lim_{(h,k)\to(0,0)} B(h,k) \). \( df(a, b) = f_x(a, b)h + f_y(a, b)k \) is called total differential of \( f \).

CHAPTER 9
APPLICATIONS OF DIFFERENTIAL CALCULUS
TANGENTS AND NORMALS TO A PLANE CURVE
Equation of the tangent to a curve in Cartesian form
The equation of tangent to a planar curve whose equation is given in Cartesian explicit form \( y = f(x) \) at a point \( P : (a, b) \) on the curve is given by: \( y - b = \frac{dy}{dx}|_{(a,b)} (x - a) \).

The equation of normal to \( y = f(x) \) at a point \( P : (a, b) \) (that is, a line through \((a, b)\) perpendicular to the tangent at \( P \)) is given by: \( y - b = -\frac{1}{\frac{dy}{dx}|_{(a,b)}} (x - a) \)
(In both cases, we assume that the line is not parallel to the y-axis; otherwise, the equation is \( y = b \).)

If the equation of the curve is in the implicit form \( f(x, y) = 0 \), we calculate \( \frac{dy}{dx} = -\frac{f_x}{f_y} \).

If the equation of the curve is in the parametric form \( x = f(t), y = g(t) \), then \( \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \).

Let \( T \) and \( G \) be the points of intersection of the tangent \( PT \) drawn at \( P \) with the x-axis and \( N \) be the foot of the perpendicular from \( P \) on the x-axis. Then subtangent at \( P \) is \( NT = \frac{y}{\frac{dy}{dx}} \) and subnormal at \( P \) is \( NG = y \frac{dy}{dx} \) (NT and NG are the signed...
distance from N to T and G and may be positive or negative). Length of the tangent PT is the absolute length PT intercepted on the tangent line by the curve and the x-axis and is given by $\frac{y_1 \sqrt{1+y_1^2}}{y_1}$, where $y_1 = \frac{dy}{dx}$.

Angle between two straight lines

If both lines are parallel to y-axis, angle between the lines is 0.

If one of the line is parallel to the y-axis while the other one has angle of inclination $\theta \neq \frac{\pi}{2}$, then the angle between the lines is $\left| \frac{\pi}{2} - \theta \right|$.

If none of the lines is parallel to y-axis, then the angle between them is $\tan^{-1} \left| \frac{m_2-m_1}{1+m_2m_1} \right|$.

Angle between two curves at their point(s) of intersection

Angle of intersection of two curves is the angle between the tangents drawn to the two curves at their point of intersection.

Rule of finding equations of tangents to a rational algebraic curve at the origin:

A rational algebraic curve of nth degree is a curve whose equation is of the form

$$(a_1x + b_1y) + (a_2x^2 + b_2xy + c_2y^2) + \cdots + (a_nx^n + b_nx^{n-1}y + \cdots + k_ny^n) = 0,$$

where at least one of $a_n, b_n, \ldots, k_n$ is nonzero.

The equations of tangent(s) to a rational algebraic curve at (0,0) are obtained by equating to zero the terms of the lowest degree in the equation.

Ex: Show that at any point of the curve $x^{m+n} = k^{m-n}y^{2n}$, the mth power of the subtangent varies as the nth power of the subnormal.

$$(\text{subtangent})^m = \left( \frac{2n}{m+n} \right)^m x^m$$ and $$(\text{subnormal})^n = \left( \frac{m+n}{2n} \right)^n \frac{x^m}{a^{m+n}}$$ (using given equation of the curve). Hence the result.

Ex: Prove that the curves $ax^2 + by^2 = 1$ and $Ax^2 + By^2 = 1$ intersect orthogonally if $\frac{1}{a} - \frac{1}{b} = \frac{1}{A} - \frac{1}{B}$. 

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Let \((h, k)\) be a point of intersection of the two curves. Then \(x^2 + by^2 = Ax^2 + By^2 = 1\), whence \(k^2 = \frac{a-A}{B-b}\). Also, for the first curve, \(\frac{dy}{dx} (h, k) = -\frac{ah}{bk}\) and for the second curve, \(\frac{dy}{dx} (h, k) = -Ah/Bk\). Product of slopes of tangents is \(\frac{aAh^2}{bBk^2} = \frac{aA}{a-A} \frac{B-b}{bB} = -1\). Hence the result.

Ex: Show that the points of the curve \(y^2 = 4a(x + a \sin \frac{x}{a})\) where the tangents are parallel to the x-axis lie on the curve \(y^2 = 4ax\).

Let \((h, k)\) be a point on the curve \(y^2 = 4a(x + a \sin \frac{x}{a})\) where the tangents are parallel to the x-axis. Thus \(k^2 = 4a(h + a \sin \frac{h}{a})\). Since tangent to the curve at \((h, k)\) is parallel to x-axis, \(\frac{dy}{dx} (h, k) = 0\), that is, \(\sin \frac{h}{a} = 0\); hence \(k^2 = 4ah\). Thus \((h, k)\) lies on \(y^2 = 4ax\).

Ex: In the curve \(x^m y^n = a^{m+n}\), prove that the portion of the tangent intercepted between the axes is divided at the point of contact into two segments which are in a constant ratio.

Let the tangent at \((h, k)\) to the curve meet the co-ordinate axes at A and B respectively. From the equation of the curve, \(mln x + nln y = (m+n)ln a\). Differentiating, \(\frac{dy}{dx} (h, k) = -\frac{m}{nh}\). The equation of tangent at \((h, k)\) is \(y - k = -\frac{mk}{nh}(x - h)\), that is, \(\frac{mx}{(m+n)h} + \frac{ny}{(m+n)k} = 1\).

Thus \(A: \left(\frac{(m+n)h}{m}, 0\right)\) and \(B: \left(0, \frac{(m+n)k}{n}\right)\). It can be easily seen that \(AP: PB :: n: m\) (ratio independent of \((h, k)\)).

Equation of the tangent to a curve in Polar form

If the equation of the curve is in polar form \(r = f(\theta)\), \(\theta\) is the vectorial angle of a point \((r, \theta)\) on the curve and \(\varphi\) is the angle between the radius vector and the tangent vector and if \(p\) is the length of the perpendicular dropped from the pole to the tangent at \((r, \theta)\), then \(\tan \varphi = \frac{r}{\frac{dr}{d\theta}}\), \(\frac{dr}{d\theta}\) is to be calculated from the equation \(r = f(\theta)\) of the curve and evaluated at \((r, \theta)\), \(p = r \sin \varphi\).
Angle of intersection between two curves whose polar equations are given

Let the two curves whose equations are \( r = f(\theta) \) and \( r = g(\theta) \) intersect at \( P: (r, \theta) \) and let \( \varphi_1 \) and \( \varphi_2 \) be the angles made by the tangent vector at \( P \) with the radii vector at \( P \) of \( r = f(\theta) \) and \( r = g(\theta) \) respectively. Then \( \tan \varphi_1 = r/f^{(1)}(\theta) \) and \( \tan \varphi_2 = r/g^{(1)}(\theta) \) at \( P \). Thus \( \tan(\varphi_1 - \varphi_2) = \frac{\tan \varphi_1 - \tan \varphi_2}{1 + \tan \varphi_1 \tan \varphi_2} \) gives the angle of intersection \( |\varphi_1 - \varphi_2| \) of the two curves at \( P \).

If \( p \) be the length of the perpendicular dropped from the pole onto the tangent to the curve \( r = f(\theta) \) at \( P: (r, \theta) \), then we have the result \( \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{\frac{dr}{d\theta}^2} \).

**Ex:** Prove that the normal at any point \( (r, \theta) \) on the curve \( r^n = a^n \cos n \theta \) makes an angle \( (n + 1) \theta \) with the initial line.

Given equation of the curve is \( n \ln r = n \ln a + n \cos n \theta \). Thus \( \frac{dr}{r \ d\theta} = -n \tan n \theta \) which gives \( \tan \varphi = -\cot n \theta \). Thus \( \varphi = \frac{\pi}{2} + n \theta \). Let \( \psi \) be the angle that the tangent makes with initial line. Then \( \psi - \frac{\pi}{2} \) is the angle made by the normal with the initial line. Now \( \psi = \theta + \varphi = (n + 1) \theta + \frac{\pi}{2} \). Thus \( \psi - \frac{\pi}{2} = (n + 1) \theta \) is the angle made by the normal with the initial line.

**Ex:** Show that the curves \( r^n = a^n \cos n \theta \) and \( r^n = b^n \cos n \theta \) intersect orthogonally.

Taking natural logarithms and differentiating w.r.t. \( \theta \), for the first curve \( \frac{n dr}{r \ d\theta} = -n \tan n \theta \), which gives \( \tan \varphi_1 = -\cot n \theta \), that is, \( \varphi_2 = \pi/2 + n \theta \). Similarly, for the second curve, \( \varphi_2 = n \theta \). Thus the angle between the curves is \( |\varphi_1 - \varphi_2| = \frac{\pi}{2} \).

**Pedal equation from Cartesian and Polar Equation**

A relation between \( p \), the length of the perpendicular from a given point \( O \) to the tangent at any point \( P \) on a curve and \( r \), the distance of \( P \) from \( O \) is called pedal
equation of the curve w.r.t. O. When nothing is mentioned, O is to be taken as the origin or pole according as the equation is Cartesian or polar.

(A) Pedal Equation from Cartesian

Let \( f(x, y) = 0 \) \( \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1) \) be the Cartesian equation of the curve.

The equation of tangent at \((x, y)\) being \( f_x x + f_y y - (xf_x + yf_y) = 0 \),

we have \( p^2 = \frac{(xf_x+yf_y)^2}{f_x^2+f_y^2} \) \( \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2) \)

Also \( r^2 = x^2 + y^2 \) \( \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3) \).

Eliminating \( x, y \) from \((1),(2)\) and \((3)\), we obtain the pedal equation.

(B) Pedal Equation from polar

Let \( f(r, \theta) = 0 \) be the polar equation. We have \( \tan \phi = \frac{r}{dr/d\theta} \) and \( p = r \sin \phi \).

We obtain pedal equation from these equations after eliminating \( \theta \) and \( \phi \).

Ex: Find the pedal equation of the asteroid \( x^{2/3} + y^{2/3} = a^{2/3}, a > 0 \).

»The equation of tangent at a point \((x_1, y_1)\) to the curve is \( xx_1^{1/3} + yy_1^{1/3} = a^{2/3} \).

Thus

\[
p^2 = \frac{a^{2/3}}{x^{2/3} + y^{2/3}} = (axy)^{2/3}, r^2 = x^2 + y^2 = (x^{2/3} + y^{2/3})^3 - 3x^{2/3}y^{2/3}(x^{2/3} + y^{2/3}) = a^2 - 3(axy)^{2/3} = a^2 - 3p^2. \]

Thus the pedal equation is \( r^2 + 3p^2 = a^2 \).

Ex: Find the pedal equation of ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) with respect to one of its foci.

»Let \((ae, 0)\) be the focus and \(P: (h, k)\) be any point on the ellipse. Equation of tangent at \(P\) is \( y - k = -\frac{b^2h}{a^2k} (x - h) \). Here, \( k^2 = b^2(1 - \frac{h^2}{a^2}) = b^2 - h^2(1 - e^2) = (a^2 - x^2)(1 - e^2) \) and \( r^2 = h^2 + k^2 = (x - ae)^2 + y^2 = (a - ex)^2 \).

Thus \( r = a - ex \). Now \((p)^2 = \text{square of the perpendicular from} (ae, 0) \) to the tangent at \(P\) \( = \frac{(b^2aex - a^2b^2)^2}{a^4y^2 + b^4x^2} = \frac{a^2b^4(a - ex)^2}{a^4y^2 + b^4x^2} = \frac{b^2r^2}{a^2 - e^2x^2} \).

Thus \( \frac{b^2}{p^2} = \frac{a + ex}{r} \).

\[
\frac{2a - r}{r} = \frac{2a}{r} - 1.
\]
Ex: Find the pedal equation of ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) with respect to the centre.

»The parametric equation of the ellipse is \( x = a \cos \varphi, y = b \sin \varphi \). Thus, \( r^2 = x^2 + y^2 = a^2 \cos^2 \varphi + b^2 \sin^2 \varphi \). Equation of the tangent at \((a \cos \varphi, b \sin \varphi)\) is \( \frac{x \cos \varphi}{a} + \frac{y \sin \varphi}{b} = 1 \). Here \( p = \frac{ab}{\sqrt{b^2 \cos^2 \varphi + a^2 \sin^2 \varphi}} \). Hence \( \frac{a^2 b^2}{p^2} = a^2 + b^2 - r^2 \) is the required pedal equation.

Ex: Find the pedal equation of the asteroid \( x = a \cos 3 \theta, y = b \sin 3 \theta \) with respect to the origin.

»Equation of the tangent at any point ‘\( \theta \)’ is \( x \sin \theta + y \cos \theta = a \sin \theta \cos \theta \).

Thus \( p = a \sin \theta \cos \theta \). Now \( r^2 = x^2 + y^2 = a^2 (\cos 6 \theta + \sin 6 \theta) = a^2 \{ (\cos 2 \theta + \sin 2 \theta)^3 - 3 \sin 2 \theta \cos 2 \theta \} = a^2 (1 - 3p^2/a^2) = a^2 - 3p^2 \).

Ex: Find the pedal equation of the parabola \( y^2 = 4a(x + a) \) with respect to the vertex.

»The vertex is at \((-a, 0)\). Let \( P: (h,k) \) be any point on the parabola.

\[ r^2 = (h + a)^2 + k^2 = (h + a)^2 + 4ah + 4a^2. \]

Equation of tangent at \((h, k)\) is \( ky = 2a(x + h) + 4a^2 \)

The length of the perpendicular from \((-a, 0)\) on the tangent at \((h, k)\) is \( p = \frac{2a(h - a) + 4a^2}{\sqrt{k^2 + 4a^2}} = \frac{2a(h + a)}{\sqrt{k^2 + 4a^2}} \)

Also \( k^2 = 4a(h + a) \)

Eliminating \( h, k \) from (1), (2) and (3), we get the required equation.

CHAPTER 10

CURVATURE

Let \( L \) be a fixed line of reference, \( \Gamma \) be a planar curve in the plane of \( L \), \( A \) is a fixed point on \( \Gamma \) with reference to which arc length is measured and \( P, Q \) be two points on \( \Gamma \). Let arc \( AP = s \) and arc \( AQ = s + \Delta s \) so that arc \( PQ \) is \( \Delta s \). Let the tangents to \( \Gamma \) at \( P \) and \( Q \) respectively make angles \( \psi \) and \( \psi + \Delta \psi \) with \( L \) respectively. Thus tangent
rotates through an angle $\Delta \psi$ for a change $\Delta s$ in arc length. The curvature (rate of bending of the curve) $k$ of $\Gamma$ at $P$ is defined as $\lim_{\Delta s \to 0} \frac{\Delta \psi}{\Delta s} = \frac{d\psi}{ds}$.

Circle: a special case

Let $PQ$ be an arc of a circle of radius $r$ subtending an angle $\Delta \psi$ (in radian) at the centre of the circle and let $\Delta s$ be the arc length of the arc. Then $\Delta s = r \Delta \psi$ and thus curvature of the circle at $P$ is

$$k = \frac{d\psi}{ds} = \lim_{\Delta s \to 0} \frac{\Delta \psi}{\Delta s} = \frac{1}{r}, \text{ independent of the point } P \text{ on the circle. The radius of the circle } r = \frac{1}{k}. \text{ This prompts the following general definition:}$$

Circle of curvature, radius of curvature

Let $\Gamma$ be a planar curve, $P$ be a point on the curve such that the curvature $k$ of $\Gamma$ at $P$ is non-zero. The circle $C$ satisfying the following properties is called the circle of curvature of $\Gamma$ at $P$; its centre and radius are called centre of curvature and radius of curvature of $\Gamma$ at $P$:

- The radius of the circle is $1/k$
- $C$ passes through $P$ and has the same tangent as $C$ at $P$
- An arc of $C$ containing $P$ and the circle lies on the same side of the tangent at $P$

Note For a point $P$ on $\Gamma$, curvature of $\Gamma$ at $P$ is equal to the curvature of its circle of curvature at $P$.

Formulae for finding radius of curvature $\rho$

- Intrinsic Equation $s = f(\psi)$: $\rho = \frac{ds}{d\psi}$
- Cartesian Equation $f(x, y) = 0$: $\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}, y_1 = \frac{dy}{dx}, y_2 = \frac{d^2y}{dx^2}$
- Polar Equation $r = f(\theta)$: $\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}$
- Pedal Equation $p = f(r)$: $\rho = r \frac{dr}{dp}$. 

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Ex: Find the radius of curvature of the curve $y = xe - x$ at its point of local maximum.

$\Rightarrow y_1 = e - x(1 - x) = 0$ gives $x = 1$ ($e - x \neq 0$ for any real $x$). $y_2(1) = -e - 1 < 0$. Thus curve has local maximum at $(1, e - 1)$. Thus $\rho = |e - 1| = e$.

Ex: Prove that the radius of curvature of the curve $x = a \cos 3\theta, y = a \sin 3\theta$, at the point $\theta$ is $\rho = 3a \sin \theta \cos \theta$.

$\Rightarrow y_1 = \frac{3a \sin^2 \theta \cos \theta}{3a \cos^2 \theta \sin \theta} = \tan \theta, y_2 = \sec 2\theta \frac{d\theta}{dx} = \frac{\sec^2 \theta}{-3a \cos^2 \theta \sin \theta} = -\frac{1}{3a} \sec^4 \theta \cosec \theta$.

Thus $\rho = \left|\frac{\sec^3 \theta}{-\frac{1}{3a} \sec^4 \theta \cosec \theta}\right| = 3a \sin \theta \cos \theta$.

Ex: In the curve $r^n = a^n \cos n\theta$, verify that $\rho = \frac{r^n}{(n+1)p} = \frac{a^n}{(n+1)r^{n-1}}$.

$\Rightarrow$ Then $n \ln r = \ln(a^n + \ln \cos n\theta). \frac{n}{r} \frac{dr}{d\theta} = -n \tan n\theta$. Thus $\tan \varphi = \tan \left(\frac{\pi}{2} + n\theta\right)$. Taking particular solution, $\varphi = \frac{\pi}{2} + n\theta$. Thus $\rho = r \sin \varphi = r \cos n\theta$ . eliminating $\cos n\theta$, the pedal equation of the curve is

$r^{n+1} = a^n p$. Differentiating w.r.t. $p$, $\rho = r \frac{dr}{dp} = \frac{a^n}{(n+1)r^{n-1}} = \frac{a^n r^2}{(n+1)r^{n+1}} = \frac{r^2}{(n+1)p}$.

Ex: Show that at any point of the cardiode $r = a(1 - \cos \theta)$, $\rho$ is numerically equal to $\frac{2}{3} \sqrt{2a r}$.

$\Rightarrow$ $r = a(1 - \cos \theta)$. $r_1 = a \sin \theta$, $r_2 = a \cos \theta$. $r^2 + r_1^2 = a^2 [(1 - \cos \theta)^2 + \sin 2\theta] = 2a^2 (1 - \cos \theta) = 4a^2 \sin^2 \frac{\theta}{2}$. Thus $(r^2 + r_1^2)^{\frac{3}{2}} = 8a^3 \sin^3 \frac{\theta}{2}$. Also

$r^2 + r_1^2 - rr_2 = 6a^2 \sin^2 \frac{\theta}{2}$. Hence $\rho = \frac{4a^3 \sin^2 \frac{\theta}{2}}{3} = \frac{2}{3} \sqrt{2ar}$ (from equation of the curve).

Newton’s Method of finding radius of curvature

Let $\Gamma$ be a planar curve, $P$ and $Q$ be two points on $\Gamma$. Let $C$ be a circle through $P$ and $Q$, and having common tangent with $\Gamma$ at $P$. If $Q$ approaches $P$ along $\Gamma$, the limiting position of $C$ is identical to the circle of curvature to $\Gamma$ at $P$. A few important results coming out of this considerations are as follows:
(1) If a curve passes through the origin and the x-axis is tangent at the origin to the curve, then \( \lim_{(x,y) \to (0,0)} \frac{x^2}{2y} \) equals the radius of curvature of the curve at the origin.

(2) If a curve passes through the origin and the y-axis is tangent at the origin to the curve, then \( \lim_{(x,y) \to (0,0)} \frac{y^2}{2x} \) equals the radius of curvature of the curve at the origin.

(3) If a curve passes through the origin and if \( ax+by=0 \) is tangent at the origin to the curve, then \( \frac{1}{2} \sqrt{a^2 + b^2} \lim_{(x,y) \to (0,0)} \frac{x^2+y^2}{ax+by} \) equals the radius of curvature of the curve at the origin.

**Ex:** Show that the radii of curvature of the curve \( y^2(a-x) = x^2(a+x) \) at the origin are \( \pm \sqrt{2a} \).

»Equating to zero the lowest degree terms of the given equation \( a(y + x)(y - x) = x^3 + xy^2 \), the tangents at the origin of the given curve are \( y + x = 0 \) and \( y - x = 0 \). The radius of curvature of the branch of the curve to which \( y - x = 0 \) is a tangent,

\[
\rho = \frac{1}{2} \sqrt{1^2 + (-1)^2} \lim_{(x,y) \to (0,0)} \frac{x^2+y^2}{y-x} = \frac{1}{2} \sqrt{2} \lim_{(x,y) \to (0,0)} \frac{(x^2+y^2)a(y+x)}{x^3+xy^2} = \sqrt{2a} \lim_{(x,y) \to (0,0)} \frac{\left\{1+\left(\frac{y}{x}\right)^2\right\}\left\{1+\frac{y}{x}\right\}}{1+\left(\frac{y}{x}\right)^2} = \sqrt{2a}, \text{ since } \lim_{(x,y) \to (0,0)} \frac{y}{x} (= \text{ slope of the tangent } y - x = 0 \text{ at the origin}) = 1. \text{ Similarly, the radius of curvature of the other branch of the curve corresponding to the tangent } y + x = 0 \text{ is } -\sqrt{2a}. 
\]

**Co-ordinates of Centre of Curvature:** Equation of Circle of Curvature, The centre of curvature \( (\bar{x}, \bar{y}) \) of a curve whose Cartesian equation is given is given by \( \bar{x} = x - \frac{y_1(1+y_2^2)}{y_2}, \bar{y} = y + \frac{(1+y_2^2)}{y_2}, \) where \( y_1 \) and \( y_2 \) are the first and second order derivative calculated from the given equation of the curve and \( y_2 \neq 0 \).

**Ex:** Find the equation of the circle of curvature of \( 2xy + x + y = 4 \) at \((1,1)\).
From the given equation of the curve, \((y_1)_{(1,1)} = -1\) and \((y_2)_{(1,1)} = 4/3\). Thus \(\rho = \frac{3\sqrt{2}}{2}, \bar{x} = 5/2, \bar{y} = 5/2\). Hence the equation of the circle of curvature at \((1,1)\) is \((x - 5/2)^2 + (y - 5/2)^2 = 9/2\).

Let \(\Gamma\) be a planar curve, \(P: (x, y)\) be a point on \(\Gamma\) where \(y_1, y_2\) exist and \(y_2 \neq 0\), and \(C: (\bar{x}, \bar{y})\) be the corresponding co-ordinate of the centre of curvature of \(\Gamma\) at \(P\). Locus \(\Gamma_1\) of \(C\) corresponding to the locus \(\Gamma\) of \(P\) is called evolute of \(\Gamma\) and \(\Gamma\) is called involute of \(\Gamma_1\).

**Ex:** Find the equation of the evolute of the parabola \(y^2 = 12x\).

\[y_1 = \frac{6}{y} = \frac{\sqrt{3}}{\sqrt{x}}, \quad y_2 = -\frac{36}{y^3} = -\frac{\sqrt{3}}{2x^{3/2}}.\] Hence \(\bar{x} = x - \frac{\sqrt{3}(1+\frac{3}{2})}{\sqrt{2}x^{3/2}} = 3x + 6, \bar{y} = -\frac{y^3}{36}\).

Thus \(x = \frac{\bar{x} - 6}{3}, y = -\frac{3}{\sqrt{36}}\bar{y}\). Substituting these values in the given equation of parabola, we get \(81\bar{y}^2 = 4(\bar{x} - 6)^3\). Changing to current co-ordinates, the equation of the envelope is \(81y^2 = 4(x - 6)^3\).

**Ex:** Show that the evolute of the asteroid \(x^{2/3} + y^{2/3} = a^{2/3}\) is \((x + y)^{2/3} + (x - y)^{2/3} = 2a^{2/3}\).

Let \(x = a\cos 3\theta, y = a\sin 3\theta\) be a point on the asteroid. \(\bar{x} = a\cos 3\theta + 3a\sin 2\theta\cos \theta, \bar{y} = a\sin 3\theta + 3a\cos 2\theta\sin \theta\). Thus \((\bar{x} + \bar{y})^{2/3} + (\bar{x} - \bar{y})^{2/3} = 2a^{2/3}\).

**Ex:** Find the equation of the evolute of the hyperbola \(xy = a^2\).

\[y = a2/x, y(1) = -a2/x2, y(2) = 2a2/x3.\] Thus \(\bar{x} = \frac{3x}{2} + \frac{y^3}{2a^2}, \quad \bar{y} = \frac{3y}{2} + \frac{x^3}{2a^2}.\)

Hence \(\bar{x} + \bar{y} = \frac{1}{2a^2} (x + y)^3\) and \(\bar{x} - \bar{y} = -\frac{1}{2a^2} (x - y)^3\). Thus \((\bar{x} + \bar{y})^{2/3} - (\bar{x} - \bar{y})^{2/3} = (4a)^{2/3}\).
CHAPTER 11

RECTILINEAR ASYMPTOTES

Let \( \Gamma \) be a planar curve whose Cartesian equation is given and let \( L \) be a line whose equation is \( ax + by + c = 0 \). Let \( P: (\alpha, \beta) \) be an arbitrary point on \( \Gamma \), whose perpendicular distance from \( L \) is \( \frac{|a\alpha + b\beta + c|}{\sqrt{a^2 + b^2}} \). If \( \frac{|a\alpha + b\beta + c|}{\sqrt{a^2 + b^2}} \) tends to zero when either \( |\alpha| \) or \( |\beta| \) or both tend to \( \infty \), then \( L \) is a rectilinear asymptote (or asymptote, in short) of \( \Gamma \).

Example: \( x = -1 \) is an asymptote of \( y = \frac{1}{x+1} \) since \( \lim_{|y| \to \infty} x + 1 = \lim_{|y| \to \infty} y = 0 \). The circle \( y = \pm \sqrt{1 - x^2} \) does not have any asymptote since for an arbitrary point \( P: (\alpha, \beta) \) on the curve, \( |\alpha| \leq 1 \) and \( |\beta| \leq 1 \) and hence neither \( |\alpha| \) nor \( |\beta| \) tends to \( \infty \). Similarly, \( x = \frac{\pi}{2} \) is an asymptote to \( y = \tan x \).

THEOREM: \( y = mx + c \) is an asymptote to a planar curve \( \Gamma: f(x, y) = 0 \) if and only if \( m = \lim_{|x| \to \infty} \frac{y}{x} \) and \( c = \lim_{|x| \to \infty} (y - mx) \) \( (x, y) \) is a point of \( \Gamma \).

Determination of asymptotes not parallel to the \( y \)-axis of an algebraic curve

Let \( \Gamma \) be a planar algebraic curve whose equation is

\[ a_0 y^n + a_1 y^{n-1} x + \cdots + a_n x^n + (b_1 y^{n-1} + b_2 y^{n-2} x + \cdots + b_n x^{n-1}) + \cdots + (c_{n-1} y + c_n x) + d_n = 0 \]

where \( f_r \left( \frac{y}{x} \right) \) is a homogeneous polynomial of degree \( r \). If (1) has an asymptote with (unknown) slope \( m \), then \( m = \lim_{|x| \to \infty} \frac{y}{x} \).

Dividing (2) by \( x^n \) and passing to limit as \( |x| \to \infty \), we get \( f_n(m) = 0 \)

whose roots give slopes of possible asymptotes not parallel to \( y \)-axis. Let \( m_1 \) be a root of \( f_n(m) = 0 \). Corresponding \( c \) value of \( y = mx + c \) is given by \( c = \lim_{|x| \to \infty} (y - mx) \). Let \( y - m_1 x = k_1 \); then \( c = \lim_{|x| \to \infty} k_1 \). Thus \( \frac{y}{x} = m_1 + \frac{k_1}{x} \). From (2) we get

\[ x^n f_n \left( m_1 + \frac{k_1}{x} \right) = 0 \]
\[x^{n-1}f_{n-1}\left(m_1 + \frac{k_1}{x}\right) + \cdots + f_0\left(m_1 + \frac{k_1}{x}\right) = 0.\] By Taylor’s Theorem (f i’s are polynomials and hence satisfy all conditions of Taylor’s Theorem),

\[x^n \left[f_n(m_1) + \frac{k_1}{x} f_n^{(1)}(m_1) + \frac{k_1^2}{2x^2} f_n^{(2)}(m_1) + \cdots \right] + x^{n-1}\left[f_{n-1}(m_1) + \frac{k_1}{x} f_{n-1}^{(1)}(m_1) + \frac{k_1^2}{2x^2} f_{n-1}^{(2)}(m_1) + \cdots \right] + \cdots \cdots = 0 \quad \cdots\cdots\cdots(4)\]

Arranging in descending powers of \(x\), we get

\[x^n f_n(m_1) + x^{n-1}\left[k_1 f_n^{(1)}(m_1) + f_{n-1}(m_1)\right] + x^{n-2}\left[\frac{k_1^2}{2} f_n^{(2)}(m_1) + k_1 f_n^{(1)}(m_1) + f_{n-2}(m_1)\right] + \cdots = 0.\]

Since \(f_n(m_1) = 0\) ( \(m_1\) is a root of \(f_n(m) = 0\), dividing by \(x^{n-1}\) and taking limit as \(|x| \to \infty\), we get

\[c_1 f_n^{(1)}(m_1) + f_{n-1}(m_1) = 0 \quad \cdots\cdots(5),\]

or, \(c_1 = -\frac{f_{n-1}(m_1)}{f_n^{(1)}(m_1)}\), if \(f_n^{(1)}(m_1) \neq 0\). Thus \(y = m_1 x - \frac{f_{n-1}(m_1)}{f_n^{(1)}(m_1)}\) is an asymptote if \(f_n^{(1)}(m_1) \neq 0\).

If \(f_n^{(1)}(m_1) = 0\) but \(f_{n-1}(m_1) \neq 0\), then no \(c\) value can be obtained from (5); hence there will be no asymptote corresponding to \(m_1\).

If \(f_n^{(1)}(m_1) = 0\) and \(f_{n-1}(m_1) = 0\), then (5) becomes an identity which is not acceptable since corresponding to a given slope \(m_1\), infinite number of \(c\) values is not possible. From (4) by dividing by \(x^{n-2}\) and allowing \(|x| \to \infty\), we have

\[\frac{c_1^2}{2} f_n^{(2)}(m_1) + c_1 f_n^{(1)}(m_1) + f_{n-2}(m_1) = 0\] from which two values of \(c_1\) are obtained. If the roots \(c_{11}, c_{12}\) are real and distinct, corresponding to the slope \(m_1\), there are two asymptotes: \(y = m_1 x + c_{11}\) and \(y = m_1 x + c_{12}\). If the roots are real and equal, say \(c_{11}\), then there is one asymptote \(y = m_1 x + c_{11}\) corresponding to \(m_1\). If the roots are conjugate complex, no asymptote corresponding to slope \(m_1\).

Similarly we proceed, if necessary, to higher powers of \(c_1\).

Note Real roots of \(f_n(m) = 0\) determines asymptotic directions. It may be that \(m_1\) is a root of \(f_n(m) = 0\) but all the corresponding values of \(c\) may be complex: then there is no asymptote with slope \(m_1\). If \(y = m_1 x + c_1\) and \(y = m_1 x + c_2\) are asymptotes to the same curve, then \(m\) is a multiple root of \(f_n(m) = 0\).

Summarising, an algebraic curve of degree \(n\) can have at most \(n\) asymptotes.
Determination of asymptotes parallel to the y-axis of an algebraic curve

Let \( F(x, y) = y^m g(x) + y^{m-1} g_1(x) + \cdots + g_m(x) = 0 \), where \( g, g_1, \ldots, g_m \) are polynomials in \( x \), be the equation of a rational algebraic curve. Dividing by \( y^m \) and letting \( y \to \infty \) (since \( F \) has asymptote parallel to y-axis, \( \lim_{|y| \to \infty} x \) should exist), we get \( g(k) = 0 \) if \( \lim_{|y| \to \infty} x = k \). Thus the vertical asymptotes to \( F(x, y) = 0 \) are obtained by equating to zero the coefficient of the highest power of \( y \). No vertical asymptote to the curve exists if the coefficient of highest power of \( y \) is a constant or not resolvable in real linear factors.

Ex: \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \) can be written in the form \( \left( \frac{x}{a} + \frac{y}{b} \right) \left( \frac{x}{a} - \frac{y}{b} \right) + (-1) = 0 \), which is of the form \( F_2 + F_0 = 0 \), where \( F_2 \) can be written as product of two real linear factors \( \frac{x}{a} + \frac{y}{b}, \frac{x}{a} - \frac{y}{b} \) such that \( \frac{x}{a} + \frac{y}{b} = 0 \) and \( \frac{x}{a} - \frac{y}{b} = 0 \) represent two non-parallel lines; hence the asymptotes are given by \( \frac{x}{a} + \frac{y}{b} = 0 \) and \( \frac{x}{a} - \frac{y}{b} = 0 \).

Ex: The parabola \( y^2 = 4ax \) has no asymptote parallel to the \( y \)-axis; the equation can be written in the form \( x^2 f_2(\frac{x}{a}) + xf_1(\frac{x}{a}) = 0 \), where \( f_2(m) = m_2, f_1(m) = -4a \cdot f_2(m) = 0 \) gives \( m = 0 \), \( c \), if it exists, corresponding to \( m = 0 \) is given by \( cf_2(1)(0) + f_1(0) = 0 \), that is, \( c \cdot 0 - 4a = 0 \), contradiction. Hence no asymptote non-parallel to \( y \)-axis either: thus no asymptote to the curve.

CHAPTER 12

ENVELOPE OF A FAMILY OF CURVES

A point \( P(a, b) \) is a singular point of a curve \( f(x, y) = 0 \) if \( f(a, b) = 0, f_x(a, b) = 0 \) and \( f_y(a, b) = 0 \) holds simultaneously. In contrast \( P \) is an ordinary point of the curve \( f(x, y) = 0 \) if at least one of \( f_x(a, b) \) and \( f_y(a, b) \) is not equal to zero.

Let \( f(x, y, \alpha) = 0 \) be a family of curves, where \( \alpha \) is a parameter (corresponding to each value of \( \alpha \), there is a curve). The characteristic points of the family of curves \( f(x, y, \alpha) = 0 \) are those ordinary points which are lying on each curve \( f(x, y, \alpha) = 0 \) of the family and at those points \( \frac{\partial f}{\partial \alpha} = 0 \) holds simultaneously.
Ex: The characteristic points of the family of circles \((x - \alpha)^2 + y^2 = a^2\) (\(\alpha\) is the parameter) can be obtained by solving simultaneously \(f(x, y, \alpha) = (x - \alpha)^2 + y^2 - a^2\) and \(\partial f / \partial \alpha = -2(x - \alpha) = 0\), which give the two points \((\alpha, \pm \alpha)\). Also \(\partial f / \partial y(\alpha, \pm \alpha) = \pm 2\alpha\) is not equal to zero; hence \((\alpha, \pm \alpha)\) are ordinary points and hence are characteristic points of the family.

Characteristic points may not exist for a family of curves: for the family \(x^2 + y^2 = a^2\) of concentric circles, there is no characteristic point.

The envelope of a family of curves \(f(x, y, \alpha) = 0\) (\(\alpha\) parameter) is the locus of isolated characteristic points of the family.

Note: If \(f(x, y, \alpha) = 0\) and \(\partial f / \partial \alpha(x, y, \alpha) = 0\) both hold for a point where \(f_x = 0\) and \(f_y = 0\), then the point is a singular point and, therefore, not a characteristic point.

Ex: Let us consider the family of curves \(x \cos \alpha + y \sin \alpha = a \sin \alpha \cos \alpha\), where \(\alpha\) is the parameter, \(a\) fixed. The characteristic points are obtained by solving the equations \(x \cos \alpha + y \sin \alpha = a \sin \alpha \cos \alpha\) and \(-x \csc \alpha \cot \alpha + y \sec \alpha \tan \alpha = 0\) simultaneously.

The envelope, that is, the locus of characteristic points is obtained by eliminating \(\alpha\) and is given by \(x^{2/3} + y^{2/3} = a^{2/3}\).

Ex: \(y = mx + a/m, m\) parameter. Differentiating partially w.r.t. the parameter \(m\), we get \(0 = x - a/m^2\), that is, \(m = \pm \sqrt{a/x}\). Substituting in the given equation, \(y = \pm 2\sqrt{ax}\), that is, \(y^2 = 4ax\).

Ex: Find the envelope of circles described on the radii vectors of the parabola \(y^2 = 4ax\) as diameter.

»Let \(P: (at^2, 2at)\) be an arbitrary point on the parabola and \(O\) be the origin. Equation of circle with \(OP\) as one of its diameter is \((x - 0)(x - at^2) + (y - 0)(y - 2at) = 0\), that is, \(x^2 + y^2 - 2ayt - at^2x = 0\). Differentiating partially w.r.t. \(t\), \(-2ay - 2atx = 0\), that is, \(t = -y/x\). Substituting in the equation, \(ay^2 + x(x^2 + y^2) = 0\) is the required envelope.
Case of two parameters

Let $f(x, y, \alpha, \beta) = 0$  (1) be the equation of a 2-parameter family of curves where the parameters $\alpha, \beta$ are connected by $g(\alpha, \beta) = 0$ (2). For a fixed point $(x, y)$ on the envelope, from given relations, we have $\frac{\partial f}{\partial \alpha} + \frac{\partial f}{\partial \beta} \frac{d\beta}{da} = 0$ and $\frac{\partial g}{\partial \alpha} + \frac{\partial g}{\partial \beta} \frac{d\beta}{da} = 0$. Eliminating $\frac{d\beta}{da}$ from the last two relations, we have $\frac{\partial f}{\partial \alpha} = \frac{\partial f}{\partial \beta} \frac{\partial g}{\partial \beta}$. 

-------- (3). Eliminating $\alpha, \beta$ from (1), (2) and (3), we obtain the required envelope.

**Ex:** Find the envelope of the family of co-axial ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where the parameters $a, b$ are connected by $a^2 + b^2 = c^2$, $c$ fixed.

»Differentiating partially w.r.t. $a$, $-2 \frac{x^2}{a^3} - 2 \frac{y^2}{b^2} \frac{db}{da} = 0$ and $2a + 2b \frac{db}{da} = 0$.

Equating values of $\frac{db}{da}$, we get $\frac{x^2}{a^4} = \frac{y^2}{b^4}$, that is, $\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{x^2 + y^2}{a^2 + b^2} = \frac{1}{c^2}$. Hence $a^2 = xc, b^2 = yc$. Since $a^2 + b^2 = c^2$, the required envelope is $x + y = c$. 


STUDY MATERIALS ON INTEGRAL CALCULUS

DEFINITE INTEGRAL

Let $f$ be a real valued **continuous** function defined on a **closed** and **bounded** interval $[a,b]$. Let us choose a **partition** (collection of finite number of points of $[a,b]$ including $a$ and $b$) $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ of $[a,b]$ (for example: $\{0,1/2,1\}$ is a partition for $[0,1]$).

Let $\delta_r = x_r - x_{r-1}$, $r = 1, \ldots, n$ and $\delta = \max \{\delta_r \mid r = 1,2,\ldots,n\}$. Choose an arbitrary point $c_r \in (x_{r-1}, x_r)$ for all $r$ and consider sum of areas of rectangles $\Sigma_1^n f(c_r)\delta_r$.

It can be seen that this sum approaches more closely the actual area under the curve if we make width of the rectangles smaller, that is, if we increase number $n$ of points of subdivision (**sum of areas of two rectangles on $\delta_r$ gives a better approximation to the area under the curve than area of a single rectangle**).

**Definition:** $\int_a^b f(x)dx = \lim_{n \to \infty} \Sigma_1^n f(c_r)\delta_r$, provided the limit exists independent of choice of points of subdivision $x_i$ and that of $c_i$, for all $i$. It can be proved that for a **continuous function** $f$ defined over a closed bounded interval $[a,b]$, $\int_a^b f(x)dx$ exists in above sense.

**Simpler equivalent expression for calculating $\int_a^b f(x)dx$:**

We can make choices of $x_i$ and $c_i$ **suitably** so as to obtain equivalent simpler expression of $\int_a^b f(x)dx$. 
Let us choose $x_i$'s equi-spaced, that is, $\delta_1 = \delta_2 = \ldots = \delta_n = (b-a)/n$. Then $\int_a^b f(x)dx = \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=1}^{n} f(c_i)$.

Let us choose $c_r = a + rh$, $r = 1,\ldots,n$, where $h = (b-a)/n$. Then $\int_a^b f(x)dx = \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=1}^{n} f(a + r \frac{b-a}{n}) = \lim_{h \to 0} h \sum_{i=1}^{n} f(a + rh)$.

As a special case, $\int_0^1 f(x)dx = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f \left( \frac{r}{n} \right) = \lim_{h \to 0} h \sum_{i=1}^{n} f(rh)$

**Example:** From definition, calculate $\int_0^1 x^2dx$.

\[ \int_0^1 x^2dx = \lim_{h \to 0} h \sum_{i=1}^{n} (rh)^2 = \lim_{h \to 0} \frac{(nh)(nh+h)(2nh+h)}{6} = 1/3, \text{ since nh = 1 holds, for every positive integer n and the corresponding h.} \]

**Fundamental Theorem of Integral Calculus**

**Theorem:** If $\int_a^b f(x)dx$ exists and if there exists a function $g:[a, b] \to \mathbb{R}$ such that $g_i(x) = f(x)$ (suffix denotes order of differentiation) on $[a, b]$, then $\int_a^b f(x)dx = g(b) - g(a)$.

**NOTE:** $g$ is called a **primitive** of $f$. A function $f$ may not possess a primitive on $[a,b]$ but $\int_a^b f(x)dx$ may exist; in that case, $\int_a^b f(x)dx$ can not be calculated using fundamental theorem. Primitives of $f$ on $[a, b]$ are given by the indefinite integral $\int f(x)dx$: that is the reason why we consider indefinite integrals.

**Example** $\int_0^1 x^2dx$ exists, since $x^2$ is continuous on $[0,1]$. Also $g(x) = \int x^2dx = \frac{x^3}{3} + c$ is a primitive of $x^2$ on $[0,1]$. Hence Fundamental Theorem gives $\int_0^1 x^2dx = (\frac{1}{3} + c) - c = \frac{1}{3}$. 

Note $\int_0^1 x^2\,dx$ is independent of $c$ though $\int x^2\,dx$ involves $c$.

**PROPERTIES OF DEFINITE INTEGRALS**

We assume below that the definite integrals exist and whenever we consider $\int_a^b f(x)\,dx$, a primitive $g$ to $f$ over $[a,b]$ exists, so that we can apply Fundamental Theorem. For $a<b$, we define $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

1. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ (irrespective of relative algebraic magnitude of $a,b,c$)
   
   Example $\int_1^3 x^2\,dx + \int_1^4 x^2\,dx = -\int_1^3 x^2\,dx + \int_1^4 x^2\,dx = -\left(9 - \frac{1}{3}\right) + \left(\frac{64}{3} - \frac{1}{3}\right) = \frac{37}{3} = \int_1^4 x^2\,dx$.

2. $\int_0^a f(x)\,dx = a - f(0)\,dx$
   
   Example $\int_0^{\pi/2} \sin x\,dx = \int_0^{\pi/2} \cos x\,dx$

3. $\int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_0^{a/2} f(a - x)\,dx$. In particular, if $f(a-x) = f(x)$ for all $x$ in $[0,a]$, then $\int_0^a f(x)\,dx = 2\int_0^{a/2} f(x)\,dx$ and if $f(a-x) = -f(x)$ for all $x$ in $[0,a]$, then $\int_0^a f(x)\,dx = 0$.

4. $\int_0^a f(x)\,dx = n\int_0^a f(x)\,dx$, if $f(a+x) = f(x)$, $n$ natural.

5. $\int_{-a}^a f(x)\,dx = \int_{-a}^a (f(x) + f(-x))\,dx$. If $f$ is even, $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$. If $f$ is odd, $\int_{-a}^a f(x)\,dx = 0$.

Example $\int_{-1}^1 x^3\cos 2x\,dx = 0$, $\int_{-1}^1 x^4\cos 2x\,dx = 2\int_0^1 x^4\cos 2x\,dx$.
REDUCTION FORMULA

In this chapter, we study how to decrease complexity of some integrals in a stepwise manner by the use of recurrence relation that we derive generally using integration by parts formula.

1. Let \( I_n = \int \sin^n x \, dx \), \( n \) natural.

\[
I_n = \int \sin^{n-1}x \sin x \, dx = \sin^{n-1}x(-\cos x) - (n-1)\int \sin^{n-2}x \cos x \, dx =
\]

\[
\sin^{n-1}x(-\cos x)+(n-1)\int \sin^{n-2}x(1 - \sin^2x)\,dx = -\sin^{n-1}x \cos x+(n-1)I_{n-2}-(n-1)I_n.
\]

hence \( I_n=\frac{\sin^{n-1}x \cos x}{n} + \frac{n-1}{n} I_{n-2} \).

If we denote \( J_n = \int_0^{\pi/2} \sin^n x \, dx \), then \( J_n = -\frac{\sin^{n-1}x \cos x}{n} \bigg|_0^{\pi/2} + \frac{n-1}{n} J_{n-2} = \frac{n-1}{n} J_{n-2} \).

By repeated application of the reduction formula, it can be proved that \( J_n=\frac{n-1}{n} J_{n-2}=...=\frac{n-1}{n} \frac{n-3}{n-2} ... 1 \cdot \frac{1}{2} \cdot \frac{\pi}{2} \), if \( n \) is even natural and \( J_n=\frac{n-1}{n} \frac{n-3}{n-2} ... 2 \cdot 3 \cdot \frac{2}{3} = \frac{n-1}{n} \frac{n-3}{n-2} ... \frac{2}{3} \), if \( n \) is odd natural.

2. Let \( I_n = \int \tan^n x \, dx \), \( n \) natural.

Then \( I_n = \int \tan^{n-2}x \cdot \tan^2x \, dx = \int \tan^{n-2}x \cdot (\sec^2x - 1) \, dx = \frac{\tan^{n-1}x}{n-1} - I_{n-2} \).

Also, \( J_n = \int_0^{\pi/4} \tan^n x \, dx = \frac{\tan^{n-1}x}{n-1} \bigg|_0^{\pi/4} - J_{n-2} = \frac{1}{n-1} - J_{n-2} \).

3. Let \( I_n = \int \sec^n x \, dx \), \( n \) natural. Then \( I_n = \int \sec^{n-2}x \cdot \sec^2x \, dx = \sec^{n-2}x \tan x - (n-2) \int \sec^{n-2}x \cdot (\sec^2x - 1) \, dx = \sec^{n-2}x \tan x - (n-2)(I_n-I_{n-2}) \). Hence \( I_n = \frac{\sec^{n-2}x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2} \).
4. Let \( I_{m,n} = \int \sin^m x \cos^n x \, dx = \int (\sin^m x \cos x) \cos^{n-1} x \, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} x \sin^m x \, dx \) 
\[ = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} x \sin^m x (1 - \cos^2 x) \, dx \]
\[ = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n} \]. Transposing and simplifying, we get a reduction formula for \( I_{m,n} \).

5. Let \( I_{m,n} = \int \cos^m x \sin nx \, dx = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} \int \cos^{m-1} x \sin x \cos nx \, dx = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} \int \cos^{m-1} x \sin nx \cos x \, dx + \frac{m}{n} \int \cos^{m-1} x \sin(n-1)x \, dx \) [ since \( \cos nx \sin x = \sin nx \cos x - \sin(n-1)x \) ] \[ = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1} \]. Transposing and simplifying, we get a reduction formula for \( I_{m,n} \).

**Illustrative examples**

1. \( I_n = \int_0^\pi x^n \sin x \, dx \) and \( n > 1 \), show that \( I_n + n(n-1)I_{n-2} = n \left( \frac{\pi}{2} \right)^{n-1} \)

\[ \Rightarrow I_n = (-x^n \cos x) \bigg|_0^{\pi/2} + n \int_0^{\pi/2} x^{n-1} \cos x \, dx = n \left[ x^{n-1} \sin x \right]_0^{\pi/2} - (n-1) \int_0^{\pi/2} x^{n-2} \sin x \, dx \] \[ = n \left( \frac{\pi}{2} \right)^{n-1} - n(n-1)I_{n-2} \]. Hence the proof.

2. \( I_{m,n} = \int x^m (1-x)^n \, dx = \frac{x^{m+1}}{m+1} (1-x)^n + \frac{n}{m+1} \int x^m (1-(1-x))(1-x)^{n-1} \, dx \) \[ = \frac{x^{m+1}}{m+1} (1-x)^n + \frac{n}{m+1} (I_{m,n-1} - I_{m,n}) \]. Hence \( I_{m,n} \) can be obtained.

3. \( I_m = \int_0^{\pi/2} \cos^m x \sin mx \, dx \). From 5 above, \( I_m = \frac{1}{2m} + \frac{1}{2} I_{m-1} = \frac{1}{2m} + \frac{1}{2} \left[ \frac{1}{2(m-1)} + \frac{1}{2} I_{m-2} \right] \) \[ = \frac{1}{2m} + \frac{1}{2^2 (m-1)} + \frac{1}{2} I_{m-2} \]. Repeating the use of the reduction formula, it can be proved that \( I_m = \frac{1}{2m+1} \left[ 2 + \frac{2^2}{2} + \frac{2^3}{3} + \cdots + \frac{2^m}{m} \right] \).
**IMPROPER INTEGRAL**

When we consider the definite integral \( \int_a^b f(x)dx \) in earlier standards, we implicitly assume two conditions to hold: (a) \( f \) is continuous on \([a, b]\) or, to that matter, at least the limit \( \int_a^b f(x)dx = \lim_{n \to \infty} \sum_1^n f(c_r)\delta_r \) exists independent of choice of points of subdivision \( x_i \) and that of \( c_i \), for all \( i \) and (b) the interval \([a,b]\) is bounded. We want to extend the definition of \( \int_a^b f(x)dx \) when either (a) or (b) or both are not met. This extended definition of definite integral is referred to as **Improper Integrals**. Improper integrals can be of two types: (a) **Type 1**: interval of integration is unbounded, (b) **Type 2**: integrand has a finite number of infinite discontinuities in the interval of integration.

**Definition of TYPE I improper integral** \( \int_a^\infty f(x)dx, \int_{-\infty}^a f(x)dx \) and \( \int_{-\infty}^\infty f(x)dx \)

Let the function \( f \) be integrable in \([a,B]\), for every \( B>a \). If \( \lim_{B \to \infty} \int_a^B f(x)dx \) exist finitely, we define \( \int_a^\infty f(x)dx = \lim_{B \to \infty} \int_a^B f(x)dx \) and we say \( \int_a^\infty f(x)dx \) exists or converges; otherwise \( \int_a^\infty f(x)dx \) diverges. Similarly, \( \int_{-\infty}^a f(x)dx = \lim_{B \to -\infty} \int_B^a f(x)dx \) (provided the limit exists) and \( \int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx \), \( a \) is any real, provided \( \int_a^\infty f(x)dx \) and \( \int_{-\infty}^a f(x)dx \) exist separately.
Example: $\int_1^\infty \frac{dx}{x^2}, \int_1^\infty \frac{dx}{\sqrt{x}}$. The range of integration of the integrals are unbounded. For $a>1$, $\int_1^a \frac{dx}{x^2} = 1- \frac{1}{a}$ and $\int_1^a \frac{dx}{\sqrt{x}} = 2(\sqrt{a}-1)$. Since $\lim_{a \to \infty} \left(1 - \frac{1}{a}\right) = 1$ exists but $\lim_{a \to \infty} \left(2(\sqrt{a} - 1)\right)$ does not exist, hence the improper integral $\int_1^\infty \frac{dx}{x^2}$ converges and $\int_1^\infty \frac{dx}{\sqrt{x}}$ diverges. (Compare areas below the curves $y = 1/x^2$ and $y = 1/\sqrt{x}$ in diagram below)

**Definition of TYPE II improper integral**

Let $f$ have an infinite discontinuity only at the point $a$ (that is, $\lim_{x \to a^+} f(x) = \pm \infty$ or $\lim_{x \to a^-} f(x) = \pm \infty$) and is continuous in $(a, b]$. Then we define $\int_a^b f(x)dx = \lim_{c \to 0^+} \int_a^{b-c} f(x)dx$, $0<c<b-a$, provided the limit exists. Similarly, if $f$ has an infinite discontinuity only at the point $b$ and is continuous in $[a, b)$, then we define $\int_a^b f(x)dx = \lim_{c \to 0^-} \int_a^b f(x)dx$, $0<c<b-a$, provided the limit exists. If $f$ has an infinite discontinuity at $d$, $a<d<b$, and is otherwise continuous in $[a, b]$, we define $\int_a^b f(x)dx = \int_a^d f(x)dx + \int_d^b f(x)dx$, provided both of $\int_a^d f(x)dx$ and $\int_d^b f(x)dx$ exist separately.

Example: $\int_0^1 \frac{dx}{\sqrt{x}}$, $\int_0^1 \frac{dx}{x^2}$. The integrands have an infinite discontinuity at $x=0$. For $0<a<1$, $\int_a^1 \frac{1}{\sqrt{x}}dx = 2(1 - \sqrt{a})$ and $\int_a^1 \frac{dx}{x^2} = \frac{1}{a} - 1$. Since $\lim_{a \to 0^+} 2(1 - \sqrt{a}) = 2$ exists but $\lim_{a \to 0^+} \left(\frac{1}{a} - 1\right)$ does not exist, so $\int_0^1 \frac{dx}{\sqrt{x}}$ converges whereas $\int_0^1 \frac{dx}{x^2}$ diverges. (Compare areas between $x = a, 0<a<1$, and $x = 1$, below the curves $y = 1/x^2$ and $y = 1/\sqrt{x}$ in diagram above)
Example: \( \int_0^\infty \frac{dx}{4+9x^2} = \frac{\pi}{12} \)

» The integrand \( \frac{1}{4+9x^2} \) is continuous everywhere but the interval of integration is unbounded. Let \( a>0 \) be fixed. \( \int_0^a \frac{dx}{4+9x^2} = \frac{1}{9} \int_0^a \frac{dx}{x^2 + \left(\frac{2}{3}\right)^2} = \frac{1}{6} \tan^{-1} \frac{3a}{2} \). Thus \( \lim_{a \to \infty} \int_0^a \frac{dx}{4+9x^2} = \frac{1}{6} \cdot \frac{\pi}{2} = \frac{\pi}{12} \).

Example: \( \int_0^3 \frac{dx}{\sqrt{9-x^2}} = \frac{\pi}{2} \).

» The integrand has an infinite discontinuity at \( x = 3 \) and is continuous on \([0, 3)\). Let \( 0<a<3 \). Then \( \int_0^a \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}(a/3) \). So \( \lim_{a \to 3^-} \int_0^a \frac{dx}{\sqrt{9-x^2}} = \lim_{a \to 3^-} \sin^{-1} \left( \frac{a}{3} \right) = \frac{\pi}{2} \). Hence \( \int_0^3 \frac{dx}{\sqrt{9-x^2}} = \frac{\pi}{2} \).

Note: we can apply standard methods of integration, in particular method of substitution, only to a proper integral and not directly to an improper integral. Thus if we substitute \( z=1/x \) directly in the improper integral \( \int_{-1}^1 \frac{dx}{x^2} \), we get a value \(-2\) of the integral whereas it can be checked from definition that the improper integral diverges.

TESTS FOR CONVERGENCE OF IMPROPER INTEGRALS

TYPE I INTEGRAL

Theorem: (Comparison test) Let \( f \) and \( g \) be integrable in \([a, B]\), for every \( B>a \). Let \( g(x)>0 \), for all \( x \geq a \). If \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = c \neq 0 \), then the integrals \( \int_a^\infty f(x)dx \) and \( \int_a^\infty g(x)dx \) either both converge or both diverge. If \( c = 0 \) and \( \int_a^\infty g(x)dx \) converges, then \( \int_a^\infty f(x)dx \) converges.
**Theorem:** (μ Test) Let \( f \) be integrable in \([a, B]\), for every \( B > a \). Then \( \int_a^\infty f(x)\,dx \) converges if \( \lim_{x \to \infty} x^\mu f(x) \) exists with \( \mu > 1 \) and \( \int_a^\infty f(x)\,dx \) diverges if \( \lim_{x \to \infty} x^\mu f(x) \) exists and \( \neq 0 \) with \( \mu \leq 1 \).

**Example:** \( \int_0^\infty \frac{dx}{e^{x+1}} \) converges by comparison test, since \( 0 \leq \frac{1}{e^x} \) for all \( x \geq 0 \), \( \lim_{x \to \infty} \frac{e^x}{e^{x+1}} = \lim_{x \to \infty} \frac{1}{1+e^{-x}} = 1 \) and \( \int_0^\infty \frac{dx}{e^x} \) converges (need to prove!).

**Example:** \( \int_0^\infty e^{-x^2} \,dx \) converges by μ Test since \( \lim_{x \to \infty} (x^2 e^{-x^2}) = 0 \) (verify using L’Hospital’s rule), \( \mu = 2 > 1 \) and \( e^{-x^2} \) is continuous, and hence integrable, in \([0,B]\) for \( B > 0 \).

**Example:** \( \int_0^\infty \frac{x^{3/2}}{3x^2 + 5} \,dx \) diverges, since \( \lim_{x \to \infty} \left( \frac{x^{3/2}}{3x^2 + 5} \right) = 1/3 \), \( \mu = 1/2 < 1 \) and \( \frac{x^{3/2}}{3x^2 + 5} \) is continuous, and hence integrable, in \([0,B]\) for \( B > 0 \).

**TYPE II INTEGRAL**

**Theorem:** (Comparison test) Let \( f \) and \( g \) be integrable in \([c, b]\), for every \( c, a < c < b \). Let \( g(x) > 0 \), for all \( x, a < x \leq b \). If \( \lim_{x \to a^+} \frac{f(x)}{g(x)} \neq 0 \), then \( \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \) both converge or both diverge. If \( \lim_{x \to a^+} \frac{f(x)}{g(x)} = 0 \) and \( \int_a^b g(x)\,dx \) converges, then \( \int_a^b f(x)\,dx \) converges.

**Theorem:** (μ Test) Let \( f \) be integrable in \([c, b]\), for every \( c, a < c < b \). Then \( \int_a^b f(x)\,dx \) converges if \( \lim_{x \to a^+} (x - a)^\mu f(x) \) exists for \( 0 < \mu < 1 \) and \( \int_a^b f(x)\,dx \) diverges if \( \lim_{x \to a^+} (x - a)^\mu f(x) \) exists \( \neq 0 \) for \( \mu \geq 1 \).
Example: \( \int_0^1 \frac{dx}{(1+x)^{\sqrt{x}}} \) converges, since \( \lim_{x \to 0^+} (x - 0)^{\frac{1}{2}} \frac{1}{(1+x)^{\sqrt{x}}} = 1 \), for \( \mu < 1 \) and \( \frac{1}{(1+x)^{\sqrt{x}}} \) is continuous, and hence integrable, in \([c,1]\) for \( 0<c<1 \).

Example: \( \int_{1/2}^1 \frac{dx}{\sqrt{x(1-x)}} \) converges, since \( \lim_{x \to 1^-} (1 - x)^{\frac{1}{2}} \frac{1}{\sqrt{x(1-x)}} = 1 \), for \( \mu = \frac{1}{2} < 1 \) and \( \frac{1}{\sqrt{x(1-x)}} \) is continuous, and hence integrable, in \([1/2, c]\) for \( 1/2<c<1 \).

THE GAMMA AND BETA FUNCTIONS

Definition (Gamma function) For \( n>0 \), \( \Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx \).

NOTE: Gamma function is an improper integral of type I. If \( 0<n<1 \), \( \Gamma(n) \) is also an improper integral of type II. We shall assume convergence of the gamma function in our course of study.

Definition (Beta function) For \( m, n>0 \), \( \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} \, dx \)

NOTE: Beta function is an improper integral of type II if either \( m \) or \( n \) or both lies between 0 and 1 strictly; otherwise it is a proper integral.

Properties of Gamma and Beta functions

1. For any \( a>0 \), \( \int_0^\infty e^{-ax}x^{n-1} \, dx = \Gamma(n)/a^n \).

   » let \( 0<c<d \). Consider the proper integral \( I = \int_c^d e^{-ax}x^{n-1} \, dx \). Let \( y = ax \).

   Then \( I = \int_{ac}^{ad} e^{-y}y^{n-1} \frac{1}{a^n} \, dy = \frac{1}{a^n} \int_{ac}^{ad} e^{-y}y^{n-1} \, dy \). Thus \( \lim I = \frac{\Gamma(n)}{a^n} \) as \( c \to 0^+ \) and \( d \to \infty \).
2. \( \Gamma(n+1) = n \Gamma(n) \)
   
   » Let \( 0 < c < d \). Using integration by parts on the proper integral
   
   \[
   I = \int_c^d e^{-x}x^n \, dx \quad \text{we get} \quad I = \left[ -\frac{x^n}{e^x} \right]_c^d + n \int_c^d e^{-x}x^{n-1} \, dx = \left( \frac{e^n}{e^c} - \frac{d^n}{e^d} \right) + n \int_c^d e^{-x}x^{n-1} \, dx,
   \]
   
   which tends to \( n \Gamma(n) \) as \( c \to 0^+ \) and \( d \to \infty \) (by use of L'Hospital's rule). Hence the result.

3. \( \Gamma(1) = 1 \) (can be verified easily)

4. \( \Gamma(n+1) = n! \), for a natural \( n \) (follows from property 2 and 3)

4. \( \beta(m,n) = \beta(n,m) \) (follows using a substitution \( y = 1-x \) after passing to a proper integral)

5. \( \beta(m,n) = 2 \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \) (follows using a substitution \( x = \sin^2\theta \) after passing to a proper integral)

6. \( \beta\left(\frac{1}{2},\frac{1}{2}\right) = \pi \) (follows from definition)

7. \( \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \)

8. \( \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \)

9. For \( 0 < m < 1 \), \( \Gamma(m) \Gamma(1-m) = \pi \csc(m \pi) \)

**Example:** \( \int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2} \).

» The range of integration of the given integral is unbounded but the integrand is continuous everywhere. For \( 0 < a \), \( \int_0^a e^{-x^2} \, dx = \frac{1}{2} \int_0^a y^{-1/2}e^{-y} \, dy \)

(substituting \( y = x^2 \) in the proper integral ). Thus \( \lim_{a \to \infty} \int_0^a e^{-x^2} \, dx = \frac{1}{2} \lim_{a \to \infty} \int_0^a y^{-1/2}e^{-y} \, dy = \frac{1}{2} \int_0^\infty e^{-y}y^{-1/2} \, dy = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}/2. \)
Example: \[ \int_0^1 \frac{dx}{(1-x^6)^{\frac{1}{6}}} \]

The integrand has an infinite discontinuity at \( x = 1 \). Let \( 0 < c < 1 \). Substituting \( x^3 = \sin \theta \) in the proper integral \( \int_0^c \frac{dx}{(1-x^6)^{\frac{1}{6}}} = \int_0^c \frac{dx}{(1-x^6)^{\frac{1}{6}}} = \frac{1}{3} \int_0^{\sin^{-1} c^3} \cos^{\frac{5}{3}-1} \theta \sin^{\frac{1}{3}-1} \theta \, d\theta \). Since \( \lim_{c \to 1^-} \int_0^c \frac{dx}{(1-x^6)^{\frac{1}{6}}} = \frac{1}{6} \beta(5/3, 1/3) = \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{r\left(\frac{2}{3}\right) r\left(\frac{1}{3}\right)}{r\left(\frac{3}{3}+\frac{1}{3}\right)} = \frac{1}{6} \cdot \frac{2}{3} \cdot \frac{1}{1} = \frac{1}{9} \Gamma \left(\frac{1}{3}\right) \Gamma \left(1 - \frac{1}{3}\right) = \frac{1}{27} \pi \csc \left(\frac{\pi}{3}\right).

**DOUBLE INTEGRAL**

Let \( f(x,y) \) be a bounded function of two independent variables \( x \) and \( y \) defined over a closed rectangular region \( R: a \leq x \leq b; c \leq y \leq d \). We take partitions \( \{a = x_0, x_1, \ldots, x_{r-1}, x_r = b\} \) of \([a,b]\) and \( \{c = y_0, y_1, \ldots, y_{s-1}, y_s = d\} \). These partitions divide the rectangle \( R \) into \( mn \) number of subrectangles \( R_{ij} (1 \leq i \leq n, 1 \leq j \leq m) \). Let us choose arbitrarily \( (\alpha_i, \beta_j) \in R_{ij} \) where \( \alpha_i \in [x_{i-1}, x_i] \) and \( \beta_j \in [y_{j-1}, y_j], 1 \leq i \leq n, 1 \leq j \leq m \). The volume of the parallelepiped with base \( R_{ij} \) and altitude \( f(\alpha_i, \beta_j) \) is \( f(\alpha_i, \beta_j)(x_i-x_{i-1})(y_j-y_{j-1}) \). \( \sum_{i,j} f(\alpha_i, \beta_j)(x_i-x_{i-1})(y_j-y_{j-1}) \), sum of the volumes of all the parallelepipeds erected over all of the \( R_{ij} \)'s, gives an approximation of the volume enclosed by the curve and the planes \( x = a, x = b, y = c, y = d \) and \( z = 0 \). The approximation can be improved by increasing number of subrectangles into which \( R \) is divided into. Thus the limit \( \lim_{m \to \infty, n \to \infty} \sum_{i,j} f(\alpha_i, \beta_j)(x_i-x_{i-1})(y_j-y_{j-1}) \), provided it exists, gives the volume and is represented by \( \iint_R f(x,y) \, dx \, dy \).
NOTE: Every continuous function is integrable over any rectangle.

**Theorem: (equivalence of double integrals with repeated integrals)** If \( \iint_R f(x, y) \, dx \, dy \) exists over a rectangle \( R: a \leq x \leq b; c \leq y \leq d \) and \( \int_a^b f(x, y) \, dx \) exists for each value of \( y \) in \([c, d]\), then the repeated integral \( \int_c^d \int_a^b f(x, y) \, dx \, dy \) exists and is equal to \( \iint_R f(x, y) \, dx \, dy \).

**Example:** Evaluate \( \iint_R \sin(x + y) \, dx \, dy \) over \( R: 0 \leq x \leq \pi, 0 \leq y \leq \frac{\pi}{2} \).

**Sol:** \( \sin(x+y) \) is continuous on \( R \), so the double integral \( \iint_R \sin(x + y) \, dx \, dy \) exists. Evaluating given double integral in terms of repeated integrals,

\[
\iint_R \sin(x + y) \, dx \, dy = \int_0^{\pi/2} dx \int_0^{\pi/2} \sin(x + y) \, dy = \int_0^{\pi/2} \left[ -\cos(x + y) \right]_0^\pi dx = \int_0^{\pi/2} (\cos x + \sin x) \, dx = 2.
\]

**EVALUATION OF AREA**

**Cartesian co-ordinate**

It has already been seen that the area of the region bounded by the curve \( y = f(x) \), lines \( x = a, x = b \) and \( y = 0 \) is given by \( \int_a^b f(x) \, dx \), provided it exists. Similarly, the area of the region bounded by the curve \( x = g(y) \), lines \( y = c, y = d \) and \( x = 0 \) is given by \( \int_c^d g(y) \, dy \), provided it exists. We can define \( F: [a, b] \rightarrow \mathbb{R} \) by \( F(t) = \int_a^t f(x) \, dx \), \( a \leq t \leq b \).
**Example:** Find the area of the bounded region bounded by the curves \( y = x^2 \) and \( x = y^2 \).

**Sol:** On solving the given equations of the curves, the point of intersection of the two curves are \((0,0)\) and \((4,4)\). Thus required area = \( \int_0^4 \sqrt{4x} \, dx - \int_0^4 \frac{x^2}{4} \, dx \).

**Example:** Find the area of the loop formed by the curve \( y^2 = x(x-2)^2 \)

**Sol:** The abscissa of points of intersection of the curve with the x-axis are given by \( y = 0 \), that is, \( x = 0, 2, 2 \). For \( x<0 \), no real value of \( y \) satisfy the equation. Hence no part of the curve exist corresponding to \( x<0 \). Corresponding to each x-value satisfying \( 0<x<2 \), there exist two values of \( y \), equal in magnitude and opposite in sign. Thus between \( x = 0 \) and \( x = 2 \), the curve is symmetric about the x-axis and a loop is formed thereby. For \( x>2 \), \( y \to \infty \) as \( x \to \infty \). the required area = \( 2\int_0^2 (x - 2)\sqrt{x} \, dx \) (by symmetry of the curve about x-axis).

**Example:** Prove that area included in a circle of radius \( r \) unit is \( \pi r^2 \) square unit.

**Sol:** We can choose two perpendicular straight lines passing through the centre of the circle as co-ordinate axes. With reference to such a co-ordinate system, equation of the circle is \( y = \pm\sqrt{r^2 - x^2} \). Curve is symmetric about the axes . Thus required area = \( 4\int_0^r \sqrt{r^2 - x^2} \, dx \).
Polar Coordinates:

The area of the region bounded by the curve \( r = f(\theta) \), the radius vector \( \theta = \alpha, \theta = \beta \) is given by \( \frac{1}{2} \int_{\alpha}^{\beta} r^2 \, d\theta \).

**Example:** Find the area enclosed by the cardioid \( r = a(1 + \cos \theta) \)

**Sol:** As \( \theta \) varies from 0 to \( \frac{\pi}{2} \), \( r \) decreases continuously from 2a to a. When \( \theta \) further increases from \( \frac{\pi}{2} \) to \( \pi \), \( r \) decreases further from a to 0. Also the curve is symmetric about the initial line (since the equation of the curve remains unaffected on replacing \( \theta \) by \(-\theta\)). Hence the area enclosed by the curve = \( 2 \cdot \frac{a^2}{2} \int_{0}^{\pi/2} (1 + \cos \theta)^2 \, d\theta \).

**Example:** Find the area enclosed by the cardioid \( r = a(1 + \cos \theta) \) and \( r = a(1 - \cos \theta) \)

**Sol:** The vectorial angle corresponding the points of intersection of the curves are \( \theta = \frac{\pi}{2} \) and \( \theta = -\frac{\pi}{2} \). Because of the symmetry of the curves about the initial line, \( \theta = \frac{\pi}{2}, \theta = -\frac{\pi}{2} \) and \( \theta = \pi \), required area is \( 4 \cdot \frac{a^2}{2} \int_{0}^{\pi/2} (1 - \cos \theta)^2 \, d\theta \).
**STUDY MATERIALS ON DIFFERENTIAL EQUATION**

**Definition 1.1** An relation between an independent variable $x$, a dependent variable $y$, and one or more derived functions $f'(x), f''(x), f'''(x), \ldots, f^n(x), \ldots$ is called an Ordinary Differential Equation.

The word ordinary is used to emphasize that only one independent variable is involved, if, on the other hand, more than one independent variables and partial derivatives appear, then the equation is called a Partial Differential Equation.

**Definition 1.2** the order of an ODE is the order of the highest ordered derivative occurring in the equation. The degree of an ODE is the largest power of the highest ordered derivative occurring in the equation after the ODE has been made free from the radicals and fractions as far as the derivatives are concerned.

**Example:** $y = x \frac{dy}{dx} + \frac{dx}{dy}$: first order, second degree

$(1+y^2)^{3/2} = y_1$: first order, sixth degree.

**Definition 1.3** An ODE in which the dependent variables and all its derivatives present occur in first degree only and no products of dependent variables and/or derivatives occur is known as a linear ODE. An ODE which is not linear is called nonlinear ODE. Thus $y_1 = \sin x + \cos x$ is linear while $y = y_1 + 1/y_1$ is non-linear.

**Definition 1.4** A solution of an ODE is a relation between the dependent and independent variables, not involving the derivatives such that this relation and the derivatives obtained from it satisfies the given ODE. For example, $y = ce^{2x}$ is a solution of the ODE $y_1 = 2y$, since $y_1 = 2ce^{2x}$ and $y = ce^{2x}$ satisfy the given ODE.
**Fundamental Theorem:** A differential equation of order n has n, and can’t have more than n independent first integrals. The equation can’t have, therefore, more than n arbitrary and independent constant in general solution.

**Ex.** Obtain the differential equation of all circles each of which touches the axis of x at the origin.

**Solution:** The equation will be of the form:

\[ x^2 + y^2 + 2fy = 0 \]

On differentiation, \( 2x + 2y\frac{dy}{dx} + 2f\frac{dy}{dx} = 0 \)

Eliminating \( f \) \( \Rightarrow (x^2 + y^2) + 2y \left\{ -x + y\frac{dy}{dx} \right\}/(dy/dx) = 0 \)

Or, \( (x^2 - y^2)\frac{dy}{dx} = 2xy \) [Required Differential Equation]

**Existence Theorem:** An ODE of first order and first degree can be written as

\[ \frac{dy}{dx} = f(x,y) \]  \hspace{1cm} (1)

Or, in the symmetric form \( M\,dx + N\,dy = 0 \), where \( M=M(x,y), N=N(x,y) \).

**Existence and Uniqueness Theorem:**

If the function \( f(x,y) \) be continuous on a domain containing the point \((x_0,y_0)\) at its interior, then \( \exists \) a function \( y=\varphi(x) \) satisfying equation (1) and taking the values \( y_0 \) for \( x=x_0 \). If, in addition, the partial derivative \( \partial f/\partial y \) is continuous, that is the solution of the equation is unique.

**Some Useful Formulae:**

1. \( dx \pm dy = d(x\pm y) \)
2. \( xdy + ydx = d(xy) \)
3. \( xdx \pm ydy = d(\frac{x^2 \pm y^2}{2}) \)
4. \( (xydy - ydx)/x^2 = d(x/y) \)
5. \( (ydx - xdy)/y^2 = d(x/y) \)
6. \( (ydx - xdy)/(x^2 + y^2) = d\{\tan^{-1}(y/x)\} \)
7. \( (x^2 - y^2)/(xy) = d\{\log(y/x)\} \)
8. \( (x^2 + y^2)/(x^2 + y^2) = d\{\log(x^2 + y^2)\}/2 \)
Linear Equations:

Taking the linear form: \( \frac{dy}{dx} + Py = Q \), where, \( P \) & \( Q \) are either functions of \( x \) or constants (including zero).

Case I: Let \( Q = 0, P \neq 0 \). Then \( \frac{dy}{dx} + Py = 0 \) \( \Rightarrow \frac{dy}{y} = -Pdx \)

So, \( y = c.e^{-\int Pdx} \), where \( c \) is the constant of integration.

If \( P = 0, Q \neq 0 \), then \( y = c + \int Qdx \).

If \( P = Q = 0 \), then \( y = c \).

Case II: Let \( Q \neq 0 \). Then \( y = v.e^{-\int Pdx} \).

Elimination of Arbitrary Constants

Suppose we are given an equation (not a differential equation) containing \( n \) parameters (arbitrary constants). By differentiating the given equation successively \( n \) times, we get \( n \) equations more containing \( n \) parameters and derivatives. By eliminating \( n \) parameters from the above \((n+1)\) equations and obtaining an equation which involves derivatives up to the \( n \)th order, we get an ODE of order \( n \).

Example: Obtain an ODE by eliminating parameters \( a,b \) from \((x-a)^2+(y-b)^2 = c^2, c \) constant.

\((x-a)+(y-b)y_1 = 0, 1+y_1^2+(y-b)y_2 = 0\). Obtain values of \( x-a \) and \( y-b \) in the given equation, obtain the ODE.

Example: Find the ODE corresponding to the family of curves \( y = c(x-c)^2 \) (2.1), where \( c \) is a parameter.

\( Y_1 = 2c(x-c) \). so \( y_1^2 = 4c^2(x-c)^2 \) (2.2). from (2.1) and (2.2), \( c = y_1^2/(4y) \).

Substituting in (2.1), required equation is \( 8y^2 = 4xyy_1-y_1^2 \).
GENERAL, PARTICULAR AND SINGULAR SOLUTIONS

A solution which contains a number of independent parameters equal to the order of the ODE is called the general solution or complete integral of the ODE. A solution obtained from the general solution by putting particular values to at least one of the parameters present in the general solution is called a particular solution. A solution which is neither the general solution nor is a particular solution of an ODE is called a singular solution of the ODE.

NOTE (1) In counting the parameters in the general solution, we must check whether they are independent and are not equivalent to a lesser number of parameters. Thus a solution of the form \( c_1 \cos x + c_2 \sin(x + c_3) \) appears to have three parameters but actually they are equivalent to two; for, \( c_1 \cos x + c_2 \sin(x + c_3) = (c_1 + c_2 \sin c_3) \cos x + c_2 \cos c_3 \sin x = A \cos x + B \sin x \).

(2) The general solution of an ODE may have more than one form, but parameters in one form will be related to parameters in another form. Thus \( y = c_1 \cos(x + c_2) \) and \( y = c_3 \sin x + c_4 \sin x \) are both solutions of \( y^2 + y = 0 \). Here \( c_4 = c_1 \cos c_2, c_3 = -c_1 \sin c_2 \).

Example: \( y = \ln x \) is a solution of \( xy^2 + y_1 = 0 \) on \((0, \infty)\) but is not a solution on \((-\infty, 0)\) since \( \ln x \) is not defined on \((-\infty, 0]\).
ODE OF FIRST ORDER AND FIRST DEGREE

An ODE of the first order and first degree is of the form $y_1 = f(x,y)$ which is sometimes conveniently written as $M(x,y)dx+N(x,y)dy = 0 \quad (2.1)$, where $M(x,y)$ and $N(x,y)$ are either functions of $x, y$ possessing partial derivatives or constants.

NOTE: First order first degree ODE whose variables are separable or which are homogeneous or which are reducible to homogeneous equations have been studied earlier and will not be repeated here. Students are advised to revise the same.

**Exact equations**

**Definition 2.1** An ODE of the form (2.1) is called exact if and only if there exists a function $f(x,y)$ with continuous partial derivatives such that (2.1) can be written in the form $df(x,y) = 0$, that is, in the form $f_x \, dx + f_y \, dy = 0$. In this case, $f(x,y) = c$ will be the general solution of given ODE.

For example, $xdx+ydy = 0$ is exact since the given equation can be written in the form $d(x^2+y^2) = 0$.

**Theorem:** (2.1) is exact iff $M_y = N_x$ holds.

**Example:** Verify whether the differential equation $(\sin x \cos y + e^{2x}) \, dx + (\cos x \sin y + \tan y) \, dy = 0$ is exact. Also find the general solution of the equation.

**Sol:** Comparing with the form $Mdx+Ndy = 0$, $M = \sin x \cos y + e^{2x}$, $N = \cos x \sin y + \tan y$. so $M_y = -\sin x \sin y = N_x$. hence given equation is exact. Let $f(x,y) = 0$ be the solution. Given equation will be identical with $f_x \, dx + f_y \, dy = 0$. Comparing, $f_x = \sin x \cos y + e^{2x} \quad (2.2)$, $f_y = \cos x \sin y + \tan y \quad (2.3)$. 
From (2.2), \( f(x,y) = -\cos x \cos y + 1/2 e^{2x} + g(y) \), where \( g(y) \) is the integration constant. Substituting in (2.3), \( \cos x \sin y + g_1(y) = f_y = \cos x \sin y + \tan y \). Thus \( g_1(y) = \tan y \) so that \( g(y) = \ln \sec y \). Hence the general solution is \( f(x,y) = -\cos x \cos y + 1/2 e^{2x} + \ln \sec y = c \), \( c \) parameter.

**Integrating factors**

In theory, a non-exact ODE can always be made exact by multiplying the equation by some function \( \mu(x,y) \) of \( x \) and \( y \). Such a function \( \mu(x,y) \) is called an **Integrating factor (I.F.)** of the ODE. Although there is always an I.F. for a non-exact ODE, there is no general method of finding the I.F. We shall now discuss methods of finding I.F. in some particular cases.

**Method 1** (by inspection) use is made of exact differentials like

\[
d\left(\frac{y}{x}\right) = \frac{x dy - y dx}{x^2}, \quad d(xy) = x \ dy + y \ dx, \quad d\left(\ln\left(\frac{y}{x}\right)\right) = \frac{x dy - y dx}{xy}, \quad d\left(\tan^{-1}\frac{y}{x}\right) = \frac{x dy - y dx}{x^2 + y^2}.
\]

**Example**: \((1+xy)y \ dx + (1-xy)x \ dy = 0.\)

**Sol**: \(d(xy) + xy(y \ dx - x \ dy) = 0.\) thus \(\frac{d(xy)}{(xy)^2} - \frac{x dy - y dx}{xy} = 0.\) So \(d \left[-\frac{1}{xy} - \ln\left(\frac{y}{x}\right)\right] = 0.\) General solution is \(1/(xy) + \ln(y/x) = c, \ c \) parameter.

**Method 2** if the ODE \(Mdx+Ndy = 0\) is homogeneous and \(Mx+Ny \neq 0\), then \(1/(Mx+Ny)\) is an IF.

**Example**: \(x^2y \ dx - (x^3+y^3)dy = 0\)
**Sol:** Here $Mx+Ny = -y^4 \neq 0$ and the given equation is homogeneous. Thus an IF is $-1/y^4$.

**Method 3** if $M = y f(xy)$, $N = x g(xy)$, then $1/(Mx-Ny)$ is an IF of the ODE $Mdx+Ndy = 0$.

**Example:** $(xy+2x^2y^2)\ y\ dx+(xy-x^2y^2)\ x\ dy = 0$.

**Sol:** Here $M = y[(xy)+2(xy)^2]$ and $N = x[(xy)-(xy)^2]$. Thus an IF = $1/[Mx-Ny] = 1/(3x^3y^3)$.

**Method 4** $e^{\int f(x)dx}$ is an IF of $Mdx+Ndy = 0$, if $f(x)$ is a function of $x$.

**Example:** $(x^2+y^2)dx - 2xy\ dy = 0$

**Sol:** $M = x^2+y^2$, $N = -2xy\ dy$, $M_y-N_x=-2/x$. Thus an IF = $e^{\int f(x)dx} = e^{-2\ln x} = x^{-2}$.

**Example:** (a very useful form of ODE: linear equation in $y$) $y_1+yP(x)\ y = f(x)$, $P(x)$, $f(x)$ are functions of $x$ alone or constants. $M = P(x)y-f(x)$, $N = 1$. Thus $\frac{M_y-N_x}{N} = P(x)$. An IF = $e^{\int P(x)dx}$.

**Method 5** $e^{\int f(y)dy}$ is an IF of $Mdx+Ndy = 0$, if $\frac{N_x-M_y}{M}$ is a function of $y$.

**Example:** $(xy^3+y)\ dx+2(x^2y^2+x+y^4)\ dy = 0$

**Sol:** $\frac{N_x-M_y}{M} = 1/y$. hence an IF = $e^{\int \frac{1}{y}dy} = y$.

**Method 6** if the equation $Mdx+Ndy = 0$ is of the form $x^ay^b(my\ dx+nx\ dy)+x^cy^d(py\ dx+qx\ dy) = 0$, where $a,b,c,d,m,n,p$ are constants, then $x^{hy^k}$ is an IF, where $h,k$ are constants and can be obtained by applying the condition that after multiplication by $x^{hy^k}$ the given ODE becomes exact.
Example: \((y^2+2x^2y)dx+(2x^3-xy)dy = 0\)

**Sol:** Let \(x^hy^k\) be an IF. Multiplying the given equation by this factor, we have \((x^hy^{k+2}+2x^{h+2}y^{k+1})dx+(2x^{h+3}-x^{h+1}y^{k+1})dy = 0\). Since (2.4) is to be exact, from the condition of exactness \((M_y=N_x)\), we obtain, 
\[(k+2)x^hy^{k+1}+2(k+1)x^{h+2}y^k = -(h+1)x^hy^{k+1}+2(h+3)x^{h+2}y^k.\] 
equating coefficients of \(x^hy^{k+1}\) and \(x^{h+2}y^k\) on both sides and solving, we get \(h = -5/2, k = -1/2\). Thus \(x^{-5/2}y^{-1/2}\) is an IF.

**FIRST ORDER LINEAR DIFFERENTIAL EQUATION**

A differential equation of first order is linear if the dependent variable \(y\) and its derivatives occur only in first degree and are not multiplied together. It is of the form \(y_1+Py = Q\ldots\ldots(2.4)\), where \(P\) and \(Q\) are either constants or functions of \(x\) alone.

**Example:** \((1+x)y_1-xy = 1-x\).

**Sol:** \(Y_1 - \frac{x}{1+x} y = \frac{1-x}{1+x}.\) Comparing with (2.4), \(P = -\frac{x}{1+x}\) and \(Q = \frac{1-x}{1+x}\). an I.F. = 
\[e^{\int \left(\frac{1}{1+x} - 1\right) dx} = e^{\ln(1+x)-x}.\] 
Multiplying given equation by the I.F., we get 
\[y(1+x)e^{-x} = \int \frac{1-x}{1+x} (1 + x)e^{-x} dx = xe^{-x}+c.\]

**Example:** \((1+y^2) dx = (\tan^{-1}y - x) dy\)

\[\frac{dx}{dy} + \frac{x}{1+y^2} = \tan^{-1}y,\] which is linear in \(x\). an I.F. = \(e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1}y}.\)

Solution is \(xe^{\tan^{-1}y} = \int e^{\tan^{-1}y} \cdot e^{\tan^{-1}y} dy.\) Put \(t = \tan^{-1}y.\) The solution is \(xe^{\tan^{-1}y} = e^{\tan^{-1}y} (\tan^{-1}y - 1) + c.\)
Example: \((x+2y^3)\ y_1 = y\)

Sol: The equation can be put as: \(x_1 - \frac{1}{y}.x = 2y^2\), which is \textbf{linear in} \(x\). An I.F. = \(e^{-\int \frac{dy}{y}} = 1/y\). Multiplying (2.6) by the I.F., \(x/y = \int 2y^2 \frac{1}{y} dy + c\). Thus \(x = y^3 + cy\).

**BERNOULLI’S EQUATION: REDUCIBLE TO LINEAR FORM**

An equation of the form \(y_1 + Py = Qy^n \quad \ldots \ldots (2.5)\) where \(P\) and \(Q\) are constants or functions of \(x\) alone and \(n\) is constant except 0 and 1, is called a Bernoulli’s equation. Dividing (2.5) by \(y^n\) and putting \(v = y^{-n+1}\), the equation reduces to \(v_1 + (1-n)pv = (1-n)Q\), which is \textbf{linear in} \(v\).

Example: \((1-x^2)y_1 + xy = xy^2\).

Sol: Given equation can be written as \(y^{-2}y_1 + \frac{x}{1-x^2}y^{-1} = \frac{x}{1-x^2}\). Let \(y^{-1} = v\), so that \(v_1 = -y^{-2}y_1\) and the given equation reduces to \(v_1 - \frac{x}{1-x^2}v = \frac{x}{1-x^2}\), which is linear in \(v\). An I.F. = \(\exp\left[- \int \frac{x}{1-x^2} dx\right] = (1-x^2)^{1/2}\). Multiplying by I.F. and integrating, \(v(1-x^2)^{1/2} = (1-x^2)^{1/2} + c\). Thus \(y\sqrt{1-x^2} = cy\).
LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

A linear ODE of order \( n \) is an equation of the form

\[
a_n(x)y_n+a_{n-1}(x)y_{n-1}+\ldots+a_1(x)y_1+a_0(x)y = Q(x) \ldots \ldots (3)
\]

where \( a_0, a_1, \ldots, a_n, Q \) are continuous real functions on an interval \( I \) and \( a_0(x) \neq 0 \) on \( I \).

**Notation:** \( a_i \) standing for \( i \) th function of \( x \) whereas \( y_i \) standing for \( i \)th derivative of \( y \).

If \( Q(x) \) is identically zero on \( I \), then (3) reduces to

\[
a_n(x)y_n+a_{n-1}(x)y_{n-1}+\ldots+a_1(x)y_1+a_0(x)y = 0 \quad (4)
\]

and is referred to as **homogeneous** ODE of order \( n \). If all the coefficients \( a_i \) of (3) are constants, it is called linear homogeneous equation with constant coefficients. The general solution of (4) is called **complementary function** (C.F.) of (3).

**Solution of the homogeneous linear differential equations with constant coefficients**

Consider the differential equation (DE) \[ a_n y_n + a_{n-1} y_{n-1} + \ldots + a_1 y_1 + a_0 y = 0 \] (5), where \( a_0, a_1, \ldots, a_n \) are all constants and \( a_n \neq 0 \).

Let \( y = e^{mx} \) be a solution of (5), where \( m \) is a suitable constant to be determined suitably. Calculating \( y_1, \ldots, y_n \), substituting in the equation and noting that \( e^{mx} \neq 0 \), we obtain the **characteristic equation** (CE) or **auxiliary equation**

\[
a_n m^n + a_{n-1} m^{n-1} + \ldots + a_1 m + a_0 = 0.
\]

While solving the CE, following three cases may arise:

- all the roots are real and distinct
- all roots are real but some are multiple roots
• some roots are real and some are conjugate complex
• all roots are imaginary.

We next explain through examples how to find general solution of (5) under different possible cases.

**Example:** \( y_3 + 6y_2 + 11y_1 + 6y = 0 \)

Sol: Roots of auxiliary equation are \(-1, -2, -3\). (Show calculations). Thus \( y = e^{-x} \), \( y = e^{-2x} \) and \( y = e^{-3x} \) are solutions of given DE. Solutions are also linearly independent since the **wronskian**

\[
\begin{vmatrix}
    e^{-x} & e^{-2x} & e^{-3x} \\
    -e^{-x} & -2e^{-2x} & -3e^{-3x} \\
    e^{-x} & 4e^{-2x} & 9e^{-3x}
\end{vmatrix} \neq 0 \text{ for all } x \text{ (calculate and verify).}
\]

Hence the general solution is \( y = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x} \), \( c_1, c_2, c_3 \) are independent parameters.

**Example:** \( y_3 - 3y_1 + 2y = 0 \).

Roots of auxiliary equation are \(1, 1, -2\). \( y = e^{x}, y = xe^{x} \) and \( y = e^{-2x} \) are linearly independent solutions. General solution is \( y = (c_1 + c_2x)e^{x} + c_3e^{-2x} \).

**Example:** \( y_3 + y = 0 \).

Roots of auxiliary equation are \(-1, 1/2 \pm \sqrt{3}/2 \) i. General solution is \( y = c_1e^{-x} + e^{x/2}(c_2 \cos \sqrt{3}x/2 + c_3 \sin \sqrt{3}x/2) \).

**Solution of the nonhomogeneous linear differential equations with constant coefficients by means of polynomial operator**

The general solution of (5.1) is given by \( y = y_c + y_p \), where \( y_c \) is the complementary function( that is, the general solution of the corresponding
homogeneous equation) and \( y = y_p(x) \) is a particular integral (P.I., that is, a particular solution) of (3). Above we have already seen how to find out \( y_c \) for a constant coefficient homogeneous linear DE. Below we explain how to find PI using **D-operator method**. We use the notation: \( Dy = y_1, \ D^2y = y_2, \ldots, P(D) = a_0 + a_1D + a_2D^2 + \ldots + a_nD^n. \) Using this notation, the linear nonhomogeneous constant coefficients DE of order \( n \): \( a_ny_n + a_{n-1}y_{n-1} + \ldots + a_1y_1 + a_0y = Q(x) \) can be written as \( P(D)y = Q(x) \). The operator \( P(D) \) has the following properties:

**Property 1** If \( f(x) \) is an \( n \) th order differentiable function of \( x \), then \( P(D)(e^{ax}f) = e^{ax}P(D+a)f \), where \( a \) is a constant.

**Property 2** Let \( P(D)y = Q(x) \). The inverse operator of \( P(D) \), written as \( P^{-1}(D) \) or \( 1/P(D) \), is then defined as an operator which, when operating on \( Q(x) \), will give the P.I. \( y_p \) of \( P(D)y = Q(x) \) that contains no constant multiples of a term in \( CF \): \( P^{-1}(D)Q(x) = y_p \), or, \( y_p = (1/P(D))Q(x) \).

**Property 3** If \( P(D)y = e^{ax} \), then \( y_p = \frac{e^{ax}}{P(D)} = e^{ax}/P(a) \), \( P(a) \neq 0 \).

**Property 4** If \( P(D)y = \sin(ax) \) or \( \cos(ax) \), then \( y_p = \frac{1}{P(D^2)} \sin(ax) \) or \( \frac{1}{P(D^2)} \cos(ax) \).

**Property 5** If \( P(D)y = e^{ax}V(x) \), then \( y_p = \frac{1}{P(D)}e^{ax}V = \frac{1}{P(D+a)}V \).

**Property 6** If \( P(D)y = xV(x) \), then \( y_p = \frac{1}{P(D)}(xV) = \frac{x}{P(D)}V - \frac{P_1(D)}{[P(D)]^2}V \).
Example: \((D^2-4)y = x^2, y = 0\) and \(y_1 = 1\) when \(x = 0\).

**Sol:** C.F. \(y_c = ae^{2x} + be^{-2x}\)

\[
P.I. \ y_p = \frac{1}{D(D+2)}x^2 = \frac{1}{D^2+4}x^2 = \frac{1}{4(1-\frac{D^2}{4})}x^2 = -\frac{1}{4}(1 - \frac{D^2}{4})^{-1}x^2 = -\frac{1}{4}(1 + \frac{D^2}{4} + \ldots) x^2 = \frac{1}{4}(x^2 + 1/2).
\]

Thus the general solution is \(y = y_c + y_p = ae^{2x} + be^{-2x} - \frac{1}{4}(x^2 + \frac{1}{2})\).

Hence \(y_1 = 2ae^{2x} - 2be^{-2x} - \frac{x}{2}\). Using the given conditions, \(a + b = \frac{1}{8}\) and \(a - b = \frac{1}{2}\). Hence \(a = \frac{5}{16}\) and \(b = -\frac{3}{16}\). Thus the PI satisfying the given condition is \(y = \frac{5}{16}e^{2x} - \frac{3}{16}e^{-2x} - \frac{1}{4}(x^2 + \frac{1}{2})\).

Example: \((D^2-2D+5)y = e^x\)

**Sol:** C.F. \(e^x(c_1 \cos 2x + c_2 \sin 2x)\)

\[
P.I. \ y_p = \frac{1}{D^2-2D+5}e^x = \frac{1}{(-1)^2-2(-1)+5}e^x = e^x/8.
\]

General solution is \(y = e^x(c_1 \cos 2x + c_2 \sin 2x) + e^x/8\).

Example: \((D^2-3D+2)y = 3 \sin 2x\)

**Sol:** C.F. \(ae^x + be^{2x}\)

\[
P.I. \ y_p = \frac{1}{(D^2-3D+2)^2}3 \sin 2x = 3 \frac{1}{(-2^2-3D+2)} \sin 2x = -3 \frac{1}{3D+2} \sin 2x = -3 \frac{3D-2}{9D^2-4} \sin 2x = \frac{-3}{9(-2)^2-4} (3D-2) \sin 2x = \frac{3}{20} (3 \cos 2x - \sin 2x).
\]
Example: \((D^2-2D+1)y = e^x x^2\).

**Sol:** C.F. \((a+bx)e^x\).

P.I. \(y_p = \frac{1}{D^2-2D+1} e^x x^2 = \frac{e^x}{(D+1)^2-2(D+1)+1} x^2 = e^x \frac{1}{D^2} x^2 = e^x D^{-2} x^2 = (e^x x^4)/12\).

Example: \((D^2+6D+9)y = 2e^{-3x}\).

**Sol:** We can not use property 3, since \(P(-3) = 0\). Instead we apply property 5.

\[Y_p = \frac{1}{D^2+6D+9}(2e^{-3x}) = 2e^{-3x} \frac{1}{(D-3)^2+6(D-3)+9}(1) = 2e^{-3x} \frac{1}{D^2}(1) = 2e^{-3x} x^2/2 = x^2 e^{-3x}\.

Example: \((D^2+2D+1)y = x \cos x\)

**Sol:** P.I. \(y_p = \frac{1}{D^2+2D+1}(x \cos x) = x \frac{1}{D^2+2D+1} \cos x - \frac{2D+2}{D^2+2D+1} \cos x = \frac{x}{2D^2} (D+1) \cos x = x \sin x/2 - \frac{1}{2} D^2 (-\sin x + \cos x) = x \sin x/2 - \frac{1}{2} (\sin x - \cos x)\).

Example: \((D^2+4)y = \cos (2x)\)

**Sol:** Let \(y = \frac{1}{D^2+4} \cos (2x)\), \(z = \frac{1}{D^2+4} \sin (2x)\). then \(y+iz = \frac{1}{D^2+4} (\cos 2x+i \sin 2x) = \frac{1}{D^2+4} e^{2ix} = e^{2ix} \frac{1}{(D+2i)^2+4}(1)\)
\[ e^{2ix} \frac{1}{D^2 + 4iD} (1) = e^{2ix} \frac{1}{4iD} (1 - i \frac{D}{4})^{-1} (1) = (\cos 2x + i \sin 2x)x/4i = x \sin(2x)/4 - i x \cos(2x)/4. \] Equating real part, \( y = x\sin(2x)/4. \)

**CAUCHY-EULER EQUATION**

An equation of the form \( a_n x^n y_n + a_{n-1} x^{n-1} y_{n-1} + \ldots + a_1 x y_1 + a_0 y = Q(x) \ldots \) (5.4)
where \( a_i \)'s are constants, is called a Cauchy-Euler equation of order \( n \).

by undertaking a transformation \( x = e^t \) of independent variable, (5.4) can be transformed to a constant coefficient linear ODE.

**Example:** \( x^2 y'' + xy' - 4y = x^2. \)

**Sol:** Putting \( x = e^t \), or \( \ln x = t \) and \( D_t = d/dt \), the given ODE can be written as \( [D_t(D_t-1) + D_t-4]y = e^{2t} \), or \( (D_t^2-4)y = e^{2t} \). C.F. is \( y_c = ae^{2t} + be^{-2t} = ax^2 + bx^{-2}. \)

P.I. \( y_p = \frac{1}{D_t^2-4} e^{2t} = \frac{1}{(D_t+2)^2-4} (1) = \frac{1}{4D_t} \left( 1 + \frac{D_t}{4} \right)^{-1} (1) = \frac{1}{4} e^{2t}t = \frac{1}{4} x^2 \ln x. \)

Hence the general solution of given equation is \( y = y_c + y_p = ax^2 + bx^{-2} + 1/4 x^2 \ln x. \)
TOP 10 PLACES FOR B.SC MATHEMATICS IN INDIA:

1. Indian Statistical Institute, Bangalore
2. Chennai Mathematical Institute, Chennai
3. Institute of Mathematics & Application, Orissa
4. National Institute of Science Education and Research (NISER), Bhubaneswar
5. Indian Institute of Science Education and Research (IISER), Kolkata.
6. Ramakrishna Mission Vidyamandira, Belur Math
7. Presidency University, Kolkata
8. Jadavpur University, Kolkata
9. Ramakrishna Mission Residential College, Narendrapur
10. St. Stephen's College, Delhi

TOP 10 PLACES FOR B.SC STATISTICS IN INDIA:

1. Indian Statistical Institute, Kolkata
2. Presidency University, Kolkata
3. Ramakrishna Mission Residential College, Narendrapur
4. St. Xavier's College, Kolkata
5. Fergusson College, Pune
6. Lady Shri Ram College for Women, Delhi
7. Asutosh College, Kolkata
8. Maulana Azad College, Kolkata
9. Lady Brabourne College, Kolkata
10. Hindu college, Delhi
TIPS TO REMEMBER

☐ CONGRUENT: Let us start by choosing a base n, then each integer \( N = qn + r \) where \( q \) is quotient and \( r \) is remainder \((0 \leq r < n)\). We write this as \( N \equiv r \pmod{n} \) and say that \( N \) is congruent to \( r \) modulo \( n \). E.g. \( 78 \equiv 66 \pmod{12} \).

Ex. January 1, 2000 falls on a saturday. What day of the week will January 1, 2020 be?
Sol. Because there are 20 years in the range 2000 to 2019 of which 5 are leap-years, January 1, 2020 falls on day \( 20 + 5 = 4 \pmod{7} \), i.e. Wednesday.

☐ MATHEMATICAL INDUCTION:

1. Weak Version: Let \( P(n) \) be a statement such that (i) \( P(n_0) \) is true and (ii) \( P(k) \) implies \( P(k+1) \) for any \( k \geq n_0 \), then \( P(n) \) is true for every \( n \geq n_0 \).

2. Strong Version: Let \( P(n) \) be a statement such that (i) \( P(n_0) \) is true and (ii) if \( P(n_0), P(n_0+1), \ldots, P(k) \) are true for any \( k \geq n_0 \), then \( P(k+1) \) is also true. Then \( P(n) \) is true for every \( n \geq n_0 \).

☐ PIGEONHOLE PRINCIPLE: If \( m \) pigeons are assigned to \( n \) pigeonholes, where \( m > n \), then at least two pigeons must occupy the same pigeonhole.

Proof: Let the given conclusion is false, that is, no two pigeons occupy the same pigeonhole. Then every pigeon must occupy a distinct pigeonhole, so \( n \geq m \), which is a contradiction. Thus two or more pigeons must occupy some pigeonhole.

☐ ROUND-ROBIN TOURNAMENT: In round-robin tournament every team plays every other team exactly once. Suppose there are \( n \) teams, labeled 1 through \( n \). Then the tournament can be represented
by a polygon with n vertices with every pair of vertices connected, every vertex represents a team and every line segment with endpoints i and j represents a game between teams i and j.

Let \( g_n \) = number of games by n teams in a tournament. It can be defined recursively:

\[
\begin{align*}
g_1 &= 0 \\
g_n &= g_{n-1} + (n-1), \text{ where } n \geq 2
\end{align*}
\]

Solve this recurrence relation, we get

\[
g_n = \frac{n(n-1)}{2} = ^nC_2.
\]

e.g. 5 teams will play 10 games shown in given figure.

**FERMAT'S LITTLE THEOREM**: Let \( p \) be a prime and \( a \) is any integer such that \( p \nmid a \), then \( a^{p-1} \equiv 1 \pmod{p} \).

Ex. Find the remainder when \( 24^{1947} \) is divided by 17.

Sol. \( 24 = 7 \pmod{17} \). \( \therefore 24^{1947} \equiv 7^{1947} \pmod{17} \).

From Fermat's little theorem,

\( 7^{16} \equiv 1 \pmod{17} \). \( \therefore 7^{1947} = (7^{16})^{121} \cdot 7^{11} = 1^{121} \cdot 7^{11} \equiv 7^{11} \pmod{17} \).

But \( 7^2 = -2 \pmod{17} \). \( \therefore 7^{11} = (7^2)^5 \cdot 7 = (-2)^5 \cdot 7 = -32 \cdot 7 \equiv 2 \cdot 7 = 14 \pmod{17} \).

Hence the reqd. remainder is 14.

**PRINCIPLE OF INCLUSION AND EXCLUSION (P.I.E.) OR SIEVE FORMULA**: It is very important principle is a generalization of the sum-rule to sets which need not be disjoint. Venn-diagrams show that \( |A \cup B| = |A| + |B| - |A \cap B| \) and \( |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C| \).

We generalize to n sets as follows:

\[
|A_1 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \ldots (-1)^{n+1} |A_1 \cap \ldots \cap A_n|.
\]
e.g. We consider all $n!$ permutations of $1,2,...,n$. If an element $i$ is on place number $i$, then we say $i$ is a fixed-point of the permutation. Let $p_n =$ number of fixed point free permutations and $q_n =$ number of permutations with at least one fixed point. Then $p_n = n! - q_n$.

Let $A_i =$ number of permutations with $i$ fixed point, then

$q_n = \left| A_1 \cup \ldots \cup A_n \right| = \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \ldots + (-1)^{n+1} \binom{n}{n}0! =
\begin{align*}
= n! \left\{ 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots + \frac{(-1)^{n-1}}{n!} \right\} \\
\therefore p_n = n! - q_n = n! \left\{ \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^{n-1}}{n!} \right\}.
\end{align*}$

DIVISION WITH REMAINDER: For polynomials $f(x)$ and $g(x)$ there exist unique polynomials $q$ and $r$ such that $f(x) = g(x)q(x) + r(x)$, $\deg r(x) < \deg g(x)$ or $r(x) = 0$, where $q(x)$ and $r(x)$ are quotient and remainder respectively on division of $f(x)$ by $g(x)$. If $r(x) = 0$, then we say that $g(x)$ divides $f(x)$ and we write $g(x) \mid f(x)$.

ARITHMETIC-GEOMETRIC-HARMONIC-MEAN INEQUALITY:

Take any $n$ different positive numbers $x_1, x_2, \ldots, x_n$ where $n$ is positive integer, then Arithmetic mean $A = (x_1 + x_2 + \ldots + x_n)/n$, Geometric mean $G = (x_1 x_2 \ldots x_n)^{1/n}$, Harmonic mean $H = \left\{ \left( \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} \right)/n \right\}^{-1}$ and Root mean square

$[\mu_2^{1/2}] = \left\{ \left( x_1^2 + x_2^2 + \ldots + x_n^2 \right)/n \right\}^{1/2}$

Verify that whatever the values of $n$ you take and whatever be the positive numbers you choose $[\mu_2^{1/2}] \geq A \geq G \geq H$, but if $x_1, x_2, \ldots, x_n$ are equal, then all there means are also equal.
FOLLOWING RESULTS OF HEIGHTS AND DISTANCES:

(i) **Appolonius theorem**: It in a \( \triangle ABC \), AD is median, then \( AB^2 + AC^2 = 2(AD^2 + BD^2) \).

(ii) **m-n theorem**: If \( BD : DC = m:n \), then 
\[
(m+n) \cot \theta = m \cot \alpha - n \cot \beta = n \cot B - m \cot C.
\]

(iii) **Angle bisector**: If AD is the angle bisector of \( \angle BAC \), then 
\[
\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}
\]

(iv) The exterior angle is equal to the sum of the interior opposite angles.
(v) If a line is perpendicular to a plane, then its perpendicular to every line in that plane.

REGULAR POLYGON: Let \( A_1, A_2, \ldots, A_n \) be a regular polygon of \( n \) sides each of length \( a \).

(i) **Inscribed circle of a regular polygon of \( n \) sides**:
Area = \( \pi r^2 \tan \frac{\pi}{n} \) and Radius \( r = \frac{a}{2 \cos \frac{\pi}{n}} \).

(ii) **Circumscribed circle of a regular polygon of \( n \) sides**:
Area = \( \frac{1}{2} nR^2 \sin \frac{2\pi}{n} \) and Radius \( R = \frac{a}{2 \csc \frac{\pi}{n}} \).

MODULUS OR ABSOLUTE VALUE FUNCTION:

It is defined as \( y = f(x) = |x| = \sqrt{x^2} = \begin{cases} x: & x \geq 0 \\ -x: & x < 0 \end{cases} \).

\( \therefore f(-x) = -x = f(x) \).

\( \therefore f(x) \) is symmetric about y-axis.

Here \( |0| = 0 \); \( |-5| = 5 \); \( |5| = 5 \). Domain: \( x \in \mathbb{R} \).
Range: \( y \in [0, \infty) \). Continuous everywhere.
Non differentiable at \( x = 0 \), elsewhere differentiable. Even as well as Many-one function. Monotonically
increasing and decreasing for all $x>0$ and $x<0$ respectively.

Note: $|x|$ is read modulus or mod x. $|x| = 2 \iff x=\pm 2$ ; $|x|<2 \iff -2<x<2$ and $|x|>2 \iff x<-2$ or $x>2$.

Properties:
(i) $|x+y| = |x| + |y| \iff x, y \geq 0$ or $x, y \leq 0$.
(ii) $|x-y| = |x| - |y| \iff x \geq 0, |x| \geq |y|$ or $x \leq 0, y \leq 0$ and $|x| \leq |y|$.
(iii) $|x+y| \leq |x| + |y|$.
(vi) $|x+y| \geq |x| - |y|$.

**SIGNUM FUNCTION**: It is defined as $y=f(x)=\begin{cases} x \text{ or } \frac{x}{|x|} ; x \neq 0 \\ 0 ; x = 0 \end{cases}$

Domain : $x \in \mathbb{R}$ . Range : $y \in \{-1,0,1\}$.

Continuous everywhere except at $x=0$ and discontinuous at $x=0$ . Odd as well as many-one function. Neither monotonically increasing nor decreasing.

Note: $0$ indicates that the points $(0,1)$ and $(0,-1)$ are not included and denote the point $(0,0)$ is included in the graph.

**GREATEST INTEGER OR FLOOR OR STEP FUNCTION**: It is defined as $y = f(x) = [x] = \lfloor x \rfloor = \text{gint}(x) = n$ where $n \leq x < n+1$.

E.g. $[\pi] = 3, [\log_{10} 3] = 0, [-3.5] = -4$, etc and $[0.7] = 0$ etc.

\therefore Real number = Integral Part (I) + fractional part (f), where I = [x] and 0 \leq f < 1. \therefore y = f(x) = [x]

For $-3 \leq x < -2$; $y = -3$

\hspace{1cm} $-2 \leq x < -1$; $y = -2$

\hspace{1cm} $-1 \leq x < 0$; $y = -1$

\hspace{1cm} $0 \leq x < 1$; $y = 0$

\hspace{1cm} $1 \leq x < 2$; $y = 1$ and so on.

- Domain : $x \in \mathbb{R}$ . Range : $y \in \mathbb{I}$
- Continuous and differentiable everywhere expect at $x=n$, $n \in \mathbb{I}$
- Many-one and into function for co-domain as all real numbers.
Properties:
(i) \([x+n]=n+[x],\ n\in I.\)
(ii) \([-x]=-\lfloor x\rfloor\ \forall\ n\in I \text{ and } [-x]=-[x]-1, \ x\notin I.\)
(iii) \(x=[x]+\{x\},\ \{\}\ \text{denote fractional part of } x.\)
(iv) \([x_1+x_2]\geq[x_1]+[x_2].\)
(v) \[\left[\frac{x}{n}\right]=\left[\frac{\lfloor x\rfloor}{n}\right] \ \forall\ n\in N.\]
(vi) If \([f(x)]\geq I,\ \text{then } f(x) \geq I \text{ and for } [f(x)]\leq I, \ \text{then } f(x)<I+1.\)
(vii) \([p+\lfloor q+r+s+t\rfloor]=[p]+[q]+[r]+[s]+[t].\)
(viii) \([x]+\left[\frac{1}{n}\right]+\left[\frac{2}{n}\right]+\ldots+\left[\frac{n-1}{n}\right]=[nx].\)

\[\text{LEAST INTEGER OR CEILING FUNCTION: It is defined as } y=f(x)=(x)=\lfloor x\rfloor=\text{lint } (x)=n+1,\]
where \(n<x\leq n+1.\)

\[\text{e.g. } (\pi)=4, \ (\log_{10}3)=1, \ (-3.5)=-3 \text{ etc.}\]

\[\text{y}=(x)=0 \text{ if } -1<x\leq0\]
\[=1 \text{ if } 0<x\leq1\]
\[=2 \text{ if } 1<x\leq2\]
\[=3 \text{ if } 2<x\leq3 \text{ and so on.}\]

- Domain: \(x\in R\)  
- Range: \(y\in I\)

- Many one into function for co-domain of all real numbers.
- Discontinuous at all integral values of \(x.\)

Properties:
(i) \((x+n)=(x)+n,\ n\in I.\)
(ii) \((-x)=-\lfloor x\rfloor,\ x\in I \text{ and } (-x)=-(x)+1, \ x\notin I.\)
(iii) \( x = (x) + \{x\} - 1, \{x\} \) denotes the fractional part of \( x \).

(iv) \((x_1 + x_2) \leq (x_1) + (x_2)\).

(v) \( \left( \frac{(x)}{n} \right) = \left( \frac{x}{n} \right), n \in \mathbb{N} \).

(vi) \( (x) + \left( x + \frac{1}{n} \right) + \left( x + \frac{2}{n} \right) + \ldots + \left( x + \frac{n-1}{n} \right) = (nx) + n - 1, n \in \mathbb{N} \).

\[ \square \] FRACTIONAL PART FUNCTION : It is defined as \( y = f(x) = \{x\} = x - [x] = f \) (where \( 0 \leq f < 1 \)) = \( x - n, n \leq x < n+1 \).

e.g. \( \{1.6\} = 0.6, \{1\} = 0, \{-3.6\} = 0.4 \)

\[ \therefore y = \{x\} = x \text{ if } 0 \leq x < 1 \]
\[ = x - 1 \text{ if } 1 \leq x < 2 \]
\[ = x - 2 \text{ if } 2 \leq x < 3 \text{ and so on.} \]

\bullet \text{Domain : } x \in \mathbb{R}, \text{ Range : } y \in [0,1), \]
\bullet \text{Discontinuous and Non-differentiable at all integers,} \]
\bullet \text{Monotonically increasing throughout} \]
\bullet \text{Many-one and into function for co-domain of all real numbers.} \]

Properties :

(i) If \( 0 \leq x < 1 \), then \( \{x\} = x \).

(ii) If \( x \in 1 \), then \( \{x\} = 0 \).

(iii) If \( x \notin 1 \) and \( x > 0 \), then \( \{-x\} = 1 - \{x\} \).

(iv) Domain and Range of \( \frac{1}{\{x\}} \) are \( R - 1 \) and \( (0,1) \) respectively.

Note : \( f(x) = f(x) \) (\( x \in \mathbb{R} \)), where \( T \) be a (+)ve real number and \( f(x) = \{x\} \).
\( \Rightarrow x + T - [x + T] = x - [x] \forall x \in \mathbb{R}, \Rightarrow [x + T] - [x] = T \forall x \in \mathbb{R}, \Rightarrow T = 1, 2, 3, \ldots \) Thus there exist \( T > 0 \) such that \( f(x + T) = f(x) \forall x \in \mathbb{R} \), so \( f(x) \) is periodic and the smallest value of \( T \) satisfies \( f(x + T) = f(x) \forall x \in \mathbb{R} \) is \( 1 \).

\[ \square \] DIRICHLET FUNCTION : Suppose \( c \) and \( d \) be real numbers (usually taken as \( c = 1 \) and \( d = 0 \)), then the Dirichlet function is defined
by \( y = f(x) \) = \begin{cases} \text{c for } x \text{ rational} \\ \text{d for } x \text{ irrational} \end{cases} \text{ and is discontinuous everywhere.}

Analytically: \( f(x) = \lim_{{m \to \infty}} \lim_{{n \to \infty}} \cos^{2n}(m! \pi x) \)

- Domain: \( x \in \mathbb{R} \)  
- Range: \( y \in \{c,d\} = \{0,1\} \)  
- Neither increasing nor decreasing  
- Not a periodic function  
- Discontinuous everywhere  
- Man yone and into function for co-domain of all real numbers.

**WORKED OUT EXAMPLES**

**Ex.1.** Let max \((x,y)\) denote the maximum of \(x, y\) and min \((x,y)\) their minimum, where \(x\) and \(y\) are any integer, prove that max \((x,y)\) - min \((x,y)\) = \(|x-y|\).

**Sol. Case 1.** Let \( x \geq y \), then \(|x-y| = x-y\), max \((x,y)\) = \(x\) and min \((x,y)\) = \(y\). Thus max \((x,y)\) - min \((x,y)\) = \(x-y\) = \(|x-y|\).

**Case 2.** Let \( x < y \), then \(|x-y| = y-x\), max \((x,y)\) = \(y\) and min \((x,y)\) = \(x\). Thus max \((x,y)\) - min \((x,y)\) = \(y-x\) = \(|x-y|\).

**Ex.2.** Solve: (i) \(|x+1|^\log(x+1)^{\left(3+2x-x^2\right)} = (x-3)|x|\)

   (ii) \(|x-1| + y = 4\), \(x - |y-1| = 2\).

**Sol.** (i) Given: \(|x+1|^\log(x+1)^{\left(3+2x-x^2\right)} = (x-3)|x|........(1)\)

   \(\because 3 + 2x - x^2 > 0\) (for \(\log x, x>0\))

   \(\therefore -1 < x < 3....(2).\)

   Let \(x+1>1\), then \(x>0....(3).\)

   \(\therefore \) From (1): \((x+1)^{\log(x+1)^{\left(3+2x-x^2\right)}} = (x-3)x\)

   \(\Rightarrow 3+2x-x^2 = x^2 - 3x\), \(\Rightarrow 2x^2 - 5x - 3 = 0.\) \(\therefore x = -\frac{1}{2}, 3.\)

Now, from (2) and (3): \(x \neq -\frac{1}{2}, 3.\)

Hence it has no solution.

(ii) Given: \(|x-1| + 3y = 4\), \(\Rightarrow \left\{ \begin{array}{l} x+3y =5, x \geq 1........(1) \\ -x+3y = 3, x < 1........(2) \end{array} \)\) and also given: \(x - |y-1| = 2\)
\( \begin{align*}
&\Rightarrow \begin{cases}
x-y=1, y \geq 1 \quad (3) \\
x+y=3, x < 1 \quad (4)
\end{cases} \text{ Solving (1), (3): } x=2, y=1 \text{ and solving (1), (4): } x=2, y=1.
\end{align*} \\
\therefore \text{ No solution as } x \geq 1, y < 1.
\text{ Solving (2), (3): } x=3, y=2, \text{ i.e. no solution as } x < 1, y \geq 1 \text{ and solving (2), (4): } x = \frac{5}{2}, y = \frac{3}{2}, \text{ i.e. no solution as } x < 1, y < 1.
\end{align*} \\
\text{ Hence the reqd. solution is unique, i.e. } x=2, y=1.
\text{ Ex. 3 If } f(x)=3[x]+5 \text{ and } f(x)=5[x-2]+7, \text{ find the value of } [x+f(x)].
\text{ Sol. given: } 3[x]+5=5[x-2]+7, \Rightarrow 3[x]+5=5[x]-10+7, \Rightarrow 2[x]=8, \Rightarrow [x]=4. \Rightarrow 4 \leq x < 5, \text{ i.e. } x=4+\text{fractional part}.
\therefore f(x)=3[x]+5=3.4+5=17. \Rightarrow [x+f(x)]=[17+\text{fractional part}+4]=21.
\text{ Ex. 4. Solve: } \{x+1\}+2x=4\{x+1\}-6 \text{ where } \{.\}=\text{f.p.}; [.]\text{=G.I.}
\text{ Sol. } x+1-[x+1]+2x=4[x+1]-6 (\therefore \{x\}=x-[x]), \Rightarrow 3x+1=5[x+1]-6
=5([x]+1)-6, \Rightarrow 3x=5[x]-2\ldots(i), \Rightarrow 3(\{x\}+[x])=5[x]-2.
\therefore 3\{x\}=2[x]-2\ldots(ii).
\text{ Now, } 0 \leq \{x\} < 1, \Rightarrow 0 \leq 3\{x\} < 3, \Rightarrow 0 \leq 2[x]-2 < 3 \text{ [using (ii)],}
\Rightarrow 2 \leq 2[x] < 5, \Rightarrow 1 \leq [x] < \frac{5}{2}, \Rightarrow [x]=1,2.
\therefore \text{ From (i): } [x]=1, \text{ i.e. } x=1 \text{ and } [x]=2, \text{ i.e. } x=\frac{8}{3}.
\text{ Hence reqd. solutions are } 1,\frac{8}{3}.
\text{ Ex. 5. Solve: } (x)^2+(x+2)^2=20, \text{ where } (.) = \text{ least integer.}
\text{ Sol. Let } x=1+f, \text{ then from the given equation, we have } (1+f)^2 + (1+f+2)^2
=20, \Rightarrow (1+1)^2 + (1+3)^2 = 20, \Rightarrow 1^2 + 21 + 1 + 1^2 + 61 + 9 = 20,
\Rightarrow 1^2 + 41 - 5 = 0, \Rightarrow 1 = 1, -5.
\therefore x = 1+f, \Rightarrow 0 < f < 1, \Rightarrow 1 < 1+f < 2 \text{ and } -5 < -5+f < -4.
\text{ Further } x = 1.
\therefore (x)^2+(x+2)^2=20, \text{ i.e. } x \in (-5,-4) \cup (1,2).
Ex. 6. Solve: \( y = \frac{1}{3} [\sin x + (\sin x + (\sin x))] \) and \( [y + |y|] = 2\cos x \).

Sol. we have
\[
y = \frac{1}{3} [\sin x + (\sin x + (\sin x))] = \frac{1}{3} [\sin x + (\sin x + (\sin x))]
\]
\( \therefore [I + x] = I + [x] \) for any integer I.
\[
\Rightarrow \frac{1}{3} ([\sin x] + [\sin x] + [\sin x]) \ldots \ldots (1)
\]
and \( [y + |y|] = 2\cos x \), \( \Rightarrow 2[y] = 2\cos x \), \( \Rightarrow [y] = \cos x \ldots \ldots (2) \).

\( \therefore \) From (1) and (2): \( [\sin x] = \cos x \), \( \Rightarrow [\sin x] = \cos x \).

Now, plotting the curves \( \sin x \) and \( \cos x \) on the same frame, we see that the two curves have no intersection points.

Hence the given equation has no solution.

Ex. 7. How many integers between 1 and 300 (inclusive) are (i) divisible by at least one of 3, 5, and 7? (ii) divisible by 3 and by 5, but not by 7? (iii) divisible by 5 but by neither 3 nor 7?

Sol. Let A, B and C be the set of those integers between 1 and 300 which are divisible by 3, 5, 7 respectively.

\[ A = \{ n | 1 \leq n \leq 300, 3 | n \}; \quad B = \{ n | 1 \leq n \leq 300, 5 | n \}; \]

\[ C = \{ n | 1 \leq n \leq 300, 7 | n \}. \]

To be divisible by 3 or 5 or 7 is to be at least in one of A, B, or C.

Now, \( |A| = \left\lfloor \frac{300}{3} \right\rfloor \), \( \lfloor . \rfloor \) is the greatest integer = 100.

Similarly, \( |B| = \left\lfloor \frac{300}{5} \right\rfloor = 60 \) and \( |C| = \left\lfloor \frac{300}{7} \right\rfloor = 42 \), where \( |A|, |B|, |C| \) being the cardinalities of sets A, B, C. Again, \( A \cap B \) is the set of integers between 1 and 300, which are divisible by both 3 and 5, since 3 and 5 are relatively prime, any number divisible by them must be divisible by product of them,

\( \therefore A \cap B \) is the set of integers divisible by 15.

Similarly, \( B \cap C, A \cap C, A \cap B \cap C \) are the sets of integers between 1 and 300 which are divisible by 35, 21 and 105.
\(|A \cap B| = \left[ \frac{300}{15} \right] = 20, |A \cap C| = \left[ \frac{300}{21} \right] = 14, |B \cap C| = \left[ \frac{300}{35} \right] = 8,\\ |A \cap B \cap C| = \left[ \frac{300}{105} \right] = 2.

(i) \(|A \cap B \cap C| = |A| + |B| + |C| - |AB| - |BC| - |CA| + |ABC| = 100 + 60 + 42 - 20 - 14 - 8 + 2 = 162.

(ii) The numbers divisible by 5 and 3, but not by 7, are precisely those numbers in \((ABC) \setminus C\), whose cardinality is \(|A \cap B| - |A \cap B \cap C|
\)

= 20 - 2 = 18.

(iii) The numbers divisible by 5 but by neither 3 nor 7 are those in \(B \setminus (A \cup C)\), having cardinality \(|B| - |B \cap (A \cup C)|\). Since \(B \cap (A \cup C) = (B \cap A) \cup (B \cap C)\), the principle of inclusion and exclusion gives

\(|B \cap (A \cup C)| = |B \cap A| + |B \cap C| - |(B \cap A) \cap (B \cap C)| =\\ |B \cap A| + (B \cap C) - |B \cap A \cap C| = 20 + 8 - 2 = 26\) and the number we are looking for is \(|B| - |B \cap (A \cup C)| = 60 - 26 = 34.

Ex. 8. Let \(n\) be a positive integer. Suppose a function \(L\) is defined recursively as follows:

\[L(n) = \begin{cases} 0 & \text{if } n = 1 \\ L\left[\frac{n}{2}\right] & \text{if } n > 1 \end{cases}\]

([.] is the floor function).

Find \(L(25)\).

Sol. \(L(25)\) is found recursively as follows:

\(L(25) = (12) + 1 = \ldots \) (i); \(L(12) = L(6) + 1 = \ldots \) (ii);

\(L(6) = L(3) + 1 = \ldots \) (iii); \(L(3) = L(1) + 1 = \ldots \) (iv);

Back calculating \(L(6) = L(3) + 1 = 1 + 1 = 2\); \(L(12) = L(6) + 1 = 2 + 1 = 3\).

\(\therefore L(25) = L(12) + 1 = 3 + 1 = 4\).

Ex. 9. If \(f(x) = x^3 - 9x^2 + 24x + c\) has three real and distinct roots \(\alpha, \beta, \gamma\) then find the possible values of \(c\). Hence, otherwise show that \(|\alpha| + |\beta| + |\gamma|\) can take only two values and determine these values, \([.]\) denotes the greatest integer function.

Sol. Suppose \(y = x^3 - 9x^2 + 24x\).

\(y' = 3(x - 2)(x - 4)\). \(\therefore\) For turning point \(y' = 0\).

\(\therefore x = 2, 4\) are turning points.
Thus \( y(2) = 8 - 36 + 48 = 20 \) and \( y(4) = 64 - 144 + 96 = 16 \).
From the graph we can see that x-axis will cut the graph 3 times, if it shifted downward by 16 to 20 units, i.e. \( c \in (-20, -16) \). For
\[ [\alpha] + [\beta] + [\gamma], \text{ if } c \in (-20, -16), \text{ then } \alpha \in (1, 2), \beta \in (3, 4), \gamma \in (4, 5). \]
\[ \therefore [\alpha] + [\beta] + [\gamma] = 1 + 3 + 4 = 8. \] Again, if \( c \in (-20, -18) \), then \( \alpha \in (1, 2, \ldots) \)
\( \beta \in (2, 3), \gamma \in (4, 5). \)
\[ \therefore [\alpha] + [\beta] + [\gamma] = 1 + 2 + 4 = 7. \]

Finally, We can say that \( [\alpha] + [\beta] + [\gamma] = \begin{cases} 8 & \text{if } -18 < c < -16 \\ 7 & \text{if } -20 < c < -18 \end{cases} \)

Ex. 10. If \( a, b, c \) are the cube root of \( p \), \( p < 0 \) then for any permissible value of \( x, y, z \) which is given by
\[
\left| \frac{xa + yb + zc}{xb + yc + za} \right| + (a^2 - 2b^2) \omega + \omega^2 \omega^2 (|x| + |y| + |z|) = 0, \]
where \( \omega \) is cube root of unity and \( a, b \), are real positive numbers, \( b \) is a prime, find the value of
\[ [x+a_1] + [y+b_1] + [z]|(.|=G.I.F.). \]

Sol. Given : \( 1 + \omega + \omega^2 = 0 \ldots (i) \) and \[ \left| \frac{xa + yb + zc}{xb + yc + za} \right| + (a^2 - 2b^2) \omega + ((|x| + |y| + |z|) \omega^2 = 0 \ldots (ii). \]

From (i) and (ii) : \[ \left| \frac{xa + yb + zc}{xb + yc + za} \right| = (a^2 - 2b^2) = |x| + |y| + |z| .......(iii). \]

Since \( a, b, c \) are the roots of \( (p)^{1/3} \) and let \( a = t, b = t^\omega \) and \( c = t^{\omega^2} \).
\[ \therefore \left| \frac{xa + yb + zc}{xb + yc + za} \right| = \left| \frac{xt + y + \omega + t\omega^2 \cdot z}{x\cdot t + y\cdot t\omega^2 + x\cdot t} \right| = \left| \frac{x + y\omega + z\omega^2}{x\omega + y\omega^2 + z} \right| = \left| \frac{1}{\omega} \right| = 1. \]
\[ \therefore \text{ From (iii) : } a^2 - 2b^2 = 1, \text{ i.e. } a^2 = 2b^2 + 1 \text{ (odd)} \ldots (iv) \]
Thus \( a_1 \) can be written in the from \( (2n+1), \Rightarrow (2n+1)^2 = 1 + 2b_1^2 \Rightarrow 4n^2 + 4n = 2b_1^2 = 2n(2n+1), \Rightarrow b_1^2 = 2n(2n+1), \) an even number and given prime, so \( b_1^2 \) is also prime. \( \therefore \) b_1 = 2 (\because 2 \text{ is only even prime} \)
\[ a^2 = 9, \Rightarrow a = 3. \text{ From (iii)}: [x] + [y] + [z] = a^2 - 2b^2 = 1. \]
\[ [x+a_i] + [y+b_i] + [z] = [x+3] + [y+2] + [z] = [x] + [y] + [z] + 5 = 1 + 5 = 6. \]

**Ex. 11.** Find the number of solutions of \(4\{x\} = x + [x]\), where \(\{\cdot\}\), \([\cdot]\) denotes the fractional part, greatest integer function respectively.

**Sol.** Given: \(4\{x\} = x + [x]\), \(\Rightarrow 4(x-[x]) = x + [x]\) (\(\therefore x = [x] + \{x\}\)), \(\Rightarrow 3x = 5[x].\)

\[ [x] = \frac{3}{5}x \ldots \text{(i)} \]

\(\therefore\) To plot the graph of both \(y = [x]\) and \(y = \frac{3}{5}x\).

Thus the two curves intersect when \([x] = 0\) and \([x] = 1\), \(\ldots \text{(ii)}\).

\(\therefore\) From (i) and (ii): \(x = \frac{5}{3} \cdot [x]\), i.e. \(x = \frac{5}{3} \cdot 0\) and \(x = \frac{5}{3} \cdot 1\).

Hence \(x = 0, \frac{5}{3}\), i.e. the only two solutions.

**Ex. 12.** Draw the graph of \([y] = \sin^{-1}\left(\frac{x}{2}\right)\), where \([\cdot]\) = G.I.F.

**Sol.** First draw the graph of

\(y = \sin^{-1}\left(\frac{x}{2}\right)\), \(\Rightarrow -1 \leq \frac{x}{2} \leq 1\), i.e. \(-2 \leq x \leq 2\)

and \(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\).

\([y] = -2, -1, 0, 1\). \(\therefore \sin^{-1}\left(\frac{x}{2}\right) = -2, -1, 0, 1\). or

\(\frac{x}{2} = -\sin 2, -\sin 1, 0, \sin 1,\)

\(\Rightarrow x = -2 \sin 2, -2 \sin 1, 0, 2 \sin 1\).

Now when \(x = -2 \sin 2, -2 \leq y < -1\).

\(\therefore x = -2 \sin 1, -1 \leq y < 0; x = 0, 0 \leq y < 1; x = 2 \sin 1, 1 \leq y < 2.\)
Ex. 13. Find the domain and range of \( f(x) = \frac{e^x}{1 + x} \) for all \( x \in \mathbb{R} \).

Sol. Let \( f(x) = \frac{g(x)}{F(x)} \), where \( g(x) = e^x \) and \( F(x) = 1 + [x] \), now domain of \( g(x) = e^x \) is set of all values of \( x \) as it is defined \( \forall x \in \mathbb{R} \).

\( \therefore \) Domain of \( e^x \) is the set of all real numbers.

Also, domain of \( f(x) = 1 + [x] \) is set of all values of \( x \in \mathbb{R} \). But \( 1 + [x] = 0 \) \( \forall -1 \leq x < 0 \).

\( \therefore \) Domain of \( f(x) \) will be \( \mathbb{R} - \{-1 \leq x < 0\} = (-\infty, -1) \cup [0, \infty) \).

For range \( x \geq 0, 1 + [x] \geq x, \forall x \in [0, \infty) \) and \( e^x \geq 1. \forall x \in [0, \infty) \).

For \( x \in [0, 1] \), \( f(x) = e^x \) and for \( x \in [1, 2] \), \( f(x) = \frac{e^x}{2} \) and so on.

Thus all values of \( y \) or \( e \geq 1. \therefore \) Range \( f(x) \in [1, \infty) \).

Ex. 14. Evaluate: \( \lim_{n \to \infty} \frac{\{x\} + \{2x\} + \{3x\} + \ldots + \{nx\}}{n^2} \) where \( \{x\} \) is the fractional part of \( x \).

Sol. \( \therefore 0 \leq \{rx\} < 1 \) where \( r = 1, 2, 3, \ldots, n \).

\[ 0 \leq \sum_{r=1}^{n} \{rx\} < \sum_{r=1}^{n} (1) = \sum_{r=1}^{n} \{rx\} < n. \]

Dividing throughout by \( n^2 \), we get

\[ 0 \leq \frac{\sum_{r=1}^{n} \{rx\}}{n^2} < \frac{1}{n}, \Rightarrow \lim_{n \to \infty} 0 \leq \lim_{n \to \infty} \frac{\sum_{r=1}^{n} \{rx\}}{n^2} < \frac{1}{n^2}, \]

\[ \Rightarrow 0 \leq \lim_{n \to \infty} \frac{\sum_{r=1}^{n} \{rx\}}{n^2} < 0. \]
\[ 0 \leq \lim_{n \to \infty} \frac{\{x\} + \{2x\} + \ldots + \{nx\}}{n^2} < 0. \]

According to Sandwich theorem or Squeeze principle

\[ \lim_{n \to \infty} \frac{\{x\} + \{2x\} + \ldots + \{nx\}}{n^2} = 0. \]

**Ex. 15.** If \([x]\) denotes the integral part of \(x\) and

\[
 f(x) = [x] \left( \sin \frac{\pi}{x+1} + \sin \pi \frac{x+1}{x+1} \right)
\]

\[ = \frac{[x]}{1+\{x\}} \sin \frac{\pi}{[x]+1}, \]

then show that \(f(x)\) is discontinuous at all integral points.

**Sol.** \(\sin \pi [x+1] = 0\) and \([x+1] = [x] + 1\).

\[ \therefore f(x) = \frac{[x]}{1+\{x\}} \sin \frac{\pi}{[x]+1}. \]

At \(x = n, n \in \mathbb{I}\), then \(f(x) = \frac{n}{1+n} \sin \frac{\pi}{n+1}\). For \(n < x < n+1, n \in \mathbb{I}\),

\[ f(x) = \frac{n}{1+n} \sin \frac{\pi}{n+1}. \therefore \lim_{x \to n+0} f(x) = \frac{n}{1+n} \sin \frac{\pi}{n+1}. \]

For \(n-1 < x < n, [x] = n-1\). \(\therefore f(x) = \frac{n-1}{n} \sin \frac{\pi}{n}\).

\[ \therefore \lim_{x \to n-0} f(x) = \frac{n-1}{n} \sin \frac{\pi}{n} \text{ and } f(x) = \frac{n}{1+n} \sin \frac{\pi}{n+1}. \]

Hence \(f(x)\) is discontinuous at all \(n \in \mathbb{I}\).

**Ex. 16.** Let \([x]\) stands for the g.i.f. Find the derivative of \(f(x) = (x + [x^3 + 1])^{x^2 \sin x}\), wherever it exists in \((1, 3/2)\). Indicate the points where it does not exist.

**Sol.** We see that \([x^3 + 1]\) takes a jump from 2 to 3 at \(2^{1/3}\) and again from 3 to 4 at \(3^{1/3}\) in \((1, 3/2)\).

\[ \therefore \text{It is discontinuous at } x = 2^{1/3} \text{ and } x = 3^{1/3}, \text{i.e. it is not differentiable at } \]
$x = 2^{1/3}$ and $x = 3^{1/3}$.

Now, $f(x) = (x + [x^3 + 1])^{x^2 + \sin x} = \begin{cases} (x+2)^{x^2+\sin x} & \text{if } 1 < x < 2^{1/3} \\ (x+3)^{x^2+\sin x} & \text{if } 2^{1/3} < x < 3^{1/3} \\ (x+4)^{x^2+\sin x} & \text{if } 3^{1/3} < x < \frac{3}{2} \end{cases}$

$f''(x) = \begin{cases} (x+2)^{x^2+\sin x} \left(2x+\cos x\right) \log\left((x+2)+\frac{x^2+\sin x}{x+2}\right) & \text{if } 1 < x < 2^{1/3} \\ (x+3)^{x^2+\sin x} \left(2x+\cos x\right) \log\left((x+3)+\frac{x^2+\sin x}{x+3}\right) & \text{if } 2^{1/3} < x < 3^{1/3} \\ (x+4)^{x^2+\sin x} \left(2x+\cos x\right) \log\left((x+4)+\frac{x^2+\sin x}{x+4}\right) & \text{if } 3^{1/3} < x < \frac{3}{2} \end{cases}$

Ex. 17. Evaluate:

(i) $\int_0^n [x] \, dx$; \hspace{1cm} (ii) $\int_0^n \left( [x] - \left[x - \frac{1}{2}\right] \right) \, dx$;

(iii) $\int_0^{\pi} \sin\left(\frac{2x}{n}\right) \, dx$; \hspace{1cm} (iv) $\int_{-10}^{2[x]} \frac{2[x]}{3x-[x]} \, dx$; \hspace{1cm} (v) $\int_{-1}^{\frac{\pi}{\sqrt{2}}} \frac{\sin^2 x}{\sqrt{2} + \frac{1}{2}} \, dx$

(vi) $\int_0^{[n\e^{-x}]} \, dx$, where $[x] = \text{G.I.F.}$; \hspace{0.5cm} $\{x\} = \text{F.P.}$ and $n$ is a natural number.

17. (i) Let $I = \frac{n \int \{x\} \, dx}{\int_0^n \{x\} \, dx}$
\[
\begin{align*}
0 + 1 \cdot \int_0^1 dx + 2 \int_1^2 dx + \ldots + (n-1) \int_{n-1}^n dx &= \frac{(n-1)n/2}{n} = (n-1).
\end{align*}
\]

(ii) \[
\int_0^1 \left( x - \left[ x + f\left( \frac{1}{2} \right) \right] \right) dx \quad \text{let } x = \left[ x \right] + f, 0 \leq f < 1
\]
\[
= \int_0^1 \left[ -f\left( \frac{1}{2} \right) \right] dx, \quad 0 \leq f < 1, \quad -\frac{1}{2} \leq f \left( \frac{1}{2} \right) < \frac{1}{2}, \quad \text{i.e.} \left[ f\left( \frac{1}{2} \right) \right] = -1, 0.
\]

Case 1. If \( \left[ f\left( \frac{1}{2} \right) \right] = 0 \), then \[
\int_0^1 \left( \int_0^1 1 \cdot dx \right) dx = 0.
\]

Case 2. If \( \left[ f\left( \frac{1}{2} \right) \right] = -1 \), then \[
\int_0^1 \left( \int_0^1 1 \cdot dx \right) dx = 0
\]

\[
\int_0^1 [x] dx = \int_0^1 [x] + \int_1^2 [x] + \int_2^3 [x] + \ldots + \int_{[x]-1}^x [x] dx = 0 + 1 + 2 + \ldots + ([x]-1) = \frac{\pi}{2}.
\]

(iii) \[
\int_0^\pi \sin \left[ \frac{2x}{\pi} \right] dx = \int_0^\pi \sin 0 dx + \int_0^{\pi/2} \sin 1 dx + \int_0^{\pi/2} \sin 2 dx + \ldots + \int_0^{\pi/2} \sin (2n-1) dx = \int_0^{\pi/2} \sin (2n-2) dx + \int_0^{\pi/2} \sin (2n-1) dx = \int_0^{\pi/2} \sin (2n-4) dx + \int_0^{\pi/2} \sin (2n-3) dx + \ldots + \int_0^{\pi/2} \sin (2n-1) dx.
\]
\[
\frac{\pi}{2} \left\{ \sin 1 + \sin 2 + \sin 3 + \ldots + \sin (2n-2) + \sin (2n-1) \right\} = \\
\frac{\pi}{2} \left[ \frac{\sin \left(a + \frac{(2n-2)}{2}\right)}{\sin \left(\frac{h}{2}\right)} \times \sin \left(\frac{(2n-1)h}{2}\right) \right] \text{ where } a = 1, \ h = 1,
\]
\[
= \frac{\pi}{2} \left[ \frac{\sin \left(1 + \frac{2(n-1)}{2}\right)}{\sin \frac{1}{2}} \times \frac{\sin \frac{(2n-1)}{2}}{2} \right] = \frac{\pi}{2} \cdot \frac{\sin \frac{2n-1}{2}}{\sin \frac{1}{2}}
\]

(iv) Let \( f(x) = \left\lfloor \frac{2x}{3x-[x]} \right\rfloor \).

Clearly \( f \) is not defined if \( x = 0 \) and when \( 3x = [x] \).

\[ \therefore \text{ In } (-10, 0), \ f \text{ is not defined at } x = -\frac{1}{3}. \]

Case 1. \( x \in \left( -10, -\frac{1}{3} \right) \), \( [x] < 0 \) and \( 3x-[x] < 0 \)

\[ \left\lfloor \frac{[x]}{3x-[x]} \right\rfloor > 0, \ \text{i.e.} \ f(x) = 1. \]

Case 2. \( x \in \left( -\frac{1}{3}, 0 \right) \), \( [x] < 0 \) and \( 3x-[x] > 0 \), i.e. \( f(x) = -1. \)
Now, \( \int_{-10}^{0} f(x) \, dx = \int_{-10}^{0} dx + \int_{-10}^{-\frac{1}{3}} (-1) \, dx = (x)_{-10}^{-\frac{1}{3}} - (x)_{-10}^{0} = \left( -\frac{1}{3} + 10 \right) \)

\[ -\left( 0 + \frac{1}{3} \right) = \frac{28}{3} \]

(v) Let \( I = \int_{-1}^{1} f(x) \, dx \), where \( f(x) = \frac{\sin^2 x}{[x/\sqrt{2}] + 1/2} \)

\[ f(-x) = \frac{\sin^2 x}{[-x/\sqrt{2}] + 1/2} = \sin^2 x \quad \therefore [x] - [-x] = -1 \forall x \in \mathbb{R} \]

\[ \therefore x/\sqrt{2} \text{ is not an integer in } (-1, 0) \text{ and } (0, 1). \]

\[ f(-x) = \frac{\sin^2 x}{[x/\sqrt{2}] - 1/2} = -f(x), \text{ i.e. } f(x) \text{ is an odd function in } x \]

\[ I = \int_{-1}^{1} f(x) \, dx = 0. \]

(vi) Let \( f(x) = n e^{-x} \forall n \in \mathbb{N}, x \in \mathbb{R}^+ \).

\[ \therefore f'(x) = -n e^{-x} < 0 \forall x \in \mathbb{R}. \]

This is decreasing \( 0 \) to \( \infty \) and in such type of problems we always break the internal for \( x > \log \ln n \), we have \([n e^{-x}] = 0\).

\[ \therefore \int_{0}^{\infty} [x e^{-x}] \, dx = \int_{0}^{\log \ln n} (x-1) \, dx + \int_{\log \ln n}^{\log \ln n - 1} (x-2) \, dx + \int_{\log \ln n - 1}^{\log \ln n - 2} (x-3) \, dx + \ldots \ldots \]

\[ \int_{\log \ln n - 1}^{\log \ln n} 1 \, dx + \int_{\log \ln n}^{\infty} 0 \, dx = (n-1) \left( \log \ln n - 0 \right) + (n-2) \left( \log \ln n - 1 - \log \ln n - 2 \right) + \ldots \ldots + (\log \ln n - 2) \left( \log \ln n - 3 - \log \ln n - 2 \right) + \ldots \ldots + \log \ln n - 0 \left( \log \ln n - \log \ln n \right) + \ldots \ldots + 0 \left( \infty - \log n \right). \]
Now writing the terms in reversed order, we have
\[ \int_{0}^{\infty} x e^{-x} dx = 1 \left( \log \frac{n}{2} - \log \frac{n}{1} \right) + 2 \left( \log \frac{n}{3} - \log \frac{n}{2} \right) + 3 \left( \log \frac{n}{4} - \log \frac{n}{3} \right) + \ldots + (n-2) \left( \log \frac{n}{n-1} - \log \frac{n}{n-2} \right) + (n-1) \log \frac{n}{n-1} \]
\[ = \log n + (2-1) \log \frac{n}{2} + (3-2) \log \frac{n}{3} + \ldots + [(n-1) - (n-2)] \log \frac{n}{n-1} \]
\[ = \log n + \log \frac{n}{2} + \log \frac{n}{3} + \ldots + \log \frac{n}{n-1} = \log \left( \frac{n \cdot n \cdot \ldots \cdot n}{1 \cdot 2 \cdot \ldots \cdot (n-1)} \right) \]
\[ = \log \left( \frac{n^{n-1}}{(n-1)!} \right) = \log \left( \frac{n^{n-1}}{(n-1)!} \right) = \log \left( \frac{n^n}{n!} \right) \]

Ex. 18. Let \( f(x) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \in I \\ 0, & \text{if } x \notin I \end{cases} \) where \([.]\) denotes the G.I.F. If \( g(x) = \max \{x^2, f(x), |x|\}, \forall x \in [-10, 10] \), then find the value of \( \int_{-2}^{2} g(x)dx \).

Sol. It is clear from the graph
\[
go(x) = \begin{cases} x^2, & \text{if } -2 \leq x \leq -1 \\ -x, & \text{if } -1 \leq x \leq -\frac{1}{4} \\ x + \frac{1}{2}, & \text{if } -\frac{1}{4} \leq x \leq 0 \\ x, & \text{if } 0 \leq x \leq 1 \\ x^2, & \text{if } 1 \leq x \leq 2 \end{cases} \]
\[ \therefore \int_{-2}^{2} f(x)dx = \int_{-2}^{-1} x^2 dx + \int_{-1}^{-\frac{1}{4}} (-x)dx + \int_{-\frac{1}{4}}^{0} \left( x + \frac{1}{2} \right) dx + \int_{0}^{1} x dx = \frac{275}{98} \]

Ex. 19. Let the curves \( C_1: y^2 = 4[\sqrt{y}]x \) and \( C_2: x^2 = 4[\sqrt{x}]y \) where \([.]\) denotes the G.I.F., find the area of the region enclosed by these two curves with in the square formed by the lines, \( x=1, y=1, x=4, y=4 \).
Sol. Since $1<x<4$, so $1<\sqrt{x} < 2$. \( \therefore \lfloor \sqrt{x} \rfloor = 1 \) and similarly \( \lfloor \sqrt{y} \rfloor = 1 \).

Now, $c_1: y^2 = 4x$ and $c_2: x^2 = 4y$.

\[ \therefore \text{Reqd. area} = \int_1^4 2\sqrt{x} \, dx - (2-1) \int_1^4 \frac{x^2}{4} \, dx \]

\[ = \frac{4}{3} x^{3/2} \bigg|_1^4 - \frac{1}{4} x^3 \bigg|_1^2 = \frac{11}{3} \text{ sq.units.} \]

Ex. 20. If \( x \) is a real number in \([0,1]\), then show that the value of \( \lim_{m \to \infty} \lim_{n \to \infty} \left\{ 1 + \cot^2 m (n! \pi x) \right\} \) is given by 2 or 1 according as \( x \) is rational or irrational.

Sol. Case I. If \( x \in \mathbb{Q} \), then \( n! \pi x \) will be an integral multiple of \( \pi \) for large values of \( n \).

\( \therefore \cos (n! \pi x) \) will be either 1 or \(-1\), i.e. \( \cos^2 m (n! \pi x) = 1 \)

\[ \therefore \lim_{m \to \infty} \lim_{n \to \infty} \{1 + \cos^2 m (n! \pi x)\} = 1 + 1 = 2. \]

Case 2. If \( x \in \mathbb{Q} \), then \( x! \pi x \) will not be an integral multiple of \( \pi \).

\( \therefore \cos (x! \pi x) \) will be between \(-1\) or 1, i.e. \( \cos^2 m (n! \pi x) = 0 \).

Hence, \( \lim_{m \to \infty} \lim_{n \to \infty} \{1 + \cos^2 m (n! \pi x)\} = 1 + 0 = 1. \)

**Liming Position of a Secant**: A secant (Latin seco, I cut) is a straight line that cuts a curve at two points. If the secant in fig (i) moves in such a way that \( P \) remains fixed in position, but \( R \) approaching \( P \) more and more closely, successively occupies the position \( R_1, R_2, R_3 \) and \( R_4 \), shown in fig. (ii). It will be seen that the length of chord \( PR \) and the length arc \( PR \) both becomes successively less and less as \( R \) approaches \( P \), while all the time the secant of which \( PR \) is part approaches closer and closer to its limiting position, which is that of the tangent to the curve at point \( P \) as shown in fig. (iii).
SIGN OF $\frac{dy}{dx}$: Let $y=f(x)$. If $\frac{\Delta y}{\Delta x}$ is (+)ve $\Delta x$ and $\Delta y$ have the same sign, i.e. an increase in $x$ produces an increase in $y$ and a decrease in $x$ produces a decrease in $y$.

If $y=f(x)$ be drawn, $P$ being the point $(x,y)$ and $Q (x+\Delta x, y+\Delta y)$. Then if $\Delta x$ and $\Delta y$ both (+)ve $P$ and $Q$ will be placed as in (1), if both negative as in (2). In each case the chord $PQ$ has a (+)ve gradient. But if $\Delta x$ be (+)ve and $\Delta y$ (-)ve, $P$ and $Q$ will be placed as in (3), if $\Delta x$ (-)ve, and $\Delta y$ (+)ve, as in (4).

So, if $\frac{dy}{dx}$ is (+)ve for a given value of $x$, $x$ and $y$ are both increasing or both decreasing, but if $\frac{dy}{dx}$ is (-)ve, $x$ is increasing and $y$ decreasing or vice-versa. In the graph if $\frac{dy}{dx}$ is (+)ve at the point $(x,y)$, the gradient of the tangent is (+)ve and the curve in the neighbourhood of $P$ is shaped like (5) or (6), but if $\frac{dy}{dx}$ is (-)ve, the shape is like (7) or (8).
MAXIMA AND MINIMA:
At A and D, the gradient is changing from (−)ve to (+)ve and these are called minimum points, FA and HD are minimum (or minima) values of y. At B and E, the gradient is changing from (+)ve to (−)ve and these are called maximum (or maxima) points, GB and JE are maximum values of y.
The words max. and min. are used in the sense of greatest and least only in the immediate vicinity of the point, this local meaning is brought out clearly in this curve, since a maximum value, JE as in fact less than a minimum value FA.
At C the gradient is zero, but is not changing sign, this point is a point of inflexion, which may be linked to the point on an S-bend at which a road stops turning left and begins to turn right or vice-versa. The gradient of a curve at a point of inflexion need not be zero (the reader should be able to spot four more in given fig.), however at this stage we are concerned only with searching for maximum and minimum. We need to bear in mind points of inflexion only as a third possibility at points where the gradient of a curve is zero, At any point where the gradient of a curve is zero, y is said to be a stationary value. Any max or min point is called a turning point and y is said to be a turning value there.
Note: While a function has at most only one greatest value or absolute max. or global max and at most only one least value or absolute min. or global min. It may have several max. value or local max and several min value or local min.

DISTINGUISHING BETWEEN MAX AND MIN POINTS:
At a max point \( \frac{dy}{dx} = 0 \). Just before the point \( \frac{dy}{dx} \) is (+)ve and just after if \( \frac{dy}{dx} \) is (−)ve. Thus in passing from one side of the point to the other \( \frac{dy}{dx} \) decrease. If a graph of \( \frac{dy}{dx} \) against x is plotted, it has a downward slope
in the region of the point under consideration. Now the slope of the graph is given by \( \frac{dy}{dx} \) \( \left( \frac{dy}{dx} - x \right) \) graph is given by \( \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2y}{dx^2} \). So, at the max. point \( \frac{d^2y}{dx^2} \) is (-)ve (except as below). In fig. the graph of y is represented by a continuous line and the graph of \( \frac{dy}{dx} \) by a dotted line. In a similar way it can be seen that at a min. point \( \frac{d^2y}{dx^2} \) will be (+)ve (except as below) This provides a method of investigating the nature of a turning point.

If, at a point, \( \frac{dy}{dx} = 0 \) and \( \frac{d^2y}{dx^2} < 0 \), i.e. max. point.

If, at a point, \( \frac{dy}{dx} = 0 \) and \( \frac{d^2y}{dx^2} > 0 \), i.e. min. point.

Ex. O is a point on a straight line. A particle moves along the line so that it is s ft. from O, t see. after a certain instant, where \( s = t(t-2)^2 \). Describe the motion before and after \( t=0 \).

Sol. Here \( s = t^3 - 4t^2 + 4t \).

\[ \frac{ds}{dt} = 3t^2 - 8t + 4 = (3t - 2)(t - 2) = v. \]

This graph has a min. point \( \left( \frac{1}{3}, -\frac{1}{3} \right) \)

and passes through \( \left( \frac{2}{3}, 0 \right), (2, 0) \)

and \( (0, 4) \), it is the middle sketch and upper sketch the max pt. \( \left( \frac{2}{3}, \frac{32}{27} \right) \)

and min. point (2,0).
Again, \( \frac{dv}{dt} = a = 6t - 8 \) and is the bottom sketch.

Notice that the max. and min. values of \( s \) occur when \( v = \frac{ds}{dt} = 0 \) and that the min. value of \( v \) occurs when \( a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 0 \).

Before \( t = 0 \), the particle is approaching 0 from the (-)ve side, at \( t = 0 \) it is passing through 0 with velocity 4 ft./Sec. and acceleration \(- 8\) ft./Sec\(^2\).

Hence its speed is decreasing and it comes momentarily to rest \( \frac{32}{27} \) ft.

from 0 (on the (+)ve side), when \( t = \frac{2}{3} \), it returns to 0, where it is momentarily at rest when \( t = 2 \), and thereafter it moves away from 0 in the (+)ve direction.

**SIGN OF AREA**: If we advance from left to right, i.e. if \( \Delta x \) is (+)ve, \( y \Delta x \) representing the area of our typical strip is (+)ve if \( y \) is (+)ve, i.e. if the strip lies above the \( x \)-axis and (-)ve if the strip lies below the \( x \)-axis is given fig.

If we advance from right to left, i.e. if \( \Delta x \) is (-)ve, \( y \Delta x \) is (+)ve if the strip lies below and (-)ve if the strip lies above the \( x \)-axis.

**DIVISION BY ZERO**: Values like \( \frac{0}{0} \) \( \frac{3}{0} \) \( \frac{a}{0} \) are inadmissible. There have no meaning. Some students are under the impression that \( \frac{5}{0} \) is
Infinity which is wrong. Infinity (∞) is not a number it is a symbol.

Nor is \( \frac{0}{0} = 1 \). If we let \( \frac{0}{0} = 1 \), then we shall be led to absurdities. Consider, for example, what would be the result of dividing by zero both side of an equation like \( 5 \times 0 = 2 \times 0 \).

Thus the statement \( x = \infty \) is meaningless unless it stands for the statement \( x \to \infty \).

Corresponding to the notion of infinity, the symbol (zero) denotes the continuous decrease without limit. The student should understand the following facts also \( \frac{1}{x} \to 0 \) if \( x \to +\infty \) and \( \frac{1}{x} \to 0 \) if \( x \to -\infty \).

Ex. Show that the number 0(zero) has no reciprocals.

Sol. Suppose that 0 has the reciprocals \( \frac{1}{0} \), then \( 0 \times \left( \frac{1}{0} \right) = 1 \).

\( \therefore \) product of two reciprocals = 1).

Also, \( 0 \times \left( \frac{1}{0} \right) = 0 \) (\( \therefore \) product of any number by zero is 0 ). It follows that 1=0, which brings contradiction.

Hence 0 has no reciprocal.
NOTES ON GRAPHS

Introduction: The graphs of a function provides many information regarding the function \( f(x) \). All the information we need to draw the graph of a function accurately can be put together to a name and that is DR. T. SAM. This means Domain (D), Range (R), Transformation (T), Symmetry (S), Asymptotes (A), Maximum and Minimum (M), Now we shall discuss some important graphs.

Draw the graphs of each of the following functions:

(a) \( f(x) = \frac{2x}{1 + x^2} \).

(i) \( f(x) \) is an odd function,
(ii) graph is symmetrical about origin,
(iii) \( f(x) \) attains minimum value \( f(x) = 0 \),
(iv) Maximum value of the function \( f(x) \) equals 1.
(v) \( f(0) = 0 \) and \( |f(x)| \leq 1 \) indeed \((1-|x|)^2 \geq 0 \) or \( 1 + x^2 \geq |x|\),

or \( 1 \geq \frac{2|x|}{1 + x^2} = |f(x)| \)

Since \( f(x) \geq 0 \) at \( x \geq 0 \) and \( f(1) = 1 \), in the interval \([0, \infty)\) the maximum value of the function \( f(x) \) equals 1, the minimum value being zero.

Domain: \(-\infty < x < \infty\).

Range: \(-1 \leq f(x) \leq +1\)

(b) \( f(x) = \sin^2 x - 2 \sin x = (\sin x - 1)^2 - 1 \).

When \( f(x) \) increases then \( \sin x \) decreases.
Similarly, \( f(x) \) decreases when \( \sin x \) increases.

When \( \sin x \) increases

\((-\pi/2 \leq x \leq \pi/2) \) \( \sin x \) decreases

\((\pi/2 \leq x \leq 3\pi/2) \).
(c) \( f(x) = x^4 - 2x^2 + 3 = (x^2 - 1)^2 + 2 \)
(i) \( f(x) \) is even function,
(ii) \( f(x) \) is symmetric about y-axis,
(iii) \( f(x) \) is attains min. value at \( x = \pm 1 \),
(iv) \( f(x) \) decreases \( x=0 \) to 1 and similarly \( x=0 \) to -1,
(v) \( f(x) \) increases \( x=1 \) to \( \infty \),
(vi) \( f(0)=3, \ D_f: -\infty < x < \infty \) and \( R_f: 2 \leq y < \infty \).

(d) \( y = f(x) = \cos^{-1}(\cos x) \)
\( f(x) \) is periodic with period \( 2\pi \).
\( f(x) = x, 0 \leq x \leq \pi \) \[\text{[from definition of } \cos^{-1} x]\]
\[= 2\pi - x, \quad \pi \leq x \leq 2\pi \]

Let \( 2\pi - x = x' \)

or \( 2\pi - x' = x \)

\[\pi \leq 2\pi - x' \leq 2\pi \]
\[-\pi \geq x' - 2\pi \geq -2\pi \]
\[\pi \geq x' \geq 0 \]
\[0 \leq x' \leq \pi \]

\( \cos^{-1} \cos(2\pi - x') = \cos^{-1} \cos x' = x' = 2\pi - x \)

Domain: \( -n\pi \leq x \leq n\pi \) \[\text{[1,2,...=n]}\]
Range: \( 0 \leq f(x) \leq \pi \)

(e) \( y = f(x) = \sqrt{\sin x} \)

Here \( \sin x \geq 0, 0 \leq x \leq \pi \)
\[2\pi \leq x \leq 3\pi \]
\[2n\pi \leq x \leq (2n + 1)\pi \]
\[n=0,1,2,...] \]

Domain: \( 2n\pi \leq x \leq (2n + 1)\pi \) \[n = 0, 1, 2... \]
Range: \( 0 \leq f(x) \leq 1 \)
(f) \( y = f(x) = x^{1/ \log x} \)

Domain: \( 0 < x < 1 \) and \( 1 < x < \infty \)

\[
\begin{align*}
f(x) &= x^{1/ \log x} \\
&= x^{(\log_{10} x / \log_{10}^2 x)} \\
&= x^{\log_{10}^2 x}
\end{align*}
\]

\[\therefore f(x) = 10.\]

(g) \( y = \begin{cases} 
\sin x & \text{at } -\pi \leq x \leq 0 \\
2 & \text{at } 0 < x \leq 1 \\
1/(x-1) & \text{at } 1 < x \leq 4 
\end{cases} \)

(h) \( y = [x]^2 \), we have

\[
\begin{align*}
y &= 0 \text{ when } 0 \leq x < 1 \\
&= 1 \text{ when } 1 \leq x < 2 \\
&= 4 \text{ when } 2 \leq x < 3 \\
&= 9 \text{ when } 2 \leq x < 4 \\
\end{align*}
\]

and so on.

Similarly in the case of negative values.

(i) \( y = x^2 + [x]^2 \), we have

\[
\begin{align*}
y &= \begin{cases} 
x^2 & \text{if } x \in [0, 1]\[0.5em]
x^2 + 1 & \text{if } x \in [1, 2] \\
x^2 + 4 & \text{if } x \in [2, 3] \\
x^2 + 1 & \text{if } x \in [-1, 0] 
\end{cases}
\end{align*}
\]

and so on.

(j) \( y = \begin{cases} 
-2 & \text{at } x > 0 \\
1/2 & \text{at } x = 0 \\
-x^3 & \text{at } x < 0
\end{cases} \)
(k) \[ y = \cos x + |\cos x| \]
\[ y = \cos x + \cos x \quad \cos x \geq 0 \]
\[ y = \cos x - \cos x \quad \cos x < 0 \]
\[ y = \cos 2x \quad -\pi/2 \leq x \leq \pi/2 \]
\[ y = 0 \quad \pi/2 \leq x \leq 3\pi/2 \]

(i) \[ y = |x + 2|x \]
\[ y = (x + 2)x \quad x \geq -2 \]
\[ = -(x + 2)x \quad x \leq -2 \]
\[ y = x^2 + 2x \]
\[ = (x + 1)^2 - 1 \quad x \geq -2 \]
\[ y = 1 - (x + 1)^2 \quad x \leq -2 \]

(m) \[ y = 2|x - 2| - |x + 1| + x \text{ at } x \geq 2 \]
\[ y = 2(x - 2) - (x + 1) + x \]
\[ = 2x - 5 \quad -1 \leq x \leq 2 \]
\[ y = 2(2-x) - (x + 1) + x \]
\[ = -2x + 3 \quad x \leq -1 \]
\[ y = 2(2-x) + (x + 1) + x \]
\[ = 5 \]

(n) \[ y = 2^x - 2^{-x} \]
Let us define domain as \((-1,1)\)
\[ y = 2^x - 2^{-x} (-1,1) \]
\[ y = 2^{-1} - 2^1 \]
\[ = -3/2 \]
\[ y = 2^1 - 2^{-1} \]
\[ = 3/2 \]
\[ y = 2^0 - 2^{-0} \]
\[ = 0 \]

(o) \[ y = x\sin x : 0 \leq x \leq \pi; \]
\[ y = x\sin x : 0 \leq y \leq \pi /2 ; \]
\[ 0 \leq x \leq \pi /2 ; \]
\[ : \pi /2 \leq y \leq 0 : \pi /2 \leq x \leq \pi ; \]
\( p \) \[ y = \frac{1}{(x^2 - 9)} \]
\[ = \frac{1}{(x+3)(x-3)} \]
At \( x = 3 \), -3 the curve is undefined, i.e., the two asymptotes are \( x = 3 \) and \( x = -3 \).

\( q \) \[ y = \begin{cases} (x^2 + x + 1) & : -1 \leq x \leq 0 \\ \sin^2 x & : 0 \leq x \leq \pi \\ (x-1)/(x+1) : \pi < x \leq 5 \end{cases} \]
\[ \therefore y = \left( x + \frac{1}{2} \right)^2 + \frac{3}{4} \]
\[ y = \sin^2 x : 0 \leq x \leq \pi \]
\[ y = \frac{x-1}{x+1} : \pi < x \leq 5\pi \]
\[ = \frac{x+1-2}{x+1} = 1 - \frac{2}{x+1} \]

\( r \) \[ y = x + \frac{1}{x} \]

\( s \) \[ y = x^2 - x^3 \]
\[ y = x^2(1-x) \]
(t) \( y = 3 \sin(2x-4) \) \[ 0 \leq 2x-4 \leq 2\pi \]
\[ 4 \leq 2x \leq 2\pi + 4 \]
\[ 2 \leq x \leq \pi + 2 \]

(u) \( y = 2\sqrt{(-3(x+1.5)) - 1.2} \)
\[ y = 0 \]
\[ 1.2 = 2\sqrt{-3(x+1.5)} \]
\[ 1.44 = -3(x+1.5) \]
\[ \frac{-12}{-3} - 1.5 = x \]
\[ -1.62 = x \]

(v) \( y = |x^2 - 2x - 1| \)
\( (x^2 - 2x - 1) > 0 \) or \( x \geq 1 + \sqrt{2} \)
\( (x^2 - 2x - 1) < 0 \) or \( x \leq 1 - \sqrt{2} \)
\[ 1 - \sqrt{2} \leq x \leq 1 + \sqrt{2} \]
\( y = (x-1)^2 - 2 \)
\( y = -(x-1)^2 + 2 \)

(w) \( y = |x-1| \)
\[ |x-1| \geq 0 \]
\[ x \geq 1 \text{ or } x \leq -1 \]
\[ = |x-1| \]
\[ -1 \leq x \leq 1 \]
\[ y = |x|-1 \]
\[ x \geq 1 \text{ or } x \leq -1 \]
\[ y = x - 1 \]
\[ x \geq 1 \]
\[ y = -x - 1 \]
\[ x \leq -1 \]
Similarly
\[ y = 1 - x \quad 0 \leq x \leq 1 \]
\[ y = 1 + x \quad -1 \leq x \leq 0 \]

Arranging all parts
\[ y = -x - 1 \quad x \leq -1 \]
\[ y = 1 + x \quad -1 \leq x \leq 0 \]
\[ y = 1 - x \quad 0 \leq x \leq 1 \]
\[ y = x - 1 \quad x \geq 1 \]

\[(x) \quad y = |\sin x| + \sin x \text{ on the interval } [0, 3\pi]\]
\[ = \sin x + \sin x \quad \sin x \geq 0 \quad 0 \leq x \leq \pi \]
\[ = 2 \sin x \quad 2\pi \leq x \leq 3\pi \]
\[ = -\sin x + \sin x \quad \sin x \leq 0 \quad \pi \leq x \leq 2\pi \]
\[ = 0 \]

\[(y) \quad y = -|x + 2|\]
\[ y = -(x + 2) \quad x + 2 \geq 0; \quad x \geq -2 \]
\[ = (x + 2) \quad x + 2 < 0; \quad x \leq -2 \]

\[(z) \quad y = |x - 1| - 2\]
\[ y = |x - 1| - 2 \quad |x - 1| \geq 2 \]
\[ = -(|x - 1| - 2) \quad |x - 1| < 2 \]
\[ y = (x - 1) - 2 \quad x \geq 3 \]
\[ y = x - 3 \]
\[ y = -x - 1 \quad x \leq -1 \]
\[ y = -x + 3 \quad +1 \leq x \leq 3 \]
\[ y = x + 1 \quad -1 \leq x \leq 1 \]
(A) \( y = |x + 2| + |x - 3| \)
\[ y = (x + 2) + (x - 3) \]
\[ y = 2x - 1 \]
\[ y = (x + 2) + (3 - x) \]
\[ y = 5 \quad x \leq 3 \]
\[ y = -(x + 2) + (3 - x) \]
\[ x \leq -2 \quad x \geq 3 \]

Not possible

\[ y = -(x + 2) + (3 - x) \quad x \leq -2 \text{ and } x \leq 3 \]
\[ y = -2x + 1 \quad x \leq -2 \]

(B) \( y = \frac{x}{|x|} \)
\[ y = x + \frac{x}{|x|} \quad x \geq 0 \]
\[ = x + 1 \quad x \geq 0 \]
\[ = x - 1 \quad x \leq 0 \]

\[ y = x + |x - 1| + \frac{|x - 2|}{x - 2} \]

(C) \( y = x + |x - 1| + 1 \quad x \geq 2 \)
\[ = x + x - 1 + 1 \quad x \geq 2 \]
\[ y = 2x \quad x \geq 2 \]
\[ y = x + x - 1 - 1 \quad 1 \leq x \leq 2 \]
\[ = 2x - 2 \]
\[ y = x + 1 - x - 1 \quad x \leq 1 \]
\[ = 0 \]

(D) \( \frac{y}{x+1} = 1 \)

At \( x = -1 \), the curve does not exist, i.e. undefined.
(E) \[|y| + x = -1\]
\[y + x = -1\]
\[y = -x - 1 \quad y \geq 0\]
\[-y + x = -1 \quad y \leq 0\]
\[= x + 1\]

(F) \[|x| + |y| = 2\]
\[x + y = 2 \quad x \geq 0; y \geq 0\]
\[x - y = 2 \quad x \geq 0; y \leq 0\]
\[-x + y = 2 \quad x \leq 0; y \geq 0\]
\[-x - y = 2 \quad x \leq 0; y \leq 0\]

(G) \[|y - 3| = |x - 1|\]
\[y - 3 = x - 1 \quad y \geq 3; x \geq 1\]
\[y = x + 2\]
\[y - 3 = 1 - x \quad y \geq 3; x \leq 1\]
\[y = 4 - x\]
\[3 - y = x - 1 \quad y \leq 3; x \geq 1\]
\[y = 4 - x\]
\[3 - y = 1 - x \quad y \leq 3; x \leq 1\]
\[y = 2 + x\]

(H) \[|x + y| + |x - y| = 4 \quad (x + y) > 0\]
\[(x + y) + (x - y) = 4 \quad x - y > 0\]
\[2x = 4 \quad x < -y\]
\[x = 2 \quad x > y\]
\[(x + y) - x + y = 4 \quad x + y > 0\]
\[2y = 4 \quad x - y < 0\]
\[y = 0\]
\[(x+y)-x+y=4\quad x+y>0\]
\[2y=4\quad x-y<0\]
\[y=2\quad x-y<0\]
\[-(x+y)-(x-y)=4\quad -(x+y)+(x-y)=4\]
\[-x-y+x+y=4(x+y)<0\quad -2y=4\]
\[-2x=4\quad y=-2\]
\[x=-2\]

(I) \[y \mid x=x\]
\[yx=x\quad y>0\quad x(y-1)=0\]
\[-yx=x\quad y<0\quad x=0; y=1\]
\[x+yx=0\quad y=-1\]
\[x=0\]

(J) \[x-y \mid +y=0\]
\[x-y+y=0\quad x-y>0\]
\[x=0\quad x>y\]
\[-x+y+y=0\quad x<y\]
\[x=2y\]
\[y=\frac{x}{2}\]

(K) \[y=2\left|\begin{array}{c}
-x-x^2 \\
-x-x^2
\end{array}\right|, x>0\]
\[=2-x-x^2\]
\[y=2+x-x^2, x<0\]
\[=-(x^2+x-2), x>0\]
\[=-\left\{(x+\frac{1}{2})^2-\frac{9}{4}\right\}, x>0\]
\[y=-(x^2-x-2)\]
\[ y = -\left[ \left( x - \frac{1}{2} \right)^2 - \frac{9}{4} \right] \quad x < 0 \]

\[(L) \quad y = |x^2 + x|\]

\[= |(x^2 + \frac{1}{2})^2 - \frac{1}{4}|\]

\[y = \left( x + \frac{1}{2} \right)^2 - \frac{1}{4}, \left( x + \frac{1}{x} \right)^2 - \frac{1}{4} \geq 0\]

\[= -\left\{ \left( x + \frac{1}{2} \right)^2 - \frac{1}{4} \right\}, \left( x + \frac{1}{x} \right) - \frac{1}{4} < 0\]

\[\frac{1}{4} - \left( x + \frac{1}{2} \right)^2 \geq 0\]

\[\left( x + \frac{1}{2} \right)^2 - \frac{1}{4} \leq 0\]

\[\left( x + \frac{1}{2} + \frac{1}{2} \right) \left( x + \frac{1}{2} - \frac{1}{2} \right) \leq 0\]

\[(M) \quad y = 3^{-|x|}\]

\[= 3^{-x} \quad x \geq 0\]

\[= 3^{+x} \quad x < 0\]

\[(N) \quad y = \log_2(1-x) : (1-x) = 2\]

\[x = -1\]

\[1 - x = 0\]

\[x = 1\]
(O) \[ y = |\log_2 (1-x)| \]
\[ = \log_2 (1-x) \quad \log_2 (1-x) \geq 0 \]
\[ \geq \log_1 \infty \leq x \leq 0 \]
\[ = -\log_2 (1-x) \log_2 (1-x) \leq 0 \]
\[ 0 \leq x < 1 \]

(P) \[ y = x \sin x \]
\[ -x \leq x \sin x \leq x \]
\[ \Rightarrow -x \leq y \leq x \]
\[ y = x \sin \pi x, \text{ same graph but instead of} \]
\[ x = 0, \frac{\pi}{2}, \pi; \quad \pi = 0, 1, \frac{1}{2} \]

(Q) \[ y = \frac{|\ln x|}{\ln x} \]
\[ 1 \ln x \geq 0 \]
\[ = \frac{\ln x}{\ln x} \quad x \geq 1 \]
\[ = 1 \]
\[ y = -\frac{\ln x}{\ln x} \quad 1 \ln x \leq 0 \]
\[ = -1 \]

(R) \[ y = x \log_2 x \]
\[ y = 2 \]

(S) \[ y = e^{\ln x} \]
\[ = e^{\ln x} : x \geq 1 \]
\[ = x \]
\[ = e^{-\ln x} = e^{-\ln x} \]
\[ = \frac{1}{x} \]
(T) \( y = \log_2 (x^2 - 2x) \)
\[ y = \log_2 ((x-1)^2 - 1) \]
\[ (x-1)^2 - 1 = 1 \]
\[ (x-1) = \pm \sqrt{2} \]
\[ x = 1 \pm \sqrt{2} \]
\[ (x-1)^2 - 1 = 0 \]
\[ x = 2, 0 \]

(U) \( y = \log_2 \left| \frac{x}{x-1} \right| \)
\[ = \log_2 \left( \frac{x}{x-1} \right) : \frac{x}{x-1} \geq 0 \]
\[ \begin{array}{l}
  x > 1 \\
  x < 0 
\end{array} \]
\[ = \log_2 \left( \frac{x}{x-1} \right) : \frac{x}{x-1} \geq 0 \]
\[ \begin{array}{l}
  0 < x < 1 \\
  \text{at } x = 1: y \to : \end{array} \]
\[ \text{at } x = 0 y \to -\infty \]

(V) \( y = \log_2 \sin x : \sin x > 0 \)
\[ 0 < x < \pi \]
\[ 2\pi < x < 3\pi \]

(W) \(|y| = \log_2 (-x)\)
\[ y = \log_2 (-x) : y > 0 \]
\[ x < 0 \]
\[ y = -\log_2 (-x) : y < 0 \]
[\]
(X) \[ y = |\sin x| \]
\[ = \begin{cases} 
\sin x & \sin x > 0 \\
0 & 0 < x \leq \pi \\
-\sin x & \sin x < 0 \\
2\pi < x < 3\pi \\
\end{cases} \]
\[ = \begin{cases} 
\sin x & x < 0 \\
0 & 0 < x \leq 2\pi \\
-\sin x & x > 2\pi \\
3\pi < x < 4\pi \\
\end{cases} \]

(Y) \[ y = \frac{|\sin x|}{\sin x} \]
\[ = \begin{cases} 
1 & \sin x > 0 \\
0 < x \leq \pi \\
-1 & \sin x < 0 \\
\pi < x < 2\pi \\
\end{cases} \]

(Z) \[ y = x + \sin x \]
\[ \therefore -1 \leq \sin x \leq 1 \]
\[ \therefore x - 1 \leq x + \sin x \leq x + 1 \]
\[ \Rightarrow x - 1 \leq y \leq x + 1 \]

(Z₁) \[ y = \sin |x| \]
\[ = \sin x & x > 0 \\
= -\sin x & x < 0 \]

(Z₂) \[ y = \sin^{-1}(\log_{1/2} x) \]
\[ x = 1/2 \]
\[ y = \log_{1/2} x \]
\[ \log_{1/2}(1/2) = 1 \]
\[ \log_{1/2} 1 = 0 \]
\[ \log_{1/2} 2 = \frac{\log_2 2}{\log_2(1/2)} = \frac{1}{-1} = -1 \]
SOME MORE GRAPHS OF THE FOLLOWING FUNCTIONS:

\[
y = \frac{2}{x-1} \quad x \neq 1 \quad y \leq \frac{1}{x} \quad y \geq x-1
\]

\[
y = \frac{1}{2+x} \quad y \geq -\frac{1}{2} \quad y \geq 2-\frac{1}{x}
\]

\[
y = \frac{1}{|x-3|} \quad y = 1/3 \quad x = 3
\]

\[
y = 2 + \frac{1}{x} \quad y = 1-2/x \quad y = 2 + 1/x
\]

\[
y = \frac{x+1}{x-1} \quad y = 1 \quad y = 1 + 1/x
\]

\[
y = \frac{1}{2-|x|} \quad y = 1/2 \quad y = 2-1/|x|
\]

\[
y = \frac{|x-1|}{|x|-1} \quad y = 1 \quad y = 1/(1-x)
\]

\[
y = \sin(2x)
\]
<table>
<thead>
<tr>
<th>y = -\sin \frac{x}{3}</th>
<th>y = \sin(x - \frac{\pi}{3})</th>
</tr>
</thead>
<tbody>
<tr>
<td>![Graph of y = -\sin \frac{x}{3}]</td>
<td>![Graph of y = \sin(x - \frac{\pi}{3})]</td>
</tr>
</tbody>
</table>

| y = \csc |x| | y = 2^\sin x |
|---|---|
| ![Graph of y = \csc |x|] | ![Graph of y = 2^\sin x] |

<table>
<thead>
<tr>
<th>y = \cos \left(-\frac{x}{2}\right)</th>
<th>y = 2 \log_2 (2-x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>![Graph of y = \cos \left(-\frac{x}{2}\right)]</td>
<td>![Graph of y = 2 \log_2 (2-x)]</td>
</tr>
</tbody>
</table>

| y = |x|(x-2) | y = (3-x)|x+1|
|---|---|
| ![Graph of y = |x|(x-2)] | ![Graph of y = (3-x)|x+1|] |
SOME MORE GRAPHS OF THE FOLLOWING INEQUALITIES:

(a) $|x-y| \leq 1$
- $-1 < (x-y) \leq 1$  \hspace{1cm} x-y \leq 1
- $-1 \leq (x-y)$ \hspace{1cm} x-1 \leq y
- $y < x+1$ \hspace{1cm} y \geq x-1

(b) $|x-y| \geq 2$
- $(x+y) \geq 2$ \hspace{1cm} (x+y) \leq -2
- $y \leq 2-x$, $y \leq 2 -x$

(c) $|x+y| \geq 1$
- $x+y \geq 1$ \hspace{1cm} x \geq 0 \hspace{1cm} y \geq 0
- $x-y \geq 1$ \hspace{1cm} x \geq 0 \hspace{1cm} y \geq 0
- $-x+y \geq 1$ \hspace{1cm} x \leq 0 \hspace{1cm} y \leq 0
- $-x-y \geq 1$ \hspace{1cm} x \leq 0 \hspace{1cm} y \geq 0

(d) $|x+y| \leq 3$
- $x+y \leq 3$ \hspace{1cm} x \leq 0 \hspace{1cm} y \leq 0
- $x-y \leq 3$ \hspace{1cm} x \leq 0 \hspace{1cm} y \leq 0
- $-x+y \geq 3$ \hspace{1cm} x \leq 0 \hspace{1cm} y \geq 0
- $-x-y \geq 3$ \hspace{1cm} x \leq 0 \hspace{1cm} y \geq 0

(e) $|x-1| + |y+1| \geq 2$
- $(x-1)^+ (y+1) \geq 2$
- $(x-1) \geq 0 \hspace{1cm} (y+1) \geq 0$

(f) $|x+y| + |x-y| \leq 2$
- $(x-1)^+ (y+1) \geq 2$
- $(x-1) \geq 0 \hspace{1cm} (y+1) \geq 0$
- $-(x-1) - (y+1) \leq 2$
- $(x-1) \leq 0 \hspace{1cm} (y+1) \leq 0$
- $-(x-1) - (y+1) \leq 2$
- $(x-1) \leq 0 \hspace{1cm} (y+1) \geq 0$
SOME MORE GRAPHS OF THE FOLLOWING INEQUALITIES:

\[
\begin{align*}
y - 2x &\leq 0 \\
x^2 + y^2 - 5 &\leq 0
\end{align*}
\]

\[
\begin{align*}
x^2 + y^2 &\geq 1 \\
x^2 + y^2 &\geq 16
\end{align*}
\]

\[
\begin{align*}
x^2 + y^2 &\leq 0 \\
2x^2 + y - 1 &\leq 0
\end{align*}
\]

\[
\begin{align*}
y + x^2 &\leq 0 \\
y - 2x + 3 &\geq 0 \\
y + 1 &\leq 0
\end{align*}
\]

\[
\begin{align*}
y &\geq \frac{x^2}{2}
\end{align*}
\]

\[
\begin{align*}
2y &\geq x^2 \\
y &\leq -2x^2 + 3x
\end{align*}
\]

\[
\begin{align*}
3y - x &< 5 \\
y + 2x &< 11 \\
4y + x &> 9
\end{align*}
\]