SOLVING MATHEMATICAL PROBLEMS

1000 Solved Problems for Master's Entrances

By Tanujit Chakraborty

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Preface

"This book is useful for all the Masters Entrances conducted by Indian Statistical Institute. Readers are also suggested to go through the websites of ***Ctanujit Classes*** (**www.ctanujit.in** and **www.ctanujit.org)** for other online resources like topic wise lecture notes and previous years solved papers. Since 2013, I am involved with teaching for different entrance exams at ISI. I tried to cover several previous year's papers of Mathematics and Statistics portions asked in **ISI MSTAT, MMATH, MSQE, MSQMS, MTECH & PGDBA** Entrance exams. If you have any queries you can mail me at tanujitisi@gmail.com ."

With Best wishes,

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Content of the Book:

1. 15 Objective Problem Sets with Solutions (Total Solved Problems : 450) for all ISI MS/MTech/PG Entrances.
2. 100 Topic-wise Objective Solved Problems (Total Solved Problems : 100) for all ISI Entrances.
3. 10 Subjective Problem Sets with Solutions (Total Solved Problems : 100) for all ISI MS/MTech/PG Entrances.
4. 100 Subjective Problems with Solutions (Total Solved Problems : 100) on Probability & Statistics (mainly for MSTAT).
5. 15 Subjective Model Papers with Solutions (Total Solved Problems : 150) for ISI MSQE/MSQMS/MTECH Exams.
6. 100 Subjective Problems with Solutions (Total Solved Problems : 100) on Mathematics (mainly for MSQE/MSQMS/MTech).

ISI OBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 1

There will be 30 questions in MMA Paper. For each question, exactly one of the four choices is correct. You get four marks for each correct answer, one mark for each unanswered question, and zero mark for each incorrect answer.

1. How many zeros are at the end of 1000! ?

(a) 240 (b) 248 (c) 249 (d) None

Ans:- (c) The number of two's is enough to match each 5 to get a 10.

So,

$5^1 \longrightarrow 200$

$5^2 \longrightarrow 40$ ∴Thus, 1000! Ends with 249 zeros

$5^3 \longrightarrow 8$

$5^4 \longrightarrow 1$

[Theorem: (de Polinac's formula) Statement: Let p be a prime and e be the largest exponent of p such that p^e divides n! , then $e=\sum [n/p^i]$, where i is running from 1 to infinity.]

So, [1000/5]+[1000/25]+[1000/125]+[1000/625]=249 .

Thus, 1000! ends with 249 zeros.

2. The product of the first 100 positive integers ends with

(a) 21 zeros (b) 22 zeros (c) 23 zeros (d) 24 zeros.

Ans:-

$5^1 \longrightarrow 20$ (d) 24 zeros .

$5^2 \longrightarrow 4$

Alternatively, put p=5,n=100,thus from above theorem we have [100/5]+[100/25]=24 zeros as the answer.

3. Let P (x) be a polynomial of degree 11 such that P (x) = $\frac{1}{x+1}$ $for\ x = 0\ (1)11.$

Then P (12) = ?

(a) 0 (b) 1 (c) $\frac{1}{13}$ (d) none of these

Ans:- (a) P (x)= $\frac{1}{x+1}$

⇒ (x+1)[P (x)]-1 = c (x-0)(x-1)….(x-11)

Putting x= -1, 0- 1= c (-1)(-2)….(-12)

$$\Rightarrow c = -\frac{1}{12!}$$

∴ [P (x)](x+1)-1= - $\frac{1}{12!}$(x-0)(x-1)….(x-11)

⇒ P (12) 13-1 = - $\frac{1}{12!}$ 12 .11. ….2.1

$\Rightarrow$ P (12) 13-1 = -1

$\Rightarrow$P (12) = 0.

4. **Let s= $\{(x_1, x_2, x_3) | 0 \le x_i \le 9 \text{ and } x_1 + x_2 + x_3$ is divisible by 3}.**

Then the number of elements in s is

(a) 334 (b) 333 (c) 327 (d) 336

Ans:- (a) with each (x_1, x_2, x_3) identify a three digit code, where reading zeros are allowed. We have a bijection between s and the set of all non-negative integers less than or equal to 999 divisible by 3. The no. of numbers between 1 and 999, inclusive, divisible by 3 is $\left(\frac{999}{3}\right) = 333$

Also, '0' is divisible by 3. Hence, the number of elements in s is = 333 + 1= 334.

5. **Let x and y be positive real number with x< y. Also 0 < b< a < 1.**

Define E =$\log_a\left(\frac{y}{x}\right) + \log_b\left(\frac{x}{y}\right)$. Then E can't take the value

(a) -2 (b) -1 (c) -$\sqrt{2}$ (d) 2

Ans :- (d) E =$\log_a\left(\frac{y}{x}\right) + \log_b(\frac{x}{y})$ = $\frac{\log \frac{y}{x}}{\log_a} - \frac{\log \frac{y}{x}}{\log_b}$

= $\log\left(\frac{y}{x}\right)\{\frac{1}{\log_a} - \frac{1}{\log_b}\}$ = $\log\left(\frac{y}{x}\right) \{\frac{\log_b - \log_a}{(\log_a)(\log_b)}\}$

= $\log\left(\frac{y}{x}\right) . \frac{\log(\frac{b}{a})}{(\log_a)(\log_b)}$ = - $\log\left(\frac{y}{x}\right) . \frac{\log(\frac{a}{b})}{(\log_a)(\log_b)}$

Log 0< a < 1, 0< b <1 $\therefore$ $\log_a$ and $\log_b$ are both negative.

Also $\frac{y}{x} > 1$ and $\frac{a}{b} > 1$. Thus $\log\left(\frac{y}{x}\right)$ and $\log\left(\frac{a}{b}\right)$ are both positive. Finally E turns out to be a negative value. So, E can't take the value '2'.

6. **Let S be the set of all 3- digits numbers. Such that**
   - **(i) The digits in each number are all from the set {1, 2, 3, …., 9}**
   - **(ii) Exactly one digit in each number is even**

The sum of all number in S is

(a) 96100 (b) 133200 (c) 66600 (d) 99800

Ans:- (b) The sum of the digits in unit place of all the numbers in s will be same as the sum in tens or hundreds place. The only even digit can have any of the three positions,

i.e. $^3c_1$ ways.

And the digit itself has 4 choices (2, 4, 6 or 8). The other two digits can be filled in 5$\times$ 4 = 20 ways.

Then the number of numbers in S = 240.

Number of numbers containing the even digits in units place = 4 $\times$ 5 $\times$ 4 = 80

The other 160 numbers have digits 1, 3, 5, 7 or 9 in unit place, with each digit appearing

$\frac{160}{5}$ = 32 times. Sum in units place = 32 (1+ 3+ 5+ 7+ 9) + 20 (2+ 4+ 6+ 8)

= $32.5^2 + 20 \times 2 \times \frac{4\times5}{2}$ = $32\times 25 + 20 \times 20$ = 1200

∴ The sum of all numbers= 1200 (1+ 10 +10^2) = 1200× 111 = 133200.

7. Let y = $\frac{x}{x^2+1}$, Then $y^4(1)$is equals

(a) 4 (b) -3 (c) 3 (d) -4

Ans:- (b) Simply differentiating would be tedious,

So we take advantage of 'i' the square root of '-1'

y = $\frac{x}{x^2+1} = \frac{1}{2}\{\frac{1}{(x-i)} + \frac{1}{(x+i)}\}$

$\frac{d^4y}{dx^4} = \frac{1}{2}\{\frac{4!}{(x-i)^5} + \frac{4!}{(x-i)^5}\}$

Note that, $\frac{d^n}{dx^n}\left\{\frac{1}{x+a}\right\} = \frac{(-1)^n\, n!}{(x+a)^{n+1}}$

So, $y^4(x) = \frac{4!}{2}\{\frac{1!}{(x-i)^5} + \frac{1!}{(x-i)^5}\}$ Then

$y^4(1) = 12\,\{\frac{1!}{(x-i)^5} + \frac{1!}{(x-i)^5}\}$ = $12\,\{\frac{1-i}{(-2i)^3} + \frac{1-i}{(2i)^3}\}$ = $12\,\{\frac{1-i}{8i} + \frac{1-i}{8i}\}$ = $12\,(-\frac{1}{8} - \frac{1}{8})$ = -3.

8. A real 2× 2 matrix. M such that $M^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1-\epsilon \end{pmatrix}$

(a) exists for all ∊ > 0

(b) does not exist for any ∊> 0

(c) exists for same ∊> 0

(d) none of the above

Ans:- (b) since M^2 is an diagonal matrix, so M= $\begin{bmatrix} i & 0 \\ 0 & \sqrt{1-\epsilon} \end{bmatrix}$,

So, M is not a real matrix, for any values of ∊

M is a non –real matrix.

9. The value of $\left(\frac{1+i\sqrt{3}}{2}\right)^{2008}$ is

(a) $\frac{1+i\sqrt{3}}{2}$ (b) $\frac{1-i\sqrt{3}}{2}$ (c) $\frac{-1-i\sqrt{3}}{2}$ (d) $\frac{-1+i\sqrt{3}}{2}$

Ans:- (c) A = ($\frac{1+i\sqrt{3}}{2}$), $A^2 = \frac{-1+i\sqrt{3}}{2}, A^4 = \frac{-1-i\sqrt{3}}{2} = -A$

∴ $A^{2008} = (A^4)^{502} = A^4 = \frac{-1-i\sqrt{3}}{2}$.

10. Let f(x) be the function f(x)=

$$\begin{cases} \frac{x^P}{(sinx)^q} & if\ x > 0 \\ 0 & if\ x = 0 \end{cases}$$

Then f(x) is continuous at x= 0 if

(a) p > q (b) p > 0 (c) q > 0 (d) p < q

Ans:- (b) |f(x)-f(0)|= $|\frac{x^P}{(sinx)^q} - 0| \le |x^P| < \epsilon$

Whenever |x-0| < $\epsilon^{\frac{1}{p}} = \delta$ if p > 0.

So, f(x)is continuous for p > 0 at x= 0.

11. The limit $\lim_{x\to\infty} \log(1-\frac{1}{n^2})^n$ equals

(a) e^{-1} (b) $e^{-\frac{1}{2}}$ (c) e^{-2} (d) 1

Ans:- (d) L= $(1-\frac{1}{n^2})^n$

⇨ $logL = nlog(1-\frac{1}{n^2})$

⇨ $\lim_{x\to\infty} logL = \lim_{x\to\infty}[-n\{\frac{1}{n^2} + \frac{1}{2n^4} + \cdots\infty\}] = 0$

∴L = $e^0 = 1$.

12. The minimum value of the function f(x, y)= $4x^2 + 9y^2 - 12x - 12y + 14$ is

(a) 1 (b) 3 (c) 14 (d) none

Ans:- (a) f(x, y) = $4x^2 + 9y^2 - 12x - 12y + 14$

= $(4x^2 - 12x + 9)$+($9y^2 - 12y + 4$)+1

= $(2x-3)^2 + (3y-2)^2 + 1 \geq 1$

So, minimum value of f(x, y) is 1.

13. From a group of 20 persons, belonging to an association, A president, a secretary and there members are to be elected for the executive committee. The number of ways this can be done is

(a) 30000 (b) 310080 (c) 300080 (d) none

Ans:- (b) $20_{c_1} \times 19_{c_1} \times 18_{c_3}$ or $\frac{20!}{1!1!3!15!}$ = 310080

14. The $\lim_{x \to 0} \frac{\cos x - \sec x}{x^2(1+x)}$ is

(a) -1 (b) 1 (c) 0 (d) does not exist

Ans:- (a) $\lim_{x\to 0} \frac{\cos x - \sec x}{x^2(1+x)} = \lim_{x \to 0} \frac{-\sin^2 x}{\cos x\,(x^2)(x+1)}$

= - $\lim_{x \to 0} \frac{1}{\cos x}(\frac{\sin x}{x})^2 . \frac{1}{(x+1)}$ = -1. 1. 1 = - 1.

15. Let R = $\frac{48^{52} - 46^{52}}{96^{26} + 92^{26}}$. Then R satisfies

(a) R < 1 (b) 23^{26}< R < 24^{26} (c) 1 < R < 23^{26} (d) R > 24^{26}

Ans:- (b) R= $\frac{(2.24)^{52} - (2.23)^{52}}{(4.24)^{26} + (4.23)^{26}} = \frac{2^{52}(24^{52} - 23^{52})}{4^{26}(24^{26} + 23^{26})}$

= $\frac{2^{52}}{2^{52}} . \frac{(24^{26} + 23^{26})(24^{26} - 23^{26})}{24^{26} + 23^{26}}$

= $24^{26} - 23^{26}$

< 24^{26}

Also, R= $24^{26} - 23^{26} = (1+23)^{26} - 23^{26}$

= $23^{26} + 26_{c_1}.23^{25} + 26_{c_2}.23^{24} + \ldots + 1 - 23^{26}$

= 26. 23^{25}+$26_{c_2}.23^{24} + \ldots + 1$ >26. 23^{25} > 23. 23^{25} = 23^{26}

∴ 23^{26} < R < 24^{26}

16. A function f is said to be odd if f (-x)= -f (x) ∀ x. Which of the following is not odd?

(a) f (x+ y)= f(x)+ f(y) ∀ x, y

(b) f (x)= $\frac{xe^{x/2}}{1+e^x}$

(c) f (x) = x - [x]

(d) f (x) = $x^2 \sin x + x^3 \cos x$

Ans:- (c) f (x+ y)= f(x)+ f(y) ∀ x, y

Let x = y = 0

⇒ f (0) = f (0) + f (0)

∴ f (0)= 0

Replacing y with –x , we have

f (x- x) = f(x) + f (-x)

⇒ f (0) = f(x) + f (-x)

⇒ f(x) + f (-x) = 0

⇒ f (-x) = -f(x)

Thus f is odd.

Again for f (x) = $\frac{xe^{x/2}}{1+e^x}$

f(-x)= $\frac{(-x)(e^{-x/2})}{1+e^{-x}}$ = $\frac{(-x)\left(e^{-x/2}\right).e^x}{1+e^x}$ = - $\frac{xe^{x/2}}{1+e^x}$ = -f (x)

∴ f is odd.

f (x) = x- [x] is not odd.

Counter example:-

f (-2.3) = -2.3 – [-2.3] =-2.3 – (-3) = 3- 2.3 = 0.7

f (2.3) = 2.3 – [2.3] = 2.3 -2 =0.3

∴ f(2.3) ≠ f(-2.3)

Thus f is not odd

f (x) = $x^2 \sin x + x^3 \cos x$

f(-x) = -$x^2 \sin x - x^3 \cos x$ = -f(x)

∴ f is odd here.

17. Consider the polynomial $x^5 + ax^4 + bx^3 + cx^2 + dx + 4$. If (1+2i) and (3-2i) are two roots of this polynomial then the value of a is

(a) -524/65 (b) 524/65 (c) -1/65 (d) 1/65

Ans:- (a) The polynomial has 5 roots. Since complex root occur in pairs, so there is one real root taking it as m.

So, m, 1+2i, 1-2i, 3+2i, 3-2i are the five roots.

Sum of the roots= $-\frac{a}{1} = 8 + m.$

Product of the roots= (1+4)(9+4)m= 65

m=$\frac{4}{65}$

∴ m = $\frac{4}{65}$.

∴ a= $-8 - \frac{4}{65} = -\frac{524}{65}$.

18. In a special version of chess, a rook moves either horizontally or vertically on the chess board. The number of ways to place 8 rooks of different colors on a 8×8 chess board such that no rook lies on the path of the other rook at the start of the game is

(a) 8× ⌊8 (b) ⌊8 × ⌊8 (c) 2^8 × ⌊8 (d) $2^8 \times \binom{64}{8}$

Ans:- The first rook can be placed in any row in 8 ways & in any column in 8 ways. So, it has 8^2 ways to be disposed off. Since no other rook can be placed in the path of the first rook, a second rook can be placed in 7^2 ways for there now remains only 7 rows and 7 columns. Counting in this manner, the number of ways = $8^2.7^2.6^2 \ldots 1^2 = (8!)^2$

19. The value of $\int_0^1 \int_0^1 \{Min\ (x, y) - xy\} dxdy$ is

(a) ½ (b) 1/3 (c) 1/6 (d) 1/12

Ans:- (d) $\int_0^1 \int_0^1 \{Min\ (x, y) - xy\} dxdy$

= $\int_0^1 \int_0^y xdxdy + \int_0^1 \int_0^x ydxdy$

= $\frac{1}{3}$

And $\int_0^1 \int_0^1 xy\ dxdy = \frac{1}{4}$

∴ I = $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

20. Given that $\sum a_n$ converges ($a_n > 0$); Then $\sum a_n{}^3 \sin n$

(a) Converges (b) Diverges

(b) Doesn't exist (d) None

Ans:- (a) Since $\sum a_n$ converges, we have $\lim_{n\to\infty} n \,.\, a_n$ converges.

i.e. $|n \,.\, a_n| \le 1 \; for \; n \ge M$ (say)

$\Rightarrow n \,.\, a_n < 1 \quad [\because a_n > 0]$

$\Rightarrow a_n < \frac{1}{n}$

$\therefore a_n{}^3 < \frac{1}{n^3}$

$\Rightarrow a_n{}^3 \sin n \le \frac{1}{n^3} \sin n \le \frac{1}{n^3}$

$\Rightarrow \sum a_n{}^3 \sin n \le \sum \frac{1}{n^3}$

$\because$ RHS converges so LHS will also converge.

21. The differential equation of all the ellipses centered at the origin is

(a) $y^2 + x(y')^2 - yy' = 0$

(b) x y y″+ $x(y')^2 - yy' = 0$

(c) y y″+ $x(y')^2 - xy' = 0$

(d) none

Ans:- (d) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, after differentiating w.r.t x, we get

$\mapsto \frac{2x}{a^2} + \frac{2yy\prime}{b^2} = 0 \mapsto \frac{yy\prime}{b^2} = -\frac{x}{a^2}$

$\mapsto \frac{(y\prime)^2}{b^2} + \frac{y(y\prime\prime)}{b^2} = -\frac{1}{a^2}$

$\mapsto (y')^2 + y(y'')^2 = -\frac{b^2}{a^2}$.

22. If f(x)= x+ sinx, then find $\frac{2}{\pi^2} \cdot \int_{\pi}^{2\pi} (f^{-1}(x) + sinx)dx$

(a) 2 (b) 3

(c) 6 (d) 9

Ans:- (b) Let x= f(t) ⇒ dx= f′(t)dt

$\Rightarrow \int_{\pi}^{2\pi} f^{-1}(x)dx = \int_{\pi}^{2\pi} t\ f'(\mathrm{t})\mathrm{dt} =$ $(\mathrm{t}\,[\mathrm{f}(\mathrm{t})])^{2\pi}{}_{\pi} - \int_{\pi}^{2\pi} f(\mathrm{t})\mathrm{dt} = (4\pi^2 - \pi^2) - \int_{\pi}^{2\pi} f(\mathrm{t})\mathrm{dt}$

I= $\int_{\pi}^{2\pi} (f^{-1}(x) + sinx)dx = \int_{\pi}^{2\pi} f^{-1}(x)dx + \int_{\pi}^{2\pi} sinxdx$

$= 3\pi^2 - \int_{\pi}^{2\pi} f(\mathrm{t})\mathrm{dt} + \int_{\pi}^{2\pi} sinxdx$

$= 3\pi^2 - \int_{\pi}^{2\pi} (f(x) - sinx)dx$

$= 3\pi^2 - \int_{\pi}^{2\pi} xdx = 3\pi^2 - \frac{1}{2}(4\pi^2 - \pi^2)$

$= \frac{3}{2}\pi^2$

$\Rightarrow \frac{2}{\pi^2} I = 3.$

23. Let P= (a, b), Q= (c, d) and 0 < a < b < c < d, L≡(a, 0), M≡(c, 0), R lies on x-axis such that PR + RQ is minimum, then R divides LM

(a) Internally in the ratio a: b
(b) internally in the ratio b: c

(c) internally in the ratio b: d
(d) internally in the ratio d: b

Ans:- (c) Let R = (α, 0). PR+RQ is least

⟹ PQR should be the path of light

⟹ Δ PRL and QRM are similar

$\Rightarrow \frac{LR}{RM} = \frac{PL}{QM} \Rightarrow \frac{\alpha - a}{c - \alpha} = \frac{b}{d}$

$\Rightarrow \alpha$d- αd= bc –αb

$\Rightarrow \alpha = \frac{ad+bc}{b+d}$

$\Rightarrow$ R divides LM internally in the ratio b : d (as $\frac{b}{d} > 0$)

24. A point (1, 1) undergoes reflection in the x-axis and then the co-ordinate axes are roated through an angle of $\frac{\pi}{4}$ in anticlockwise direction. The final position of the point in the new co-ordinate system is-

(a) (0, $\sqrt{2}$) **(b) (0, $-\sqrt{2}$)**
(c) $-\sqrt{2}, 0$ **(d) none of these**

Ans:- . (b) Image of (1, 1) in the x-axis is (1, -1). If (x, y) be the co-ordinates of any point and (x' , y') be its new co-ordinates, then x' = x cos θ+ y sin θ,

y'= y cosθ – x sin θ, where θ is the angle through which the axes have been roated.

Here $\theta = \frac{\pi}{4}$, x= 1, y= -1

∴ x'= 0, y'= -$\sqrt{2}$

25. If a, $x_1, x_2, \ldots, x_k$ and b, $y_1, y_2, \ldots, y_k$ from two A.P. with common difference m and n respectively, then the locus of point (x, y) where x= $\frac{\sum_{i=1}^{k} x_1}{k}$ $is\ and\ y = \frac{\sum_{i=1}^{k} y_1}{k}$ is

(a) (x-a)m= (y-b)n
(b) (x-m) a= (y-n) b

(c)(x-n)a = (y-m)b
(d) (x-a) n-(y-b) m

Ans:- (d)

X= $\frac{\frac{k}{2}(x_1+x_k)}{k} = \frac{x_1+x_k}{2} = \frac{a+m+a\ mk}{2}$

$or,\ x = a + \frac{(k+1)m}{2}$

or, 2(x-a)= (k+1)m ……………..(1)

Similarly,

2(y-b)= (k+ 1)n ……………….(2)

We have to eliminate k

From (1) and (2)

$\frac{x-a}{y-b} = \frac{m}{n}$

or, (x- a)n = (y -b)m

26. An unbiased die with faces marked 1, 2, 3, 4, 5 and 6 is rolled four times. Out of four face values obtained the probability that the minimum face value is not less than 2 and the maximum face value is not greater than 5 is-

(a) $\frac{16}{81}$ **(b) $\frac{1}{81}$**
(c) $\frac{80}{81}$ **(d) $\frac{65}{81}$**

Ans. (a)

For minimum face value not to be less than 2 and maximum face value not to be greater than 5, a number out of 2, 3, 4, 5 must occur in each toss.

Probability of occurrence of 2, 3, 4, 5 in one toss = $\frac{4}{6} = \frac{2}{3}$

∴ Required probability = $\left(\frac{2}{3}\right)^4 = \frac{14}{81}$

27. The probability of India winning test match against west Indies is ½ . Assuming independence from match to match, the probability that in a 5 match series India's second win occurs at the third test, is

(a) $\frac{2}{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{1}{8}$

Ans. (c)

Let E_r denotes the probability that india wine the rth match. Required probability

$= P(E_1)P(E_2')P(E_3)+P(E_1')P(E_2)P(E_3) = \frac{1}{2}\left(1-\frac{1}{2}\right)\frac{1}{2}+\left(1-\frac{1}{2}\right).\frac{1}{2}.\frac{1}{2}=\frac{1}{8}.\frac{1}{8}=\frac{1}{4}$

28. The remainder on dividing $\mathbf{1234^{567}} + \mathbf{89^{1011}}$ by 12 is

(a) 1 **(b) 7**
(c) 9 **(d) none**

Ans:- (c) $1234\equiv 1 \pmod 3 \Rightarrow 1234^{567} \equiv 1\ (mod\ 3) and\ 89 \equiv -1 (mod 3)$

$\Rightarrow 89^{1011} \equiv -1\ (mod\ 3)$

$\therefore 1234^{567} + 89^{1011} \equiv 0\ (mod\ 3)$

Here 1234 is even, so $1234^{567} \equiv 0 (mod\ 4) and\ 89 \equiv 1\ (mod\ 4)$

$\Rightarrow 89^{1011} \equiv 1\ (mod\ 4)$

Thus $1234^{567} + 89^{1011} \equiv 1\ (mod\ 4)$

Hence it is 9 (mod 12)

29. Given that $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$, then the value of

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+xy+y^2)}dxdy\ , where\ D = \{(x,y) \in \mathbb{R}^2; x^2+y^2 \ \leq 1\}$ **is**

(a) $\frac{\pi}{3}$ **(b) $\frac{2\pi}{3}$**
(c) $\frac{2\pi}{\sqrt{3}}$ **(d) $\frac{\pi}{2}$**

Ans:- (c) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+xy+y^2)}dxdy$

$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\{(x-\frac{y}{2})^2+\frac{3}{4}y^2)\}}dxdy$

$= \int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty} e^{-(x-\frac{y}{2})^2}dx\ \}\, e^{-\frac{3}{4}y^2}dy$

$= \int_{-\infty}^{\infty}\sqrt{\pi}\ e^{-\frac{3}{4}y^2}dy = \frac{2\sqrt{\pi}}{\sqrt{3}}\int_{-\infty}^{\infty} e^{-u^2}du$

[let $\frac{\sqrt{3}}{2}y = u$]

$= 2\sqrt{\frac{\pi}{3}} \times \sqrt{\pi}$

$= \frac{2\pi}{\sqrt{3}}$.

30. The value of $\int_1^2\int_1^2\int_1^2\int_1^2 \frac{x_1+x_2+x_3-x_4}{x_1+x_2+x_3+x_4}\, dx_1dx_2dx_3dx_4$ is

(a) ½ **(b) 1/3**
(c) ¼ **(d) 1**

Ans:- (a)

$$\int_1^2\int_1^2\int_1^2\int_1^2 \frac{x_i\, dx_1dx_2dx_3dx_4}{x_1+x_2+x_3+x_4}$$
$$= \frac{1}{4}\ \ as \int_1^2\int_1^2\int_1^2\int_1^2 \frac{x_1+x_2+x_3+x_4}{x_1+x_2+x_3+x_4}\ dx_1dx_2dx_3dx_4$$
$= 1.$

$\therefore I = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$.

ISI OBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 2

There will be 30 questions in MMA Paper. For each question, exactly one of the four choices is correct. You get four marks for each correct answer, one mark for each unanswered question, and zero mark for each incorrect answer.

1. $\boldsymbol{a_1 = a_2 = 1, a_3 = -1, a_n = a_{n-1} \cdot a_{n-3}}$. **The value of** $\boldsymbol{a_{1964}}$ **is**

(a) 1 **(b) -1**
(c) 0 **(d) none**

Ans:- (b) $a_1 = a_2 = 1, a_3 = a_4 = a_5 = -1, a_6 = 1, a_7 = -1$

$\underbrace{+1, +1, -1, -1, -1, 1, -1}$,
$\underbrace{1, 1, -1, -1, -1, 1, -1,}$ …………

Since 1964= (7×280)+4= 7×280 +4. Thus we have a_{1964}= -1.

2. **If a, b are positive real variables whose sum is a constant $\boldsymbol{\lambda}$, then the minimum value of** $\sqrt{\left(\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{a}}\right)\left(\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{b}}\right)}$ **is**

(a) $\boldsymbol{\lambda - \frac{1}{\lambda}}$ **(b)** $\boldsymbol{\lambda + \frac{2}{\lambda}}$
(c) $\mathbf{1+\frac{2}{\boldsymbol{\lambda}}}$ **(d) none**

Ans:- (c) $E^2 = 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{ab} = \frac{a+b+1}{ab} + 1 = 1 + \frac{\lambda+1}{ab}$, it will be minimum when ab is maximum. Now we know that if sum of two quantities is constant, then their product is maximum when the quantities are equal.

$\therefore$ a+ b = λ ⇒a = b = $\frac{\lambda}{2}$

$\therefore E^2 = 1 + \frac{\lambda+1}{\frac{\lambda^2}{4}} = (\frac{\lambda+2}{\lambda})^2 \quad \Rightarrow E = 1+\frac{2}{\lambda}$, which is the required result.

Alternative: (c) $\sqrt{\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)}$ will minimum when a and b will take the maximum value.

a+b = λ, then the max. Value of a and b is a= b=$\frac{\lambda}{2}$,

Putting these, we get, $\sqrt{\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)}$

min = $\sqrt{\left(1+\frac{2}{\lambda}\right)(1+\frac{2}{\lambda})} = 1+\frac{2}{\lambda}$.

3. **The number of pairs of integers (m, n) satisfying** $\boldsymbol{m^2 + mn + n^2 = 1}$ **is**

(a) 8 **(b) 6**
(c) 4 **(d) 2**

Ans:- (b) Consider $m^2 + mn + n^2 - 1$

The equation is symmetric in m and n, we make the substitution

u = m + n and v= m- n

So that $u^2 + v^2 = 2\,(m^2 + n^2), u^2 - v^2 = 4mn$

Multiplying the given equation by 4, we have

$4m^2 + 4mn + 4n^2 = 4$

$\Rightarrow 4\,(m^2 + n^2) + 4mn = 4$

$\Rightarrow 2\ (u^2 +\ v^2) + u^2 -\ v^2 = 4$

$\Rightarrow 3u^2 + v^2 = 4$

Set $u^2 = x, v^2 = y$ with x, y ≥ 0, then we get 3x+ y= 4

The ordered pairs (x, y) satisfying the above equation in integers are (0, 4) and (1, 1).

We have,

u^2= 0 and $u^2 = 1$ & $v^2 = 4$, and $v^2 = 1$

I.e. u= 0, v= 2; u=0, v= -2;

u=1, v= 1; u =1, v= -1;

u =-1, v =1; u = -1, v= -1;

Giving 6 ordered pair solutions (m, n) viz (1, -1), (-1, 1), (1, 0), (0, 1), (0, -1), (-1, 0)

4. **The sum of the digits of the number $\mathbf{100^{13} - 26}$, written in decimal notation is**
   (a) 227 **(b) 218**
   (c)228 **(d) 219**

Ans:- (a) $10^{26} - 26 = \underbrace{100 \ldots..0} - 26$

$= \underbrace{999 \ldots.974}$

with 24 9's

∴ The sum of the digits = $24 \times 9 + 7 + 4 =$ 227.

5. **The great common divisor (gcd) of $\mathbf{2^{2^{22}} + 1}$ *and* $\mathbf{2^{2^{222}} + 1}$ is**
   (a) 1 **(b) $\mathbf{2^{2^{22}} + 1}$**
   (c) $\mathbf{2^{2^{11}} - 1}$ **(d) $\mathbf{2^{2^{21}} - 1}$**

Ans:- (a) let $F_n = 2^{2^n} + 1$, with m > n

$F_n - 2 = 2^{2^n} + 1 - 2 = 2^{2^n} - 1 =$ $(2^{2^{n-1}})^2 - 1 = (2^{2^{n-1}} + 1)(2^{2^{n-1}} - 1)$

$= (2^{2^{n-1}} + 1)\ (2^{2^{n-2}} -$ $1)(2^{2^{n-2}} + 1) =(2^{2^m} + 1)\ (2^{2^m} - 1)$ $(2^{2^{m-n-1}} + 1)$

$= \lambda\ F_m$; Now, $F_n - \lambda\ F_m$=2

Let d | F_n and d |F_m then d|2. Then d= 1 or 2. But F_m & F_n are both odd, hence gcd = 1.

6. **The number of real roots of the equation $\mathbf{1+ \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^7}{7} = 0}$**

(without factorial) is

(a) 7 **(b) 5**
(c) 3 **(d) 1**

Ans:- (d) let f has a minimum at x = x_0 , where then f'(x_0) = 0

f (x)= $1+ \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^6}{6}$;

$\Rightarrow 1+\ x_0 + {x_0}^2 + {x_0}^3 + {x_0}^4 + {x_0}^5 = 0$

$\Rightarrow \frac{{x_0}^6 - 1}{x_0 - 1} = 0$

$\Rightarrow \frac{({x_0}^3 - 1)({x_0}^3 + 1)}{x_0 - 1} = 0$

$\Rightarrow ({x_0}^2 + x_0 +\ 1)(\ {x_0}^2 - x_0 +\ 1)(\ x_0 +$ $1) = 0$

Which has a real root x_0 = -1

But, f (-1)= 1-1 + $(\frac{1}{2} - \frac{1}{3})$+ $(\frac{1}{4} - \frac{1}{5})$ +$\frac{1}{6}$ > 0

The f (x)> 0 and hence f has no real zeros.

Now let, g (x)= $1+ \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^7}{7}$

An odd degree polynomial has at least one real root.

If our polynomial g has more than one zero, say x_1, x_2

Then by Role's theorem in (x_1, x_2) we have 'x_3' such that g′ (x_3) = 0

⇒ $1+ x_3 + {x_3}^2 + \cdots + {x_3}^6 = 0$

But this has no real zeros. Hence the given polynomial has exactly one real zero.

7. Number of roots between $-\pi$ and π of the equation $\frac{2}{3}\sin x . x= 1$ is
(a) 1 (b) 2 (c) 3 (d) 4

Ans:- (d) $\sin x = \frac{3}{2x}$

Now, draw the curve of y= $\sin x$ and y = $\frac{3}{2x}$ or xy = $3/2$

∴ there are 4 real roots. (Draw the graph yourself)

8. The number a^{7389b}, a, b are digits, is divisible by 72 , Then a+ b equals
(a) 10 (b) 9 (c) 11 (d) 12

Ans:- (b) 72 = 8× 9, and 8 and 9 are co prime. As the number a^{7389b} is divisible by 72, it is divisible by 9 and 8 both. For divisibility by 8, the last three digits must be divisible by 8.

i.e. 800+ 90+b |8.

so (b+2)| 8 ; ∴ b = 6

For divisibility by 9, the sum of the digits A + 7 +3 +8+ 9 +b should be divisible by 9.

i.e. a+ 7+ 3+ 8+ 9+ 6≡ 0 (mod 9)

⇒ a+ 6 ≡ 0 (mod 9)

⇒ a ≡ -6(mod 9)

⇒ a ≡ 3 (mod 9)

∴ a= 3 only. Hence a+ b= 9

9. 3 balls are distributed to 3 boxes at random. Number of way in which we set at most 1 box empty is (a) 20 (b) 6 (c) 24 (d) none

Ans:- (c) zero box empty + 1 box empty

= 3 balls in 3 boxes + {$^3C_1$ × 3 balls in 2 boxes}

= 3! + 3× $(3)_2 = 24$.

10. The value of $\max_a \boldsymbol{I(a)}, \boldsymbol{where\ I(a) = \int_{a-1}^{a+1} e^{-|x|}dx}$ is
(a) e^a (b) 2- $e^{a-1} - e^{a+1}$ (c) $e^{a-1} - e^{a+1}$ (d) none

Ans:- (b) $I(a) = \int_{a-1}^{a} e^{-|x|}dx + \int_{a}^{a+1} e^{-|x|}dx$

Let 0< a< 1 , then $\int_{a-1}^{a} e^{-|x|}dx = \int_{a-1}^{0} e^{x}dx + \int_{0}^{a-1} e^{-x}dx$

= 1- $e^{a-1} - (e^{-a} - 1)$

And, $\int_{a}^{a+1} e^{-|x|}dx = e^{-a} - e^{-a-1}$

$\therefore$ I (a) = 2 - ($e^{a-1} + e^{-a-1}$)

$\therefore \frac{d}{dx} I(a) = 0$

$\Rightarrow e^{a-1} = e^{-a+1}$

$\Rightarrow$ a= 0

Also, -1 < a < 0 ; $\int_{a-1}^{a} e^x dx = e^a - e^{a-1}$

And $\int_a^{a+1} e^{-|x|} dx = \int_a^0 e^x dx + \int_0^{a+1} e^{-x} dx$

= (1- e^a) – ($e^{a+1} - 1$)

= 2 -$e^a - e^{a+1}$

$\therefore$ I (a) = 2- $e^{a-1} - e^{a+1}$

$\therefore$ I(a) is maximum at a = 0.

11. The value of $\prod_{n=2}^{\infty}(1 - \frac{1}{n^2})$ is

(a) 1 **(b) 0** **(c) ½**
(d) none

Ans:- (c) $\prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right) = \prod_{n=2}^{\infty}\left(1 + \frac{1}{n}\right)(1 - \frac{1}{n}) = \prod_{n=2}^{\infty}\frac{n+1}{n} \cdot \frac{n-1}{n}$

= $(\frac{2+1}{2} \cdot \frac{2-1}{2})(\frac{3+1}{3} \cdot \frac{3-1}{3})$……… = $\frac{3}{2} \cdot \frac{1}{2} \cdot \frac{4}{3} \cdot \frac{2}{3}$……. = $\frac{1}{2}$.

12. $cos^8\theta - sin^8\theta$= 1. Number of roots are there in between [0, 2π] is

(a) 1 **(b) 2** **(c) 3** **(d) 0**

Ans:- (c) Note that , $cos^8\theta = 1 + sin^8\theta$ is possible only if,

$cos^8\theta = 1$ *and* $sin^8\theta$ =0

$\therefore \theta$= 0, π, 2π

Hence 3 roots are three between [0, 2π]

13. If $u_n = \frac{1}{1.n} + \frac{1}{2(n-1)} + \frac{1}{3(n-2)} + \ldots + \frac{1}{(n-1)}$; Then $\lim_{n\to\infty} u_n$ equals

(a) 0 **(b) 1**
(c) ∞ **(d) π**

Ans:- (a) $u_n = \frac{1}{(n+1)} [\left(1 + \frac{1}{n}\right) + \left(\frac{1}{2} + \frac{1}{n-1}\right) + \left(\frac{1}{3} + \frac{1}{n-2}\right) + \cdots + (\frac{1}{n} + 1)]$

$= \frac{1}{(n+1)} 2 (1 + \frac{1}{2} + \cdots + \frac{1}{n})$

$\therefore \lim_{n\to\infty} u_n = 2 \lim_{n\to\infty} \frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{n} . \lim_{n\to\infty} \frac{n}{n+1} =$ 2. 0.1= 0.

14. If x+ $\left(\frac{1}{x}\right)$ = -1, The value of $x^{99} + \left(\frac{1}{x^{99}}\right)$ is

(a) 1 **(b) 2**
(c) 0 **(d) none**

Ans:- (b) If $a_n = x^n + \frac{1}{x^n}$

Then, $a_{n+1} = a_n.a_1 - a_{n-1}$ *for* $n \geq 1$

$a_0 = 2 , a_2 = -a_1 - a_0 = -1, a_3 = 2, a_4 = -1, a_5 = 2, a_1 = -1 ; a_n = -1, a_{n-1} = -1.$

$so , a_{99} = 2$

or, $\therefore a_{n+1} = x^{99} + \frac{1}{x^{99}}$

$= \left(x^{98} + \frac{1}{x^{98}}\right)\left(x + \frac{1}{x}\right) - (x^{97} + \frac{1}{x^{97}})$

= - $a_n - a_{n-1} = +1 + 1 = 2$

15. Consider the equation of the form$x^2 + bx + c = 0$. The number of such, equations that have real roots and have coefficients b and c in the set {1, 2, 3, 4, 5,6} , (b may be equal to c) is

(a) 16 **(b) 19**
(c) 21 **(d) none**

Ans:- (b) Let $x^2 + bx + c = 0$ has real roots, then $b^2 - 4c \geq 0$, and also , s= {1, 2, 3, 4, 5, 6}.

Now $s_1 = \{4, 8, 12, 16, 20, 24\}$= set of possible values of 4c.

Thus the number of equations will be same as the number of pairs of elements (a_1, a_2),

$a_1 \in$ s, $a_2 \in s_1$ such that

${a_1}^2 - 4a_2 \geq 0$, i.e. 1+ 2+ 4+ 6+6 = 19

16. If f: R ⟶R, satisfies f(x +y)= f(x)+f(y)∀ $\boldsymbol{x, y} \in \mathbb{R}$ and f(1)=7, then$\sum_{r=1}^{n} \boldsymbol{f(r)}$ is

(a) $\frac{7\,(n+1)}{2}$ **(b) 7n (n+1)**
(c) $\frac{7\,n(n+1)}{2}$ **(d)none**

Ans:- (c) putting x= 1, y=0, then f(1)= f(1)+f(0)

⇒f(0)=0, ⇒ f(1)=7

Again , putting x=1, y=1, then f(2)= 2f(1)= 14, similarly,

f(3)=21 and so on.

$\sum_{r=1}^{n} f(r)$= 7 {1+ 2+ 3+…..+ n}= $\frac{7\,n(n+1)}{2}$.

17. Let f(0)= 1, $\lim_{x\to\infty} \boldsymbol{f''(x) = 4}$ *and* $\boldsymbol{f(x) \geq f(1)}$. Let f(x) is polynomial ∀ x∈ℝ. The value of f(2) is

(a) 4 **(b) 0**
(c) 1 **(d) none**

Ans:- (c) $f''(x) = 4 = constant$

⇒ f(x)= $2x^2 + ax + b$

f(0)= 1 ⇒ b= 1

f(1)= 3+ a

f(x) ≥f(1) ⇒ f′(1)=0

⇒ 4+ a= 0

⇒ a = -4

∴ f(x)= $2x^2 + 4x + 1$

∴ f(2)= 1.

18. Let $\frac{1-3P}{2}, \frac{1+4P}{3}, \frac{1+P}{6}$ are the probabilities of 3 mutually exclusive and exhaustive events, then the set of all values of P is

(a) [-1/4, 1/3] **(b) (0, 1)**
(c) (0, ∞) **(d) none**

Ans:- (a) $\frac{1-3P}{2} \geq 0, \frac{1+4P}{3} \geq 0, \frac{1+P}{6} \geq 0$ and

$$\frac{1-3P}{2} + \frac{1+4P}{3} + \frac{1+P}{6} = 1$$

$\Rightarrow -\frac{1}{4} \leq P \leq \frac{1}{3} \Rightarrow P \in [-\frac{1}{4}, \frac{1}{3}]$

19. If $\sqrt{5x - x^2 - 6} + \frac{\pi}{2}\int_0^x dt >$ $x\int_0^{\pi} \sin^2 t dt$, then x ∈

(a) (2, 3) **(b) (-∞, 2) ∪ (3, ∞)**
(c) (5/2 , 3) **(d) none**

Ans:- (a) $\sqrt{5x - x^2 - 6} + \frac{\pi x}{2} > x\{\frac{1}{2}\int_0^{\pi}(1 - cos2t)dt\}$

$\Rightarrow \sqrt{5x - x^2 - 6} + \frac{\pi x}{2} > x\{\frac{1}{2}(t - \frac{1}{2}\sin 2t)_0^{\pi}\}$

$\Rightarrow \sqrt{5x - x^2 - 6} + \frac{\pi x}{2} > \frac{\pi x}{2}$

$\Rightarrow \sqrt{5x - x^2 - 6} > 0$

$\Rightarrow x^2 - 5x + 6 < 0,$

$\Rightarrow$ (x-2)(x-3)< 0, i.e. , x ∈ (2, 3).

20. If f(x)= $(4+x)^n$, n ∈ N and $f^r(0)$ represents the r^{th} derivative of f(x) at x= o, then the value of $\sum_{r=0}^{\infty} \frac{f^r(0)}{r!} =$

(a) 2^n (b) e^n

(c) 5^n (d) none

Ans:- (c) f′(x)= n $(4+x)^{n-1}$

f″(x)= n (n-1) $(4+x)^{n-2}$

$f^r(x)$= n (n-1)….. (n- r+1) . $(4+x)^{n-r}$, r $\leq n$

$f^r(0) = \frac{n!}{(n-r)!} . 4^{n-r}$, r $\leq n$

= 0, r $> n$

$\therefore \sum_{r=0}^{\infty} \frac{f^r(0)}{r!} = \sum_{r=0}^{n} \binom{n}{r} . 4^{n-r} = (1+4)^n = 5^n$.

21. The two lines $\vec{r} = \vec{a} + \lambda(\vec{b} + \vec{c})$ and $\vec{r} = \vec{b} + \mu(\vec{c} + \vec{a})$ intersects at a point, where λ and μ are scalars, then

(a) $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar
(b) $|\vec{a}| = |\vec{b}| = |\vec{c}|$

(c) $\vec{a}.\vec{c} = \vec{b}.\vec{c}$
(d) $\lambda(\vec{b}\times\vec{c}) + \mu(\vec{c}\times\vec{a}) = \vec{c}$

Ans. (c)

The two lines intersect

$\therefore \vec{a} + \lambda(\vec{b} \times \vec{c}) = \vec{b} + \mu(\vec{c} \times \vec{a})$

Taking dot product with $\vec{c}$ on both sides, we get

$\vec{a}. \vec{c} = \vec{b} .\vec{c}$

22. Let f(x)=

$$\begin{cases} \boldsymbol{x|x|}; & \boldsymbol{x \leq -1} \\ \boldsymbol{[x+1] + [1-x]}; & \boldsymbol{-1} < x < 1 \\ \boldsymbol{-x\,|x|}; & \boldsymbol{x \geq 1} \end{cases}$$

Then the value of $\int_{-2}^{2} f(x)dx$ is

(a) - $\frac{8}{3}$ (b) - $\frac{7}{3}$

(c) $\frac{7}{3}$ (d) none

Ans:- (a) f(x) = $\begin{cases} -x^2 , & x \leq -1 \\ 1 , & -1 < x < 0 \\ 2 , & x = 0 \\ 1 , & 0 < x < 1 \\ x^2 , & x \geq 1 \end{cases}$

∴ f(x) is an even function, i.e. $\int_{-2}^{2} f(x)dx = 2\int_{0}^{2} f(x)dx$

$= 2 \{\int_0^1 f(x)dx + \int_1^2 f(x)dx\}$

$= 2\,(1 - \frac{x^3}{3})_1^2 = -\frac{8}{3}$.

23. Area bounded by y = g(x), x-axis and the lines x= - 2,

Where g (x)=
$$\begin{cases} \textbf{max}\{\boldsymbol{f(t)}: \boldsymbol{-2} \leq \boldsymbol{t} \leq \boldsymbol{x}\}, & \boldsymbol{where -2} \leq \boldsymbol{x} < 0; \\ \textbf{min}\{\boldsymbol{f(t)}: \boldsymbol{0} \leq \boldsymbol{t} \leq \boldsymbol{x}\}, & \boldsymbol{where\ 0} \leq \boldsymbol{x} \leq \boldsymbol{3} \end{cases}$$

And f(x)= $x^2 - |x|$, is equal to

(a) $\frac{113}{24}$ (b) $\frac{111}{24}$

(c) $\frac{117}{24}$ (d) none

Ans:- (a) g(x)=
$$\begin{cases} 2\ ; & -2 \leq x < 0 \\ x^2 - x\ ; & 0 \leq x \leq \frac{1}{2} \\ -\frac{1}{4}\ ; & \frac{1}{2} < x \leq 3 \end{cases}$$

$\therefore$ Required area = $\int_{-2}^{0} 2dx + \int_{0}^{\frac{1}{2}}(x - x^2)dx + \int_{\frac{1}{2}}^{3}\left(\frac{1}{4}\right)dx = \frac{113}{24}unit^2$

24. Total number of positive integral values of n such that the equations $\cos^{-1} x + (\sin^{-1} y)^2 = \frac{n\pi^2}{4}$ *and* $(\sin^{-1} y)^2 - \cos^{-1} x = \frac{\pi^2}{16}$ are constant, is equal to

(a) 1 **(b) 2**

(c) 3 **(d) none**

Ans:- (a) Here 2 $(\sin^{-1} y)^2 = \frac{4n+1}{16}\pi^2$

$\Rightarrow 0 \leq \frac{4n+1}{16}\pi^2 \leq \frac{\pi^2}{4}$,

$\Rightarrow -\frac{1}{4} \leq n \leq \frac{7}{4}$.

Also. 2($\cos^{-1} x$)= $\frac{4n-1}{16}\pi^2$

$\Rightarrow 0 \leq \frac{4n-1}{16}\pi^2 \leq \pi,$

$\Rightarrow \frac{1}{4} \leq n \leq \frac{8}{\pi} + 1$.

Hence, the least positive integral value of n is 1.

25. Radius of bigger circle touching the circle $x^2 + y^2 - 4x - 4y + 4 = 0$ and both the co-ordinate axis is

(a) 3+2$\sqrt{2}$ **(b) 2(3+2$\sqrt{2}$)**

(c) 3- 2$\sqrt{2}$ **(d) none**

Ans:- (b) Let (h, h) be the centre of the required circle.

$\therefore \angle$COD= $\angle$CBE=$\frac{\pi}{4}$, CB= h+ 2 AND BD= h- 2.

$\therefore \frac{h-2}{h+2} = cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$,

$\Rightarrow$ h= $\frac{2(\sqrt{2}+1)}{(\sqrt{2}-1)} = 2(3 + 2\sqrt{2})$.

26. Tangents and normal drawn to parabola at A $(at^2, 2at), t \neq 0$ meet the X- axis at point B and D, respectively. If the rectangle ABCD is

(a) y-2a= 0 **(b) y+ 2a= 0**

(c) x-2a= 0 **(d) none**

Ans:- (c) Evolution of tangent & normal at A are $y^t = X + at^2, y = -tx + 2at + at^3$.

$\therefore$ B = ($-at^2$, 0) and D= (2a+ at^2, 0)

Suppose ABCD is rectangle,

Then midpoints of BD and AC will be coincident,

$\therefore$ h+$at^2 = 2a + at^2 - at^2$ *and* $k + 2at = 0$

i.e. h= 2a, k= -2at.

Hence, the locus is X= 2a, i.e. X-2a=0.

27. The series $\sum_{k=2}^{\infty}\left(\frac{1}{k(k-1)}\right)$ converges to

(a) -1 **(b) 1** **(c) 0** **(d) does not converges**

Ans:- (b) $s_n = \sum_{k=2}^{\infty}\frac{1}{k(k-1)} = \sum_{k=2}^{n}(\frac{1}{(k-1)} - \frac{1}{k}) = \left(1 - \frac{1}{n}\right)$

$\therefore \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = 1.$

28. The limit $\lim_{x\to\infty}\left(\frac{3x-1}{3x+1}\right)^{4x}$ equaqls

(a) 1 **(b) 0** **(c) $e^{-\frac{8}{3}}$** **(d) $e^{\frac{4}{9}}$**

Ans:- (c) $\lim_{x\to\infty}\{(\frac{1-\frac{1}{3x}}{1+\frac{1}{3x}})^x\}^4 = (\frac{e^{-\frac{1}{3}}}{e^{\frac{1}{3}}})^4 = e^{-\frac{8}{3}}$

[since $\lim_{x\to\infty}(1+\frac{k}{x})^x = e^k$].

29. $\lim_{x\to\infty}\frac{1}{n}\left(\frac{n}{n+1}+\frac{n}{n+2}+\cdots+\frac{n}{2n}\right)$ equals

(a) ∞ **(b) 0** **(c) $\log_e 2$** **(d) 1**

Ans:- (c) $\lim_{x\to\infty}\frac{1}{n}(\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\cdots+\frac{1}{1+\frac{n}{n}})$

$= \int_0^1 \frac{1}{1+x}dx = [\log(1+x)]_0^1 = \log_e 2.$

30. Let k be an integer greater than 1. Then $\lim_{n\to\infty}\left[\frac{n}{n+1}+\frac{n}{n+2}+\cdots\right]$ is

(a) $\log_e k$ **(b) (k-1) $\log_e k$**
(c) 0 **(d) ∞**

Ans:- (a) $\lim_{n\to\infty}\left[\sum_{r=1}^{n(k-1)}\frac{1}{n+r}\right] = \int_0^{k-1}\frac{dx}{1+x} = \log(1+x)]_0^{k-1} = \log_e k$.

ISI OBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 3

There will be 30 questions in MMA Paper. For each question, exactly one of the four choices is correct. You get four marks for each correct answer, one mark for each unanswered question, and zero mark for each incorrect answer.

1. Number of solutions are possible in $0 \le x \le 99$ for the equation

$|3 - 3^x| + |1 - 3^x| = 1 - 3^x - \frac{3^{-x}}{4}$ is

(a) 1 **(b) 0** **(c) 2** **(d) none**

Ans:- (b) LHS= $|3 - 3^x| + |3^x - 1| \ge |(3 - 3^x) + (3^x - 1)| \ge 2$

But RHS= 1 - $(3^x + \frac{3^{-x}}{4})$

=1- $\{(3^{\frac{x}{2}} + \frac{3^{-\frac{x}{2}}}{2})$ - 2. $3^{\frac{x}{2}}.\frac{3^{-\frac{x}{2}}}{2}\}$

= 2 - $(3^{\frac{x}{2}} + \frac{3^{-\frac{x}{2}}}{2})^2 < 2$

$\therefore$ given equation has no solution for any real x.

2. If f(x) = $\log_e(6 - |x^2 + x - 6|)$, then domain of f(x) has how many integral values of x?

(a) 5 **(b) 4** **(c) infinite** **(d) none of these**

Ans:- (b) f(x) is defined only when $6 - |x^2 + x - 6| > 0$

i.e. $|x^2 + x - 6| < 6$

⇒ -6 < $x^2 + x - 6 < 6$

⇒ $x^2 + x > 0$ and $x^2 + x - 12 < 0$

⇒ x (x+1)>0 and (x+4)(x-3)< 0

⇒ (x< -1 or x > 0) and (-4 < x < 3)

⇒ x ∈ (-4, -1) ∪(0, 3) ⇒ x= -3, -2, 1, 2 as integral values.

3. The sum of the real solution of $2|x|^2$+ 51= |1+ 20x| is
(a) 5 (b) 0
(c) 24 (d) none of these

Ans:- (d) $2x^2 + 51 = \pm(1 + 20x)$

⇒ $x^2 - 10x + 25 = 0 \ or \ x^2 + 10x + 26 = 0$

⇒ $(x - 5)^2 = 0 \ or (x + 5)^2 + 1 = 0$(impossible)

⇒ x= 5, 5

∴ Sum of the real solution = 5+ 5= 10.

4. The solution set of ||x- 1|-1| + x ≤ 2 is
(a) $(-\infty, 2]$ (b) [0, 1)
(c) [0, 2) (d) [1, 2)

Ans:- (a) (i) If x < 0, then |1- x- 1|+x ≤ 2

⇒ |x|+x ≤ 2

⇒ -x +x ≤ 2

⇒ 0 ≤ 2 (true)

∴ x < 0

(ii) If $0 \le x \le 1$, then |1- x- 1|+x ≤ 2
⇒ |x|+x ≤ 2
⇒ 2x ≤ 2
⇒ x ≤ 1, ∴$0 \le x < 1$

(iii) If $1 \le x < 2$, then |x-1 -1|+x ≤ 2
⇒ |x-2|+x ≤ 2
⇒ 2- x +x ≤ 2
⇒ 2 ≤ 2 (true)

(iv) If x ≥ 2, then |x-1 -1|+x ≤ 2
⇒ x-2 +x ≤ 2
⇒ x ≤ 2
∴ x=2 [∵ x≥ 2]
∴ Required solution set is $(-\infty, 2]$

5. If domain of f(x)= $\sqrt{\frac{1}{|x-1|+[x]}}$ be (a, b), then ([.] denotes greatest integer function)
(a) a= 1, b=∞ (b) a= -∞, b= 0
(c) a= -∞, b= 1 (d) none of these

Ans:- (c) we must have, |x-1|> [x].......(1)

∴ x-1 < [x]≤ x, i.e. [x]> x – 1..........(2)

∴ on combining (1) and (2), we have |x -1|> x -1

This is true only if x-1< 0, i.e. if x < 1, i.e. if x ∈ (-∞, 1)

∴$D_f = (-\infty, 1) \Rightarrow$ a = $-\infty$, b= 1

6. If there are 4 distinct solutions of ||x - 2012| + $\log_2 a$| = 3, then a ∈

(a) $(-\infty, -6)$ (b) $(-\infty, \frac{1}{8})$

(c) $(-\infty, -\frac{1}{8})$ (d) none of these

Ans:- (b) we have | x -2012|+$\log_2 a = \pm 3$

⇒ | x -2012| = -$\log_2 a$+ 3, -$\log_2 a - 3$

∴ If there are 4 distinct solutions of the above equation, then we must have

-$\log_2 a$+ 3> 0 and -$\log_2 a - 3 > 0$

i.e. $\log_2 a$ < 3 and $\log_2 a$ < -3 ⇒ a < 2^{-3}

∴ a ∈ $(-\infty, \frac{1}{8})$

7. The number of value of k for which the equation $x^3 - 3x + k = 0$ has two distinct roots lying in the interval (0, 1) are

(a) 3 (b) 2 (c) infinitely many (d) no value of k satisfies the requirement

Ans:- (d) Let there be a value of k for which $x^3 - 3x + k = 0$ has two distinct roots between 0 and 1. Let, a, b are two distinct roots of $x^3 - 3x + k = 0$ lying between 0 and 1 such that a < b

Let f (a)= f(b)= 0. Since between any two roots of a polynomial f(x) there exist at least one roots of its derivative f′(x).

Therefore, f′(x) = $3x^3 - 3$ has at least one root between a and b

But f′(x) =0 has two roots equal to $\pm$ 1 which don't lie between 0 and 1 for any value of k.

8. If $\frac{dy}{dx} = f(x) + \int_0^1 f(x)dx$ then the equation of the curve y=f(x) passing through (0, 1) is

(a) f(x)= $\frac{2e^x - e + 1}{3 - e}$ (b) f(x)= $\frac{3e^x - 2e + 1}{2(x - e)}$ (c) f(x)= $\frac{e^x - 2e + 1}{e + 1}$

(d) none of these

Ans:- (a) f″(x)= f′(x)

⇒ $\frac{f''(x)}{f'(x)} = 1$

On integrating f′(x) = c e^x

Which gives f(x)= c e^x+D

But f(0) = 1 ⇒ c+ D = 1

∴ f(x)= c $e^x + 1 - c$

So, f′(x)= c e^x putting it in f′(x)= f(x)+ $\int_0^1 f(x)dx$

⇒ c e^x= c $e^x + 1 - c + \int_0^1 (c\, e^x + 1 - c)dx$

⇒ c = $\frac{2}{3-e}$

So, f(x)= $\frac{2e^x - e + 1}{3 - e}$

9. A staircase has 10 steps, a person can go up the steps one at a time, or any combination of 1's and 2's . The number of ways in which the person can go up the stairs is

(a) 89 (b) 144

(c) 132 (d) 211

Ans:- (a)

x+ 2y= 10, where x is the number of times he takes single steps, and y is the number times he takes two steps

	Case	Total no. of ways
1	X=0, y=5	5! /5!=1
2	X=2, y= 4	6! /2! 4! =15
3	X=4, y=3	7! /3!4!= 35
4	X=6, y=2	8! /2! 6!= 28
5	X=8, y=1	9! /8! = 9
6	X=10, y=0	10! /10! =1

∴ P= 89

10. The remainder when $1690^{2608} + 2608^{1690}$ is divided by 7 is

(a) 1 **(b) 2** **(c) 3** **(d) none**

Ans:- (a) 1690= $7\times 241 + 3$;
2608= 7×372 +4

Let s =$1690^{2608} + 2608^{1690}$

= $(7 \times 241 + 3)^{2608} + (7 \times 372 + 4)^{1690}$

= a number multiple of 7+$3^{2608} + 4^{1690}$

Let s′= $3^{2608} + 4^{1690}$

Clearly remainder in s and s′ will be same when divided by 7.

s′= $3\times 3^{3\times 867} + 4 \times 4^{3\times 563}$

= $3\times 27^{867} + 4 \times 64^{563}$

= $3(28-1)^{867} + 4\,(63+1)^{563}$

= 3[multiple of 7- 1]+ 4[multiple of 7+ 1]

= multiple of 7+ 1

∴ Hence remainder is 1.

11. The value of $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i3^j3^k}$ is $(i \neq j \neq k)$

(a) 80/207 **(b) 81/208** **(c) 1/208** **(d) none**

Ans:- (b) Let us first of all find the sum without any restriction i, j, k.

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i3^j3^k} = \left(\sum_{i=0}^{\infty}\frac{1}{3^i}\right)^3 = \frac{27}{8}$$

For the requirement sum we have to remove the cases when i= j= k or when any two of them are equal and not equal to other variable (say, i= j≠ k).

Case –I:- when i= j= k

In this case $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i3^j3^k} = \sum_{i=0}^{\infty}\frac{1}{3^i} = \frac{27}{16}$

Case – II:- i= j≠ k

In this case, $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i3^j3^k} = \left(\sum_{i=0}^{\infty}\frac{1}{3^{2i}}\right)\left(\sum_{k=0}^{\infty}\frac{1}{3^k}\right)$

= $\sum_{i=0}^{\infty}\frac{1}{3^{2i}}\left(\frac{3}{2}-\frac{1}{3^i}\right)$

= $\frac{3}{2}.\frac{9}{8} - \frac{27}{26} = \frac{135}{8.26}$

Hence required sum =$\frac{27}{8} - \frac{27}{26} - \left(\frac{135}{8.26}\right).3 = \frac{81}{208}$

12. The solution of the differential equation f(x) $\frac{dy}{dx} + f'(x)y = 1$ is given by f(x) =

(a) yx + c **(b) $\frac{x+c}{y}$**

(c) yc **(d) none**

Ans:- (b) f(x)dy +f′(x)ydx = dx

i.e. d (f(x), y) = d (x)

Integrating we get, y. f(x) = x+ c

or, f(x) = $\frac{x+c}{y}$

13. If $\int_0^x f(x)sintdt$ = constant, 0 < x < 2π and f(π)= 2 Then find the value of f$\left(\frac{\pi}{2}\right)$

(a) 2 **(b) 4** **(c) 6**

(d) 8

Ans:- (b) Differentiable both sides, we get

f′(x) (1- cosx)+ f(x)sinx= 0

$\Rightarrow \int \frac{f\prime(x)}{f(x)} dx = \int \frac{sinx}{1-cosx} dx$

$\Rightarrow$ ln(f (x)| = -2ln sin$\frac{x}{2} + lnc$

$\Rightarrow$ f (x)= $\frac{c}{(\sin\frac{x}{2})^2}$ $\Rightarrow$ f(π)=2 $\Rightarrow$ c=2: f($\frac{\pi}{2}$)=4

14. For a ∊ R if |x+ a-3| + |x- 2a|= |2x –a - 3| is three for all x ∊R, then exhaustive set of a is

(a) a ∊ [-4, 4] **(b) a ∊ [-3, 2]**

(c) a ∊ {-2, 2} **(d) a ∊ {1}**

Ans:- (d) |x|+ |y| = |x+y|

$\Rightarrow$ xy ≥ 0, therefore (x- (3- a)) (x- 2a)≥ 0 $\forall\, x \in R$

$\Rightarrow x^2 - x(3+a) + 2a(3-a) \geq 0\ \forall\, x \in R$

$\Rightarrow (a+3)^2 - 8a(3-a) \leq 0 \Rightarrow (a-1)^2 \leq 0 \Rightarrow a = 1$ which is true $\forall\, x \in R$

15. If A is skew–symmetric matrix, then B = (I- A) $(I + A)^{-1}$ is (where I is the identity matrix of the same order as A)

(a) idempotent matrix **(b) symmetric matrix** **(c)orthogonal matrix** **(d) none**

Ans:- (c) B= (I- A) $(I + A)^{-1}$

$\Rightarrow B^T = (I + A^T)^{-1}(I + A^T) = (I - A)^{-1}$(I+ A)

$B\, B^t$= I as (I- A) (I+ A)= (I+ A) (I-A)

16. If f(x)= max ($\frac{1}{\pi}\cos^{-1}(\cos\pi x), \{x\}$) and g (x) min ($\frac{1}{\pi}\cos^{-1}(\cos\pi x)$, $\{x\}$) (where { .} represents fractional part of x). Then find the value of $\int_1^2 f(x)dx /$ $\int_1^2 g(x)dx$ is

(a) 1 **(b) 3** **(c) 5**

(d) 7

Ans:- (b)

$\Rightarrow \int_1^2 f(x)dx = \frac{3}{4}\ \ and\ \ \int_1^2 g(x)dx = \frac{1}{4}$

$\Rightarrow$ Ratio = 3

17. If sin (sinx +cosx)= cos (cosx- sinx) and largest possible value of sinx is $\frac{\pi}{k}$, then the value of k is

(a) 2 **(b) 3**
(c) 4 **(d) none**

Ans:- (c) sin (sinx +cosx)= cos (cosx- sinx)

cos (cosx- sinx) = cos $(\frac{\pi}{2} - (sinx + cosx))$

$\therefore cosx - sinx = 2n\pi \pm (\frac{\pi}{2} - (sinx + cosx))$

Taking + ve sign

$cosx - sinx = 2n\pi + \frac{\pi}{2} - sinx - cosx$

$cosx = n\pi + \frac{\pi}{4}$, for n= 0, $cosx = \frac{\pi}{4}$, which is the only possible value

$\Rightarrow sinx = \frac{\sqrt{16-\pi^2}}{4}$ ………………..(i)

Taking –ve sign

$sinx = \frac{\pi}{4}$ ……………..(ii)

From (i) & (ii) , we get $\frac{\pi}{4}$ as the largest value. Hence k= 4.

18. The number of solution(s) of the equation $z^2 - z - |z|^2 - \frac{64}{|z|^5} = 0$ is / are

(a) 0 **(b) 1** **(c) 2**
(d) 3

Ans:- (b) z = 2 is the only solution.

So there is only one solution of the given equation.

19. If function f(x) = cos(nx)×sin($\frac{5x}{n}$), satisfies f(x+ 3π)= f(x), then find the number of integral value of n

(a) 8 **(b) 9** **(c) 10**
(d) 11

Ans:- (a) f(x+ λ) = f(x)

$\Rightarrow$ cosn(x+ λ) sin$\left(\frac{5(x+\lambda)}{n}\right)$ = $\cos(nx)$ sin$(\frac{5x}{n})$

At x = 0, cos(nλ)sin $(\frac{5\lambda}{n})$= 0

If cos(nλ)= 0, nλ= rπ+$\frac{\pi}{2}$, r ∈ I

n (3π)= rπ+$\frac{\pi}{2}$ ($\because\lambda$= 3π)

(3n- r)= ½ [not possible]

$\therefore$ cosn$\lambda \neq 0 \therefore$ sin $(\frac{5\lambda}{n})= 0 \Rightarrow \frac{5\lambda}{n} = P\pi(P \in I) \Rightarrow n = \frac{15}{P}$

For P= $\pm1, \pm3, \pm5, \pm15$

n=$\pm15, \pm5, \pm3, \pm1$

20. Let a, b, c be any real numbers such that $a^2 + b^2 + c^2 = 1$ then the quantity ab +bc+ ca satisfies the conditions

(a) ab+ bc+ ac = constant
(b) - ½ ≤ $ab + bc + ca \leq 1$
(c) - ¼ ≤ $ab + bc + ca \leq 1$
(d) -1 ≤ $ab + bc + ca \leq \frac{1}{2}$

Ans:- (a) $(a + b + c)^2 \geq 0$

$\Rightarrow a^2 + b^2 + c^2 \geq -2(ab + bc + ca)$

$\Rightarrow \frac{1}{2} \geq -(ab + bc + ca) \Rightarrow (ab + bc + ca) \geq -\frac{1}{2}$

21. The maximum value of xyz for +ve x, y, z subject to condition that xy + yz+ zx= 12 is
(a) 9 (b) 6 (c) 8 (d) none

Ans:- (c) $\frac{xy + yz + zx}{3} \geq (xy\,.yz.\ zx)^{1/3}$

$\Rightarrow (xyz) \leq 8$

22. Let a, b, c are 3 positive real numbers such that a+ b+ c= 2, then the value of $\frac{a}{1-a}.\frac{b}{1-b}.\frac{c}{1-c}$ is always
(a) > 8 (b) < 8 (c) 8 (d) none

Ans:- (a) Let 1- a= x, 1- b= y, 1- c = z

3- (a+ b+ c) = x+ y+ z = 1(∵ a+ b+ c= 2)

Now, $\frac{1-x}{x}.\frac{1-y}{y}.\frac{1-z}{z}$

$= \frac{y+z}{x}.\frac{z+x}{y}.\frac{y+x}{z}$

$= \left(\frac{y+z}{2}\right)\left(\frac{z+x}{2}\right)\left(\frac{y+x}{2}\right).\frac{8}{xyz} > \sqrt{yz}\sqrt{zx}\sqrt{zy}\,.\frac{8}{xyz}$ (By AM> GM inequality)

$\Rightarrow \frac{a}{1-a}.\frac{b}{1-b}.\frac{c}{1-c} > 8$

23. Let a+ b +c = 1 then the value of the quantity is always $\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1}$
(a) equals 21 (b) ≤ 21 (c) > 21 (d) none

Ans:- (b) 4a+ 4b+ 4c = 4

⇒(4a+ 1)+ (4b+ 1)+ (4c +1) = 7

Applying c-s inequality:- $a_1 = \sqrt{4a+1}$, $a_2 = \sqrt{4b+1}$, $a_3 = \sqrt{4c+1}$ & $b_i = 1$

$\therefore \left(\sum_{i=1}^{3} a_i\,.1\right)^2 \leq \left(\sum_{i=1}^{3} {a_i}^2\right)\left(\sum_{i=1}^{3} 1\right)$; where $a_i = a_1, a_2, a_3$

$\Rightarrow (\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1})^2 \leq (4a+1+4b+1+4c+1) \times (1+1+1)$

$= 3 \times 7 = 21$

24. If f(x)is a polynomial function satisfying f(x)f$\left(\frac{1}{x}\right)$= f(x)+ f$\left(\frac{1}{x}\right)$ and f(3)=28 then f(4) is
(a) 28 (b) 65 (c) 78 (d) none

Ans:- (c) The given functional equation is satisfied by f(x)= $\pm x^n + 1$

f(3)= $+3^3 + 1 = 28$

Hence, n= 3

So, f(4)= $4^3 + 1$ = 65.

25. If 2x+ 4y= 1, then prove that the quantity $x^2 + y^2$ is always greater than equal to
(a) 1/20 (b) 5/64 (c) 1 (d) none

Ans:- (a) Maximize $x^2 + y^2$ subject to 2x+ 4y -1=0 by

Method of Lagrange multiplier⟶

F= $x^2 + y^2 + \lambda(2x + 4y - 1)$

$\frac{\partial F}{\partial x} = 2x + 2\lambda = 0$; $\frac{\partial F}{\partial y} = 2y + 2\lambda = 0$

∴ x= $-\lambda$ ∴ y= -2λ

2x+ 4y = 1 $x_{max} = +\frac{1}{10}$, $y_{max} = \frac{1}{5}$

$\Rightarrow \lambda = \frac{-1}{10}$;

$\therefore x^2 + y^2 \geq \frac{1}{100} + \frac{4}{100} = \frac{5}{100} = \frac{1}{20}$.

26. If a, b, c are positive real numbers ∋ a+ b+ c= 1. The value of $a^2 + b^2 + c^2$ is always

(a) ½ **(b) 1/3**

(c) ¼ **(d) none**

Ans:- (b) Using C-S inequality,

$\left(\sum_{i=1}^{3} x_i y_i\right)^2 \leq \left(\sum {x_i}^2\right)\left(\sum {y_i}^2\right)$ Taking $y_i = 1$ & x_i = a, b, c.

$\Rightarrow (a+b+c)^2 \leq (a^2+b^2+c^2).3$

$\Rightarrow a^2 + b^2 + c^2 \geq \frac{1}{3}$

27. If a, b, c , x are real numbers such that abc≠ 0 $\boldsymbol{and}$ $\frac{xb+(1-x)c}{a} = \frac{xc+(1-x)a}{b} = \frac{xa+(1-x)b}{c}$

Then prove that a+ b+ c equals to

(a) 1 **(b) 2**

(c) 0 **(d) none**

Ans:- (c) $\frac{xb+(1-x)c}{a} = \frac{xc+(1-x)a}{b} = \frac{xa+(1-x)b}{c} = 1$

$\therefore$ x= $\frac{a-c}{b-c}, x = \frac{b-a}{c-a}, x = \frac{c-b}{a-b}$

The only solution of these are: a= b= c or a+ b+ c= 0.

28. If f : R ⟶R is given by f(x)= $\frac{4^x}{4^x+2}$ $\forall x \in R$, check f(x)+f(1-x)= 1.

Hence the value of f$\left(\frac{1}{1997}\right) + f\left(\frac{2}{1997}\right) + \cdots + f\left(\frac{1996}{1997}\right)$ is

(a) 998 **(b) 1996**

(c) 1997 **(d) none**

Ans:- (a) f(1- x)= $\frac{4^{1-x}}{4^{-x+1}+2} = \frac{4/4^x}{4/4^x+2}$ = f(x)+ f(1- x)= 1.

Now, putting x = $\frac{1}{1997}, \frac{2}{1997}, \frac{3}{1997}, \ldots, \frac{998}{1997}$

So, f$\left(\frac{1}{1997}\right) + f\left(\frac{2}{1997}\right) + \cdots + f\left(\frac{1996}{1997}\right)$

$= \underbrace{(1+1+\cdots+1)}_{998 \text{ terms}}$

= 998

29. If gcd (a, b)=1, then gcd (a+b, a-b) is

(a) a or b **(b) 1 or 2**

(c) 1 or 3 **(d) none**

Ans:- (b) let d = gcd (a+ b, a- b)then

d | (a+ b) and d |(a -b).

∴ d | (a +b+ a -b) , ⇒ d| 2a and

∴ d | (a +b- a +b) , ⇒ d| 2b

Thus d |(2a, 2b), ⇒ d|2(a,b)

Hence d= 1 or 2, because gcd(a, b)= 1

30. The number of solution (positive integers) of the equation 3x+ 5y = 1008 is

(a) 61 **(b) 67**

(c)79 **(d) none**

Ans:- (b) x, y ∈ ℕ, then 3 |5y ⇒ 3|y, y = 3k $\forall k \in \mathbb{N}$

Thus 3x + 15k = 1008

⇒ x + 5k= 336

⇒ 5k $\leq$ 335

ISI OBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 4

There will be 30 questions in MMA Paper. For each question, exactly one of the four choices is correct. You get four marks for each correct answer, one mark for each unanswered question, and zero mark for each incorrect answer.

1. If S_n denotes the sum of first n terms of an A.P. whose

(a) P$\sum_{r=1}^{n} r$ (b) n$\sum_{p=1}^{n} p$
(c) a$\sum_{r=1}^{p} r$ (d) none of these

Ans. (d)

$\frac{S_{nx}}{S_x} = \frac{\frac{nx}{2}[2a+(nx-1)d]}{\frac{x}{2}[2a+(x-1)d]} = \frac{n[2a-1]+nxd}{(2a-d)+xd}$

For $\frac{S_{nx}}{S_x}$ to be independent of x

2a - d= 0

∴ 2a= d

Now, $S_p = \frac{p}{2}[2a+(p-1)d] = p^2a$

2. if $a_{n=} \int_0^{\pi} \frac{\sin(2n-1)}{sinx}$ dx, then $a_1, a_2, a_3,$……..are in

(a) A.P. and H.P.
(b) A.P. and G.P. but not in H.P.

(c) G.P. and H.P.
(d) A.P., G.P. and H.P.

Ans. (b)

$$a_{n+1} - a_n$$
$$= \int_0^{\pi} \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin x} dx$$
$$= \int_0^{\pi} \frac{2\cos 2nx.sinx}{sinx} dx = \left[\frac{2\sin 2nx}{2n}\right]_0^{\pi}$$
$$= 0$$

$\therefore a_{n+1} = a_n \Longrightarrow a_1 = a_2 = a_3 = \cdots$
Also $a_1 = \pi \neq 0$

Hence $a_1, a_2, \ldots a_n$ are in A.P. and G.P. but not in H.P. (Equal numbers cannot be in H.P)

3. If a, b, c are proper fractions and are in H.P. and x=$\sum_{n=1}^{\infty} a^n$, y=$\sum_{n=1}^{\infty} b^n$, z=$\sum_{n=1}^{\infty} c^n$,

then x, y, z are in

(a) A.P. (b) G.P. (c) H.P. (d) none of these

Ans. (c)

X = $\frac{a}{1-a} \Longrightarrow a = \frac{x}{1-x}$

Similarly, b = $\frac{y}{1-y}, c = \frac{z}{1-z}$

Now, a, b, c are in H.P.

$\Rightarrow \frac{1+x}{x}, \frac{1+y}{y}, \frac{1+z}{z}$ are in A.P.

$\Rightarrow \frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P.

$\Rightarrow$ x, y, z are in H.P.

4. If a, b, c be the p^{th}, q^{th} and r^{th} terms respectively of an A.P. and G.P. both, then the product of the roots of equation $a^b\, b^c c^a x^2 - abcx + a^c b^a c^b$=0 is equal to

(a) -1 **(b) 1** **(c) 2**
(d) (b-c)(c-a)(a-b)

Ans. (b)

a= x+ (p- 1)d, b= x+ (q-1)d, c= x+ (r-1)d

a=$mn^{p-1}, b = mn^{q-1}, c = mn^{r-1}$

∴ Product of roots =
$(mn^{p-1})^{(r-q)d}, (mn^{q-1})^{(p-r)d}, (mn^{r-1})^{(q-p)d} = m^0 . n^0$= 1.

5. If a, b, c, be the p^{th}, q^{th} and r^{th} terms respectively of a G.P. then the equation-

$a^q b^r c^p x^2$ + pqrx + $a^r b^n c^q$ = 0 has

(a) both roots zero
(b) at least one root zero
(c) no root zero
(d) both roots unity

Ans. (c)

Product of roots= $a^{r-q} b^{p-r} c^{q-p} = 1 \neq 0$

$\Rightarrow$ no root is equal to zero.

6. If $(r)_n$denotes the number rrr….. (n digits), where r=1, 2, 3,…9 and a=$(6)_n$, b=$(8)_n$, c=$(4)_{2n}$, then

(a) a^2+b+c=0
(b) a^2+b-c=0
(c) a^2+b-2c=0
(d) a^2+b-9c=0

Ans. (b)

A = $(6)_n = 6\,6\,6 \ldots 6(n\ digits) = 6 \times 1 + 6 \times 10 + 6 \times 10^2 + \cdots 6 \times 10^{n-1}$

= $\frac{6}{9}(10^n - 1) = \frac{2}{3}(10^n - 1)$

b= $\frac{8}{9}(10^n - 1), c = \frac{4}{9}(10^{2n} - 1)$

Now $a^2 + b$= $\frac{4}{9}(10^n - 1)^2 + \frac{8}{9}(10^n - 1) = \frac{4}{9}(10^n - 1)^2(10^n - 1 + 2) = \frac{4}{9}(10^{2n} - 1) = c$

7. Let a=1 1 1.....1(55 digits), b=1+10+10^2+…10^4, c=1+10^5+10^{10}+10^{15}+…+10^{50}, then

(a) a=b+c **(b) a=bc**
(c) b=ac **(d) c=ab**

Ans. (b)

a= 1+ 10+ $10^2 + \cdots + 10^{54} = \frac{10^{55}}{10-1} = \frac{10^{55}-1}{10^5-1} . \frac{10^5-1}{10-1} = bc$

8. If $\sum_{r=1}^{n} t_r = \sum_{k=1}^{n} \sum_{j=1}^{k} \sum_{i=1}^{j} 2$, then $\sum_{r=1}^{n} \frac{1}{t_r} =$

(a) $\frac{n+1}{n}$ **(b) $\frac{n}{n+1}$**
(c) $\frac{n-1}{n}$ **(d) $\frac{n}{n-1}$**

Ans. (b)

$\sum_{k=1}^{n}\sum_{j=1}^{h}\sum_{i=1}^{j} 2 = \sum_{k=1}^{n}\sum_{j=1}^{k} 2j =$ $2\sum_{k=1}^{n}\sum_{j=1}^{k} j = 2\sum_{k=1}^{n}\frac{k(k+1)}{2} = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$

$=\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{3}$

$\therefore S_n = \frac{n(n+1)(n+2)}{3}$

$\Longrightarrow t_r = S_r - S_{r-1} = \frac{r(r+1)(r+2)}{3} - \frac{(r-1)r(r+1)}{3} = r(r+1)$

$\frac{1}{t_r} = \frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$ $\therefore \sum_{r=1}^{n}\frac{1}{t_r} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$

9. If a=$\sum_{r=1}^{\infty}\frac{1}{r^4}$, then $\sum_{r=1}^{\infty}\frac{1}{(2r-1)^4}$ =

(a) $\frac{16}{15}$a **(b) $\frac{a}{2}$**
(c) $\frac{15}{16}$a **(d) $\frac{14}{15}$a**

Ans. (c)

$\sum_{r=1}^{\infty}\frac{1}{(2r-1)^4} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots + to\ \infty$

$= \left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + to\ \infty\right) - \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \cdots + to\ \infty\right)$

$= a - \frac{1}{2^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + to\ \infty\right) = a - \frac{a}{16} = \frac{15}{16}a$

10. If a_1, a_2, a_3,…are in G.P. having common ratio r such that $\sum_{k=1}^{n} a_{2k-1}$=$\sum_{k=1}^{n} a_{2k+2} \neq 0$, then number of possible values of r is

(a) 1 **(b) 2**
(c) 3 **(d) none of these**

Ans. (c)

Given $a_1 + a_3 + a_5 + \cdots + a_{2n-1} = a^4 + a^6 + a^8 + \cdots + a_{2n+2}$

$= r^3(a_1 + a_3 + a_5 + \cdots + a_{2n-1})$

$\Rightarrow r^3 = 1 \Rightarrow r = 1, \omega, \omega^2$

11. If x^2- x + a - 3= 0 has at least one negative value of x, then complete set of values of ‘a’ is

(a) $(-\infty, 1)$ **(b) $(-\infty, 2)$**
(c) $(-\infty, 3)$ **(d) none**

Ans. (c) x^2- x +a - 3= 0 has at least one negative root and for real roots,

1- 4(a - 3) ≥ 0

⇒ a ≤ $\frac{13}{4}$

⇒ a ∈ $(-\infty, \frac{13}{4})$

Now, both root will be non-negative of D ≥ 0, ⇒ $a - 3 \geq 0$ ⇒ $a \geq 3$

∴ a ∈ $(3, \frac{13}{4})$

∴ a ∈ $(-\infty, \frac{13}{4})$∪ a ∈ $(3, \frac{13}{4})$

∴$(-\infty, 3)$

12. Let α, β are the roots of the equation x^2+ax +b=0, then maximum value of the expression - (x^2+ax +b) - $(\frac{\alpha-\beta}{2})^2$ will be

(a) $\frac{1}{4}(a^2 - 4b)$ **(b) 0**
(c) 1 **(d) none**

Ans. (b) let z= - (x^2-ax +b)

Now, $z_{max.} = -\frac{D}{4a} = -\frac{a^2-4b}{4} = \frac{4b-a^2}{4} = +(\frac{\alpha-\beta}{2})^2$

∴Thus the maximum value of the given equation is 0.

13. Let P (x) = x^2+bx +c, where b and c are integers and P(x) is a factor of

both x^4+6x^2+25 *and* $3x^4+4x^2+28x+5$, then P(1) is

(a) 4 **(b) 8**
(c) 24 **(d) none**

Ans. (a) ∴ P(x) is a factor of 3 (x^4+6x^2+25)- $(3x^4+4x^2+28x+5)$= $14(x^2-2x+5)$

∴ P(x) = x^2-2x+5

⇒ P(1)= 4.

14. The value of a for which $(a^2-1)x^2+2(a-1)x+2>0\ \forall\ x$ are

(a) a≥ 1 **(b) a≤ 1**
(c) a > - 3 **(d) none**

Ans. (d) we know, $Px^2+qx+c>0$ if P > 0, and $q^2-4Pc<0$,

∴ $(a^2-1)x^2+2(a-1)x+2>0\ \forall\ x$

Now, $a^2-1>0$ and 4 $(a-1)^2-8(a^2-1)\le 0$

⇒ $a^2-1\ge 0$ and -4(a-1)(a+3) ≤ 0

⇒ a ≤ -1 or a ≥ 1 $and\ a\le -3\ or\ a\ \ge 1$

i.e., $a\le -3\ or\ a\ \ge 1$.

15. The sum of real roots of the equation $x^2-2^{2007}.x+|x-2^{2006}|+2(2^{4011}-1)=0$ is

(a) 2^{2006} **(b) 2^{2007}**
(c) 2^{2006}+2^{2007} **(d) none**

Ans. (b) ∴ $(x-2^{2006})^2+|x-2^{2006}|-2=0$

⇒ $|x-2^{2006}|^2$+ $|x-2^{2006}|-2$=0 ⇒ x= 2^{2006}+1, 2^{2006} -1.

∴The sum of real roots are =2^{2007}

16. Consider an expression x^2+y^2+2x+y= constant. If for two constants α, β, the conditions x> α and x > β imply the same limits for the value of y, then α +β is

(a) -2 **(b) -4**
(c) 1 **(d) none**

Ans. (a) $x^2+y^2+2x+y=k$

⇒ $(x+1)^2+(y+\frac{1}{2})^2=k+\frac{5}{4}$

⇒ x= -1$\pm\sqrt{(k+\frac{5}{4})^2-(y+\frac{1}{2})^2}$

Now, the two values of x corresponds to α and β as y takes the same limits of values.

Hence α +β= -2.

17. $\frac{a^4+b^4}{a^2+b^2}+\frac{b^4+c^4}{b^2+c^2}+\frac{c^4+a^4}{c^2+a^2}\ge$

(a) a+ b+ c **(b) $a^2+b^2+c^2$**
(c) ab+ bc+ ca
(d) none

Ans. (b) $(a^2-b^2)^2\ge 0$

$\Rightarrow a^4 + b^4 \geq 2a^2.b^2$

$\Rightarrow 2a^4 + 2b^4 \geq a^4 + b^4 + 2a^2.b^2 = (a^2 + b^2)^2$

$\Rightarrow \frac{a^4+b^4}{a^2+b^2} \geq \frac{a^2+b^2}{2}$ ……………(1)

Similarly, $\frac{b^4+c^4}{b^2+c^2} \geq \frac{b^2+c^2}{2}$ ……………(2)

And, $\frac{c^4+a^4}{c^2+a^2} \geq \frac{c^2+a^2}{2}$ ……………………(3)

(1)+(2)+(3) implies

$\frac{a^4+b^4}{a^2+b^2} + \frac{b^4+c^4}{b^2+c^2} + \frac{c^4+a^4}{c^2+a^2} \geq a^2 + b^2 + c^2$.

18. Let m > 1, n ∈ℕ, then $\mathbf{1^m + 2^m + 2^{2m} + 2^{3m} + \cdots + 2^{nm-m}}$>

(a) $\boldsymbol{n^{1-m}}$ **(b) $\boldsymbol{(1-m)^n}$**
(c) $\boldsymbol{n^{1-m}(2^{n-1})}$ **(d) none**

Ans. (c) $\frac{1^m+2^m+2^{2m}+2^{3m}+\cdots+(2^{n-1})^m}{n} > (\frac{1+2+4+\cdots+2^{n-1}}{n})^m$

[∵ m> 0 and AM of mth power > mth power of AM]

$\Rightarrow 1^m + 2^m + 2^{2m} + 2^{3m} + \cdots + (2^{n-1})^m > n(\frac{2^{n-1}}{n})^m > n^{1-m}(2^{n-1})^m$

19. Let $\boldsymbol{x^2 + y^2 = c^2}$, then the least value of $\boldsymbol{x^{-2} + y^{-2}}$ is

(a) c **(b) $\boldsymbol{c^2}$**
(c) $\boldsymbol{c^3}$ **(d) none**

Ans. (d) Let z= $x^{-2} + y^{-2} = \frac{x^2+y^2}{x^2y^2} = \frac{c^2}{x^2y^2}$ and

It will be minimum when x^2y^2 will be maximum.

As $x^2 + y^2 = c^2$, then x^2y^2 is maximum when $x^2 = y^2 = \frac{c^2}{2}$

$\therefore z_{min.} = \frac{c^2}{\frac{c^4}{4}} = \frac{4}{c^2}$.

20. $\boldsymbol{n^n(\frac{n+1}{2})^{2n}}$ >

(a) n! **(b) $\boldsymbol{(n!)^2}$**
(c) $\boldsymbol{(n!)^3}$ **(d) none.**

Ans. (c) $\frac{1^3+2^3+\cdots+n^3}{n} > (1^3.2^3.\ldots.n^3)^{\frac{1}{n}}$
[∵AM> GM]

$\Rightarrow \frac{n(n+1)^2}{4} > \{(n!)^3\}^{\frac{1}{n}}$

$\Rightarrow n^n(\frac{n+1}{2})^{2n} > \{(n!)^3\}$.

21. If $\boldsymbol{a_1, a_2, a_3, \ldots, a_n}$ are non- negative and $\boldsymbol{a_1, a_2, a_3, \ldots, a_n = 1}$,

then (1+$\boldsymbol{a_1}$)(1+$\boldsymbol{a_2}$) …(1+$\boldsymbol{a_n}$)≥

(a) $\boldsymbol{2^n}$ **(b) $\boldsymbol{3^n}$**
(c) $\boldsymbol{4^n}$ **(d) none**

Ans. (a) $\left(\frac{1+a_i}{2}\right) \geq \sqrt{ai}$, where i= 1(1)n. (AM ≥ GM)

Putting the all I value and then multiplies the in equations,

(1+a_1)(1+a_2) … (1+a_n)≥ $2^n\sqrt{a_1, a_2, a_3, \ldots, a_n}$

⇒ (1+a_1)(1+a_2) … (1+a_n)≥ 2^n(∵$a_1, a_2, a_3, \ldots, a_n = 1$)

22. If $a_1, \dots, a_n$ are positive real nos. whose product is a fixed number c, then the minimum value of $a_1 + a_2 + \dots + a_{n-1} + 2_{an}$ is

(a) $n(2c)^{\frac{1}{n}}$ **(b) $(n+1)c^{\frac{1}{n}}$**
(c) $2nc^{\frac{1}{n}}$ **(d) done**

Ans. (a) AM ≥ GM

So, LHS ≥ $n(a_1 \dots 2_{an})^{\frac{1}{n}} = n(2c)^{\frac{1}{n}}$

23. If f(x) = $\int_0^x \frac{e^{\cos t}}{e^{\cos t}+e^{-\cos t}}dt$, then 2f($\pi$) =

(a) 0 **(b) π** **(c) $-\pi$**
(d) none of these

Ans. (b)

$f(\pi) = \int_0^\pi \frac{e^{\cos t}}{e^{\cos t}+e^{-\cos t}}dt$..........(1)

$f(\pi) = \int_0^\pi \frac{e^{-\cos t}}{e^{-\cos t}+e^{\cos t}}dt$ ....(2)

[since cos (π-t)= - cost]

∴ 2f(π)= $\int_0^\pi dt = \pi$

24. Let [x] denotes the greatest integer less than or equal to x, then $\int_0^{\frac{\pi}{4}} \sin x \, d(x - [x])$=

(a) ½ **(b) 1 - $\frac{1}{\sqrt{2}}$**
(c) 1 **(d) none of these**

Ans. (b) $\int_0^{\pi/4} \sin x \, d(x - [x]) = \int_0^{\pi/4} \sin x \, dx = -[\cos x]_0^{\frac{\pi}{4}} = -\left[\frac{1}{\sqrt{2}} - 1\right] = 1 - \frac{1}{\sqrt{2}}$

[∵ 0 < x <$\frac{\pi}{4}$ ∴ $[x] = 0$]

25. Let g(x) = $\int_0^x f(t)dt$, where $\frac{1}{2} \le f(t) \le 1, t \in [0,1]$ and $0 \le f(t) \le \frac{1}{2}$ for $t \in (1,2]$. Then

(a) $-\frac{3}{2} \le g(2) < \frac{1}{2}$ **(b) $0 \le g(2) < 2$**
(c) $\frac{3}{2} < g(2) \le \frac{5}{2}$ **(d) 2 < g(2)< 4**

Ans. (b) $g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt$ as $\frac{1}{2} \le f(t) \le 1$ for $0 \le t \le 1$,

$$\therefore \int_0^1 \frac{1}{2}dt \le \int_0^1 f(t)dt \le \int_0^1 1\,dt$$

or, $\frac{1}{2} \le \int_0^1 f(t)dt \le 1$ .... (1)

$$as\ 0 \le f(t) \le \frac{1}{2}\ for\ 1 < t \le 2,$$

$$\therefore \int_1^2 0\ dt \le \int_1^2 f(t)\,dt \le \int_1^2 \frac{1}{2}dt$$

or, $0 \le \int_1^2 f(t)\,dt \le \frac{1}{2}$ .... (2)

(1) + (2) ⟹ $\frac{1}{2} \le g(2) \le \frac{3}{2}$

∴ g(2) satisfies the inequality 0≤ $g(2) < 2$.

26. The tangent at point P of a curve meets the y- axis at B, the line through P parallel to y-axis meets the x-axis at A. If the area of ΔAOB is constant, the curve is

(a) parabola **(b) hyperbola**
(c) ellipse **(d) circle**

Ans. (b)

Let P= (x, y)

Equation of tangent to the curve at P(x, y) is

Y- y = $\frac{dy}{dx}(X-x)$

When X= 0, Y= y – x $\frac{dy}{dx}$

$\therefore$B≡ $\left(0, y - x\frac{dy}{dx}\right)$

Area of ΔAOB ant=k

$\therefore \frac{1}{2}x\left(y - x\frac{dy}{dx}\right) \Rightarrow xy - \frac{x^2 dy}{dx} = \pm 2k$

$$\Rightarrow \frac{x^2 dy}{dx} - xy = \pm 2k = c$$

$$\Rightarrow \frac{dy}{dx} + y\left(-\frac{1}{x}\right) = \frac{2}{x^2}$$

$$I.F. = e^{-\log x} = \frac{1}{x}$$

$$solution\ is\ given\ by\ y, \frac{1}{x} = \int \frac{C}{x^3}dx + a$$

$$or\ y = x\left(-\frac{C}{2x^2}\right) + ax$$

$$or\ 2xy = -C + 2ax^2$$

$or\ 2ax^2 - 2xy - C = 0$ …………(1)

Here h= -1, a= a, b= 0

$\therefore h^2$> ab. Hence curve (1) is a hyperbola

27. The function f(k) = $\frac{d}{dk}\int_0^k \frac{dx}{1-\cos k.\cos k}$ satisfies the differentiable equation

(a) $\frac{df}{dk} + 2f(k).\cot k = 0$
(b) $\frac{df}{dk} + 2f(x).\cos k = 0$
(c) $\frac{df}{dk} - 2f(k).cos^2 k = 0$
(d) none of these

Ans. (a) $f(x) = \frac{1}{1-\cos k \cot k} = cosec^2 k$

$$\frac{df}{dk} = 2\ cosec\ k\ (-cosec\ k \cot k)$$
$$= -2f(k)\cot k$$

$$or \frac{df}{dk} + 2f(k)cotk = 0$$

28. The largest value of 'c' such that there exists a differentiable function f(x) for –c< x < c that satisfies the equation $y_1 = 1 + y^2$ with f(0)= 0 is

(a) 1 **(b) π**
(c) $\frac{\pi}{3}$ **(d) $\frac{\pi}{2}$**

Ans. (d) $\frac{dy}{dx} = 1 + y^2 \Rightarrow \tan^{-1} y = x + k$

$\because$ f(x) satisfies the equation

$\therefore \tan^{-1} f(x) = x + k$

Now, f(0)= 0= k= 0

$\Rightarrow$ x= $\tan^{-1} f(x)$ $\therefore -\frac{\pi}{2} < x < \frac{\pi}{2}$

29. If y= (x) and $\frac{2+\sin x}{y+1}\left(\frac{dy}{dx}\right) = -\cos x$, $y(0) = 1$, $y\left(\frac{\pi}{2}\right)$ equals:

(a) 1/3 **(b) 2/3** **(c) -1/3** **(d) 1**

Ans. (a) Given, $\frac{2+\sin x}{y+1}\frac{dy}{dx} = -\cos x$ ………..(1) & $y(0) =$ 1 ………….(2)

$$(1) \Rightarrow \int \frac{dy}{y+1} = -\int \frac{\cos x}{2+\sin x}dx$$

$$\Rightarrow \log(y+1) = -\log(2+\sin x) + \log c$$

$$\Rightarrow y+1 = \frac{c}{2+\sin x}$$

$\Rightarrow y = \frac{c}{2+\sin x} - 1$ …………..(3)

$$Also\ given\ y(0) = 1 \Rightarrow 1 = \frac{c}{2} - 1 \Rightarrow c = 4$$

$$\therefore from\ (3), y(x) = \frac{2-\sin x}{2+\sin x}$$

$$\therefore y = \left(\frac{\pi}{2}\right) = \frac{2-1}{2+1} = \frac{1}{3}$$

30. A permutation of 1, 2, …., n is chosen at random. Then the probability that the numbers 1, 2 appear as neighbor equals

(a) $\frac{1}{n}$ **(b)** $\frac{2}{n}$ **(c)** $\frac{1}{n-1}$ **(d)** $\frac{1}{n-2}$

Ans. (a) P(A)=$\frac{(n-1)!}{n!} = \frac{1}{n}$.

Since there are n! Permutations total. Since 1, 2 appear as neighbour, so taking it as a group, so there are total (n-1)! as number of favorable cases.

ISI OBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 5

There will be 30 questions in MMA Paper. For each question, exactly one of the four choices is correct. You get four marks for each correct answer, one mark for each unanswered question, and zero mark for each incorrect answer.

1. If $a_n = \int_0^{\frac{\pi}{2}} \frac{\sin^2 nx}{\sin^2 x}dx$**, then** $\begin{bmatrix} a_1 & a_{51} & a_{101} \\ a_2 & a_{52} & a_{102} \\ a_3 & a_{53} & a_{103} \end{bmatrix} =$

(a) 1 **(b) 0** **(c) -1** **(d) none of these**

Ans. (b) $a_{n+2} + a_n - 2a_{n+1} = 0$

$\Rightarrow a_1, a_2, a_3, \ldots are\ in\ A.P.$

$\therefore a_1 + a_{101} = 2a_1 + 2a_1 + 100d = 2(a_1 + 50d) = 2a_{51}$

$a_2 + a_{102} = 2a_{52}, a_3 + a_{103} = 2a_{53}$

2. If $t_r = 2^{r/3} + 2^{-r/3}$**, then** $\sum_{r=1}^{100} t_r^{\ 3} - 3\sum_{r=1}^{100} t_r + 1 =$

(a) $\frac{2^{101+1}}{2^{100}}$ **(b)** $\frac{2^{101-1}}{2^{100}}$ **(c)** $\frac{2^{201-1}}{2^{100}}$ **(d) None of these**

Ans. (c)

$t_r^{\ 3} = 2^r + 2^{-r} + 3t_r$

$\therefore \sum_{r=1}^{100} t_r{}^3 = \sum_{r=1}^{100} 2^r + \sum_{r=1}^{100} \frac{1}{2^r} + 3\sum_{r=1}^{100} t_r$

$= 2(2^{100} - 1 + \frac{\frac{1}{2}\left(1-\frac{1}{2^{100}}\right)}{1-\frac{1}{2}} + 3\sum_{r=1}^{100} t_r =$

$2^{101} - 2 + 1 - \frac{1}{2^{100}} + 3\sum_{r=1}^{100} t_r$

$= \frac{2^{201}-1}{2^{100}} - 1 + 3\sum_{r=1}^{100} t_r$

3. If $\sum_{r=1}^{n} r$.r! = 100! - 1, then n equals

(a) 100 **(b) 101**
(c) 99 **(d) none of these**

Ans. (c)

t_r= r. r != (r+1 -1)r!= (r+1)! –r!

$\therefore \sum_{r=1}^{n} t_r = (n+1)! - 1! = (n+1)! - 1$

4. If m = $\sum_{r=0}^{\propto} a^r$, n=$\sum_{r=0}^{\propto} b^r$ where 0<a<1,0<b<1,then the quadratic equation whose

roots are a and b is

(a) mnx^2+(m+n-2mn)x+mn-m-n+1=0
(b) mnx^2+(2mn-m-n)x+mn-m-n+1=0

(c) mnx^2+(2mn+m+n)x+mn+m+n+1=0
(d) mnx^2-(2mn+m+n)x+mn+m+n+1=0

Ans. (a)

m = $\frac{1}{1-a} \Rightarrow a = \frac{m-1}{m}, similarly\ b = \frac{n-1}{n}$.

Required quadratic equation is

$x^2 - (a+b)x + ab = 0$

or, $x^2 - \left(\frac{m-1}{m} + \frac{n-1}{n}\right)x + \frac{(m-1)(n-1)}{mn} = 0$

or, mn$x^2 - (2mn - m - n)x + mn - m - n + 1 = 0$

5. If $\sum_{r=1}^{n} r^4 = a_n$, then $\sum_{r=1}^{n} r^4\,(2r - 1)^4$=

(a) a_{2n}+a_n **(b) a_{2n}-a_n**
(c) a_{2n}-16a_n **(d)a_{2n}+16b_n**

Ans. (c) $\sum_{r=1}^{n}(2r-1)^4 = 1^4 + 3^4 + 5^4 + \cdots + (2n-1)^4$

$= [1^4 + 2^4 + 3^4 + \cdots + (2n)^4] - [2^4 + 4^4 + 6^4 + \cdots + (2n)^4]$

$= a_{2n} - 2^4(1^4 + 2^4 + 3^4 + \cdots + n^4) = a_{2n} - 16a_n$.

6. If positive numbers a, b, c be in H.P., then equation $x^2 - kx + 2b^{101} - a^{101} = 0$ ($k \in R$) has

(a) both roots positive
(b) both roots negative

(c) one positive and one negative root
(d) both roots imaginary.

Ans. (c)

a, b, c are in H.P.

⟹H.M. of a and c= b⟹$\sqrt{ac}$ > b (∵G.M. > H.M.)

Since A .M. > G.M.

$\therefore \frac{a^{101}+c^{101}}{2} > (\sqrt{ac})^{101} > b^{101}$ [$\because \sqrt{ac} > b$]

$\Rightarrow 2b^{101} - a^{101} - c^{101} < 0$

Let f(x)= $x^2 - kx$+2$b^{101} - a^{101}$-c^{101}

Then $f(-\infty) = \infty > 0, f(0) = 2b^{101} - a^{101} - c^{101} < 0, f(\infty) = \infty > 0$.

Hence equation f(x)= 0 has one root in $(-\infty, 0)$and other in$(0, \infty)$.

7. If the sum of the series $\sum_{n=0}^{\infty} r^n$, |r|<1, is s, then sum of the series $\sum_{n=0}^{\infty} r^{2n}$ is

(a) s^2 **(b) $\frac{2s}{s^2-1}$**

(c)$\frac{s^2}{2s+1}$ **(d)$\frac{s^2}{2s-1}$**

Ans. (d)

s= $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots to\ \infty = \frac{1}{1-r}$

∴ r=1- $\frac{1}{s} = \frac{s-1}{s}$

$\sum_{n=0}^{\infty} r^{2n} = \frac{1}{1-r^2} = \frac{1}{1-\frac{(s-1)^2}{s^2}} = \frac{s^2}{2s-1}$

8. The limit of the product $\sqrt[2]{5}, \sqrt[4]{5}, \ldots. \sqrt[2n]{5}$ as n→∞ is

(a) $\frac{1}{5}$ **(b) $\log_{10} 5$**

(c) 1 **(d) 5**

Ans. (d) Required limit= $\underset{n\to\infty}{\text{Lt}}\ 5^{\frac{1}{2}}.5^{\frac{1}{4}}.5^{\frac{1}{8}} \ldots.5^{\frac{1}{2^n}} = \underset{n\to\infty}{\text{Lt}}\ 5^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}} = 5^{\frac{\frac{1}{2}}{1-\frac{1}{2}}} = 5$

9. If numbers p, q, r are in A.P. , then m^{7p}, m^{7q}, m^{7r} (m>0) are in

(a) A.P. **(b) G.P.**

(c) H.P. **(d)none of these**

Ans. (b) $\frac{m^{7q}}{m^{7p}} = m^{7(q-p)}, \frac{m^{7r}}{m^{7q}} = m^{7(r-q)}$

∴ q- p= r- q

∴ m^{7p}, m^{7q}, m^{7r} are in G.P.

10. Let n be a positive integer and $(1 + x + x^2)^n = a_0 + a_1x + \cdots + a_{2n}x^{2n}$, then the value of $a_0{}^2 - a_1{}^2 + a_2{}^2 \ldots\ldots + a_{2n}{}^2$ is

(a) 0 **(b) a_0**

(c) a_n **(d) a_{2n}**

Ans. (c) Replacing x by (- 1/x), we get

$$(1 - \frac{1}{x} + \frac{1}{x^2})^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots - a_{2n-1}.\frac{1}{x^{2n-1}} + \frac{a_{2n}}{x^{2n}}$$

or, $(1 - x + x^2)^n = a_0x^{2n} - a_1x^{2n-1} + a_2x^{2n-2} + \cdots + a_{2n}$ ……………….(1)

And given $(1 + x + x^2)^n = a_0 + a_1x + \cdots + a_{2n}x^{2n}$…………………(2)

Multiplying corresponding sides of (1) and (2), we have

$(1 + x^2 + x^4)^n = (a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}) \times (a_0x^{2n} - a_1x^{2n-1} + a_2x^{2n-2} + \cdots + a_{2n})$ …….. (3)

$(1 + x^2 + x^4)^n = (a_0 + a_1x^2 + a_2x^4 + \cdots + a_nx^n + \cdots + a_{2n}x^{4n})$ ……………..(4)

Equating coefficient of x^{2n} on both sides of (3) and (4)

$a_0{}^2 - a_1{}^2 + a_2{}^2 \ldots\ldots + a_{2n}{}^2 = a_n$.

11. The set of all real number x such that ||3-x|-|x+2||=5 is

(a) [3, ∞) **(b) (-∞, −2]**

(c) (-∞, −2] ∪[3, ∞) **(d) (-∞, −3] ∪[2, ∞)**

Ans. (c) $(||3-x|-|x+2||)^2 = 25$

$\Rightarrow (3-x)^2 + (x+2)^2 - 2|3-x||x+2| = 25$

$\Rightarrow x^2 - x - |-x^2 + x + 6| = 6$

So, it is clear that $-x^2 + x + 6 < 0$, i.e. $-x^2 + x + 6 \geq 0$

$(x-3)(x+2) \geq 0$

So, $x \leq -2$ & $x \geq 3$

$\therefore x \in (-\infty, -2] \cup [3, \infty)$

12. The differential equation of the system of circle touch the y – axis at the origin is

(a) $x^2 + y^2 - 2xy\frac{dy}{dx} = 0$
(b) $x^2 + y^2 + 2xy\frac{dy}{dx} = 0$

(c) $x^2 - y^2 - 2xy\frac{dy}{dx} = 0$
(d) $x^2 - y^2 + 2xy\frac{dy}{dx} = 0$

Ans. (d) $x^2 + y^2 - 2ax = 0$

$2x + 2y\frac{dy}{dx} - 2a = 0$

$\Rightarrow 2(x + y\frac{dy}{dx}) = 2\left(\frac{x^2+y^2}{2x}\right)$

$\Rightarrow 2x^2 + 2xy\frac{dy}{dx} = x^2 + y^2$

$\Rightarrow x^2 - y^2 + 2xy\frac{dy}{dx} = 0.$

13. Let y(x) be a non-trivial solution of the second order liner differential equation

$\frac{d^2y}{dx^2} + 2c\frac{dy}{dx} + ky = 0, where\ c < 0, k > 0, c^2 - k$**. Then**

(a) $|y(x)| \longrightarrow \infty$ as $x \to \infty$
(b) $|y(x)| \longrightarrow 0$ as $x \to \infty$

(c) $\lim_{x \to \pm\infty} |y(x)|$ **exists & is finite**
(d) none

Ans. (a) $m^2 + 2cm + k = 0$

$\therefore m = \frac{-2c \pm \sqrt{4c^2 - 4k}}{2} = \frac{-2c \pm \sqrt{4(c^2 - k)}}{2} = \frac{-2c \pm 2a}{2}$ $[\because c^2 - k = a^2 = c^2 - k]$

$= \frac{-c-a}{2}, \frac{-c+a}{2}$

The general solution of the given L.D.E. is y $= c_1e^{m_1x} + c_2e^{m_2x} = c_1e^{-(\frac{c+a}{2})x} + c_2e^{-(\frac{c-a}{2})x}$

So, $|y(x)| \longrightarrow \infty$ as $x \to \infty$

14. Let y be a function of x satisfying $\frac{dy}{dx} = 2x^3\sqrt{y} - 4xy$**. If y(0)= o and then y(1)equals**

(a) $\frac{1}{4e^2}$ **(b) 1/e**
(c) $e^{1/2}$ **(d)** $e^{3/2}$

Ans. (a) $\frac{dy}{dx}(4x)y = 2\sqrt{y}x^3$ (Bernoulli's Equation)

Putting $\sqrt{y} = z$, the equation reduces to

$\frac{dz}{dx} + (2x)z = x^3$ (linear in z)

$\therefore$ I. F.$= e^{\int 2x dx} = e^{x^2}$

Multiplying and integrating

$ze^{x^2} = \int x^3 e^{x^2} dx$ (put $x^2 = u$)

$= \frac{1}{2}(x^2 - 1)e^{x^2} + c$

$\therefore$ General solution is given by:- $\sqrt{y} = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$

Since y(0)=0, so, c= $\frac{1}{2}$

$\therefore$y(1)= $(\frac{1}{2e})^2 = \frac{1}{4e^2}$.

15. Let x_i are non -ve reals and s= $x_1 + x_2 + \cdots + x_n$, $then\ x_1x_2 + x_2x_3 + \ ... + x_{n-1}x_n \leq$

(a) $\frac{s^2}{2}$ **(b) $\frac{s^2}{3}$**
(c) $\frac{s^2}{4}$ **(d) none**

Ans. (c) $(x_1 + x_3 + x_5 + \cdots)(x_2 + x_4 + x_6 + \cdots) \geq \ x_1x_2 + x_2x_3 + \ ... + x_{n-1}x_n$

As when expanding LHS, we must get RHS and many additional non- negative terms since $x_i \neq 0$.

Thus maximum achieved by taking $x_1 = x, x_2 = s - a$ and all other terms 0, but

x(s-x) $\leq \frac{s^2}{4}$ with equality when x = $\frac{s}{2}$ (using AM $\geq$ GM)

16. For any positive reals x, y, z and a is the arithmetic mean of x, y, z then $x^xy^yz^z$ is

(a) $\geq (xyz)^a$ **(b) < $(xyz)^a$**
(c) >$(xyz)^a$ **(d) none**

Ans. (a) Let $\geq y \geq z$, then $x^xy^y \geq x^yy^x$, as $\left(\frac{x}{y}\right)^x \geq \left(\frac{x}{y}\right)^y$ is obviously true.

Similarly, $y^yz^z \geq y^zz^y and z^zx^x \geq \ z^xx^z$

Multiplying all these, $(x^xy^yz^z)^2 \geq x^{y+z}.y^{z+x}.z^{x+y}$

$\Rightarrow x^xy^yz^z \times (x^xy^yz^z)^2 \geq x^{x+y+z}.y^{x+y+z}.z^{x+y+z}$

$\Rightarrow (x^xy^yz^z)^3 \geq (xyz)^{3a}$

$\Rightarrow x^xy^yz^z \geq (xyz)^a$

17. The number of integers between 1 and 567 are divisible by either 3 or 5, is

(a) 200 **(b) 250**
(c) 300 **(d) none**

Ans. (d) Let z= {1, 2, 3, …., 566, 567}

P = {x ∈ $\frac{z}{3}$ $devides\ x$} and

Q = { x ∈ $\frac{z}{5}$ $devides\ x$ }

Here, |P|= 189 [$\because$ 567= 189$\times$ 3]

And |Q|= 113 [$\because$ 567= 113$\times$ 5 + 2]

P $\cap Q$= set of multiple of both 3 and 5,

| P $\cap Q$ |=37; |P $\cup Q$|= 189+ 113- 37= 265.

18. Sets A and B have 3 and 6 elements respectively. The minimum number of elements in A $\cup$B is

(a) 3 **(b) 6**
(c) 9 **(d) none**

Ans. (b) n(A $\cup$ B)$\geq$ max$\{n\ (A), n\ (B)\}$

Thus n (A $\cup$ B)$\geq$ max {3, 6} = 6.

19. A has n elements. How many (B, C) are such that $\neq B \subseteq C \subseteq A$?

(a)2^n **(b) 3^n** **(c) 4^n** **(d) none**

Ans. (b) There are $\binom{n}{m}$ choices for a subject B with m elements.

Then each of the remaining n-m elements can be in C or not, so there are 2^{n-m} choices for C

Thus the total no of pairs (B, C) is $\sum 2^{n-m} . n_{c_m} = \sum 2^m . n_{c_m} = (1+2)^n = 3^n$ (from binomial theorem) [$\because n_{c_m} = n_{c_{n-m}}$]

20. The value of the integral $\int_{-10}^{0} \frac{\left|\frac{2[x]}{3x-[x]}\right|}{\frac{2[x]}{3x-[x]}}$ dx, where [.] denotes greatest integer function is

(a) 0 **(b) 10**
(c) -10 **(d) none of these**

Ans. (d)

Let $f(x) = \frac{\left|\frac{2[x]}{3x-[x]}\right|}{\frac{2[x]}{3x-[x]}}$

Clearly f is not defined if x= 0 and when 3x= [x]

So in (-10, 0), f is not defined at x= $-\frac{1}{3}$.

When x$\in(-10, -\frac{1}{3})$

[x] < 0 and 3x- [x]< 0

So, $\frac{[x]}{3x-[x]} > 0 \implies f(x) = 1$

When x $\in (-\frac{1}{3}, 0)$

[x] < 0 and 3x- [y] > 0$\implies$ f(x)= -1

$$\int_{-10}^{0} f(x)\,dx = \int_{-10}^{-1/3} dx + \int_{-1/3}^{0} (-1)\,dx$$
$$= [x]\begin{matrix} -\frac{1}{3} \\ -10 \end{matrix} - (x)\begin{matrix} 0 \\ -\frac{1}{3} \end{matrix}$$
$$= \left(-\frac{1}{3} + 10\right) - \left(0 + \frac{1}{3}\right)$$
$$= 10 - \frac{2}{3}$$

21. The equation $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(a|\sin x| + \frac{b\sin x}{1+\cos x} + c\right) dx$= 0 gives a relation between

(a) a, b and c **(b) a and b**
(c) b and c **(d) a and c**

Ans. (d) I = 2a$\int_0^{\frac{\pi}{4}} |\sin x|\,dx + 0 + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} c\,dx = 2a\int_0^{\frac{\pi}{4}} sinx\,dx + c.\frac{\pi}{2}$

$$= -2a[cosx]\begin{matrix} \frac{\pi}{4} \\ 0 \end{matrix} + \frac{\pi}{2}c$$
$$= -2a\left(\frac{1}{\sqrt{2}} - 1\right) + \frac{\pi}{2}c$$

22. Let f(x) = max. {2- x, 2, 1+ x} then $\int_{-1}^{1} f(x)\,dx$=

(a) 0 **(b) 2** **(c) 9/2** **(d) none of these**

Ans. (c) ∴ f(x) = 2-x, x≤ 0

= 2, 0≤x≤1

= 1+ x, x ≥1

I= $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)dx + \int_0^1 f(x)dx = \int_{-1}^{0}(2-x)dx + \int_0^1 2\,dx$

$$= \left[2x - \frac{x^2}{2}\right]\begin{matrix}0\\-1\end{matrix} + 2[x]\begin{matrix}1\\0\end{matrix}$$

$$= 0 - \left(-2 - \frac{1}{2}\right) + 2(1-0)$$

$$= \frac{9}{2}$$

23. Let f(x) be a continuous function such that f(a-x)+f(x)=0 for all x ∊[0, a].

Then $\int_0^a \frac{dx}{1+e^{f(x)}}$ equals

(a) a **(b) a/2** **(c) ½ f(a)** **(d) none of these**

Ans. (b) Given, f(a- x)= - f(x)

Now 2I= $\int_0^a \frac{dx}{1+e^{f(x)}} + \int_0^a \frac{dx}{1+e^{f(a-x)}} = \int_0^a \frac{dx}{1+e^{f(x)}} + \int_0^a \frac{dx}{1+e^{-f(x)}} = \int_0^a dx = a$

$$\therefore I = \frac{a}{2}$$

24. Let f(x) be an integrable odd function in [-5, 5] such that f(10+ x)= f(x),

then $\int_x^{10+x} f(t)dt$ equals

(a) 0 **(b) $2\int_x^5 f(x)dx$**
(c) > 0 **(d) none of these**

Ans. (a) Let y= $\int_x^{x+10} f(t)\,dt$…… (1)

Then, $\frac{dy}{dx} = f(x+10).1 - f(x) = 0$ [∵ f(10+x)= f(x)]

∴ y is independent of x.

Putting x= -5 in (1), we get

y= $\int_{-5}^{5} f(t)\,dt = 0$ ………… (2)

Since y is independent of x, therefore y has same value for all x.

∴$\int_x^{x+10} f(x)\,dx$= 0

25. If $\int_0^1 xe^{x^2}\,dx = k\int_0^1 e^{x^2}\,dx$, then

(a) k > 1 **(b) 0 < k < 1**
(c) k=1 **(d) none**

Ans. (b) Here 0 < x < 1

$\Rightarrow 0 < xe^{x^2} < e^{x^2} \Rightarrow 0 < \int_0^1 xe^{x^2}dx < \int_0^1 e^{x^2}dx \Rightarrow 0 < k\int_0^1 e^{x^2}dx < \int_0^1 e^{x^2}dx$

$\Rightarrow 0 < k$

< 1 $\quad [dividing\ by \int_0^1 e^{x^2}dx]$

26. Consider the parabola $3y^2 - 4y - 6x + 8$=0. The points on the axis of this parabola from where 3 distinct normals can be drawn are given by

(a) $\left(\frac{2}{3}, h\right)$, *where h >* $\frac{29}{18}$
(b) $\left(h, \frac{1}{3}\right)$, *where h >* $\frac{19}{18}$

(c) $\left(h, \frac{2}{3}\right)$, *where h >* $\frac{29}{18}$
(d) none of these

Ans. (c)

Given parabola is $\left(y - \frac{2}{3}\right)^2 = 2\left(x - \frac{10}{9}\right)$

Let X= $x - \frac{10}{9}, Y = y - \frac{2}{3}$

$\therefore Y^2 = 2x$ becomes the equation of parabola with reference to the new origin.

Hence equation of normal will be

Y= mX – m- $x - \frac{m^3}{2}$

[∵ three normals are drawn from point on the axis (H, 0) (say)]

∴ H= $1 + \frac{m^2}{2} \implies$ m= $\pm\sqrt{2H-1}$

For m to be real, H > ½

$\implies h - \frac{10}{9} > \frac{1}{2} \implies h > \frac{29}{18}$

[where h is the abscissa w.r.t. the previous co-ordinate system]

Hence the points are given by $\left(h, \frac{2}{3}\right), where\ h > \frac{29}{18}$.

27. A (x_1, y_1) and B (x_2, y_2) are any two points on the parabola y= c$x^2 + bx + a$. If P(x_3, y_3) be the point on the arc AB where the tangent is parallel to the chord AB, then

(a) x_2 is the A.M. between $x_1 and\ x_3$
(b) x_2 is the G.M. between $x_1 and\ x_3$

(c) x_2 is the H.M. between $x_1 and\ x_3$
(d) none of these

Ans. (d) Slope of tangent at p= $\frac{dy}{dx} at\ (x_3, y_3) = 2ax_3 + b = \frac{y_2 - y_1}{x_2 - x_1}$ [given]……….(A)

∵ A and B lie on the parabola,

∴$y_1 = a{x_1}^2 + bx_1 + c$ ………………(1)

And $y_2 = a{x_2}^2 + bx_2 + c$ ……………..(2)

∴ $y_1 - y_2 = [a(x_1 + x_2)(x_1 - x_2) + b](x_1 - x_2)$

∴ $\frac{y_2 - y_1}{x_2 - x_1} = a\ (x_1 + x_2) + b$

∴ From (A), a$(x_1 + x_2) + b = 2ax_3 + b$

$\implies \frac{x_1 + x_2}{2} = x_3$

28. Let P (α,β) be any point on parabola $y^2 = 4x(0 \le \beta \le 2)$. M is the foot of perpendicular from the focus S to the tangent at P, then the maximum value of area of

(a) 1 **(b) 2** **(c) $\frac{\pi}{3}$** **(d) $\frac{\pi}{6}$**

Ans. (a) Let α= $t^2, \beta = 2t$

∴ 0≤ 2t ≤ 2 $\implies$ 0≤ t ≤ 1

Equation of tangent at $(t^2, 2t)\ is\ yt = x + t^2$

If S be the focus, then S ≡(1, 0)

SM= $\frac{|1+t^2|}{\sqrt{1+t^2}} = \sqrt{1+t^2}$

PS= $\sqrt{(t^2-1)^2 + 4t^2} = (t^2+1)$

PM= $\sqrt{PS^2 - SM^2} = t\sqrt{t^2+1}$

Area of Δ PMS= ½ .PM.SM= ½ .t$\sqrt{t^2+1}.\sqrt{t^2+1}$

$= \frac{t(t^2+1)}{2}$

Which is an increasing function hence its maximum value occurs at t= 1

∴ Maximum area= 1 sq. unit.

29. The point A on the parabola $y^2 = 4x$ for which |AC-AB | is maximum, where

B≡ $(0, a)$ and $C \equiv (-a, 0)$ is

(a) (a, 2a) **(b) (4a, 4a)**
(c) (a- 2a) **(d) none of these**

Ans. (a)

For any three points A, B, and C

|AC - AB|≤ BC

∴ required point A will be on the intersection of BC and the parabola.

∴ A≡ (a, 2a) [∵ AB ia tangent to the parabola]

30. The mean and variance of a binomial variable X are 2 and 1 respectively. If X takes values greater than 1, then its probability will be

(a) $\frac{5}{16}$ **(b) $\frac{11}{16}$**
(c) $\frac{1}{4}$ **(d) none of these**

Ans. (b)

Given, np= 2, npq= 1

∴ q= ½ , p= ½ , n= 4

Now p(X> 1)= 1- P(X≤ 1)

= 1- [P(X=0)+P(X= 1)]

= 1- $[4_{C_0}p^0q^4 + 4_{C_1}p^1q^3] = 1 - \left[\left(\frac{1}{2}\right)^4 + 4.\frac{1}{2}\left(\frac{1}{2}\right)^3\right] = 1 - \frac{5}{16} = \frac{11}{16}$

ISI OBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 6

There will be 30 questions in MMA Paper. For each question, exactly one of the four choices is correct. You get four marks for each correct answer, one mark for each unanswered question, and zero mark for each incorrect answer.

1. If x ∈={1, 2, 3,……, 9}and f_n(x)=xxx……x (n digits),then $f_n^2(3)+f_n(2)$ =

(a) $2f_{2n}(1)$
(b) $f_n^2(1)$

(c) $f_{2n}(1)$
(d)$-f_{2n}(4)$

Ans. (c)

$f_n(x) = x.1 + x.10^2 + x.10^3 + \cdots x.10^{n-1} = x\frac{(10^n-1)}{10-1} = \frac{x}{9}(10^n - 1)$

$\therefore f_n{}^2(3) + f_n(2) = \left[\frac{3}{9}(10^n - 1)\right]^2 + \frac{2}{9}(10^n - 1) = \frac{1}{9}(10^n - 1)(10^n - 1 + 2) = \frac{10^n-1}{9} = f_{2n}(1)$

2. If a_i∈R-{0}, i=1, 2, 3, 4 and x∈R and $(\sum_{i=1}^{3} a_i^2)x^2$ - 2x($\sum_{i=1}^{3} a_i\, a_i$+1) + $\sum_{i=2}^{4} a_i^2 \geq$ 0,

Then a_1,a_2,a_3,a_4 are in

(a) A.P. **(b) G.P.**
(c) H.P. **(d) none of these**

Ans. (b)

Given quadratic expression≥ 0

$\therefore D \leq 0$

$\Rightarrow (\sum_{i=1}^{3} a_i a_i + 1)^2 - (\sum_{i=1}^{3} {a_i}^2)(\sum_{i=1}^{4} {a_i}^2) \leq 0$

$\Rightarrow (a_1 a_2 + a_2 a_3 + a_3 a_4)^2 - ({a_1}^2 + {a_2}^2 + {a_3}^2)({a_2}^2 + {a_3}^2 + {a_4}^2) \leq 0$

$\Rightarrow ({a_2}^2 - a_1 a_3)^2 + ({a_3}^2 - a_2 a_4)^2 + (a_2 a_3 - a_1 a_4)^2 = 0$

$\Rightarrow ({a_2}^2 - a_1 a_3)^2 = 0, ({a_3}^2 - a_2 a_4)^2 = 0, (a_2 a_3 - a_1 a_4)^2 = 0$

$\Rightarrow \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3}$

3. Let a = $\frac{1}{n!} + \sum_{r=1}^{n-1} \frac{r}{(r+1)!}$**, b =** $\frac{1}{m!} + \sum_{r=1}^{m-1} \frac{r}{(r+1)!}$ **then a+b equals**

(a) 0 **(b) 1** **(c)2**
(d) none of these

Ans. (c)

$\frac{r}{(r+1)!} = \frac{r+1-1}{(r+1)!} = \frac{1}{r!} - \frac{1}{(r+1)!}$

$\therefore \sum_{r=1}^{n-1} \frac{r}{(r+1)!} = 1 - \frac{1}{n!} \Rightarrow a = 1$

Similarly, $\frac{1}{m!} + \sum_{r=1}^{m} \frac{r}{(r+1)!} = 1$

$\therefore$ a= 1, b=1 $\Rightarrow$a+ b= 2

4. If $\sum_{n=1}^{k}[\frac{1}{3} + \frac{n}{90}]$ **=21, where [x] denotes the integral part of x, then k=**

(a) 84 **(b) 80**
(c) 85 **(d) none of these**

Ans. (b)

21= $\sum_{n=1}^{k} \left[\frac{1}{3} + \frac{n}{90}\right], where\ m = k!$

$= \left[\frac{1}{3} + \frac{1}{90}\right] + \left[\frac{1}{3} + \frac{2}{90}\right] + \cdots + \left[\frac{1}{3} + \frac{59}{90}\right] + \left[\frac{1}{3} + \frac{60}{90}\right] + \left[\frac{1}{3} + \frac{61}{90}\right] + \cdots + \left[\frac{1}{3} + \frac{k}{90}\right]$

$= (0 + 0 + \cdots to\ 59\ terms). +(1 + 1 + \cdots to\ (k - 59) terms)$

$\therefore$ 21= k- 59$\Rightarrow$ k= 80.

5. Let f: R→R such that f(x) is continuous and attains only rational value at all real x and f(3)=4. If a_1**,**a_2**,**a_3**,**a_4**,**a_5 **are in H.P., then** $\sum_{r=1}^{4} a_r\, a_{r+1}$**=**

(a) f(5).$a_1 a_5$ **(b) f(3).**$a_4 a_5$
(c) f(3).$a_1 a_2$ **(d) f(2).**$a_1 a_3$

Ans. (a)

Since f(x) is continuous and attains only rational values

$\therefore$ f(x)= constant= 4

$\therefore$ f(2) = f(3) = f(5) = 4

Since a_1, a_2, a_3, a_4, a_5 are in H.P.

$\therefore\ a_1\, a_2 + a_2 a_3 + a_3\, a_4 + a_4\, a_5 = 4a_1\, a_5 = f(5). a_1\, a_5$

6. If three successive terms of a G.P. with common ratio r >1 from the sides of a triangle and [r] denotes the integral part of x, then [r] + [-r]=

(a) 0 **(b) 1**
(c) -1 **(d) none**

Ans. (b)

Since root of equation

F(x)= $x^2+2(a-3)x+9=0$ lie between-6 and 1

∴(i) D≥ 0 (ii) f(-6)> 0 (iii) f(1) > 0 (iv) -6 <$\frac{\alpha+\beta}{2}$ (v) 1 > $\frac{\alpha+\beta}{2}$

Hence $6 \le \alpha < \frac{27}{4}$

∴|a| =6

$a_3 = 2+3d = 2+3.\ \frac{6-2}{21} = 2+\frac{4}{7} = \frac{18}{7}$

$\frac{1}{h_{18}} = \frac{1}{2} + 18.\left(\frac{\frac{1}{6}-\frac{1}{2}}{21}\right) = \frac{1}{2}-\frac{2}{7} = \frac{3}{14}$

$\therefore a_3 h_{18} = \frac{18}{7}.\frac{14}{3} = 12$

7. If x_1, x_2, x_3, x_4, x_5 are in H.P. then $\frac{1}{x_1x_5}(\sum_{k=1}^{4} x_k\ x_{k+1})$ is a root of equation

(a) $x^2-3x+2=0$
(b) $x^2-5x-4=0$

(c) $x^2-9x+20=0$
(d) $x^2-6x-8=0$

Ans.(c)

x_1, x_2, x_3, x_4, x_5 are in H.P.

$\therefore \sum_{k=1}^{4} x_k x_{k+1} = x_1x_2 + x_2\ x_3 + x_3x_4 + x_4\ x_5 = 4x_1x_5$

$\therefore \frac{1}{x_1x_5}\sum_{k=1}^{4} x_k x_{k+1} = 4$

Clearly, 4 is a root of equation

$x^2-9x+20$ =0.

8. Let f : (0, ∞)⟶R and F(x)= $\int_0^x f(t)\,dt$
If F$(x^2) = x^2(1+x)$, then $f(4)$ =

(a) 5/4 **(b) 7**
(c) 4 **(d) 2**

Ans. (c)

Given, F(x)= $\int_0^x f(t)\,dt$ ………(1)

$F(x^2) = x^2(1+x)$ ………..(2)

From (1), F'(x)= f(x)

∴ f(4)= F'(4)…………..(3)

From (1),

F'(x^2).2x= 2x+ 3x^2

⟹ F'$(x^2) = \frac{2+3x}{2}$ [$\because 0 < x < \infty \therefore x \neq 0$]

⟹ $F'(4) = \frac{2+6}{2}$ = 4 [Put x= 2]

∴ from (3), f(4)= 4

9. If n > 1 then $\int_0^\infty \frac{dx}{(x+\sqrt{1+x^2})^n}$ =

(a) $\frac{n}{n^2-1}$ **(b) $\frac{n^2-1}{n}$**
(c) $-\frac{n}{n^2-1}$ **(d) $\frac{1-n^2}{n}$**

Ans. (a)

Put z= x+$\sqrt{1+x^2}$

∴ z- x= $\sqrt{1+x^2}$ => $z^2+x^2-2zx = 1+x^2$ => $x = \frac{z^2-1}{2z}$

$$\therefore dx = \frac{1}{2}\frac{[z.2z-(z^2-1).1]}{z^2}dz$$
$$= \frac{z^2+1}{2z^2}dz$$

When x= 0, z= 1 and when x= ∞, $z = \infty$

$\therefore$ I= $\int_1^\infty \frac{1}{z^n}\frac{z^2+1}{2z^2}dz = \frac{1}{2}\int_1^\infty (z^{-n} + z^{-n-2})\,dz = \frac{1}{2}\left[\frac{z^{-n+1}}{-n+1} + \frac{z^{-n-1}}{-n-1}\right]_1^\infty$

$$= \frac{1}{2}\left[0 - \left(\frac{1}{1-n} - \frac{1}{1+n}\right)\right] = \frac{1}{2}\left(-\frac{2n}{1+n^2}\right)$$
$$= \frac{n}{n^2-1}$$

10. If f(x)= ae^{2x}+ be^x +cx satisfies the conditions f(0)= -1, f'(log 2)= 28,

$\int_0^{\log 4}[\boldsymbol{f(x) - cx}]\boldsymbol{dx} = \frac{39}{2}$**, then**

(a) a= 5, b=6, c= 3 (b) a= 5, b= - 6, c= 0 (c) a= -5, b=6, c= 3 (d) none

Ans. (b)

Given f(x)= $ae^{2x} + be^x + cx$ ……(1)

$given, f(0) = -1 \Rightarrow a + b = -1$…..(2)

f'(x)= 2 $ae^{2x} + be^x + cx$

∴f' (log 2)= $ae^{\log_e 4} + be^{\log_e 2} + c$

Given 8a+ 2b+c= 28……….(3)

Given, $\int_0^{\log 4}(ae^{2x} + be^x)\,dx = \frac{39}{2}$

$$\Rightarrow \left[\frac{a}{2}e^{2x} + be^x\right]_0^{\log 4} = \frac{39}{2}$$

$\Rightarrow \frac{a}{2}e^{\log 16} + be^{\log 4} - \left(\frac{a}{2} + b\right) = \frac{39}{2}$

$\Rightarrow$ 15a + 6b= 39……….(4)

Thus a= 5, b= -6, c= 0

11. Let $\boldsymbol{\frac{d}{dx}f(x) = \frac{e^{\sin x}}{x}, x >}$ **0.** $\boldsymbol{If\ \int_1^4 \frac{2e^{\sin x^2}}{x}dx = f(k) - f(1)}$**, then one of the possible value of k is**

(a) -4 (b) 0
(c) 2 (d) 16

Ans. (d)

Given, $\frac{d}{dx}(f(x)) = \frac{e^{\sin x}}{x}, x > 0$

$now\ I = \int_1^4 \frac{2e^{sinx^2}}{x}dx$ [put z= $x^2, dz = 2x\,dx$]

$\therefore$ I= $\int_1^4 \frac{2e^{sinx^2}}{x^2}dx = \int_1^{16}\frac{e^{\sin z}}{z}dz = [f(z)]_1^{16} = f(16) - f(1)$

∴ f(k)= f(16)

∴ one possible value of k= 16

12. All the values of a for which $\boldsymbol{\int_1^2[a^2 + (4-4a)x + 4x^3]\ dx \le 12}$ **are given by**

(a) a= 3 (b) a ≤ 4
(c) 0≤ $\boldsymbol{a}$ ≤ 3 (d) none of these

Ans. (a)

$$\int_1^2 [a^2 + (4-4a)x + 4x^3]\,dx$$
$$= a^2[x]_1^2 + (2-2a)[x^2]_1^2 + [x^4]_1^2$$
$$= a^2 + (2-2a)3 + 15, Given, a^2 - 6a + 21 \le 12$$
$$\Rightarrow a^2 - 6a + 9 \le 0 \Rightarrow (a-3)^2 \le 0 \Rightarrow (a-3)^2 = 0 \Rightarrow a = 3$$

13. $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{(2r)^k}{n^{k+1}}, k \neq -1$, **is equal to**

(a) $\frac{2^k}{k-1}$ **(b)** $\frac{2^k}{k}$

(c) $\frac{1}{k-1}$ **(d)** $\frac{2^k}{k+1}$

Ans. (d)

Reqd. limit= $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{(2r)^k}{n^{k+1}} = \lim_{n\to\infty} 2^k\sum_{r=1}^{n}\frac{r^k}{n^k.n} = \lim_{n\to\infty} 2^k\sum_{r=1}^{n}\left(\frac{r}{n}\right)^k = \frac{1}{n} = 2^k\int_0^1 x^k dx = 2^k.\left[\frac{x^{k+1}}{k+1}\right]_0^1 = \frac{2^k}{k+1}$

14. $\lim_{n\to\infty}\{\frac{n!}{(kn)^n}\}^{\frac{1}{n}}, k \neq 0$, **is equal to**

(a) $\frac{k}{e}$ **(b)** $\frac{e}{k}$

(c) $\frac{1}{ke}$ **(d) none of these**

Ans. (c)

Let P= $\lim_{n\to\infty}\frac{1}{k}\left(\frac{n!}{n^n}\right)^{1/n} = \frac{1}{k}\lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{1/n} = \frac{1}{ke}$

15. $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{n}}{(\sqrt{r}\,(3\sqrt{r}+4\sqrt{n})^2}$=

(a) $\frac{1}{7}$ **(b)** $\frac{1}{10}$

(c) $\frac{1}{14}$ **(d) none of these**

Ans. (c)

Required limit

$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{n}}{\sqrt{r}.n\left(3\sqrt{\frac{r}{n}}+4\right)^2} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{\frac{r}{n}}\left(3+\sqrt{\frac{r}{n}}+4\right)^2}.\frac{1}{n}$

$$= \int_0^1 \frac{1}{\sqrt{x}\,(3\sqrt{x}+4)^2}\,dx$$

Put z= $3\sqrt{x}+4$, then dz= $\frac{3}{2\sqrt{x}}\,dx$

When x= 0, z= 4, when x= 1, z= 7

∴Reqd. limit= $\frac{2}{3}\int_4^7\frac{dz}{z^2} = \frac{2}{3}\left[-\frac{1}{z}\right]_4^7 = -\frac{2}{3}\left[\frac{1}{7}-\frac{1}{4}\right] = -\frac{2}{3}\left(-\frac{3}{28}\right) = \frac{1}{14}$

16. If f(x)= e^xcosx.sinx, |x|≤ $2 = 2, otherwise\ then$ $\int_2^3 f(x)dx$ is equal to

(a) 0 **(b) 1**

(c) 2 **(d) 3**

Ans. (c)

$$\int_{-2}^{3} f(x)\,dx = \int_{-2}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx$$
$$= \int_{-2}^{2} e^{\cos x}.\sin x\ dx + \int_2^3 2\,dx = 0 + 2(3-2),$$

$= 2$ $[\because e^{\cos x}.\sin x\ \ is\ an\ odd\ function = 2]$

18. The area of the region enclosed by the curves y= xe^x and y= xe^{-x} and the line x= 1, is

(a) 1/e **(b) 1- 1/e**

(c) 2/e **(d) 1- 2/e**

Ans. (c)

y=xe^x …….(1)

y= xe^{-x}……(2)

equating y from (1) and (2) we get

x$e^{x} = xe^{-x} \Longrightarrow x(e^{x} - e^{-x}) = 0$

$\Longrightarrow$x= 0

∴ Required area= $\int_0^1 (y_1 - y_2)\,dx = \int_0^1 (xe^{x} - xe^{-x})\,dx = [xe^{x} - e^{x} — xe^{-x} - e^{-x})_0^1$

$$= (e - e) - (0 - e^0) + [(e^{-1} + e^{-1}) - (0 + 1)] = \frac{2}{e}$$

19. The area bounded by y = xe$^{|x|}$ and the lines |x|=1, y= 0 is

(a) 1 **(b) 2**
(c) 4 **(d) 6**

Ans. (b)

For x ≥ 0, curve is y= xe^{x}……..(1)

For curve (1), $\frac{dy}{dx} = e^{x}(1 + x) > 0$

∴ y is increasing.

$$\frac{d^2y}{dx^2} = e^{x}(2 + x) > 0$$

∴ curve is convex downward.

For x ≤ 0, y= xe^{-x}

∴$\frac{dy}{dx} = e^{-x}(1 - x) > 0$

∴ y is increasing

$$\frac{d^2y}{dx^2} = e^{-x} - e^{-x}(1 - x) > 0 = -e^{-x}(2 - x) < 0$$

∴ curve is concave downward.

Required area = $2\int_0^1 xe^{x}x\,dx = 2[xe^{x} - e^{x}]_0^1 = 2[(e - e) - (0 - e^0)] = 2$

20. A bag contains unlimited number of white, red, black, and blue balls. The number of ways of selecting 10 balls so that there is at least one ball of each colour is

(a) 180 **(b0 270**
(c) 192 **(d) none**

Ans. (d) Number of ways= coefficient of $X^{10} in\ (X + X^2 + X^3 + \cdots)^4$

= coefficient of $X^{10} in\ X^4(1 - X)^{-4}$

= coefficient of $X^6 in\ (1 - X)^{-4}$

= $\frac{(6+1)(6+2)(6+3)}{1.2.3}$ [∵coefficient of $x^r in (1 - X)^{-4} = \frac{(r+1)(r+2)(r+3)}{1.2.3}$]

= $\frac{7\times 8\times 9}{1\times 2\times 3}$ = 84.

21. The number of ways of selecting r balls with replacement out of n balls numbered

1, 2, 3, …., 100 such that the largest numbered selected is 10 is 271, then r=

(a) 3 **(b) 4**
(c) 5 **(d) none**

Ans. (a) from the given condition, we can write

$10^r - 9^r$= 271,

Applying Trial and error method:-

r= 1,   10-9= 1

r =2, $10^2 - 9^2$= 19

r = 3, $10^3 - 9^3$ = 271

∴ r= 3.

22. N men and n women sit along a line alternatively in x ways and along a circle in y ways such that x= 10y, then the number of ways in which n men can sit at a round table so that all shall not have same neighbours is

(a) 6 **(b) 12** **(c) 36** **(d) none**

Ans. (b) $\frac{x}{y} = \frac{2.\lfloor n \lfloor n}{\lfloor n-1 \ \lfloor n} = 2n$

⇒ x = 2ny = 10y ⇒ n = 5

Hence the required number = $\frac{1}{2} \times$ ⌊4= 12.

23. A contest consists of predicting the result (win, draw or defeat) of 10 matches. The number of ways in which one entry contains at least 6 incorrect results is

(a) $\sum_{r=6}^{10} 10_{c_r}.3^r$ **(b)** $\sum_{r=1}^{5} 10_{c_r}.2^r$ **(c)** $\sum_{r=6}^{10} 10_{c_r}$ **(d) none**

Ans. (d) Since total number of ways predicting the results of one match is 3 , so results of 10 match is 3^{10}, now number of ways that the result of one match is correct is 1 and also number of ways to predict wrongly of one match is 2 .

No. of ways to predict wrongly exactly r matches =$10_{c_r}.2^r\ 1^{10-r}$

∴ The required number is $3^{10} - \sum_{r=1}^{4} 10_{c_r}.2^r$

24. Let 1 to 20 are placed in any around a circle. Then the sum of some 3 consecutive numbers must be at least

(a) 30 **(b) 31** **(c) 32** **(d) none**

Ans. (c) Suppose $x_1, x_2, \dots, x_{20}$ be the numbers placed around the circle. Now the mean of the 20 sums of 3 consecutive numbers such as $(x_1 + x_2 + x_3)$, $(x_2 + x_3 + x_4)$, …..,

$(x_{19} + x_{20} + x_{21})$, $(x_{20} + x_1 + x_2)$ is $\frac{1}{20}\{3(x_1 + x_2 + \cdots + x_{20})\} = \frac{3\times20\times21}{2\times20} =$ 31.5

Thus from Pigon hole principle that at least one of the sums must be ≥ 32.

25. The number of different seven–digit numbers can be written using only there digits 1, 2, 3 under the condition that the digit 2 occurs twice in each number is

(a) 512 **(b) 640** **(c) 672** **(d) none**

Ans. (c) We have to put 2 twice in each numbers, so any 2 out of the 7 places can be chosen in $7c_2$ ways. The remaining 5 places can be filled with the other two numbers in 2^5 ways.

The required numbers of numbers are $7c_2 \times 2^5$= 672.

26. The value of $\{\sum_{i=0}^{100}\binom{k}{i}\binom{M-k}{100-i}(\frac{M-k}{100-i})\}$ / $\binom{M}{100}$, where M - k > 100, k >100, is

(a) $\frac{k}{M}$ **(b) $\frac{M}{k}$**
(c) $\frac{k}{M^2}$ **(d) none**

Ans. (a) $\{\sum_{i=0}^{100}\binom{k}{i}\binom{M-k}{100-i}(\frac{M-k}{100-i})\}/\binom{M}{100}$

$= (\frac{k}{M-100})\sum_{i=0}^{100}[\frac{\binom{k}{i}\binom{M-k}{100-i}}{\binom{M}{100}} - \sum_{i=0}^{100}\frac{i\binom{k}{i}\binom{M-k}{100-i}}{(M-100)\binom{M}{100}}]$

$= \frac{k}{M-100}.\frac{\binom{M}{100}}{\binom{M}{100}} - \frac{\frac{k}{M}.100\binom{M}{100}}{(M-100)\binom{M}{100}} = \frac{k}{M}.$

27. Let n be an odd positive integer. If $i_1, i_2, \ldots, i_n$ is a permutation of 1, 2, 3, …., n.

Then (1-i_1)(2-i_2)….(n-i_n)is

(a) Odd **(b) even**
(c) prime **(d) none**

Ans. (b) since n is odd, let n= 2m+ 1, where m is a non-negative integer.

Then set s ={1, 2, …, n} contains m+ 1 odd nos, namely 2, 4, …, 2m.

This is also true for the permutation $i_1, i_2, \ldots, i_n$ of s.

Consider m+ 1 numbers$1 - i_1$, $3 - i_3$, …. n - i_n which are of the from r - i_r, where r is odd.

Since i_5 is even for only m values of s, by P.H.P. , one of the m+ 1 , numbers, $i_1, i_2, \ldots, i_n$, say it is odd, where t is also odd. Hence t-i_t is even and the product (1-i_1)(2-i_2)….(n-i_n) is even.

28. The value of $\sum_{n=1}^{\infty}\sin^{-1}\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}}$ =

(a) $\frac{\pi}{4}$ **(b) $\frac{\pi}{2}$**
(c) -$\frac{\pi}{2}$ **(d) $\frac{\pi}{3}$**

Ans. (b)

$t_n = \sin^{-1}\left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}}\right)$

$\therefore t_n = \sin^{-1}\frac{1}{\sqrt{n}} - \sin^{-1}\frac{1}{\sqrt{n+1}}$

$\therefore S_n = \sin^{-1}(1) - \sin^{-1}\frac{1}{\sqrt{n+1}}$

$\therefore S_\infty = \underset{n\to\infty}{\text{Lt}} S_n = \sin^{-1}(1) - \sin^{-1}(0)$

$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$

29. The number of ways to give 16 different things to 3 persons, according as A< B < C so that B gets 1 more than A and C get 2 more than B, is

(a) 4!5!7! **(b) $\frac{4!5!7!}{16!}$**
(c) $\frac{16!}{4!5!7!}$ **(d) none**

Ans. (c) Here x+ y +z=16, x= y+1, y= z+2

∴x=4, y=5, z= 7

$\therefore$ Required number of ways = $16c_4 \times 12c_5 \times 7c_7 = \frac{16!}{4!5!7!}$

30. For how many positive integers n less than 17, ⌊n+ ⌊n+1+ ⌊n+2 is an integral multiple of 49? (a) 4
(b) 5 (c) 6
(d) none

Ans. (b) ⌊n+ ⌊n+1+ ⌊n+2= ⌊n{1+(n+1)+(n+2)(n+1)}

= ⌊n$(n+2)^2$

Since 49 divides $(n+2)^2$ ⌊n, so either 7 devides (n+2) or 49 divides ⌊n. Thus n=5, 12, 14, 15, 16, i.e. number of integers are 5.

ISI OBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 7

There will be 30 questions in MMA Paper. For each question, exactly one of the four choices is correct. You get four marks for each correct answer, one mark for each unanswered question, and zero mark for each incorrect answer.

1. Let x, y, z be different from 1 satisfying x+ y +z = 2007,

Then the value of $\frac{1}{1-x} + \frac{1}{1-y} + \frac{1}{1-z}$ is

(a) 0 (b) 1 (c) 2008 (d) $\frac{1}{2008}$

Ans:- (a) $\frac{1}{1-x} + \frac{1}{1-y} + \frac{1}{1-z}$

$= \frac{3-2(x+y+z)+(xy+yz+zx)}{(1-x)(1-y)(1-z)}$

$= \frac{3-2\times2007+4011}{(1-x)(1-y)(1-z)} = 0$

2. In a ΔABC, if r= $r_2 + r_3 + r_1$, *and angle*$(A) > \frac{\pi}{3}$ *then the range of* $\frac{s}{a}$ is equal to

(a) (½, 2) **(b) (½, ∞)**
(c) (½, 3) **(d) (3, ∞)**

Ans. (a)

$$r = r_2 + r_3 - r_1$$

$$\frac{\Delta}{s} = \frac{\Delta}{s-b} + \frac{\Delta}{s-c} - \frac{\Delta}{s-a}$$

$$\Rightarrow \frac{1}{s} + \frac{1}{s-a} = \frac{1}{s-b} + \frac{1}{s-c}$$

$$\Rightarrow \frac{2s-a}{2s-b-c} = \frac{s(s-a)}{(s-b)(s-c)}$$

$$\Rightarrow \frac{2s-a}{a} = \cot^2\frac{A}{2} \Rightarrow \frac{s}{a} = \frac{1}{2}\left(\cot^2\frac{A}{2}+1\right) \Rightarrow \frac{s}{a} \in \left(\frac{1}{2}, 2\right)$$

3. If $a_1, a_2, \dots, a_n$ are positive real nos, then $\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$ is always

i) $\geq n$ ii) $\leq n$ iii) $n^{1/n}$ iv) none of these.

Ans:- AM ≥ GM gives

$\frac{a_1}{a_2} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} \geq n\sqrt{\frac{a_1}{a_2} \cdot \ldots \cdot \frac{a_{n-1}}{a_n} \cdot \frac{a_n}{a_1}} = 1$

$\therefore \frac{a_1}{a_2} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} \geq n$

4. The maximum possible value of xy^2z^3 subject to the condition xyz ≥ 0 and

x+y+z = 3 is

i) 1 ii) $^1/_8$ iii) $^1/_4$ iv) $^{27}/_{16}$

Ans:- x+ y+ z = 3

$\Rightarrow$ x. $\frac{2y}{2} + 3.\frac{z}{3} = 3$

Applying AM $\geq$ GM,

So, $\frac{x+2.\frac{2y}{2}+3.\frac{z}{3}}{1+2+3} \geq 6\sqrt{x(\frac{y}{2})^2(\frac{z}{3})^3}$

$\Rightarrow (\frac{3}{6})^6 \geq \frac{xy^2z^3}{2^2.3^3}$

$\Rightarrow xy^2z^3 \leq \frac{27}{16}$.

5. If y(t) is a solution of (1+t)$\frac{dy}{dt} - ty = 1$ and y(t) then y(1) equals

(a) ½ **(b) e + ½** **(c) e+ ½** **(d)- ½**

Ans:- (d) $\frac{dy}{dt} - \frac{t}{1+t}y = \frac{1}{1+t}$

$\therefore$I.F. $= e^{-\int\frac{t}{1+t}dt} = e^{-(t-\log(1+t))} = e^{-t}.(1+t)$

Multiplying and integrating

$ye^{-t}.(1+t) = \int e^{-t}.(1+t)\frac{dt}{(1+t)} = e^{-t} + c$

When y(0) = -1, ⤇ c= 0.

$\therefore ye^{-t}.(1+t) = -e^{-t}$

$\therefore$y= $-\frac{1}{1+t}$ $\therefore$y(1)= - ½.

6. If the quadratic equation $x^2 + ax + b + 1 = 0$ has non- zero

Integer solutions, then

a) **$a^2 + b^2$ is a prime number**
b) **ab is prime number**
c) **Both a) and b)**
d) **Neither a) nor b)**

Ans:- (d) α+β= -a, $\alpha\beta$= (b+ 1)

$\therefore a^2 + b^2 = (\alpha+\beta)^2 + (\alpha\beta - 1)^2$

$= (\alpha^2 + 1)(\beta^2 + 1)$

7**. Let u = $(\sqrt{5} - 2)^{1/3} - (\sqrt{5} + 2)^{1/3}$ and v= $(\sqrt{189} - 8)^{1/3} - (\sqrt{189} + 8)^{1/3}$,**

Then for each positive integer n, $u^n + v^{n+1}$ = ?

(a) -1 **(b) 0** **(c) 1** **(d) 2**

Ans:- (b) $u^3 = (\sqrt{5} - 2) - (\sqrt{5} + 2) - 3(\sqrt{5} - 2)^{1/3}(\sqrt{5} + 2)^{1/3}.(u)$

i.e. $u^3 = -4 - 3u$

$\Rightarrow$ (u-1)($u^2 - u + 4$)= 0

$u^2 - u + 4$ is always +ve. So, u= 1

Similarly $v^3 + 15v + 16 = 0$

$\Rightarrow$ (v +1)($v^2 - v + 16$)= 0

$\Rightarrow$ v= -1

So, for each n, $u^n + v^{n+1}$ = 0

8. The number of real values of x satisfying the equation

$\boldsymbol{x.2^{1/x} + \frac{1}{x}.2^x}$**= 4 is / are**

(a) 1 **(b) 2** **(c) 3** **(d)4**

Ans:- (a) if x < 0, LHS = -ve but RHS= +ve

If x = 0, LHS= not defined.

If x > 0, use AM ≥ GM inequality

$$x.2^{1/x} + \frac{1}{x}.2^x \geq 2\sqrt{2^{1/x+x}}$$

$$\geq 2.\sqrt{2^2} = 4$$

$\Rightarrow x.2^{1/x} = \frac{1}{x}.2^x$; so, x= 1.

9. Let f (x) and g (x) be functions, which take integers as arguments. Let f (x+ y) = f (x) + f (y) + 8 for all integers x and y. Let f (x) = x for all negative numbers x and let g(8)=17, then f (0)=?

(a) 8 **(b) 9** **(c) 17** **(d) 72**

Ans:- (c) put x = -8, y= 8 in the given functional equation,

10. Let x = [$\boldsymbol{\frac{2007.2006.2004.2003}{\frac{1}{3}\times(2005)^4}}$], where [x] denotes the greatest integer integer less than or equal to x. then $\boldsymbol{\frac{((x+1).x^2)+1}{(x^2+1)}}$ is

(a) 80 **(b) 80.2** **(c) 80.5** **(d) 81**

Ans:- (b) x= [3.$\frac{2007}{2005}.\frac{2006}{2005}.\frac{2004}{2005}.\frac{2003}{2005}$]

= [3(1+$\frac{2}{2005}$)(1+$\frac{1}{2005}$)(1-$\frac{1}{2005}$)(1-$\frac{2}{2005}$)]

= [3(1-$\frac{4}{(2005)^2}$)(1-$\frac{1}{(2005)^2}$)]

$\Rightarrow$ x=2.

11. A graph defined in polar co – ordinates by r ($\boldsymbol{\theta}$) = cos $\boldsymbol{\theta}$+$\boldsymbol{\frac{1}{2}}$. The smallest x –co- ordinates of any point on this graph is

(a) 1/16 **(b) -1/16** **(c) 1/8** **(d) -1/8**

Ans:- (b) x =r cosθ

= $\cos^2\theta + \frac{1}{2}cos\theta$

= $(cos\theta + \frac{1}{4})^2$ = 1/16

12. A monic polynomial is one in which the coefficient of the highest order term is 1. The monic polynomial P(x) (with integer coefficient) of least degree that satisfies P ($\boldsymbol{\sqrt{2}+\sqrt{5}}$)= 0 is

(a) $\boldsymbol{x^4 - x^3 - 14x^2 + 9}$=0
(b) $\boldsymbol{x^4 - 14x^2 + 9}$= 0

(c) $\boldsymbol{x^4 + x^3 - 14x^2 + 9}$=0
(d) $\boldsymbol{x^4 + 14x^2 - 9}$

ANS:- (b) Let x=$\sqrt{2}+\sqrt{5}$. Squaring, $x^2 = 7 + 2\sqrt{10}$

$\Rightarrow x^2 - 7 = 2\sqrt{10}$. Squaring again, $x^4 - 14x^2 + 9$= 0

13. The number of distinct real roots of the equation $x^4 + 8x^2 + 16 = 4x^2 - 12x + 9$ is

(a) 1 **(b) 2**
(c) 3 **(d) 4**

Ans:- (a) $(x^2 + 4)^2 = (2x - 3)^2 \quad \Rightarrow x^2 + 4 = \pm(2x - 3)$

Giving $x^2 - 2x + 7 = 0$ and $x^2 + 2x + 1 = 0$.

Solving x = - 1 only one real root.

14. If in an isosceles triangle with base 'a', vertical angle 20° and lateral side of each wih length 'b' is given then the value of $a^3 + b^3$ equals

(a) 3ab **(b) $3ab^2$**
(c) $3a^2b$ **(d) 3**

Ans:- (b) $\sin 10° = \frac{a}{2b} \Rightarrow \sin 30° = 3\,sin 10° - 4 sin^3 10°$

$\Rightarrow \frac{1}{2} = \frac{3a}{2b} - \frac{4a^3}{8b^3}$

$\Rightarrow 1 = \frac{3a}{b} - \frac{4a^3}{8b^3}$

$\Rightarrow a^3 + b^3 = 3ab^2$.

15. If $a^2 + b^2 + c^2 - 2ab = 0$, then the point of concurrency of family of lies ax+ by+ c= 0 lies on the line

(a) y= x **(b) y = x+ 1**
(c) y = -x **(d)3x= y**

Ans:- (c) $(a - b)^2 - c^2 = 0$

⇒ (a-b -c)(a- b+ c)=0

If a- b= c ⇒ ax + by+ (a-b)=0

⇒ a (x+1)+b(y-1)=0⇒ x=-1 , y= 1

If a-b = -c ⇒ ax+ by+ (b-a) =0

⇒ a (x- 1)+ b(y +1)=0

⇒ x=1, y= -1.

16. The value of k for which the inequality $k\cos^2 x - k\cos x + 1 \geq 0\ \forall\, x \in (-\infty, \infty)$ holds is

(a) $k < -\frac{1}{2}$ **(b) $k > 4$**
(c) $-\frac{1}{2} \leq k \leq 4$ **(d) $\frac{1}{2} \leq k \leq 5$**

Ans:- (c) $k\cos^2 x - k\cos x + 1 \geq 0\ \forall\, x \in (-\infty, \infty)$

$\Rightarrow k(\cos^2 x - \cos x) + 1 \geq 0$…………(i)

But $\cos^2 x - \cos x = (\cos x - \frac{1}{2})^2 - \frac{1}{4}$

$\Rightarrow -\frac{1}{4} \leq \cos^2 x - \cos x \leq 2$

From (i) we get $2k+1 \geq 0 \Rightarrow k \geq -\frac{1}{2}$

$\Rightarrow -\frac{k}{4} + 1 \geq 0$

$\Rightarrow k \leq 4$

$\Rightarrow -\frac{1}{2} \leq k \leq 4$

17. Consider two series (i) $\sum_{n=1}^{\infty} \sin\frac{\pi}{n}$ (ii) $\sum_{n=1}^{\infty} (-1)^n \cos\frac{\pi}{n}$, then

(a) Both (i) and (ii) converge
(b) (i) converges, (ii) diverges

(c) (i) diverges, (ii) converges
(d) both (i) and (ii) diverges.

Ans:- (d) $\sum_{n=1}^{\infty} sin\frac{\pi}{n} = \frac{\pi}{n} - \frac{1}{3!}(\frac{\pi}{n})^3 + \cdots = U_n$

Let $V_n = \frac{1}{n}, \therefore \lim_{n\to\infty} \frac{U_n}{V_n} = \pi \neq 0$

Since $\sum U_n\ diverges, so\ does \sum V_n.$

$$(ii) \sum_{n=1}^{\infty} (-1)^n cos\frac{\pi}{n}\ converges$$

$$if \sum_{n=1}^{\infty} \left|(-1)^n cos\frac{\pi}{n}\right|$$

$$= \sum_{n=1}^{\infty} cos\frac{\pi}{n}\ converges.$$

But $\lim_{n\to\infty} cos\frac{\pi}{n} = cos\theta = 1 \neq 0$

So, $\sum_{n=1}^{\infty} cos\frac{\pi}{n}$ diverges.

18. If a= $\lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$ $and\ b = \lim_{n\to\infty}\frac{1}{n}(1 + \frac{1}{2} + \cdots + \frac{1}{n})$ then

(a) both a= ∞ $and\ b = \infty$

(b) a= ∞ and b= 0

(c) a= ∞ and b=1

(d) none.

Ans:- (d) $\{u_n\}= \{\frac{1}{n}\}$, $\lim_{n\to\infty} u_n = 0, \therefore a = 0$

So, By Cauchy's first theorem $\lim_{n\to\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n} = 0$

$\therefore$ b= 0.

19. Let $\{a_n\}$ be a sequence of non-negative real numbers such that the series $\sum_{n=1}^{\infty} a_n$ is convergent. If p is a real number such that the series $\sum \frac{\sqrt{a_n}}{n^p}$ diverges, the

(a) P< ½ (b) P≤ $\frac{1}{2}$
(c) $\frac{1}{2}$ <P≤ 1 (d) $\frac{1}{2}$ ≤ P ≤ 1

Ans:- (a) $\sum a_n$ is convergent

$\Rightarrow \sum \frac{a_n}{n}$ is convergent.

$\sum(a_n + \frac{1}{n})$ is convergent. By AM≥ GM

$\Rightarrow \sum \frac{\sqrt{a_n}}{n^{\frac{1}{2}}}$ is converges

S0, for P < ½ , the series is divergent.

20. A rigid body is spinning about a fixed point (3, -2, -1) with angular velocity of 4 rad/sec., the axis of rotation being in the direction of (1, 2, -2), then the velocity of the particle at the point (4, 1, 1) is

(a) 4/3 (1, -4, 10) (b) 4/3 (4, -10, 1)
(c) 4/3 (10, -4, 1) (d) 4/3(10, 4, 1)

Ans:- (c)

$\vec{\omega} = 4\left(\frac{\hat{\imath}+2\hat{\jmath}-2\hat{k}}{\sqrt{1+4+4}}\right) = \frac{4}{3}\left(\hat{\imath} + 2\hat{\jmath} - 2\hat{k}\right)$

$\vec{r} = \vec{OP} - \vec{OA}$

$= (4\hat{\imath}+\hat{\jmath}+\hat{k})-(3\hat{\imath}-2\hat{\jmath}-\hat{k})= \hat{\imath}+3\hat{\jmath}+ 2\hat{k}$

$\hat{v}= \hat{\omega}\times \vec{r} = \frac{4}{3}\left(\hat{\imath} + 2\hat{\jmath} - 2\hat{k}\right) \times \left(\hat{\imath} + 3\hat{\jmath} + 2\hat{k}\right) = \frac{4}{3}(10\hat{\imath} - 4\hat{\jmath} + \hat{k})$

21. A particle has an angular speed of 3 rad /sec and the axis of rotation passes through the point (1, 2, 2) and (1, 2, -2), then the velocity of the particle at the point P(3, 6, 4) is

(a) $\frac{3}{\sqrt{17}}$**(22, 8, -2)**

(b) $\frac{3}{\sqrt{17}}(\mathbf{22, 0, 2})$

(c) $\frac{3}{\sqrt{17}}(\mathbf{22, -8, -2})$

(d) $\frac{3}{\sqrt{17}}(\mathbf{22, -8, 2})$

Ans:- (c)

$\vec{OA} = \hat{\imath} + \hat{\jmath} + 2\hat{k}$

$\vec{OB} = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$

$\therefore \vec{AB} = \hat{\jmath} - 4\hat{k}$

$\Rightarrow |\vec{AB}| = \sqrt{17}$

$\vec{AP} = (3\hat{\imath}+6\hat{\jmath}+4\hat{k}) - (\hat{\imath}+\hat{\jmath}+2\hat{k})$

$= 2\hat{\imath}+5\hat{\jmath}+2\hat{k}$

$\therefore \vec{\omega} = \frac{3}{\sqrt{17}}(\hat{\jmath}-4\hat{k})$

$\vec{v} = \omega \times \vec{r} = \frac{3}{\sqrt{17}}(\hat{\jmath}-4\hat{k}) \times (2\hat{\imath} + 5\hat{\jmath} + 2\hat{k}) = \frac{3}{\sqrt{17}}(22\hat{\imath} - 8\hat{\jmath} - 2\hat{k})$

22. In a group of equal number of boys and girls, 20% girls and 35% boys are graduate. If a member of the group is selected at random, then the probability of this member not being a graduate is

(a) $\frac{11}{40}$ **(b)** $\frac{9}{20}$ **(c)** $\frac{11}{20}$ **(d)** $\frac{29}{40}$

Ans. (d) Let A and B denotes the events that the member selected at random is a boy and a girl respectively. Let E denotes the event that the member selected is a graduate. Reqd. prob.

=1- [P(A).P(E/A)+P(B).P(E/B)]

$=1- \left[\frac{1}{2}.\frac{35}{100} + \frac{1}{2}.\frac{20}{100} = \frac{55}{200}\right] = \left(1 - \frac{11}{40}\right) = \frac{29}{40}$

23. for any two events A and B in a sample space

(a) P (A/B)≥ $\frac{P(A)+P(B)-1}{P(B)}, P(B) \neq 0$ **is always true.**

(b) P (A∩B̅) = P (A)- P(A∩B) does not hold

(c) P (A∪B) =1-P (A̅).P(B̅) if A and B are independent

(d) P (A∪B) =1-P (A̅).P(B̅) if A and B are disjoint

Ans. (c)

$P(A/\bar{B})+P(\bar{A}/\bar{B}) = 1,$

$\therefore P(\bar{A}+\bar{B}) = 1-P(A/\bar{B})$

$$\frac{1 - P(A \cup B)}{P(B')} = \frac{P(A \cup B)'}{P(B')} = \frac{P(A' \cap B')}{P(B')} = P\left(\frac{A'}{B'}\right)$$

24. one hundred identical coins, each with probability P, of showing up heads are tossed. If 0 < P < 1 and the probability of heads showing on 50 coins is equal to that of the heads showing on 51 coins, then p=

(a) $\frac{1}{2}$ **(b)** $\frac{49}{101}$ **(c)** $\frac{50}{101}$ **(d)** $\frac{51}{101}$

Ans. (d)

Here n= 100, p= p, q= 1-p

Given, p(50) = p(51)

$\Rightarrow 100_{C_{50}} p^{50}(1-p)^{50} = 100_{C_{51}} p^{51}(1-p)^{49}$

$$\Rightarrow \frac{100!}{50!\,50!}(1-p) = \frac{100!}{51!\,49!}p$$

$$\Rightarrow 51(1-p) = 50p \Rightarrow p = \frac{51}{101}$$

25. A box contains 24 identical balls of which 12 are white and 12 are black. The balls are drawn at random from the box one at a time with replacement. The probability that a white ball is drawn for the 4th time on the 7th draw is

(a) $\frac{5}{64}$ **(b) $\frac{27}{32}$**
(c) $\frac{5}{32}$ **(d) $\frac{1}{2}$**

Ans. (c) Probability of drawing a white ball in any draw$=\frac{12}{24} = \frac{1}{2}$

A white ball will be drawn for the 4th time on the 7th draw ball is drawn in the 7th draw and 3 white balls are drawn in the first 6 draws.

∴ Required probability

$= 6_{C_3} p^3 q^3 . p = 20.\left(\frac{1}{2}\right)^3.\left(\frac{1}{2}\right)^3.\frac{1}{2} = \frac{5}{32}$

26. If [x]denotes the integral part of x, then the domain of the function

f(x)= $\sin^{-1}[2x^2-3] + \log_2\{\log_{1/2}(x^2 - 5x + 5)\}$ is

(a) $\left(-\sqrt{\frac{5}{2}}, -1\right)$ **(b) $\left(1, \sqrt{\frac{5}{2}}\right)$**
(c) $\left(-\sqrt{\frac{5}{2}}, -1\right) \cup \left(1, \sqrt{\frac{5}{2}}\right)$
(d) none of these

Ans. (d)

For f(x) to be defined

(i) $[2x^2 - 3]$= -1, 0, 1

$\Rightarrow -1 \le 2x^2 - 3 < 2 \Rightarrow 2 \le 2x^2 < 5$

$\Rightarrow 1 \le x^2 < \frac{5}{2}$

$$\Rightarrow \begin{cases} 1 \le x^2 \Rightarrow x \le -1 \text{ or } x \ge 1 \\ x^2 < \frac{5}{2} \Rightarrow -\sqrt{\frac{5}{2}} < x < \sqrt{\frac{5}{2}} \end{cases}$$

$\Rightarrow -\sqrt{\frac{5}{2}} < x \le -1$ or $1 \le x < \sqrt{\frac{5}{2}}$ ………..(A)

(ii) $x^2 - 5x + 5 > 0 \Rightarrow x < \frac{5-\sqrt{5}}{2}$ or $x > \frac{5+\sqrt{5}}{2}$ …….(B)

(iii) $\log_{\frac{1}{2}}(x^2 - 5x + 5) > 0$

$\Rightarrow x^2 - 5x + 5 < \left(\frac{1}{2}\right)^0$

$\Rightarrow x^2 - 5x + 5 < 1 \Rightarrow x^2 - 5x + 4 < 0$

$\Rightarrow$ 1< x< 4 ……………(C)

From (A), (B) and (C), $1 \le x < \frac{5-\sqrt{5}}{2}$

27. If f(x)= $\underset{m\to\infty}{\text{Lt}}\ \underset{n\to\infty}{\text{Lt}}\ cos^{2m} n!\,\pi x$, then range of f(x)is

(a) [0, 1] **(b) [0, 1]**
(c) (0, 1) **(d) {0}**

Ans. (b)

When x is rational say p/q , then n! $x\pi$ is a multiple of π and $cos^2 n!\,x\pi = 1$

$\therefore \underset{m\to\infty}{\text{Lt}}\ cos^{2m}\, n!\,x\pi = \underset{m\to\infty}{\text{Lt}}\ 1^m = 1$

$\therefore$ f(x)= 1

When x is irrational,

n! $x\pi \neq$a multiple of π

$\therefore cos^2 n!\, x\pi \neq 1$

$\therefore$ 0$\leq cos^2 n!\, x\pi < 1$

$\therefore \lim\limits_{m\to\infty} cos^{2m}\, n!\, x\pi = \lim\limits_{m\to\infty} (cos^2 n!\, x\pi)^m = 0$

Thus f(x)= 0, when x is rational

= 1, when x is irrational

$\therefore$ Range f= {0, 1}

28. The normal at any point P $(t^2, 2t)$ on the parabola $y^2 = 4x$ meets the curve again at Q, the area of ΔPOQ, O being the origin is $\frac{k}{|t|}(1+t^2)(2+t^2)$ then

(a) k > 2 **(b) k=2**
(c) k < 2 **(d) k= 1**

Ans. (b)

Given P ≡ $(t^2, 2t)$

Given parabola is $y^2 = 4x$ ……….. (1)

Here a= 1.

Let Q=($t_1^2, 2t_1$)

Since normal at P meet the curve again at Q.

$\therefore t_1 = -t - \frac{2}{t} = \frac{t^2+2}{t}$ ………… (2)

Now O≡ (0, 0), P≡ $(t^2, 2t)$, Q=($t_1^2, 2t_1$)

Given,

$\frac{k}{|t|}(1+t^2)(2+t^2) = area\ of\ \Delta POQ$

$= \frac{1}{2}|t^2.2t_1 - 2t.t_1^2|$

$= |t^2 t_1 - t t_1^2|$

$= \left|-t^2\left(\frac{t^2+2}{2}\right) - t\frac{(t^2+2)^2}{t^2}\right|$

$= (t^2+2)\left|t + \frac{(t^2+2)}{t}\right|$

$= (t^2+2)\left|t + \frac{(t^2+2)}{t}\right| = (t^2+2)2\,\frac{(1+t)^2}{|t|}$

$\therefore$ k= 2

29. If {x} denotes the fractional part of x, then $\left\{\frac{3^{200}}{8}\right\} =$

(a) $\frac{1}{8}$ **(b) $\frac{3}{8}$**
(c) $\frac{5}{8}$ **(d) none of these**

Ans. (d)

$$\frac{3^{200}}{8} = \frac{9^{100}}{8} = \frac{(1+8)^{100}}{8}$$
$$= \frac{1 + 100_{C_1}8 + 100_{C_1}8^2 + \cdots + 8^{100}}{8}$$
$$= \frac{1}{8} + an\ integer$$

$\therefore \left\{\frac{3^{200}}{8}\right\} = \frac{1}{8}$

30. The number of distinct terms in the expression of $(x_1 + x_2 + \cdots + x_p)^n$

(a) $n+p-1_{C_n}$ **(b) n+ p+ 1**
(c) n+1 **(d) $n+p-1_{C_{p+1}}$**

Ans. (d) Number of terms

$n+p-1_{C_n} = n+p-1_{C_{p-1}}$

ISI OBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 8

There will be 30 questions in MMA Paper. For each question, exactly one of the four choices is correct. You get four marks for each correct answer, one mark for each unanswered question, and zero mark for each incorrect answer.

1. If [x] denotes the integral part of x, then $\underset{x \to 0}{\text{Lt}} \frac{\sin[cosx]}{1+[cosx]} =$

(a) 0 **(b) 1** **(c) $\frac{\sin 1}{2}$**
(d) does not exist

Ans. (a) $\underset{x \to 0-0}{\text{Lt}} [\cos x] = 0$

[∵ when x ⟶0- 0, 0 < cosx < 1] and

$\underset{x \to 0+0}{\text{Lt}} [\cos x] = 0$

[∵ when x ⟶0+ 0, 0 < cosx < 1]

$\therefore \underset{x \to 0-0}{\text{Lt}} \frac{\sin[\cos x]}{1+[\cos x]} = \frac{\sin 0}{1+0} =$
$0 \underset{x \to 0+0}{\text{Lt}} \frac{\sin[\cos x]}{1+[\cos x]} = \frac{\sin 0}{1+0} = 0$

∴Required limit = 0

2. Let f(x)= $\underset{n \to \infty}{\text{Lt}} \sum_{r=0}^{n-1} \frac{x}{(rx+1)\{(r+1)x+1\}}$ then

(a) f(x) is continuous but not differentiable at x= 0

(b) f(x) is both continuous and differentiable at x= 0

(c) f(x) is neither continuous nor differentiable at x=0

(d) f(x) is a periodic function

Ans. (c)

$$t_{r+1} = \frac{x}{(rx+1)\{(r+1)x+1\}}$$
$$= \frac{(r+1)x+1-(rx+1)}{(rx+1)[(r+1)x+1]}$$

$$= \frac{1}{(rx+1)} - \frac{1}{(r+1)x+1}$$

$$\therefore S_n = \sum_{r=0}^{n-1} t_{r+1} \frac{1}{nx+1} = 1, x \neq 0 = 0, x$$
$$= 0$$

$\therefore \underset{n \to \infty}{\text{Lt}} S_n = \underset{n \to \infty}{\text{Lt}} \left(1 - \frac{1}{nx+1}\right)$

Thus, f(x)= $\begin{cases} 1, x \neq 0 \\ 0, x = 0 \end{cases}$

$\therefore \underset{x \to 0}{\text{Lt}} f(x) = 1 \; and \; f(0) = 0$

Hence f(x) is neither continuous nor differentiable at x= 0

Clearly f(x) is not a periodic function.

3. Let f(x)= $\underset{n \to \infty}{\text{Lt}} \frac{\log(2+x)-x^{2n}sinx}{1+x^{2n}}$ then f(x) is discontinuous at

(a) x= 1 only **(b) x=-1 only**
(c) x= -1, 1 only **(d) no point**

Ans. (c)

$$\underset{n\to\infty}{\text{Lt}} X^{2n} = \underset{n\to\infty}{\text{Lt}} (x^2)^n$$
$$= \begin{cases} \infty, & x^2 > 1 \\ 0, & 0 \le x^2 < 1 \\ 1, & x^2 = 1 \end{cases}$$
$$= \begin{cases} \infty, & x < -1 \text{ or } x > 1 \\ 0, & -1 < x < 1 \\ 1, & x = \pm 1 \end{cases}$$

$\therefore f(x)$
$$= \begin{cases} -sinx, & x < -1 \text{ or } x > 1 \\ \log(2+x), & -1 < x < 1 \\ \dfrac{\log(2+x) - sinx}{2}, & x = \pm 1 \end{cases}$$

$$\underset{x\to 1+0}{\text{Lt}} f(x) = -sin1, \underset{x\to 1-0}{\text{Lt}} f(x)$$
$$= log3, \underset{x\to 1+0}{\text{Lt}} f(x)$$
$$= \underset{x\to -1}{\text{Lt}} \log(2+x) = 0,$$

$$\underset{x\to 1-0}{\text{Lt}} f(x) = \underset{x\to -1}{\text{Lt}} (-\sin x) = sin1$$

Clearly f(x) is discontinuous only at two points x= -1, 1

4. The function f(x) = max {(1-x), (1+x), 2} is, where x ∈(-∞, ∞)

(a) discontinuous at all points
(b) differentiable at all points

(c) differentiable at all points except -1 and 1 (d) continuous at all points except -1 and 1

Ans. (c)

We draw the graph of y= 1- x, y= 1+ x and y = 2

f(x)= max.{1-x, 1+x, 2}

∴ f(x)= 1- x, x ≤ -1= 2, -1≤ x≤ 2 = 1+x, x ≥ 2

From graph it is clear that f(x) is continuous at all x and differentiable at all x except x= -1 and x= 1

5. If f(x) = p |sin x|+$qe^{|x|}$ + $r|x|^3$ *and* $f(x)$ is differentiable at x=0, then

(a) p= q= r=0
(b) p=0, q=0, r== any real number

(c) q=0, r=0, p is any real number
(d) r=0, p=0, q is any real number

Ans. (b)

f'(0- 0)= $\underset{h\to 0-0}{\text{Lt}} \frac{f(h)-f(0)}{h} =$
$\underset{h\to 0-0}{\text{Lt}} \frac{p|\sin h|+qe^{|h|}+r|h|^3-q}{h}$

$$= \underset{h\to 0-0}{\text{Lt}} \frac{-p\sin h + qe^h - rh^3 - q}{h}$$

$$= \underset{h\to 0-0}{\text{Lt}} \left\{-p\frac{sinh}{h} - \frac{q(e^h-1)}{-h} - rh^2\right\}$$
$$= -p - q$$

Similarly, f' (0+0)= p+q

Since f(x) is differentiable at x= 0

∴ f'(0- 0) = f'(0+ 0)⟹ - p- q= p+ q

⟹ p+ q= 0

Here r may be any real number.

6. Let f(x)= $x^3 - x^2 + x + 1$, $g(x) = max.\{f(t), 0 \le t \le x\}, 0 \le x \le 1$

$$= 3 - x, 1 < x \le 2$$

then in [0, 2] the points where g(x) is not differentiable is (are)

(a) 1 **(b) 2**
(c) 1 and 2 **(d) none of these**

Ans. (a) $f(t) = t^3 - t^2 + t + 1$

∴ f' (t)= $3t^2 - 2t + 1 > 0$

∴ f(t) is an increasing function.

Since 0≤ t ≤ x

∴ max f(t)= f(x)= $x^3 - x^2 + x + 1$

Thus g(x)= $x^3 - x^2 + x + 1, 0 \leq x \leq 1$= 3-x, 1 < x ≤ 2

The only doubtful point for differentiability of g(x) in [0, 2] is x = 1

Clearly, $\underset{x \to 1+0}{\text{Lt}} g(x) = 1^3 - 1^2 + 1 + 1 = 2$

$$\underset{x \to 1+0}{\text{Lt}} g(x) = \underset{x \to 1}{\text{Lt}} (3 - x) = 2 \; and \; g(1) = 2$$

∴ g(x) is continuous at x= 1

Also g' (x)= $3x^2 - 2x + 1, 0 \leq x < 1 = -1, 1 < x \leq 2$

∴ g'(1-0)= 3. $1^2 - 2.1 + 1 = 2$ and g'(1+0)= -1

Hence g(x) is not differentiable at x= 1.

7. If [x] denotes the integral part of x and f(x)= [x]$\left\{\frac{\sin\frac{\pi}{[x+1]} + \sin \pi[x+1]}{1+[x]}\right\}$, then

(a) f(x) is continuous in R
(b) f(x) is continuous but not differentiable in R
(c) f"(x) exists for all x in R
(d) f(x) is discontinuous at all integer points in R

Ans. (d)

Sin π [x+ 1]= 0

Also [x+ 1]= [x]+ 1

∴ f(x)= $\frac{[x]}{1+[x]} \sin \frac{\pi}{[x]+1} \; at \; x = n, n \; \in I, f(x) = \frac{n}{1+n} \sin \frac{\pi}{n+1} \; for \; n - 1 < x < n, [x] = n - 1$

∴ f(x)= $\frac{n-1}{n} \sin \frac{\pi}{4}$

$$hence \; \underset{x \to n-0}{\text{Lt}} f(x) \frac{n-1}{n} \sin \frac{\pi}{4},$$

∴ f(n)= $\frac{n}{1+n} \sin \frac{\pi}{n+1}$

∴ f(x) is discontinuous at all n ∊ I

8. Let f(x) = $\frac{[tan^2 x] - 1}{tan^2 x - 1}, x \neq n\pi \pm \frac{\pi}{4} = 0, x = n\pi \pm \frac{\pi}{4}$ then f(x) is

(a) continuous at all x **(b) continuous at x= $\frac{\pi}{4}$** **(c) discontinuous at x= $\frac{\pi}{4}$** **(d) none**

Ans. (c)

Since tan x is not defined at

X= (2n+1) $\frac{\pi}{2}, n \; \in I, therefore \; f(x) is \; discontinuous \; at \; x = (2n + 1)\frac{\pi}{2}, n \; \in I$

Now f $\left(\frac{\pi}{4}\right) = 0$

$$\underset{x\to\frac{\pi}{4}-0}{\text{Lt}} f(x) = \underset{x\to\frac{\pi}{4}-0}{\text{Lt}} \frac{[tan^2x]-1}{tan^2x-1}$$
$$= \underset{x\to\frac{\pi}{4}-0}{\text{Lt}} \frac{0-1}{tan^2x-1} = \infty$$

Hence f(x) is discontinuous at x= $\frac{\pi}{4}$

9. Let f(x)= $\int_0^x t\, sin\frac{1}{t}\, dt$, then the number of points of discontinuity of f(x) in (0, π) is

(a) 0 (b) 1
(c) 2 (d) more than 2

Ans. (a)

$$f(x) = \int_0^x t\sin\frac{1}{t}dt$$

∴ f'(x)= x sin $\frac{1}{x}$

Clearly f'(x) is a finite number at all x in (0, π).

∴ f(x) is differentiable and hence continuous at all x in (0, π)

10. if [x] denotes the integral part of x and in (0, π), we define

f(x)= $\left[\frac{2(sinx-sin^nx)+|sinx-sin^nx|}{2(sinx-sin^nx)-|sinx-sin^nx|}\right] = 3, x = \frac{\pi}{2} x \neq \frac{\pi}{2}$ then for n > 1

- **(a) f(x) is continuous but not differentiable at x= $\frac{\pi}{2}$**
- **(b) both continuous and differentiable at x= $\frac{\pi}{2}$**
- **(c) (c) neither continuous nor differentiable at x= $\frac{\pi}{2}$**
- **(d) $\underset{x\to\frac{\pi}{2}}{\text{Lt}} f(x)$ *exist but* $\underset{x\to\frac{\pi}{2}}{\text{Lt}} f(x) \neq f(\frac{\pi}{2})$**

Ans. (b)

For $0 < x < \frac{\pi}{2}$ *or* $\frac{\pi}{2} < x < \pi$ $0 < \sin x < 1$

∴ for n > 1, sin x > sin^4x

∴ f(x)=$\left[\frac{3(\sin x-sin^4x)}{\sin x-sin^nx}\right] = 3, x \neq \frac{\pi}{2} = 3, x = \frac{\pi}{2}$

Thus in (0, π) , f(x) = 3

Hence f(x)is continuous and differentiable at x= $\frac{\pi}{2}$

11. If[x] denotes the integral part of x and f(x) = [n+ psin x], 0< x < π, n∈I and p is a prime number, then the number of points where f(x) is not differentiable is

(a) p-1 (b) p
(c) 2p-1 (d) 2p +1

Ans. (c) [x] is not differentiable at integral points.

Also [n+ p sin x]= n+ [p sin x]

∴ [p sin x] is not differentiable, where p sin x is an integer. But p is prime and 0 < sin x ≤1 [∵ 0 < x < π]

∴ p sin x is an integer only when

sinx = $\frac{r}{p}$, where 0< r ≤ p and r ∈ N

For r= p, sin x= 1 ⟹ x= $\frac{\pi}{2}$ in (0, π)

For 0 < r < p, sin x= $\frac{r}{p}$

∴ x= $\sin^{-1}\frac{r}{p}$ *or* $\pi - \sin^{-1}\frac{r}{p}$

Number of such values of x= p- 1+p- 1= 2p -2

∴ Total number of points where f(x) is not differentiable

= 1+ 2p – 2= 2p – 1

12. If $\frac{sec^4\theta}{a}+\frac{tan^4\theta}{b}=\frac{1}{a+b}$, *then*

(a) |b|=|a| **(b) |b|≤ $|a|$**
(c) |b|≥ $|a|$ **(d) none of these**

Ans. (b)

$$\frac{sec^4\theta}{a}+\frac{tan^4\theta}{b}=\frac{1}{a+b}=\frac{sec^2\theta-tan^2\theta}{a+b}$$

$\Rightarrow \frac{sec^2\theta}{a(a+b)}[(a+b)sec^2\theta-a]+\frac{tan^2\theta}{(a+b)b}[(a+b)tan^2\theta+b]=0$

$\Rightarrow a tan^2\theta+b sec^2\theta=0$

$\Rightarrow sin^2\theta=-\frac{b}{a}$ *is non*-negative and ≤ 1⟹ $\left|\frac{b}{a}\right|\leq 1$

13. If c be a positive constant and |f(y)-f(x)|≤ $c(y-x)^2$ for all real x and y, then

(a) f(x)= 0 for all x **(b) f(x)= x for all x**
(c) f'(x)= 0 for all x **(d) f'(x)= c for all x**

Ans. (c)

Given, |f(y)- f(x)| ≤$c(y-x)^2, c>0$

⟹|f(y)- f(x)| ≤$c|y-x|^2 \Rightarrow \left|\frac{f(y)-f(x)}{y-x}\right|\leq c|y-x| \Rightarrow \underset{y\to x}{\text{Lt}}\left|\frac{f(y)-f(x)}{y-x}\right|\leq \underset{y\to x}{\text{Lt}}\ c|y-x|$

$$\Rightarrow |f'(x)|\leq 0$$

⟹$|f'(x)|\leq 0$ for all real x

⟹ f'(x)= 0∀x∊ R

14. Let f(t)in t. then $\frac{d}{dx}\left\{\int_{x^2}^{x^3} f(t)dt\right\}$

(a) has a value 0 when x= 0
(b) has a value 0 when x=1 , x= 4/9

(c) has a value $9e^2-4e$ when x=e
(d) has a differential coefficient 27e -8 when x= e

Ans. (c) $\frac{d}{dx}\int_{x^2}^{x^3} f(t)\,dt=f(x^3).3x^2-f(x^2).2x$

$$=logx^3.3x^2-\log x^2.2x$$
$$=9x^2\,logx-4x\,logx$$
$$=x\,logx(9x-4)$$

$$let\ z=x\log x\,(9x-4) then\ \frac{dz}{dx}=(1+\log x)(9x-4)+9x\log x$$

$$at\ x=e, \frac{dz}{dx}=2(9e-4)+9e=27e-8$$

15. If a, $\alpha_1,\alpha_2,...\alpha_{2n-1},b$ *are in A.P.*, $a,\beta_1,\beta_2,...\beta_{2n-1},b$ are in G.P. and

a, $\gamma_1,\gamma_2,...\gamma_{2n-1},b$ are in H.P., where a, b are positive, then the equation

$\alpha_n x^2-\beta_n x+\gamma_n=0$ has

(a) real and equal roots
(b) real and unequal roots

(c) imaginary roots
(d) roots which are in A.P.

Ans. (c)

The middle terms of the A.P., G.P. and H.P. are $\alpha_n, \beta_n\ and\ \gamma_n$ respectively

$\therefore\ \alpha_n$= A.M. of a and b,

β_n= G.M. of a and b,

γ_n= H.M. of a and b,

$\because$ AH=G^2

$\therefore\ \alpha_n\gamma_n = {\beta_n}^2$

Now, discriminant of given equation =${\beta_n}^2 - 4\alpha_n\gamma_n = -3\alpha_n\gamma_n < 0$ ($\because \alpha_n, \gamma_n$ are positive)

16. If a_n = the digit at units place in the number 1! +2! +3! +…+n! for n≥ 4,

then $a_4, a_5, a_6, \ldots$ are in

(a) A.P. only (b) G.P. only (c) A.P. and G.P. only (d) A.P., G.P., and H.P.

Ans.(c)

1 ! +2 ! +3 ! +4! = 33

The digits at units place in each of 5 !, 6 !,… is 0

$\therefore a_4 = a_5 = a_6 = \cdots = 3$

Clearly $a_4, a_5, a_6, \ldots$ are in A.P. and G.P. but not in H.P. as they are equal.

17. Let p, q, r ∊ R^+ and 27pqr ≥ $(p + q + r)^3$ and 3p +4q +5r = 12 then $p^3 + q^4 + r^5$ is equal to

(a) 3 (b) 6

(c) 2 (d) none of these.

Ans. (c)

1 ! +2 ! +3 ! +4! = 33

The digits at units place in each of 5 !, 6 !,… is 0

$\therefore a_4 = a_5 = a_6 = \cdots = 3$

Clearly $a_4, a_5, a_6, \ldots$ are in A.P. and G.P. but not in H.P. as they are equal.

18. If (2+ x)(2+x^2)$(2 + x^3) \ldots (2 + x^{100})$ = $\sum_{r=0}^{n} x^r$, then n equals

(a) 2550 (b) 5050
(c) 2^8 (d) none of these.

Ans. (b) $x^n = x^{1+2+3+\cdots+100} = x^{\frac{100\times101}{2}} = x^{5050}$

$\Longrightarrow$ n= 5050

19. If p, q, r, s ∊ R, then equation $(x^2 + px + 3q)$(- $x^2 + rx + q$)(- $x^2 + sx - 2q$)= 0 has

(a) 6 real roots
(b) at least two real roots

(c) 2 real and 4 imaginary roots
(d) 4 real and 2 imaginary roots.

Ans. (b)

$D_1 + D_2 + D_3 = p^2 - 12q + r^2 + 4q + s^2 + 8q = p^2 + r^2 + s^2 \geq 0$

$\Longrightarrow$ at least one of $D_1, D_2, D_3 \geq 0$

20. If a, b, c, d, are four non-zero real numbers such that $(d + a - b)^2 + (d + b - c)^2$= 0 and roots of the equation

a(b-c) x^2 + b(c-a)x + c(a-b)= 0 and real and equal, then a, b, c

(a) are equal (b) are not equal (c) are zero (d) none of the above

Ans. (a) Equation $a(b-c)x^2 + b(c-a)x + c(a-b)$= 0 has equal roots

$\Rightarrow$ b= $\frac{2ac}{a+c}$ …….(1)

$(d + a - b)^2 + (d + b - c)^2$ =0

$\Rightarrow$ a-b = b- c = -d

$\Rightarrow$ 2b= a+ c ……(2)

$\Rightarrow \frac{4ac}{a+c}$= a +c

$\Rightarrow (a-c)^2 = 0 \Rightarrow$ a= c

From (2), b= a

thus a= b= c.

21. If p, q be non-zero real numbers and f(x)≠ 0 in [0, 2] and $\int_0^1 f(x).(x^2 + px + q)dx = \int_0^2 f(x).(x^2 + px + q)dx = 0$ then equation $x^2 + px + q$= 0 has

(a) two imaginary roots
(b) no root in (0, 2)

(c) one root in (0,1) and other in (1,2)
(d) one root in (-∞, 0) and other in (2,∞)

Ans. (c)

Let F(x) = $\int f(x)(x^2 + px + q)dx$,

Then according to question

F(1) –F(0) = 0, F(2) – F(1) = 0

∴ F(0) = F(1) and F(1)= F(2)

Hence, equation F'(x)= 0 i.e. equation f(x). $(x^2 + px + q) = 0$

i.e. equation $x^2 + px + q$ = 0 has at least one root (here exactly one root) in (0, 1) and exactly one root in (1, 2).

22. If a, b, c, ∊ R, a ≠ 0 and $(b-1)^2 < 4ac$, then the number of roots of the system of equation (in three unknowns x_1, x_2, x_3)

$\boldsymbol{a{x_1}^2 + bx_1 + c = x_2}$

$\boldsymbol{a{x_2}^2 + bx_2 + c = x_3}$

$\boldsymbol{a{x_3}^2 + bx_3 + c = 1}$ **is**

(a) 0 **(b) 1**

(c) 2 **(d) 3**

Ans. (a) Let f(x) = $ax^2 + (b-1)x$+c

Given system of equation is equivalent

$to \left.\begin{matrix} f(x_1)=x_2-x_1 \\ f(x_2)=x_3-x_2 \\ f(x_3)=x_1-x_3 \end{matrix}\right\}$

$\Rightarrow f(x_1) + f(x_2) + f(x_3) = 0$

∴ $af(x_1) + af(x_2) + af(x_3) = 0$ (not possible)

As $(b-1)^2 - 4ac < 0$.

∴ $af(x_1), af(x_2), af(x_3) > 0$.

Hence given system of equation has no real root.

23. If α, β are the roots of the equation x^2-ax +b= 0 and $A_n = \alpha^n + \beta^n$ then which of the following is true?

(a) $A_{n+1} = aA_n + bA_{n-1}$
(b) $A_{n+1} = bA_n + aA_{n-1}$

(c) $A_{n+1} = aA_n - bA_{n-1}$
(d) $A_{n+1} = bA_n - aA_{n-1}$

Ans.(a)

α +β= a, $\alpha\beta$= b

Given, $A_n = \alpha^n + \beta^n$

Now, $A_{n+1} = \alpha^{n+1} + \beta^{n+1}$

= $(\alpha^n + \beta^n)$(α +β)-$\alpha\beta(\alpha^{n-1} + \beta^{n-1})$

= a$A_n + bA_{n-1}$

24. If x satisfies |x-1|+|x-2|+|x-3|≥ 6, then

(a) 0≤ $x \leq 4$ (b) $x \leq -2$ or $x \geq 4$
(c) $x \leq 0$ or $x \geq 4$ (d) $x \geq 0$

Ans. (c)

For x $\leq 1, -3x + 6 \geq 6 \Rightarrow x \leq 0$ ……(A)

For 1≤ $x \leq 2, -x + 4 \geq 6 \Rightarrow$ x≤ -2

(not acceptable as 1≤ $x \leq 2$)

For x≥ $3, 3x - 6 \geq 6 \Rightarrow x \geq 4$ ………(B)

From (A) and (B) all positive value of x are given by x≤ 0 *or* $x \geq 4$

25. $2^{sinx} + 2^{cosx} \geq 2^{1} - \frac{1}{\sqrt{2}}$

(a) only for x ≥0 (b) only for x≤ 0
(c) for all real x (d) only for x ≠0

Ans. (c)

Since A.M. ≥ G.M

$\therefore \frac{2^{\sin x}+2^{\cos x}}{2} \geq \sqrt{2^{\sin x}.2^{\cos x}} =$
$2^{\left(\frac{1}{2}\right)(\sin x+\cos x)} = 2^{\left(\frac{1}{\sqrt{2}}\right)\sin\left(x+\frac{\pi}{4}\right)}$

$\Rightarrow 2^{\sin x} + 2^{\cos x} \geq 2^{1+\frac{1}{\sqrt{2}}\sin(x+\frac{\pi}{4})} \geq 2^{1-\frac{1}{\sqrt{2}}}$

[∵ least value of $\sin\left(x+\frac{\pi}{4}\right) = -1$]

26. How many different nine digit numbers can be formed from the number 223355888 by rearranging its digits so that the odd digits occupy even positions?

(a) 16 (b) 36
(c) 60 (d) 180

Ans. (c)

Number of digits= 9

Number of odd digits = 4, number of even digits= 5

Number of even places= 4

Odd digits can be arranged in even paces in $\frac{\underline{|4}}{\underline{|2}\underline{|2}}$ ways. Even digits can be arranged in remaining 5 places in $\frac{\underline{|5}}{\underline{|2}\underline{|3}}$ ways

∴ Required number = $\frac{\underline{|4}}{\underline{|2}\underline{|2}}.\frac{\underline{|5}}{\underline{|2}\underline{|3}} = 60$

27. For 2≤ $r \leq n$, $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} =$

(a) $\binom{n+1}{r-1}$ **(b)** $2\binom{n+1}{r+1}$
(c) $2\binom{n+2}{r}$ **(d)** $\binom{n+2}{r}$

Ans. (d)

$\binom{n}{r}$ *stands for* n_{C_r}

Now $n_{C_r} + 2n_{C_{r-1}} + n_{C_{r-2}}$

$= \left(n_{C_r} + n_{C_{r-1}}\right) + \left(n_{C_{r-1}} + n_{C_{r-2}}\right) =$
$n + 1_{C_r} + n + 1_{C_{r-1}} = n + 2_{C_r}$

28. If $\sum_{i=1}^{10} \sin^{-1} X_i = 5\pi, then \sum_{i=1}^{10} {X_i}^2 =$

(a) 0 **(b) 5**
(c) 10 **(d) none of these**

Ans. (c)

$$\sum_{i=1}^{10} \sin^{-1} x_i = 5\pi = 10.\frac{\pi}{2}$$

$\Rightarrow \sin^{-1} x_i = \frac{\pi}{2}, \forall i \Rightarrow x_i = 1 \forall i$

$\Rightarrow \sum_{i=1}^{10} {x_i}^2 = 1$

29. Range of f(x) = $sin^{20}x + cos^{48}x$ is

(a) [0, 1] **(b) (0, 1)**
(c) (0, ∞) **(d) none of these**

Ans. (b)

$0 \leq sin^2x \leq 1 \Rightarrow sin^{20}x \leq sin^2x$

Thus $0 \leq sin^{20}x \leq sin^2x$ ………. (1)

Again $0 \leq cos^{48}x \leq cos^2x$ ………..(2)

$since\ sin^{20}x\ and\ cos^{48}x\ can't\ be\ zero\ at\ a\ time$

$\Rightarrow 0 < f(x) \leq 1$

Hence range of f(x) = (0, 1)

30. Let x, y, z = 105, where x, y, z ∈N. Then number of ordered triplets (x, y, z) satisfying the given equation is:

(a) 15 **(b) 27**
(c) 6 **(d) none of these**

Ans. (b)

105= $3\times 5 \times 7$

When no 1 is taken as a solution, number of solutions= $\underline{|3} = 6$

When only 1' s taken, number of solutions= $3_{C_2}.\underline{|3} = 18$

When two 1's are taken, number of solutions = $3_{C_1}.\frac{\underline{|3}}{\underline{|2}} = 3$

∴ Reqd. number= 6+18+3 =27

Second method: xyz = $3\times 5 \times 7$

3 will be a factor of x or y or z in 3 ways

5 will be a factor of x or y or z in 3 ways

7 will be a factor of x or y or z in 3 ways

∴Total number of ways=$3\times 3 \times 3 = 27$

ISI OBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 9

There will be 30 questions in MMA Paper. For each question, exactly one of the four choices is correct. You get four marks for each correct answer, one mark for each unanswered question, and zero mark for each incorrect answer.

1. If f(x) = $(p\alpha - \alpha^2 - 2)x - \int_0^x (cos^4 t + sin^2 t - 2)\, dt$ is a decreasing function of x for all x ∈R and α∈ R, where α being independent of x, then

(a) p∈ (-∞, 1) **(b) p∈ (-1, $\sqrt{3}$)**
(c) p∈ (1, ∞) **(d) none of these**

Ans. (b) Given, $f(x) = (p\alpha - \alpha^2 - 2)x - \int_0^x (cos^4 t + sin^2 t - 2)\, dt$ ……….(1)

$\therefore f'(x) = p\alpha - \alpha^2 - 2 - (cos^4 x + sin^2 x - 2) = -\alpha^2 + p\alpha - (cos^4 x + sin^2 x)$

$= -\alpha^2 + p\alpha - (cos^4 x + cos^2 x + 1)$

$= -\alpha^2 + p\alpha - \left[\left(cos^2 x - \frac{1}{2}\right)^2 + \frac{3}{4}\right] = -\alpha^2 + p\alpha - \frac{3}{4} - \left(cos^2 x - \frac{1}{2}\right)^2$

Clearly f'(x) ≤ $-\alpha^2 + p\alpha - \frac{3}{4}$ [∵ $Min.\, value\, of\, \left(cos^2 x - \frac{1}{2}\right)^2 = 0$]

For f(x) to be decreasing for all real x,

f'(x) ≤ 0

$\therefore -\alpha^2 + p\alpha - \frac{3}{4} \le 0 \Rightarrow 4\alpha^2 - 4p\alpha + 3 \ge 0, \forall\, \alpha \in R$

∴ D ≤ 0⟹ $16p^2 - 48 \le 0 \Rightarrow -\sqrt{3} \le p \le \sqrt{3}$

2. Consider the following statements S and R. S: both sinx and cosx are decreasing function in $(\frac{\pi}{2}, \pi)$ & R: If a differentiable function decreases in (a, b) then its derivative also decreases in (a, b). Which of the following are true?

(a) both S and R are wrong
(b) S is correct and R is wrong

(c) both S and R are correct but R is not the correct expiation for S

(d) S is correct and R is the correct explanation for S

Ans. (b) From the trend of value of sin x and cos x we know sin x and cosx decrease in $\frac{\pi}{2} < x < \pi$. So, the statement S is correct.

The statement R is incorrect cos x is a differentiable function which decreases in $\left(\frac{\pi}{2}, \pi\right)$ but its d.c. – sin x is increasing in $\left(\frac{\pi}{2}, \pi\right)$

3. If f(x)= $\int_{x^2}^{x^2+1} e^{-t^2}\, dt$, then the interval in which f(x) is increasing is

(a) (0, ∞) **(b) (-∞, 0)**
(c) [-2, 2] **(d) none of these**

Ans. (b)

$$f(x) = \int_{x^2}^{x^2+1} e^{-t^2}\, dt\; f'(x)$$

$$= e^{-(x^2+1)^2}.2x - e^{-x^4}.2x$$

$$= \frac{2x}{e^{(x^2+1)^2}}\Big[1 - e^{-x^4+(x^2+1)^2}\Big]$$

$$= \frac{2x}{e^{(x^2+1)^2}}\left[1 - e^{2x^2+1}\right]$$

$$= \frac{2\left(e^{2x^2+1} - 1\right)}{e^{(x^2+1)^2}}(-x)$$

But $e^{2x^2+1} > 1$

∴ f'(x)> 0 in (-∞, 0) and hence f(x) is increasing in (-∞, 0)

4. The value of $\int_0^x \frac{(t-|t|)^2}{1+t^2} dt$ is equal to

(a) 4(x-$\tan^{-1} x$) if x < 0 **(b) 0 if > 0**
(c) $\log(1+x^2)$ if x > 0 **(d) none of these**

Ans. (a)

$$I = \int_0^x \frac{(t-|t|)^2}{1+t^2} dt$$

$$case\ I: x > 0\,, then\ 0 < t < x, |t| = t$$

∴ I= $\int_0^x \frac{(t-t)^2}{1+t^2} dt = 0$

Case II: x < 0, then x < t< 0⟹ |t|= -t

∴ I= $\int_0^x \frac{(t+t)^2}{1+t^2} dt = \int_0^x \frac{4t^2}{1+t^2} dt = 4\int_0^x \left(1 - \frac{1}{1+t^2}\right) dt = 4[t - \tan^{-1} t]_0^x = 4(x - \tan^{-1} x)$

5. If $\int \frac{x^2-2}{(x^4+5x^2+4)\tan^{-1}\left(\frac{x^2+2}{x}\right)} dx = \log|f(z)| + c$, then

(a) f(z) = $\tan^{-1} z\,, where\ z = \sqrt{x+2}$
(b) f(z) = $\tan^{-1} z\,, where\ z = x + \frac{2}{x}$
(c) f(z)= $\sin^{-1} z\,, where\ z = \frac{x+2}{x}$
(d) none of these

Ans. (b)

I= $\int \frac{x^2-2}{(x^4+5x^2+4)\tan^{-1}(\frac{x^2+2}{x})} dx$

Dividing numerator & denominator by x^2 we have

$$= \int \frac{1-\frac{2}{x^2}}{\left(x^2+5+\frac{4}{x^2}\right)\tan^{-1}\left(x+\frac{2}{x}\right)} dx = \int \frac{1-\frac{2}{x^2}}{\left[\left(x+\frac{2}{x}\right)^2+1\right]\tan^{-1}\left(x+\frac{2}{x}\right)} dx$$

$$let\ \tan^{-1}\left(x+\frac{2}{x}\right) = u$$

$$\Rightarrow \frac{1}{1+(x+\frac{2}{x})^2}.\left(1-\frac{2}{x^2}\right) = dx = du$$

$$now\ I = \int \frac{1}{u} du = \log|u| + c$$

$$= log\left|\tan^{-1}\left(x+\frac{2}{x}\right)\right|$$

$$= \tan^{-1} z, where\ z = \left(x+\frac{2}{x}\right)$$

6. $\int x \log\left(1+\frac{1}{x}\right) dx = f(x)\log(x+1) + g(x)x^2 + Lx + c$, then

(a) L= 1 **(b) f(x) = $\frac{1}{2}x^2$**
(c) g(x) = log x **(d) none of these**

Ans. (d)

I= $\int x \log\left(1+\frac{1}{x}\right)\ dx = \int x\ log\ (x+1)dx - \int x\ log\ x\ dx$

$$= \frac{x^2}{2}\log(x+1) - \frac{1}{2}\int \frac{x^2}{1+x}dx - \frac{x^2}{2}logx + \frac{1}{2}\int x\ dx$$

$$= \frac{x^2}{2}\log(x+1) - \frac{x^2}{2}logx - \frac{1}{2}\int \left(x-1+\frac{1}{x+1}\right)dx + \frac{1}{2}\int x\ dx$$

$$= \frac{x^2}{2}\log(x+1) - \frac{x^2}{2}\log x - \frac{1}{2}\log(x+1) + \frac{x}{2} + c$$

$$f(x) = \frac{x^2}{2} - \frac{1}{2}, g(x) = -\frac{1}{2}\log x$$

$$L = \frac{1}{2}.$$

7. $\int \frac{dx}{(x-1)^{\frac{3}{4}}(x+2)^{\frac{5}{4}}} =$

(a) $\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{\frac{1}{4}} + c$ **(b)** $\frac{4}{3}\sqrt{\frac{x-1}{x+2}} + c$

(c) $\left(\frac{x+2}{x-1}\right)^{\frac{1}{4}} + c$ **(d) none**

Ans. (a)

$$I = \int \frac{dx}{(x-1)^2\left(\frac{x+2}{x-1}\right)^{\frac{5}{4}}}$$

$$put\ z = \frac{x+2}{x-1}, then$$

$$dx = \frac{(x-1).1-(x-2).1}{(x-1)^2}dx = -\frac{3}{(x-1)^2}dx$$

$$now\ I = \frac{1}{3}\int z^{-5/4}\,dz = \frac{4}{3}z^{-1/4} + C = \frac{4}{3}.\left(\frac{x-1}{x+2}\right)^{1/4} + C$$

8. $\int e^x \frac{1+n.x^{n-1}-x^{2n}}{(1-x^n)\sqrt{1-x^{2n}}}$ dx=

(a) $\frac{e^x\sqrt{1-x^{2n}}}{1-x^{2n}} + c$ **(b)** $\frac{e^x\sqrt{1-x^n}}{1-x^n} + c$

(c) $\frac{e^x\sqrt{1-x^{2n}}}{1-x^n} + c$ **(d) none of these**

Ans. (c)

I= $\int e^x\left[\frac{1-x^{2n}+nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}}\right]dx = \int e^x\left[\frac{\sqrt{1-x^{2n}}}{1-x^n} + \frac{nx^{n-1}}{(1-x^n)^2}\sqrt{\frac{1-x^n}{1+x^n}}\right]dx = \int e^x\{f(x) + f'(x)]dx,$

$$where\ f(x) = \sqrt{\frac{1-x^{2n}}{1-x^n}} = e^x f(x) + C = e^x\frac{1-x^{2n}}{1-x^n} + C$$

9. $\int \frac{(x+1)}{x(1+xe^x)^2}dx = log|-f(x)| + f(x) + c\ then\ f(x) =$

(a) $\frac{1}{x+e^x}$ **(b)** $\frac{1}{x+xe^x}$

(c) $\frac{1}{(1+xe^x)^2}$ **(d) none**

Ans. (b)

Put z= xe^x, then dz = ($e^x + xe^x$) dx

I= $\int \frac{dz}{z(1+z)^2} = \int \left[\frac{1}{z} - \frac{1}{1+z} - \frac{1}{(1+z)^2}\right] dz =$
$\log \frac{z}{1+z} + \frac{1}{1+z} + C = log\left|\frac{xe^x}{1+xe^x}\right| + \frac{1}{1+xe^x} + c$

$$= \log\left|1 - \frac{1}{1+xe^x}\right| + \frac{1}{1+xe^x} + C$$

10. If $I_n = \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x}{sinx} dx, and\ a_n = \int_0^{\frac{\pi}{2}} (\frac{\sin n\theta}{\sin\theta})^2\, d\theta, then\ a_{n+1} - a_n$=

(a) I_n **(b) $2I_n$**
(c) $I_n + 1$ **(d) 0**

Ans. (c) $a_{n+1} - a_n = \int_0^{\frac{\pi}{2}} \frac{sin^2(n+1)x - sin^2 nx}{sin^2 x} dx$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)\, xsin\, x}{sin^2 x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x} dx$$

$$= I_{n+1}$$

11. If n ≠ 1, $\int_0^{\frac{\pi}{4}} (tan^n x + tan^{n-2} x)\, d(x - [x])$=

(a) $\frac{1}{n-1}$ **(b) $\frac{1}{n+1}$**
(c) $\frac{1}{n}$ **(d) $\frac{2}{n-1}$**

Ans. (a)

Let $I_n = \int_0^{\frac{\pi}{4}} (tan^n x + tan^{n-2} x)\, d(x - [x])$

$$here\ 0 < x < \frac{\pi}{4} \therefore [x] = 0 \ \therefore x - [x] = x$$

$$now\ I_n = \int_0^{\frac{\pi}{4}} tan^{n-2} x sec^2 x\, dx$$

$$= \int_0^1 z^{n-2}\, dz, putting\ z$$
$$= \tan x$$

$$= \left[\frac{z^{n-1}}{n-1}\right]_0^1 = \frac{1}{n-1}$$

12. If f(α)= f(β) and n ∈N, then the value of $\int_\alpha^\beta \left(g\left(f(x)\right)\right)'' g'\left(f(x)\right). f'(x)\, dx$ =

(a) 1 **(b) 0**
(c) $\frac{\beta^{n+1} - \alpha^{n+1}}{n+1}$ **(d) none of these**

Ans. (b)

Put z = g(f(x)), then dz = g'(f(x)) f'(x) dx

= $\int z^n dz = \frac{z^{n+1}}{n+1}$

$I = \frac{1}{n+1} [\{g\left(f(x)\right)]^{n+1}]\ {}_\alpha^\beta =$
$\frac{1}{n+1}\left[[\{g\left(f(\alpha)\right)]^{n+1} - [\{g\left(f(\beta)\right)]^{n+1}\right] = 0$
[∵ f(α)= f(β)]

13. Let [x] denotes the integral part of a real number x and {x} = x- [x], then solution of 4{x}= x+ [x] are

(a) $\pm\frac{2}{3}, 0$ **(b) $\pm\frac{4}{3}, 0$**
(c) 0, $\frac{5}{3}$ **(d) $\pm 2, 0$**

Ans. (c)

4{x}= x+ [x]= [x]+ {x} +[x]

$\Longrightarrow$ {x}= $\frac{2}{3}[x]$ …….(1)

Since $0 \le \{x\} < 1$

$\therefore 0 \leq \frac{2}{3}[x] < 1 \Rightarrow 0 \leq [x] < \frac{3}{2}$

Hence [x]= 0, 1

$\therefore \{x\} = 0, \frac{2}{3}$ [from (1)]

$\therefore$ x= [x]+ {x}= 0, $\frac{5}{3}$

14. The maximum number of real roots of the equation $x^{2n} - 1 = 0$ (n ∈N) is

(a) 2 **(b) 3**
(c) n **(d) 2n**

Ans. (a)

$x^{2n} - 1 = 0 \Rightarrow x^{2n} = 1 = \cos 0 + i \sin 0$

$\therefore x = \cos\frac{2r\pi}{2n} + i\sin\frac{2r\pi}{2n} = \cos\frac{r\pi}{n} + i\sin\frac{r\pi}{n}, r = 0, 1, \ldots, (2n-1)$

x will be real only when $\sin\frac{r\pi}{n} = 0$

or $\frac{r\pi}{n} = m\pi$

or r = mn = a multiple of n

But, r= 0, 1, 2,, …, 2n- 1

$\therefore$ r = 0, n

$\therefore x^{2n} - 1 = 0$ has only two real root 1, -1.

Second method: Let f(x) = $x^{2n} - 1$

Then, f'(x) = $2nx^{2n-1}$

Sign scheme for f'(x) is

Hence graph, of y= f(x) will either intersect x –axis at two points or touch x-axis or will not interest x-axis or will not interest x-axis. Therefore eqn. f(x)= 0 has two distinct real roots or two equal real roots or no real root.

15. The roots of equation $7^{\log_7(x^2-4x+5)}$ are

(a) 4, 5 **(b) 2, -3**
(c) 2, 3 **(d) 3, 5**

Ans. (c)

Given, $x^2 - 4x + 5 = x - 1$

$\Rightarrow x^2 - 5x + 6 = 0 \Rightarrow$ x= 2, 3

16. Equation $\frac{a^2}{x-\alpha} + \frac{b^2}{x-\beta} + \frac{c^2}{x-\gamma} = m - n^2x$ (a, b, c, m, n ∈ r) has necessarily

(a) all the roots real
(b) all the roots imaginary

(c) two real and two imaginary roots
(d) two rational and two irrational roots

Ans.(a)

Let p + iq be a root of given equation, then

$$\frac{a^2}{p-\alpha+iq} + \frac{b^2}{p-\beta+iq} + \frac{c^2}{p-\gamma+iq} = m - n^2(p + iq)$$

$$\Rightarrow \frac{a^2[p-\alpha-iq]}{(p-\alpha)^2+q^2} + \frac{b^2[(p-\beta)-iq]}{(p-\beta)^2+q^2} + \frac{c^2[(p-\gamma)-iq]}{(p-\gamma)^2+q^2} = m - n^2p - in^2q$$

Equating imaginary parts we get

$$q\left[\left\{\frac{a^2}{(p-\alpha)^2+q^2} + \frac{b^2}{(p-\beta)^2+q^2} + \frac{c^2}{(p-\gamma)^2+q^2}\right\} + n^2\right] = 0$$

$\therefore$ q= 0.

Hence p +iq= p= a real number.

17. If a, b, c ∈ { 1, 2, 3, 4, 5}, the number of equations of the form $ax^2 + bx + c = 0$ which have real roots is

(a) 25 **(b) 26**
(c) 207 **(d) 24**

Ans. (d)

For real roots ac $\leq \frac{b^2}{4}$

B	$\frac{b^2}{4}$	Possible value of ac such that ac $\leq \frac{b^2}{4}$	No. of possible pairs (a, c)
2	1	1	1
3	2.25	1.2	3
4	4	1, 2, 3, 4	8
5	6.25	1, 2, 3, 4, 5, 6	12
		Total	24

Value of ac	Possible pairs (a, c)
1	(1, 1)
2	(1, 2), (2, 1)
3	(1, 3), (3, 1)
4	(1, 4), (4, 1), (2, 2)
5	(1, 5), (5, 1)
6	(2, 3), (3, 2)

Hence number of quadratic equations having real roots = 24

18. If x, $\boldsymbol{a_1, a_2, a_3, \ldots, a_n \in R}$ *and* $\boldsymbol{(x - a_1 + a_2)2 + (x - a_2 + a_3)^2 + \cdots}$

$\boldsymbol{+(x - a_{n-1} + a_n)^2 = 0}$,
then $\boldsymbol{a_1, a_2, a_3, \ldots, a_n}$ are in

(a) A.P. (b) G.P. (c) H.P. (d) none of these.

Ans. (a) $(x - a_1 + a_2)^2 + (x - a_2 + a_3)^2 + \cdots + (x - a_{n-1} + a_n)^2 = 0$

$\Rightarrow a_1 - a_2 = a_2 - a_3 = \cdots = a_{n-1} - a_n = x$

$\Rightarrow a_1, a_2, a_3, \ldots . a_n$ are in A.P. with common difference x.

19. Let f(x) = $\boldsymbol{ax^2 + bx + c}$ and g(x) = af(x) + bf′(x) + cf″(x) If f(x) > 0 for all x , then the sufficient condition for g(x) to be > 0 v x is

(a) c > 0 **(b) b > 0**
(c) b< 0 **(d) c < 0**

Ans. (d)

g(x)= $a(ax^2 + bx + c) + b(2ax + b) + c + 2a = a^2x^2 + 3abx + b^2 + 3ac$

discriminant of its corresponding equation ,

D = $9a^2b^2 - 12a^3c$

$= 9a^2b^2 - 36a^3c + 24a^3c$

$= 9a^2(b^2 - 4ac) + 24a^3c$ ……(1)

Since f(x)> 0, $\forall\, x \in R$

∴ a > 0 and $b^2 - 4ac$< 0

For g(x)> 0 $\forall\, x \in R, a^2 > 0$ *and* $D < 0$

But from (1), D < 0 when c < 0

21. The constant term of the quadratic expression $\boldsymbol{\sum_{k=1}^{n}\left(x - \frac{1}{k+1}\right)\left(x - \frac{1}{k}\right)}$ *as n* $\rightarrow \infty$ is

(a) -1 **(b) 0**
(c) 1 **(d) none of these**

Ans. (c) Constant term

c = $\frac{1}{1.2}+\frac{1}{2.3}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$

$$\lim_{n\to\infty} c = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1$$

21. If $\boldsymbol{\theta_i} \in \left[\mathbf{0},\frac{\boldsymbol{\pi}}{\mathbf{6}}\right], \boldsymbol{i}=\mathbf{1,2,3,4,5}$ and $\boldsymbol{sin\theta_1 z^4+sin\theta_2 z^3+sin\theta_3 z^2+sin\theta_4 z+sin\theta_5}=\mathbf{2}$ then z satisfies

(a) |z|> $\frac{3}{4}$ (b) |z|< $\frac{1}{2}$ (c) $\frac{1}{2}<$ |z|< $\frac{3}{4}$ (d) none of these

Ans.(a)

Since 0≤ $\theta_i, \le \frac{\pi}{6}$

∴ 0≤ $sin\theta_i, \le \frac{1}{2}$

From given condition

|2|= $|sin\theta_5+zsin\theta_4+z^2sin\theta_3+z^3sin\theta_2+z^4sin\theta_1|$

$\Rightarrow 2\le\ |sin\theta_5|+|z||sin\theta_4|+|z|^2|sin\theta_3|+|z|^3|sin\theta_2|+|z|^4|sin\theta_1|\le\frac{1}{2}+\frac{1}{2}|z|+\frac{1}{2}|z|^2+\frac{1}{2}|z|^3+\frac{1}{2}|z|^4|<\frac{1}{2}|z|+|z|^2+\cdots\ to\ \infty$ ….(1)

When$|z|<1$, from (1),

2 < $\frac{1}{2}.\frac{1}{1-|z|}$

∴ $1-|z|<\frac{1}{4}\Rightarrow\ |z|>\frac{3}{4}$

When |z|> 1, clearly $|z|>\frac{3}{4}$

Thus $|z|>1, clearly\ |z|>\frac{3}{4}$

22. Number of solutions of $\mathbf{3}^{|\boldsymbol{x}|}=|\mathbf{2}-|\boldsymbol{x}||$ is

(a) 0 **(b) 2**
(c) 4 **(d) infinite**

Ans. (b)

Given equation is

$\left(\frac{1}{3}\right)^x=2-x,\quad -\infty<x\le\ -2$

= 2+ x, -2 ≤ $x\le 0$

$3^x=2-x, 0\le x\le 2$

= x – 2, 2≤ $x<\infty$

At x = 2, $3^x-x+2=9$

For x >2, $3^x-x+2>9$

(As 3^x-x+2 is an increasing function for x > 2)

For x= -2, $\left(\frac{1}{3}\right)^x+2+x=9$

For x < -2, $\left(\frac{1}{3}\right)^x+2+x<9$

[as $\left(\frac{1}{3}\right)^x+2+x\ is\ decreasing$]

Hence given equation has only two solutions -2 and 2.

23. The number of real roots of the equation $(\mathbf{9}+\boldsymbol{sinx})^{\frac{1}{1-x}}+(\mathbf{10}+\boldsymbol{sinx})^{\frac{1}{1-x}}=(\mathbf{11}+\boldsymbol{sinx})^{\frac{1}{1-x}}$ for x ∈ (0, 1) is

(a) exactly one (b) at least one (c) at most one (d) none of these

Ans. (a)

Given eqn. is f(x) = 1,

where f(x) = $\left(1+\frac{1}{10+\sin x}\right)^{\frac{1}{1-x}}-\left(1-\frac{1}{10+\sin x}\right)^{\frac{1}{1-x}}$

Clearly

f(0)= $\frac{1}{5}<1$ and $f(1-0)=\infty$

Also f(x) is an increasing function

∴ f(x)= 1 only for one value of x.

24. If 0 < αr < 1 for r= 1, 2, 3, …, k and m be the number of real solutions of equation

$\sum_{r=1}^{k}(a_r)^x=1$ & n be the number of real solution of equation $\sum_{r=1}^{k}(x-a_r)^{101}=0$, then

(a) m= n (b) m ≤ n (c) m ≥ n (d) m > n

Ans. (b)

Let α be a root of eqn.

$a_1{}^x+a_2{}^x+\cdots+a_k{}^x=1$ …………….(1)

Then when x < α, L.H.S. of (1)> 1

And when x > α, L.H.S. of (1)< 1

Hence, eqn. (1) cannot have more than one root.

$\therefore m\le 1$

Let f(x) = $(x-a_1)^{101}+(x-a_2)^{101}+\cdots+(x-a_n)^{101}$

∴ f'(x) > 0 ⟹ f(x) is an increasing function

Also f$(-\infty)=-\infty<0$ and $f(\infty)=\infty>0$

∴ f(x) = 0 has exactly one real root

∴ n= 1

Hence m ≤ n.

25. If m be number of integral solutions of equation $2x^2-3xy-9y^2-11=0$ and n be the number of real solutions of equation $x^3-[x]-3$= 0, then m =

(a) n (b) 2n (c) n/2 (d) 3n

Ans. (b)

Given, $2x^2-3xy-9y^2-11=0$

⟹ (2x +3y)(x- 3y)= 11

$\therefore \left.\begin{matrix}2x+3y=1\\x-3y=11\end{matrix}\right\},\left.\begin{matrix}2x+3y=11\\x-3y=1\end{matrix}\right\}$

$\left.\begin{matrix}2x+3y=-1\\x-3y=-11\end{matrix}\right\},\left.\begin{matrix}2x+3y=-11\\x-3y=-1\end{matrix}\right\}$

∴ x= 4, y= 1, x= -4, y= -1

∴ m= 2

Again, given

$x^3-[x]-3=0\Rightarrow x^3-(x-\alpha)-3=0,$

Where α= {x}= x-[x]

$\Rightarrow x^3-x=3-\alpha$. But $0\le\alpha<1$

$\therefore 2 < x^3 - x \leq 3$

For x ≥ 2,

$x^3 - x = x(x^2 - 1) \geq 2(2^2 - 1) = 6$

For x$\leq -1, x^3 - x = x(x^2 - 1) < 0$

For -1 < x< 0, $x^3 - x < 1 < 2$

For 0 < x $\leq 1, x^3 - x < x^3 < 1 < 2$

For x= 0, $x^3 - x = 0 < 2$

$\therefore$1 < x< 2 $\therefore$ [x]=1

$\therefore$Given equation becomes

$x^3 - 4$= 0$\Rightarrow$x= $4^{\frac{1}{3}}$

$\therefore$ n= 1

Thus m= 2, n=1

26. If [x] denotes the integral part of x and k = $sin^{-1}\frac{1+t^2}{2t} > 0$, then integral value of α for which the equation (x-[k])(x+α) - 1 = 0 has integral roots is

(a) 1 **(b) 2**
(c) 4 **(d) none of these**

Ans. (d)

For $\sin^{-1}\frac{1+t^2}{2t}$ *to be defined*, $\left|\frac{1+t^2}{2t}\right| \leq 1$

$\Rightarrow \frac{1+t^2}{2t} < 1$

$\Rightarrow 1+|t|^2 \leq 2|t|$

$\Rightarrow (1+|t|)^2 \leq 0$

$\Rightarrow (1+|t|)^2 = 0 \Rightarrow |t| = 1$

$\Rightarrow$ t= ± 1

$\therefore$ k= $\sin^{-1} 1 = \frac{\pi}{2}$ ($\because$k> 0)

$\therefore$ [k] = $[\frac{\pi}{2}]$= 1

Given equation is (x-1)(x-α)-1= 0

$\Rightarrow$ (x- 1)(x+ α)= 1 ……. (1)

We have to find integral value of α for which equation (1) has integral roots.

$\therefore$ x and α are integers.

From (1), (i) x- 1 = 1$\Rightarrow$ x= 2

X+ α= 1$\Rightarrow$ α= 1- x= -1

(ii) x- 1 = -1$\Rightarrow$ x= 0

X+ α= -1 $\Rightarrow$ α= -1

Thus, α= -1.

27. If [x] denotes the integral part of x and m= $\left[\frac{|x|}{1+x^2}\right]$, $\boldsymbol{n} =$ *integral values of* $\frac{1}{2-sin3x}$, then

(a) m$\neq$ n **(b) m > n**
(c) m + n = 0 **(d) $n^m = 0$**

Ans. (a)

$0 \leq \frac{|x|}{1+x^2} < 1$;

$\therefore$ m= $\left[\frac{|x|}{1+x^2}\right] = 0$

Again 1$\leq 2 - \sin 3x \leq 3$

$\therefore \frac{1}{3} \leq \frac{1}{2-\sin 3x} \leq 1$

$\therefore$ n= integral value of $\frac{1}{2-\sin 3x}$= 1

$\therefore m \neq n$ is the correct choice.

28. If 1 lies between the roots of equation $y^2 - my + 1 = 0$ and [x] denotes the integral part of x, then $\left[\left(\frac{4|x|}{x^2+16}\right)^m\right]$ =

(a) 1 (b) 0 (c) undefined (c) 2

Ans. (b)

Since 1 lies between the roots of equation

$y^2 - my + 1 = 0,$

$\therefore$ f(1)< 0

$\Rightarrow$ 2-m < 0 $\Rightarrow$ m > 2 …… (1)

Let y= $\frac{4|x|}{x^2+16} = \frac{4|x|}{|x|^2+16} = \frac{4z}{z^2+16}$, where z= |x|

$\therefore yz^2 - 4z + 16y = 0$

Since z is real,

$\therefore 16 - 64y^2 \geq 0 \Rightarrow -\frac{1}{2} \leq y \leq \frac{1}{2}$

$\therefore 0 \leq y \leq \frac{1}{2}$ [$\because$ y > 0]

$\therefore 0 \leq y^m \leq \frac{1}{2^m} < 1$

$\therefore [y^m] = 0$ …….. (2)

29. Let $\{f_n(x)\}$ be a sequence of polynomials defined inductively as

$$\mathbf{f_1(x) = (x - 2)^2}$$

$$\mathbf{f_{n+1}(x) = (f_n(x) - 2)^2 , n \geq 1.}$$

Let a_n and b_n respectively denote the constant term and the coefficient of x in $f_n(x)$. Then

(a) $a_n = 4, b_n = -4^n$

(b) $a_n = 4, b_n = -4n^2$

(c) $a_n = 4^{(n-1)!}$, $b_n = -4^n$

(d) $a_n = 4^{(n-1)!}$, $b_n = -4n^2$

Ans. (a) $a_n = (a_{n-1} - 2)^2$; $a_{n-1} = 4$; $a_n = 4$

$f_n(x) = (f_{n-1}(x) - 2)^2 = (f_{n-1}(x))^2 - 4 f_{n-1}(x) + 4$

So, $b_n = 2a_{n-1}b_{n-1} - 4b_{n-1} = 4b_{n-1} = 4^n b_0 = - 4^n$

30. Let x be a positive real number. Then

(a) $x^2 + \pi^2 + x^{2\pi} > x\pi + (\pi + x)x^\pi$
(b) $x^\pi + \pi^x > x^{2\pi} + \pi^{2x}$

(c) $\pi x + (\pi + x)x^\pi > x^2 + \pi^2 + x^{2\pi}$
(d) none of the above

Ans. (a) By A.M > GM inequality, we have

(i) $\Pi^2 + x^{2\pi} > 2\pi x^\pi$
(ii) $x^2 + x^{2\pi} > 2\pi x^\pi$
(iii) $x^2 + \pi^2 > 2\pi x$

So we have $x^2 + \pi^2 + x^{2\pi} > x\pi + (\pi + x)x^\pi$

ISI OBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 10

There will be 30 questions in MMA Paper. For each question, exactly one of the four choices is correct. You get four marks for each correct answer, one mark for each unanswered question, and zero mark for each incorrect answer.

1. If the tangents at points P and Q of the parabola $y^2 = 4ax$ meet at R and b, c, d be the length of perpendiculars from P, Q, R to any tangent to the parabola, then the roots of equations bx^2+ 2dx+ c = 0 are necessarily

(a) imaginary **(b) real and equal**
(c) real and unequal **(d) rational**

Ans. (b)

Let P ≡ (a$t_1{}^2, 2at_1$), Q ≡ (a$t_2{}^2, 2at_2$)

Then R ≡ [a$t_1 t_2, a(t_1 + t_2)$]

Let any tangent to the parabola be yt = x + at^2

or, x- yt+ a$t^2 = 0$ ………….(4)

b= length of perp. from P to line (4) = $\frac{|at_1{}^2 - 2at_1 t + at^2|}{\sqrt{1+t^2}} = \frac{|a|}{\sqrt{1+t^2}}(t_1 - t)^2$

c = $\frac{|a|}{\sqrt{1+t^2}}(t_2 - t)^2$

d = $\frac{|at_1{}^2 - a(t_1+t_2) + at^2|}{\sqrt{1+t^2}} = \frac{|a|}{\sqrt{1+t^2}}|(t_1 - t)^2(t_2 - t)|$

Clearly, bc = $D^2 \Rightarrow roots\ of\ eqn.$

b$x^2 + 2dx + c = 0$ are real and equal.

2. If n$c_0 - nc_1 + nc_2 - nc_3 + \cdots + (-1)^{rnc_r} = 28$, then n is equal to

(a) 6 **(b) 7**
(c) 8 **(d) 9**

Ans. (d) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^r\binom{n}{r}$

= $\binom{n-1}{0} - \left(\binom{n-1}{0} + \binom{n-1}{1}\right) + \left(\binom{n-1}{1} + \binom{n-1}{2}\right) - \left(\binom{n-1}{2} + \binom{n-1}{3}\right) + \cdots + (-1)^r(\left(\binom{n-1}{r-1} + \binom{n-1}{r}\right)$

= $(-1)^r\binom{n-1}{r}$

∴$(-1)^r\ \binom{n-1}{r}$ = 28 ⇒ r is even

∴ $\binom{n-1}{r}$ = 28 = 7× 4 $= \frac{7\times 8}{2} = 8c_2$

⇒ n= 9.

3. If x ∈ $\left(0, \frac{\pi}{2}\right)$ and cosx= ½ then the value of $\sum_{n=0}^{\infty}\frac{cosnx}{3^n}$ is equal to

(a) 1 **(b) -1**
(c) 2 **(d) -2**

Ans. (a) let c= 1+$\frac{cosx}{3} + \frac{cos2x}{3^2} + \cdots$

And S = $\frac{sinx}{3} + \frac{sin2x}{3^2} + \cdots$

⇒c + iS= 1+$\frac{e^{ix}}{3} + \frac{e^{ix}}{3^2} + \cdots$

= $\frac{1}{1-\frac{e^{ix}}{3}} = \frac{3}{3 - cosx - i\ sinx}$

Comparing real parts

c = $\frac{3(3-cosx)}{(3-cosx)^2+sin^2x}$

⇒ c =1 (cosx = $\frac{1}{3}$)

4. The number of value of n for which $3^9 + 3^{12} + 3^{15} + 3^n$ a perfect cube, is

(a) 2 **(b) 6**
(c) 8 **(d) none**

Ans. (d) $3^9 + 3^{12} + 3^{15} + 3^n$

= $3^9(1 + 3^4 + 3^6 + 3^{n-9})$

= $(3^3)^3\{1 + 3\,.\,3^2 + 3.(3^2)^2 + (3^2)^3 + 3^{n-9} - 3.(3^2)^2\}$

= $(3^3)^3(1 + 3^2)^3$,

⇒$3^{n-9} - 3^5 = 0$

⇒ n-9 =5

⇒ n = 14.

5. The number of integral solution of xy = $2^2.3^4.5^7(x + y)$is

(a) 675 **(b) 680**
(c) 685 **(d) none**

Ans. (a) Let N = $2^2.3^4.5^7$

Then, xy = N (x+ y)

⇒ xy- Nx- Ny= 0

⇒(x- N)(y- N)=$N^2 = 2^4.3^8.5^{14}$

∴ The number of integral solutions = (4+ 1)(8+ 1)(14+ 1) = 675.

6. A printer numbers the pages of a book starting with 1 and uses 3189 digit in all, then the number of pages are

(a) 200 **(b) 300**
(c) 400 **(d) none**

Ans. (d) No. of digits used for numbering pages 1 to 9 = 1 ×9 = 9.

Similarly, 10 to 99 = 90×2 = 180, 100 to 999 = 900×3 = 2700

Number of digits will remain after using 2889 (=9+ 180+ 2700)digits = 3189- 2889= 300,

The digits can be used for numbering 300 ÷4 = 75 pages, i.e. from 1000 to 1074.

Hence the book has 1074 pages.

7. The unit's digits of $3^{1001}.7^{1002}.13^{1003}$ is

(a) 1 **(b) 3**
(c) 5 **(d) none**

Ans. (d) unit digit in 3^{1001} is 3 ;

7^{1002} is 9;

And 3^{1001} is 7;

∴Ans. is = 3× $9 \times 7 = 9\ (unit\ digit)$

∴ 9 is in unit place.

8. The number of pairs of positive integer (x, y) which satisfy the equation $x^2 + y^2 = x^3$ is

(a) 0 **(b) 1**
(c) 2 **(d) none**

Ans. (d) $y^2 = x^2(x-1)$

So, if k is an integer satisfying x - 1 =k^2

⇒ x= $k^2 + 1$

Thus there are infinitely many solutions.

9. If 16−$x^2 >$ $|x-a|$ is to be satisfied by at least one non- negative values of x, then complete set of values of 'a' is

(a) (-8, 8) **(b) (-16,$\frac{65}{4}$)**
(c) (-8, $\frac{65}{4}$) **(d) none**

Ans. (b) 16−$x^2 > |x-a|$

⇒$x^2-16<x-a<16$−x^2

⇒$x^2-16-x<-a<16-x^2-x$

⇒x + 16-$x^2 > a > -16+x^2+x$

⇒$\frac{65}{4}-(x-\frac{1}{2})^2 > a > -16+x^2+x; x \leq 0$

∴ a ∈ (-16, $\frac{65}{4}$) (∵x ∈ℝ)

10. Number of positive solutions for $x^2-2-2[x]=0$, where [.]= the greatest integer, is

(a) 0 **(b) 1**
(c) 2 **(d) none**

Ans. (b) $x^2-2-2[x]\geq 0 \Rightarrow [x] = -1$

When [x]=-1, then $x^2-2 = -2 \Rightarrow x = 0$, which is not possible.

When [x]=1 ⇒ $x^2 = 4 \Rightarrow x = \pm 2$(impossible)

When [x]= 2 ⇒ $x^2 = 6 \Rightarrow x = \pm\sqrt{6}$, i.e. only one possible value , i.e. $\sqrt{6}$.

11. The solution set of |$\frac{x+1}{x}$|+|x+1|=$\frac{(x+1)^2}{|x|}$ is

(a) {x:x ≥ 0} **(b) {x: x > 0}∪ {−1}** **(c) {-1, 1}**
(d) none

Ans. (b) |$\frac{x+1}{x}$|+|x+1|=$\frac{|x+1|^2}{|x|}$

⇒ |x+1| ($\frac{1}{|x|}+1-\frac{|x+1|}{|x|}$)= 0

⇒ |x+1|= 0 or, 1+ $|x|-|x+1| = 0$

⇒|x+1|= 0

⇒ x+ 1 > 0 and x ≠ 0

i.e. x = -1 ,or >0 i.e. {x:x > 0}∪ {−1} .

12. The sum of the cubes of the root of equation

$\boldsymbol{x^4+ax^3+bx^2+cx+d=0}$ **is**

(a) a^3-3c **(b) 3ab - a^3** **(c) 3ab- c**
(d) none

Ans. (d) let $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ be the root of the equation

Here $a_0=1, a_1=a, a_2=b, a_3=c, a_4=d$

∴$a_0s_1+a_1=0 \Rightarrow s_1+a=0 \Rightarrow s_1=-a,$

Now, $a_0s_2+a_1s_1+2a_2=0$

⇒ $s_2+\ a\,(-a)+2b=0$

⇒ $s_2=\ \alpha^2-2b,$

⇒ ${\alpha_1}^2+{\alpha_2}^2+{\alpha_3}^2+{\alpha_4}^2=a-2b,$

∴$a_0s_3+a_1s_2+a_1s_1$+$3a_3=0$

⇒ $s_3=\ -a^3+3ab-3c$

$\Rightarrow {\alpha_1}^3 + {\alpha_2}^3 + {\alpha_3}^3 + {\alpha_4}^3 = 3ab - a^3 - 3c$.

13. The integral roots of $5x^3 - 11x^2 + 12x - 2 = 0$ are

(a) (1, 2, 3) **(b) (-1, -2, -3)**
(c) (0, 1, 2) **(d) none**

Ans. (d) f(x)= $5x^3 - 11x^2 + 12x - 2 = 0$

Where constant term = -2, and divisors of constant term are $\pm1, \pm2$, i.e. the possible value of integral roots are $\pm1, \pm2$

Now, f(1)$\neq 0, f(-1) \neq 0; f(2) \neq 0; f(-2) \neq 0$.so it has no integral roots.

14. If f(x) = $x^2 + 2bx + 2c^2$ and g(x)= $-x^2 - 2cx + b^2$ are such that min f(x), then relation between b and c, is

(a) |c|> $\sqrt{2}$ **(b) 0 < c < $\frac{b}{2}$**
(c) |c|< $\sqrt{2}$|b| **(d) none**

Ans. (a) f(x) = $(x+b)^2 + 2c^2 - b^2$

$\Rightarrow$ min f(x) = $2c^2 - b^2$

g(x) = $b^2 + c^2 - (x+c)^2$

$\Rightarrow$ max g(x) = $b^2 + c^2$

Thus, min f(x) > max g(x)

$\Rightarrow 2c^2 - b^2 > b^2 + c^2$

$\Rightarrow |c| > \sqrt{2}|b|$

15. The number of positive integers which are less than or equal to 1000 and are divisible by none of 17, 19 and 23 equals

(a) 854 **(b) 153**
(c) 160 **(d) none.**

Ans. (a) A: integers divisible by 17

B: integers divisible by 19

C: integers divisible by 23

$n(A \cup B \cup C)^c = 1000 - n(A) - n(B) - n(C) + n(A \cap B) + n(B \cap C) + n(C \cap A) - n(A \cap B \cap C)$

$= 1000 - \left[\frac{1000}{17}\right] - \left[\frac{1000}{19}\right] - \left[\frac{1000}{23}\right] + \left[\frac{1000}{17\times19}\right] + \left[\frac{1000}{17\times23}\right] + \left[\frac{1000}{19\times23}\right] - \left[\frac{1000}{17\times19\times23}\right]$

= 1000- 58- 52- 43+3+2+2-0

= 854

16. Let $\{a_n\}$ be a sequence of real numbers. Then $\lim_{n\to\infty} a_n$ exists if and only if

(a) $\lim_{n\to\infty} a_{2n}$ and $\lim_{n\to\infty} a_{2n+1}$ exist
(b) $\lim_{n\to\infty} a_{2n}$ and $\lim_{n\to\infty} a_{2n+2}$ exist
(c) $\lim_{n\to\infty} a_{2n}$ and $\lim_{n\to\infty} a_{2n+1}$ and $\lim_{n\to\infty} a_{3n}$ exist
(d) None of the above

Ans. (a) If a sequence converges then all of its subsequences converges.

$\therefore \lim_{n\to\infty} a_n$ converges $\Rightarrow \lim_{n\to\infty} a_{2n}$ and $\lim_{n\to\infty} a_{2n+1}$ exist

a_{2n} & a_{2n+1} cover all the terms in a_n

So, converse is also true.

17. In the Taylor expansion of the function f(x) = $e^{\frac{x}{2}}$ about x=3, the coefficient of $(x-3)^5$ is

(a) $e^{\frac{3}{2}}.\frac{1}{5!}$ **(b) $e^{\frac{3}{2}}.\frac{1}{2^5 5!}$**
(c) $e^{-\frac{3}{2}}.\frac{1}{2^5 5!}$ **(d) none**

Ans. (b) $\frac{f^{(r)}(x_0)}{r!}(x-x_0)^r = \frac{f^5(3)(x-3)^5}{5!} = \frac{e^{\frac{3}{2}}(x-3)^5}{2^5 5!}$

18. Let f(x, y) =
$$\begin{cases} e^{-1/(x^2+y^2)} & if\ (x,y) \neq (0,0) \\ 0 & if\ (x,y) = (0,0) \end{cases}$$

Then f(x, y) is

(a) Not continuous at (0, 0)
(b) Differentiable at (0, 0)

(c) Continuous at (0, 0) but does not have first order partial derivatives

(d) Continuous at (0, 0) and has first order partial derivatives but not differentiable at (0, 0)

Ans. (a) check yourself.

19. The maximum value of
$$\begin{bmatrix} 1+sin^2x & cos^2x & 4sin2x \\ sin^2x & 1+cos^2x & 4sin2x \\ sin^2x & cos^2x & 1+4sin2x \end{bmatrix}$$ **is**

(a) 0 (b) 2 (c) 4 (d) 6

Ans. (d) $\Delta = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ sin^2x & cos^2x & 1+4sin2x \end{bmatrix}$
$[R_1' = R_1 - R_2][R_2' = R_2 - R_3]$

= (1+ $4sin2x + cos^2x$) + (sin^2x)

= 2+ $4sin2x$

Since | $sin2x$ |≤ 1 $, so, 2 + 4sin2x \leq$ 2+4 = 6

20. The pages of book are numbered consecutively starting from pages. A total of 2989 digits was used to number the pages. Then the number of pages is divisible by

(a) 2 **(b) 3**
(c) 5 **(d) 7**

Ans. (a) 2989 = 189 + 2800= 189+2700+100 = $1\times 9 + 20\times 9 + 300\times 9 + 25\times 4$

From 1 to 9 pages, there are 9 digits

From 10 to 99 pages, there are 20×9 digits

From 100 to 999 pages, there are 300×9 digits.

From 1000 to 1024 pages, there are 25×4 digits.

So, there are total 1024 pages in the book.

21. Let A be a set of n elements. The number of ways, we can choose an ordered pair (B, C), where B, C are disjoint subsets of A, equals

(a) n^2 **(b) n^3**
(c) 2^n **(d) 3^n**

Ans. (d) 3^n (Give reason)

22. Consider the following system of equivalences of integers.

x ≡ 2 mod 15 **&**
x ≡ 4 mod 21.

The number of solutions in x, where 1 ≤ x ≤ 315, to the above system of equivalences is

(a) 0 **(b) 1**
(c) 2 **(d) 3**

Ans. (a) Use Chinese Remainder Theorem, there will be no solution.

23. The set of complex numbers z satisfying the equation (3 + 7i)z + (10 − 2i)$\bar{z}$ + 100 = 0 represents, in the complex plane,

(a) a straight line
(b) a pair of intersecting straight lines

(c) a point
(d) a pair of distinct parallel straight lines

Ans. (c) Hint: Put z = x + iy, $\bar{z}$ = x – iy

24. The $\lim_{n\to\infty} \sum_{k=1}^{n} \mid e^{\frac{2\pi ik}{n}} - e^{\frac{2\pi i(k-1)}{n}} \mid is$

(a) 2 **(b) 2e**
(c) 2π **(d) 2i**

Ans. (c) $\lim_{n\to\infty} \sum_{k=1}^{n} \mid e^{\frac{2\pi ik}{n}} \mid\mid e^{\frac{2\pi i}{n}} - 1 \mid =$

$\lim_{n\to\infty} n \mid e^{\frac{2\pi i}{n}} - 1 \mid = 2\pi$

25. For non-negative integers m, n define a function as follows

f(m, n) =
$$\begin{cases} n+1 & if\ m=0 \\ f(m-1,1) & if\ m\neq 0, n=0 \\ f(m-1, f(m, n-1)) & if\ m\neq 0, n\neq 0 \end{cases}$$

Then the value of f(1, 1) is

(a) 4 **(b) 3**
(c) 2 **(d) 1**

Ans. (b) From 1st equation, f(0, 1) = 2

From 3rd equation, f(1, 1) = f(0, f(1, 0)) = f(1, 0) + 1 (from 1st equation) = 3

26. Let S be the set of real numbers x for which the power series $\sum_{n=1}^{\infty}[1-(-2)^n]x^n$ converges. Then s equals

(a) {0} **(b) ($-\frac{1}{2}, \frac{1}{2}$)**
(c) ($-\frac{1}{2}, \frac{1}{2}$) **(d) (-1, 1)**

Ans. (b) $\sum_{n=1}^{\infty} a_n x^n$ converges if |x| < $\frac{1}{\underset{n\to\infty}{\text{lt}} |a_n|^{\frac{1}{n}}}$

i.e. |x| < $\frac{1}{\underset{n\to\infty}{\text{lt}} [1-(-2)^n]^{\frac{1}{n}}}$

i.e. |x| <- $\frac{1}{2}$.

27. There are 30 questions in a multiple choice test. A student gets 1 mark for each unattempted question, 0 mark for e wrong answer and 4 marks for each correct answer. A student has answered x question correctly and has score then the number of possible values of x is

(a) 15 **(b) 10**
(c) 6 **(d) 5**

Ans. (c)

60 =10 × 4 + 20 × 1, x =10

= 11 × 4 + 16 × 1 + 3 × 0 , x= 11

= 12× 4 + 12 × 1 + 6 × 0 , x= 12

= 13× 4 + 8 × 1 + 9 × 0 , x= 13

= 14 × 4 + 4 × 1 + 12 × 0 , x= 14

$= 15 \times 4 + 0 \times 1 + 15 \times 0$, x= 15

Total, 6 possible values are there, $10 \leq x \leq 15$.

28. The number of permutations of {1, 2, 3, 4, 5} that keep at least one integer fixed is

(a) 81 **(b) 76**
(c) 120 **(d) 60**

Ans. (c) There are total 5! Permutations of these 5 digits.

29. If f(x) is a real value function such that 2f(x) + 3f(-x) = 15- 4x, ∀ x ∈ℝ, then f(2) is

(a) -15 **(b) 22**
(c) 11 **(d) 0**

Ans. (c) put x= -x , 2f (-x) + 3f(x) = 15+ 4x

2f(x) + 3f(-x) = 15- 4x

Solving equations, we get f(x) = 3+4x, f(2) = 11.

30. If M is a matrix of 3×3 order such that

[0 1 2] M = [1 0 0] and [3 4 5] M = [0 1 0]. Then [6 7 8] M is equal to

(a) [2 1 -2] **(b) [0 1 0]**
(c) [-1 2 0] **(d) [9 10 8]**

Ans. (b) Do yourself.

ISI OBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 11

There will be 30 questions in MMA Paper. For each question, exactly one of the four choices is correct. You get four marks for each correct answer, one mark for each unanswered question, and zero mark for each incorrect answer.

1. Let f(x) = cosx(sinx + $\sqrt{\sin^2 x + \sin^2\theta}$), where '$\theta$' is a given constant,

then maximum value of f(x) is

(a) $\sqrt{1+\cos^2\theta}$ **(b) $\sqrt{1+\sin^2\theta}$**
(c) $|\cos\theta|$ **(d) none**

Ans. (b) $\{f(x)secx - sinx\}^2 = sin^2x + sin^2\theta$,

$\Rightarrow f^2(x)(1+tan^2x) - 2f(xtanx) = sin^2\theta$

$\Rightarrow f^2(x)tan^2x - 2f(xtanx) + f^2(x) - sin^2\theta = 0$

$\Rightarrow 4f^2(x) \geq 4f^2(x)\{f^2(x) - sin^2\theta\}$

$\Rightarrow f^2(x) \leq 1 + sin^2\theta$

i.e. $|f(x)| \leq \sqrt{1+sin^2\theta}$.

2. $\lim_{n\to\infty} \frac{1+\sqrt{2}+3\sqrt{3}+\cdots+n\sqrt{n}}{n}$

(a) equals 0 **(b) equals 1**
(c) equals ∞ **(d) none**

Ans. (b) Cauchy's First limit theorem:-
If $\lim\limits_{n\to\infty} u_n = l, then \lim\limits_{n\to\infty} \frac{u_1+u_2+\cdots+u_n}{n} = l$.

Here $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1, so, by$ Cauchy's first limit theorem

$$\lim_{n\to\infty} \frac{u_1+u_2+\cdots+u_n}{n} = \lim_{n\to\infty} \frac{1+\sqrt{2}+3\sqrt{3}+\cdots+n\sqrt{n}}{n} = 1.$$

3. The sum of the series 1+$\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+\cdots$ is

(a) e^2 (b) 3 (c) $\sqrt{5}$ (d) $\sqrt{8}$

Ans. (d) $\sqrt{8} = 2^{\frac{3}{2}} = (\frac{1}{2})^{-\frac{3}{2}} = (1-\frac{1}{2})^{-\frac{3}{2}}$

$= 1+(\frac{1}{2})\left(\frac{3}{2}\right) + \frac{\left(-\frac{3}{2}\right)(-\frac{3}{2}-1)}{2!}(-\frac{1}{2})^2 + \cdots$

$= 1+\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+\cdots$

4. If 0 < x < 1, then the sum of the infinite series $\frac{1}{2}x^2+\frac{2}{3}x^3+\frac{3}{4}x^4+\cdots$ is

(a) $\log\frac{1+x}{1-x}$ (b) $\frac{x}{1-x}+\log(1+x)$ (c) $\frac{1}{1-x}+\log(1-x)$ (d) $\frac{x}{1-x}+\log(1-x)$

Ans. (b) $\frac{1}{2}x^2+\frac{2}{3}x^3+\frac{3}{4}x^4+\cdots$

$= (1-\frac{1}{2})\,x^2 + (1-\frac{1}{3})\,x^3 + (1-\frac{1}{4})x^4 + \ldots.$

$= \{x^2+x^3+x^4+\ldots.\ \}-\{\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\cdots\}$

$= \{1+x+x^2+\ldots.\}-\{x+\frac{1}{2}x^2+\frac{1}{3}x^3+\ldots\}-1$

$= \frac{1}{1-x}+\log(1-x)-1$

$= \frac{x}{1-x}+\log(1-x)$

5. The polar equation r= acosθ represents

(a) a spiral (b) a parabola (c) a circle (d) none

Ans. (c) $r^2 = arcos\theta$

$\therefore x^2 + y^2 = ax$ (since r = $x^2 + y^2, x = acos\theta$)

$\therefore x^2 + ax + y^2 = 0$

$\therefore (x+\frac{a}{2})^2 + y^2 = \frac{a^2}{4}$

This is a circle of radius $\frac{a}{2}$ and centre ($-\frac{a}{2}$, 0).

6. The value of the infinite product P = $\frac{7}{9}\times\frac{26}{28}\times\frac{63}{65}\times\ldots\times\frac{n^3-1}{n^3+1}\times\ldots$ is

(a) 1 (b) 2/3 (c) 7/3 (d) none

Ans. (b)

P= $\frac{2^3-1}{2^3+1}.\frac{3^3-1}{3^3+1}\ldots\ldots$

$= (\frac{(2-1)(2^2+1+2)}{(2+1)(2^2+1-2)})(\frac{(3-1)(3^2+1+3)}{(3+1)(3^2+1-3})\ldots..$

$= (\frac{1.2.3.4\ldots..}{3.4.5.6\ldots\ldots})(\frac{7.13.21\ldots\ldots}{3.7.13\ldots\ldots})$

$= \frac{1.2}{3} = \frac{2}{3}.$

7. If f(x) = $\underset{i\to\infty}{\text{Lt}}\frac{(1+sin\pi x)^t-1}{(1+sin\pi x)^t+1}$, then range of f(x) is

(a) {-1, 1} (b) {0, 1}
(c) {-1, 1} (d) {-1, 0, 1}

Ans. (d)

f(x)= $\underset{t\to\infty}{\text{Lt}}\frac{(1+\sin\pi x)^t-1}{(1+\sin\pi x)^t+1}=$

$$\begin{cases}\dfrac{1-\frac{1}{(1+\sin\pi x)^t}}{1+\left(\frac{1}{1+\sin\pi x}\right)^{t,}} & \sin\pi x>0\\ \dfrac{0-1}{0+1}, & \sin\pi x<0\\ \dfrac{1-1}{1+1}, & \sin\pi x=0\end{cases} =$$

$$\begin{cases}1, & \sin\pi x>0\\ -1, & sin\pi x<0\\ 0, & sin\pi x=0\end{cases}$$

∴ Range f= {-1, 0, 1}

8. If f: $\left(0,\frac{\pi}{n}\right)\to \boldsymbol{R}, \boldsymbol{defined\ by\ f(x)} =$ $\sum_{k=1}^{n}[1+\sin kx]$, where [x] denotes the integral part of x, then range of f(x) is

(a) {n-1, n+1} **(b) {n-1, n, n+1}**
(c) {n, n+1} **(d) none of these**

Ans. (c)

f(x)= $\sum_{k=1}^{n}(1+[\sin kx])=n+[\sin x]+[\sin 2x]+\ ...+[\sin nx]$ …….(1)

case 1: when kx ≠ $\frac{\pi}{2}$ for k= 1, 2, 3, …, n

since 0 < kx < π and kx ≠ $\frac{\pi}{2}$

∴ 0 < sin kx < 1, for k= 1, 2, …., n

∴ [sin kx]= 0, for k= 1, 2, 3, …., n

∴ from (1), f(x)= n

When exactly one of x, 2x, 3x, …, nx is $\frac{\pi}{2}$. Here not more than one of x, 2x, 3x, …, nx can be $\frac{\pi}{2}$. In this case one of sin x, sin 2x, …, sinmnx is 1 and other lie between 0 and 1

∴ from (1), f(x)= n+1

Hence range of f= {n, n+ 1}

9. If f(x) = $\underset{n\to\infty}{\text{Lt}}\frac{x}{x+1}+\frac{x}{(x+1)(2x+1)}+\frac{x}{(2x+1)(3x+1)}+\cdots+$ to n terms, then range of f(x) is

(a) {0, 1} **(b) {-1, 0}** **(c) {-1, 1}** **(d) none of these**

Ans. (a)

$$S_n=\left(1-\frac{1}{1+x}\right)+\left(\frac{1}{1+x}-\frac{1}{1+2x}\right)+\left(\frac{1}{1+2x}-\frac{1}{1+3x}\right)+\cdots+\left(\frac{1}{1+(n-1)x}-\frac{1}{1+nx}\right)=1-\frac{1}{1+nx}$$

$but\ \underset{n\to\infty}{\text{Lt}}\ nx=\ \infty, x>0=\ -\infty, x<0=0, x=0$

$$\therefore f(x)=\underset{n\to\infty}{\text{Lt}}S_n=1, when\ x\ \neq 0$$
$$=0, \quad when\ x=0$$

Hence range f= {0, 1}

10. Period of f(x) = sin $\frac{\pi x}{(n-1)!}$ + cos $\frac{\pi x}{n!}$ is

(a) n ! **(b) 2 (n!)**
(c) 2 (n-1)! **(d) none of these**

Ans. (b) sin $\frac{\pi x}{(n-1)!}$ is a periodic function with period 2π+ $\frac{\pi}{(n-1)!}=2(n-1)!$

$$Period\ of\ \cos\frac{\pi x}{n!}=2\pi+\frac{\pi}{n!}=2n!$$

$$L.C.M. of\ 2(n-1)!\ and\ 2(n!) is\ 2(n!)$$

$$\therefore Period\ of\ f(x) is\ 2(n!)$$

11. Period of the function cos {(x+3)-[x+3]}, where [x] denotes the integral part of x is

(a) 1 (b) 2 (c) π (d) 2π

Ans. (a)

x- [x] is a periodic function with period 1.

∴ (x+ 3)- [x+ 3] is a periodic function with period 1.

∴ cos {(x+ 3)}- [x+ 3]] is a periodic function with period 1.

12. If f(x) = $2^{sin^3\pi x+x-[x]}$, where [x] denotes the integral part of x is a periodic function with period

(a) 1 (b) 2 (c) π (d) none of these

Ans. (b)

Period of x- [x] is 1 and period of $sin^3\pi x$ is 2.

L.C.M. of 1 and 2 is 2

∴ f(x) is a periodic function with period 2.

13. If f(x)= cos x+ cos ax is a periodic function, then a is necessarily

(a) an integer (b) a rational number (c) an irrational number (d) an event number

Ans. (b) Period of cos x= 2π and period of cos ax= $\frac{2\pi}{|a|}$

Period of f(x) = L.C.M. of $\frac{2\pi}{1}$ and $\frac{2\pi}{|a|} = \frac{L.C.M. of\ 2\pi\ and\ 2\pi}{H.C.F. of\ 1\ and\ |a|}$

Since k= H.C.F. of 1 and |a|

∴ $\frac{1}{k}$= an integer= m (say) and $\frac{|a|}{k}$= an integer = n (say)

∴ |a|=$\frac{n}{m} \Longrightarrow a = \pm\frac{n}{m}$ = a rational number.

14. Let f : R⟶ R defined by f(x)= $x^3 + x^2 + 100x + 5\sin x$, then f is

(a) many-one onto (b) many-one into (c) one-one onto (d) one-one into

Ans. (c)

$f(x) = x^3 + x^2 + 100x + 5\sin x$

∴ f'(x)= $3x^2 + 2x + 100 + 5\cos x$

= $3x^2 + 2x + 94 + (6 + 5\,cosx) > 0$

∴ f is an increasing function and consequently a one –one function.

Clearly f($-\infty$)=$-\infty$, f(∞)=∞ and f(x) is continuous, therefore range f= R= co domain f. Hence f is onto.

15. Let f(x) = $\frac{sin^{101}x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}$, where [x] denotes the integral part of x is

(a) an odd function
(b) an even function

(c) neither odd nor even function
(d) both odd and even function

Ans. (a) when x= nπ, n ϵ I, sin x = 0 and $\left[\frac{x}{\pi}\right]+\frac{1}{2}\neq 0$

∴ f(x) = 0

∴ when x = nπ, f(x)= 0 and f(-x)= 0

∴ f(-x)= f(x)

When x $\neq n\pi, n\,\epsilon\, I, \frac{x}{\pi}\neq an\ integer$

$$\therefore \left[\frac{x}{\pi}\right]+\left[-\frac{x}{\pi}\right] = -1 \ \therefore \left[-\frac{x}{\pi}\right] = -1-\left[\frac{x}{\pi}\right]$$

$$\Rightarrow \left[-\frac{x}{\pi}\right]+\frac{1}{2} = -\left[\frac{x}{\pi}\right]-\frac{1}{2} = -\left(\left[\frac{x}{\pi}\right]+\frac{1}{2}\right)$$

Now $f(-x) = \frac{sin^{101}(-x)}{\left[-\frac{x}{\pi}\right]+\frac{1}{2}} = \frac{-\sin x}{-(\left[\frac{x}{\pi}\right]+\frac{1}{2})} = \frac{\sin x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}} = f(x)$

Hence in all cases f(-x)= f(x)

16. . If k be the value of x at which the function

f(x) =$\int_{-1}^{x} t(e^t-1)(t-1)(t-2)^3(t-3)^5\,dt$ has maximum value and sinx + cosecx = k, then for n ∈N, $sin^n x + cosec^n x = \ldots$

(a) 2 **(b) -2**
(c) $\frac{\pi}{2}$ **(d) π**

Ans. (a) f'(x) = $x(e^x-1)(x-1)(x-2)^3(x-3)^5$

By Sign Rule we get

f(x) has max. at x = 2

∴ k = 2

Now sin x + cosec x = k ⟹ sin x + cosec x = 2

$\Rightarrow(\sin x-1)^2 = 0 \Rightarrow \sin x = 1$

∴ cosec x = 1

Hence $sin^n x + cosec^n x = 2$

17. If f(x+ y) = f(x) + f(y) – xy – 1for all x, y ∈R and f(1)=1, then the number of solutions of f(n)= n, n ∈N is

(a) 0 **(b) 1** **(c) 2** **(d) more than 2**

Ans. (b)

Given

f(x+ y)= f(x)+f(y)- xy- 1$\forall x, y, \epsilon R$ ……………(1)

f(1)= 1 ……………..(2)

f(2) = f(1+1)= f(1)+f(-1)-1-1= 0

f(3)= f(2+1)= f(2)+f(1)-2.1-1= -2

f(n+1) = f(n) +f(1) – n – 1 = f(n)- n< f(n)

Thus f(1) > f(2) > f(3)> …and f(1)= 1

∴f(1)= 1 and f(n)< 1, for n> 1

Hence f(n)= n, n ϵ N has only one solution n= 1

18. If f is an increasing function and g is a decreasing function such that g(f(x)) exists, then

(a) g(f(x)) is an increasing function
(b) g(f(x)) is an decreasing

(c) nothing can be said
(d) g(f(x)) is a constant function

Ans. (b)

f' (x)> 0 (∵ f(x) is an increasing function)

g'(x)< 0

(∵g(x) is a decreasing function)

(g(f(x)))'= g' (f(x)).f'(x) < 0

∴g(f(x)) is a decreasing function

19. f: R⟶ R, f(x)=x|x| is

(a) one-one and onto
(b) one-one but not onto

(c) not one-one but onto
(d) neither one-one nor onto

Ans. (a)

$$f(x) = \begin{cases} -x^2, x \leq 0 \\ x^2, x \geq 0 \end{cases}$$

$$\therefore \; f'(x) = \begin{cases} -2x^2, x \leq 0 \\ 2x, x \geq 0 \end{cases}$$

∴ f'(x)=>0 ∴ f(x) is an increasing function and consequently it is a one –one function.

Also f(−∞)= -∞, f(∞)=∞,

Hence range f= R

20. Let f(r) = 1+ $\frac{1}{2}$ +$\frac{1}{3}$ + ⋯ + $\frac{1}{r}$, *then* $\sum_{i=1}^{n} f(i)$=

(a) (n+1) f(n)-(n- 1) **(b) (n+ 1) f(n)- n**
(c) n f(n)- (n-1) **(d) (n-1) f(n)**

Ans. (b)

f(1)+ f(2)+ …+f(n)= 1+ $\left(1+\frac{1}{2}\right)+\left(1+\frac{1}{2}+\frac{1}{3}\right)+\cdots\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$

$$= n + \frac{(n-1)}{2} + \frac{(n-2)}{3} + \cdots + \frac{[n-(n-1)]}{n}$$

$$= n\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) - \left(\frac{1}{2}+\frac{2}{3}+\cdots+\frac{n-1}{n}\right)$$

$$= nf(n) - \left[\left(1-\frac{1}{2}\right)+\left(1-\frac{1}{3}\right)+\cdots + \left(1-\frac{1}{n}\right)\right]$$

$$= nf(n) - (n-1) + f(n) - 1$$
$$= (n+1)f(n) - n$$

21. The period of f(x) = $e^{\sin\{x\}}$ + $\sin\left(\frac{\pi}{2}[x]\right)$ is ([.]) and {.} are the greatest integer function and fraction function

(a) 1 **(b) 4**
(c) 2 **(d) not periodic**

Ans. (b)

Period of {x} i.e. x- [x] is 1 and period of $\sin\left(\frac{\pi}{2}[x]\right)$ *is* 4

L.C.M. of 1 and 4 is 4

∴ Period of f(x) is 4

22. If f(x) = $(a - x^n)^{1/n}$, x > 0 and g(x)> x ∀x ∊R, then for all x > 0

(a) g(g(x))= f(f(x)) **(b) g(g(x))>2 f(f(x))** **(c) g(g(x))< f(f(x))** **(d) g(g(x))> f(f(x))**

Ans. (d)

f(x)= $(a - x^n)^{1/n}, x > 0$

∴ f(f(x))= f(y), where y= f(x)= $(1-y^n)^{\frac{1}{n}} = \{1-(a-x^n)\}^{\frac{1}{n}} = x, x > 0$

Given, g(x)-> 0 ∀ x ϵ R

∴ g(g(x))-g(x)> 0 ∀ x ϵ R

[Putting g(x) in place of x]

Adding we get, g(g(x))-x> 0

⟹ g(g(x))> x ⟹g(g(x))> f(f(x)), x > 0

[∵ f(f(x))= x, x > 0]

23. Let f : R ⟶R be a function defined by f(x)=$\frac{e^{|x|}-e^{-x}}{e^x+e^{-x}}$, then

(a) 'f' is one-one and onto
(b) 'f' is one-one but not onto

(c) 'f' is not one-one but onto
(d) 'f' is neither one-one nor onto

Ans. (d)

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}, x \geq 0 = 0, x \leq 0$$

Since f(x)= 0, for all x ≤ 0

∴ f(x) is a many –one function

Let y= $\frac{e^x-e^{-x}}{e^x+e^{-x}}, x \geq 0$

⟹ $\frac{e^x-e^{-x}}{e^x+e^{-x}} = \frac{1}{y}$

⟹$\frac{e^x}{e^{-x}} = \frac{1+y}{1-y}$

⟹$e^{2x} = \frac{1+y}{1-y}$ ⟹ $x = \frac{1}{2}\log\frac{1+y}{1-y}$

$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1}, x \geq 0$$

Clearly $e^{2x} \geq 1 \; for \; x \geq 0$

∴y≥ 0 for x ≥ 0

∴ Range f = [0, ∞) ≠ co domain f.

Hence f is not onto.

Thus f is a many-one into mapping.

24. Given, y= sgn(x), then

(a) |x|= x sgn(x)
(b) sgn(sgn(x)) = sgn(x)

(c) x= |x|sgn(x)
(d) all of (a), (b), (c)

Ans. (d)

$$f(x) = sgn(x) = \begin{cases} 1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{cases}$$

$$x\, sgn\,(x) = \begin{cases} x, x > 0 \\ 0, x = 0 \\ -x, x < 0 \end{cases} = |x|$$

$$sgn\,(sgn(x)) = \begin{cases} sgn(1), x > 0 \\ sgn(0), x = 0 \\ sgn(-1), x < 0 \end{cases}$$
$$= \begin{cases} 1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{cases} = sgn(x)$$

$$|x|sgn(x) = \begin{cases} x, x > 0 \\ 0, x = 0 \\ x, x < 0 \end{cases} = x, \forall\, x \,\epsilon\, R$$

25. If l_1 *and* l_2 are the side length of two variables squares s_1 *and* s_2, respectively. If $l_1 = l_2 + {l_2}^3 + 6$, then the rate of change of the area of s_2 with respect to rate of change of the area of s_1 when $l_2 = 1$ is

(a) 3/2 **(b) 2/3** **(c) 4/3** **(d) none**

Ans. (d) Let Δ_1 *and* Δ_2 be the area of the sequences s_1 *and* s_2,

$\Delta_1 = {l_1}^2$ *and* $\Delta_2 = {l_2}^2$

$\therefore \frac{d\Delta_1}{dl_1} = 2l_1$ *and* $\frac{d\Delta_2}{dl_2} = 2l_2$.

$\Rightarrow \frac{d\Delta_2}{d\Delta_1} = \frac{l_2}{l_1} \cdot \frac{dl_2}{dl_1} = \frac{l_2}{l_1} \cdot \frac{1}{1+3l_2}$

When $l_2 = 1, l_1 = 8$, *then* $\frac{d\Delta_2}{d\Delta_1} = \frac{1}{32}$.

26. The limit $\lim_{x\to\infty}(\frac{4x}{5+4x})^{2x}$ is

(a) $e^{-5/2}$ **(b) $e^{-2/5}$** **(c) e^{-5}** **(d) none**

Ans. (a) $\lim_{x\to\infty}(\frac{1}{1+\frac{5}{4x}})^{2x} = \frac{1}{\{\lim_{x\to\infty}(1+\frac{5}{4x})^x\}^2} = e^{-\frac{5}{4}\times 2} = e^{-5/2}$.

27. Let a and b be real numbers such that $\lim_{x\to\infty} \sqrt{x^2 - x - 1} - ax - b = 0$.

Then the value of b is

(a) -1 **(b) - ½** **(c) 0** **(d) ½**

Ans. (b) $\lim_{x\to\infty}\left(\sqrt{x^2 - x - 1} - ax - b\right) = \lim_{x\to\infty} \frac{(x^2-x-1)-(ax+b)}{\sqrt{x^2-x-1}+ax^2+b}$

$=$

$\lim_{x\to\infty} \frac{(1-a^2)^{x^2}-(1+2ab)x-(1+b^2)}{\sqrt{x^2-x-1}+ax^2+b} = 0$

$\Rightarrow 1 - a^2 = 0 \Rightarrow$ a= 1, 1+2ab= 0 1+2b= 0 $\Rightarrow$ b = - ½ .

28. $\lim_{x\to 0} sin \frac{e^x - x - \frac{x^2}{2}}{x^2}$ is equal to

(a) 0 **(b)** ½ **(c) 1** **(d) does not exist**

Ans. (a) For x⟶ 0, $e^x \doteq 1 + x + {x^2}/{2!}$

So, $\lim_{x\to 0} sin \frac{(1+x+{x^2}/{2})-(1+x+{x^2}/{2})}{x^2} = \lim_{x\to 0} \sin(0) = 0$.

29. The limit $\lim_{x\to 0+} \log(\frac{1+x}{1-x})^{\frac{1}{x}}$ equals

(a) **0** **(b) 1** **(c) 2** **(d) does not exist**

Ans. (c) L= $\log(\frac{1+x}{1-x})^{\frac{1}{x}}$

$logL = \frac{1}{x}\log(\frac{1+x}{1-x})$

$\operatorname*{lt}_{x\to 0+} \log L = \operatorname*{lt}_{x\to 0+} \frac{1}{x}\{2(x + \frac{x^3}{3} + \cdots)\} = 2$.

30. If 0 < c < d, then the sequence $a_n = (c^n + d^n)^{\frac{1}{n}}$ is

(a) Bounded & monotone decreasing
(b) bounded & monotone increasing

(c) monotone increasing & unbounded for 1 < c < d

(d) monotone decreasing & unbounded for 1 < c < d.

Ans. (b) 0 < c < d ⇒ 0 < $c^n < d^n \; \forall\, n \in \mathbb{N}$

$\therefore c^n + d^n < 2d^n$

or, $a_n < 2^{\frac{1}{n}}.d \forall\, n \in \mathbb{N}$

or, d < $a_n < d.2^{\frac{1}{n}}$

By squeeze theorem, lim(a_n)= d. So, the sequence is bounded and monotone increasing.

ISI OBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 12

There will be 30 questions in MMA Paper. For each question, exactly one of the four choices is correct. You get four marks for each correct answer, one mark for each unanswered question, and zero mark for each incorrect answer.

1. For positive real numbers $a_1, a_2, \ldots\ldots a_{100}$, let P= $\sum_{i=1}^{100} a_i$ and $q = \sum_{1\le i\le j\le 100} a_i a_j$, then

(a) q =$\frac{P^2}{2}$ **(b) $q^2 \le \frac{P^2}{2}$**

(c) q < $\frac{P^2}{2}$ **(d) none**

Ans. (c) $a_1 + a_2 + \cdots \ldots + a_{100}$ = P,

$P^2 = (a_1 + a_2 + \cdots \ldots + a_{100})^2 = \sum_{i=1}^{100} {a_i}^2 + 2\sum_{i<j}^{100} a_i a_j$

$\therefore P^2 - 2q \ge 0$ [$\because \sum_{i=1}^{100} {a_i}^2 \ge 0$]

$\therefore q \le \frac{P^2}{2}$.

2. A club with x members is organized into four committees such that (a) each member is in exactly two committees, (b) only two committees have exactly one member in common, then x has

- **(a) Exactly two values both between 4 and 8**
- **(b) Exactly one value between 4 and 8**
- **(c) Exactly two values both between 8 and 16**
- **(d) Exactly one value between 8 and 16**

Ans. (b) Four committees are there, let us denote member

A , B, C, D, E, F, …..

1st combination :- ADE
[(a) each member is exactly in two committees

2nd combination:- ABF (b) only two committees exactly one member common]

3rd combination:- BCE

4th combination:- CDF

3. Let A = $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. Then

(a) there exists a matrix C such that A= BC= CD.

(b) there is no matrix C such that A= BC.

(c) there exists a matrix C such that A= BC, but A $\neq$ CB

(d) there is no matrix C such that A= CB.

Ans. (c) B=$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ is an lower triangular mtx.

Take, C = $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is an upper triangular mtx.

$\therefore$BC= $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}, CB \neq \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

4. If the matrix A = $\begin{bmatrix} \boldsymbol{a} & \mathbf{1} \\ \mathbf{2} & \mathbf{3} \end{bmatrix}$ has 1 as an eigenval then trace (A) is

(a) 4 **(b) 5**
(c) 6 **(d) 7**

Ans. (b) |A- λI| = 0 ⇒a = 2

∴ trace (A) = 2+ 3 = 5= sum of diagonal elements.

5. The eigen values of the matrix X= $\begin{pmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} \end{pmatrix}$ are

(a) 1, 1, 4 **(b) 1, 4, 4**
(c) 0, 1, 4 **(d) 0, 4, 4**

Ans. (a) |X-λI|= 0

$\Rightarrow \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{bmatrix} = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

⤇ (λ-1)(λ-1)(λ-4)= 0

⤇ λ= 1, 1, 4 are the eigen value of X.

6. If f(x) = x^2 and g(x) = $x sin x + cos x$ then

(a) f and g agree at no points
(b) f and g agree at exactly one points

(c) f and g agree at exactly two point
(d) f and g agree at more than two points

Ans. (c) So, we can two graphs meet at exactly two points.

7. A subset S of the set of numbers {2, 3, 4, 5, 6, 7, 8, 9, 10} is said to be good if it has exactly 4 elements and their gcd = 1. Then the number of good subset is

(a) 0126 **(b) 125**
(c) 123 **(d) 121**

Ans. (d) Total number of subsets containing exactly 4 elements from 9 elements are

= $9_{c_4} = \frac{9!}{4!5!} = 126.$

Now, gcd = 1, so we need not to take into count these subset: {2, 4, 6, 8}, {2, 4, 6, 10}, {2, 4, 8, 10},{2, 6, 8, 10}, {4, 6, 8, 10}.

So, there are total (126 - 5) = 121 good subsets.

8. In how many ways can there persons, each throwing a single die once,

make a score of 11?

(a) 22 (b) 27 (c) 24 (d) 38

Ans. (b) we can use a tree diagram here:-

$1 < \begin{matrix} \to 6\ ,4 & :3!ways \\ \to 5\ ,\ 5 & :3\ ways \end{matrix}$

$2 < \begin{matrix} \to 4\ ,\ 5 & :3!ways \\ \to 3\ ,\ 6 & :3!\ ways \end{matrix}$

$3 < \begin{matrix} \to 4\ ,\ 5 & :3\ ways \\ \to 3\ ,\ 6 & :3\ ways \end{matrix}$

So, there are total (3!+3 +3!+3!+3+3)= 27 ways.

Explanation: - For {1, 6, 4} there are total 3! Ways, since{1, 6, 4}, {4, 6, 1}, {1, 4, 6}, {4, 1, 6}, {6, 4, 1} are six possibilities, but in case of {1, 5, 5} there are total 3 ways, since there are {1, 5, 5}, {5, 1, 5}, {5, 5, 1} only 3 such permutations.

9. $x^2 + x + 1$ is a factor of $(x+1)^n - x^n - 1$, wherever

(a) n is odd **(b) n is odd and a multiple of 3**

(c) n is even multiple of 3 **(d) n is odd and not a multiple 3**

Ans. (c) Let n= 1, (x+1)-x-1= 0

n=2, $(x+1)^2 - x^2 - 1 = +2x$

n= 3, $(x+1)^3 - x^3 - 1 = 3x^2 + 3x$

⋮

10. The map f(x) = $a_0 cos|x| + a_1 sin|x| + a_2|x|^3$ is different at x=o if and only if

(a) $a_1 = 0\ and\ a_2 = 0$
(b) $a_0 = 0\ and\ a_1 = 0$

(c) $a_1 = 0$
(d) a_0, a_1, a_2 can take any real value

Ans. (c) $|x|^3$ is differentiable at x=0.

$cos|x| = cosx$ is differentiable at x= 0.

So, $a_1 = 0$ is the only criteria for f(x) to be differentiable.

11. f(x)is a differentiable function on the real line such that $\lim_{x\to\infty} f(x) = 1\ and\ \lim_{x\to\infty} f'(x) = \alpha$, then

(a) α must be 0 **(b) α need not to be 0, but $|\alpha|$<1** **(c) α > 1** **(d) α < -1**

Ans. (a) Let f(x)=1+$\frac{k}{x^P}$, $p > 0$

So, $\operatorname*{lt}_{x\to\infty} f(x) = 1$,

$\therefore \lim_{x\to\infty} f'^{(x)} = \operatorname*{lt}_{x\to\infty} \frac{k(-P)}{x^{P+1}} = 0 = \alpha$.

12. $\int_0^\pi \min(sinx, cosx)dx$ equals

(a) 1-2$\sqrt{2}$ **(b) 1** **(c) 0** **(d) 1-$\sqrt{2}$**

Ans. (d) f(x)= $\begin{cases} sinx, 0 \le x \le \frac{\pi}{4} \\ cosx, \frac{\pi}{4} \le x \le \pi \end{cases}$

$\therefore \int_0^\pi \min(sinx, cosx)dx = \int_0^{\frac{\pi}{4}} sinxdx + \int_{\frac{\pi}{4}}^{\pi} cosdx = \left(1 - \frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right) = 1 - \sqrt{2}$

13. The value of the integral $\int_{-2}^{2} \min\{|x-1|, |x+2|\}dx$, is

(a) $\frac{11}{4}$ **(b) $\frac{9}{4}$** **(c) $\frac{11}{2}$** **(d) $\frac{9}{2}$**

Ans. $\int_{-2}^{2} \min\{|x-1|, |x+2|\}dx$

$= \int_{-2}^{-\frac{1}{2}} |x+2|dx + \int_{-\frac{1}{2}}^{2} |x-1|dx$

14. Let f(x) be a given differentiable function. Consider the following differential equation in y f(x)$\frac{dy}{dx} =$ $yf'(x) - y^2$.

The general solution of the equation is given by

(a) y= $-\frac{x+c}{f(x)}$ **(b)** $y^2 = \frac{f(x)}{x+c}$
(c) y = $\frac{f(x)}{x+c}$ **(d)** y = $\frac{[f(x)]^2}{x+c}$

Ans. (c) $\frac{dy}{dx} = \frac{yf'(x)-y^2}{f(x)}$

$\Rightarrow$ -$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\frac{f\prime(x)}{f(x)} = \frac{1}{f(x)}$ [put $\frac{1}{y}$ = v, -$\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$]

$\Rightarrow \frac{dv}{dx} + v.\frac{f\prime(x)}{f(x)} = \frac{1}{f(x)}$ which is a linear equation in v

$\therefore$ I.F. = $e\int\frac{f\prime(x)}{f(x)}dx = e^{\log f(x)} = f(x)$.

Hence solution is: v. f(x)= $\int\frac{1}{f(x)}.f(x)dx + c = x + c \;\; i.e. y = \frac{f(x)}{x+c}$.

15. Number of integral terms in the expansion of $(\sqrt{6}+\sqrt{7})^{32}$=

(a) 15 **(b) 17**
(c) 19 **(d) none**

Ans. (b) $(\sqrt{6}+\sqrt{7})^{32} = \sum_{r=0}^{32} 32_{c_r}.6^{\frac{r}{2}}.7^{\frac{32-r}{2}}$

For integral terms $\frac{r}{2}$ and $\frac{32-r}{2}$ both are integers and w is in turn possible if $\frac{r}{2}$ is an integer.

$\therefore$ r = 0, 2, 4,…, 32 means r can take 17 different values.

16. Let P is an odd prime and n= 1+p!, then total number of prime in the list n+1, n+2, n+3, ……, n+p-1 is equal to

(a) P- 3 **(b) P- 5**
(c) 0 **(d) none**

Ans. (c) $\because$ n= 1+p!

$\therefore$ n+ r= (r+1)+p!

If $1 \le r \le p-1, then\ 2 \le r+1 \le p$ and clearly,

(n+ r) is divisible by r+1. $\therefore$ n+ r can't be a prime

Hence, there is no prime in the given list.

17. The remainder obtained when 1! +2! +3!+….+ 95! is divided by 15, is

(a) 3 **(b) 5**
(c) 7 **(d) none**

Ans. (a) here 1! +2! +3! +4! = 33 and n! is divisible by 15 where n $\ge$ 5.

The remainder is same as the remainder obtained by dividing 33 with 15, i.e., 3.

18. The value of $\lim\limits_{x\to 0}\frac{e^{tanx}-e^x}{tanx-x}$ is

(a) 0 **(b) 1**
(c) e **(d) none**

Ans. (b) $\underset{x\to 0}{\mathrm{Lt}} \frac{e^{\tan x}-e^x}{\tan x - x} = \underset{x\to 0}{\mathrm{Lt}}\, e^x \left(\frac{e^{\tan x}-1}{\tan x - x}\right) =$ $e^0 . \log_e e = 1$

19. Total number of solutions of sinx = $\frac{|x|}{10}$ is

(a) 0 ; (b) 3 ; (c) 4 ; (d) none

Ans. (d) Two graphs meet exactly 6 times, hence, it has 6 solutions. Draw graph yourself.

20. Let f(x)= $\begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$ ***where p is a constant , then at x =*** **0,** $\frac{d^3(f(x))}{dx^3} =$

(a) p **(b) 0**
(c) p^2 **(d) –p**

Ans. (b) $\frac{d^3}{dx^3}(x^3) = 3! = 6, \frac{d^3}{dx^3}(\sin x) = -\cos x, \frac{d^3}{dx^3}(cosx) = sinx$

$$\therefore \frac{d^3 f(x)}{dx^3} = \begin{vmatrix} 6 & -\cos x & \sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} at\ x = 0,$$

$$\frac{d^3 f(x)}{dx^3} = \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} = 0$$

21. Let f : (0, +∞) → R *and* $F(x) = \int_0^x f(t)dt$ *if* $F(x^2) = x^2(1+x)$, then f(4) equals

(a) 5/4 **(b) 7**
(c) 4 **(d) 2**

Ans. (c)

We have, f(x^2)= $\int_0^{x^2} f(t)\ dt = x^2 + x^3$

Differentiating both sides, we get;

f(x^2).$2x = 2x + 3x^2$

⟹ f(x^2)= 1+(3/2)x

⟹ f(4)= 1+3/2(2)= 4

22. The equation of a curve is y= f(x). The tangents at (α, f(α)), (β,f(β)) and (γ, f(γ)) make angles $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}$ respectively with the positive direction of the x- axis. Then the value of

$\int_\beta^\gamma f'(x).f''(x)dx + \int_\alpha^\gamma f''(x)dx$ **is equal to**

(a) $-\frac{1}{\sqrt{3}}$ **(b) $\frac{1}{\sqrt{3}}$**
(c) 0 **(d) none of these**

Ans. (a)

Given, $f'(\alpha) = \frac{1}{\sqrt{3}}, f'(\beta) = \sqrt{3}, f'(\gamma) = 1$

Now $\int_0^\gamma f'(x)f''(x)\,dx + \int_\alpha^\gamma f''(x)\ dx = \left[\frac{1}{2}\left(f'(x)\right)^2\right]_\beta^\gamma + [f'(x)]_\alpha^\gamma = \frac{1}{2}\left(f'(\gamma)\right)^2 - \frac{1}{2}\{f'(\beta)\}^2 + f'(\gamma) - f'(\alpha) = \frac{1}{2}(1-3) + 1 - \frac{1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$

23. A rod of length 10ft sides with ends on the co-ordinates axes. If the end on x-axis moves with constant velocity of 2ft/ minute, then the magnitude of the velocity of the middle point at the instant the rod makes an angle of 30° with x-axis is

(a) 2ft / sec **(b) 3 ft / sec**
(c) $\sqrt{3}$ ft / sec **(d) none of these**

Ans. (a)

Let AB be the position of rod at any time t and p be its middle point.

Let OA= x, OB= y, then P $\equiv \left(\frac{x}{2}, \frac{y}{2}\right)$

$$x^2 + y^2 = 10^2$$

$\therefore 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{x}{y}.2$$

$$when\ \theta = 30°, \frac{x}{y} = \cot 30° = \sqrt{3}$$

$$\therefore \frac{dy}{dt} = -2\sqrt{3}ft/sec$$

$$now\ V = \sqrt{\left(\frac{1}{2}\frac{dx}{dt}\right)^2 + \left(\frac{1}{2}\frac{dy}{dt}\right)^2}$$
$$= \frac{1}{2}\sqrt{2^2 + 12} = 2\ ft/sec$$

24. Two persons are moving on the curve $x^3 + y^3 = a^3$. When the position of first and second persons are (α, β) and (γ, δ) the second persons is in the direction of the instantaneous motion, then

(a) $\frac{\gamma}{\alpha} + \frac{\delta}{\beta} + 1 = 0$ (b) $\frac{\alpha}{\gamma} + \frac{\beta}{\delta} - 1 = 0$ (c) $\alpha\gamma + \beta\delta = 1$ (d) none of these

Ans. (a)

Given curve is $x^3 + y^3 = a^3$……….(1)

Let P≡ $(\alpha, \beta), Q \equiv (\gamma, \delta)$

Since P and Q lie on(1)

$\therefore \alpha^3 + \beta^3 = a^2\ and\ \gamma^3 + \delta^3 = a^2$

$\alpha^3 - \gamma^3 = \delta^3 - \beta^3$……………(2)

From(1), $\frac{dy}{dx} = -\frac{x^2}{y^2}$

Equation of tangent at P$(\alpha, \beta)\ is\ y - \beta = \frac{\alpha^2}{\beta^2}(x - \alpha)$………..(3)

According to question, (3) passes through

Q $\therefore \alpha^2(\alpha - \gamma) = \beta^2(\delta - \beta)$ ……(4)

$$\frac{\alpha^2 + \gamma^2 + \square\gamma}{\alpha^2} = \frac{\beta^2 + \delta^2 + \beta\beta}{\beta^2}$$
$$\Rightarrow 1 + \frac{\gamma^2}{\alpha^2} + \frac{\gamma}{\alpha}$$
$$= 1 + \frac{\delta^2}{\beta^2} + \frac{\delta}{\beta}$$

$$\Rightarrow \left(\frac{\gamma}{\alpha}\right)^2 - \left(\frac{\delta}{\beta}\right)^2 = \left(\frac{\gamma}{\alpha} - \frac{\delta}{\beta}\right) \Rightarrow \frac{\gamma}{\alpha} - \frac{\delta}{\beta} + 1$$
$$= 0[\because \frac{\gamma}{\alpha} \neq \frac{\delta}{\beta}]$$

25. The triangle formed by the tangents to the curve f(x) = $x^2 + bx - b$ at the point (1, 1) and the co-ordinate on the first quadrant. If its area is 2 then the value of b is

(a) -1 (b) 3
(c) -3 (d) 1

Ans. (c) $\frac{dy}{dx} = 2x + b$

$\therefore$ The equation of the tangent at (1, 1) is

y-1= (2+b)(x-1)

or (2+b)x- y= 1+ b

$\therefore$ OA= $\frac{1+b}{2+b}$ $and\ OB = -(1+b)$

Since Δ AOB lies in the first quadrant,

$\therefore \frac{1+b}{2+b} > 0 and\ 1+b < 0$

$\therefore$1+b < 0, 2+b < 0$\Longrightarrow$ b < -2………..(1)

Now, area (ΔAOB)= 2

$\therefore$ 2= $\frac{1}{2} . \frac{1+b}{2+b}\{-(1+b)\}$

or, 4(2+b)+ $(1+b)^2 = 0$

or, $b^2 + 6b + 9 = 0$

Or $(b+3)^2 = 0$

$\therefore$ b= -3 > 1

26. If 2a+ 3b +6c = 0, then equation $ax^2 + bx + c = 0$ has at least one root in

(a) (-1, 1) **(b) (1, 2)**
(c) (-1, 0) **(d) (2, 3)**

Ans. (a)

Let f'(x) = a$x^2 + bx + c$, then

f(x) = $\frac{ax^3}{3} + \frac{bx^2}{2} + cx = \frac{2ax^3+3bx^2+6cx}{6}$

$$f(1) = \frac{2a+3b+6c}{6} = 0, f(0) = 0$$

$\therefore$ f(0)= f(1)

$\therefore$ there exists α, 0< α < 1 such that f' (α)= 0

i.e., equation a$x^2 + bx + c = 0$ has at least one root in (0, 1).

27. Equation sin x + 2sin 2x +3 sin 3x= $\frac{8}{\pi}$ has at least one root in

(a) $\left(\pi, \frac{3\pi}{2}\right)$ **(b) $\left(0, \frac{\pi}{2}\right)$**
(c) $\left(\frac{\pi}{2}, \pi\right)$ **(d) none of these**

Ans. (b)

Let f(x) = $\frac{8}{\pi}x + \cos x + \cos 2x + \cos 3x$

$$then\ f'(x) = \frac{8}{\pi}x - sinx - 2\sin 2x - 3\sin 3x$$

f(x) is continuous and differentiable at every point

Also f(0)= f$\left(\frac{\pi}{2}\right)$ $\therefore$ By Rolle's theorem

f'(c)= 0 for at least one c in $\left(0, \frac{\pi}{2}\right)$

28. Let f(x) = 1+|x- 2|+|sin x|, then Lagrange's mean value theorem is applicable for f(x) in

(a) $[0, \pi]$ **(b) $[\pi, 2\pi]$**
(c)$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ **(d) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$**

Ans. (b)

f(x) is not differentiable at x= 2 and x= nπ,

n $\in$I

29. let f(x) and g(x) be differentiable functions for 0$\leq x \leq$ 1 such that f(0) = 2, g(0)= 0, f(1)= 6.Let there exist a real number c in (0, 1) such that f'(c)= 2g'(c), then g(1)=

(a) 1 **(b) 2**
(c) -2 **(d) -1**

Ans. (b)

Let ϕ(x)= f(x)+Ag(x)

Then ϕ'(x)= f'(x)+ Ag'(x)

Choosing A such that φ(0)= ϕ(1),

We have A= - $\frac{f(1)-f(0)}{g(1)-g(0)}$

for this value of A using Rolle's theorem for ϕ(x) in (0, 1), we have

ϕ'(c)= 0 for some c ∊ (0, 1)

$$\Rightarrow \frac{f'(c)}{g'(c)} = -A = \frac{f(1)-f(0)}{g(1)-g(0)}$$
$$= \frac{6-2}{g(1)-0} \Rightarrow 2 = \frac{4}{g(1)}$$
$$\Rightarrow g(1) = 2$$

30. Which of the following function does not obey mean value theorem in [0, 1]

(a) f(x)= $\frac{1}{2}$ **- x, x < ½; f(x)=**$\left(\frac{1}{2}-x\right)^2, x \geq \frac{1}{2}$
(b) f(x) = $\frac{\sin x}{x}, x \neq 0; f(x) = 1, x = 0$

(c) f(x)= x |x|
(d) f(x)= |x|

Ans. (a)

Let f(x) = $x^3 - 3x + k$

Then f'(x) = 3$(x^2 - 1) < 0 \; in \; (0, 1)$

⟹ f'(x) has no root in (0, 1)

But f(x) = 0 has two distinct roots $\alpha$and β in (0, 1)

⟹ f'(x)= 0 has at least one root in (α, β).

ISI OBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 13

There will be 30 questions in MMA Paper. For each question, exactly one of the four choices is correct. You get four marks for each correct answer, one mark for each unanswered question, and zero mark for each incorrect answer.

1. When m, n are positive integers, then (m+n)! is divisible by

(a) m!n! **(b) m! +n!**
(c) m!-n! **(d) none**

Ans. (a) (m+n)!=
1.2…….m(m+1)(m+2)….(m+n)

= $\frac{m!\,.(m+1)(m+2)....(m+n)n!}{n!}$

= m! n! × an integer

⇒ $\frac{(m+n)!}{m!n!} = an\ integer.$

2. If n and r are positive integers such that 0 < r< n, then the roots of the quadratic equation $\boldsymbol{n_{c_{r-1}}x^2 + 2.n_{c_r}x + n_{r+1} = 0}$ **are**

(a) rational **(b) imaginary**
(c) real & distinct **(d) none**

Ans. (c) Discriminant(D) = 4 {$(n_{c_r})^2 - n_{c_{r-1}}.\,n_{c_{r+1}}$} = 4 (a - b),

Where, a = $(n_{c_r})^2$ and b = $n_{c_{r-1}}.\,n_{c_{r+1}}$

$\therefore \frac{a}{b} = \frac{r+1}{r} . \frac{n-r+1}{n-r} = \left(1 + \frac{1}{r}\right)\left(1 + \frac{1}{n-r}\right) > 1$
as, n > r

$\therefore$ a > b

$\Rightarrow$ D > 0.

3. Given, f(x+y) = f(x)+ f(y) $\forall\ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}$ and f(1)=3, then the value of $\sum_{r=1}^{n} f(r)$is

(a) $3^n - 1$ **(b) $\frac{3}{2}(3^n - 1)$** **(c) $\frac{3}{2}.3^n - 1$**
(d) none

Ans. (b) let f(x)= $a^{\lambda x}$, λ is a constant

f(1)= a^{λ} = 3

$\therefore\ \sum_{r=1}^{n} f(r) = \sum_{r=1}^{n} a^{\lambda r} = a^{\lambda} + a^{2\lambda} + a^{3\lambda} + \ldots. + a^{n\lambda}$

$= \frac{a^{\lambda}(a^{n\lambda}-1)}{(a^{\lambda}-1)} = \frac{3(3^n-1)}{3-1} = \frac{3}{2}(3^n - 1)$

4. If f(x). $f\left(\frac{1}{x}\right)$ = f(x) + $f\left(\frac{1}{x}\right)$ and f(4)= 65, then f(6)is

(a) 201 **(b) 205**
(c) 215 **(d) none**

Ans. (d) f(x)= $x^n \pm 1$, where n ∈ I

1st case: f(4)= $4^n + 1$= 65 $\Rightarrow$ 4^n= 64 $\Rightarrow$ n=3

2nd case: f(4)= $4^n - 1$=65 $\Rightarrow$ 4^n =66, impossible as n ∈ I

$\therefore$ f(x)= $x^3 + 1$

$\therefore$ f(6) =$6^3 + 1$ =217.

5. Let f be a function of a real variable such that it satisfies f(x +y) = f(x)+f(y))$\forall\ \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{I}$, then $f\left(\frac{m}{n}\right)$

(a) $\frac{m}{n}$ **(b) $\frac{f(m)}{f(n)}$**
(c) $\frac{m}{n} f(1)$ **(d) none**

Ans. (c) since f(x+ y)= f(x)+f(y) ∀ x, y

So, f(x) =ax is the functional form, where a= constant.

$\therefore$f(1)=a

i.e. f($\frac{m}{n}$)= a($\frac{m}{n}$)= f(1). $\frac{m}{n}$

6. Let f(x)= $(x-a)^3 + (x-b)^3 + (x-c)^3$, a < b < c. Then no. of real roots of f(x) = 0 is

(a) 3 **(b) 2**
(c) 1 **(d) none**

Ans. (c) f′(x)= 3$\{(x-a)^2 + (x-b)^2 + (x-c)^2\}$> 0

$\therefore$ f(x) is an increasing function

Note that f(x)< 0 if x< a

f(x) > 0 if x > c

$\therefore$ there is one root.

7. A, B are two square matrix such that AB = A and BA = B, then

(a) A, B both are idempotent
(b) only A is idempotent.

(c) only B is idempotent
(d) none

Ans. (a) (AB)A = A× $A = A^2$

$\Rightarrow$ A (BA)= A^2

$\Rightarrow$ AB= A^2

$\Rightarrow$A = A^2

Hence A is idempotent

Similarly, B is idempotent

8. Let B is non- singular matrix and A is a square matrix, then det($B^{-1}AB$) =

(a) det B **(b) det A**
(c) det(AB) **(d) none**

Ans. (b) det$(B^{-1}AB) =$
$\det(B^{-1})\det(A)\det(B) =$
$\frac{1}{\det(B)}.\det(A).\det(B) = \det A$

9. A subset A of the set X= {1, 2, 3, …..,100}is chosen at random. The set X is reconstructed by replacing the elements of A and another subset B of x is chosen at random. The probability that A∩B contains exactly 10 elements is

(a) $\binom{100}{10}(\frac{3}{4})^{90}$ **(b)** $\binom{100}{10}(\frac{1}{2})^{100}$
(c) $\binom{100}{10}(\frac{2}{3})^{100}$ **(d) none**

Ans. (d) A and B can be chosen in general in $=\{\sum_{r=0}^{90} 90_{c_r} 2^{90-r}\}.100_{c_{10}}$ Ways

$= (1+2)^{90}.\ 100_{c_{10}} = 100_{c_{10}}.3^{90}$ ways.

10. Let A be a 2× 2 matrix to be written down using the numbers 1, -1 as elements. The probability that the matrix is non-singular is

(a)1/2 **(b) 3/8**
(c) 5/8 **(d) none**

Ans. (a) A 2× 2 matrix has 4 elements each of which can be chosen in 2 ways, so, total number of 2× 2 square matrices with elements 1, -1 is $2^4 = 16$.

Out of these 16 matrices, following matrices are singular

$\begin{bmatrix}1 & 1\\-1 & -1\end{bmatrix}, \begin{bmatrix}-1 & -1\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}-1 & -1\\-1 & -1\end{bmatrix},$

$\begin{bmatrix}-1 & 1\\-1 & 1\end{bmatrix}, \begin{bmatrix}1 & -1\\1 & -1\end{bmatrix}, \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}, \begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix}.$

Thus the number of non-singular matrices = 16-8 =8

∴ required probability is = $\frac{8}{16} = \frac{1}{2}$.

11. Let $A_1, A_2, \ldots, A_n$ are n independent events with P(A_j) $= \frac{1}{1+j}(1 \le j \le n)$. The probability that not one of $A_1, A_2, \ldots, A_n$ occur is

(a) $\frac{1}{(n+1)!}$ **(b)** $\frac{1}{n+1}$
(c) $\frac{n!}{(n-1)!}$ **(d) none**

Ans. (c) Required probability = P $(\bar{A_1} \cap \bar{A_2} \cap \ldots\ldots \cap \bar{A_n})$

= P $(\bar{A_1})$P$(\bar{A_2})$…. P$(\bar{A_n})$

$= \frac{1}{2} \times \frac{2}{3} \times \ldots . \times \frac{n}{n+1} = \frac{n!}{(n+1)!}$ (Ans.)

[∵P $(\bar{A_j})$= 1-$A_j) = 1 - \frac{1}{1+j} = \frac{j}{1+j}$]

12. Given a circle of radius r. Tangents are drawn from points A and B lying on one of its diameters which meet at a point P lying on another diameter perpendicular to the other diameter. The minimum area of the triangle PAB is

(a) r^2 **(b) $2r^2$**
(c) πr^2 **(d) $\frac{r^2}{2}$**

Ans. (b)

OP = r sec α

OA = r cosec α

Area of Δ APB = $\frac{1}{2}.OP.2OA = OP.OA = \frac{2r^2}{\sin 2\theta}$

∴Minimum area = $2r^2$

13. Solution set of $\frac{(x-1)(x-2)^2(x+4)}{(x+2)(x-3)} \geq 0$ *is*

(a) $(-\infty,-4] \cup (-2,1] \cup \{2\} \cup (3,\infty)$
(b) [-4, -2)$\cup$ $[1,2] \cup (3,\infty)$

(c) $(-\infty,-4] \cup \{2\} \cup (3,\infty)$
(d) none

Ans. (a) G(x) = $\frac{(x-1)(x-2)^2(x+4)}{(x+2)(x-3)} \geq 0$

So, sign change is like this :- $\frac{+\;-\;+\;-\;-\;+}{-4\;-2\;\;1\;\;2\;\;3}$,

⤇ x ∊ $(-\infty,-4] \cup (-2,1] \cup \{2\} \cup (3,\infty)$.

14. If $X_1, X_2, \ldots, X_n$ are the roots of $x^n + ax + b = 0$, then the value of

$(X_1 - X_2)(X_1 - X_3)(X_1 - X_4)\ldots(X - X_n)$ =

(a) n$X_1 + b$ **(b) n${X_1}^{n-1}$**
(c) n${X_1}^{n-1} + a$ **(d) none**

Ans. (c) $x^n + ax + b = (X - X_1)(X - X_2)(X - X_3)\ldots(X - X_n)$

⤇ $(X - X_2)(X - X_3)\ldots(X - X_n) = \frac{X^n+aX+b}{(X-X_1)}$

⤇ $(X_1 - X_2)(X_1 - X_3)(X_1 - X_4)\ldots(X_1 - X_n) = \lim_{X\to X_1} \frac{X^n+aX+b}{(X-X_1)}$, $[By\ L'hospital\ rule]$

= n${X_1}^{n-1} + a$.

15. Let these three values of X such that X, [X], {X} are in H.P. then the number of values of x are

(a) 1 **(b) 2**
(c) 3 **(d) none**

Ans. (a) [X]= $\frac{2\{X\}X}{\{X\}+X}$; Now, we know X = [X]+ {X}, putting value of X , we get

⤇ $[X]^2 = 2\{X\}^2$

⤇ $\{X\}^2 = \frac{1}{2}[X]^2$

⤇ 0 < $\frac{[X]^2}{2} < 1$,

⤇ 0 < $[X]^2 < 2$, ⤇ 0 < [X] < $\sqrt{2}$ ⤇ [X]= 1
⤇ {X}= $\frac{1}{\sqrt{2}}$,

So, X = 1+$\frac{1}{\sqrt{2}}$ is the only value.

16. Let f(x) = $\sin^{-1}\frac{2g(x)}{1+(g(x))^2}$, then

(a) f(x) is decreasing if g(x) is increasing and |g(x)|> 1

(b) f(x) is an increasing function if g(x) is increasing and |g(x)|≤ 1

(c) f(x) is decreasing if g(x) is decreasing and |g(x)|> 1

(d) none of these

Ans. (c)

$$f(x) = \sin^{-1}\frac{\frac{2}{g(x)}}{1+\left(\frac{1}{g(x)^2}\right)}$$

$$= \sin^{-1}(\sin 2\theta)\,, where \tan\theta = \frac{1}{g(x)}$$

$$= 2\,\theta = 2\cot^{-1}\big(g(x)\big), |g(x)| > 1$$

$thus\ f'(x) = \frac{2}{1+(g(x))^2}g'(x), |g(x)| \leq 1 = -\frac{2}{1+(g(x))^2}g'(x), |g(x)| > 1$.

17. Let f(x)– $\int_0^x (\boldsymbol{coc^4\theta + sin^4\theta})\boldsymbol{d\theta} +$ $(\boldsymbol{pq - q^2 - 1})\boldsymbol{x}$ **then for f(x) to be a decreasing function in R for all real values of q independent of x, the set of values of p is**

(a) [10, $\sqrt{\mathbf{10}}$**]** **(b)** $[-\sqrt{\mathbf{10}}, \sqrt{\mathbf{10}}]$

(c) [-$\sqrt{\mathbf{10}}$**, 0]** **(d) none of these**

Ans. (d)

$$f'(x) = -(cos^4x + sin^4x) + (pq - q^2 - 1) = -(cos^2x + sin^2x)^2 + 2sin^2xcos^2x + pq - q^2 - 1$$

$$= pq - q^2 + \frac{1}{2}sin^2 2x - 2$$

But f'(x) $\leq$ pq - $q^2 - 2 + \frac{1}{2} = pq - q^2 - \frac{3}{2}$

For f(x) to be decreasing in R, f'(x) ≤ 0

This will be true if $pq - q^2 - \frac{3}{2} \leq 0\ or\ 2q^2 - 2pq + 3 \geq 0\ for\ all\ q\ \in R\ \Rightarrow D \leq 0$

$or\ 4p^2 - 24\ \leq 0\ or - \sqrt{6} \leq p \leq \sqrt{6}.$

18. The lengths of the sides of a triangle are $\boldsymbol{\alpha}$**-** $\boldsymbol{\beta}$**,** $\boldsymbol{\alpha}$ **+**$\boldsymbol{\beta}$ **and** $\sqrt{\boldsymbol{3\alpha^2 + \beta^2}}, (\boldsymbol{\alpha} > \beta > 0)$**. Its largest angle is**

(a) $\frac{3\pi}{4}$ **(b)** $\frac{\pi}{2}$

(c) $\frac{2\pi}{3}$ **(d)** $\frac{5\pi}{6}$

Ans. (c)

$\alpha, \beta > 0$

$\therefore\ \alpha + \beta > \alpha - \beta$

Also $\alpha - \beta > 0$

$\therefore \alpha > \beta$ then $3\alpha^2 + \beta^2 - (\alpha+\beta)^2 = 2\alpha(\alpha - \beta) >\ 0$

$\Rightarrow\sqrt{3\alpha^2 + \beta^2} >\ \alpha + \beta$

Let a = $\alpha - \beta, b =\ \alpha + \beta, c = \sqrt{3\alpha^2 + \beta^2}$

Now $\cos C\ = \frac{a^2+b^2-c^2}{2ab} = \frac{2\alpha^2+2\beta^2-3\alpha^2-\beta^2}{2(\alpha^2-\beta^2)} = -\frac{1}{2}$

$\therefore C = \frac{2\pi}{3}$

19. The set of critical points of the function

f(x) = $\boldsymbol{x - In\ x} + \int_2^x \left(\frac{\mathbf{1}}{\boldsymbol{t}} - \mathbf{2} - \mathbf{2}\cos\mathbf{4}\boldsymbol{t}\right)\boldsymbol{dt}$ **is**

(a) $\left\{\frac{\pi}{2}, n\pi + \frac{\pi}{6}\right\}, n \in I$ (b) $\left\{\frac{\pi}{6}, \frac{n\pi}{2} \pm \frac{\pi}{6}, n \in N\right\}$ (c) {nπ}, n ∈I (d) none of these

Ans. (b)

f'(x)= $1 - \frac{1}{x} + \frac{1}{x} - 2 - 2\cos 4x \ or\ f'(x) = -1 - 2\cos 4x$

for critical points f'(x)= 0

$\Rightarrow \cos 4x = -\frac{1}{2} = \cos\frac{2\pi}{3} \Rightarrow 4x = 2n\pi \pm \frac{2\pi}{3} \Rightarrow x = \frac{n\pi}{2} \pm \frac{\pi}{6}, n \in I$

But for log x to be defined, x > 0

∴ x = $\frac{\pi}{6}, \frac{n\pi}{2} \pm \frac{\pi}{6}, n \in N$

20. If 0 < α< β < γ< $\frac{\pi}{2}$, then the equation $\frac{1}{x-\sin\alpha} + \frac{1}{x-\sin\beta} + \frac{1}{x-\sin\gamma} = 0$ has

(a) imaginary roots
(b) real and equal roots

(c) real and unequal roots
(d) rational roots

Ans. (c)

0 < α < β < γ < $\frac{\pi}{2}$

$\Rightarrow$ sin α< sin β < sin γ

Given equation is

(x- sinβ)(x- sinγ)+(x- sinα) (x- sinγ) +(x- sinα)(x- sinβ)= 0

Or, f(x)= 0

f(sin α) = (sin α- sin β)(sin α- sin γ)> 0

f(sin β) = (sin β- sin α)(sin β- sin γ)< 0

f(sin γ) = (sin γ- sin α)(sin γ- sin β)> 0

Hence equation f(x) = 0 has one root between sin α and sin β and other between sin β and sin γ.

21. If f(x)= $\sum_{r=0}^{100} a_r x^r$ and f(0) and f(1) are odd numbers, then for any integer x

(a) f(x) is odd or even according as x is odd or even

(b) f(x) is even or odd according as x is odd or even

(c) f(x) is even for all integral value of x

(d) f(x) is odd for all integral values of x

Ans. (d) Given f(x)= $a_0 + a_1 x + a_2 x^2 + \cdots + a_{100} x^{100}$

∴ f(2m) = $a_0 + a_n$ (even number)

= f(0) + an even number

= an odd number

f(2m +1) = $a_0 + a_1(1+2m) + a_2(1+2m)^2 + \cdots + a_{100}(1+2m)^{100}$

= $a_0 + a_1 + \cdots + a_{100} + a_n$ even number

= f(1) + an even number

= an odd number

22. Let P= $\sqrt{3}\, e^{i\pi/3}, Q = \sqrt{3}\, e^{-i\pi/3}$ *and* $R = \sqrt{3}\, e^{-i\pi}$ form a triangle PQR in the Argand plane. Then ΔPQR is

(a) isosceles (b) equilateral
(c) scalene (d) none of these

Ans. (b)

The points P, Q, R lie on a circle |z|= $\sqrt{3}$ with a difference in argument = $\frac{2\pi}{3}$ for any two complex numbers.

∴ Δ PQR is equilateral.

23. A, b and C are points represented by complex numbers $z_1, z_2, and\ z_3$. If the circumcentre of the ΔABC is at the origin and the altitude AD of the triangle meets the circumcircle again at P, then P represents the complex number

(a) $\frac{-z_1z_2}{z_3}$ **(b)** $\frac{z_1z_3}{z_2}$

(c) $\frac{-z_2z_3}{z_1}$ **(d)** $\frac{z_1z_2}{z_3}$

Ans. (c)

$|z_1|= |z_2|z_3| =|z|=$ r (say)

$\therefore |z_1|^2 = |z_2|^2 = |z_3|^2 = |z|^2 = r^2$

$\Rightarrow z_1\overline{z_1} = z_2\overline{z_2} = z_3\overline{z_3} = z\bar{z} = r^2$

$\therefore \frac{z-z_1}{\bar{z}-\overline{z_1}} = \frac{z_3-z_3}{\overline{z_2}-\overline{z_3}} \Rightarrow \frac{z-z_1}{\frac{r^2}{z}-\frac{r^2}{z_1}} = -\frac{z_3-z_3}{\frac{r^2}{z_2}-\frac{r^2}{z_3}} =$

$-zz_1 = z_2z_3$

$\Rightarrow$ z = - $\frac{z_3}{z_1}$

24. If |2z- 1|= |z- 2| and z_1, z_2, z_3 are complex numbers such that $|z_1 - \alpha|< \alpha$, $|z_2 - \beta|< \beta$, then $\left|\frac{z_1+ z_2}{\alpha+\beta}\right|$

(a) < |z| **(b) <2|z|**

(c) > |z| **(d) > 2|z|**

Ans. (b)

|2z- 1|= |z- 2|

$\Rightarrow |2z - 1|^2 = |z - 2|^2$

$\Rightarrow$ (2z -1) (2$\bar{z}$ -1)= (z- 2) ($\bar{z}$ -2)

$\Rightarrow$ 4z$\bar{z}$- 2$\bar{z}$- 2z +1 = z$\bar{z}$ -2$\bar{z}$ -2z +4

$\Rightarrow 3|z|^2 = 3 \Rightarrow |z|= 1$

Again

$|z_1 - z_2| = |z_1 - \alpha + z_2 - \beta + \alpha + \beta| \leq |z_1 - \alpha| + |z_2 - \beta| + |\alpha+\beta| < \alpha+\beta+ |\alpha+ \beta|$

$= 2|\alpha+ \beta|$[$\because \alpha, \beta > 0$]

$\therefore \left|\frac{z_1+z_2}{\alpha+\beta}\right| < 2 \Rightarrow \left|\frac{z_1+z_2}{\alpha+\beta}\right| < 2|z|$

25. If 1, $\alpha_1, \alpha_2, \ldots, \alpha_{3n}$ be the roots of equation $x^{3n+1} - 1 = 0$, and ω be an imaginary cube root of unity, then $\frac{(\omega^2-\alpha_1)(\omega^2-\alpha_2)\ldots(\omega^2-\alpha_{3n})}{(\omega-\alpha_1)(\omega-\alpha_2)\ldots(\omega-\alpha_{3n})} =$

(a) ω **(b) $-\omega$**

(c) 1 **(d) ω^2**

Ans. (c)

Since 1, $\alpha_1, \alpha_2, \ldots, \alpha_{3n}$ are the roots of equation

$x^{3n+1} - 1 = 0$

$\therefore x^{3n+1} - 1$ = (x- 1) (x-α_1) (x-α_2) … (x-α_{3n})

$\Rightarrow$ (x-α_1) (x-α_2) … (x- α_{3n}) = $\frac{x^{3n+1}-1}{x-1}$

$\Rightarrow$ (x-α_1) (x-α_2) … (x - α_{3n}) = 1+ x+$x^2 + \cdots + x^{3n}$ ……………(1)

$\therefore \frac{(\omega^2-\alpha_1)(\omega^2-\alpha_2)\ldots(\omega^2-\alpha_{3n})}{(\omega-\alpha_1)(\omega-\alpha_2)\ldots(\omega-\alpha_{3n})} = \frac{\omega^{6n+2}-1}{\omega^2-1} . \frac{\omega-1}{\omega^{3n+1}-1} = \frac{(\omega^2-1)(\omega-1)}{(\omega^2-1)(\omega-1)} = 1$

26. If ω be an imaginary cube root of unity and $x_1, x_2, x_3 \in R, \sum_{r=1}^{3} \frac{1}{x_r+\omega} = 2\omega^2, \sum_{r=1}^{3} \frac{1}{x_r+\omega^2} = 2\omega$ then $\sum_{r=1}^{3} \frac{1}{x_r+1} =$

(a) -2 **(b) 2**
(c) 0 **(d) none of these**

Ans. (b)

Consider the equation

$\frac{1}{x_1+x} + \frac{1}{x_2+x} + \frac{1}{x_3+x} = \frac{2}{x}$ ………………..(1)

Equation (1) is

x (x+x_2) (x+x_3) $+ x(x+x_1)(x+x_3) + x(x+x_1)(x+x_2) - 2(x+x_1)(x+x_2)(x+x_3)$= 0

or, $x^3 - (x_1x_2 + x_2x_3 + x_3x_1) - 2x_1x_2x_3 = 0$ ……………(2)

from given conditions it is clear that ω and ω^2 are the roots of equation (1) i.e. of eqn. (2).

Therefore, if γ be its third roots, then

ω+ $\omega^2 + \gamma$= 0 $\Longrightarrow \gamma$= 1

∴ 1 is a root of equation (1)

$\Longrightarrow \sum_{r=1}^{3} \frac{1}{x_r+1} = 2$

27. If 1, $\alpha, \alpha^2, \dots, \alpha^{n-1}$ be the nth roots of unity, then $\left(\frac{3^n-1}{3^{n-1}}\right)\left(\sum_{r=1}^{n-1} \frac{1}{3-\alpha^r} + \frac{1}{2}\right) =$

(a) –n **(b) 0**
(c) n **(d) 1**

Ans. (c)

According to question,

$x^n - 1$= (x- 1) (x- α) (x- α^2)… (x - α^{n-1})

Taking logarithm, we get

$\log(x^n - 1) = \log(x-1) + \log(x-\alpha) + \dots + \log(\alpha^{n-1})$

Differentiating w.r.t.x, we get

$\frac{nx^{n-1}}{x^n-1} = \frac{1}{x-1} + \frac{1}{x-\alpha} + \frac{1}{x-\alpha^2} + \cdots + \frac{1}{x-\alpha^{n-1}}$

$\Longrightarrow \frac{x^n-1}{x^{n-1}} (\frac{1}{x-1} + \frac{1}{x-\alpha} + \cdots + \frac{1}{x-\alpha^{n-1}})$= n

$\Longrightarrow \frac{3^n-1}{x^{n-1}}\left(\frac{1}{2} + \sum_{r=1}^{n-1} \frac{1}{3-\alpha^r}\right) = n,$

[Putting x= 3]

28. If $\begin{vmatrix} x^2+x & x-1 & x+1 \\ x & 2x & 3x-1 \\ 4x+1 & x-2 & x+2 \end{vmatrix} = px^4 + qx^3 + rx^2 + sx + 1$ be an identity in x and ω be an imaginary cube root of unity, then $\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} + \frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} =$

(a) p **(b) 2p**
(c) -2p **(d) –p**

Ans. (a)

Equating the coefficient of x^4, we get

P=1. (2-3)= -1

Given expression

$= \frac{\omega^2(c+a\omega+b\omega^2)}{c+a\omega+b\omega^2} + \frac{\omega\,(b+c\omega+a\omega^2)}{b+c\omega+a\omega^2} = \omega^2 + \omega = -1 = p$

29. If the equation- $(p+iq)x^2 + (m+in)x + r = 0$ has real roots where p, q, m, n and r are real (r≠ 0) then $pn^2 + rq^2 - mnq$ =

(a) 1 **(b) 0**
(c) -1 **(d) none of these**

Ans. (b)

Let α be the real root satisfying the given equation, then

(p$\alpha^2 + m\alpha + r) + i\ (q\alpha^2 + n\alpha)$= 0

Equating real and imaginary parts, we have
$\alpha = -\frac{n}{q}$ $and\ \frac{pn^2}{q^2} - \frac{mn}{q} + r = 0\ \left[\alpha = -\frac{n}{q}\right]$

$\Rightarrow$ pn^2 + r$q^2 - mnq = 0$

30. If n(A)= p, n(B)= q and total number of mappings from A to 343, then p - q =

(a) 3 **(b) -3**
(c) 4 **(d) none of these**

Ans. (d)

Total number of mapping from A to B = q^p

Given $q^p = 343 = 7^3 \Rightarrow q = 7, p = 3$

∴ p – q= - 4

ISI OBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 14

There will be 30 questions in MMA Paper. For each question, exactly one of the four choices is correct. You get four marks for each correct answer, one mark for each unanswered question, and zero mark for each incorrect answer.

1. Let X be a non-empty set and let P(X) denotes the collection of all subsets of X. Define

F: X× $\boldsymbol{P(X) \to R}$ by

f(x, A) = $\begin{cases} \mathbf{1} & \boldsymbol{if\ x \in A} \\ \mathbf{0} & \boldsymbol{if\ x \notin A} \end{cases}$

Then f(x, A∪B) equals

(a) f(x, A) + f(x, B)
(b) f(x, A) + f(x, B) – 1

(c) f(x, A) + f(x, B) – f(x, A)f(x, B)
(d) none

Ans. (c) n(A∪B) = n(A) + n(B) – n(A∩B)

So, f(x, A∪B) = f(x, A) + f(x, B) – f(x, A)f(x, B)

2. The set $\left\{ \boldsymbol{x} :\ \left| \boldsymbol{x} + \frac{\mathbf{1}}{\boldsymbol{x}} \right| > 6 \right\}$ **equals the set**

(a) (0, $\mathbf{3 - 2\sqrt{2}}) \cup (\mathbf{3 + 2\sqrt{2}}, \infty)$
(b) (-∞, $\mathbf{-3 - 2\sqrt{2}}) \cup (\mathbf{-3 + 2\sqrt{2}}, \infty)$

(c) (-∞, $\mathbf{3 - 2\sqrt{2}}) \cup (\mathbf{3 + 2\sqrt{2}}, \infty)$
(d) none

Ans. (c) $x^2 - 6x + 1 = 0$ gives us x = $\frac{-6\pm\sqrt{36-4}}{2} = 3 \pm 2\sqrt{2}$

So, set is (-∞, $3 - 2\sqrt{2}) \cup (3 + 2\sqrt{2}, \infty)$

3. If f : (3, 6) ⟶(2, 5) is a function defined by f(x)= x-$\left[\frac{x}{3}\right]$ **(where [.] denotes the greatest integer function), then** $\boldsymbol{f^{-1}x}$

(a) x -1 (b) x +1 (c) x (d) none of these

Ans. (b)

3 < x < 6

$\Rightarrow \left[\frac{x}{3}\right] = 1 \therefore f(x) = x - 1 \therefore f^{-1}(x) = x + 1$

4. The possible values of ‘a’ for which the function f(x) = $e^{x-[x]} + \cos ax$ (where [.] denotes the greatest integer function) is periodic with finite fundamental period is

(a) π **(b) 0**
(c) 1 **(d) 2**

Ans. (a)

Period of x- [x] is 1.

∴ Period of $e^{x-[x]}$is 1 and period of cos ax is $\frac{2\pi}{|a|}$

When a= π, period of cos ax = 2

When a= 2π, period of cos ax= 1

In first case period of f(x) will be 2 and in second case it will be 1.

But when a = 1 or 2.

Period of cos ax will be an irrational number T and L.C.M. of 1 and T will not exist.

5. If f(x)= $[x]^2 - [x^2]$, (where [.] denotes the greatest integer function), and x ∊[0, 2], then the range of f(x) is

(a) {-1, 0, 2} (b) {0, 1, 2} (c) {-2, -1, 0} (d) {-2, 0, 1}

Ans. (c)

$$f(x) = [x]^2 - [x^2]$$
$$= \begin{cases} 0, & 0 \le x < \sqrt{2} \\ -1, & \sqrt{2} \le x < \sqrt{3} \\ -2, & \sqrt{3} \le x < 2 \\ 0, & x = 2 \end{cases}$$

∴ Range of f(x) is {-2, -1, 0}

6. Let f: R⟶R be a function defined by f(x)= max [x, x^3]. The set of all points where f(x) is not differentiable is

(a) {-1, 1} **(b) {-1, 0}** **(c) {0, 1}** **(d) {-1, 0, 1}**

Ans. (d) $f(x) = \max\{x, x^3\}$

$$\Rightarrow f(x) = \begin{cases} x, & x \le -1 \\ x^3, & -1 < x \le 0 \\ x, & 0 < x \le 1 \\ x^3, & x \ge 1 \end{cases}$$

Clearly f(x) is non-differentiable at x= -1, 0, 1

7. Let f: R⟶R be a function defined by f(x)= 2x + sinx, x ∊R. Then f is

(a) one-to-one and onto
(b) one-to-one but not onto

(c) onto but not one-to-one
(d) neither one-to-one nor onto

Ans. (a)

f’(x)= 2+ cosx > 0∀ x ϵ R

⟹ f(x) is increasing

⟹ f is one-one

Also, as x⟶∞ , f(x)⟶ ∞ *and as* $x \to -\infty, f(x) \to -\infty$

More over f is continuous

$\Longrightarrow$ Range of R $\Longrightarrow$ f is onto.

8. f : [0, ∞) $\rightarrow [\mathbf{0},\infty)$, $\boldsymbol{defined\ by\ f(x)} = \frac{\boldsymbol{x}}{\mathbf{1}+\boldsymbol{x}}$ is

(a) one-one and onto
(b) one-one but not onto

(c) onto but not one-one
(d) neither one-one nor onto

Ans. (b)

f : [0, ∞)⟶ [0, ∞) ; $f(x) = \frac{x}{1+x} = 1 - \frac{1}{1+x}$

∴f'(x)= $\frac{1}{(1+x)^2} > 0$

Hence f(x) is increasing and therefore one-one.

Also 0≤ x < ∞ ∴ f(0)≤ f(x) < f(∞)

⟹0 ≤ f(x) < $\underset{x\to\infty}{\mathrm{Lt}} \frac{x}{1+x}$

⟹ 0≤ f(x) < 1

Hence range f(x) = [0, 1), therefore f is not onto.

9. The value of $\lim\limits_{\boldsymbol{x}\to\mathbf{0}} \frac{\begin{vmatrix} \frac{\boldsymbol{x}}{\boldsymbol{x}-\mathbf{1}} & \frac{\mathbf{1}}{\boldsymbol{logx}} \end{vmatrix}}{(\boldsymbol{x}-\mathbf{1})\boldsymbol{logx}}$ is

(a) 1 **(b) -1**
(c) ½ **(d) -½**

Ans. (c)

Required limit= $\underset{x\to 1}{\mathrm{Lt}} \frac{x\log x-(x-1)}{(x-1)\log x} =$ $\underset{h\to 0}{\mathrm{Lt}} \frac{(1+h)\log(1+h)-h}{h\log(1+h)}$ $[putting\ x =\ 1 + h]$

$$= \underset{h\to 0}{\mathrm{Lt}} \frac{\log(1+h) - h + h\log(1+h)}{h\log(1+h)}$$

$$= \underset{h\to 0}{\mathrm{Lt}} \frac{\left(h - \frac{h^2}{2} + \frac{h^3}{3} - \cdots\right) - h + h\log(1+h)}{h\log(1+h)}$$

$$= \underset{h\to 0}{\mathrm{Lt}} \frac{-\frac{h^2}{2}\left(1 - \frac{h}{3} + \frac{h^2}{4} - \cdots\right) + \frac{h^2\log(1+h)}{h}}{\frac{h^2\log(1+h)}{h}}$$

$$= \underset{h\to 0}{\mathrm{Lt}} \frac{-\frac{1}{2}\left(1 - \frac{h}{3} + \frac{h^2}{4} - \cdots\right) + \frac{\log(1+h)}{h}}{\frac{\log(1+h)}{h}}$$

$$= \frac{-\frac{1}{2} + 1}{1} = \frac{1}{2}$$

10. The value of $\lim\limits_{\boldsymbol{n}\to\infty} \frac{\mathbf{1}}{\boldsymbol{n}^{\mathbf{4}}}\left[\mathbf{1}\left(\sum_{\boldsymbol{k}=\mathbf{1}}^{\boldsymbol{n}} \boldsymbol{k}\right) + \mathbf{2}\left(\sum_{\boldsymbol{k}=\mathbf{1}}^{\boldsymbol{n}-\mathbf{1}} \boldsymbol{k}\right) + \mathbf{3}\left(\sum_{\boldsymbol{k}=\mathbf{1}}^{\boldsymbol{n}-\mathbf{2}} \boldsymbol{k}\right)\right] + \cdots + \boldsymbol{n}.\mathbf{1}]$ will be

(a) $\frac{\mathbf{1}}{\mathbf{24}}$ **(b) $\frac{\mathbf{1}}{\mathbf{12}}$**
(c) $\frac{\mathbf{1}}{\mathbf{6}}$ **(d) $\frac{\mathbf{1}}{\mathbf{3}}$**

Ans. (a)

(r+1) th term of the series

$$t_{r+1} = (r+1).\sum_{k-1}^{n-r} k$$
$$= \frac{(r+1)(n-r)(n-r+1)}{2}$$
$$= \frac{r[(n-r)^2 + n - r]}{2} + P(n),$$

$$where\ P(n) = \frac{n^2 - (2n+1)r + n + r^2}{2}$$
$$= \frac{r}{2}(n^2 - (2n+1)r + r^2 + n) + P(n)$$

$$= \frac{1}{2}[r^3 - (2n+1)r^2 + (n^2+n)r] + P(n)$$

∴sum of the series= $\sum_{r=0}^{n-1} t_{r+1} =$
$\frac{1}{2}[\sum_{r=0}^{n-1} r^3 - (2n+1)\sum_{r=0}^{n-1} r^2 + n(n+1)\sum_{r=0}^{n-1} r] + q(n)$

Where q(n) is a polynomial in n of degree 3

$$= \frac{1}{2}\left[\left\{\frac{(n-1)n}{2}\right\}^2 - (2n+1)\frac{(n-1)n(2n-1)}{6} + n(n+1)\frac{(n-1)n}{2}\right] + q(n)$$

$$= \frac{1}{2}\left[\frac{(n^2-2n+1)n^2}{4} - \frac{(n^2-n)(4n^2-1)}{6} + \frac{n^2(n^2-1)}{2}\right] + q(n)$$

$$\therefore Reqd.limit = \frac{1}{2}\left(\frac{1}{4} - \frac{4}{6} + \frac{1}{2}\right) = \frac{1}{2}\left(\frac{3}{4} - \frac{2}{3}\right) = \frac{1}{24}$$

11. If $\operatorname*{Lt}_{x\to\infty}[1 + x\frac{f(x)}{x}]^{1/x} = e^3$, then the value of the function f(x) may be

(a) $\frac{x^2}{2}$ **(b) x^2**
(c) $2x^2$ **(d) $3x^2$**

Ans. (c)

When $f(x) = kx^2$, $\operatorname*{Lt}_{x\to 0}\left[1 + x + \frac{f(x)}{x}\right]^{1/x} = \operatorname*{Lt}_{x\to 0}[1 + x(1+k)]^{1/x} = e^{k+1}$

For $e^{k+1} = e^3, k = 2$

∴ f(x) = $2x^2$

12. $\operatorname*{Lt}_{n\to\infty} n_{c_x}(\frac{m}{n})^x\left(1 - \frac{m}{n}\right)^{n-x}$ equals to

(a)$\frac{m^x}{x!}.e^{-m}$ **(b) $\frac{m^x}{x!}.e^{m}$**
(c) 1 **(d) 0**

Ans. (a)

$$\operatorname*{Lt}_{n\to\infty} n_{C_x}\left(\frac{m}{n}\right)^x\left(1 - \frac{m}{n}\right)^{n-x}$$

$$= \operatorname*{Lt}_{n\to\infty}\frac{n!}{x!\,(n-x)!}\left(\frac{m}{n}\right)^x \frac{\left(1-\frac{m}{n}\right)^x}{\left(1-\frac{m}{n}\right)^x}$$

$$= \operatorname*{Lt}_{n\to\infty}\frac{m^x}{x!}\frac{\left(1-\frac{m}{n}\right)^x}{\left(1-\frac{m}{n}\right)^x}.\frac{n(n-1)\dots(n-x+1)}{n^x}$$

$$= \frac{m^x}{x!}\; e^{-m}.\operatorname*{Lt}_{n\to\infty}\left[1\left(1-\frac{1}{n}\right)\dots\left(1-\frac{x-1}{n}\right)\right] = e^{-m}\frac{m^x}{x!}$$

13. If f(x) = $\lim_{n\to\infty} n^2\left(x^{\frac{1}{n}} - x^{\frac{1}{n+1}}\right)$, x > 0 then $\int$ x f(x)dx equals to

(a) $\frac{x^2}{2}$ **(b) 0**
(c) $\frac{x^3}{2}$ **(d) none of these**

Ans. (d)

$$f(x) = \operatorname*{Lt}_{n\to\infty} n^2\left(x^{\frac{1}{n}} - x^{\frac{1}{n+1}}\right), x > 0[\infty/\,0\; from]$$

$$= \operatorname*{Lt}_{n\to\infty} n^2\left[c^{\frac{1}{n}\log x} - e^{\frac{1}{n+1}\log x}\right]$$

$$= \operatorname*{Lt}_{n\to\infty} n^2\left[\left\{1 + \frac{logx}{n} + \left(\frac{logx}{n}\right)^2 + \cdots\right\} - \left\{1 + \frac{logx}{n+1} + \left(\frac{logx}{n=1}\right)^2 + \cdots\right\}\right]$$

$$= \underset{n\to\infty}{\text{Lt}}\, n^2 \left[\left(\frac{1}{n} - \frac{1}{n+1}\right)\log x + (\log x)^2\left\{\frac{1}{n^2} - \frac{1}{(n+1)^2}\right\} + (\log x)^3\left\{\frac{1}{n^3} - \frac{1}{(n+1)^3}\right\} + \cdots\right]$$

$$= \underset{n\to\infty}{\text{Lt}}\, \frac{n^2}{n(n+1)}\log x + n^2\left\{\frac{1}{n^2} - \frac{1}{(n+1)^2}\right\}(\log x)^2 + \cdots$$

$$= \log x$$

14. If f(x) = $\lim\limits_{n\to\infty}\frac{x^n - x^{-n}}{x^n + x^{-n}}$, $0 < x < 1$, $n \in N$, then $\int(\sin^{-1} x)\, f(x)dx$ is equal to

(a) $-\left[x\sin^{-1} x + \sqrt{1-x^2}\right] + c$
(b) $x\sin^{-1} x + \sqrt{1-x^2} + c$

(c) constant
(d) none of these

Ans. (a)

$$f(x) = \underset{n\to\infty}{\text{Lt}}\, \frac{x^n - x^{-n}}{x^n + x^{-n}}, 0 < x < 1, n\epsilon\, N$$

$$= \underset{n\to\infty}{\text{Lt}}\, \frac{x^{2n}-1}{x^{2n}+1} = \underset{n\to\infty}{\text{Lt}}\, \frac{(x^2)^n - 1}{(x^2)^n + 1} = -1$$

$$\therefore \int \sin^{-1} x\,.\,f(x)dx = -\int 1.\sin^{-1} x\, dx$$

$$= -\left[x\sin^{-1} x - \int \frac{2x}{2\sqrt{1-x^2}}dx\right]$$

$$= -\left[x\ \sin^{-1} x + \sqrt{1-x^2}\right] + c$$

15. $\underset{x\to 1}{\text{Lt}}\, \frac{x\sin\{x\}}{x-1}$, where {x} denotes the fractional part of x, is equal to

(a) -1 **(b) 0**
(c) 1 **(d) does not exist**

Ans. (d)

$$\underset{x\to 1-0}{\text{Lt}}\{x\} = \underset{x\to 1-0}{\text{Lt}}(x - [x]) = 1 - 0 = 1$$

$$\underset{x\to 1-0}{\text{Lt}}\{x\} = \underset{x\to 1-0}{\text{Lt}}(x - [x]) = 1 - 1 = 0$$

$$\therefore \underset{x\to 1-0}{\text{Lt}}\, \frac{x\sin\{x\}}{x-1} = \underset{x\to 1-0}{\text{Lt}}\, \frac{x}{x-1}\sin\{x\} = -\infty.\sin(1) = -\infty$$

$$\underset{x\to 1-0}{\text{Lt}}\, \frac{x\sin\{x\}}{x-1} = \underset{x\to 1-0}{\text{Lt}}\, \frac{x\sin\{x\}}{\{x\}}.\frac{\{x\}}{x-1}$$

$$= \underset{x\to 1-0}{\text{Lt}}\, \frac{x\sin\{x\}}{\{x\}}.\frac{x-[x]}{x-1}$$

$$= \underset{x\to 1-0}{\text{Lt}}\, \frac{x\sin\{x\}}{\{x\}}.\frac{x-1}{x-1}$$

$$= 1 \times 1 \times 1 = 1$$

Since L.H. limit ≠R.H. limit

∴ limit does not exist.

16. If a = $\underset{n\to\infty}{\text{Lt}}\sum_{r=1}^{n}\frac{1}{(r+2)r!}$ *and b* = $\underset{x\to 0}{\text{Lt}}\frac{e^{\sin x} - e^x}{\sin x - x}$, then

(a) a= b **(b) a= 2b**
(c) 2a = b **(d) a+ b = 0**

Ans. (c)

$$t_r = \frac{1}{(r+2)r!} = \frac{r+1}{(r+2)!} = \frac{(r+2)-1}{(r+2)!}$$

$$= \frac{1}{(r+1)!} - \frac{1}{(r+2)}$$

$$\sum_{r=1}^{n} t_r = \frac{1}{2!} - \frac{1}{(n+2)!}$$

$$\therefore a = \underset{n\to\infty}{\text{Lt}} \left[\frac{1}{2} - \frac{1}{(n+2)!}\right] = \frac{1}{2}$$

$$b = \underset{x\to 0}{\text{Lt}} \frac{e^{sinx} - e^x}{sinx - x}$$

$$= \underset{x\to 0}{\text{Lt}}\, e^x . \left(\frac{e^{sinx-x} - 1}{sinx - 1}\right)$$

$$= e^0 . 1 = 1$$

$\therefore$ 2a = b

17. If $\underset{x\to\infty}{\text{Lt}}\left(\sqrt{x^2 - x + 1} - ax - b\right) = 0$, *then for* $k \geq 2$, $\underset{n\to\infty}{\text{Lt}}\, sec^{2n}(k!\ \pi b)$=

(a) a **(b) –a**
(c) 2a **(d) b**

Ans. (a)

$$\underset{x\to\infty}{\text{Lt}}\left(\sqrt{x^2 - x + 1} - ax - b\right) = 0[here\ a > 0\ for\ if\ a \leq 0, then\ limit = \infty]$$

$$\Rightarrow \underset{n\to\infty}{\text{Lt}} \frac{x^2 - x + 1 - (ax+b)^2}{\sqrt{x^2 - x + 1} + ax + b}$$

$$\Rightarrow \underset{n\to\infty}{\text{Lt}} \frac{(1-a^2)x^2 - (1+2ab)x + 1 - b}{\sqrt{x^2 - x + 1} + ax + b} = 0$$

This is possible only when 1- $a^2 = 0$ and 1+ 2ab= 0

$\therefore$ a= 1($\because$ a > 0) and b= - ½

Now k! π b= k! π (- ½)= an integer multiple of π as k ≥ 2

$\therefore sec^2 k!\, \pi b = 1$

$$\therefore \underset{n\to\infty}{\text{Lt}} (sec^{2n} k!\, \pi b) = 1 = a$$

18. $\underset{n\to\infty}{\text{Lt}}\ \underset{x\to 0}{\text{Lt}} \frac{1}{\left(1^{cot^2 x} + 2^{cot^2 x} + \cdots + n^{cot^2 x}\right)^{tan^2 x}}$ is equal to

(a) 0 **(b) ∞**
(c) n **(d) none of these**

Ans. (a)

$$\underset{x\to 0}{\text{Lt}}\left(1^{cot^2 x} + 2^{cot^2 x} + \cdots + n^{cot^2 x}\right)^{tan^2 x} [\infty^0 form]$$

$$= \underset{y\to\infty}{\text{Lt}} (1^y + 2^y + \cdots + n^y)^{\frac{1}{y}} [where\ y = cot^2 x\ ]$$

$$= \underset{y\to\infty}{\text{Lt}}\, n \left[\left(\left(\frac{1}{n}\right)^y + \left(\frac{2}{n}\right)^y + \cdots + \left(\frac{n-1}{n}\right)^y + 1\right)^{\frac{1}{y}}\right]$$

$$= n. \underset{y\to\infty}{\text{Lt}} \frac{1}{e^y}\left[\left(\frac{1}{n}\right)^y + \left(\frac{2}{n}\right)^y + \cdots + \left(\frac{n-1}{n}\right)^y\right]$$

$$= n. e^0 = n$$

$$\therefore Reqd.limit = \underset{n\to\infty}{\text{Lt}} \frac{1}{n} = 0$$

19. $\underset{x\to\infty}{\text{Lt}} \frac{logx^n - [x]}{[x]}, n \in N$, where [x] denotes the integral part of x, is equal to

(a) 0 **(b) 1**
(c) -1 **(d) ∞**

Ans. (c)

$$\underset{x\to\infty}{\text{Lt}} \frac{logx^n - [x]}{[x]} = \underset{x\to\infty}{\text{Lt}} \frac{n \log x - [x]}{[\boldsymbol{x}]}$$

$$= \underset{x\to\infty}{\text{Lt}} \frac{n\frac{logx}{x} - \frac{[x]}{x}}{\frac{[x]}{x}} \ldots\ldots\ldots(1)$$

$but \underset{x\to\infty}{\text{Lt}} \frac{logx}{x}\left[\frac{\infty}{\infty}\text{from}\right] = \underset{x\to\infty}{\text{Lt}} \frac{\frac{1}{x}}{1} = 0$ and $\underset{x\to\infty}{\text{Lt}} \frac{[x]}{x}$

$$= \underset{x\to\infty}{\text{Lt}} \frac{x-[x]}{x} = \underset{x\to\infty}{\text{Lt}} \left(1 - \frac{[x]}{x}\right) = 1 - 0$$
$$= 1$$

$$\therefore from, reqd.\,limit = \frac{0-1}{1} = -1$$

20. If $\boldsymbol{\alpha}$ = min. of $(\boldsymbol{x^2+2x+3})$ and $\boldsymbol{\beta}$= $\underset{n\to\infty}{\textbf{Lt}} \sum_{r=1}^{n} \frac{1}{(r+2)r!}$, $\boldsymbol{then\ \sum_{r=0}^{n} \alpha^r \beta^{n-r}} =$

(a) $\frac{2^{n+1}+1}{3.2^n}$ **(b)** $\frac{2^{n+1}-1}{3.2^n}$
(c) $\frac{4^{n+1}-1}{3.2^n}$ **(d) none of these**

Ans. (c)

$$\alpha = min.of\ x^2+2x+3 = \frac{4.1.3-2^2}{4}$$
$$= 2$$

$$\beta = \underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{n} \frac{1}{(r+2)r!} = \underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{n} \frac{r+1}{(r+2)!}$$
$$= \underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{n} \frac{(r+2)-1}{(r+2)!}$$

$$= \underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{n} \left(\frac{1}{(r+1)!} - \frac{1}{(r+2)!}\right)$$
$$= \underset{n\to\infty}{\text{Lt}} \left[\frac{1}{2} - \frac{1}{(n+2)!}\right] = \frac{1}{2}$$

$$now \sum_{r=0}^{n} \alpha^r \beta^{n-r} = \beta^n + \alpha\beta^{n-1} + \cdots + \alpha^n$$

$$= \beta^n \left[1 + \frac{\alpha}{\beta} + \left(\frac{\alpha}{\beta}\right)^2 + \cdots + \left(\frac{\alpha}{\beta}\right)^n\right]$$
$$= \beta^n \frac{\left(\frac{\alpha}{\beta}\right)^{n+1} - 1}{\frac{\alpha}{\beta} - 1}$$

$$= \frac{1}{2^n}\left(\frac{4^{n+1}-1}{3.2^n}\right)$$

21. If [x]denotes the integral part of x, then

$\underset{x\to\infty}{\textbf{Lt}} \frac{\log_e[x]}{x}$=

(a)-1 **(b) 1**
(c) 0 **(d) none of these**

Ans. (c)

$$x - 1 < [x] \le x$$

$$\Rightarrow log_e(x-1) < log_e[x] \le log_e x$$

$$\Rightarrow \underset{x\to\infty}{\text{Lt}} \frac{\log_e(x-1)}{x} \le \underset{x\to\infty}{\text{Lt}} \frac{\log_e[x]}{x}$$
$$\le \underset{x\to\infty}{\text{Lt}} \frac{\log_e x}{x}$$

$$\Rightarrow \underset{x\to\infty}{\text{Lt}} \frac{\frac{1}{x-1}}{1} \le \underset{x\to\infty}{\text{Lt}} \frac{\log_e[x]}{x} \le \frac{\frac{1}{x}}{1}$$

$$\Rightarrow 0 \le \underset{x\to\infty}{\text{Lt}} \frac{\log_e[x]}{x} \le 0 \Rightarrow \underset{x\to\infty}{\text{Lt}} \frac{\log_e[x]}{x}$$
$$= 0$$

22. $\underset{x\to 1+0}{\textbf{Lt}} \frac{\int_1^x |t-1|dt}{\sin(x-1)}$=

(a) -1 **(b) 0**
(c) 1 **(d) none of these**

Ans. (b)

Reqd. limit = $\underset{x\to 1+0}{\mathrm{Lt}} \frac{\int_1^x (t-1)dt}{\sin(x-1)} =$

$\underset{x\to 1+0}{\mathrm{Lt}} \frac{\left[\frac{(t-1)^2}{2}\right]_1^x}{\sin(x-1)} = \underset{x\to 1+0}{\mathrm{Lt}} \frac{\frac{(x-1)^2}{2}}{\sin(x-1)} =$

$\underset{x\to 1+0}{\mathrm{Lt}} \frac{x-1}{\sin(x-1)}.\left(\frac{x-1}{2}\right) = 1 \times 0 = 0$

23. $\underset{n\to\infty}{\mathbf{Lt}} \prod_{r=n}^{n^4} \log_{r-1}(r) =$

(a) 0 **(b)** ∞
(c) k **(d) k!**

Ans. (c)

$$\underset{x\to\infty}{\mathrm{Lt}} \prod_{r=n}^{n^k} \log_{r-1} r = \underset{n\to\infty}{\mathrm{Lt}} \prod_{r=n}^{n^k} \frac{logr}{\log(r-1)}$$

$$= \underset{n\to\infty}{\mathrm{Lt}} \frac{\log n}{\log(n-1)}.\frac{\log(n+1)}{\log n}.\frac{\log(n+2)}{\log(n+1)}$$

$$\cdots \frac{\log n^k}{\log(n^k-1)}$$

$$= \underset{n\to\infty}{\mathrm{Lt}} \frac{\log n^k}{\log(n-1)}$$

$$= \underset{n\to\infty}{\mathrm{Lt}} \frac{k\log n}{\log(n-1)} \left[\frac{\infty}{\infty} from\right]$$

$$= \underset{n\to\infty}{\mathrm{Lt}} \frac{k.\frac{1}{n}}{\frac{1}{n-1}} = \underset{n\to\infty}{\mathrm{Lt}}\, k\left(1-\frac{1}{n}\right) = k$$

24. If $\boldsymbol{\alpha_r = \frac{x-a_r}{|x-a_r|}, r = 1,2,3,..,n}$ **&** $\boldsymbol{\alpha_1 < \alpha_2 < \cdots < \alpha_n}$**. Then** $\underset{x\to\alpha_m}{\mathbf{Lt}} (\boldsymbol{\alpha_1\, \alpha_2 \ldots \alpha_n})$,

$\mathbf{1} \le \boldsymbol{m} \le \boldsymbol{n}$ **is equal to**

(a) $(-\mathbf{1})^{m-1}$ **(b)** $(-\mathbf{1})^{m}$
(c) $(-\mathbf{1})^{m+1}$ **(d) none**

Ans. (d)

When x $\longrightarrow a_m + 0, \alpha_r = 1, r = 1,2,3,\ldots,m = -1, r = m+1, m+2,\ldots,n$

$$\therefore \underset{x\to a_m+0}{\mathrm{Lt}} (\alpha_1,\alpha_2,\ldots,\alpha_n) = 1^m(-1)^{n-m}$$
$$= (-1)^{n-m}$$

$$when\ x \longrightarrow a_m - 0,$$

$$\alpha_r = 1, for\ r = 1,2,\ldots,m-1$$
$$= -1, for\ r$$
$$= m, m+1,\ldots,n$$

$$\therefore \underset{x\to a_m-0}{\mathrm{Lt}} (\alpha_1,\alpha_2,\ldots,\alpha_n)$$
$$= 1^{m-1}(-1)^{n-m+1}$$
$$= (-1)^{n-m+1}$$

Since L.H. limit ≠ R.H. limit

$$\underset{x\to a_m}{\mathrm{Lt}} (\alpha_1,\alpha_2,\ldots,\alpha_n)\ does\ not\ exist$$

25. If $\boldsymbol{S_n = \sum_{r=1}^{n} a_r}$ $\boldsymbol{and}$ $\underset{n\to\infty}{\mathbf{Lt}}\, \boldsymbol{a_n = a, then}$ $\underset{n\to\infty}{\mathbf{Lt}} \frac{\boldsymbol{S_{n+1}-S_n}}{\sqrt{\sum_{r=1}^{n} r}}$

(a) a **(b) 2a**
(c) –a **(d) 0**

Ans. (d)

$$\underset{n\to\infty}{\mathrm{Lt}} \frac{S_{n+1}-S_n}{\sqrt{\sum_{r=1}^{n} r}} = \underset{n\to\infty}{\mathrm{Lt}} \frac{a_{n+1}}{\sqrt{\frac{n(n+1)}{2}}}$$

$$= \frac{\underset{n\to\infty}{\mathrm{Lt}} \frac{a_{n+1}}{n}}{\sqrt{\frac{1\left(1+\frac{1}{n}\right)}{2}}} = 0\ \left[\because \underset{n\to\infty}{\mathrm{Lt}}\, a_{n+1} = a\right]$$

26. If $\underset{n\to\infty}{\text{Lt}} \frac{n^k sin^2 n!}{n+1} = 0$ for

(a) all k **(b) 0≤k < 1**

(c) k= 1 **(d) for k > 1**

Ans. (b)

$$\underset{n\to\infty}{\text{Lt}} \frac{n^k sin^2\, n!}{n+1} = \underset{n\to\infty}{\text{Lt}} \frac{n^k}{n+1}.(sin^2 n!)$$

$\underset{n\to\infty}{\text{Lt}}\ sin^2 n!$ does not exist but if

$$\underset{n\to\infty}{\text{Lt}} \frac{n^k}{n+1} = 0, then\ \underset{n\to\infty}{\text{Lt}} \frac{n^k sin^2\, n!}{n+1} = 0$$

$$but\ \underset{n\to\infty}{\text{Lt}} \frac{n^k}{n+1} = 0 \Leftrightarrow 0\ \le k \le 1$$

27. $\underset{n\to\infty}{\text{Lt}} \frac{\{x\}+\{2x\}+\{3x\}+\cdots+\{nx\}}{n^2}$ where {x} = x-[x] denotes the fractional part of x,

(a) 1 **(b) 0**

(c) ½ **(d) none of these**

Ans. (b)

0≤ {nx} < 1, for n= 1, 2, 3,…, n

$\Rightarrow 0 \le \sum_{n=1}^{n}\{nx\} < n \ \Rightarrow \frac{0}{n^2} \le \frac{\sum_{n=1}^{n}\{nx\}}{n^2} < \frac{1}{n}$

$$\underset{n\to\infty}{\text{Lt}} \frac{0}{n^2} \le \underset{n\to\propto}{\text{Lt}} \le \frac{\sum_{n=1}^{n}\{nx\}}{n^2} < \underset{n\to\infty}{\text{Lt}} \frac{1}{n}$$

$$\Rightarrow 0 \le \underset{n\to\infty}{\text{Lt}} \frac{\sum_{n=1}^{n}\{nx\}}{n^2} \le 0$$

$$\Rightarrow \underset{n\to\infty}{\text{Lt}} \frac{\{x\}+\{2x\}+\cdots+\{nx\}}{n^2} = 0$$

28. If [x] denotes the integral part of x, then $\underset{n\to\infty}{\text{Lt}} \frac{[x]+[2x]+[3x]+\cdots+[nx]}{n^2}$ is

(a) x **(b) x/2**

(c) x/3 **(d) independent of x**

Ans. (b)

nx – 1 < [nx] ≤ nx, for n= 1, 2, …, n

$\Rightarrow \sum_{n=1}^{n}(nx-1) < \sum_{n=1}^{n}[nx] \le \sum_{n=1}^{n} nx$

$$\Rightarrow \frac{n(n+1)}{2}x - n < \sum_{n=1}^{n}[nx] \le \frac{n(n+1)}{2}$$

$$\Rightarrow \frac{n(n+1)}{2n^2}x - \frac{1}{n} < \frac{\sum_{n=1}^{n}[nx]}{n^2} \le \frac{n(n+1)}{2n^2}x$$

$$\Rightarrow \underset{n\to\infty}{\text{Lt}}\left(\frac{n(n+1)}{2n^2}x - \frac{1}{n}\right) \le l\ \underset{n\to\infty}{\text{Lt}} \frac{n(n+1)}{2n^2}x \Rightarrow \frac{x}{2} \le l \le \frac{x}{2} \Rightarrow l = \frac{x}{2}$$

29. If [x] denotes the integral part of x, and $a_n = \sum_{r=0}^{n-1}\left[x + \frac{r}{n}\right]$, then $\underset{n\to\infty}{\text{Lt}} \frac{a_1+a_2+\cdots+a_n}{n^2} =$

(a) $\frac{x}{2}$ **(b) $\frac{x}{3}$**

(c) x **(d) none of these**

Ans. (a)

$$[x] + \left[x+\frac{1}{n}\right] + \left[x+\frac{2}{n}\right] + \cdots + \left[x+\frac{n-1}{n}\right] = [nx]$$

$$\therefore a_n = [nx]$$

$$\therefore \lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n^2}$$

$$= \lim_{n\to\infty} \frac{[x] + [2x] + [3x] + \cdots + [nx]}{n^2} = \frac{x}{2}$$

30. $\lim_{n\to\infty}\left(\tan\theta + \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{2^2}\tan.\frac{\theta}{2^2} + \cdots + \frac{1}{2^n}\tan\frac{\theta}{2^n}\right) =$

(a) $\frac{1}{\theta}$ **(b)** $\frac{1}{\theta} - 2\cot 2\theta$
(c) 2 $\cot 2\theta$ **(d) none**

Ans. (b) $\tan\theta = \cot\theta - 2\cot 2\theta$

$\therefore \frac{1}{2}\tan\frac{\theta}{2} = \frac{1}{2}\cot\frac{\theta}{2} - \cot\theta$

… … …

$$\frac{1}{2^n}\tan\frac{\theta}{2^n} = \frac{1}{2^n}\cot\frac{\theta}{2^n} - \frac{1}{2^{n-1}}\cot\frac{\theta}{2^{n-1}}$$

$\therefore S_n = \frac{1}{2^n}\cot\frac{\theta}{2^n} - 2\cot 2\theta$

$\therefore$ Reqd. limit = $\lim_{n\to\infty} S_n =$

$$\lim_{n\to\infty}\left(\frac{1}{\frac{2^n\tan\frac{\theta}{2^n}}{\frac{\theta}{2^n}}\cdot\frac{\theta}{2^n}} - 2\cot 2\theta\right)$$

$$= \frac{1}{\theta} - \cot 2\theta$$

ISI OBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 15

1. Let f(x) = $\det\begin{pmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{pmatrix}$

where p is a constant. Then $\frac{d^3}{dx^3}\{f(x)\}$ **at x = 0 is**

(a) p **(b) p+** p^2
(c) p + p^3 **(d) independent of p**

Ans. (d)

$$f'''(x) = \begin{vmatrix} 6 & -\cos x & \sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$\therefore f'''(0) = \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} = 0$$

2. The number of solutions of the equations $x_2 - x_3 = 1$
$-x_1 + 2x_3 = 2$
$x_1 - 2x_2 = 3$ **is**

(a) zero **(b) one**
(c) two **(d) infinite**

Ans. (b)

$$\text{Hence } \Delta = \begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{vmatrix}$$

= -1 (-2)-1 (2)=0

Hence given system of equations is consistent and since have no two equations are identical, therefore number of solutions is one.

3. If p(x), q(x), r(x) be three quadratic expressions in x and f(x) = $\begin{vmatrix} p(x) & q(x) & r(x) \\ p'(x) & q'(x) & r'(x) \\ p''(x) & q''(x) & r''(x) \end{vmatrix}$ **, where dash denotes the derivative with respect to x, then**

(a) f(x) = f(0) **(b) f(x) = 2f(0)**
(c) f(x) = -f(0) **(d) none of these**

Ans. (b) Let p(x) = $ax^2 + bx + c,\ q(x) = mx^2 + nx + k,\ r(x) = ux^2 + vx + w$

f(x) =
2

$$\begin{vmatrix} ax^2 + bx + c & mx^2nx + k & ux^2 + vx + w \\ 2ax + b & 2mx + n & 2ux + v \\ a & m & u \end{vmatrix} =$$

$$2\begin{vmatrix} a & k & w \\ b & n & v \\ a & m & u \end{vmatrix}\ \begin{matrix} R_1 \to R_1 - xR_2 + x^2R_3 \\ R_2 \to R_2 - 2xR_3 \end{matrix}$$

= 2f(0)

4. If $(1 + ax + bx^2)^4 = a_0 + a_1x + a_2x^2 + \cdots + a_8x^8$, where a, b, $a_0, a_1, \ldots, a_8 \in R$ such that $a_0 + a_1 + a_2 \neq 0$ and $\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_0 \\ a_2 & a_0 & a_1 \end{vmatrix} = 0$ then

(a) a= 3/4, b= 5/8 **(b) a= 1/4, b= 5/32** **(c) a= 1, b= 2/3** **(d) none**

Ans. (b)

Putting x= 0, $a_n = 1$

Differentiating both sides and putting X= 0, we get $a_1 = 4a$

Differentiating again and putting x= 0

We get $a_2 = 6a^2 + 4b$

Clearly, $\Delta = -({a_0}^3 + {a_1}^3 + {a_2}^3 - 3a_0a_1a_2)$

$\because a_0 + a_1 + a_2 \neq 0 \therefore a_0 = a_1 = a_2$

$\therefore$ 1= 4a= $6a^2 + 4b \Rightarrow a = \frac{1}{4},\ b = \frac{5}{32}$

5. If $\alpha, \beta, \gamma \in$ R, then $\begin{vmatrix} (e^{i\alpha} + e^{-i\alpha})^2 & (e^{i\alpha} - e^{-i\alpha})^2 & 1 \\ (e^{i\beta} + e^{-i\beta})^2 & (e^{i\beta} - e^{-i\beta})^2 & 1 \\ (e^{i\gamma} + e^{-i\gamma})^2 & (e^{i\gamma} - e^{-i\gamma})^2 & 1 \end{vmatrix}$ **equals**

(a) $e^{i(\alpha+\beta+\gamma)}$ **(b) $e^{-i(\alpha+\beta+\gamma)}$**
(c) $e^{i\alpha\beta\gamma}$ **(d) none of these**

Ans. (d)

Applying

$C_1 \to C_1 - C_2$, we have Δ= 0

6. If $f_n(x) = (n^x + n^{-x})^2, g_n(x) = (n^x - n^{-x})^2$ then $\begin{vmatrix} f_2(x) & g_2(x) & 1 \\ f_3(y) & g_3(y) & 1 \\ f_5(z) & g_5(z) & 1 \end{vmatrix}$ **is**

(a) equal to zero **(b) independent of x, y, z** **(c) $2^x3^y5^z$** **(d) $\frac{1}{2^x3^y5^z}$**

Ans.(a)

$$\Delta = \begin{vmatrix} (2^x + 2^{-x})^2 & (2^x - 2^{-x})^2 & 1 \\ (3^y + 3^{-y})^2 & (3^y - 3^{-y})^2 & 1 \\ (5^z + 5^{-z})^2 & (5^z - 5^{-z})^2 & 1 \end{vmatrix} =$$

$$\begin{vmatrix} 4 & (2^x - 2^{-x})^2 & 1 \\ 4 & (3^y - 3^{-y})^2 & 1 \\ 4 & (5^z - 5^{-z})^2 & 1 \end{vmatrix} = 0$$

7. If $\sum_{n=1}^{n} \alpha_n = pn^2 + qn + r$ and $\alpha_1, \alpha_2, \alpha_3 \in \{1, 2, 3, \ldots, 9\}$ and $25\alpha_1, 37\alpha_2, 49\alpha_3$ be three digit numbers, then $\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ 5 & 7 & 9 \\ 25\alpha_1 & 37\alpha_2 & 49\alpha_3 \end{vmatrix} =$

(a) $\alpha_1 + \alpha_2 + \alpha_3$ (b) $\alpha_1 - \alpha_2 + \alpha_3$ (c) 7
(d) 0

Ans. (d)

$\alpha_n = 2pn + q$

$\Delta = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ 5 & 7 & 9 \\ 25\alpha_1 & 37\alpha_2 & 49\alpha_3 \end{vmatrix} =$
$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ 5 & 7 & 9 \\ 200 & 300 & 400 \end{vmatrix} [R_3 \to R_3 - R_1 - 10R_2]$

$= \begin{vmatrix} \alpha_1 & 2p & 2p \\ 5 & 2 & 2 \\ 200 & 100 & 100 \end{vmatrix} = 0 \begin{bmatrix} C_3 \to C_3 - C_2 \\ C_2 \to C_2 - C_1 \end{bmatrix}$

8. If f(x) = $\begin{vmatrix} p(x) & q(x) & r(x) \\ p(\alpha) & q(\alpha) & r(\alpha) \\ p'(\alpha) & q'(\alpha) & r'(\alpha) \end{vmatrix}$, where p(x), q(x), r(x) are polynomials of degree 3, 4, 5 respectively, then $(x - \alpha)^m$ is a factor of F(x), where

(a) m ≥ 1 (b) m ≥ 2
(c) m= 2 (d) m ≥ 3

Ans. (b) F(α) = 0, F'(α) = 0

∴ $(x - \alpha)^2$ will necessarily be a factor or F(x) ∴ m≥ 2

9. If Y= SX, Z= tX, all the variables being differentiable functions of x, and lower suffices denotes the derivative w.r.t. to x and $\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} + \begin{vmatrix} S_1 & t_1 \\ S_2 & t_2 \end{vmatrix} = x^n$, then n=

(a) 1 (b) 2
(c) 3 (d) none of these

Ans. (c)

$\Delta=$
$\begin{vmatrix} X & SX & tX \\ X_1 & SX_1 + S_1X & tX_1 + t_1X \\ X_2 & SX_2 + 2S_1X_1 + S_2X & tX_2 + 2t_1X_1 + t_2X \end{vmatrix}$

$=$
$\begin{vmatrix} X & 0 & 0 \\ X_1 & S_1X & t_1X \\ X_2 & 2S_1X_1 + S_2X & 2t_1X_1 + t_2X \end{vmatrix} \begin{bmatrix} C_3 \to C_3 - tC_1 \\ C_2 \to C_2 - SC_1 \end{bmatrix}$

$= X^2 \begin{vmatrix} S_1 & t_1 \\ 2S_1X_1 + S_2X & 2t_1X_1 + t_2X \end{vmatrix} =$
$X^3 \begin{vmatrix} S_1 & t_1 \\ S_2 & t_2 \end{vmatrix} [R_3 \to R_2 - 2X_1R_1]$

∴ n= 3

10. If f(x) = cos[π]x+ cos[πx], where [y]is the greatest integer function of y, then f$\left(\frac{\pi}{2}\right)$=

(a) 0 (b) cos3
(c) cos4 (d) none of these

Ans. (c)

f(x) = cos [π]x + cos [πx]

$\therefore f\left(\frac{\pi}{2}\right) = cos[\pi]\frac{\pi}{2} + \cos\left[\frac{\pi^2}{2}\right] = \cos\frac{3\pi}{2} + \cos 4 = 0 + \cos 4 = \cos 4$

11. Let g(x) =

$$\begin{vmatrix} f(x+\alpha) & f(x+2\alpha) & f(x+3\alpha) \\ f(x) & f(2\alpha) & f(3\alpha) \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix},$$

where α is a constant, then $\mathrm{lt}_{x\to 0} \frac{g(x)}{x} =$

(a) 0 **(b) 1**
(c) -1 **(d) none of these**

Ans. (a)

$\underset{x\to 0}{\mathrm{Lt}} \frac{g(x)}{x} \left[\frac{0}{0}\ from\ as\ g(0) = 0\right]$

$= \underset{x\to 0}{\mathrm{Lt}} \frac{g'(x)}{1} = g'(0) = 0$

12. Choose any 9 distinct integers. These 9 integers can be arranged to from 9! determinants each of order 3. The sum of these 9! determinants is

(a) 0 **(b) > 0**
(c) < 0 **(d) 9!**

Ans. (a) Let the nine distinct digits be $a_1, a_2, \dots, a_9$

Let Δ_1= one of the 9! Determinates then there exists

α Determinant Δ_2= adding 9! Determinants obtained by interchanging R_1 *and* R_2 *in* Δ_1 such that

$$\Delta_1 + \Delta_2 = 0$$

Thus 9! /2 pairs of determinants will be there such that sum of each pair is zero.

∴ Required sum = 0

13. If a complex number z satisfies $\log_{\frac{1}{2}}\left(\frac{|z|^2+2|z|+6}{2|z|^2-2|z|+1}\right) < 0$, then locus of point represented by z is

(a) |z|< 5 **(b) |z| > 6**
(c) 1 <|z|< 3 **(d) |z|=5**

Ans. (a)

Since $\log_{\frac{1}{2}}\left(\frac{|z|^2+2|z|+6}{2|z|^2-2|z|+1}\right) < 0$

$\therefore \frac{|z|^2+2|z|+6}{2|z|^2-2|z|+1} > 1$

$\Rightarrow |z|^2 - 2|z| - 5 < 0$

⟹ (|z|- 5)(|z| +1) < 0 ⟹ |z| < 5

14. If |z|= maximum {|z+2|, |z-2|}, then

(a) $|z-\bar{z}|$= ½ **(b) $|z+\bar{z}|$=2**
(c) $|z+\bar{z}|$= ½ **(d) $|z-\bar{z}|$= 2**

Ans. (b)

|z| =|z +2|

⟹ $z\bar{z} = (z+2)(\bar{z}+2)$

⟹ $z+\bar{z} = -2$ ⟹ $|z+\bar{z}| = 2$

|z|= | z -2| ⟹ $z\bar{z} = (z-2)(\bar{z}-2)$

⟹ $z+\bar{z} = 2$ ⟹ $|z+\bar{z}| = 2$

15. Let a be a complex number such that |a|< 1 and $z_1, z_2, z_3, \dots$ be the vertices of a polygon ∋ $z_k = 1 + a + a_2 + \cdots + a^{k-1}$ for all k= 1, 2, 3, … then $z_1, z_2, \dots$ lie within the circle

(a) $\left|z - \frac{1}{1-a}\right| = \frac{1}{|a-1|}$

(b) $\left|z + \frac{1}{a+1}\right| = \frac{1}{|a+1|}$

(c) $\left|z - \frac{1}{1-a}\right| = |a-1|$

(d) $\left|z + \frac{1}{a+1}\right| = |a+1|$

Ans. (a)

Given $z_k = 1 + a + a^2 + \cdots + a^{k-1} = \frac{1-a^k}{1-a}$

$\Rightarrow z_k - \frac{1}{1-a} = -\frac{a^k}{1-a}$

$\Rightarrow \left|z_k - \frac{1}{1-a}\right| = \frac{|a|^k}{|1-a|} < \frac{1}{|1-a|}$ [$\because$ $|a| < 1$]

$\Rightarrow z_k$ lies within the circle

$\left|z - \frac{1}{1-a}\right| = \frac{1}{|1-a|}$

16. The number of ordered triplets of positive integers which satisfy the inequality $20 \le x + y + z \le 50$ is

(a) $\binom{50}{3} - \binom{19}{2}$ **(b)** $\binom{50}{2} - \binom{19}{3}$ **(c)** $\binom{50}{3} - \binom{20}{3}$
(d) none

Ans. (a) $20 \le x + y + z \le 50, where\ x, y, z, \in N$

$\Rightarrow 17 \le a + b + c \le 47,$

Where a = x - 1, b = y - 1, c = z - 1

$\therefore$ Reqd. number = $\binom{19}{17} + \binom{20}{18} + \cdots + \binom{49}{47}$

= $\binom{19}{2} + \binom{20}{2} + \cdots + \binom{49}{2}$

= $\binom{50}{3} - \binom{19}{2}$

17. Let A= $\{x_1, x_2, x_3, x_4, x_5\}$. Then the number B= $\{y_1, y_2, y_3, y_4, y_5\}$. Then the number of one-one mappings, from A to B such that $f(x)_i \ne y_i, i = 1, 2, \dots, 5$ is

(a) 40 **(b) 44**
(c) 6 **(d) 24**

Ans. (b)

Required number= $5!\left(\frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \frac{1}{\lfloor 5}\right)$

$= 120\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right)$

$= 60 - 20 + 5 - 1 = 44$

18. Number of divisors of $2^2.3^3.5^3.7^5$ of the form 4n+1, n ∈N is

(a) 46 **(b) 47**
(c) 96 **(d) none of these**

Ans. (b) 4n + 1 is an odd number

Divisor will be of the from 4n+1 if and only if it is the product of

(i) All numbers of the from 4k +1, or
(ii) 2 or 4 numbers of the from 4n +3

$\therefore$ Number of divisors of

N= $2^2.3^3.5^3.7^3$, which are of the from 4n +1 excluding 1

= number of terms in the product

$(1+3^2)(1 + 5 + 5^2 + 5^3)(1 + 7^2 + 7^4)$ + number of terms in the product

$(3 + 3^3)(7 + 7^3 + 7^5)(1 + 5 + 5^2 + 5^3) - 1$

$= 2 \times 4 \times 3 + 2 \times 3 \times 4 - 1 = 47$

19. If m = number of distinct rational numbers $\frac{p}{q} \in (0, 1)$ such that p, q ∈ {1, 2,

3, 4, 5} and n = number of mappings from {1, 2, 3} onto {1, 2}, then m –n is

(a) 1 **(b) -1**
(c) 0 **(d) none of these**

Ans. (c)

n= $2^3 - 2 = 6$

$m = 4 + 3 + 2 + 1 - 1 = 9\left(as\ \frac{2}{4} = \frac{1}{2}\right)$
∴ m – n = 3

20. There are n different books each having m copies. If the total number of ways of making a selection from them is 255 and m-n+1 = 0. Then distance of point (m, n) from the origin is

(a) 3 **(b) 4**
(c) 5 **(d) none of these**

Ans. (c) Total number of selections = $(m + 1)^n - 1$ (number of ways in which no book is selected = 1)

Given, $(m + 1)^n - 1 = 255 \Longrightarrow (m + 1)^n = 256$

∴ $n^n = 256 = 4^4$ [∵ m+ 1= n]

⟹ n= 4, m= 3

∴ Required distance= $\sqrt{m^2 + n^2}$= 5

21. Let f(x) = $\boldsymbol{sin^2\frac{x}{2} + cos^2\frac{x}{2}\ and\ g(x) = sec^2x - tan^2x\ also\ f(x) = g(x)}$,then

(a) x ∈R
(b) x ∈Φ (empty set)

(c) x ∈R - {x: x = (2n+1)$\frac{\pi}{2}$where n ∈ I}
(d) none of these

Ans. (c)

Clearly f(x) = g(x)

But here sec x and tan x should be defined

$\therefore x \neq (2n + 1)\frac{\pi}{2}, n \in I$

∴ x ∈ R – {x : x=$(2n + 1)\frac{\pi}{2}, n \in I$ }

22. Consider a set P consisting of n elements. A subset 'A' of P is chosen thereafter set P is reconstructed by replacing the elements of A and finally another subset 'B' of P is chosen. The number of ways of choosing 'A' and 'B' such that (A∪B) is a proper subset of P is

(a) 4^n **(b) $4^n - 3^n$**
(c) $4^n - 2^n$ **(d) none of these**

Ans. (b)

For any element a_i of P, there are 4 possibilities:

(i) $a_i \in A\ and\ a_i \in B$
(ii) $a_i \in A\ but\ a_i \notin B$
(iii) $a_i \notin A\ \ but\ a_i \in B$
(iv) $a_i \notin A\ and\ a_i \notin B$

Total number of ways for one element of P and two subsets A and B = 4

∴ Total number of ways for n elements $a_1, a_2, \ldots, a_n$ of P and two subsets A and B= 4^n

Number of ways in which one particular element $a_i \in A \cup B = 3$

∴ number of ways in which all elements $a_1, a_2, \ldots, a_n \in A \cup B$ =3^n

∴ number of ways in which at least one element of p does not belong to

A∪ B = $4^n - 3^n$

∴ Required number = $4^n - 3^n$.

23. Suppose $A_1, A_2, A_3, A_4, \ldots, A_{30}$ are thirty sets each with five elements and $B_1, B_2, \ldots B_n$ are n sets each with three elements such that $\cup_{i=1}^{30} A_i = \cup_{i=1}^{n} B_i = s$.

If each element of S belongs to exactly ten of the A_i's and exactly 9 of $B_i's$ then the value of n is

(a) 15 **(b) 135**
(c) 45 **(d) 90**

Ans. (c)

Since A_i has 5 elements, we have

$\sum_{i=1}^{30} n(A_i) = 5 \times 30 = 150$ ……………..(1)

Suppose S has ‘m’ distinct elements.

Since each element of S belongs to exactly 10 of $A_i's$ we also have

$$\sum_{i=1}^{30} n(A_i) = 10m,$$

From (1) and (2) , we get 10m = 150

⟹ m= 15

Since B_i has 3 elements and each element of S belongs to exactly 9 of $B_i's$, we have

$$\sum_{i=1}^{n} n(B_i) = 3n \text{ and } \sum_{i=1}^{n} n(B_i) = 9m$$

⟹ 3n= 9m

⟹ n= 3m

⟹ n= 45

24. Triplet (x, y, z) is chosen from the set {1, 2, 3, …, n}, such that x ≤y < z. The number of such triplets is

(a) n^3 **(b) n_{C_3}**
(c) n_{C_2} **(d) $n_{C_2} + n_{C_3}$**

Ans. (d)

Number of selections when x < y < z = $\binom{n}{3}$

Number of selections when x= y < z = $\binom{n}{2}$

(Here in fact we have to select only two numbers out of n numbers).

∴ required number= $\binom{n}{3}$+$\binom{n}{2}$

25. There are n locks and n keys. If all locks and keys are to be matched 100% then maximum numbers of trials required are

(a) $\frac{n(n-1)}{2}$ **(b) $\sum_{r=1}^{n} r$**
(c) $\binom{n+1}{2}$ **(d) $\binom{n-1}{2}$**

Ans. (a)

Maximum number of trials to match first lock and key is (n- 1).

Max. number of trails to match second lock and key is (n- 2) and so on.

∴ Required number = $\sum_{r=1}^{n-1} r = \frac{(n-1)n}{2}$

26. If $\sin x + \cos x + \tan x + \cot x + \sec x + cosec\, x = 7$ and $sin2x = a - b\sqrt{7}$ ordered pair (a, b) can be

(a) (6, 2) **(b) (8, 3)**
(c) (22, 8) **(d) (11, 4)**

Ans. (c)

$$\sin x + \cos x + \tan x + \cot x + \sec x + cosec\ x = 7$$

$$\Rightarrow (\sin x + \cos x) + \left(\frac{1}{\sin x \,.\, \cos x}\right) + \frac{(\sin x + \cos x)}{\sin x \,.\, \cos x} = 7$$

$$\Rightarrow (\sin x + \cos x)\left(1 + \frac{2}{\sin 2x}\right) = \left(7 - \frac{2}{\sin 2x}\right)$$

$$\Rightarrow (1 + \sin 2x)\left(1 + \frac{4}{sin^2 2x} + \frac{4}{\sin 2x}\right) = 49 + \frac{4}{sin^2 2x} - \frac{28}{\sin 2x}$$ (Squaring both sides)

$$\Rightarrow sin^2 2x - 44 sin^2 2x + 36 sin2x = 0$$

$\Rightarrow$ sin2x = 22 - $8\sqrt{7}$

27. If u_r denotes the number of one–one functions from $\{x_1, x_2, \ldots, x_r\}$ *to* $\{y_1, y_2, \ldots, y_r\}$ such that f(x_i) $\neq y_i$, for i= 1, 2, 3, …, r then u_4 =

(a) 9 **(b) 44**
(c) 265 **(d) none of these**

Ans. (a)

u_r = number of ways of putting $x_1, x_2, \ldots, x_r$ in r corresponding place so that no x_1 is put in the corresponding place

$$= \underline{|r}\left(\frac{1}{\underline{|2}} - \frac{1}{\underline{|3}} + \cdots + \frac{(-1)^r}{\underline{|r}}\right)$$

$$\therefore u_r = \underline{|4}\left(\frac{1}{\underline{|2}} - \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}}\right) = 12 - 4 + 1$$

28. Number of positive unequal integral solutions of equation x+ y+ z = 6 is

(a) 4! **(b) 3!**
(c) 6! **(d) 2×4!**

Ans. (b)

Given x+ y+ z = 6 …….(1)

x, y, z ∈ N and are unequal.

$\Rightarrow$ x, y, z ∈ {1, 2, 3} and are unequal

∴ Required number of solutions= 3! = 6

29. The plain containing the two straight lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{b} + \mu\vec{a}$ is

(a) $[\vec{r}\ \ \vec{a}\ \ \vec{b}] = 0$
(b) $[\vec{r}\ \ \vec{a}\ \ \vec{a} \times \vec{b}] = 0$

(c) $[\vec{r}\ \ \vec{b}\ \ \vec{a} \times \vec{b}] = 0$
(d) none

Ans. (a)

Given lines are $\vec{r} = \vec{a} + \lambda\vec{b}$ …………..(1)

$\vec{r} = \vec{b} + \mu\vec{a}$ ……………..(2)

lines (1) and (2) intersect at $(\vec{a} + \vec{b})$

Then the plane passes through $(\vec{a} + \vec{b})$

Also, line (1) is parallel to $\vec{b}$ and line (2) is parallel to $\vec{a} \Rightarrow (\vec{a} \times \vec{b})$ is normal to plane containing these lines.

$\therefore$ Eqn. of reqd. plane is

$[\vec{r} - (\vec{a} + \vec{b})].(\vec{a} \times \vec{b}) = 0$

$\vec{r}(\vec{a} \times \vec{b}) - (\vec{a} + \vec{b}).(\vec{a} \times \vec{b}) = 0$

$\Longrightarrow [\vec{r}\ \vec{a}\ \vec{b}] = 0$

30. Let $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$. If $\vec{c}$ is a vector such that $\vec{a}.\vec{c} = |\vec{c}|$, $|\vec{c} - \vec{a}| = 2\sqrt{2}$ and angle between $|(\vec{a} \times \vec{b}) \times \vec{c}| =$

(a) 2/3 **(b) 1/3**
(c) 3/2 **(d) 1**

Ans. (c) Given $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$

$\vec{b} = \hat{i} + \hat{j}$

$\vec{a}.\vec{c} = |\vec{c}|$……………(1)

$|\vec{c} - \vec{a}| = 2\sqrt{2}$ …………….(2)

Angle between $(\vec{a} \times \vec{b})$ and $\vec{c} = \frac{\pi}{6}$

Now, $|(\vec{a} \times \vec{b}) \times \vec{c}| = |\vec{a} \times \vec{b}||\vec{c}|\sin\frac{\pi}{6} = \frac{3}{2}|\vec{c}|$……(3)

From (3), $|\vec{c} - \vec{a}|^2 = 8$

$\Longrightarrow (\vec{c} - \vec{a}).(\vec{c} - \vec{a}) = 8$

$\Longrightarrow |\vec{c}|^2 + |\vec{a}|^2 - 2\vec{a}.\vec{c} = 8$

$\Longrightarrow |\vec{c}|^2 + 9 - 2|\vec{c}| = 8$ (from (1))

$\therefore\ |\vec{c}| = 1$

From (3), $|(\vec{a} \times \vec{b}) \times \vec{c}| = \frac{3}{2}$.

TOPIC-WISE OBJECTIVE SOLVED PROBLEMS

THEORY OF EQUATIONS

1. The sum of the roots of equation $\boldsymbol{x^7 + 9x^6 - 2 = 0}$ is

   (a) 0 (b) 3
   (c) -9 (d) 7

Solution:- (c)

Sum of the roots = $\left(-\frac{coefficient\ of\ x^6}{coefficient\ of\ x^7}\right)$

$= -9.$

2. The sum of all the roots of the equation

   $\boldsymbol{|\ x - 2\ |^2 + |\ x - 2\ | - 2 = 0}$ is

   $(\boldsymbol{a})$ 4 (b) 2
   (c) 6 (d) none.

Solution:- (a) $|x - 2| = t$

$t^2 + t - 2 = 0$

$\therefore$ t = 1, -1, but -1 is not acceptable.

So, x = 3, 1.

Sum of the roots = 4.

3. The equation $\boldsymbol{3x^{10} + 7x^6 + 5x^4 + 2x^2 + 1 = 0}$ has

   (a) 10 (b) 6 (c) 2
   (d) zero real roots.

Solution:-

Let $f(x) = 3x^{10} + 7x^6 + 5x^4 + 2x^2 + 1$
$f(-x) = 3x^{10} + 7x^6 + 5x^4 + 2x^2 + 1$

Here f(x) and f(-x) has no sign change.

By sign rule, f(x) = 0 has no real roots.

4. The equations $\boldsymbol{x^2 - kx - 21 = 0 \text{ and } x^2 - 3kx + 35 = 0}$ have a common root then the value of k is equal to
   (a) -6 (b) 4
   (c) 5 (d) 6

Solution:- (b) From the above two equations, we have

$$\frac{x^2}{-35k - 63k} = -\frac{x}{35 + 21} = \frac{1}{-3k + k}$$

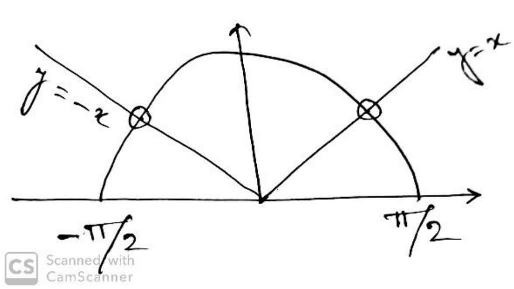
$$\therefore x = \frac{7k}{4}, x = \frac{28}{k}$$

$$So, \ k^2 = 16$$

$$\Rightarrow k = \pm 4.$$

5. The numbers of solutions of the equation $\boldsymbol{|x| = \cos x}$ is
   (a) 1 (b) 2
   (c) 3 (d) none

Solution: (b) Two graphs below intersect at two points only.



6. The equation $\boldsymbol{(x - a)^3 + (x - b)^3 + (x - c)^3}$ has
   (a) All real roots (b) one real two imaginary roots (c) three real roots (d) none

Solutions:- (b) Differentiating the function w.r.t. x, we get

$$f'(x) = 3\{(x - a)^2 + (x - b)^2 + (x - c)^2\} > 0 \ \forall \ x$$

$So, f'(x) = 0$ has no repeated roots.

7. If $\boldsymbol{\sqrt{a} + \sqrt{b}}$ be one of the roots of the given equation (a and b are not perfect squares) with rational coefficients, then lowest degree of such an equation must be
   (a) 2 (b) 3
   (c) 4 (d) none.

Solutions:- (c) Irrational roots occur in pairs.

If one root is $\sqrt{a} + \sqrt{b}$, then the other roots are: $\sqrt{a} - \sqrt{b}, -\sqrt{a} + \sqrt{b}, \sqrt{a} - \sqrt{b}$.

∴ Number of roots are = 4.

So, the lowest degree is 4.

8. If $\boldsymbol{\alpha_1, \alpha_2, \ldots, \alpha_n}$ are the roots of the equations $\boldsymbol{x^n - nax - b = c \text{ and then}}$ $\boldsymbol{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \ldots (\alpha_1 - \alpha_n)}$ equals
   (a) nα (b) n $\boldsymbol{({\alpha_1}^{n-1} + a)}$
   (c) na. $\boldsymbol{{\alpha_1}^{n-1}}$ (d) none

Solution:- (b) $(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n) = x^n - nax - b$

Differentiating w. r. t. x,

$(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)+(x-\alpha_1)(x-\alpha_3)\ldots.(x-\alpha_n)+\cdots+(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_{n-1})=nx^{n-1}+na$

Putting x = α_1, we get ______

$(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\ldots(\alpha_1-\alpha_n)={\alpha_1}^{n-1}.n+na.$

9. Let p, q ∈{1, 2, 3, 4}, the number of equations of $\boldsymbol{px^2+qx+1=0}$ having real roots are
   (a) 7 (b) 8
   (c) 9 (d) none

Solutions:- (a) Δ= $q^2-4p\geq 0$ for real roots.

i.e. $q^2\geq 4p$

Now, if p =1, then $q^2\geq 4$, i.e. q= 2, 3, 4.

If p = 2, then $q^2\geq 8$, i.e. q = 3, 4.

If p = 3, then $q^2\geq 12$, q = 4

If p =4, then $q^2\geq 16, i.e. q=4$

There are 7 such favourable cases.

10. If 0 <a < b <c and the roots α, β of the equation $\boldsymbol{ax^2+bx+c=}$ are imaginary, then
    (a) |α| <1 (b) |β| <1
    (c) |α| = |β| (d) none

Solution:- Since roots are imaginary, so Δ= $b^2-4ac<0$

∴The roots α and β are given by, α= $\frac{-b+i\sqrt{4ac-b^2}}{2a}$ $and\ \beta=\frac{-b-i\sqrt{4ac-b^2}}{2a}$.

And α = $\overline{\beta}$, so | α| = | β| .

Moreover, | α| = $\sqrt{\frac{b^2}{4a^2}+\frac{4ac-b^2}{4a^2}}=\sqrt{\frac{c}{a}}$ ⇒| α |>1 ($\because c>a$).

∴ | α| = | β| & | α| >1.

NUMBER THEORY

11. The congruence 35x ≡14 (mod 21) has
    (a) 7 solutions (b) 6 solutions
    (c) Unique solution (d) No solution

Solution:- (a) 35x -14 is divisible by 21

gcd (35, 21) = 7

and 7 divides 14; hence the given congruence has 7 solution.

12. The maximum value of f(x) = $\boldsymbol{(x-2)^n\,(3-x)^n}$ for a natural number n≥ 1 and 2 ≤ x ≤ 3 is
    (a) $\frac{1}{2^n}$ (b) $\frac{1}{4^n}$
    (c) $\frac{1}{8^n}$ (d) $\frac{1}{16^n}$

Solution:- (b) [If a + b = λ is given then ab is maximum when a = b = $\frac{\lambda}{2}$.]

Here (x -2) + (3- x) = 1 then(x- 2) = ½ = (3- x)

$\therefore [f(x)]_{max} = \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^n = \left(\frac{1}{4}\right)^n$.

13. If a ≡ b (mod n). Prove that gcd(a, n) is
    (a) gcd (b, n) (b) b
    (c) n (d) none

Solution:- (a) gcd (a, n) = d

⇒d | a, d| n but n| (a-b)

⇒d | a –b , d | a

⇒d | a –(a –b)= b

⇒d | a, d | b.

Same as gcd (b, n) = d. So, gcd (a, n) = gcd (b, n).

14. The highest power of 3 contained in 1000! Is
    (a) 493 (b) 494
    (c) 495 (d) 496

Solution:- (d) P= 3, n = 1000.

The highest power of 3 contained in n! Is given by

$$k(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{P^2}\right] + \left[\frac{n}{P^3}\right] + \cdots$$

$$So, Ans = \left[\frac{1000}{3}\right] + \left[\frac{1000}{9}\right] + \left[\frac{1000}{3^3}\right] + \left[\frac{1000}{3^4}\right] + \left[\frac{1000}{3^5}\right] + \left[\frac{1000}{3^6}\right]$$

$$= 496.$$

15. The sum if the series $\frac{\mathbf{1}}{\mathbf{5.6}} + \frac{\mathbf{1}}{\mathbf{6.7}} + \cdots + \frac{\mathbf{1}}{\mathbf{104.105}}$ is
    (a) $\frac{\mathbf{22}}{\mathbf{21}}$ (b) $\frac{\mathbf{20}}{\mathbf{21}}$
    (c) 1 (d) None

Solution:- (b) $\frac{1}{5.6} + \frac{1}{6.7} + \cdots + \frac{1}{104.105} = \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{104} - \frac{1}{105}\right) = \frac{1}{5} - \frac{1}{105} = \frac{20}{21}$.

16. The sum of the series $\frac{\mathbf{1}}{\mathbf{1+\sqrt{2}}} + \frac{\mathbf{1}}{\mathbf{\sqrt{2}+\sqrt{3}}} + \frac{\mathbf{1}}{\mathbf{\sqrt{3}+\sqrt{4}}} + \cdots + \frac{\mathbf{1}}{\mathbf{\sqrt{99}+\sqrt{100}}}$ is
    (a) 1 (b) 9
    (c) 10 (d) None

Solution:- (b) S = $\frac{1-\sqrt{2}}{1-2} + \frac{\sqrt{2}-\sqrt{3}}{2-3} + \frac{\sqrt{3}-\sqrt{4}}{3-4} + \cdots + \frac{\sqrt{99}-\sqrt{100}}{99-100} = -1 + 10 = 9.$

17. The last digit of $\mathbf{43^{17}}$ is
    (a) 3 (b) 7
    (c) 1 (d) None

Solution:- (a) 43 ≡ 3 (mod 10)

$(43)^{17} \equiv 3^{17}$(mod 10) ;

i.e. last digit of 43^{17} is the last digit of 3^{17} .

Now, $3^4 \equiv 1\ (mod\ 10)$

$(3^4)^4 \equiv 1\ (mod\ 10)$

So, the last is 3.

18. The remainder when $(\mathbf{2222})^{\mathbf{5555}}$ is divisible by 7 is
    (a) 3 (b) 5
    (c) 7 (d) 9

Solution:- (b) 2222 ≡ 3 mod 7

$(2222)^3$≡ 27 mod 7

$(2222)^3$≡ (-1) mod 7

$(2222)^{5553} \equiv (-1)^{1851}$ mod 7

$(2222)^2$ ≡ 9 mod 7

$(2222)^{5553}$ ≡ -9 mod 7$-$ 9 ≡ 5 mod 7.

19. The unit digit of $(\mathbf{2337})^{\mathbf{2337}}$ is

(a) 3 (b) 5

(c) 7 (d) 9

Solution:- (c) Last digits

$$(2337)^1 \equiv 7\ (mod\ 10)$$

$$(2337)^2 \equiv 9\ (mod\ 10)$$

$$(2337)^3 \equiv 3\ (mod\ 10)$$

$$(2337)^4 \equiv 1\ (mod\ 10)$$

$$(2337)^5 \equiv 5\ (mod\ 10)$$

2337 ≡ 1 (mod 4)

$((2337)^{584})^4.2337 \equiv 7(mod\ 10)$

Unit digit is 7.

20. The least positive residue in $\mathbf{2^{41}}$(mod 23) is

(a) 3 (b) 5

(c) 7 (d) 9

Solution:- If a ≡ b (mod m) then b is said to be the residue of a modulo m.

If a ≡ b (mod m) then $a^n \equiv b^n \forall\, n \in I^+$. But the 23 is a prime & 2 is a prime to 23.

By Fermat's theorem, $2^{22} \equiv 1\ (mod\ 23)$

$$2^{44} \equiv 1\ (mod\ 23)$$

$$2^{44} \equiv 24\ (mod\ 23)$$

$$2^{41}.8 \equiv 3 \times 8\ (mod\ 23)$$

$$2^{41} \equiv 3\ (mod\ 23)$$

So, the least positive residue is 3.

21. The remainder when 4(29)! +5! Is divisible by 31 is

(a) 3 (b) 5

(c) 7 (d) None

Solution :-(d) Wilson's theorem states that " If p is a prime then (p -1)!+ 1 ≡ 0(mod p). The converse of this theorem is also true.

So, by Wilson's theorem,

(30)! +1 ≡ 0 (mod 31), since 31 is prime.

(31 -1)(29)! +1 ≡ 0 (mod 31)

⇨29! +1 ≡ 0(mod 31)

⤇29! -1 ≡ 0 (mod 31)

⤇ 4(29)! -4≡ 0 (mod 31)

⤇ 4(29)! -4 +124 ≡ 0 (mod 31)

⤇ 4(29)! +120 ≡ 0 (mod 31)

⤇ 4(29)! +5! ≡ 0(mod 31)

22. The smallest positive integer that has remainder 4, 3 and 1 when divided by 5, 7, and 9, respectively, is

(a) 211 (b) 201

(c) 199 (d) 189

Solution:- X ≡ 4 (mod 5)

X ≡ 3 (mod 7)

X ≡ 1 (mod 9)

Let X = 4 +5t, t ∈ℤ

So, 4 + 5t ≡ 3 (mod 7) ⇨ 5t +1 ≡ 0 (mod 7)

Let, t = 4 + 7u, u ∈ℤ

So, X = 4 +5 (4+ 7u) = 24+ 35u.

24 +35u ≡ 1(mod 9) ⤇ 35u + 23 ≡ 0 (mod 9)
⤇ 36u –u ≡ -23 (mod 9) ⤇- u ≡ -23 (mod 9)

⤇ 23 –u ≡ 0 (mod 9)

∴ u = 5 +9v (let), v ∈ ℤ

∴ X = 24 + 35 (5+ 9v) = 199+ 315v.

So, the smallest positive solution is 199.

23. The last digit of 3^{80} is

(a) 3 (b) 9
(c) 7 (d) 1

Solution:- (d) $3^1 \equiv 3\ (mod\ 10)$

$3^2 \ \equiv 9\ (mod\ 10)$

3^3 ≡ 7 (mod 10)

3^4 ≡ 1 (mod 10)

3^5 ≡ 3 (mod 10)

3^{80}=$(3^{16})^5 = (3^{3\times5}.3)^5 = (3^{25})^3.3^5 \equiv 3^3.3\ (mod\ 10)$ ≡ 1 (mod 10)

More explicitly, we can write :-

Ends in 9 : $3^{10}, 3^{30}, 3^{50}, 3^{70}, 3^{90}, ...$;

Ends in 1 : $3^{20}, 3^{40}, 3^{60}, 3^{80}, 3^{100}, ...$

24. The last digit of 9^{9^9} is

(a) 1 (b) 7 (c) 9 (d) none

Solution:- The last digit of 9^3 is 9.

The last digit of 9^9 is 9.

Thus, the last digit of 9^{9^9} is 9.

LINEAR ALGEBRA

25. Which of the following sets is not LIN?
   (a) $\{\mathbf{1}, \boldsymbol{X}, \mathbf{1}+\boldsymbol{X}+\boldsymbol{X}^{\mathbf{2}}\}$ in a vector space of all polynomials over the field of real numbers.
   (b) $\{\mathbf{1}, \boldsymbol{X}, \boldsymbol{X}^{\mathbf{2}}, \dots\}$ in a vector space of all polynomials over the field of real numbers.
   (c) $\{(\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{0}), (\mathbf{0},\mathbf{1},-\mathbf{1},\mathbf{0}), (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{3})\}$ in $\boldsymbol{V}_{\mathbf{4}}(\boldsymbol{R})$.
   (d) $\{(\mathbf{1},\mathbf{2},\mathbf{1}), (\mathbf{3},\mathbf{1},\mathbf{5}), (\mathbf{3},-\mathbf{4},\mathbf{7})\}$ in $\boldsymbol{V}_{\mathbf{3}}(\boldsymbol{R})$.

Solution:- (a) Let a, b, c be scalars such that

a(1) + bX +c $(1+X+X^2)$ = 0 ⇒(a+ c) + (b +c)X + cX^2= 0 ⇒a+ c = 0, b+ c = 0, c = 0

⇒a = 0, b = 0, c = 0.

∴ The vectors 1, X, $1+X+X^2$are LIN over the field of real no's.

$$(b) a_0 1 + a_1 X + a_2 X^2 + \cdots = 0$$

By definition of equality of two polynomials $a_0 = a_1 = a_2 = \cdots = 0$

∴ The vectors are LIN.

(c) a(1,1, 0, 0) + b(0, I, -1, 0) + c(0, 0, 0, 3) = 0

$\Rightarrow \begin{cases} a = 0 \\ a + b = 0 \\ -b = 0 \\ 3c = 0 \end{cases} \Rightarrow a = b = c = 0$ is the only solution.

∴The vectors are LIN.

$(d) \begin{vmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{vmatrix} = 0 \Rightarrow$ rank (A) <3 ⇒ the set of vectors are linearly dependent.

26. The eigenvalue of $\boldsymbol{A^4}$, where A = $\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{9} & \mathbf{4} & \mathbf{1} \\ \mathbf{3} & \mathbf{1} & \mathbf{1} \end{pmatrix}$, is

(a) 3, 4, 5, (b) 1, 2, 3, (c) 5, 6, 7, (d) none

Solution:- (d) | A –λI| = 0

$\Rightarrow det \begin{bmatrix} 1-\lambda & 0 & -1 \\ 9 & 4-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} = 0 \Rightarrow$
$(1-\lambda)(2-\lambda)(3-\lambda) = 0 \Rightarrow \lambda = 1, 2, 3$

∴For the matrix A^4, the eigen values are: $(1)^4 = 1, (2)^4 = 16, (3)^4 = 81.$

27. Let, $\boldsymbol{M_{2\times 2}(R)}$ be the vector space of all $2\times \mathbf{2}$ matrices over R and

Let $\boldsymbol{w_1} = \left\{\begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \\ \mathbf{0} & \boldsymbol{x} \end{bmatrix} : \boldsymbol{x}, \boldsymbol{y}, \in \mathbb{R}\right\}$ & $\boldsymbol{w_2} = \left\{\begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \\ \boldsymbol{z} & \mathbf{0} \end{bmatrix} : \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}\right\}$ then dim $(\boldsymbol{w_1} \cap \boldsymbol{w_2})$ is

(a) 2 (b) 3
(c) 4 (d) 1

Solution:- (d) $w_1 \cap w_2 = \left\{\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} : x, y \in \mathbb{R}\right\}$; dim $(w_1 \cap w_2)$ = 1.

28. Let T:$\boldsymbol{R^n}$ ⟶> $\boldsymbol{R^m}$, m >n, be a linear transformation.

Consider the following statements about T:

(i)T can be one to one (ii) T can onto (iii) dim (T($\boldsymbol{R^n}$))≥ n

(A) Only (i) is true (B) only (ii) is false
(C) only (ii) is true (D) only (iii) is true

Solution:- (B) T:R^n ⇛ R^m, m >n.

i.e. no. Of elements in domain < no. of elements in range.

i.e. T can be one to one is true statement.

But T can be onto is false as m > n.

dim(T(R^n))≥ n is also a true statement as n < m.

29. If A = $\begin{bmatrix} \boldsymbol{a} & \boldsymbol{-b} \\ \boldsymbol{b} & \boldsymbol{a} \end{bmatrix} \in \boldsymbol{k^{2\times 2}}, \boldsymbol{b} \neq \mathbf{0}$ then A has eigen values if k is

(A) R (B) C
(C) Q (D) All above

Solution:- (D) Ch. Equation | A- λI| = 0

$\Rightarrow \begin{bmatrix} a-\lambda & -b \\ b & a-\lambda \end{bmatrix} = 0$

$\Rightarrow \lambda^2 - 2a\lambda + (a^2 + b^2) = 0$

$$\Rightarrow \lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2+b^2)}}{2} = a \pm b$$

Hence the eigenvalues of A be a + b and a – b.

Then A has eigenvalues when k = R, C, Q

30. The no. of solution of the system of equation

2x +y –z = 7

X -3y +2z = 1

X +4y -3z = 5

(a) Unique solution (b) no solution (c) many solution (d) exactly two solution

Solution:- (b) $\begin{pmatrix} 2 & 1 & -1 \\ 1 & -3 & 2 \\ 1 & 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 5 \end{pmatrix}$

$$\Rightarrow A\underset{\sim}{X} = \underset{\sim}{b}$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -3 & 2 \\ 1 & 4 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -1 \\ -1 & -4 & 3 \\ 1 & 4 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -1 \\ 1 & 4 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank (A) = 2

$$\begin{bmatrix} 2 & 1 & -1 & :7 \\ 1 & -3 & 2 & :1 \\ 1 & 4 & -3 & :5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 & :7 \\ -1 & -4 & 3 & :-6 \\ 1 & 4 & -3 & :5 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 1 & -1 & :+7 \\ -1 & -4 & 3 & :-6 \\ 0 & 0 & 0 & :-1 \end{bmatrix}$$

Rank (A ⋮ b) = 3

∴ rank (A) ≠ rank (A ⋮ b)

The system has no solution.

31. The equations have:

x –y +2z =4

3x +y +4z = 6

x +y +z = 1

(a) Unique solution (b) infinite solution (c) no solution (d) none of these

Solution:- (b) A = $\left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 3 & 1 & 4 & 6 \\ 1 & 1 & 1 & 1 \end{array}\right] \sim$
$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 4 & -2 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$

∴ rank (A ⋮ B) = 2 = rank (A) < 3

∴ the given system of equations are consistent & have infinite number of solutions.

32. If the matrix A = $\begin{bmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} \end{bmatrix}$, then find the matrix $\boldsymbol{A^{n+1}}$

(a) $2\boldsymbol{A^n}$ (b) $\boldsymbol{2^n A}$ (c) $\boldsymbol{2^{n-1} A}$ (d) A

Solution:- (b) |A- λI| = 0

$$\Rightarrow \begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 - 2\lambda = 0$$

So $A^2 - 2A = 0$

$\Rightarrow A^3 = 2A^2 = 4A$

$\therefore A^{n+1} = 2^n A$

33. The following system of linear equation is constant if α, β don't equal to

x +3y +z = 3

2x +3y +5z = 4

4x +9y +αz = β

(a) (1, 3) (b) (5, 10)
(c) (7, 10) (d) None

Solution:- (c) Given system of equation can be expressed as AX = B

Argumented matrix [A | B] =

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 2 & 3 & 5 & 4 \\ 4 & 9 & \alpha & \beta \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & -3 & 3 & -2 \\ 0 & -3 & \alpha-4 & \beta-12 \end{array}\right] \begin{matrix} R_2 \leftrightarrow R_2 - 2R \\ R_3 \leftrightarrow R_3 - 4R \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & -3 & 3 & -2 \\ 0 & 0 & \alpha-7 & \beta-10 \end{array}\right] R_3 \leftrightarrow R_3 - R_2$$

Given system of linear equation be constant

i.e. rank (A) = rank (A| B) = 3

i.e. $\alpha - 7 \neq 0$, $\beta - 10 \neq 0$

$\Rightarrow \alpha \neq 7$ & $\beta \neq 10$

34. Let x & y in $\boldsymbol{R^n}$ be non zero row vectors from the matrix A = x$\boldsymbol{y^T}$, wherer $\boldsymbol{y^T}$ denote the transpose of y. Then the rank of A is
(a) 0 or 1 (b) 2
(3) at least n/2 (d) none

Solution:- (a) A = xy^T

Then A is a matrix of order 1×1.

If A is non -zero then rank (A) = 1

If A i zero mtx. Then rank (A) = 0.

35. Which of the following is false?
(a) The eigenvalues of Hermition matrix are real.
(b) The eigenvalues of real symmetric matrix are real.
(c) The eigenvalues of skew Hermition matrix are real.
(d) The eigenvalues of unitary matrix may be real.

Solution:- (c) As the eigenvalues of skew-Hermition matrix are either pure imaginary or zero.

36. Let $\boldsymbol{M_n(R)}$ be the set of n×m matrices with real entries, if all A ∈ $\boldsymbol{M_n(R)}$ have both negative and positive eigen values then the set is having
(a) Positive semi definite matrices only
(b) Positive & negative semi definite matrices
(c) Negative definite matrices only
(d) Indefinite matrices

Solution:- (d) A positive and positive semi definite matrices have positive eigenvalues only.

Negative definite matrices have negative eigenvalues only.

But indefinite matrices have both positive and negative eigenvalues.

37. Which of the following is true?
(a) The matrix $\begin{bmatrix} \mathbf{2} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} \end{bmatrix}$ is diagonalisable
(b) The matrix $\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{5} \end{bmatrix}$ is diagonalisable
(c) The matrix $\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{5} \end{bmatrix}$ is diagonalisable
(d) The matrix $\begin{bmatrix} \mathbf{5} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}$ is not diagonalisable

Solution:- (b) We know that any matrix (2 ×2) is said to be diagonalisable if it has two different eigen values.

$\begin{bmatrix} 1 & 0 \\ 1 & 5 \end{bmatrix}$ is lower triangular matrix, its diagonal entries are its eigen values since they are distinct .

⇒the matrix is said to be diagonalisable.

38. If V is vector space on the field ℤ / 3ℤ and $\boldsymbol{dim}_{\frac{\mathbb{Z}}{3\mathbb{Z}}}(\boldsymbol{V}) = \mathbf{3}$ then V has
    (a) 27 elements (b) 9 elements
    (c) 30 elements (d) 15 elements

Solution:- (a) since, dim V = 3 and ℤ / 3ℤ = {0, 1, 2}

So, they are 3^3 elements in V which can be expressed as the linear combination of elements of basis.

39. If V is a vector space over the field ℤ /7ℤ and $\boldsymbol{dim}_{\frac{\mathbb{Z}}{7\mathbb{Z}}}(\boldsymbol{Y}) = \mathbf{2}$ then V has
    (a) 49 elements (b) 14 elements
    (c) 128 elements (d) None

Solution:- (a) Since, dim V = 2 and ℤ /7ℤ = {0, 1, 2, 3, 4, 5, 6}

So, there may be 7^2 elements which can be expressed which can be expressed as the linear combination of elements of basis & their coefficient could only be {0, 1, 2, 3, 4, 5, 6}.

40. If V is the real vector space of all mapping from ℝ to ℝ, $\boldsymbol{V_1} = \{\boldsymbol{f} \in \boldsymbol{V} \mid \boldsymbol{f}(-\boldsymbol{X}) = \boldsymbol{f}(\boldsymbol{X})\}$ **and** $\boldsymbol{V_2} = \{\boldsymbol{f} \in \boldsymbol{V} \mid \boldsymbol{f}(-\boldsymbol{X}) = -\boldsymbol{f}(\boldsymbol{X})\}$,
    (A) Neither $\boldsymbol{V_1}$ nor $\boldsymbol{V_2}$ is a subspace of V.
    (B) $\boldsymbol{V_1}$ is a subspace of V, but not $\boldsymbol{V_2}$.
    (C) $\boldsymbol{V_2}$ is a subspace of V, but not $\boldsymbol{V_2}$.
    (D) $\boldsymbol{Both}\ \boldsymbol{V_1}, \boldsymbol{V_2}$ are subspaces of V.

Solution:- (D) The necessary and sufficient condition for a non-empty subset W of a vector space V(F) to be subspace of V is

a, b ϵ F and for all α, β ∈W ⇒ aα + bβ ∈W

λa ∈W

Hence both V_1, V_2 satisfied these conditions.

41. Let P be an n × m idempotent matrix, i.e. $\boldsymbol{P^2} = \boldsymbol{P}$. Which of the following is FALSE?
    (a) $\boldsymbol{P^T}$ is idempotent.
    (b) The possible eigenvalues of P can be zero.
    (c) The non-diagonal entries of P can be zero.
    (d) There may be infinite no. of n× m non- singular matrices that one idempotent.

Solution:- (d) Given P be an n × m idempotent matrix s. t $P^2 = P$

If P is idempotent then P^T is also idempotent P (P –I) = 0

⇒the possible eigenvalues of P are 0 and 1.

⇒non-diagonal entries of P can be zero.

So, (d) is false.

42. If A is a 3 ×3 non-zero matrix such that A^2= 0, then the number of non-zero eigen values of A is
    (a) 0 (b) 1
    (c) 2 (d) 3

Solution:- (a) $A^2 = 0$

A is nilpotent matrix, so all of its eigenvalues will be zero.

43. Let T : $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be a linear transformation defined by T (x, y, z) = (x +)
    (a) 0 (b) 1
    (c) 2 (d) 3

Solution:- (b) T(x, y, z) = (x +y, x –z) Now, (x +y, x –z) = (0, 0)

⇒ (x +y) = 0; (x –z) = 0 ⇒x = -y = z

⇒ N(T) = (a, -a, a) | a∈ Rζ

i.e. nullity = 1.

ABSTRACT ALGEBRA

44. Let G be the set of cube roots of unity. Then under multiplication of complex numbers
   (a) G is a group of finite order
   (b) G is an abelian group
   (c) G is a cyclic group
   (d) None of the above.

Solution:-

(c) Here G = {1, W, W^2}

So, G is an abelian cyclic group of order 3, since

	1	w	w^2
1	1	w	w^2
W	w	w^2	1
w^2	w^2	1	w

45. The irreducible polynomials in C[X] are the polynomials of degree
   (a) 0 (b) 1
   (c) 2
   (d)None

Solution:- (b) The polynomials of degree 0 are the invertible element of C[X].

By the fundamental theorem of algebra, any polynomial of positive degree has a root in C and hence a linear factor. Therefore, any polynomial of degree greater than 1 are reducible and those of degree 1 are irreducible.

46. Which of the following statement is false :
   (a) The polynomial $X^3 - X + 1$ is irreducible in $\mathbb{Z}$ / 2$\mathbb{Z}$ [X]
   (b) The polynomial $X^2 - 3$ is irreducible in Q[X].
   (c) The polynomial $X^2 + 1$ is irreducible in $\mathbb{Z}$ / 5$\mathbb{Z}$ [X].
   (d) The polynomial $X^2 + 1$ is irreducible in $\mathbb{Z}/7\mathbb{Z}$ [X].

Solution:- (c) The polynomial $X^2 + 1$ is reducible in $\mathbb{Z}/5\mathbb{Z}$[X].

As $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\} and\ f(x) = x^2 + 1$.

f(3) = 10 ≡ 0 (mod 5)

$\Longrightarrow X^2 + 1$ is reducible in $\mathbb{Z}/5\mathbb{Z}[X]$.

47. Let f : G ⟶ H be a group homomorphism from a group G into a group H with kernel K. If O(G) = 75, O(H) = 45, O(K) = 15. Then the order of the image f(G) is :

(a) 3 (b)5
(c) 15 (d) 45

Solution:- (b) $f(G) \cong \frac{G}{K}$.

$O\{f(G)\} = O\left(\frac{G}{K}\right) = \frac{O(G)}{O(K)} = \frac{75}{15} = 5.$

48. Which of the following is a cyclic group?

(i) $\mathbb{Z}_{12} \times \mathbb{Z}_9$ (ii) $\mathbb{Z}_{10} \times \mathbb{Z}_{85}$ (iii) $\mathbb{Z}_{22} \times \mathbb{Z}_{21} \times \mathbb{Z}_{65}$ (iv) None

Solution:- (iii) Any group $\mathbb{Z}_p \times \mathbb{Z}_r$ is said to be cyclic if the greatest common divisor (GCD) of p and r is equal to 1. Similarly, for $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ is called cyclic if GCD between any two is equal to one.

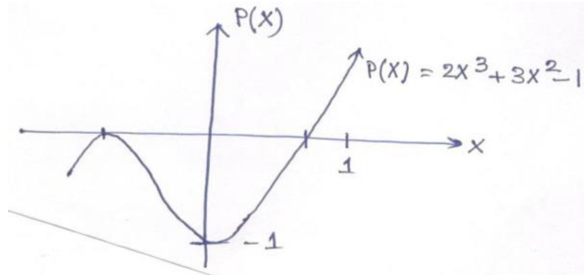
(i) GCD of (12, 9) = 3
(ii) GCD of (10, 85) = 5
(iii) GCD of (22, 21) & (21, 65) & (22, 65) = 1.

$\Longrightarrow \mathbb{Z}_{22} \times \mathbb{Z}_{21} \times \mathbb{Z}_{65}$ is a cyclic group.

49. Factor P(X) = $2X^3 + 3X^2 - 1$ in Q[X]

(i) $(X-1)^2(2X+1)$ (ii) $(X+1)^2(2X-1)$ (iii) $(X-1)^2(2X-1)$ (iv) $(X+1)^2(2X+1)$

Solution:- (ii)



X	-1	0	½	1	2
P(X)	0	-1	0	4	27

∴ P(X) = $(X+1)^2(2X-1)$.

50. Any group of 2p, where p is a prime number has a normal subgroup of order p, then the index of subgroup H in G is

(a) P (b) 2
(c) $\frac{P^2}{2}$ (d) none

Solution:- (b) Given that O(G) = 2p.

Since p is prime, and

By Cauchy's theorem, G has an element of order p then the cyclic group, H = {a, $a^2, \ldots, a^P$}is a subgroup of order p.

∴ The index of H in G is = $\frac{O(G)}{O(H)} = \frac{2p}{p} = 2.$

51. If 1, $\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_{11}$ are the 12 roots of unity forming the cyclic group under multiplication. Then $\mathbb{Z}_9$ generates a cyclic sub group of the above containing:

(a) 12 elements (b) 9 elements (c) 8 elements (d) none

Solution:- (d) The integral divisors of 9 are 1, 3, 9.

∴ All the elements of order 1, 3, 9 will give subgroups.

So, $\{\mathbb{Z}_9\}$ has the subgroups {e}, $(\mathbb{Z}_1)$, $(\mathbb{Z}_3)$, $(\mathbb{Z}_9)$.

So there is 4 elements.

52. The n equal rotations of a regular polygon of n sides
    (a) From an abelian but not cyclic group
    (b) From a cyclic group
    (c) Don't from a cyclic group
    (d) From non-abelian non-cyclic group.

Solution:- (b) The rotations are the generators of the group G.

Hence, G must be a cyclic group.

53. Let G = {z ∈c : | z| = 1}, then under multiplication of complex numbers
    (a) G is a group of order (finite)
    (b) G is a group of infinite
    (c) G is a cyclic group
    (d) None of the above.

Solution:- (c) Let G ={z ∈c : | z| = 1} and $z_1 z_2 \in G$.

Then $z_1 z_2$∈G ⤇ z_1| = 1, | z_2| = 1.

⤇ $z_1 z_2$| = | z_1| | z_2| = 1.

∴ G is closed for multiplication.

And ∃ inverse of every element in G. Hence, G is multiplicative group.

REAL ANALYSIS

54. If f and F be both continuous in [a, b] and are derivable in (a, b) and F'(x) = f'(x) ∀ x in (a, b) then f(x) and F(x) differ b
    (A) 1 in [a, b] (B) x in [a, b] (C)constant in [a, b] (D) none

Solution:- (c) Since they are continuous

f'(x) = F'(x)

Let $\phi(\mathrm{x})= \mathrm{f(x)} - \mathrm{F(x)}$

$\Rightarrow \phi'(\mathrm{x}) = \mathrm{f'(x)}\text{-}\mathrm{F'(x)}= 0$ i.e. $\phi\,(\mathrm{x}) =$ constant.

55. Let $\boldsymbol{f_X(x) = n\, sin^{2n+1}\, x \cos x}$, then the value of $\boldsymbol{\underset{n\to\infty}{\mathrm{Lt}} \int_0^{\frac{n}{2}} f_n(x)\, dx -}$ $\boldsymbol{\int_0^{\pi/2} \underset{n\to\infty}{\mathrm{Lt}}\ \big(f_n(x)\big) dx}$ is
    (a) ½ (b) 0
    (c) -½ (d) -∞

Solution:- (d) Value =

$$\underset{n\to\infty}{\mathrm{Lt}} \left[\frac{n\ \Gamma\left(\frac{2n+1+1}{2}\right)\Gamma\left(\frac{1+1}{2}\right)}{2\ \Gamma\left(\frac{2n+1+1+2}{2}\right)}\right] - \int_0^{\pi/2} \underset{n\to\infty}{\mathrm{Lt}}\ \ n sin^{2n+1}\, x \cos x\, dx$$

$$= \underset{n\to\infty}{\mathrm{Lt}} \frac{n}{2}\frac{\Gamma n+1}{\Gamma n+2} - \int_0^{\pi/2} \infty \ = \underset{n\to\infty}{\mathrm{Lt}} \frac{n}{2}\frac{1}{n+1} - \infty = -\infty\,.$$

56. If $\{\boldsymbol{s_n}\}$ be a convergent sequence of positive numbers ∋ $\boldsymbol{s_n = \frac{1}{2}(s_{n-1} + s_{n-2})} \forall \boldsymbol{n} \geq$ **2 then** $\boldsymbol{\underset{n\to\infty}{\mathrm{Lt}}\ s_n} = ?$
    (a) $\boldsymbol{\frac{1}{3}(s_2 + s_3)}$ (b) $\boldsymbol{\frac{1}{3}(s_3 + \frac{1}{2}s_1)}$ (c) $\boldsymbol{\frac{1}{3}(s_1 + 2s_2)}$ (d) None

Solution:- (c) Let $\underset{n\to\infty}{\text{Lt}}\ s_n = l$

$$s_n = \frac{1}{2}(s_{n-1} + s_{n-2})$$

$$s_3 = \frac{1}{2}(s_2 + s_1)$$

$$s_4 = \frac{1}{2}(s_3 + s_2)$$

$$s_{k-1} = \frac{1}{2}(s_{k-2} + s_{k-3})$$

$$s_k = \frac{1}{2}(s_{k-1} + s_{k-2})$$

Adding all these, $s_k + \frac{1}{2}s_{k-1} = \frac{1}{2}(s_1 + 2s_2)$

$\Rightarrow\ l + \frac{1}{2}l = \frac{1}{2}(s_1 + 2s_2)$

$\Rightarrow\ l = \frac{1}{3}(s_1 + 2s_2)$

57. For every function f : [0, 1] ⟶ℝ, which is twice differentiable and satisfies f' (x) ≥1 ∀ x ∈ [0, 1], we must have
    (a) f''(x) ≥ 0∀ x∈[0, 1
    (b) f(x) ≥ x ∀ x ∈ [0, 1]
    (c) f$(\boldsymbol{x_2}) - \boldsymbol{x_2} \leq \boldsymbol{f}(\boldsymbol{x_1}) - \boldsymbol{x_1}\ \forall\ \boldsymbol{x_1}, \boldsymbol{x_2}\ \in [\boldsymbol{0}, \boldsymbol{1}]\boldsymbol{with}\ \boldsymbol{x_2} \geq \boldsymbol{x_1}$
    (d) f$(\boldsymbol{x_2})$ -$\boldsymbol{x_2}\ \geq \boldsymbol{f}(\boldsymbol{x_1}) - \boldsymbol{x_1}\ \forall\ \boldsymbol{x_1}, \boldsymbol{x_2}\ \in [\boldsymbol{0}, \boldsymbol{1}]$ with $\boldsymbol{x_2} \geq \boldsymbol{x_1}$

Solution:- (d) Taylor's formula gives

f(x_2)- f$(x_1) > (x_2 - x_1)f'\left(\frac{x_1+x_2}{2}\right) \forall\ x_1, x_2 \in [0,1]\ with\ x_2 >\ x_1$

if $x_2 >\ x_1$, then $f'\left(\frac{x_1+x_2}{2}\right) = f'(x_1) \geq 1\ \forall x_1\ \in [0,1]$

$$\begin{aligned} f(x_2) - f(x_1) &> (x_2 - x_1)f'(x_1) \\ &\geq (x_2 - x_1)\forall x_2 \\ &\geq x_1\ \forall\ x_1, x_2 \in [0,1] \end{aligned}$$

$$\Rightarrow f(x_2) - x_2 \geq f(x_1) - x_1.$$

58. A function f is defined as {0, 1}, by f(x) = $\frac{\boldsymbol{1}}{\boldsymbol{x}}\ \forall\ \frac{\boldsymbol{1}}{\boldsymbol{x}} > \boldsymbol{x} > \frac{\boldsymbol{1}}{\boldsymbol{n+1}}$ for all n = 1, 2, 3, ... if given that f ∈ R {0, 1} then evaluate $\int_{\boldsymbol{0}}^{\boldsymbol{1}} \boldsymbol{f}(\boldsymbol{x})\ \boldsymbol{dx}$

$(\boldsymbol{A})\ \frac{\boldsymbol{\pi^2}}{\boldsymbol{6}}$	(B) $\frac{\boldsymbol{\pi^2}}{\boldsymbol{6}} + \boldsymbol{1}$
(C) $\frac{\boldsymbol{\pi^2}}{\boldsymbol{6}} - \boldsymbol{1}$	(D) none

Solution:- (C) $\int_0^1 f(x)\ dx = \underset{n\to\infty}{\text{Lt}}\ \sum_{r=1}^{n} \frac{1}{r}\left\{\frac{1}{r} - \frac{1}{r+1}\right\}$

$$= \underset{n\to\infty}{\text{Lt}}\ \left\{\left(1 - \frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \frac{1}{n}\left(\frac{1}{n} - \frac{1}{n+1}\right)\right\}$$

$$= \underset{n\to\infty}{\text{Lt}}\ \left\{\left(1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right) - \left(1 - \frac{1}{n+1}\right)\right\}$$

$$= \frac{\pi^2}{6} - 1.$$

59. Let f be a differentiable function defined on [0, 1],
    k ∈ (0, 1) ∋ f(x) < f(k) = f(0) ∀ x ∈ [0, 1], x ≠ k, then
(a) f'(k) = 0 and f' (0)= 0
    (b) f'(k) = 0 and f(0) = 0

(c) $f'(k) = 0$ and $f'(0) \leq 0$
(d) $f'(k) > 0$ and $f'(0) \leq 0$.

Solution:- (c) f(k) = f(0) is maximum of f in [0, 1]

$\therefore f'(k) = 0$

$f'(0) = \underset{h\to 0}{\mathrm{Lt}} \frac{f(h)-f(0)}{h}$

Since $f(h) < f(0)\ \forall h \in [0, 1]$

$\Rightarrow f(h) - f(0) < 0$

$\Rightarrow \frac{f(h)-f(0)}{h} < 0 \quad [\because h > 0]$

i.e. $f'(0) \leq 0$.

60. Let $\{\boldsymbol{a_n}\}$ ***and*** $\{\boldsymbol{b_n}\}$ be sequence of real nos. Defined as $\boldsymbol{a_1 = 1}$ ***and for*** $\boldsymbol{n \geq 1, a_{n+1} = a_n + (-1)^n 2^n, b_n = \frac{2a_{n+1}-a_n}{a_n}}$ then

(a) $\{\boldsymbol{a_n}\}$ converges to zero and $\{\boldsymbol{b_n}\}$ is a Cauchy sequence

(b) $\{\boldsymbol{a_n}\}$ converges to non-zero and $\{\boldsymbol{b_n}\}$ is a Cauchy sequence.

Solution :- (b) $a_1 = 1$

$$a_2 = a_1 + \frac{(-1)}{2} = \frac{1}{2}$$

$$a_3 = a_2 + \frac{1}{2^2} = \frac{3}{4} \quad \left[a_2 = \frac{1}{2}\right]$$

$$a_4 = a_3 - \frac{1}{2^3} = \frac{5}{8} \quad \left[a_3 = \frac{3}{4}\right]$$

$$a_5 = a_4 + \frac{1}{2^4} = \frac{11}{16} \quad \left[a_4 = \frac{5}{8}\right]$$

$$\vdots \qquad \vdots$$

And so on.

Since $a_1, a_3, a_5, \ldots\ldots$ is a decreasing sequence and $a_2, a_4, a_6, \ldots.$ is a increasing sequence.

$\{a_n\}$ converges to $\frac{1}{3}(a_1 + 2a_2) = \frac{1}{3}\left(1 + \frac{2}{2}\right) = \frac{2}{3}$.

Again, $b_n = \frac{2a_{n+1}-a_n}{a_n}$

$$b_1 = 0$$

$$b_2 = 2$$

$$b_3 = \frac{2}{3}$$

$$b_4 = \frac{6}{5}$$

$$b_5 = \frac{10}{11}$$

$$\vdots$$

$$\vdots$$

And so on.

$b_1, b_3, b_5 \ldots.$ are increasing sequence & $b_2, b_4, b_6, \ldots.$ are decreasing sequence the $\{b_n\}$ converges to limit

$$\frac{1}{3}(b_1 + 2b_2) = \frac{1}{3}(0 + 2.2) = 4$$

$\therefore \{b_n\}$ is a Cauchy sequence.

61. On x = c [0, 1] define T: x ⟶ > x by T(f(x)) $= \int_0^x \boldsymbol{f(t)\ dt}\ \forall$ f in x then
   a) T is one-one and onto b) T is one- one but not onto c) T is not one-one but onto d) T is neither one-one nor onto.

Solution:- (a) T(f(x)) = $\int_0^x f(t)\ dt$ ∀ f in x

⇒ Let $x_1, x_2 \ \in X$

S.t., T(f(x_1)) = T (f(x_2))

$\Rightarrow \int_0^{x_1} f(t)\, dt = \int_0^{x_2} f(t)\, dt.$

$\Rightarrow x_1 = x_2$

i.e. T is one-one

for each T(f(x)) , ∃ only one x ∈ X,

S.t, T(f(x)) = $\int_0^x f(t)\ dt$

⇒T is onto.

62. Evaluate $\int_0^5 f(x)\, dx, if\ f(x) = \begin{cases} 0, if\ 0 \le x \le 1 \\ 1, \{1 \le x < 2\} \cup \{3 \le x < 4\} \\ 2, \{2 \le x < 3\} \cup \{4 \le x < 5\} \end{cases}$

By using Riemann & Lebesgue definition of the integral

a) R $\int_0^5 f(x)\, dx > L \int_0^5 f(x)\, dx$

b) R$\int_0^5 f(x)\, dx < L \int_0^5 f(x)\, dx$

c) R$\int_0^5 f(x)\, dx = L \int_0^5 f(x)\, dx$

d) None.

Solution:- c) Using Riemann definition of the integral (where the subdivision is taken of the segment [0, 5] by the division points $x_0, x_1, \dots, x_n$ on X-axis) the upper & lower Riemann sums tend to the common value.

0(1- 0) +1 (2- 1) + 2(3- 2) + 1 (4 -3) +2 (5 -4) = 6

(Since the function is constant an each of the subintervals)

∴ R $\int_0^5 f(x)\, dx$= 6.

Evaluating the Lebergue integral [0, 2+ δ[, δ >0],

We get

∴ L$\int_0^5 f(x)\, dx = 6.$

0[1- 0]+ 1 [(2 -1)+ (4 -3)]+ 2[(3- 2) + (5 -4)] = 6.

63. Let f be an one to one function from the closed interval [-1, 1] to the set of real numbers ℝ, then

a) f must not be onto

b) Range of f must contain a rational number.

c) Range of f must contain an irrational no.

d) Range of f must contain both rational and irrational nos.

Solution:- d) y = $\sin^{-1} x$ (let)

x ∈ [-1 , 1]

y ∈ [$-\frac{\pi}{2}, \frac{\pi}{2}$]

⇒ range of f must contain both rational and irrational nos. f is onto here.

64. The sequence $\sqrt{11}, \sqrt{11 + \sqrt{11}}, \sqrt{11 + \sqrt{11} + \sqrt{11}}, \dots.$ converges to

a) $\frac{1+\sqrt{43}}{2}$ b) $\frac{1+\sqrt{45}}{2}$ c) $\frac{1+\sqrt{23}}{2}$ d) $\frac{1+\sqrt{29}}{2}$

Solution:- b) $S^2 = 2 + S \Rightarrow S^2 - S - 11 = 0 \Rightarrow S = \frac{1+\sqrt{45}}{2}$

65. The sequence

$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2}+\sqrt{2}}$, ...

converges to

a) 2 b) 3

c) $\frac{2+\sqrt{2}}{3}$ d) $\frac{\sqrt{3}+1}{2}$

Solution:- a) $S^2 = 2 + S \Rightarrow S^2 - S - 2 = 0 \Rightarrow (S+1)(S-2) = 0 \Rightarrow S = 2.$

66. For x > 0, $\underset{x\to 0}{\mathbf{Lt}}\left[(\sin x)^{\frac{1}{x}} + \left(\frac{1}{x}\right)^{\sin x}\right]$ is

a) 0 b) -1

c) 1 d) 2

Solution:- c) L = $\underset{x\to 0}{\text{Lt}}(\sin x)^{\frac{1}{x}} + \underset{x\to 0}{\text{Lt}}\left(\frac{1}{x}\right)^{\sin x}$

$$= 0 + \underset{x\to 0}{\text{Lt}}\, e^{\log\left(\frac{1}{x}\right)^{\sin x}} \quad \left[\underset{x\to 0}{\text{Lt}}\,(decimal))^{\infty} = 0\right]$$

$$= e^{\underset{x\to 0}{\text{Lt}}\frac{\log\left(\frac{1}{x}\right)}{cosec\, x}}$$

Applying L' Hospital's rule, we get

$$\text{L} = e^{\underset{x\to 0}{\text{Lt}}\frac{x\left(\frac{-1}{x^2}\right)}{-cosec\, x\cot x}} = e^{\underset{x\to 0}{\text{Lt}}\frac{\sin x}{x}.\tan x}$$

$$= e^0 = 1.$$

67. The function f(x) = $\frac{\log(1+ax)-\log(1-bx)}{x}$ is not defined at x = 0. The value which should be assigned to & at x = 0, so that f(x) is continuous at x = 0, is

a) a –b b) a +b

c) $\log a + \log b$ d) none

Solution:- b)

f(0)= $\underset{x\to 0}{\text{Lt}}\, f(x) = \underset{x\to 0}{\text{Lt}}\frac{\log(1+ax)-\log(1-bx)}{x}$

$$= \underset{x\to 0}{\text{Lt}}\frac{a\log(1+ax)}{ax} + \underset{x\to 0}{\text{Lt}}\frac{-b\log(1-bx)}{-bx}$$

$$= a.1 + b.1$$

$$= a + b.$$

68. If a is a real number then

$\underset{n\to\infty}{\mathbf{Lt}}\begin{bmatrix}\mathbf{1} & \boldsymbol{a/n}\\ \boldsymbol{-a/n} & \mathbf{1}\end{bmatrix}^{\boldsymbol{n}}$ is equal to

a) $\boldsymbol{I}_{2\times 2}$ b) $\boldsymbol{O}_{2\times 2}$

c) 1' d) None.

Solution:- b) A = $\begin{bmatrix}1 & a/n\\ -a/n & 1\end{bmatrix} = \frac{1}{n}\begin{bmatrix}n & a\\ -a & n\end{bmatrix}$

Let n = $r\cos\theta$, $a = r\sin\theta$.

$\Rightarrow$r = $\sqrt{n^2+a^2}$; $\theta = \tan^{-1}\left(\frac{a}{n}\right)$

$$A = \frac{r}{n}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} \Rightarrow A^n = \frac{r^n}{n^n}\begin{pmatrix}\cos n\theta & \sin n\theta\\ -\sin n\theta & \cos n\theta\end{pmatrix}$$

$$\Rightarrow A^n = \left(\sqrt{1+\frac{a^2}{n^2}}\right)^n\begin{pmatrix}\cos n\theta & \sin n\theta\\ -\sin n\theta & \cos n\theta\end{pmatrix}$$

$$\Rightarrow \frac{A^n}{n} = \left(1+\frac{a^2}{n^2}\right)^{\frac{n}{2}}\begin{pmatrix}\frac{\cos n\theta}{n} & \frac{\sin n\theta}{n}\\ -\frac{\sin n\theta}{n} & \frac{\cos n\theta}{n}\end{pmatrix}$$

$$\Rightarrow \underset{n\to\infty}{\text{Lt}}\frac{A^n}{n} = (1+0)\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

$$\Rightarrow \underset{n\to\infty}{\text{Lt}} A^n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O_{2\times 2}.$$

69. The series $\sum \frac{(-1)^n}{(n+1)^p}$ is

a) Conditionally convergent if 0<p ≤ 1
b) Absolutely convergent if p >1
c) Oscillatory if p ≤0
d) All above

Solution: d) Case I: p ≤ 0

p = -q

Then the given series becomes

$$\sum (-1)^n (n+1)^q$$
$$= -2^q + 3^q - 4^q + 5^q - \cdots$$

This is an Oscillatory series.

Case II:- 0 < p < 1

This series is :

$$\sum U_n = -\frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \frac{1}{5^p} - \cdots$$

By Leibnitz's test this series is convergent.

Also, $|\sum U_n| = \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$

This series is P-series and p ≤ 1.

So, it is divergent.

Case III: p >1 , the series is

$$\sum U_n = -\frac{1}{2^p} - \frac{1}{3^p} - \frac{1}{4^p} - \cdots$$

By Leibnitz's test this series is convergent.

$|\sum U_n| = \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$

The series is also convergent.

So, the series is absolutely convergent.

70. $\underset{x\to 0}{\textbf{Lt}} \left[\frac{\sin x}{x}\right]$ is equal to

a) 1 b) 0
c) does not exist d) none

Solution:- b) since, $|\frac{\sin x}{x}|$ <1

$\Rightarrow \frac{\sin x}{x}$ tends to 1 forms the values that are less than one as x ⟶ 0. Thus, $\underset{x\to 0}{\text{Lt}} \left[\frac{\sin x}{x}\right]$ = 0

71. The net profit of an industry in a year is given by y = 2ax -x^2, where x denotes the input.
Then the profit increases in relation to x if

a) 0 < a < x b) x = a
c) a < x < 2a d) x <a

Solution:- d) y = 2ax -x^2 $\Rightarrow \frac{dy}{dx} = 2a — 2x$

Profit will increase if 2a -2x > 0 i.e. x < a

72. If $a_1 = 1$ *and* $a_{n+1} = \frac{4+3a_n}{3+2a_n}, n \geq 1$; then $\underset{n\to\infty}{\textbf{Lt}}\, a_n = l$. The l is equal to

a) $-\sqrt{2}$ b) $\sqrt{2}$ c) 2 d) none

Solution: b) $l = \frac{4+3l}{3+2l} \Rightarrow l^2 = 2 \Rightarrow l = \sqrt{2}$.

Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = l$.

73. $\underset{n\to\infty}{\textbf{Lt}} \frac{n^P \sin^2(n!)}{n+1}, 0 < P < 1$, is equal to

a) 0 b) ∞
c) 1 d) none

Solution : a) $\underset{n\to\infty}{\text{Lt}} \frac{sin^2(n!)}{n^{1-P}\left(1++\frac{1}{n}\right)}$ $(\because 0 < P < 1)$

$= \frac{same\ real\ no.in\ [0,1]}{\infty}$ $[\because 1-P > n0]$

$= 0$

74. The series $\sum_{n=1}^{\infty} \sin\frac{\pi}{n^P}$ is
    a) Convergent for all values of P
    b) Convergent for p ≤ 1 and divergent for P >1
    c) Convergent for P > 1 and divergent for p ≤1
    d) Divergent for all values of P.

Solution: c) $U_n = \sin\frac{\pi}{n^P} = \frac{\pi}{n^P} - \frac{\pi^3}{3!n^3 P} + \cdots and\ \ V_n = \frac{1}{n^P}$

$\therefore \underset{n\to\infty}{\text{Lt}} \frac{U_n}{V_n} = \pi \neq 0$

$\sum V_n$ is convergent for P > 1 and divergent for P≤ 1

$\therefore \sum U_n$ is convergent for P > 1 and divergent for P ≤ 1.

75. Let $U_n = \sin(1/n)$ and consider the series $\sum U_n$.
    Which of the following statement is true?
    a) $\sum U_n$ is convergent
    b) $U_n \longrightarrow 0\ as\ n \longrightarrow \infty$
    c) $\sum U_n$ is divergent
    d) $\sum U_n$ is absolutely convergent.

Solution:- c) $U_n = \sin(1/n)\,, V_n = \frac{1}{n}$

$\therefore \lim_{n\to\infty}\frac{U_n}{V_n} = \lim_{n\to\infty}\frac{\sin 1/n}{1/n} = 1$

So, $\sum V_n\ diverges.$

∴ By limit comparison test, $\sum U_n$ is also diverges.

76. If $f_n(x)$ be a function defined on [0, 1] and then the sequence $\{f_n(x)\}$, where $f_n(x) = x^n$, is ___________
    a) Uniformly convergent in [0, 1]
    b) Uniformly convergent in (0, 1)
    c) Uniformly convergent in $\mathbb{R}$
    d) None

Solution:- b) $\lim_{n\to\infty} f_n(x) \lim_{n\to\infty} x^n = \begin{cases} 0, & where\ 0 \le x < 1 \\ 1, & where\ x = 1. \end{cases}$

Then the sequence is point wise convergent in [0, 1] and uniformly convergent in (0, 1).

77. Which of the following functions is uniformly continuous on the domain as stated?
    (a) f(x) = x^2, x ∈ $\mathbb{R}$ **(b) f(x) =** $\frac{1}{x}, x \in [1,\infty)$ **(c) f(x) =** $\tan x, x \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ **(d) f(x) = [x], x ∈ [0, 1]**

Solution:- (b) $f(x) = \frac{1}{x}$ is uniformly continuous in $[1,\infty)$.

SET, COMBINATORICS, PROBABILITY

78. The number of non–empty of a set consisting 6 elements is

(a) 63 **(b) 64** **(c) 65** **(d) none**

<u>Solution</u>:- (a) The no. of non-empty subset of a set consisting n element is

$= 2^n$.

79. Le A and B be two sets having 7 common elements, then the number of elements common to $\boldsymbol{A \times B}$ *and* $\boldsymbol{B \times A}$ is
(a) 0 **(b) 2^7**
(c) 49 **(d) none.**

Sol. (c) The no. of common elements to $\times B$ *and* $B \times A$ is $= n^2$.

80. The number of squares that can be formed on a chess board is
(a) 204 **(b) 224**
(c) 230 **(d) None**

<u>Solution</u>:- (a)A chess board has 9 equi-spaced horizontal and vertical lines we need to choose two consecutive horizontal and vertical lines to make a 1×1 square from among these which is done in $8 \times 8 =$ ways.

Similarly, 2×2 square needs 3 consecutive horizontal and vertical lines, i.e. in $7 \times 7 = 7^2$ ways.

$\therefore$ Total number of squares $= 8^2 + 7^2 + 6^2 + 5^2 + \cdots + 1^2$

$$= \sum_{i=1}^{8} i^2 = \frac{8(8+1)(16+1)}{6} = 204$$

81. How many friends must you have to guarantee that it least five of them will have birthdays in the same month?

(a) $50 \leq n \leq 60$ **(b) $49 \leq n \leq 59$**
(c) $40 \leq n \leq 60$ **(d) $49 \leq n \leq 60$**

Sol. (d) No of friends = n

Months (holes) (m) = 12.

By extended pigeon –Hole principle,

$$\left[\frac{n-1}{m}\right] + 1 = 5$$

$$\Longrightarrow \left[\frac{n-1}{12}\right] + 1 = 5$$

$\Longrightarrow 49 \leq n \leq 60.$

82. Let U be the set of positive integers not exceeding 1000 then the number of sets of such integers which are not divisible by 3, 5, 7 is
(a) 255 **(b) 456**
(c) 457 **(d) 256**

Sol. (c)

A : integers divisible by 3

B : integers divisible by 5

C : integers divisible by 7

n(A)$= \left[\frac{1000}{3}\right] = 333, n(B) = \left[\frac{1000}{5}\right] = 200, n(C) = \left[\frac{1000}{7}\right] = 142$

$$n(A \cap B) = \left[\frac{1000}{15}\right] = 66, n(B \cap C) = \left[\frac{1000}{35}\right] = 28, n(A \cap C) = \left[\frac{1000}{21}\right] = 47$$

$$n(A \cap B \cap C) = \left[\frac{1000}{105}\right] = 9.$$

By inclusion –exclusion principle,

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C) = 543$.

So, required answer is $= n(A \cup B \cup C)^c$

$$= 1000 - n(A \cup B \cup C) = 457.$$

83. A and B toss a fair win each simultaneously 50 times. The probability that both of them will not get tail in the same toss is

(a) $\left(\frac{3}{4}\right)^{50}$ (b) $\left(\frac{2}{7}\right)^{50}$
(c) $\left(\frac{1}{8}\right)^{50}$ (d) none.

Sol. (a) There are four possibilities in each toss, i.e..

A = tail B = Head

A = Head B = tail

A = Head B = Head

A = tail B= tail

Total number of cases = 4^{50}

In each case there are 3 possibilities of not getting tail on the same toss,

$\therefore$ Favourable cases = 3^{50}.

Hence the required probability is $\left(\frac{3}{4}\right)^{50}$.

84. If the integers m and n are chosen at random between 1 to 100. Then the probability that a number of the form $7^m + 7^n$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{7}$
(c) $\frac{1}{8}$ (d) None.

Sol. (a) The unit place of $7^{k,}$where k is an integer will be 9, 3, 1, 7.

$$7^1 = 7, 7^2 = 49, 7^3 = 343, 7^4 = 2407, \dots, 7^7 \equiv 1 \ (last\ digit)$$

Now $7^m + 7^n$ is divisible by 5 if m = 3 or 7 and n = 7 or 3.

Also, $7^m + 7^n$ is divisible by 5 if m = 9 or 1 and n = 1 or 9.

Now, $7^m + 7^n$ is divisible by 5 only when the last digit in the unit place is zero.

$\therefore$ Required probability is $= \frac{4}{2^4} = \frac{1}{2^2} = \frac{1}{4}$.

85. The total number of subsets of a set of 12 elements are

(a) 144 (b) 12^{12}
(c) 47900 (d) 4096

Sol. (d) Answer = 2^{12}= 4096.

86. The total number of non-empty even subsets of a set having n elements is

(a) 2^{n-1} (b) $2^{n-1} - 1$ (c) 2^n
(d) $2^{n+1} + 1$

Sol. (b) If a set having n elements then total no. of subsets is = 2^n

Total no. of even subsets is = 2^{n-1}.

Excluding the empty set ϕ, we have $2^{n-1} - 1$ as total number of non-empty subsets.

87. A bar of unit length is broken into 3 parts x, y, z. The probability that a triangle can be formed from the resulting parts is

(a) $\frac{1}{2}$ **(b) $\frac{1}{3}$**
(c) $\frac{1}{4}$ **(d) None**

Sol. (c) Let z = 1- (x +y)

x > 0, y > 0, (x +y) <1.

The sample space is $\frac{1}{2}X \mid X \mid = \frac{1}{2}$ = interior of a unit triangle with unit legs.

Then two conditions are needed to satisfy to from a triangle:

(a) The sum of the two sides is greater than the third side
(b) The difference between any two sides is smaller than the third one.

The area of the new triangle domain is = $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$

∴ Prob. is $= \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$.

88. Total number of non-negative integer solutions of $x_1 + x_2 + x_3 = 10$ is

(a) 10_{C_2} **(b) 10_{C_3}**
(c) 1_{C_2} **(d) none.**

Sol. (c) $n + r - 1_{C_{r-1}} = 10 + 3 - 1_{C_{3-1}} = 12_{C_2}$

89. A point is selected at random from the interior of a circle.

The probability that the point is closed to the centre than the boundary of the circle is

(a) $\frac{1}{2}$ **(b) $\frac{1}{4}$**
(c) $\frac{1}{6}$ **(d) none**

Sol. (b) AB = r ; CD = $\frac{r}{2}$

n(S) = the area of the circle of radius r = $\Pi(r^2)$

n(E) = the area of the circle of radius $\frac{r}{2}$ = $\Pi\left(\frac{r}{2}\right)^2$

∴ P(E) $= \frac{n(E)}{n(s)} = \frac{\Pi\left(\frac{r}{2}\right)^2}{\Pi r^2} = \frac{1}{4}$.

90. Two finite sets have m and elements. The total number of subsets of the first set is 56, more than the total number of subsets of the second set, The value of m, n are

(a) 7, 6 **(b) 6, 3**
(c) 5, 1 **(d) none**

Sol. (b) We know that $2^m - 2^n = 56$.

By trial, m = 6, n = 3

So, (b) is correct.

91. Total number of polynomials of the form $x^3 + ax^2 + bx + c = 0$ which is divisible by $x^2 + 1$, where $a, b, c \in \{1, 2, 3, \dots, 10\}$ is

(a) 15 **(b) 10** **(c) 5** **(d) none**

Solution: (b)

Take $i^3 + ai^2 + bi + c = 0$

and $-i^3 + ai^2 - bi + c = 0$

implying $(b - 1)i + (c - a) = 0$

Thus, b = 1 and a = c.

So, total number of polynomials equals to $\binom{10}{1} = 10$.

92. Let $x + y = 2a$, where a is a constant, and all values of x lying between 0 and 2a are equally likely. Then the chance that $xy > \frac{3a^2}{4}$ is

(a) ½ (b) 1/3 (c) ¼ (d) none

Solution:- (a)

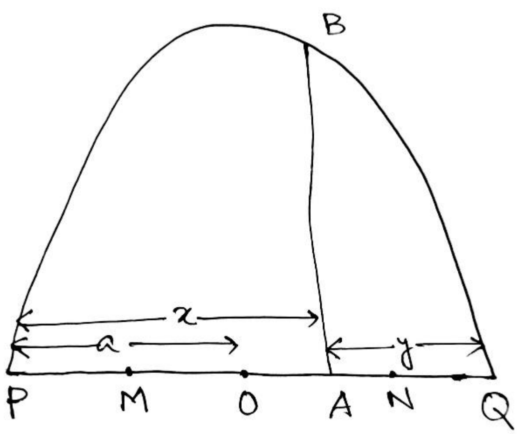
Let OP = a, AP = x, AQ = y, and x + y = 2a.

Now, AB^2 = AP, PQ = xy.

Also, MP = MO, and NO = NQ.

If A lies in MN then AB > a$\sqrt{\frac{3}{4}}$

Thus, $P\left(xy > \frac{3a^2}{4}\right) = \frac{MN}{AB} = \frac{1}{2}$



DIFFERENTIAL EQUATIONS

93. The differential equation $\left|\frac{dy}{dx}\right| = |y|, y(0) = 1, y \neq 0$ has
(a) Unique solution (b) non-trivial solution (c) finite number of solution (d) infinite number of solution

Solution:- (b) The equation is $|\text{y}| = e^{x+c}$

y(0) = 1, gives $e^c = 1 = e^0 \Rightarrow c = 0$.

So, y = e^x and y = - e^x are two solutions.

So, the ODE has non-trivial solution.

[Note: Trivial solution:- A solution in which ever variable has zero value is called trivial solution.

Infinite solution:- If the constant(s) of the solution of the ODE remain undetermined then the equation has infinite number of solutions.]

94. Number of solution of the ODE $\frac{d^2y}{dx^2} + 4y = 0,\ y(0) = 0, y\left(\frac{\pi}{4}\right) = 1$ is
(a) 0 (b) 1 (c) 2 (d) None

Solution:- (b)

$\frac{d^2y}{dx^2} + \lambda y = 0\ (\lambda > 0)$ has the general solution

y = $c_1 \cos\sqrt{\lambda x} + c_2 \sin\sqrt{\lambda x}$; where λ > 0.

So, $\frac{d^2y}{dx^2} + 4y = 0$ has the solution

y = $c_1 \cos 2x + c_2 \sin 2x$

$y(0) = 0 \Rightarrow c_1 = 0$

$y\left(\frac{\Pi}{4}\right) = 1 \implies c_2 = 1.$

∴ y = sin 2x is the unique solution of the given differential equation.

95. The solution of the ODE $\frac{dy}{dx} = x$, $y(0) = 0$ is

(a) Unbound (b) positive

(c) negative (d) zero

Solution:- (b)

$\int dy = \int x\, dx$

$\implies y = x^2 + c$

$y(0) = 0 \; giving \; c = 0$

∴ y $= x^2$ is the solution which is always positive.

96. Number of solutions of the ODE $\frac{d^2y}{dx^2} = 0$, $y(0) = 1$ is

(a) 0 (b) 1 (c) infinite no. of solutions

(d) none

Solution:- (c)

$\frac{d^2y}{dx^2} = 0$ has the solution y $= c_1x + c_2$

y(0)= 1 $\implies c_2 = 1.$

∴ y $= c_1x + 1$, c_1 is arbitrary constant.

⟹ The ODE has infinite number of solutions.

97. One of the integrating factors of the ODE

$(y^2 - 3xy)dx + (x^2 - xy)dy = 0$

is

(a) $\frac{1}{(x^2y^2)}$ (b) $\frac{1}{(x^2y)}$

(c) $\frac{1}{(xy^2)}$ (d) $\frac{1}{(xy)}$

Solution:- (b)

M $= y^2 - 3xy$

N $= x^2 - xy$

As M dx + N dy = 0 is homogeneous.

An I.F. is $= \frac{1}{Mx+Ny} = \frac{1}{(-2x^2y)}$.

So, $\frac{1}{x^2y}$ is an I. F. by ignoring the constant.

98. General solution of $\left(xysin(xy) + cos(xy)\right)ydx + \left(xysin(xy) - cos(xy)\right)dy = 0$ is

(a) ysin(xy) = cx (b) xsec(xy) = cy

(c) ytan(xy) = cx (d) none.

Solution: (b)

xysin(xy)(ydx+xdy)+cos(xy)(ydx-xdy)=0

implies tan(xy).d(xy)+dx/x – dy/y = 0

implies log |secxy | + log |x/y | = c'

implies xsec(xy) = cy

99. The solution of the curve y = f(x) satisfying the differential equation

$\sqrt{(x-y)\frac{dy}{dx}} = |x^2 - y^2|$ **and passing through the point (1,0) is**

(a) $(x-1) = y^2(x^2 - y^2)$
(b) $y^2 = x - 1$
(c) $(2x-3) + \frac{1}{(x^2-y^2)} = 0$
(d) **None**

Solution: (c)

$$\sqrt{(x-y)\frac{dy}{dx}} = |x^2 - y^2|$$

$$\rightarrow \frac{d(x^2 - y^2)}{2(x^2 - y^2)^2} = dx$$

$$\rightarrow \frac{-1}{(x^2 - y^2)} = 2x + c$$

Which passes through (1,0).

Thus, - 1 = 2x + c which gives c = - 3.

Hence the curve is

$$(2x-3) + \frac{1}{(x^2 - y^2)} = 0$$

100. The solution of the equation $xdy - ydx = \sqrt{(x^2 - y^2)}dx$ subject to the condition y(1)=0, is

(a) y = xsin(logx)
(b) y = x^2sin(logx)
(c) y = x^2(x – 1)
(d) None

Solution: (a)

$$\frac{xdy - ydx}{x^2} = \frac{1}{x}\sqrt{1 - \left(\frac{y}{x}\right)^2}\,dx$$

$$\rightarrow d\left(\frac{y}{x}\right) = \frac{1}{x}\sqrt{1 - \left(\frac{y}{x}\right)^2}\,dx$$

$$\rightarrow \frac{d\left(\frac{y}{x}\right)}{\sqrt{1 - \left(\frac{y}{x}\right)^2}} = \frac{1}{x}dx$$

$$\rightarrow sin^{-1}\left(\frac{y}{x}\right) = \log(x) + c$$

ISI SUBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 1

1. Find all real numbers satisfying $6^x + 2^{2x} + 24^x - 36^x - 16^x = 1.$

Ans:- Rewrite the given relation as:

$6^x + 4^x - 36^x + 24^x - 16^x = 1$

Let $6^x = a, 4^x = b$, we have

a+ b -$a^2 + ab - b^2$ = 1

⇒$a^2 - ab + b^2 - a - b + 1$=0

⇒ $2a^2 - 2ab + 2b^2 - 2a - 2b + 2$=0

⇒ $(a^2 - 2ab + b^2)$+ $(a^2 - 2a + 1)$+ $(b^2 - 2b + 1)$=0

$\Rightarrow (a-b)^2+(a-1)^2+(b-1)^2$=0

∴ a= 1 and b= 1 when a= b.

$\Rightarrow 4^x = 1$ and $6^x = 1$, giving x= 0 only.

2. Two boxes contain between them 65 balls of several different sizes. Each ball is white, black, red, or yellow. If you take any five balls of the same colour, at least two of them will always be of the same size (radius). Prove that there are at least three balls which lie in the same box, have the same colour and are of the same size.

Ans:- we will make repeated use of pigon–hole- principle (PHP). As there are 65 balls and 2 boxes , one of these boxes must contain at least $\left[\frac{65}{2}\right]$+1 = 33 balls.

Consider that box, now we have four colours (white, black, red, yellow) and hence there must be at least $\left(\frac{33}{4}\right)$+1 = 9 balls of the same colour.

There can be at most 4 different sizes available for these 9 balls of the same colour, For if there were 5 (or more)different sizes, then collection of 5 balls, all of different sizes, would not satisfy the given property.

Thus of these 9 balls there must be at least 3 balls of the same size.

3. Find all continuous function f : (0, ∞)⟶(0, ∞) ∋ f (1)= 1 and

$$\frac{1}{2}\int_0^x (f(t))^2 dt = \frac{1}{x}\left(\int_0^x f(t)dt\right)^2$$

Ans:- Define , F (x) = $\int_0^x f(t)dt$ and G (x)= $\int_0^x (f(t))^2 dt$

Since f: (0, ∞) ⟶ (0, ∞)

we have F (x)> 0 ∀ $x > 0$

Also, $\frac{1}{2}G(x) = \frac{1}{x}\{F(x)\}^2$, from the given condition on differentiation, we have

$\frac{1}{2}G'(x) = \frac{1}{x}.2F(x).F'(x) - \frac{1}{x^2}(F(x))^2$

This means that $\frac{1}{2}(F(x))^2 = \frac{2}{x}F(x)\,F'(x) - \frac{1}{x^2(F(x))^2}$

or, $\frac{1}{2}\left(\frac{xF'(x)}{F(x)}\right)^2 = 2\frac{xF'(x)}{F(x)} - 1$

Solving this equation as a quadratic in $\frac{xF'(x)}{F(x)}$ we have

$\frac{xF'(x)}{F(x)} = 2 \pm 2 = k$(say)

On integration, we obtain $\int \frac{dF(x)}{F(x)} = k \int \frac{dx}{x}$

$\Rightarrow$ ln $F(x)$= klnx + ln$\lambda \Rightarrow F(x) = \lambda x^k$

$\Rightarrow$ f (x)=λk$x^{k-1} \Rightarrow$ f (1) = 1

$\Rightarrow \lambda$k=1

∴ f(x)= $x^{k-1} = x^{1+\sqrt{2}}/x^{1-\sqrt{2}}$

4. Let x ≥ 1, $f(x) = \frac{\sqrt{[x]}+\sqrt{\{x\}}}{\sqrt{x}}$, where [.] denotes G.I.F. and { } denotes fractional part. Determine the smallest number k ∋ f(x)≤ k for each x ≥ 1

Ans:- Let x = a+ b where a= [x], b= {x}

f(x)= $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a+b}}$

$(\mathrm{f}(x))^2 = \frac{a+b+2\sqrt{ab}}{a+b} = 1 + \frac{2\sqrt{ab}}{a+b}$

Using AM$\geq GM, \leq 1+1 \Rightarrow f(x) \leq \sqrt{2}$.

5. Solve the equation $(\sqrt{2}+\sqrt{2})^x + (\sqrt{2}-\sqrt{2})^x = 2^x$

Ans:- 1+$\frac{\sqrt{2}}{2} = 1 + cos\frac{\pi}{4} = 2\cos^2\frac{\pi}{8}$

$(\frac{2+\sqrt{2}}{4})^{x/2} + (\frac{2-\sqrt{2}}{4})^{x/2}$

= $(cos\frac{\pi}{8})^x + (sin\frac{\pi}{8})^x$

⇒ x= 2

6. Let f(x) be a polynomial with real coefficient for which the equation f(x)= x has no real solution. Prove that the equation f(f(x))= x has no real solution, either.

Ans:- Suppose, if possible that f(f(a))=a, let b= f(a), then f(b)=a by hypothesis b ≠ a. Assume that a < b then f(a)- a > 0 and f(b)-b < 0. So, by intermediate value theorem f(x) - x = 0 should be a root between (a, b).

But this contradicts our assumption. Hence, f(f(x)) can have no real solution.

7. Let a ∈ [0, 4]. Prove that the area bounded by the curves y=1 -|x-1| and y= |2x -a| can't exceed 1/3.

Ans:- when a ∈ [0, 1].the area is a triangle formed by (0, 0), ($^1/_2$, 0) and (1, 1) with area equals 1/4.

When a ∈ [1, 3] , the area is a quadrilateral with vertices at ($\frac{a}{3}, \frac{a}{3}$), ($\frac{a}{2}$, 0), ($\frac{a+2}{3}, \frac{4-a}{3}$) and (1, 1)

So, the net area is $\frac{1}{3} - \frac{(a-2)^2}{6}$ which also does not exceed 1/3.

When a ∈ [3, 4], the area is same as when a ∈ [0, 1]

8. Determine a value of the parameter θ ∋ f(x)= $\cos^2 x + \cos^2(x+\theta) - cosxcos(x+\theta)$.

Is a constant function of x?

Ans:- f(x)=$\sin^2\theta + (2\,cos\theta - 1)(\cos^2 xcos\theta - sin2xsin\theta)$

The function f(x)is constant when $(2\,cos\theta - 1)$= 0

i.e. θ= $\frac{\pi}{3}$ and the constant value is $\frac{3}{4}$.

9. If $\frac{{}^nc_0}{2} - \frac{{}^nc_1}{3} + \frac{{}^nc_2}{4} - \cdots + (-1)^n\frac{{}^nc_n}{n+2} = \frac{1}{1999\times2000}$, then what is the value of n?

Ans:- (a)

$(1-x)^n = c_0 - c_1x + c_2x^2 - \cdots + (-1)^n.c_nx^n$

⟹ x$(1-x)^n = c_0x - c_1x^2 + c_2x^3$ ... $+ (-1)^n.c_nx^{n+1}$

Integrating between the limits 0 and 1,

We get $\left[c_0\frac{x^2}{2} - c_1\frac{x^3}{3} + c_2\frac{x^4}{2} - \cdots + (-1)^n.c_n\frac{x^{n+2}}{n+2}\right]_0^1 = \left[-x.\frac{(1-x)^{n+1}}{n+1}\right]_0^1 + \int_0^1 1.\frac{(1-x)^{n+1}}{n+1}dx$

$= 0 - \left[\frac{(1-x)^{n+2}}{(n+1)(n+2)}\right]_0^1 = \frac{1}{(n+1)(n+2)}$

$\therefore \frac{c_0}{2} - \frac{c_1}{3} + \cdots + (-1)^n \frac{c_n}{n+2} = \frac{1}{(n+1)(n+2)}$

Given, $\frac{1}{(n+1)(n+2)} = \frac{1}{1999\times 2000}$

So n = 1998.

10. For what value of m the sum $\sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i}$, *where* $\binom{p}{q} = 0$ if p< q, is maximum?

Ans:- (c)

$\sum_{i=0}^{10} 10_{C_i} = 20_{C_{m-i}} = 10_{C_0}20_{C_m} + 10_{C_1}20_{C_{m-1}} + \cdots + 10_{C_m}20_{C_0}$ ……..(1)

$(1+x)^{20} = 20_{C_0} + 20_{C_1}x + \cdots + 20_{C_{m-1}}x^{m-1} + 20_{C_m}x^m + \cdots + 20_{C_{20}}x^{20}$ ……………….(2)

$(1+x)^{10} = 10_{C_0} + 10_{C_1}x + \cdots + 10_{C_{10}}x^{10}$ ……………….(3)

Multiplying (2) and (3) and equating the co-efficient of x^m, we get

$10_{C_0}20_{C_m} + 10_{C_1}20_{C_{m-1}} + \cdots + 10_{C_m}20_{C_0} = 30_{C_m}$, 30_{C_m} will be greatest when m= 15.

ISI SUBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 2

1. Find $\frac{dy}{dx}$ *if* $x^{cosy} + y^{cosx} = 1$

Ans:- u= x^{cosy}

Log u=cos y logx

$\frac{1}{u}.\frac{du}{dx} = cosy.\frac{1}{x} - logx.siny.\frac{dy}{dx}$

$\therefore \frac{du}{dx} = x^{cosy}(\frac{cosy}{x} - logx.siny.\frac{dy}{dx})$ …..(1)

v=y^{cosx} => $logv = cosxlogy$

$\frac{dv}{dx} = y^{cosx}\{-logy.sinx + cosx.\frac{1}{y}.\frac{dy}{dx}\}$ ………………(2)

Now, u+ v=1

$\frac{du}{dx} + \frac{dv}{dx} = 0.$

$\Rightarrow y^{cosx}.cosx.\frac{1}{y}.\frac{dy}{dx} - x^{cosy}.logxsiny\frac{dy}{dx} = y^{cosx}.logy.sinx - x^{cosy}.\frac{cosy}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{y^{cosx}.logy.sinx - x^{cosy}.\frac{cosy}{x}}{y^{cosx}.cosx.\frac{1}{y} - x^{cosy}.logx.siny}$.

2. Find the inverse of the following matrix with $R_1 = (c_0 , c_1 , c_2 , c_3)$; $R_2 = (c_2 , c_3 , c_0 , c_1)$;

$R_3 = (c_3 , -c_2 , c_1 , -c_0)$; $R_4 = (c_1 , -c_0 , c_3 , -c_2)$,

where $c_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}, c_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}, c_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}, c_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}$

Ans:- Put ${c_0}^2 + {c_1}^2 + {c_2}^2 + {c_3}^2 = 1$

$c_0\, c_3 = -\frac{1}{16},\ c_2c_1 = \frac{3}{16},\ c_0c_2 = \frac{-3}{16},\ c_1c_3 = \frac{-3}{16}$.

Here $A^2 = I \Rightarrow A = A^{-1}$

3. True/False: If f is a continuous function on $\mathbb{R}$ ∋ f(x+ y)= f(x)+ f(y) ∀ x, y∈ $\mathbb{R}$.

Then f(0)= 0 ∀ x∈$\mathbb{R}$.

Ans:- False

f(x+ y)= f(x) .f(y)

Let f(x) = a^x , f(y)=a^y ∀ x, y∈ $\mathbb{R}$

f(x+ y)= a^x + y

f(0)=1 ≠0.

4. $\underset{x\to 0}{\mathbf{lt}}(\boldsymbol{cosx})^{\mathbf{1}/\boldsymbol{x^2}} = ?$

Ans:- $(cosx)^{1/x^2} = k, say$

∴ lnk= $\frac{1}{x^2}ln(cosx)$

∴ $\underset{x\to 0}{\text{lt}}(lnk) = \underset{x\to 0}{\text{lt}}\frac{ln\cos x}{x^2}\left(\frac{0}{0}\right) = \underset{x\to 0}{\text{lt}}\frac{-\tan x}{2x}\left(\frac{0}{0}\right) = \underset{x\to 0}{\text{lt}}\frac{-cosec^2x}{2} = -\frac{1}{2}$

∴ $\underset{x\to 0}{\text{lt}}\ k = e^{-\frac{1}{2}}$

5. Maximize x+ y subject to the condition that $2x^2 + 3y^2 \le 1$.

Ans:- $\frac{x^2}{1/2} + \frac{y^2}{1/3} \le 1$

Let z= x+ y

Now, 4x + 6y$\frac{dy}{dx} = 0$ ⇒$\frac{dy}{dx} = -\frac{2x}{3y}$

At the touching point. -$\frac{2x}{3y}$ = -1

⇒ 2x= 3y and $2x^2 + 3y^2$=1 ⇒ 2 $(\frac{3y}{2})^2 +$ $(3y^2)$ = 1

⇒ $15y^2 = 2$ ⇒ y= $\pm\sqrt{\frac{2}{15}}$

∴ x= $\frac{3}{2}\left(\pm\sqrt{\frac{2}{15}}\right) = \pm\sqrt{\frac{3}{10}}$ ∴ Max (z)= $\sqrt{\frac{3}{10}} + \sqrt{\frac{2}{15}} = \frac{5}{\sqrt{30}}$.

6. For any positive a, b prove that $(\boldsymbol{a} + \frac{\mathbf{1}}{\boldsymbol{a}})^{\mathbf{2}} + (\boldsymbol{b} + \frac{\mathbf{1}}{\boldsymbol{b}})^{\mathbf{2}} \ge \mathbf{8}$.

Ans: AM ≥ GM

$(a + \frac{1}{a})^2 + (a + \frac{1}{a})^2 \ge$ $2\sqrt{(a + \frac{1}{a})^2 + (b + \frac{1}{b})^2}$

≥ 2(ab+$\frac{1}{ab} + \frac{a}{b} + \frac{b}{a}$)

≥ 2(2+2) [∵ ab + $\frac{1}{ab}$ ≥ 2]

7. Let A & B be two invertible n ×n real matrices. Assume that A + B is invertible.

Show that $\boldsymbol{A^{-1}} + \boldsymbol{B^{-1}}$ is also invertible.

Ans:- A, B are invertible

A + B is invertible.

|A||$A^{-1} + B^{-1}$||B| = |B+ A|≠0

⇒ |$A^{-1} + B^{-1}$| ≠0 as |A|, |B|≠ 0

⇒ $A^{-1} + B^{-1}$is invertible.

8. Let A be n ×n orthogonal mtx where A is even and suppose |A| =-1. S.T. |I−A|= 0, where I denotes n ×n identity mtx.

Ans:- $A^{-1} = A^T$ |A| $= -1$

$\Rightarrow \frac{1}{\lambda} = \lambda \qquad \Rightarrow \prod_{i=1}^{n} \lambda_i = -1$ then at least one $\lambda_i = -1$

$\Rightarrow \lambda = \pm 1$

$\therefore$ Characteristics Equation is $|\lambda I_n - A| = 0$

$\Rightarrow |I_n - A| = 0$ for $\lambda_i = +1$

9. If f(x+ y)=f(x).f(y)for all x and y f(1)= 2 and $\boldsymbol{\alpha_n = f(n), n \in N}$, then find equation of the circle having $(\boldsymbol{\alpha_1, \alpha_2})$ and $(\boldsymbol{\alpha_3, \alpha_4})$ as the ends of its one diameter?

Ans:- (a)

Given f(x+ y) = f(x)f(y), for all x, y ………………….(1)

f(1)= 2 ……………..(2)

Putting x = 1, y = 1 in (1), we get

f(2) = $(f(1))^2 = 2^2$

Putting x = 2, y = 1 in (1), we get

f (3) = f(2)f(1) = $2^2 . 2 = 2^3$

Similarly, f(n) = 2^n, n ∈ N

Given $\alpha_n = f(n)$

$\therefore \alpha_n = 2^n$, n ∈ N $\qquad \therefore \alpha_1 = 2, \alpha_2 = 4, \alpha_3 = 8, \alpha_4 = 16$

Let P ≡ (2, 4), Q ≡ (8, 16)

∴ Equation of circle having PQ as a diameter is

(x - 2)(x - 8)+ (y - 4)(y - 16)= 0

10. If $\boldsymbol{0 < x < \pi}$, and f(x)

$$= \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2(1+\cos x)}}}}$$

there being n number of 2's, then $\boldsymbol{\lim_{n\to\infty} f(x) = \cdots}$

Ans:- (b)

Let y= $\lim_{n\to\infty} f(x)$

$$when\ n \to \infty,\ y^2 = 2 + y$$

$\Rightarrow y^2 - y - 2 = 0$

$\Rightarrow$ y = 2, -1

$\Rightarrow$ y = 2 ($\because$ y > 0)

ISI SUBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 3

1. Let s= $\{(\boldsymbol{a_1, a_2, a_3, a_4}): \boldsymbol{a_i} \in \mathbb{R}, \boldsymbol{i = 1, 2, 3, 4}\ ; \boldsymbol{a_1 + a_2 + a_3 + a_4 = 0}\}$

And T= $\{(\boldsymbol{a_1, a_2, a_3, a_4}): \boldsymbol{a_i} \in \mathbb{R}, \boldsymbol{i = 1, 2, 3, 4}\ ; \boldsymbol{a_1 - a_2 + a_3 - a_4 = 0}\}$

Find a basis for S∩T. Also find its dimension.

Ans:- S∩T=$\{(a_1, a_2, a_3, a_4): a_i \in \mathbb{R}, i = 1, 2, 3, 4\ ; a_1 - a_2 + a_3 - a_4 = 0\}$

Let $\underset{\sim}{x} \in S \cap T$, then

$a_1 + a_2 = -a_3 - a_4$……………..(1)

$a_1 - a_2 = -a_3 + a_4$………………(2)

$a_1 = -a_3$

$a_2 = -2a_3 - a_4$

Here, $\underset{\sim}{x} = (-a_3, -2a_3 - a_4, a_3, a_4)$

$= a_3(-1,-2,1,0) + a_4(0,-1,0,1)$

Here, $\{(-1,-2,1,0), (0,-1,0,1)\}$ forms a basis.

And also dim(S∩T)= 2.

2. Find the following limit: $\lim\limits_{x\to\infty}\left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}\right)$

Ans:- Let $u_n = \frac{n}{\sqrt{n^2+n}}$

$\therefore \lim\limits_{x\to\infty} u_n = \lim\limits_{x\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim\limits_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} = 1$.

By Cauchy's first theorem:- $\lim\limits_{x\to\infty}\left(\frac{u_1+\cdots+u_n}{n}\right) = 1.$

So, $\lim\limits_{x\to\infty}\left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}\right) = 1.$

3. for any real number x and for any positive integer n show that

$[x]+[x+\frac{1}{n}] + \left[x + \frac{2}{n}\right] + \cdots + \left[x + \frac{n-1}{n}\right] = [nx]$

Ans:- Let x= [x]+y, where $0 \le y < 1$,

Let p be an integer such that P-1 $\le ny < P$

Now, $x+\frac{k}{n} = [x] + y + \frac{k}{n}$

Also, $\frac{P+k-1}{n} < y + \frac{k}{n} < \frac{P+k}{n}$

So, long as $\frac{P-1+k}{n} < 1$, i.e. , k < n-(P-1)

So, $y + \frac{k}{n} < 1$ and consequently

$[x+\frac{k}{n}] = [x]$ for k= 0, 1, 2, …, n-P.

But $[x+\frac{k}{n}] = [x]+1$ for k= n-P+1, …..n-1.

$\therefore [x]+ [x+\frac{1}{n}]+\ldots+[x+\frac{n-1}{n}]$

$= \underbrace{([x] + [x] + \cdots + [x])} + \underbrace{(([x]+1) + ([x]+1) + \cdots + ([x+1]))]}$

= n[x]+(P-1)……………………(1)

Also, [nx]=[n[x]+ny]= n[x]+(P-1)

Since P-1 ≤ ny < P……(2)

From equation (1) & (2),

$[x]+[x+\frac{1}{n}]+\ldots+[x+\frac{n-1}{n}] = [nx].$

4. Prove that for n > 1, $1+\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$

Ans:- $P(1)= 1+\frac{1}{2^2} = \frac{5}{4} < 2 - \frac{1}{2} = \frac{3}{2} = \frac{6}{4}.$

The statement is true for n= 2.

Let , the statement is true for n= m.

$\therefore P(m)= 1+\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{m^2} < 2 - \frac{1}{m}.$

Now, we need to show that the statement is also true for n=m+1.

P(m+1)= $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{m^2}<2-\frac{1}{(m+1)^2}$

$<2-\frac{1}{m}+\frac{1}{m(m+1)}$. [$\because\frac{1}{(m+1)^2}<\frac{1}{m(m+1)}\forall\, m>1$]

$<2-\frac{1}{m+1}$.

∴ The statement is true for n= m+1

So, for all n ∊ ℕ the statement is true.

Hence proved.

5. Show that (a) *Φ*(p)= p-1;

(b) *Φ*(pq)=*Φ*(p)*Φ*(q); where p and q are prime numbers.

Ans:- (a) Let us take k as a positive integer and p be prime. The positive integer $\le p^k$ which are not prime to $p^k$are p, 2p, 3p,…, (p^{k-1})p. therefore , the number of positive integers less than p^k and prime to p^k is p^k-p^{k-1}

Hence (p^k)= $p^k-p^{k-1}=p^k\left(1-\frac{1}{p}\right)$.

For k= 1, (p)= p-1.

To prove this , we use:

(i) a is prime to pq if and only if a is prime to p and a is prime to q.

(ii) If r be the residue of a modulo q and r is prime to q then a is prime to q.

(iii) If c be an integer and a is prime to q then the number of integers in the set {c, c+a, c+2a, …, c+(n-1)a}that are prime to q is *Φ*(q)

(b) Since (1)=1, the theorem is trivial true when p and q equals 1.

Let us assume p > 1 and q > 1. We arrange pq integers in q rows of p columns as follows:

1	2	……	r	p
p+1	p+2	…….	p+r	2p
2p+1	2p+2	…..	2p+r	3p
…	….	….	…….	
(q-1)p+1	(q-1)p+2	…..	(q-1)p+r	qp

The number of integers among these, that are prime to pq is (pq) [By lemma].

The number of integers in the first row that the prime to p is (p) [Lemma]

Each column in the arrangements contain(q)integers prime to q [By lemma 3]

Hence (pq) = *Φ*(p)*Φ*(q).

6. Determine x, y, and z so that the 3× 3 matrix with the following row vectors is orthogonal:
$(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}), (1/\sqrt{2}, -1/\sqrt{2}, 0)$, (x, y,z).

Ans:- $\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix}$

$is\ an\ orthogonal\ matrix.$

7. Solve: dy/dx= (y+2)/(x-2)

Ans:- $\int \frac{dy}{y+2} = \int \frac{dx}{x-2}$

$\Rightarrow \log|y+2| = \log|x-2| + \log|c|$

$\Rightarrow (y+2)^2 = k(x-2)^2$ is the required solution.

8. If w is a complex cube root of unity then show that

$$\boldsymbol{a^3 + b^3 + c^3 - 3abc = (a+b+c)(a+bw+cw^2)(a+bw^2+cw).}$$

Ans:- $a^3 + b^3 + c^3 - 3abc$

= (a+ b +c)($a^2 + b^2 + c^2 - ab - bc - ca$)

= (a+ b +c){ $a^2 + b^2w^3 + c^2w^3 + (w+w^2)ab + (w+w^2)bc + (w+w^2)ca$}

= (a+ b +c){ $a^2 + b^2w^3 + c^2w^3 + wab + w^2ab + wbc + w^2bc + w^2bc + wca + w^2ca$}

= (a+ b +c){a(a +b$w^2 + cw$)+bw(a+b$w^2 + cw$)+c$w^2(a + bw^2 + cw)$)

= (a+ b+ c)(a+ b$w^2 + cw$)(a+ bw+ cw^2).

9. In a ΔPQR, $\angle$R = $\frac{\pi}{2}$. If $\boldsymbol{tan\frac{P}{2}}$ *and* $\boldsymbol{tan\frac{Q}{2}}$ are the roots of equation a$x^2 + bx + c$= 0 (a≠0), then show that a + b = c.

Ans:- (a) $\tan\frac{p}{2} + \tan\frac{Q}{2} = -\frac{b}{a}$, $\tan\frac{P}{2}\tan\frac{Q}{2} = \frac{c}{2}$ $\quad \therefore \frac{P}{2} + \frac{Q}{2} = \frac{\pi}{4}$

$$\therefore \frac{\tan\frac{p}{2} + \tan\frac{Q}{2}}{1 - \tan\frac{p}{2}\tan\frac{Q}{2}} = \tan\frac{\pi}{4} = 1$$

$\Rightarrow \frac{-\frac{b}{a}}{1-\frac{c}{a}} = 1 \Rightarrow b = c - a \Rightarrow a + b = c$

10. If $\boldsymbol{(1+x)^n = \sum_{r=0}^{n} a_r X^r}$ *and* $\boldsymbol{b_r = 1 + \frac{a_r}{a_{r-1}}}$ *and* $\boldsymbol{\prod_{r=1}^{n} b_r = \frac{(101)^{100}}{100!}}$, then n = ?

Ans:- (b) $(1+x)^n = \sum_{r=0}^{n} a_r\, a_r = n_{C_r}$

$b_r = 1 + \frac{a_r}{a_r - 1} = \frac{a_{r+}a_{r-1}}{a_{r-1}} = \frac{n_{C_r} + n_{C_{r-1}}}{n_{C_{r-1}}} = \frac{n+1_{C_r}}{n_{C_{r-1}}} = \frac{\frac{n+1}{r}.n_{C_{r-1}}}{n_{C_{r-1}}} = \frac{n+1}{r}$

$$\prod_{r=1}^{n} b_r = \frac{n+1}{1}.\frac{n+1}{2}\ldots\frac{n+1}{n} = \frac{(n+1)^2}{n!}$$

Given, $\frac{(n+1)^n}{n!} = \frac{(101)^{100}}{100!}$

$\therefore$ n = 100

ISI SUBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 4

1. If A and B are real orthogonal matrices of the same order and |B|+|A|= 0.

Prove that |A+ B|= 0

Ans:- |A|+|B|=0

$\Rightarrow$ |A|= $-$|B|

$|A|.|B|=-1$ [∵ $|B|=|B^{-1}|$ as they are orthogonal]

Let, C = A $(A^T + B^T)$B

⇒ $|C|= |AA^TB + AB^TB| = |B + A|$ ………..(i)

And $|C|= |A||A^T + B^T||B| = -|A^T + B^T|$

⇒ - $|(A + B)^T| = -|A + B|$…………..(ii)

$|A + B| = -|A + B|$

⇒ 2 |A +B|=0

⇒ |A+ B|=0

2. Determine whether there is a one –to – one function f: ℝ⟶ ℝ such that

$f(x^2)-[f(x)]^2 \geq \frac{1}{4} \forall x$

Ans:- Take x= 0, then $f(0) – (f(0))^2 \geq \frac{1}{4}$

⇒ $(f(0))^2 + (\frac{1}{2})^2 - 2.\frac{1}{2}.f(0) \leq 0$

⇒ $(f(0) - \frac{1}{2})^2 \leq 0$

⇒ $f(0) - \frac{1}{2} = 0$ ⇒ $f(0) = \frac{1}{2}$

Also, taking x=1 we have $f(1) - \frac{1}{2} = 0$ ∴ $f(0) = f(1)= \frac{1}{2}$

∴ This is not one -to –one function.

3. if 0 < u < 1 and $u_{n+1} = 1 - \sqrt{1 - u_n}$ ∀ $n > 1$,

Prove that (i) $\{u_n\}$ converges to zero

(ii) $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{2}$

Ans:- (i) $0 < u_1 < 1$

⇒ $0 < \sqrt{1 - u_1} < 1$

⇒ $0 < 1 - \sqrt{1 - u_1} < 1$

i.e. $0 < u_2 < 1$

Similarly, $0 < u_3 < 1$ ….. and so on.

Let $0 < u_n < 1$, then $0 < 1- \sqrt{1 - u_n} < 1$, i.e. $0 < u_{n+1} < 1$

Thus $\{u_n\}$ is bounded.

Again, $u_{n+1} - u_n = 1-\sqrt{1 - u_n} - u_n$

$=(1- u_n)- \sqrt{1 - u_n}$

$= (\sqrt{1 - u_n})^2 - \sqrt{1 - u_n}$

$= \sqrt{1 - u_n} (\sqrt{1 - u_n} - 1)$

< 0 as $0 < \sqrt{1 - u_n} < 1$

∴ $u_{n+1} < u_n$ as $0 < \sqrt{1 - u_n} < 1$

∴ $\{u_n\}$ is monotonically decreasing. ∴ $\{u_n\}$ converges two zero.

(ii) Let $\lim\limits_{n\to\infty} u_n = l$, then $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n}$

∴ $\lim\limits_{n\to\infty} \frac{1-\sqrt{1-l}}{l} = \lim\limits_{n\to\infty} \frac{l}{l(1+\sqrt{1-l})} = \frac{1}{1+\sqrt{1-0}} = \frac{1}{2}$

; Since u_n converges to zero.

4. Let g: ℝ⟶ ℝ be a continuous function ∋ g(x) = g $(\frac{x-1}{2})$ ∀ x.

Show that g must be a constant function.

Ans:- g(x) = g $(\frac{x-1}{2})$

$\Rightarrow$ g $(\frac{x-1}{2})$ = g $(\frac{\frac{x-1}{2}-1}{2})$ = g $(\frac{x-3}{4})$

Again putting x= $\frac{x-1}{2}$

g $(\frac{x-1}{2})$= g $(\frac{x-7}{2})$ and so on

Generally we have, g$(\frac{x-1}{2})$= g $(\frac{x-(2^n-1)}{2^n})$

$\therefore$ g (x)= g $(\frac{x}{2^n} - 1 + \frac{1}{2^n})$

$\therefore \lim_{n \to \infty} g\ (x) = g\ (-1)$

$\Rightarrow$ g (x)= g (-1)= constant $\forall$ x.

5. Find the greatest and least value of the function f (x)= $x^3 - 3x^2 + 2x + 1$ in [2, 3].

Ans:- f (x)= $x^3 - 3x^2 + 2x + 1$

f′(x)= $3x^2 - 6x + 2$;   f′ (2) = 2 > 0;   f′ (3) =11 > f′ (2)

$\therefore$ f(x) is an increasing function

Note that f″(x) = 6x -6 > 0 $\forall$ x ∈ [2, 3]

$\therefore$ f(x) is concave.

Thus the function has min. value at x= 2 and max. Value at x= 3

$\therefore$ minimum value= f(2)=1 $\therefore$ maximum value = f(3) = 7

6. Let F (x) = $\sum_{k=0}^{n} a_k x^k$, where a_k satisfy $\sum_{k=0}^{n} \frac{a_k}{k+1} = 0$ so that there exists a real root of   f (x) = 0 in the interval (0, 1)

Ans:- F (x) = $\int_0^x f(t)dt =$ $\int_0^x (\sum_{k=0}^{n} a_k t^k)dt$ = $\sum_{k=0}^{n} a_k \cdot \frac{x^{k+1}}{k+1}$

Clearly F (x) satisfies the conditions of Rolle’s Theorem as F (0) = 0 and

F (1) =$\sum_{k=o}^{n} \frac{a_k}{k+1} = 0$

Hence ∃ a ‘c’ ∈ (0, 1) ∋ F′(c) =0 ⇒ F (c) =0

7. Show that 1+ $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots . + \frac{1}{n}$ can never be an integer value.

Ans:- We are to show:- 1+ $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots . + \frac{1}{n} = \frac{an\ odd\ number}{an\ even\ number}\ \forall\ n > 1$

Let, P (n): 1+ $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots . + \frac{1}{n}$= $\frac{an\ odd\ number}{an\ even\ number}\ \forall\ n > 1$

When n=2, LHS= 1+ $\frac{1}{2} = \frac{3}{2} = \frac{an\ odd\ number}{an\ even\ number}$

$\therefore$ P (2) is true. Let P (m)be true

$\Rightarrow$ 1+ $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots . + \frac{1}{m} =$ $\frac{an\ odd\ number}{an\ even\ number} = \frac{k}{r}(say)$

Now, P (m+1) = $\frac{k}{r} + \frac{1}{m+1}$

m is odd or even,

But in case, it can be shown that

P (m)= $\frac{an\ odd\ number}{an\ even\ number}$

$\therefore$P (n) is true for all n ∈ℕ

8. Give an example of a function f:[a, b]⟶ ℝ ∋ |f (x)- f (y)| < (x-y)∀ x, y ∈ [a, b].

Prove that any function satisfying the above condition also satisfies

$\left|\int_a^b f(x)dx - (b-a)f(a)\right| \le \frac{1}{2}(b-a)^2$**, provided f (x) is integrable on [a, b].**

Ans:- f(x)= $\sin x$

| f (x)- f (y)| = $|\sin x - \sin y|$

= $|2\cos(\frac{x+y}{2})\sin(\frac{x-y}{2})| \le 2|\sin(\frac{x-y}{2})| \le 2|\frac{x-y}{2}|$=|x- y|

[∵ for x > 0, $\sin x \le x$]

$[|\frac{f(x)-f(y)}{x-y}| \le 1 \quad \therefore |\underset{x\to y}{\text{lt}} \frac{f(x)-f(y)}{x-y}| \le 1 \Rightarrow |f'(y)| \le 1]$

Let $\int_{-\infty}^{u} f(x)dx = F(u)$

By Taylor's theorem:- F(b)= F (a)+ (b-a)f' (a)+ $\frac{(b-a)^2}{2} f''(c*)$

⇒ F (b)-F (a) = (b-a) f (a) + $\frac{(b-a)^2}{\lfloor 2}$f'(c*)

$\Rightarrow \left|\int_a^b f(x)dx - (b-a)f(a)\right| = \frac{(b-a)^2}{2}\left|f'(c*)\right| \le \frac{(b-a)^2}{2}$.

9. Let f: ℝ → ℝ be differentiable and assume there is no x *in* ℝ ∋ f(x) = f'(x) = 0. Show that S = {x| 0≤ $x \le 1$; $f(x) = 0$} is finite.

Ans:- Consider $f^{-1}(\{0\})$. Since {0} is closed and f is continuous $f^{-1}(\{0\})$ is closed. Therefore, S=[0,1]∩ $f^{-1}(\{0\})$ is closed and bounded subset of ℝ. Hence, S is complete.

Assume S is infinite.

Then there is a limit point $x \in S$;

i.e. there is a sequence $\{x_n\}$ of distinct points in S which converges to x.

Also, as all points are in S, $f(x_n) = f(x) = 0 \ \forall\, n \in N$.

We now show that f'(x) =0.

Since.$|x_n - x|$⟶0 ,

so

$$f'(x) = \lim_{n\to\infty} \frac{f(x+(x_n-x))-f(x)}{x_n-x}$$

$$= \lim_{n\to\infty} \frac{f(x_n)-f(x)}{x_n-x}$$

$$= 0$$

The last equality holds since f(x) = f(x_n) =0 holds $\forall\, n \in N$.

10. If f'(a) = f''(a) = f'''(a)=0 but $f^{(iv)}$(a)>0 and $f^{(iv)}$(x) is continuous at x=a. Then show that f(a) is local minimum.

Ans:- Toylor's theorem states that

f(x) = f(a) + (x-a)f'(a) +$\frac{(x-a)^2}{2!}$f''(a) +……+$\frac{(x-a)^n}{n!} f^n(c)$, a < c < x

Now, applying Taylor's theorem,

f(x) = f(a) +$\frac{(x-a)^4}{4!} f^{(iv)}$(c)

=> f(x) –f(a)= $\frac{(x-a)^4}{4!} f^{(iv)} \geq 0 \ \forall\ a \in (a-\in, a+\in)$

i.e. f(x) ≥ f(a).

So, f(a) is the local minimum.

ISI SUBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 5

1. The four digit number aabb is a square. Find the number.

Ans:- aabb = n^2

Then n^2 = 1100a + 11b

= 11(100a + b)

= 11(99a + a + b)

Since, n^2 is divisible by11^2, we see that 11 | (a+b)

i.e. a+b=11. Since n^2 *is* a square , bcan't be 0,1,2,3,5,7 or 8 . Checking the remaining we see that 7744= 88^2

2. Find the maximum value of $\boldsymbol{cos\alpha_1 . cos\alpha_2 . cos\alpha_n}$, under the restrictions

0 <$\boldsymbol{\alpha_1, \alpha_2, \ldots \alpha_n \leq 2 \ \ cot\alpha_1 . cot\alpha_2 . cot\alpha_n}$ = 1.

Ans.

Given, $(\cot\alpha_1)$. $(\cot\alpha_2)$ .... $(\cot\alpha_n) = 1$

$\Rightarrow \cos\alpha_1 . \cos\alpha_2 \ldots . \cos\alpha_n =$ $\sin\alpha_1 . \sin\alpha_2 \ldots . \sin\alpha_1$ ……………(1)

Now, $(\cos\alpha_1 . \cos\alpha_2 \ldots . \cos\alpha_n)^2 =$ $(\cos\alpha_1 . \cos\alpha_2 \ldots . \cos\alpha_n)(\cos\alpha_1 . \cos\alpha_2 \ldots . \cos\alpha_n)$

$=$ $(\cos\alpha_1 . \cos\alpha_2 \ldots . \cos\alpha_n)($ $\sin\alpha_1 . \sin\alpha_2 \ldots . \sin\alpha_n)$ [from (1)]

$= \frac{1}{2^n} \sin 2\alpha_1 . \sin 2\alpha_2 \ldots . \sin 2\alpha_n \leq \frac{1}{2^n}$

$\therefore (\cos\alpha_1 . \cos\alpha_2 \ldots . \cos\alpha_n)^2 \leq \frac{1}{2^n}$

$\therefore \cos\alpha_1 . \cos\alpha_2 \ldots . \cos\alpha_n \leq \sqrt{\frac{1}{2^n}} \leq \frac{1}{2^{\frac{n}{2}}}$

$[\because \cos\alpha_i \geq 0]$

3. f(x, y) = 0 is a circle such that f(0, $\boldsymbol{\lambda}$) = 0 and f($\boldsymbol{\lambda}$,0) = 0 have equal roots and f(1,1)= - 2 then the radius of the circle is

(a) 4 **(b) 8**
(c) 2 **(d) 1**

Ans. (c)

Let f(x, y) = $x^2 + y^2 + 2gx + 2fy + c =$ 0 *be the required circle*

f(0, λ) = $\lambda^2 + 2f\lambda + c = 0$ …………(1)

f(λ, 0) = $\lambda^2 + 2g\lambda + c$= 0…………….(2)

∵ (1) and (2) have equal roots.

∴ D= 0

$\Rightarrow f^2 = g^2 = c$

∴ f(x, y) = $x^2 + y^2 + 2gx \pm 2fy + g^2 = 0$

f(1, 1) = - 2 $\Rightarrow g^2 + 2g \pm 2g + 4 = 0$

when f = - g, $g^2 = -4$ (not possible)

∴ f = g and g = f = -2 and c = 4

∴ Radius of circle is 2.

4. If $\frac{1}{1!9!} + \frac{1}{3!7!} + \frac{1}{5!5!} = \frac{2^m}{n!}$, then orthocenter of the triangle having sides x - y + 1 = 0, x + y + 3 = 0 and 2x + 5y - 2= 0 is

(a) (2m- 2n, m- n) (b) (2m- 2n, n- m) (c) (2m- n, n+ m) (d) none

Ans. (a)

$\frac{2^m}{\underline{|n}} = \frac{2}{\underline{|1}\underline{|9}} + \frac{2}{\underline{|3}\underline{|7}} + \frac{1}{\underline{|5}\underline{|5}}$

$= \frac{1}{\underline{|10}}\left[2.10_{C_1} + 2.10_{C_3} + 10_{C_5}\right]$

$= \frac{1}{\underline{|10}}\left[10_{C_1} + 10_{C_3} + 10_{C_5} + 10_{C_7} + 10_{C_9}\right]$ $\left[\because\ 10_{C_1} = 10_{C_9}, 10_{C_3} = 10_{C_7}\right]$

$= \frac{2^9}{\underline{|10}}$

∴ m= 9, n= 10

Two sides of the given triangle are x - y + 1 = 0 and x + y + 3 = 0.

Clearly they are perpendicular. Therefore orthocenter of the triangle will be point of intersection of these two lines which is (-2, -1).

5. If the algebraic sum of the perpendicular distance from the points (3, 1), (-1, 2), and (1, 3) to a variable line be zero, and $\begin{vmatrix} x^2+1 & x+1 & x+2 \\ 2x+3 & 3x+2 & x+4 \\ x+4 & 4x+3 & 2x+5 \end{vmatrix} = mx^4 + nx^3 + px^2 + qx + r$ be an identity in x, then the variable line always passes through the point

(a) (-r, m) (b) (-m, r)
(c) (r, m) (d) (2r, m)

Ans. (c)

Let the variable line be ax + by + c= 0

Given,

$\frac{3a+b+c}{\sqrt{a^2+b^2}} + \frac{-a+2b+c}{\sqrt{a^2+b^2}} + \frac{a+3b+c}{\sqrt{a^2+b^2}} = 0$

⟹ 3a+ 6b+ 3c⟹ 0 ⟹ a+ 2b +c = 0

⟹ line ax + by +c = 0 passes through the point (1, 2)

Also in the given identity putting x= 0, we get r= 1 and equating the coefficient of x^4, we get m = 6 – 4 = 2.

6. If a, b, c are positive integers such that abc + ab + bc + ca + a + b + c = 1000.

Find the value of a + b +c ?

Ans:- abc +ab + ac + bc +a + b+ c = 1000

⇒ a (bc +b+ c)+ a+ b+ c+ bc +1 = 1000+1

⇒ a (bc +b+ c+ 1)+ (b+ c+ bc+1) = 1001

⇒ (a+ 1) (b+ 1) (c+ 1) = 1001 = 13× 7 × 11

⇒ a= 12, b= 6, c = 10,

∴ a + b + c = 28.

7. Find minimum value of $2^{\cos x}+2^{\sin x}, 0 \leq x \leq 2\pi$?

Ans:- By AM $\geq GM$ inequality, we have

$2^{\cos x}+2^{\sin x} \geq 2\sqrt{2^{\cos x+\sin x}}$

$= 2.\ 2^{\frac{\sin x+\cos x}{2}}$

Note that the RHS will be minimum if $\sin x+\cos x$ is minimum, i.e. when both sin x and cos x are minimum, i.e. at x= 225^0, sin x = $-\frac{1}{\sqrt{2}} = \cos x$

$\therefore\ 2^{\cos x}+2^{\sin x} \geq 2.2^{\frac{-\frac{2}{\sqrt{2}}}{2}} = 2.2^{-\frac{1}{\sqrt{2}}}$

$\therefore$ The minimum value is $2^{1-\frac{1}{\sqrt{2}}}$

8. f (x) = $\frac{e^{2x-1}}{1+e^{2x-1}}$. Then f($\frac{1}{1234}$)+ f($\frac{3}{1234}$)+ … + f ($\frac{1231}{1234}$)+ f ($\frac{1233}{1234}$)= ?

Ans:- f ($\frac{1}{2}$)= $\frac{e^{1-1}}{1+e^{1-1}} = \frac{1}{2}$

Now, f (x) + f(1-x)=1

So, f ($\frac{1}{1234}$)+ f(1- $\frac{1}{1234}$)=1

⋮

Now there are 308 terms up to f ($\frac{615}{1234}$)+ f (1-$\frac{615}{1234}$)= 1

Now, f($\frac{617}{1234}$)+ f(1-$\frac{617}{1234}$) =1

$\Rightarrow$ f ($\frac{1}{2}$)+ f ($\frac{1}{2}$) = 1

i.e. f ($\frac{617}{1234}$) = ½

$\therefore$ f ($\frac{1}{1234}$) + f ($\frac{3}{1234}$) + …+ f ($\frac{615}{1234}$) +f ($\frac{617}{1234}$) + f ($\frac{619}{1234}$) + …+ f ($\frac{1233}{1234}$)

= 308 + 0.5 = 308.5

9. Show that $\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$.

Ans:- Let f(x)=$\tan^{-1} x$.

From Mean-value theorem,$\frac{f(b)-f(a)}{b-a} = f'(c)$

$\rightarrow \frac{\tan^{-1} b-\tan^{-1} a}{b-a} = f'(c) = \frac{1}{1+c^2}$; a < c < b.

$\rightarrow \tan^{-1} b - \tan^{-1} a = \frac{b-a}{1+c^2}$.

$\therefore\ \frac{b-a}{1+b^2} < \frac{\tan^{-1} b-\tan^{-1} a}{1} < \frac{b-a}{1+a^2}$.

10. Let s=$\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{10000}$ and I=$\int_0^{1000}\sqrt{x}\,dx$. Show that I $\leq s \leq I+100$.

Ans:- I=$\int_0^{1000}\sqrt{x}\,dx = \int_0^1\sqrt{x}\,dx + \int_1^2\sqrt{x}\,dx+\cdots+\int_{9999}^{10000}\sqrt{x}\,dx$

$\rightarrow\int_0^1 0.dx + \int_1^2\sqrt{1}$ dx +….+$\int_{9999}^{10000}\sqrt{9999}\,dx \leq I \leq \int_0^1 1.dx + \int_1^2\sqrt{2}\,dx+\cdots+\int_{9999}^{10000}\sqrt{10000}$ dx

$\rightarrow\sqrt{1}$ +$\sqrt{2}$ +…..+$\sqrt{9999} \leq I \leq \sqrt{1}$ +$\sqrt{2}$ +…..+$\sqrt{10000}$

$\rightarrow$I $\leq S$

Also, S $\leq I+\sqrt{10000}$

$\therefore$ I $\leq S \leq I + 100$

ISI SUBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 6

1. Prove that

$\mathbf{1 < \frac{1}{1001} + \frac{1}{1002} + \cdots + \frac{1}{3001} < \frac{4}{3}}$

Ans:- consider 2001 numbers $\frac{1}{k}, 1001 \leq k \leq 3001$

Using AM- HM inequality, we get

$(\sum_{k=1001}^{3001} k)(\sum_{k=1001}^{3001} \frac{1}{k} \geq (2001)^2$

But $\sum_{k=1001}^{3001} k = (2001)^2$

Hence we get the inequality $\sum_{k=1001}^{3001} \frac{1}{k} > 1$

On the other hand grouping 500 terms at a time, we also have

S= $\sum_{k=n+1}^{3n+1} \frac{1}{k} < \frac{500}{1000} + \frac{500}{1500} + \frac{500}{2000} + \frac{500}{2500} + \frac{1}{3001} < \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{3001} = \frac{3851}{3000} < \frac{4}{3}$

[Remarks:- if S =$\sum_{k=1001}^{3001} \frac{1}{k}$, there are (2n+ 1) terms in the sum and the middle term is $\frac{1}{2n+1}$; then $\frac{29}{27} < S < \frac{7}{6}$].

2. How many ordered triplet (x, y, z) of non zero real numbers have the property that each number is the product of the other two?

Ans:- x = yz, y = zx , z = xy

$\therefore$ xyz = $(xyz)^2$

i.e. xyz= 0 or 1

now, xyz = $x^2 = y^2 = z^2$

$\Rightarrow$ |x|= |y|= |z|= 1

However the remaining 4 cases are: (1, 1, 1), (-1, -1, -1), (-1, 1, -1) or (1, -1, -1); i.e. it has 4 solutions.

3. If X= {n: n is a positive integer, n$\leq$ 50}, A = {n ∊ X:n is even} and B= {n ∊ X: n is a multiple of 7}, then what is the number of elements in the smallest subset of X containing both A and B ?

Ans:- The number of integers ≤n and divisible by k is given by $[\frac{n}{k}]$, where [.] denotes the greatest integer function.

Accordingly, n(A)= $[\frac{50}{2}]$ = 25 , n(B)= $[\frac{50}{7}]$ = 7

n (A$\cap B$)= $[\frac{50}{14}]$=3.

n (A ∪ B)= $n\ (A) +\ n\ (B) -$ n (A ∩ B)= 25+7-3
=29.

4. If $\frac{1}{x(x+1)(x+2)....(x+n)} = \sum_{r=0}^{n} \frac{A_r}{x+r}$, then A$_r$ = ?

Ans:- By method of representation for partial fraction Ar , is obtained by putting x+ r= 0,

i.e. x= - r in all factors, except (x+ r).

∴ Ar =
$$\frac{1}{-r(-r+1)\dots(-r+r+1)(-r+r+1)(-r+r+2)\dots\dots(-r+n)}$$

$$= \frac{1}{(-1)^r\{r!\}\{(n-r)!\}}$$

5. If $a_1, a_2, \dots, a_n$ are non-ve and $a_1, a_2, \dots, a_n = 1$. Show (1+a_1) (1+a_2)….. (1+a_n)$\geq 2^n$

Ans:- By AM ≥ GM

$(\frac{1+a_i}{2}) \geq \sqrt{a_i}\ \forall\ i = 1(1)n$

Multiplying all these

(1+a_1) (1+a_2)….. (1+a_n)≥ $2^n\sqrt{a_1, a_2, \dots, a_n}$ = 2^n

6. If $a_i > 0\ \forall\ i = 1(1)n$. Prove that $(a_1 + a_2 + \dots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}\right) > n^2$

Ans:- AM> GM

$$\frac{a_1, a_2, \dots, a_n}{n} > (a_1, a_2, \dots, a_n)^{\frac{1}{n}}$$

And $\quad \frac{\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}}{n} > (\frac{1}{a_1}.\frac{1}{a_2}.\dots.\frac{1}{a_n})^{\frac{1}{n}}$

$\Rightarrow (a_1 + a_2 + \dots + a_n) > n(a_1, a_2, \dots, a_n)^{\frac{1}{n}}$

And $\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}$> n$(\frac{1}{a_1 \dots\dots a_n})^{\frac{1}{n}}$

$\Rightarrow (a_1 + a_2 + \dots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}\right) > n^2$

7. Show that $\frac{a_1}{a_2} + \frac{a_2}{a_3} + \dots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} > n$

Ans:- AM > GM

$$\frac{1}{n}\left(\frac{a_1}{a_2} + \frac{a_2}{a_3} + \dots + \frac{a_n}{a_1}\right) > \left(\frac{a_1}{a_2}.\frac{a_2}{a_3}.\dots.\frac{a_n}{a_1}\right)$$

or, $\left(\frac{a_1}{a_2} + \frac{a_2}{a_3} + \dots + \frac{a_n}{a_1}\right) > n$

8. If $a_1, a_2, \dots, a_n$ be non- negative real numbers such that

$a_1 + a_2 + a_3 + \dots + a_n = m$, then prove that $\sum_{i<j} a_i a_j \leq \frac{m^2}{2}$

Ans:- $m^2 = (a_1 + \dots + a_n)^2$

$m^2 = {a_1}^2 + \dots + {a_n}^2 + 2\sum_{i<j} a_i a_j$

$\Rightarrow \sum_{i<j} a_i a_j \leq \frac{m^2}{2}$. $\quad$ [$\because {a_1}^2 + \dots + {a_n}^2 \geq 0$]

9. Let A be a set containing n elements. If the number of elements in the set,

B = {(x, y, z) : x ∈A, y ∈A, z ∈A and x, y, z, are not all distinct} is equal to 280,

then find the value of n?

Ans:- (b)

According to question $n^3 - \binom{n}{3}.3! = 280$

$\therefore n^3 - n(n-1)(n-2) = 280$

⟹ n($n^2 - n^2 + 3n - 2$)= 280

⟹ n(3n- 2)= 280= 10(3.10- 2)

∴ n= 10.

10. If the maximum number of trails required to open all locks when there are n locks and n keys is 105, then what is the value of n?

Ans:- Max. number of trails to open the first lock = n – 1

Max. number of trails to open the second lock = n – 2 and so on.

∴ Maximum number of trails to open all the locks = (n- 1) + (n -2) +…+1 = $\frac{(n-1)n}{2}$

Given, $\frac{(n-1)n}{2} = 105$

⟹ n(n -1)= 210= 15× 14

∴ n= 15

ISI SUBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 7

1. Find the no. of real roots of the equation

2cos $\left(\frac{x^2+x}{6}\right) = 2^x + 2^{-x}$

Ans:- cos $\left(\frac{x^2+x}{6}\right) = \frac{2^x+2^{-x}}{2} \geq \sqrt{2^x + 2^{-x}} =$ 1, by AM ≥ GM

But cos $\left(\frac{x^2+x}{6}\right) \leq 1$

∴ cos $\left(\frac{x^2+x}{6}\right) = 1 = \cos(\frac{n\pi}{2})$

∴ $\frac{x^2+x}{6} = \frac{n\pi}{2}$

$\Rightarrow x^2 + x - 3n\pi = 0$

Here $b^2 - 4ac = 1 - 4.1(-3n\pi)$

= 1+12 $n\pi \geq 0 \ \forall\ n = 0, 1, 2, \dots\dots$

∴there are infinitely many roots.

2. Find the no. of real roots of the polynomial f(x) = $x^5 + x^3 - 2x + 1$.

Ans:- Descarte's sign rule:-

f(x)= 0 has two sign changes.

∴ No. of +ve roots ≤ 2

f(-x)= 0 = -$x^5 - x^3 + 2x + 1$

- - + +

∴f(-x) has the one sign change.

∴no. of –ve roots ≤1

⇒ there is no negative roots,

∴ As complex roots occure in pair, so, there is one +ve roots.

3. Let f(x)= $x^3 + 3x - 2$, x ∊ ℝ, Show that f(x)=0 has only one real root.

Ans:- f'(x)= $3(x+1)^2 > 0$

f(-1)= -6 < 0, f(2)> 0

∴ It has one root.

4. If $\lim\limits_{x\to\infty} f(x) = 1$ and $\lim\limits_{x\to\infty} f'(x) = \alpha$, find α .

Ans:- Let f(x)=1+$\frac{k}{x^p}$, p> 0

Then $\underset{x\to\infty}{\text{lt}} f(x) = 1$

$\therefore \alpha = \underset{x\to\infty}{\text{lt}} f'^{(x)} = \underset{x\to\infty}{\text{lt}} \frac{k(-p)}{x^{p+1}} = 0$

5. Let f (x)= $\begin{cases} \boldsymbol{x \ \ if \ x \ \in [0,2]} \\ \boldsymbol{0 \ \ if \ x \ \notin [0,2]} \end{cases}$ **and g(x)=** $\begin{cases} \boldsymbol{1 \ \ if \ x \ \in [0,2]} \\ \boldsymbol{0 \ \ if \ x \ \notin [0,2]} \end{cases}$

Let A = {(x, y): x+ y ≤ 3}, then find the value of the integral $\boldsymbol{\int_A \int f(x)g(y)dxdy}$

ANS:- $\int_A \int f(x)g(y)dxdy$

$= \int \int x\,.1.\,dxdy$

{(x, y): 0 < x < y < 2 & x+ y≤ 3} where 0 < x < 1 0 < y < 2 & 1≤ x < 2 , 0 ≤ $y < 3 - x$

$= \int_0^1 \int_0^2 xdydx + \int_1^2 (\int_0^{3-x} xdy)dx$ = 1+ $\left[\frac{3}{2}x^2 - \frac{\alpha^3}{3}\right]_1^2 = \frac{19}{6}$.

6. Using the change of variable evaluate $\boldsymbol{\int_R \int xydxdy}$**, when the region R is bounded by the curves xy = 1, xy = 3, y = 3x, y = 5x in the 1st coordinate.**

Ans:-

Then transformation domain is D = {(u, v): $1 \le u \le 3, 3 \le v \le 5\}$

$\therefore$ y = $\sqrt{uv}\,, x = \sqrt{\frac{u}{v}}$

Jacobian of the transformation is

J = $\begin{vmatrix} \frac{1}{\sqrt{v}}.\frac{1}{2\sqrt{u}} & \sqrt{u}(-\frac{1}{2}.\frac{1}{v\sqrt{v}}) \\ \sqrt{v}.\frac{1}{2\sqrt{u}} & \sqrt{u}.\frac{1}{2\sqrt{v}} \end{vmatrix}$

$= \frac{1}{4v} + \frac{1}{4}.\frac{1}{v} = \frac{1}{2v}$.

I = $\int_3^5 \int_1^3 u.\frac{1}{2v}dudv$ = $[\left[\frac{u^2}{4}\right]_1^3 \ [lnv]_3^5$ = $\log\left(\frac{25}{9}\right)$.

7. Find the value of $\boldsymbol{\int_{x^2+y^2\le 1} \int \frac{2}{1+x^2+y^2} dxdy}$ **.**

Ans:- Let x = rcosθ, y= r sinθ

As $x^2 + y^2 \le 1$

$\therefore$ 0< r < 1

& 0 < θ < 2π

J= $\begin{vmatrix} cos\theta & sin\theta \\ rsin\theta & rcos\theta \end{vmatrix} = r$

$\therefore$ I = $\int_0^{2\pi} \int_0^1 \frac{2}{1+r^2} r.drd\theta$

= [ln $(1 + r^2)]_0^1 \ [\theta]_0^{2\pi}$

= 2πlog e^2.

8. Give an example of each of the types of functions

(a) The function which is continuous but not differentiable at only
- **(i) One point (ii) two points**
- **(iii) ten points**

(b) The function which is discontinuous at
- **(i) Four points (ii) 15 points.**

(c) The function which is differentiable once but not twice at
- **(i) One points (ii) three points.**

Ans:- (a) (i) f(x)= |x|

(ii) f(x)= |x|+|x-1|

(iii) f(x)=$\sum_{k=1}^{10} |x-k|$

(b) (i) f(x)= [x], 0< x < 5

(ii) f(x)= [x], 0 < x < 16

(c) (i) f(x)= $\begin{cases} -x^2, x \le 0 \\ x^2, x > 0 \end{cases}$

$= x^2 sign(x), where\ sign(x) = \begin{cases} -1\ , if\ x \le 0 \\ 1\ , if\ x > 0 \end{cases}$

(ii) f(x)= $x^2 sign(x) + (x-1)^2 sign(x-1) + (x-2)^2 sign(x-2)$.

9. If the angles of a triangle are in the ratio 1: 2: 3, then the sides opposite to the respective angles are in the ratio

(a) 1 : $\sqrt{2}$: $\sqrt{3}$ **(b) 1 : $\sqrt{3}$: 2** **(c) 1 : $\sqrt{2}$:3**
(d) 1: 2: 3

Ans:- (b)

Let the angles be x, 2x, 3x, then

6x = 180° $\Longrightarrow x = 30°$

∴ Angles are 30°, 60°, 90°

∴ Ratio of sides is

Sin 30° ∶ sin 60° : sin 90° $i.e. 1 : \sqrt{3} : 2$

10. If $\vec{a}$ + $\vec{b}$ + $\vec{c}$ are non-coplanar, then $\frac{[\vec{a}+2\vec{b}\vec{b}+2\vec{c}\vec{c}+2\vec{a}]}{[\vec{a}\vec{b}\vec{c}]}$ =

(a) 3 **(b) 9**
(c) 8 **(d) 6**

Ans:- (b)

$$\frac{[\vec{a} + 2\,\vec{b}\vec{b} + 2\,\vec{c}\vec{c} + \ 2\,\vec{a}]}{[\vec{a}\vec{b}\vec{c}]}$$

$$= \frac{[(\vec{a} + 2\,\vec{b}) \times (\vec{b} + 2\,\vec{c}).(\vec{c} + \ 2\,\vec{a})]}{[\vec{a}\vec{b}\vec{c}]}$$

$$= \frac{9[\vec{a}\vec{b}\vec{c}]}{[\vec{a}\vec{b}\vec{c}]} = 9$$

ISI SUBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 8

1. Find the maximum value of $(1 + 2x)^2(3 - 2x)$.

Ans:- AM ≥ GM

$\Rightarrow \frac{2.\frac{1+2x}{2}+(3-2x)}{2+1} \ge 3\sqrt{\left(\frac{1+2x}{2}\right)^2(3-2x)}$

$\Rightarrow (1+2x)^2(3-2x) \le 2^2(\frac{4}{3})^3$

2. Minimize 3x+ 4y subject to $x^2y^3 = 6$.

Ans:- AM ≥ GM

$\Rightarrow \frac{\frac{3x}{2}+\frac{3x}{2}+\frac{4y}{3}+\frac{4y}{3}+\frac{4y}{3}}{5} \ge 5\sqrt{(\frac{3x}{2})^2(\frac{4y}{3})^3}$

$\Rightarrow \frac{3x+4y}{5} \ge 2, as\ x^2y^3 = 6$

⇒ 3x + 4y ≥ 10

(3x+ 4y) attains its minimum sub. To $x^2y^3 = 6$ when '='holds in AM ≥ GM

i.e. iff $\frac{3x}{2} = \frac{4y}{3} \Rightarrow x = \frac{8}{9}y$

$\therefore x^2y^3 = 6 \Rightarrow y = \frac{3}{2}, x = \frac{4}{3}$

3. Evaluate the determinant of the matrix

$\Delta = \begin{pmatrix} \mathbf{1+x_1y_1} & \cdots & \mathbf{1+x_ny_n} \\ \vdots & \ddots & \vdots \\ \mathbf{1+x_ny_1} & \cdots & \mathbf{1+x_ny_n} \end{pmatrix}$

Ans:- (a) $|\Delta| = det\begin{pmatrix} 1 & x_1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 1 & x_n & 0 & \cdots & 0 \end{pmatrix} \times$ $det\begin{pmatrix} 1 & y_1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 1 & y_n & 0 & \cdots & 0 \end{pmatrix}$ = 0

4. Do there exist function f : R⟶R and g: R⟶R ∋

f (g (x))= $\boldsymbol{x^2}$ *and* $\boldsymbol{g\left(f(x)\right) = x^3}$ ∀ $\boldsymbol{x}$ ∋ ℝ

Ans:- f(x^3)= f (g(f(x))= $\{f(x)\}^2$

Now x ∊ {-1, 0} ⇒x^3= x ⇒ f(x) = $\{f(x)\}^2$

⇒ x ∊ {0, 1}

Hence ∃ different a, b ∊ {-1, 0, 1} such that f(a)= f(b) But then $a^3 = g\left(f(a)\right) = g\left(f(b)\right) = b^3$, a contradiction . Thus the function f and g satisfying the given conditions don't exist.

5. Let f be a twice differentiable function such that

f″(x) = -f (x) ; f′ (x) =g (x) and h (x) =$\boldsymbol{f^2}$ (x) +$\boldsymbol{g^2}$ (x). Given that h (5) =1 and find h(10).

Ans:- Take f (x) =sinx

f′ (x) = -cosx=g (x)

f″ (x) =- sinx= f (x)

h(x) = f^2 (x) +g^2 (x)

= sin^2 x+cos^2 x =1

∴ h (5) = h (10) = 10.

6. Test the convergence of the series x+ $\boldsymbol{\frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!}}$ + [Assume x> 0 and examaine all possible cases]

Ans: - we have $u_n = \frac{n^n.x^n}{n!}$

$u_{n+1} = \frac{(n+1)^{n+1}.x^{n+1}}{(n+1\_!}$

$\Rightarrow \frac{u_n}{u_{n+1}} = \frac{1}{x}.\frac{n^n}{n!}.\frac{(n+1)!}{(n+1)^{n+1}}$

$= \frac{1}{x}.\frac{n^n}{(n+1)^n} = \frac{1}{x}.\frac{n^n}{n^n\{1+(1/n)\}^n}$

$\Rightarrow \lim \frac{u_n}{u_{n+1}} = \frac{1}{x} lim \frac{1}{[1+\frac{1}{n}]^n} = \frac{1}{ex}$.

∴by ratio test, $\sum u_n$ is convergent if $\frac{1}{ex} > 1 \Rightarrow x < \frac{1}{e}$ and $\sum u_n$ is divergent if x >$\frac{1}{e}$

But ratio test fails when x = $\frac{1}{e}$. Here we can use logarithm test.

When x = $\frac{1}{e}$, we have $\frac{u_n}{u_{n+1}} = e.\frac{1}{(1+\frac{1}{n})^n}$

$\Rightarrow nlog \frac{u_n}{u_{n+1}} = n\ [14 - nblog\{1+1/n\}]$

=n[1-n($\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3}$ ... ...)]

= n[$\frac{1}{2n}$-$\frac{1}{3n^2} + \cdots$.]

= $\frac{1}{2} - \frac{1}{3n} + \cdots$

$\therefore \lim_{n\to\infty} nlog \frac{u_n}{u_{n+1}} = \frac{1}{2} < 1,$

∴ $\sum u_n$ is divergent at x= $\frac{1}{e}$.

7. Find the maximum value of $\iint dxdy$ as a function of m , 0 < m< 1, where, limit of integration is Δ= {(x, y): $\frac{x^2}{m}+\frac{y^2}{1-m}\leq 1$}.

Ans:- $\iint$ dxdy =

$\sqrt{m}\sqrt{1-m}\int_0^{2\pi}\left(\int_0^1 rdr\right)d\theta$; where $\frac{x}{\sqrt{m}}=r\cos\theta, \frac{y}{\sqrt{1-m}}$ = r$\sin\theta$

$=\pi\sqrt{m}\sqrt{1-m}$

$\leq \pi.\frac{m+1-m}{2}$ ($\because$ AM$\geq GM$)

$=\frac{\pi}{2}$.

8. Find maximum value of xyz subject to $x^2+2y^2+9z^2=6$.

Ans:- AM$\geq GM$ => $\frac{x^2+2y^2+9z^2}{3}\geq\sqrt[3]{x^2.2y^2.9z^2}$

$\Rightarrow (\frac{x^2+2y^2+9z^2}{3})^3\geq 18x^2y^2z^2$

$\Rightarrow 2^3\geq 18x^2y^2z^2$

$\Rightarrow (xyz)^2\leq\frac{8}{18}\Rightarrow$ xyz $\leq\frac{2}{3}$.

9. If in a $\Delta ABC,\sum\cos 3A=1$, then show that ΔABC is an obtuse angled triangle.

Ans:-

We have A+ B + C= π and $\sum\cos 3A=1$ but $\sum\cos 3A=1+4\sin\frac{3A}{2}.\sin\frac{3B}{2}.\sin\frac{3c}{2}$

$$\Rightarrow \sin\frac{3A}{2}.\sin\frac{3B}{2}.\sin\frac{3c}{2}=0$$

$$i.e. \sin\frac{3A}{2}=0\ or\ \sin\frac{3B}{2}=0\ or\ \sin\frac{3c}{2}=0$$

$$or\ A=\frac{2\pi}{3}\ or\ B=\frac{2\pi}{3}\ or\ C=\frac{2\pi}{3}$$

$$Also, r=(s-a)\tan\frac{A}{2}$$
$$=(s-b)\tan\frac{B}{2}$$
$$=(s-c)\tan\frac{C}{2}$$

$$\Rightarrow r=\sqrt{3}(s-a) or\ r=\sqrt{3}(s-b) or\ r=\sqrt{3}(s-c)$$

10. If $(1+x+x^2+x^3)^{100}=\sum_{r=0}^{300}b_r x^r$ and $k=\sum_{r=0}^{300}b_r$, then $\sum_{r=0}^{300}r.b_r$ is

(a) 50.4^{100} **(b) $150.\ 4^{100}$**
(c) $300.\ 4^{100}$ **(d) none of these**

Ans:- (b)

Given $(1+x+x^2+x^3)^n=\sum_{r=0}^{3n}b_r x^r$

$or(1+x+x^2+x^3)^n=b_0+b_1x+b_2x^2+\cdots+b_{3n}x^{3n}$ …….(1)

Putting (1/x)

In place of x, we get

$(1+x+x^2+x^3)^n=b_0x^{3n}+b_1x^{3n-1}+b_{3n}$ ………..(2)

Equating the co- efficient of similar powers of x on the R.H.S. of (1) and (2) we have

$b_0=b_{3n}b_1=b_{3n-1}, b_r=b_{3n-r}$

Given, k= $\sum_{r=0}^{3n}b_r$ …..(3)

Let y= $\sum_{r=0}^{3n} r\ b_r$

Then y = $\sum_{r=0}^{3n}[3n-(3n-r)]b_r =$
$3n\ \sum_{r=0}^{3n}\ b_r - \sum_{r=0}^{3n}(3n-r)b_r$

$$= 3nk - \sum_{r=0}^{3n}(3n-r)b_{3n-r}\ [\because\ b_r =\ b_{3n-r}]$$

$$= 3nk - \sum_{r=0}^{3n} r\ b_r = 3nk - y$$

$\therefore 2y = 3nk \Longrightarrow y = \frac{3nk}{2}$

Putting x= 1 in the given expansion,

We get $\sum_{r=1}^{n}\ b_r =\ 4^n \Longrightarrow k =\ 4^n$

Putting n= 100, we get

y= $\frac{300.4^{100}}{2} = 150.4^{100}$

ISI SUBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 9

1. Evaluate $\int_0^{\pi} |\frac{1}{2} + \cos x\ |dx$

Ans:- note that, $\left|\frac{1}{2} + \cos x\right| =$
$\begin{cases} \frac{1}{2} + \cos x\ \ if\ \ \cos x > -\frac{1}{2} \\ -\frac{1}{2} - \cos x\ \ if\ \ \cos x < -\frac{1}{2} \end{cases}$

$=$
$\begin{cases} \frac{1}{2} + \cos x\ \ if x < \frac{2\pi}{3} \\ -(\frac{1}{2} + \cos x)\ if\ x > \frac{2\pi}{3} \end{cases}$

I = $\int_0^{\frac{2\pi}{3}}(\frac{1}{2} + \cos x)dx - \int_{\frac{2\pi}{3}}^{\pi}(\frac{1}{2} + \cos x)dx$

$= \frac{1}{2}.\frac{2\pi}{3} + \sin\frac{2\pi}{3} - [\frac{1}{2}\left(\frac{\pi}{3}\right) + \sin\pi - \sin\frac{2\pi}{3}]$

$= \frac{\pi}{6} + \sqrt{3}.$

2. If a circle intersects a hyperbola y = $\frac{1}{x}$ at 4 distinct points $\{x_i, y_i : i = 1(1)4\}$,

then prove that $x_1x_2 = y_3y_4$

Ans:- let the circle be $x^2 + y^2 = a^2$
hyperbola be y = $\frac{1}{x}$ is given.

For points of contact, $x^2 + \frac{1}{x^2} = a^2$

$\Rightarrow x^4 - x^2a^2 + 1 = 0$

It has 4 roots, so, Now, we know x = $\frac{1}{y}$

$\therefore$ i.e. $x_3 = \frac{1}{y_3}$ & $x_4 = \frac{1}{y_4}$

$\therefore x_1x_2 = y_3y_4$

3. If α_1, α_2,…………..α_n be the roots of x^n +1 =0, then (1- α_1) (1-α_2)…(1-α_n)=?

Ans:- x^n +1=(x- α_1) (x-α_2)…(x-α_n)

Putting x=1,

$\therefore$ 2=(1- α_1) (1-α_2)…(1-α_n)

4. The equation $\frac{1}{3}+\frac{1}{2}s^2+\frac{1}{6}s^3=s$ has exactly _____ solution(s) in [0,1].

Ans:- f(s)= $\frac{1}{3}+\frac{1}{2}s^2+\frac{1}{6}s^3-s$

f(1)=0

f(0)= $\frac{1}{3}$

f'(s)=s+$\frac{1}{2}s^2-1=\frac{s^2+2s-2}{2}=\frac{1}{2}(s-\alpha)(s-\beta)$;

here α= -1 -$\sqrt{3}$, β= -1+$\sqrt{3}$.

∴f'(s)=$\begin{cases} >0 & if\ x<\ \alpha\ or\ x>\beta \\ <0 & ,\alpha<x<\beta. \end{cases}$

There are two roots.

5. The number of terms free from radical sign in the expression of $(1+3^{\frac{1}{3}}+7^{\frac{1}{7}})^{10}$ is -----------

Ans:- The term from radical sign must be of the from (constant)$(3^{\frac{1}{3}})^{3m}.(7^{\frac{1}{7}})^{7n}$

where 3m+ 7n≤ 10 and m, n are non-negative integers.

This inequality has 6 solutions(m, n) ∊ {(0, 0), (1, 0), (0, 1), (1, 1), (2, 0), (3, 0)}

∴ Number of terms free from radical sign is 6.

<u>Second method</u>: General term in the given expansion= $\frac{\underline{|10}}{\underline{|\alpha}\underline{|\beta}\underline{|\Upsilon}}1^{\alpha}.3^{\frac{\beta}{3}}.7^{\frac{\Upsilon}{7}}$,

Where α+β+Υ= 10

∴ Possible value of β are: 0, 3, 6, 9

Possible values of Υ are: 0, 7

Possible values of α are: 0, 1, 2, …, 10

Since α+β+Υ= 10

∴ Possible triplets (α, β, Υ) will be

(0, 0, 10), (0, 7, 3), (3, 0, 7), (3, 7, 0), (6, 0, 4), (9, 0, 1)

6. Find all pairs of prime numbers p, q such that p+q = 18(p−q). Justify your answer.

Ans:- Solving the equation we have 19q = 17p.

Pair of prime number is (17, 19)

7. In a group of n persons, each person is asked to write down the sum of the ages of all the other (n − 1) persons. Suppose the sums so obtained are $S_1, \ldots, S_n$. It is now desired to find the actual ages of the persons from these values. Formulate the problem in the form of a system of linear equations.

Ans:- Let P_i be the actual age of person i.

Then the equation is given by

$P_i = S_i - \sum_i \frac{S_i}{n-1}$.

8. For n ≥ 4, prove that 1! + 2! + · · · + n! cannot be the square of a positive integer.

Ans:- 1! + 2! + 3! = 9 ≡ 3 $(mod\ 4)$

1! + 2! + 3! + …. + n! = Odd Number

By induction, we can say the given sum can't be the square of a positive number.

9. If f is continuous in [0, 1], then $\lim_{n\to\infty} \sum_{j=0}^{[\frac{n}{2}]} \frac{1}{n} f\left(\frac{j}{n}\right)$ = ?

Ans:- Express limiting sum in the form of definite integral. i.e. $\int_0^{1/2} f(x)dx$

10. Let α= $\underset{m\to\infty}{\text{Lt}}\ \underset{n\to\infty}{\text{Lt}}\ cos^{2m}\, n!\,\pi x$, where x is rational, β= $\underset{m\to\infty}{\text{Lt}}\ \underset{n\to\infty}{\text{Lt}}\ cos^{2m}\, n!\,\pi x$, where x is irrational, then the area of the triangle having vertices (α, β), (-2, 1) and (2, 1) is

(a) 2 **(b) 4**
(c) 1 **(d) none of these**

Ans. (a)

When x is rational: let x= $\frac{p}{q}$.

Then $|\underline{n}$ x= (1 . 2…q …n) $\frac{p}{q} = an\ integer$

$\therefore cos^2|\underline{n}x\pi = 1$

$\therefore \underset{m\to\infty}{\text{Lt}}\ \underset{n\to\infty}{\text{Lt}}\ cos^{2m}|\underline{n}x\pi = 1$

When x is irrational $|\underline{n}x$ is not an integer

$\therefore 0 \le cos^2|\underline{n}x\pi < 1$

$\Rightarrow \underset{m\to\infty}{\text{Lt}}\ \underset{n\to\infty}{\text{Lt}}\ cos^{2m}|\underline{n}x\pi = 0$

Thus α= 1, β= 0

Area of triangle =| ½ [1(1-1) -2 (1- 0)+ 2(0-1)]| = 2

ISI SUBJECTIVE SAMPLE PAPER WITH SOLUTIONS

SET – 10

1. Evaluate $\lim_{x\to 0}\{x^2(1+2+3+\cdots+\left[\frac{1}{|x|}\right])\}$

Ans.

$$\lim_{x\to 0}\{x^2(1+2+3+\cdots+\left[\frac{1}{|x|}\right])\}$$

$$= \lim_{x\to 0} x^2 \frac{\left[\frac{1}{|x|}\right]\left[\frac{1}{|x|}+1\right]}{2} = \lim_{x\to 0} \frac{|x|+1}{2} = \frac{1}{2}$$

2. Find $\lim_{x\to 0} \frac{|x|}{\sqrt{x^4+4x^2+7}} \sin\left(\frac{1}{3\sqrt{x}}\right)$.

Ans. $-1 \le \sin\left(\frac{1}{3\sqrt{x}}\right) \le 1$

$$\frac{-|x|}{\sqrt{x^4+4x^2+7}} \le \frac{|x|}{\sqrt{x^4+4x^2+7}} \sin\left(\frac{1}{3\sqrt{x}}\right) \le \frac{|x|}{\sqrt{x^4+4x^2+7}}$$

Taking limits, we will have

$$0 \le \frac{|x|}{\sqrt{x^4+4x^2+7}} \sin\left(\frac{1}{3\sqrt{x}}\right) \le 0$$

So answer is 0.

3. Evaluate the following two integrals directly and compare them

$\iint_{\sqrt{a}|x|\le 1,\sqrt{b}|y|\le 1} dxdy$ **and**

$\iint_{ax^2+by^{22}\le 1} dxdy$ **.**

Ans. $\iint_{\sqrt{a}|x|\le 1,\sqrt{b}|y|\le 1} dxdy$

$= \int_{\frac{-1}{\sqrt{a}}}^{\frac{1}{\sqrt{a}}}\int_{\frac{-1}{\sqrt{b}}}^{\frac{1}{\sqrt{b}}} dxdy$

$= \frac{4}{\sqrt{ab}}$

$> \iint_{ax^2+by^{22}\le 1} dxdy$

$= \frac{\pi}{\sqrt{ab}}$

4. Find $\frac{dy}{dx}$ where y = $(x^{logx})(logx)^x$, when x > 1.

Ans. Take log both sides,

$logy = (logx)^2 + xlog(logx)$

Now differentiate w.r.t. x both sides,

$$\frac{dy}{dx} = y\left(\frac{2}{x}logx + \frac{1}{logx} + \log(logx)\right)$$

Put the value of y and get the answer.

5. Show that

$$\lim_{n\to\infty} n^2\left\{\sqrt{\left(1-cos\frac{1}{n}\right)\sqrt{\left(1-cos\frac{1}{n}\right)\sqrt{\left(1-cos\frac{1}{n}\right)\ldots..\infty}}}\right\} = \frac{1}{2}.$$

Ans. Take y =

$$\sqrt{\left(1-cos\frac{1}{n}\right)\sqrt{\left(1-cos\frac{1}{n}\right)\sqrt{\left(1-cos\frac{1}{n}\right)\ldots..\infty}}}$$

⇨ $y^2 = y\left(1-cos\frac{1}{n}\right)$

⇨ $y = \left(1-cos\frac{1}{n}\right)$

So $\lim_{n\to\infty} n^2 y = \lim_{n\to\infty} n^2\left(1-cos\frac{1}{n}\right) =$

$\lim_{n\to\infty} n^2 y = \lim_{n\to\infty} n^2\left(1-1+\frac{1}{2n^2}\right) = \frac{1}{2}$

6. Show that $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.

Ans.

$\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2 =$

$(\int_{-\infty}^{\infty} e^{-x^2}dx)(\int_{-\infty}^{\infty} e^{-x^2}dx) =$

$(\int_{-\infty}^{\infty} e^{-x^2}dx)(\int_{-\infty}^{\infty} e^{-y^2}dy)$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy$

$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}rdrd\theta = \int_0^{2\pi}\frac{d\theta}{2} = \pi$

7. A total of n balls, numbered 1 through n, are put into n urns, also numbered 1 through n in such a way that ball i is equally likely to go into any of the urns 1, 2, 3, …, i. Find the expected number of urns that are empty.

Ans. $X_i = \begin{cases} 1 & if\ urn\ is\ empty \\ 0 & otherwise \end{cases}$

$E(X_i) = P(X_i = 1)$

= $\left(1-\frac{1}{i}\right)\left(1-\frac{1}{i+1}\right)\ldots\left(1-\frac{1}{n}\right)=\frac{i-1}{n}$

E(X) = $\sum_{i=1}^{n} E(X_i) = \frac{n-1}{n}$.

8. If $\frac{\sum_{r=0}^{k-1} x^{2r}}{\sum_{r=0}^{k-1} x^{r}}$ is a polynomial in x for two values p and q of k, then roots of equation x^2+px+q = 0 cannot be rational. Justify.

Ans.

$\frac{\sum_{r=0}^{k-1} x^{2r}}{\sum_{r=0}^{k-1} x^{r}}$ is a polynomial,

$\Rightarrow$ 1+$x^2+x^4+\cdots+x^{2(k-1)}$ *is divisible by* $1+x+x^2+\cdots+x^{k+1}$

$\Rightarrow \frac{\frac{1-x^{2k}}{1-x^2}}{\frac{1-x^k}{1-x}}$ $i.e. \frac{1+x^k}{1+x}$

is a polynomial in x.

$\Rightarrow$ 1+x^k is divisible by 1+ x

$\Rightarrow$ 1+$(-1)^k = 0$

$\Rightarrow k = an\ odd\ positive\ integer$

$\Rightarrow$ p, q are odd integers

(since p, q are value of k)

Now since coefficients of the quadratic equation $x^2+px+q=0$ are odd integers and hence its roots cannot be rational.

9. Let G be a group with identity element e. If x and y are elements in G satisfy $x^5y^3 = x^8y^5$ = e, then what is the relation between x & y?

Ans.

$x^5y^3 = x^8y^5 = x^3y^2 = x^2$ e = e

(from the given relation)

i.e. x = e, so y = x^{-1} = e.

10. Let ∂ be the permutation:

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
3	**5**	**6**	**2**	**4**	**9**	**8**	**7**	**1**

I be the identity permutation and m be the order of ∂, i.e., m = min {positive integers n: $\partial^n = 1$}. Then what is the value of m?

Ans.

1→3→6→9→1 : order 4

2→5→4→2 : order 3

7→8→7 : order 2

LCM (4, 3, 2) = 12 = m.

SHORT ANSWER TYPE QUESTIONS (MAINLY FOR MSTAT EXAM)

Q1. If a_1<a_2<………< a_m, b_1< b_2< ……<, b_n and also $\sum_{i=1}^{m}|ai-x|$=$\sum_{j=1}^{n}|bj-x|$, where x is any real number then prove that ai = bj for all i and n = m.

Solution: let f(x)=|a_1-x|+|a_2-x|+…..+|a_m-x|

And g(x) = $|b_1$-x|+$|b_2$-x|+…..+$|b_n$-x|.

Then we know only points of non-differentiability of f(x) is a_1,a_2,………, a_m,and only points of non-differentiability of g(x) is b_1, b_2,………, b_n,

Since, m, n are finite numbers and also given that f(x)=g(x)

So, we may write, $\frac{f(ai+h)-f(ai)}{h} = \frac{g(ai+h)-g(ai)}{h}$ ∀ h.

So, RHL{f'(ai)}=RHL {g'(ai)}

And also, LHL{f'(ai)}=LHL {g'(ai)}

But as f(x) is non-differentiable at x=ai,

So, LHL{f'(ai)}≠RHL {f'(ai)}, LHL{g'(ai)}≠RHL {g'(ai)→g(x) is also not differentiable at x=ai.

Now, since both the functions are equal so the points of discontinuity are same so m=n.

To show the another part, we need to show ai=bi.

In a similar way, we can say, for any given $b_r$there exists a_p

Such that b_r=a_p.

So, {a_1,a_2,………, a_m} and {b_1, b_2,………, b_n} has one –to-one and onto correspondence.

Therefore, m=n and every a_i = b_j if i = j.

Q2. Suppose w_1 and w_2 are subspaces of Ψ^4 spanned by {(1, 2, 3, 4) , (2, 1, 1, 2)} and {(1, 0, 1, 0), (3, 0, 1, 0)} respectively. Find a basic of $w_1 \cap w_2$. Also find a basis of w_1 + w_2 containing {(1, 0, 1, 0), (3, 0, 1, 0)}. (Ψ : The set of all real numbers)

Solution: w_1 ={(1, 2, 3, 4) , (2, 1, 1, 2)}
$w_2 = \{(1, 0, 1, 0), (3, 0, 1, 0)\}$

Now we will calculate dim($w_{1\upsilon}w_2$) which is equal to number of independent rows in

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 \end{matrix}$$

i.e. Rank(A)=4.

Now, dim($w_{1\cup}w_2$)=dimw_1+dimw_2-dim($w_1 \cap w_2$)

⇨ 4 =2 + 2 - dim ($w_1 \cap w_2$)

⇨ dim ($w_1 \cap w_2$)=0.

i.e. basis of ($w_1 \cap w_2$)={(0, 0, 0, 0)}

⇨Ψ^4 = $w_{1\oplus}w_2$

⇨ basis of w_2 can be extended to form basis of $w_1 + w_2$ which is given by

= {(1, 0, 1, 0), (3, 0, 1, 0), (0, 1, 0, 0), (0, 0, 0, 1)}

Q3. Two players p_1 and p_2 are playing the final of a chess championship, which consist of a series of matches. Probability of p_1 winning a match is $^2/_3$and for p_2 is $^1/_3$. The winner will be one who is ahead by 2 games as compared to the other player and wins at least 6 games.

Now, if the player p_2 wins first four matches, find the probability of p_1 winning the championship.

Solution:- p_1 can win in the following mutually exclusive ways:

(a) p_1 wins the next six matches.
(b) p_1 wins five out of next six matches, so that after next six matches score of p_1 and p_2 are tied up. This is continued up to '2_n' matches (n≥ 0) and finally p_1 wins 2 consecutive matches.

Now, probability of case (a) =$(\frac{2}{3})^6$ and probability of tie after 6 matches (in case (b)) =$(\frac{2}{3})^5(\frac{1}{3})$ =$\frac{2^6}{3^5}$.

Now probability that scores are still tied up after another next two matches=$\frac{2}{3}.\frac{1}{3}+\frac{1}{3}.\frac{2}{3}=\frac{4}{9}$.

[1st match is own by p_1 and 2nd by p_2, or , by reversively]

Similarly probability that scores are still tied up after another '2n' matches=$(\frac{4}{9})^n$.

⇨Total probability of p_1 winning the championship

$$= (\frac{2}{3})^6 + \frac{2^6}{3^5}(\sum_{n=0}^{\infty}(\frac{4}{9})^2\ (\frac{2}{3})^2)$$

$$= (\frac{2}{3})^6 + \frac{2^6}{3^5}(\frac{2}{3})^2(\frac{1}{1-\frac{4}{9}})$$

$$= \frac{17}{5}(\frac{2}{3})^6$$

$$= \frac{1088}{3645}.$$

Q4. Let $X_1, X_2, \ldots, X_n$ be a random sample drawn from a continuous distribution. The random variables are ranked in the increasing order of magnitude. R_i be the rank of the ith sample. Find the correlation coefficient between R_1 and R_2.

Solution:- R_i be the rank of X_i.

R_i be the random variable such that P($R_i = r_i$) =$\frac{1}{n}$; ri=1(1)n .

$\therefore \sum_{i=1}^{n} R_i = \frac{n(n+1)}{2}$, a constant quantity.

And since $R_1, R_2, \ldots, R_n$ is identical random variable, now R_i is th random variable and $\sum_{i=1}^{n} R_i$ is a constant.

$\therefore$ cov($R_1, \sum_{i=1}^{n} R_i$)=0

⇨ cov($R_1, R_1 + R_2 + \cdots + R_n$,)=0

⇨var(R_1) +cov(R_1, R_2) + ….+cov(R_1, R_n)=0

⇨var(R_1)+(n-1). Cov(R_1, R_2)=0
[∵R_i's are identically distributed; cov(R_i, R_j)= cov(R_i)]

⇨cov(R_1, R_2)=-$\frac{var(R_i)}{(n-1)}$

$$=-\frac{\frac{n^2-1}{12}}{(n-1)} = -\frac{(n+1)}{12}$$

$$\therefore \rho = \frac{cov(R_1,R_2)}{\sqrt{var(R_1)var(R_2)}} = \frac{-\frac{n+1}{12}}{\frac{(n+1)(n-1)}{12}} = -\frac{1}{(n+1)}.$$

Q5. Let X and Y be two random variables with joint P. D. F.

f(x, y) = 1 if –y< x<y , 0<y <1

= 0 elsewhere

Find the regression equation of Y on X and that of probability density function.

Solution:-

Here –y< x<y and 0<y <1

⇨ -1< x <1 , which is the marginal range of x.

Again, y> -x and y > x

∴ y > max (x_1 -x)

∴ max (x_1 -x) < y < 1

∴ Marginal PDF of X is given by,

$f_X(x) = \int_{\max(x_1 - x)}^{1} (x, y)\, dy$, -1 < x < 1

<u>Case –I</u>:- -1 < x <0

∴ max (x_1 -x) = -x

∴ $f_X(x) = \int_{-x}^{1} dy$, if -1 < x < 0

= 1+ x if -1 < x <0

<u>Case –II</u>:- 0 < x< 1

Max(x, -x) = x,

∴ $f_X(x) = \int_{x}^{1} dy$, if 0 < x < 1

= 1- x if 0< x <1

Marginal PDF of Y is given by.

$f_y(y) = \int_{-y}^{1} dx$, if 0 <y < 1

= 2y if 0 <y < 1

<u>Case 1</u>. -1 <x < 0,

The conditional distribution of Y given X= x is given by,

$f_{y/x}(y) = \frac{f(x,y)}{f_x(x)} = \frac{1}{1+x}$ if –x < y < 1

∴ E (Y|X) = $\int_{-x}^{1} \frac{y\, dy}{(1+x)} = \frac{1}{(1+x)} . \frac{1}{2}(1 - x^2) = \frac{1-x}{2}$.

<u>Case 2.</u> 0 < x < 1,

The conditional distribution of Y given X=X is given by,

$f_{Y|X}(y) = \frac{f(x,y)}{f(x)} = \frac{1}{1-x}$ if x < y < 1

Similarly, E (Y|X) = $\frac{1+x}{2}$

∴ If -1 < x < 1, then regression equation of Y on X is given by,

Y= $\frac{1-|x|}{2}$.

The conditional distribution of X given Y = y is given by,

$f_{X|Y}(x) = \frac{f(x,y)}{f_Y(y)} = \frac{1}{2y}$ if –y < x < y

∴ E (X|Y)=0

∴ Regression equation of X and Y is given by x =0.

Q6. (a) Let f_n be a sequence of continuous real valued functions on, [0,1] which converges uniformly to f . Prove that $\lim_{n\to\infty} f_n(x_n) = f\left(\frac{1}{2}\right)$ for any sequence $\{x_n\}$ converges to ½.

(b) Must the conclusion still hold if the convergence is only point wise? Explain.

Solution:- (a) Let $\{x_n\}$ be a sequence in [0,1] with $x_{n\to 1/2}$ as n→ Fix ∈>0 and let $N_0 \in N$ be such that n ≥ N_0 implies $|f_n(x) - f(x)|$< for all x∈[0,1]. Let δ>0 be such that $|f(x) - f(y)|$∈/2 ∀ x, y ∈ [0,1] with $|x - y| < \delta$. Finally, let $N_1 \in N$ be such that n ≥ N_1 *implies* $|x_n - 1/2|$<δ. Then n ≥ Max { N_0, N_1} inplise

$|f_n(x_n) - f(\frac{1}{2})| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(\frac{1}{2})|$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(b) Suppose the convergence is only pointwise.

Then the conclusion is false, demonstrating by an counter example:

Defining $f_n(x)$ to be the function,

$$f(x) = 0 \quad , \text{if } 0 \leq x < \frac{1}{2} - \frac{1}{2n}$$

$$= 2nx-(n-1), \text{ if } \frac{1}{2} - \frac{1}{2n} \leq x < \frac{1}{2}$$

i.e. $f_n(x)$is constantly zero for, x< ½ - $\frac{1}{2n}$, then it increases linearly until it reaches '1' at x= ½ , and then it remains constantly '1' for x> ½ .

Now, define the sequence, $x_n = \frac{1}{2} - \frac{1}{n}$,

Then f(x_n)=0 ∀ $n \in N$ and $x_{n \to \frac{1}{2}}$ for n→∞.

Therefore, f(½)=1≠0 =$\lim_{n\to\infty} f_n(x_n)$.

Q7. Let $\{x_n : n \geq 0\}$ be a sequence of real numbers such that $x_{n+1} = \lambda x_n + (1 - \lambda)x_{n-1}, n \geq 1$, for some 0< $\lambda < 1$.

(a) show that $x_n = x_0 + (x_1 - x_0)\sum_{k=0}^{n=1}(\lambda - 1)^k$.

(b) Hence, or, otherwise, show that x_n converges and find the limit.

Solution :- $x_{n+1} - x_n = \lambda x_n + (1 - \lambda)x_{n-1} - \lambda x_{n-1} + \lambda x_{n-1} - x_n$

$$= (\lambda - 1)x_n + x_{n-1}(1 - \lambda - \lambda + \lambda)$$

$$= (\lambda - 1)[x_n - x_{n-1}]$$

$$= (\lambda - 1)^2[x_{n-1} - x_{n-2}]$$

$$\vdots$$

$$= (\lambda - 1)^n(x_{1-}x_0)$$

$\therefore x_n - x_{n-1} = (\lambda - 1)^{n-1}(x_{1-}x_0)$

$$x_{n-1} - x_{n-2} = (\lambda - 1)^{n-2}(x_{1-}x_0)$$

$$\vdots$$

$$x_{1-}x_0 = (\lambda - 1)^0(x_{1-}x_0)$$

Adding we get, $x_{n-}x_0 = (x_{1-}x_0)\sum_{k=0}^{n=1}(\lambda - 1)^k$

$\therefore x_n = x_0 + (x_{1-}x_0) \sum_{k=0}^{n=1}(\lambda - 1)^k$

$\therefore \underset{n\to\infty}{\text{Lt}}\, x_n = x_0 + (x_{1-}x_0).\frac{1}{1-\lambda+1}$, as n→∞.

$= x_0 + (x_{1-}x_0).\frac{1}{2-\lambda}$.

Q8. Let f:ℝ→ℝ be a continuous function with |f(x)-f(y)|≥ $|x - y$ for every x,y,∈ℝ. Is f one –to-one? Show that there can't

exist three points a, b, c ∈ ℝ with a< b< c such that f(a)<f(c)< f(b).

Solution:- f: ℝ→ℝ such that |f(x)-f(y)|≥ $|x-y|$

or, $|\frac{f(x)-f9y)}{x-y}| \geq 0 \; \forall \; x, y \in$ ℝ.

⇨$|\frac{f(y+h)-f(y)}{h} \geq 0$, taning x=y+h.|

⇨|f'(y)|≥ 0

⇨f is either an increasing or decreasing function.

⇨$\forall \; a < b < c \rightarrow f(a) < f(b) < f(c)$

Or, a> b> c→ $f(a) < f(b) < f(c)$

i.e. a< b<c s.t. $f(a) < f(b) < f(c)$.

Q9. (a) Let ṵ and ṽ eigenvectors of A corresponding to the eigenvalues 1 and 3, respectively. Prove that ṵ +ṽ is not an eigenvector of A.

(b) Let A and B be real matrices such that the sum of each row of A is 1 and the sum of each row of B is 2. Then show that 2 is an eigenvalue of AB.

Solution:-As ṵ and ṽ are given eigen vectors corresponding to eigen values 1 and 3, so,

$A_u = 1.u \,; A_v = 3.v = 3v$.

⇨ A(u+v)=u+3v.

As, RHS is not multiple of u+v, so, u+v can't be eigen vector of A.

(b) A=$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$, B=$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

⇨AB=$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

|AB -λI|=$(1-\lambda)^2 - 1$=0

⇨$1-\lambda = \pm 1$

⇨ $\lambda = 0, 2$.

So, 2 is an eigen value of AB.

Q10. Let A and B be n×n real matrices such that $A^2 = A$, $B^2 = B$. Let I-(A+B) is invertible. Show that R(A)=R(B).

Solution:- A[I –(A+B)]=A -A^2-AB

= $A^2 - A^2$-AB

= -AB.

And,[I –(A+B)]B=B -AB - B^2

=B^2- AB - B^2

=-AB

∴ rank(A)= rank[A(I – A-B)]

=rank(-AB)=rank(b).

Q11. Let P be a matrix of order n>1 and entries are positive integers. Suppose P^{-1} exists and has integer entries, then what are the set of possible values of |p| ?

Solution:- P has integer entries,

⇨$\lambda_1 + \lambda_2 + \cdots . + \lambda_n$=trace(P)=integer.

⇨ $\sum_{i<j} \lambda i \; \lambda j$ = sum of minors

⋮

⇨$\prod_{i=1}^{n} \lambda i$|p| = integer, Then the eigen-values of P^{-1} are $\frac{1}{\lambda i}$ and they are also integers

⇨ $\lambda i = \frac{1}{\lambda i}$

⇨ $\lambda i = \pm 1$

So, |P|=$\prod_{i=1}^{n} \lambda i = \pm 1$.

Q12. Let X, Y be a bivariate normal vector such that E(X)=E(Y)=0 and V(X)=V(Y)=1. Let s be a subset of $\mathbb{R}^2$ and defined by S={(a, b) : (ax+ by) is independent of Y}.

(i) show that S be a sub space,

(ii) Find its dimension.

Solution:- S={(a, b) : (ax+by) is independent of Y }

(i) (a_1, b_1), (a_2, b_2)∈S.

Then $a_1x + b_1y$ is independent of y, similarly,

$a_2x + b_2y$ is independent of y.

⇨ ($\boldsymbol{\alpha} a_1$+$\boldsymbol{\beta} a_2$)x +($\boldsymbol{\alpha} b_1 + \boldsymbol{\beta b_2}$)y is independent of y.

⇨ **($\boldsymbol{\alpha} a_1$+$\boldsymbol{\beta} a_2$, $\boldsymbol{\alpha} b_1 + \boldsymbol{\beta b_2}$ ∈s).**

⇨ (a_1, b_1) +**β**(a_2, b_2)∈ s ∀ $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ ∈ ℝ

Hence, S is a subspace.

(ii) (a, b)∈S.

⇨ax +by is independent of y.

⇨ cov(ax +by, y) =0

⇨ acov(x,y) + bcov(y, y) =0

⇨a**ρ** +b=0, since, cov(x, y)=**ρ**, cov(y,y)=var(y)=1 as,E(x)=E(y)=0 &v(x)=v(y)=1'

⇨ b= -a**ρ**.

∴ (a, b)=a(1-**ρ**), a∈ℝ.

∴ S={(a, b) : (a,b)=a(1-**ρ**); a∈ℝ }

∴ dim(s)=1.

Q13. In a knockout tournament, 2^n equally skilled players namely, $s_1, s_2, s_3, \ldots\ldots, s_{2n}$ are participating. In each round, player are divided in pairs at random and winner from each pair moves in the next round. If s_2 reaches semi-final , then find the probability that s_1 will win the tournament.

Solution:- In a knockout tournament, 2^n equally skilled players namely, $s_1, s_2, s_3, \ldots\ldots, s_{2n}$ are participating.

Let E_1 be the event that s_1 wins the tournament and

E_2 be the event that s_2 reaches the semifinal.

We are to obtain P(E_1/E_2).

Since all the players are of equal skill and there will be four person in the semifinal.

So, P(E_2) =$\frac{2^n-1\, c_3}{2^n\, c_4}$=$\frac{4}{2^n}$.

P($E_1 \cap E_2$) =probability that s_1 and s_2 both are in the semifinal & then s_1 wins the semifinal and also in final

=$\frac{2^n-2c_2}{2^n\, c^4}$.$\frac{1}{2}$.$\frac{1}{2} = \frac{3}{2^n(2^n-1)}$

Hence, P$[E_1/E_2]$=$\frac{P(E_1 \cap E_2)}{P(E_2)}$

=$\frac{3 \cdot 2^n}{2^n(2^n-1) \cdot 4}$=$\frac{3}{4(2^n-1)}$.

Q14. Let Y, Y_2, Y_3 be i.i.d. continuous r.v.s for i=1, 2. Define U_i as U_i =1 if $Y_{i+1} > Y_i$

=0 ow

Find the mean and variance of $U_1 + U_2$.

Solution:- E(U_i)=1.P$[Y_{i+1} > Y_i]$=$\frac{1}{2}$

E $(U_i)^2$=1^2.P$[Y_{i+1} > Y_i]$=$\frac{1}{2}$

V (U_i) =$\frac{1}{2} - \frac{1}{4}$ =$\frac{1}{4}$

E $(U_1 + U_2)$= $\frac{1}{2} + \frac{1}{2}$ =1.

E $(U_1 U_2)$ =1.1.P$[Y_2 > Y_1, Y_3 > Y_2]$ = P$[Y_3$>$Y_2 > Y_1]$=$\frac{1}{6}$.

Cov(U_1, U_2)= E$(U_1 U_2)$-E(U_1)E(U_2) = -$\frac{1}{12}$

∴ V$(U_1 + U_2)$=V(U_1)+V(U_2)+2cov(U_1, U_2) =$\frac{1}{3}$.

Q15. A and B have respectively (n+1) and n coins. If they toss their coin simultaneously. What is the probability that, ____

i>A will have more heads than B.

ii> A and B will have an equal number of heads.

iii> B will have more heads than A.

Soln:- Let us define the random variable as follows,

X= no. of heads obtained by A.

Y= No. of heads obtained by B.

X ~bin (n+1, $\frac{1}{2}$)

Y ~ bin (n, $\frac{1}{2}$)

Then, (n+1-X) ~ bin (n+1, $\frac{1}{2}$)

(n-Y) ~ bin (n, $\frac{1}{2}$)

i> P(A will have more heads than b)
=(x>Y)
=P(n+1-X > n-Y)
=P (Y >X -1)
=P(Y≥ X)
=1 –P(X>Y)

∴ 2P(X>Y)=1

⇨ P(X>Y)=$\frac{1}{2}$.

ii> P(A and b have equal number of heads)
=P(X=Y)
= $\sum_{i=1}^{n} P(X = i0\ P(Y = i)$
=
$\sum_{i=1}^{n} \binom{n+1}{i} \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{n+1-i} .\binom{n}{i} \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{n-i}$
=$\sum_{i=1}^{n} \binom{n+1}{i} . \binom{n}{i} \left(\frac{1}{2}\right)^{2n+1}$ = $\left(\frac{1}{2}\right)^{2n+1} \sum_{i=1}^{n} \binom{n+1}{i} \binom{n}{i}$
= $\left(\frac{1}{2}\right)^{2n+1} \sum_{i=1}^{n} \frac{(n+1)!}{i!(n-i+1)!} . \frac{n!}{i!(n-i)!}$
= $\left(\frac{1}{2}\right)^{2n+1} \binom{2n+1}{n}$

iii> P(B have more heads than A)

$=P(y>x)$
$=1-P(X\geq Y)$
$= 1-P(X=Y)-P(X>Y)$
$=1- \left(\frac{1}{2}\right)^{2n+1} \binom{2n+1}{n} - \frac{1}{2}$
$=\frac{1}{2}[1- \left(\frac{1}{2}\right)^{2n} \binom{2n+1}{n}]$

Q16. A book of N pages contains on the average λ misprints per page. Estimate the probability that a page drawn at random contains,

(a) at least one misprints.

(b) More than k misprints.

Solution:- Let us define the random variable X as follows,

X= no. of misprints per page,

The book contains λ misprints per page on an average.

Since the number of trials i.e. the no. of words is very large and probability of a misprint is very small, hence according to the definition of poisson distribution,

$X \sim p(\lambda)$

$\therefore P(X = x) = \frac{e^{-\lambda}.\lambda^x}{x1}$, x=0, 1, 2, …; λ>0

=0 , otherwise

(a)P(at least one misprint)

$= P(X\geq 1)$

$= 1-P(X<1)$

$= 1- P(X=0)$

$= (1- e^{-\lambda})$

(b) P(more than k misprints)

$=P(x>k)$

$=P(X\geq k-1)$

$= 1 – P(X\leq k-1)$

$= 1- \sum_{x=0}^{k=1} \frac{e^{-\lambda}.\lambda^x}{x!}$

Q17. A certain mathematician carries two match boxes in his pocket, each time he wants to use a match, he selects one of boxes at random. Each pocket contain n matchsticks.

(a) Find the distribution of the number of sticks in one box, while the other is found empty.

(b) Also find the distribution of the number of sticks remaining in one box become empty.

Solution:- (a) Let us define a r.v. X denoting the number of the matchsticks remaining in the match box when the other box is found empty.

LetX_{ij}, I, j, i≠j denotes the number of matchsticks remaining in the ith box cohen the jth box is found to be empty.

The mass points of X are 0, 1, …., N

For any such mass point x,

$P[X=x] =P(X_{12}=x) +P(X_{21}=x)$

We consider the distribution of X_{12}.

The second box will be found empty if the box is chosen for the (N+1) th time. At that time the first box contain x matches if (N-x) matches have already taken from it. If the selection of the second box is regarded as success, then the event.

P[X_{12}=x]=P[(N-x) failures occur preceeding the (N+1)th success].

=p[Z=N-x], where Z ~ NB(N+1,$\frac{1}{2}$)

Similarly for P[X_{21}=x]

(b) Let us define a random variable Y denoting the number of matchsticks remaining in a matchbox when the other match box becomes empty.

Let Y_{ij}, I, j, i$\neq j$denotes the number of matchsticks remaining in the ith box when jth box becomes empty.

The mass points of Y are 0, 1, 2, …, n.

P[Y=y]=p[Y_{21}=y]+P[Y_{12}=y]

Now, P[Y_{12}=y]p[Z=n-y] , Z ~ N.B(N, $\frac{1}{2}$)

=($^{N+N-y-1}C_{N-y}$) $(\frac{1}{2})^{N}(\frac{1}{2})^{N-y}$

=($^{2N-y-1}C_{N-y}$) $(\frac{1}{2})^{2N-y}$

Similarly, P[Y_{21}=y]=($^{2N-y-1}C_{N-y}$) $(\frac{1}{2})^{2N-y}$

∴ P[Y=y]]=($^{2N-y-1}C_{N-y}$) $(\frac{1}{2})^{2N-y+1}$

Q18. A drunk man performed a random walk over the position 0, ±1, ±2,…. The drunk man stars from the point o. He takes successive unit steps with probability p at right and probability (1-p) at left. His steps are independent. X be a location of the drunk man after taking n- steps,

Find the distribution of $\frac{(n+X)}{2}$ and find out E(X).

Solution:- R denotes no. of steps at right after taking n steps.

∴ R~ Bin(n, p)

L denotes no. of steps at left after taking n steps.

∴ L~ Bin(n,1-p)

Let us define,

X: the position of the drunk and after n steps.

R + l= n,

r- l=X.

∴ 2R= n+X ⇨ R=$\frac{n+X}{2}$ ~ Bin(n,p)

∴ E($\frac{n+X}{2}$)=nP

⇨ E(X) =2[np - $\frac{n}{2}$]

=2n (p - $\frac{1}{2}$)

=n (2p-1).

Q19. Let X be an R.V. with mean μ and variance σ^2>0.

If ξ_q denotes the q^{th} quantile of X, show that

$$\mu\text{-}\sigma\sqrt{\frac{1-q}{q}} \leq \xi_q \leq \mu+\sigma\sqrt{\frac{q}{1-q}}.$$

ANS:- We know that ξ_q statistics the incauality p(X≤ ξ_q)≥q

$\therefore$ P$(\frac{X-\mu}{\sigma} \leq \frac{\xi_q-\mu}{\sigma}) \geq q$

If $\xi_q < \mu$, i.e. $\frac{\xi_q-\mu}{\sigma}$<0, we have from one sided chebyshev's inequality,

q$\leq P[\frac{X-\mu}{\sigma} \leq \frac{\xi_q-\mu}{\sigma}] \leq \frac{1}{1+(\frac{\xi_q-\mu}{\sigma})^2}$

$\therefore$ q$\leq \frac{1}{1+(\frac{\xi_q-\mu}{\sigma})^2}$

⇨ $(\frac{\xi_q-\mu}{\sigma})^2 \leq \frac{1-q}{q}$

⇨ $-\sqrt{\frac{1-q}{q}} \leq \frac{\xi_q-\mu}{\sigma} \leq \sqrt{\frac{1-q}{q}}$

⇨ μ-$\sigma\sqrt{\frac{1-q}{q}} \leq \xi_q \leq \mu+\sigma\sqrt{\frac{1-q}{q}}$.(Proved)

Q20. Let g be a non – negative non decreasing function, prove that if $\mathbf{E(g(|X-\mu|))}$ exists, where μ= E(X), then prove that if P$[|X-\mu| > t]$< $\frac{(g(|X-\mu|))}{g(t)}$

<u>ANS</u>:- P[g$|X-\mu| > g(t)$]$\leq \frac{E\{g|X-\mu|\}}{g(t)}$

But, g$|X-\mu| > g(t)$

⇔$|X-\mu|$>t   [∵ g is non decreasing & non-negative function]

$\therefore$ P [$|X-\mu|$>t] < $\frac{E\{g|X-\mu|\}}{g(t)}$. (Proved)

Q21. For a Laplaces distribution with PDF f(x)= $\frac{1}{2}$ $e^{-|x|}$, -$\infty < x < \infty$.

Find the minimum probability of an observation lying with in the mean $\pm$ 3 s. d. interval.

<u>ANS</u>:- P ($|X-\mu| \leq 3\sigma$

=P (μ-3$\sigma \leq X \leq \mu + 3\sigma$)

= $\frac{1}{2}\int_{\mu-3\sigma}^{\mu+3\sigma} e^{-|x|}\,dx$

= $\int_{0}^{\mu+3\sigma} e^{-x}dx$ [Since the integrand is an even function]

= 1-$e^{-(\mu+3\sigma)}$

=.95   [X~ Laplace (0, 1)]

By Chebyshev's inequality,

P [$|X-\mu| \geq 3\sigma$]$\leq \frac{1}{3^2}$

⇨ P [$|X-\mu| \leq 3\sigma] \geq 1-\frac{1}{9}$=$\frac{8}{9}$=.88

Hence the given probability and the Chebyshev's upperbound is nearer to each other.

Q22. For the r. v. X having the following PDF

f(x)=$\frac{e^{-x}.x^{\lambda}}{\sqrt{\lambda+1}}$, x> 0 show that P (0< X <2($\lambda+1$))> $\frac{\lambda}{\lambda+1}$

<u>ANS</u>:- E(X)= $(\lambda+1)$=Y(X)

From Chebyshev's inequality,

P [$|\frac{X-\mu}{\sigma}|$< t]> 1-$\frac{1}{t^2}$

1-$\frac{1}{t^2}$=$\frac{\lambda}{\lambda+1}$

⇨ P[-σt < $(X-\mu)$<σt]> 1-$\frac{1}{t^2}$

⇨ t= $\sqrt{\lambda+1}$

⇨ P[-($\sqrt{\lambda+1}$) ($\sqrt{\lambda+1}$)< (X-$\sqrt{\lambda+1}$)<λ+1]> 1- $\frac{1}{\lambda+1}$

⇨ P [0, X, 2$(\lambda+1)$]> $\frac{\lambda}{\lambda+1}$

Q23. Let the random variables X and Y have the joint probability density function(x, y) given by

f(x, y)= $\boldsymbol{y^2 e - y^{(x+1)}; x \geq 0, y \geq 0}$

= 0 ;otherwise

Are the random variables x and Y independent? Justify your answer .

<u>Solution</u>:- The marginal p d f of x is given by :

F(x) $= \int_0^{\infty} y^2\, e^{-y(x+1)} dy$
let, $y(x+1)$=t, (x+1) dy =dt.

$= \int_0^{\infty} \frac{t^2}{(x+1)^2}\; e^{-t} . \frac{1}{(x+1)}\; dt$

$= \frac{1}{(x+1)^3} \int_0^{\infty} t^2 e^{-t}\, dt$

$= \frac{\sqrt{(3)}}{(x+1)^3}\; = \frac{2}{(x+1)^3}$; $x \geq 0$

The marginal p d f of Y is given by :

F(y) = $\int_0^{\infty} y^2\, e^{-y(x+1)} dx$

$= \int_0^{\infty} \frac{y^2}{-y} . e^{-y(x+1)}$

$= ye^{-y}$;y≥0

As, f(x, y)≠f(x), f(y)

So, X and Y are not independent.

Q24. Let X and Y are i.i.d. with P[X= x] = $\boldsymbol{\frac{1}{x} - \frac{1}{x+1}}$**, x=1, 2, …**

Find E[Min(X, Y)].

<u>Solution</u>:- Let T= min (x, Y).

P[T= t]= P[X=t, Y>t]+ P[Y=t, X>t] + P[x=t, Y=t]

=P[x=t]P[Y> t]+ P[X> t]P[Y=t]+p [X=t]P[Y= t]

Now, P[Y≤t]= P[Y=1]+ P[Y=2]+…+P[Y=t]

$=(1-\frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \ldots + (\frac{1}{t} - \frac{1}{t+1})$

$= 1- \frac{1}{t+1}$

⇨ P[Y>t]= $\frac{1}{t+1}$

Similarly, P[X> t]= $\frac{1}{t+1}$

Hence, P[T=t]= $(\frac{1}{t} - \frac{1}{t+1}). \frac{1}{t+1} + \frac{1}{t} - \frac{1}{t+1}). \frac{1}{t+1} + (\frac{1}{t} - \frac{1}{t+1})^2$

$= \frac{1}{t(t+1)^2} + \frac{1}{t^2(t+1)}$

∴ E(T) $= \sum_{t=1}^{\infty} \frac{1}{(t+1)^2} + \sum_{t=1}^{\infty} \frac{1}{t\,(t+1)}$

$=(\frac{\pi^2}{6} - 1) + \sum_{t=1}^{\infty} (\frac{1}{t} - \frac{1}{t+1})$

$= \frac{\pi^2}{6}$ -1 + 1

$= \frac{\pi^2}{6}$

Q25. Suppose a random vector (X, Y) has joint probability density function

f(x, y) =3y on the triangle bounded by the lines y=0, y=1 –x and y= 1 +x

find the marginal PDF of X and Y. Compute (Y| X≤ $\boldsymbol{\frac{1}{2}}$**).**

<u>Solution</u>:- The joint PDF of the random vector

(X, Y) is given by

F(x, y) = 3y, on the shaded triangle of figure 1.

From the figure, the range of the marginal

Distributions of X and Y are given by,

$0 < x < 1$, $-1 < y < 1$, respectively,

Now, $f(x, y) = 3y$ if $1 - x < y < 1 + x$, $0 < x < 1$

Now, $1 + x > y$

$\Rightarrow x > y - 1$

And $1 - x < y$

$\Rightarrow x - 1 > -y$

$\Rightarrow x > 1 - y$

$\therefore x > \max\{(1-y), (y-1)\}$

$\therefore \max\{(1-y), (y-1)\} < x < 1$

<u>Case I</u>:- $-1 < y < 0$

If $-1 < y < 0$

$\Rightarrow -2 < y - 1 < -1$

And $1 > -y > 0$

$\Rightarrow 2 > 1 - y > 1$

$\therefore \max\{(y-1), (1-y)\} = 1 - y$

$\therefore 1 - y < x < 1$

Q26. Let x be a continuous random variable with distribution function f(x), which is such that F(a+x) +F(a-x) =1 for some fixed a.

i> Show that E(X) =a

ii> If y be an another r. v. defined as

Y=0 if X < a ; 1 if X > a

Then S.T. Y and Z=|X-a| will be independently distributed.

<u>ANS</u>:- i> It is given that, $F(a+x) + F(a-x) = 1$

From the above equation it is clear that the disth of X is symmetric about 'a',

Hence, $E(X-a) = 0$

$\Rightarrow E(X) = a$

Ii > It is given that,

$Y = \{ 0$ if $X < a$; 1 if $X \geq a$

And $Z = |X-a|$

Now form the equation, $F(x+a) + F(a-x) = 1$,

It is clear that $F(a) = \frac{1}{2}$ [since the distribution is symmetric about 'a']

$\therefore Y = \{ 0$ with prob. $\frac{1}{2}$; 1, with prob. $\frac{1}{2}$

Now, for same $Z > 0$

$P[Z \leq z , Y=0]$

$= P[|X-a| \leq z , X < a]$

$= P[-x + a \leq z \leq X - a , X < a]$

$= P[a - z \leq X \leq a + z, X < a]$

$= P[a - z \leq X \leq \min(a+z, a)]$

$= P[a - z \leq X \leq a]$

$= F(a) - F(a-z)$

$= \frac{1}{2} - F(a-z)$

$= \frac{1}{2}$ [F(a+z) –F(a- z)]

$= \frac{1}{2}$ P [a-z ≤ X ≤ a+ z]

$= \frac{1}{2}$ P [|X|≤ a+z]

$= \frac{1}{2}$ P [|X - a|≤z] , since a > 0

$= \frac{1}{2}$ P [Z≤z]

=P [Z≤z]. F (a)

=P [Z≤z]. P [Y=0]

Similarly, it can be shown that,

P [Z≤z , Y=1]=P [Z≤z]. P [Y=1]

Hence,

P [Z≤z]. P [Y= y]= P [Z ≤z, Y=y]

Hence, Y and Z are independently distributed.

Q27. A bag contains a coin of value M and a number of other coins whose aggregate value is m. A person draws coins one at a time till the draws the coin of value M. Find the value of his expectation.

ANS:- Let the coins be A, $B_1, B_2, \dots, B_n$.

Value of A= M & value of $B_i = m_i$(say)

Such that $\sum_{i=1}^{n} m_i = m$.

Let Y_x be the value of the coins if x drawings are needed, x= 0, 1, 2, …., n+1.

And Y is the total value of the coins eventually.

Now, E(Y)=E {E(Y_x|X = x)}

Now, E (Y_x|X = 1)=M

E (Y_x|X = 2) $= \frac{M+m_1}{n} + \frac{M+m_2}{n} + \dots + \frac{M+m_n}{n}$

$= M + \frac{m}{n}$

E (Y_x|X = 3) = $M + \frac{2m}{n}$

In general, E (Y_x|X = x) = $M + \frac{(x-1)m}{n}$

Now, E (Y)= $\sum_{x=1}^{n+1}[M + (x-1)\frac{m}{n}]$.P (X=x)

Now, P (X= x)= P(X drawings are required to get the coin A)

$= \frac{n}{n+1} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n-1} \dots \frac{n-x+1}{n-x+2} \cdot \frac{1}{n-x+1}$

$= \frac{1}{n+1}$

∴ E(Y)= $\sum_{x=1}^{n+1}[M + (x-1)\frac{m}{n}] \cdot \frac{1}{n+1}$

$= \frac{1}{(n+1)}[(n+1)M + \frac{m}{n} \cdot \frac{n(n+1)}{2}]$

$= M + \frac{m}{2}$.

So the required value of the expectation is $(M + \frac{m}{2})$.

Q28. Let X & Y be two joining distributed continuous random variable with joint PDF,

$$\boldsymbol{f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-P^2}} \exp [- \frac{1}{2(1-P^2)} \{x^2 - 2P_{xy} + y^2\}], x \in \mathbb{R}, y \in \mathbb{R}}$$

I > find the marginal PDF of X

ii> Find the conditional PDF of Y for given X = x.

Solution :- i> $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$

$=\frac{1}{2\pi\sqrt{1-P^2}} \int_{-\infty}^{\infty} \exp\,[-\frac{1}{2\,(1-P^2)}\,\{x^2 - 2P_{xy} + y^2\}]dy$

$=\frac{1}{2\pi\sqrt{1-P^2}} \int_{-\infty}^{\infty} \exp\,[-\frac{1}{2\,(1-P^2)}\,\{(y-Px)^2 + (1-P^2)x^2\}]dy$

$=\frac{1}{2\pi\sqrt{1-P^2}}\, e^{-x^2/2} \int_{-\infty}^{\infty} \exp[-\frac{(y-Px)^2}{2\,(1-P^2)}]\, dy$

Since, $\frac{1}{\sqrt{2\pi}\sigma_Y} \int_{-\infty}^{\infty} \exp[-\frac{1}{2\sigma_Y{}^2}\,(y-\ \mu Y)^2]= 1$

Here, Y =Px and $\sigma^2{}_Y = (1-P^2)$

$\therefore \frac{1}{\sqrt{2\pi}\sqrt{(1-P^2)}} \int_{-\infty}^{\infty} \exp[-\frac{1}{2(1-P^2)}\,(y - Px)^2]\ dy$

$\therefore \int_{-\infty}^{\infty} \exp[-\frac{1}{2(1-P^2)}\,(y - Px)^2]\ dy = \sqrt{2\pi}(1-P^2)$

$\therefore f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, x∈ℝ,

∴ X~ N(0, 1).

ii> Conditional PDF of Y for Given X= x is

$f_{\frac{y}{x}=x}(\frac{y}{x}) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{\frac{1}{2\pi\sqrt{1-P^2}}\, e^{-x^2/2}\, e^{-\frac{(y-Px)^2}{2(1-P^2)}}}{\frac{1}{\sqrt{2\pi}} e^{-x^2/2}}$

$=\frac{1}{\sqrt{2\pi}(1-P^2)}\ e^{-\frac{(y-Px)^2}{2(1-P^2)}}$

i.e. $\frac{Y}{X}$=x~ N $(P_x, (\sqrt{1-P^2})^2)$, $\infty < y < \infty$

E $[\frac{Y}{X} = x] = \int_{-\infty}^{\infty} f_{XY}(\frac{y}{x}) dy$ =Px.

Q29. Let x and Y have the circular normal distribution with zero mean, i.e. X & Y ~ $N_2(0, 0, \sigma^2, \sigma^2, 0)$. Cosider a circle C and a square S of equal area both with ac (0, 0). Prove that, P [(X , Y)∈s].

<u>Solution:-</u> The joint PDF of X & y is given by

F(x,y)=$\frac{1}{2\pi\sigma^2}\ e^{-\frac{1}{2\sigma^2}(x^2+y^2)}$, x∈ℝ, y ∈ ℝ, $\sigma > 0$

Let us consider a square S, with vertices (a, -a), (a, a), (-a, -a)

The area of the square =$4a^2$= s

Consider a circle C with radius = r, and the centre at (0, 0)

Area of C = πr^2

Hence, πr^2=$4a^2$[given]

⇒r =$\frac{2a}{\sqrt{\pi}}$

Therefore, a < r < $\sqrt{2a}$

Now, P [XY∈ S] =$\iint_{x,y \in S} f\ (x,y)dxdy$

= $4 \int_0^a \int_0^a f\ (x,y)dxdy$ [By symetry]

P [X, Y∈ C]= $\iint_{x,y \in C} f\ (x,y)dxdy$

Now in the first quadrant,

P [X, Y∈ C] - P [X, Y∈ S]

= $\iint_{x,y \in A} f\ (x,y)dxdy$ - $\iint_{x,y \in B} f\ (x,y)dxdy$ [From the figure canceling the common region]

A = shaded region,

B = dotted region.

Now, if (x, Y)∈ A, then,

$x^2 + y^2 < r^2$

$\Rightarrow -\frac{(x^2+y^2)}{2\,\sigma^2} > -\frac{r^2}{2\,\sigma^2}$

$\Rightarrow$ f(x, y)> $\frac{1}{2\pi\sigma^2}\, e^{-\frac{r^2}{2\,\sigma^2}}$ ………………<i>

If (X, Y)∈ B

$x^2 + y^2 > r^2$

$\Rightarrow$ f(x, y)< $\frac{1}{2\pi\sigma^2}\, e^{-\frac{r^2}{2\,\sigma^2}}$ ………………<ii>

From <i> & <ii> we get,

$\iint_{x,y\in A} f\,(x,y)dxdy > \iint_{x,y\in B} f\,(x,y)dxdy$

∴P [X, Y∈ C] > P [X, Y∈ S]

This inequality similarly holds for the other quadrants.

Q30. Show that, $\frac{1}{\sqrt{2\pi}}\int_0^a e^{-\frac{x^2}{2}}dx < \sqrt[\frac{1}{2}]{1 - e^{-\frac{2a^2}{\pi}}}$

Solution:- P (x, Y∈ S)=4 $\int_0^a\int_0^a \frac{1}{2\pi\sigma^2}\, e^{-\frac{1}{2}(x^2+y^2)}\, dxdy$

$= \frac{4}{2\pi}\,[\int_0^a e^{-\frac{x^2}{2}}dx]^2$

P (x, Y∈ C) = 4 $\int_0^{\pi/2}\int_0^r \frac{1}{2\pi}\, e^{-\frac{1}{2\sigma^2}}\, R^2$. RdRd0

$=\frac{4}{4}\,(1-e^{-\frac{r^2}{2}})$

∴ P (x, Y∈ C) > P (x, Y∈ S)

$\Rightarrow \frac{1}{4}\left(1 - e^{-\frac{r^2}{2}}\right) > \frac{1}{2\pi}\,[\int_0^a e^{-\frac{x^2}{2}}dx]^2$

$\Rightarrow$, $\frac{1}{\sqrt{2\pi}}\int_0^a e^{-\frac{x^2}{2}}dx < \sqrt[\frac{1}{2}]{1 - e^{-\frac{2a^2}{\pi}}}$

Q31. Let X and y be two r. v.'s with means zero, variance unity and correlation coefficient P, then S. T. E[Max(X^2, Y^2)]$\leq 1 + \sqrt{1 - P^2}$

Solution:- Max(X^2, Y^2) + Min$(X^2, Y^2) = X^2 + Y^2$

Max(X^2, Y^2) − Min$(X^2, Y^2) = X^2 - Y^2$

Max$(X^2, Y^2) = \frac{1}{2}\,[\,(X^2 + Y^2) + |X^2 - Y^2|]$

E [Max(X^2, Y^2)] $= \frac{1}{2}\,[E\,(X^2) + E(Y^2) + E|(X+Y)(x-Y)|]$

$=\frac{1}{2}\,[1 + 1\, E|(X+Y)(x-Y)|]$

By C – S inequality,

$$E^2[|(X+Y)(x-Y)|] \leq E(X+Y)^2 E(X-Y)^2$$

i.e. $E^2|X^2 - Y^2| \leq \left(2 + 2E(XY)\right)\left(2 - 2E(XY)\right)$

$\Rightarrow$ E $|X^2 - Y^2| \leq \sqrt[2]{1 - P^2}$

∴ E [max(X^2, Y^2)]$\leq 1 + \frac{1}{2} \cdot \sqrt[2]{1 - P^2}$

$\leq 1 + \sqrt{1 - P^2}$

Q32. (a) S.T. for a r.s. $X_1, X_2, \ldots, X_n$ $from\ n(\mu, \sigma^2)$, show that,

$$\frac{\sqrt{\frac{n}{n-1}}\,(X_1 - \bar{x})}{\sqrt{\frac{(n-1)S^2 - \frac{n}{n-1}(X_1 - \bar{x}^2}{n-2}}} \sim t_{n-2}$$

Solution:- (a) Let $Y_i = \sqrt{\frac{i}{i-1}}\ (\bar{x}_i - X_i) = \sqrt{\frac{i}{i-1}}(\frac{X_1+X_2,\ldots,X_i}{i} - X_i)$

$=\sqrt{\frac{i}{i-1}}(\frac{X_1+\ldots+x_{i-1}-(i-1)Xi}{i})$

$=\frac{X_1+\ldots+x_{i-1}-(i-1)Xi}{\sqrt{i(i-1)}}$ ∀ i=2(1)n.

Let $Y_1 = \frac{1}{\sqrt{n}}\ X_1 + \cdots + \frac{1}{\sqrt{n}}X_n$.

Hence the transformation reduces Y=AX, where

$$A=\begin{matrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \ldots\ldots\ldots\ldots & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{1.2}} & \frac{1}{\sqrt{1.2}} & 0 & \ldots\ldots\ldots & 0 \\ \vdots & \vdots & \vdots & \ldots\ldots\ldots & \vdots \\ \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \ldots & \frac{(n-1)}{\sqrt{n(n-1)}} \end{matrix}$$

is orthogonal

[Helmert's transformation]

The PDF of $(X_1,\ldots\ldots,X_n)$ is

$F(x_1,\ldots\ldots,x_n) = (\frac{1}{\sigma\sqrt{2\pi}})^n e^{-\frac{1}{2}\sum_{i=1}^{n}\frac{(X_i-\mu)^2}{\sigma^2}}$, $x_i \in$ R

Here $y^1 y = x^1 x$ and $y_1 = \sqrt{n}\bar{x}$

∴ |J|=1 is the Jacobian of the transformation.

The PDF of $(y_1,\ldots\ldots,y)$ is

$g\ (y_1, y_{2,}\ \ldots\ldots, y_n) = (\frac{1}{\sigma\sqrt{2\pi}})^n e^{-\frac{+\sum_{i=1}^{n} y_i{}^2 - 2\mu\sqrt{n}\,y_1 n\mu^2}{2\sigma^2}}$

$= \{\frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{(y_1-\mu\sqrt{n})^2}{2\sigma^2}}\}\prod_{i=2}^{n}\{\frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{y_i{}^2}{2\sigma^2}}\}$

Hence $y_1 \sim$ N $(\mu\sqrt{n}\,, \sigma^2\ )$and $y_i \sim$ N $(0\,, \sigma^2)$ I = 1(1)n independently distributed.

Now, $\sum_{i=2}^{n}\frac{i}{i-1}\ (X_i - \bar{x}_i)^2 / \sigma^2$

$=\sum_{i=2}^{n}(\frac{y_i}{\sigma})^2$, which is the sum of squares of (n-1)

i.i. d. N (0, 1) R.V.'s , follows $\lambda_{n-1}{}^2$

$[y_i \sim$ N $(0\,, \sigma^2)\,, i = 2(1)n$

$\Rightarrow \frac{y_i}{\sigma} \sim$ N $(0\,,1)]$

$\Rightarrow\ \sum_{i=2}^{n}\frac{i}{i-1}(X_i - \bar{x}_i)^2\ \sim \sigma^2\lambda_{n-1}{}^2$

Note :- $y_1 = \sqrt{n}\,\bar{x} \sim$ N $(\mu\sqrt{n}\,, \sigma^2)$

(c) Consider the transformation

Y= A(X $-\mu$1), where

$$A=\begin{matrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \ldots\ldots\ldots\ldots & \frac{1}{\sqrt{n}} \\ \frac{-(n-1)}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \ldots\ldots\ldots\ldots & \frac{1}{\sqrt{n(n-1)}} \\ a_{31} & a_{32} & a_{33} & \ldots\ldots\ldots\ldots & a_{3n} \\ a_{n1} & a_{n2} & a_{n3} & \ldots\ldots\ldots\ldots & a_{nn} \end{matrix}$$

Note that y' y= (X_1- μ)' (X $-\mu$)

$\Rightarrow \sum_{i=1}^{n} y^2{}_i = \sum_{i=1}^{n}(X_i - \mu)^2$

And |J=1|

The PDF of X is

$f_X(x) = (\frac{1}{\sigma\sqrt{2\pi}})^n.\ e^{-\frac{1}{2\sigma^2}\ \sum_{i=1}^{n}(X_i-\mu)^2}$, $x_i \in$ R

The PDF of Y is

$f_Y(y) = (\frac{1}{\sigma\sqrt{2\pi}})^n . e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} {y^2}_i}$, $y_i \in$ R

$\Rightarrow y_i \sim$ N $(0 , \sigma^2)$, i= 1(1) n

Here, $y_1 = \frac{1}{\sqrt{n}} \sum_{i=1}^{n}(X_i - \mu) = \sqrt{n}$ (x̄- μ)

and $y_2 = \frac{-(n-1)(X_1 - \mu) + (X_2 - \mu) + \ldots + (X_n - \mu)}{\sqrt{n(n-1)}}$

$= \frac{n\,\bar{x} - nX_1}{\sqrt{n(n-1)}} = -\sqrt{\frac{n}{n-1}}\ (X_1 - \bar{x})$

Hence, $\sqrt{\frac{n}{n-1}}(X_1 - \bar{x}) = -y_2 \sim$ N $(0 , \sigma^2)$

And $\sum_{i=3}^{n} {y^2}_i = \sum_{i=1}^{n}(X_i - \mu)^2 - {y^2}_1 - {y^2}_2$

$=\{ \sum_{i=1}^{n}(X_i - \mu)^2 - n(\bar{x} - \mu)^2\} - \frac{n}{n-1}(X_1 - \bar{x})^2$

$= \sum_{i=1}^{n}(X_i - \bar{x})^2 - \frac{n}{n-1}(X_1 - \bar{x})^2$

Therefore, $\sqrt{\frac{n}{n-1}}(\frac{X_1 - \bar{x}}{\sigma}) = -\frac{Y_2}{\sigma} \sim$ N(0, 1)

And, $\frac{\sum_{i=1}^{n}(X_i - \bar{x})^2 - \frac{n}{n-1}(X_1 - \bar{x})^2}{\sigma^2} = \sum_{i=3}^{n}(\frac{Y_i}{\sigma})^2$,

The sum of squares of (n-2) iid N(0, 1) R.V.'S, follows ${\lambda_{n-2}}^2$, independent 1 by defn of t- distn.

$$\frac{\sqrt{\frac{n}{n-1}}\,(\frac{X_1 - \bar{x}}{\sigma})}{\sqrt{\{\frac{\sum_{i=1}^{n}(X_i - \bar{x})^2 - \frac{n}{n-1}(X_1 - \bar{x})^2}{\sigma^2}\}\Big/(n-2)}} \sim t_{n-2}$$

$$\Rightarrow \frac{\sqrt{\frac{n}{n-1}}\,(\frac{X_1 - \bar{x}}{\sigma})}{\frac{\{(n-1)S^2 - \frac{n}{n-1}(X_1 - \bar{x})^2\}}{(n-2)}} \sim t_{n-2}$$

Q33. Suppose (X, Y)~ BN(0, 0, 1, 1, P). S.T.

i) $\boldsymbol{\frac{X^2 - 2PXY + Y^2}{1 - P^2} \sim {\lambda_2}^2}$

ii) $\boldsymbol{M_z(t) = [\{1 - (1 + P)t\}\{1 + (1 - P)t\}]^{-1/2}}$

<u>Solution :-</u>i) (X, Y)~ BN(0, 0, 1, 1, P)

∴ the joint PDF of (X, Y) is given by,

$f_{XY}(x, y) = \frac{1}{\sigma_x \sigma_y \sqrt{1 - P^2}.2\pi}\ e^{-\frac{1}{2(1 - P^2)}(X^2 - 2PXY + Y^2)}$; (x, y)$\in \mathbb{R}^2$

Let, U= X+ Y

V= X- Y

∴ $|J| = \frac{1}{2}$, $X = \frac{U+V}{2}$, $Y = \frac{U-V}{2}$

Now, note that,

$$\frac{1}{1 - P^2}\{\frac{(U + V)^2}{4} + \frac{(U - V)^2}{4} - P\frac{(U + V)(U - V)}{2}\}$$

$= \frac{1}{1 - P^2}\{\frac{U^2 + V^2}{4} - P\frac{(U^2 - V^2)}{2}\}$

$= \frac{1}{2(1 - P^2)}\{U^2 - PU^2 + V^2 - PV^2\}$

$= \frac{1}{2(1 + P)(1 - P)}\{U^2(1 - P) + V^2(1 + P)\}$

$= \frac{U^2}{2(1 + P)} + \frac{V^2}{2(1 - P)}$, ______________<i>

∴ Joint PDF of U and V is given by,

$f_{UV}(u,v)$=
$\frac{1}{\sqrt{2}\sqrt{2\pi}\sqrt{1+P}}\ e^{-\frac{1}{4(1+P)}u^2} . \frac{1}{\sqrt{2}\sqrt{2\pi}\sqrt{1+P}}\ e^{-\frac{1}{4(1+P)}v^2}$
, (u,v)∈$\mathbb{R}^2$

∴ U and V are independent.

U ~ N(0, 2(1+P))

$\therefore \frac{U}{\sqrt{2(1+P)}} \sim N(0,1) \Rightarrow \frac{U^2}{2(1+P)} \sim \lambda_1{}^2$

Similarly, $\frac{V^2}{2(1-P)} \sim \lambda_1{}^2$.

$\therefore \frac{U^2}{2(1+P)} + \frac{V^2}{2(1-P)} \sim \lambda_1{}^2$ [By the reproductive property of λ^2-distribution]

ii) MGF of X, Y is given By,

$M_{XY}(t)$=E(e^{tXY})

= E[E(e^{tXY}| X)]

Y| X ~ N(P_X, (1-P^2))

$\therefore$ E $[e^{tX.PX} + \frac{1}{2}t^2X^2(1-P^2)]$

= E $[e^{tX.PX^2} + \frac{1}{2}P^2(1-P^2)X^2]$

= E $[e^{\{tP+\frac{1}{2}t^2(1-P^2)\}X^2}]$

$= \frac{1}{[1-2(tP+\frac{1}{2}t^2(1-P^2))]^{1/2}}$ $[\because X^2 \sim \lambda_1{}^2\ ]$

$= \frac{1}{[1-2tP-t^2(1-P^2)]^{1/2}}$

$= \frac{1}{\sqrt{(1-tP)^2-t^2}}$

$= \frac{1}{\sqrt{\{(1-P)t+1\}\{(1-t(1+P)\}}}$

$= [\{1-(1+P)t\}\{1+(1-P)t\}]^{-1/2}$

Q34. X ~ R(0, 1) find

i) the distn of $X_{(r)}$.

ii) The m.g.f. , mean (E($X_{(r)}$)) and variance (var($X_{(r)}$))

<u>ANS</u>:- i) If X ~ R (0, 1), then the p.d.f of X is given by,

F(x)=1, 0 < x < 1

The distn function of X is given by,

F(x)= $\int_0^x dx$=x

∴ The PDF of r^{th} order statistic is given by,

g(x) = $\frac{n!}{(r-1)!(n-r)!}\ x^{r-1}\ (1-x)^{n-r}$, 0< x < 1

$\therefore X_{(r)}$ ~ B (r, n-r+1)

ii) Let $X_{(r)}$ be denoted as U, then

U ~ B(r-1, n-r)

$M_U(t)$= E(e^{Ut})

=E [1+Ut+$\frac{U^2t^2}{2!}+\cdots$]

$= \sum_{r=0}^{\infty} \frac{t^r}{r!}\ E(U^r)$

$=\sum_{r=0}^{\infty} \frac{t^r}{r!} .\mu_r{}^1$ [$\because \mu_r{}^1$ = rth order raw moment about zero]

E(U)=$\frac{1}{B(r,n-r+1)}\ \int_0^1 u.u^{r-1}(1-u)^{n-r}\,du$, 0<x <1

$=\frac{B(r+1,n-r+1)}{B(r,n-r+1)}$

$=\frac{r}{n+1}$.

E $(U^2)=\frac{r(r+1)}{(n+1)(n+2)}$

V(U) $=\frac{r(r+1)}{(n+1)(n+2)}$ - $\frac{r^2}{(n+1)^2}$ = $\frac{r}{n+1}$ $[\frac{r+1}{n+2}-\frac{r}{n+1}]$

$=\frac{r(n-r+1)}{(n+1)^2\ (n+2)}$

Q35. If X_1, X_2 be a random sample of size 2 drawn from a population having p.d.f. f(x)=λe $^{-\lambda x}$, x> 0, λ>0. Then find the distn of the sample range. Is the distn independent from the sampling distribution of simple AM?

ANS:- X_1, X_2 be a random sample drawn from a population with pdf f(x)= λe $^{-\lambda x}$, x>0, λ> 0.

Let us consider the following transformation

$(X_1, X_2) \rightarrow (X_{(1)}, X_{(2)})$, where $X_{(i)}$=ith order statistic.

∴ Joint distn of $X_{(1)}, X_{(2)}$ is given by,

$f_{X_{(1)},X_{(2)}}{}^{(x_1,x_2)} = 2\lambda^2\ e^{-\lambda(X_{(1)}+\ X_{(2)})}$

Let us define a variable,

$U_i = X_i - X_{(i-1)}$ ∀ i=1, 2.

U_1=$X_{(1)}$ [Assuming$X_{(0)}=0$]

$U_2 = X_{(2)} - X_{(1)}$

∴ X_2=u_1+u_2

∴ |J|=|J$(\frac{X_{(1)},X_{(2)}}{u_1,u_2})$|=$\begin{vmatrix}1 & 0\\ 1 & 1\end{vmatrix}$=1

∴ Joint pdf of u_1, u_2 is given by,

$f_{U_1,U_2}{}^{(u_1,u_2)}$=$2\lambda^2 e^{-\lambda(2u_1+u_2)}$

=2 λ .$e^{-2\lambda u_1}$, λ .$e^{-2\lambda u_2}$, (u_1,u_2)>0

∴ U_1 , U_2 are independently distributed,

∴ sample range(R)= $X_{(2)} - X_{(1)}$

$=u_2$

∴ E(R)=E(u_2)= λ$\int_0^\infty u\ e^{-\lambda u_2}\ du$

$=\frac{\lambda}{\lambda^2}=\frac{1}{\lambda}$

∴ PDF of sample range (R) is f_R(R)=λ $e^{-\lambda R}$, R> 0

Now, simple AM = $\frac{X_{(1)}+\ X_{(2)}}{2}$ = $\frac{2u_1+u_2}{2}$ = u_1+$\frac{1}{2}u_2$= Z, say,

∴ Joint PDF of (Z, u_2) is given by,

f_{Z,u_2}(Z, u_2)= $2\lambda^2\ e^{-2\lambda Z}$, Z>0

So, the distn of sample range & simple AM are different.

Q36. F(x, y) be a joint distribution function of X and Y. G(ξ,)be a function ∋ξ= max(X, Y). Show that G(x, y){$F(x,x)$ $if\ x<y$

F(x, y)+F(x,x)-F(y, y) if x≥ y

Solution:- G(x, y)= P [Max(X, Y)≤x, Min(X, Y)≤ y] if x < y

= P[P [Max(X, Y)≤x, Min(X, Y)≤ x]

=P [X≤ x , Y ≤ x]

= F(x, x)

Now, G(x, y)= P [Max(X, Y)≤x, Min(X, Y)≤ y] if x ≤ y

= P[$X_{(2)}$ ≤x, $X_{(1)} \leq y$]; A= $X_{(2)}$ ≤x, B=$X_{(1)} \leq y$

=P (A) –P(A ∩BC)

= P[$X_{(2)}$ ≤x] –[P[$X_{(2)}$ ≤x, $X_{(1)} \geq y$]

= P [X≤ x, Y≤ x]- P [y ≤ $X_{(1)} \leq X_{(2)} \leq x$]

= F(x,x)-(F(y,y)-F(x, y))

= F(x,x) +(F(x ,y)-F(y , y)).

Q37. f(x,y)= {$\frac{1}{\pi}$ $if\ x^2 + y^2 \leq 1$ (a)Are X and Y uncorrelated?

=0 if $x^2 + y^2 > 1$ (b)Are X and y independent?

Solution:- f(x,y)= {$\frac{1}{\pi} I\ x^2 + y^2 \leq 1$

$f_X(x) = \frac{1}{\pi}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dX = \frac{2}{\pi}\sqrt{1-x^2}$; -1 < x < 1

$f_Y(y)$=$\frac{2}{\pi}\sqrt{1-y^2}$; -1 < y < 1

E(X)= $\frac{2}{\pi}\int_{-1}^{1} x\sqrt{1-x^2}$ dx = 0 [∵ the function is odd]

Similarly, E(Y)=0

E(XY)=$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} xy\ \frac{1}{\pi}\ dxdy = 0$ [∵ $\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{x}{\pi} dx = 0$]

∴ X and Y are uncorrelated.

Note that, f(x,y)=$\frac{4}{\pi}\sqrt{(1-x^2)(1-y^2)} \neq$ f(x). f(y).

∴ X and Y are not independent.

Q38. X_1, ………, X_n , …… be i.i.d r. v. 's satisfying P [X_1=2^j]= $\frac{1}{2^j}$,

j = 1, 2, 3, ….. show that WLLN dose not hold for {X_n}.

<u>Solution</u>:- WLLN holds iff $\underset{n\to\infty}{Lt}[|x| > n] = 0$

Let, $2^k \leq n \leq 2^{k+1}$

i.e. n=2^k +r

P[X > n]=P [$X \geq 2^{k+1}$]= P [$X = 2^{k+1}$]+ $X \geq 2^{k+2}$+ ……

=$\frac{1}{2^{k+1}}$+$\frac{1}{2^{k+2}}$ + …….

= $\frac{1}{2^{k+1}}$.(1+ $\frac{1}{2}$ + $\frac{1}{4}$+ …..)= $\frac{1}{2^k}$

∴ nP [X > n]= $\frac{2^{k+r}}{2^k}$

= 1+$\frac{r}{2^k}$

∴ $\underset{n\to\infty}{Lt}\ nP[x > n]$= $\underset{k\to\infty}{Lt}\ (1 + \frac{r}{2^k})$=1.

∴ WLLN does not hold

Q39. There are 10 balls in an urn numbered 1 through 10.You randomly select 3 of those balls. Let the random variable Y denotes the maximum of the three numbers on the extracted balls. Find the probability mass function of y. You should simplify your answer to a fraction that does not involve binomial coefficients. Then calculate:

P [Y ≥7].

Solution:- The random variable Y can take the values in the set {3, 4, …. , 10}. For any I, the triplet resulting in Y attaining the value i must consist of the ball numbered i and a pair of balls with lower numbers. So ,

Pi =P[Y =i]= $\frac{\binom{i-1}{2}}{\binom{10}{3}} = \frac{\frac{(i-1)(i-2)}{2}}{\frac{10.9.8}{3.2.1}} = \frac{(i-1)(i-2)}{240}$

Since the balls are numbered 1 through 10, we have

P [Y ≥7] =P [Y= 7] + P [Y= 8] + P[Y=9] + P [Y= 10]

So, P [Y ≥7]= $\frac{6.5}{240} + \frac{7.6}{240} + \frac{8.7}{240} + \frac{9.8}{240}$

= $\frac{5}{6}$.

Q40. The number of misprints per page of text is commonly modeled by a poisson distribution . It is given that the parameter of this distribution is λ = 0.6 for a particular book. Find the probability that there are exactly two misprints on a given page of the book. How about the prob. that there are two or more misprints?

Solution:- Let X denote the random variable which stands for the number of misprints on a given page. Then

P [X =2]= $\frac{0.6^2}{2!}\, e^{-0.6}$≈ 0.0988

P [X≥ 2] = 1- P [X < 2]

= 1- P [X=1] – P[X=0]

=1-$e^{-0.6}$ - 0.6$e^{-0.6}$

≈ 0.122.

Q41. The unit interval (0,1)is divided into two subintervals picking a point at random from inside the interval. Denoting by Y and Z, the lengths of the larger and the shorter subintervals respectively. Show that Y/Z does not have finite expectations.

Solution:- Let X ∼ U(0, 1) and U be the length of the intervals (0, X) and (X, 1); i.e. Z = min(X, 1-X)

And let, Y = 1- U.

We can write Y and Z as Y = Y(X), Z = Z(X); as functions of X ∼ U (0, 1).

To show the expectation is infinite; we need to show that

$\int_1^0 \frac{Y(x)}{Z(x)}\, dx$=∞

Now, Z = min (X, 1-X) , Y = max(X, 1-X)

But, $\int_1^{\frac{1}{2}} \frac{Y(x)}{Z(x)}\, dx = \int_1^{\frac{1}{2}} \frac{\max(X,1-X)}{\min(X,1-X)}\, dx =$ $\int_1^{\frac{1}{2}} \frac{1-x}{x}\, dx = \infty$

One get the same thing on the interval [$\frac{1}{2}$, 1] as well after a substitution, so the integral on [0, 1] is ∞.

Aliter:- let T (x)= $\frac{\max(X,1-X)}{\min(X,1-X)}$

To calculate P [T (X)≤], for some t, If t < 1, then trivially we get 0.

Otherwise, P [T (X)≤]= P [$\frac{1-X}{X} \le t$, X∈ (0, $\frac{1}{2}$)] + P [$\frac{X}{1-X} \le t$, X∈ ($\frac{1}{2}$, 1)]

= 2P [X≥ $\frac{1}{1+t}$, X∈ (0, $\frac{1}{2}$)]

=2 $\int_{1/_{1+t}}^{1/2} dx$ = 1- $\frac{2}{t+1}$.

Differentiating, we get f(t) = $\frac{2}{(t+1)^2}$ for t≥ 1.

Now, $\int_1^{\infty} \frac{2}{(t+1)^2}\, dt = \infty$.

Q42. Let $X_1, X_2 \sim$ R(0, 1). Show that_____________

$\boldsymbol{U_1} = \sqrt{-21nX_1}\ \mathbf{cos}(2\pi X_2)$

$\boldsymbol{U_2} = \sqrt{-21nX_1}\ \mathbf{sin}(2\pi X_2)$

Are standard normal variables.

Solution:- The PDF of (X_1, X_2) is $f_{X_1,X_2}(x_1, x_2) = f(x) = \begin{cases} 1 & , \ 0 < x_1, x_2 < 1 \\ 0 & , \ ow \end{cases}$

Here, $u_1 = \sqrt{-21nx_1}\ \cos(2\pi x_2)$

$u_2 = \sqrt{-21nx_1}\ \sin(2\pi x_2)$

$\therefore {u_1}^2 + {u_2}^2$ = -21nx_1

$\Rightarrow x_1 = e^{-\frac{1}{2}({u_1}^2 + {u_2}^2)}$

And $\tan(2\pi x_2) = \frac{u_2}{u_1}$

$\Rightarrow x_2 = \frac{1}{2\pi} \tan^{-1}\frac{u_2}{u_1}$

Note that, 0 < x_1 < 1

$\Rightarrow$ -21nx_1 > 0 , 0<$2\pi x_2$ <2π

$\Rightarrow \sqrt{-21nx_1}$ > 0 , -1 $\leq \cos(2\pi x_2), \sin(2\pi x_2) \leq 1$

$\Rightarrow u_1, u_2 \in \mathbb{R}$

The Jacobian is J = $\begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_1} \end{vmatrix}$

= $\begin{vmatrix} e^{-\frac{1}{2}({u_1}^2 + {u_2}^2)}.(-u_1) & e^{-\frac{1}{2}({u_1}^2 + {u_2}^2)}.(-u_2) \\ \frac{1}{2\pi\{1+(\frac{u_2}{u_1})^2\}}.(-\frac{u_2}{u_1}) & \frac{1}{2\pi\{1+(\frac{u_2}{u_1})^2\}}.\frac{1}{u_1} \end{vmatrix}$

= $\frac{e^{-\frac{1}{2}({u_1}^2 + {u_2}^2)}}{2\pi\{1+(\frac{u_2}{u_1})^2\}} \begin{vmatrix} -u_1 & -u_2 \\ -\frac{u_2}{{u_1}^2} & \frac{1}{u_1} \end{vmatrix}$

= - $\frac{1}{2\pi}$. $e^{-\frac{1}{2}({u_1}^2 + {u_2}^2)}$

The PDF of (u_1, u_2) is

$f_{U_1,U_2}(u_1, u_2)$= 1 . | $-\frac{1}{2\pi} . e^{-\frac{1}{2}({u_1}^2 + {u_2}^2)}$ |. $(u_1, u_2) \in R^2$

= $\frac{1}{\sqrt{2\pi}}.e^{-\frac{1}{2}{u_1}^2}.\frac{1}{\sqrt{2\pi}}.e^{-\frac{1}{2}{u_2}^2}$, $(u_1, u_2) \in \mathbb{R}^2$

= $f_{u_1}(u_1).f_{u_2}(u_2)$, $u_1, u_2 \in \mathbb{R}$

Hence, $U_1, U_2 \sim$ N(0, 1).

Q43. Let X, Y ~ N(0, 1). Show that U = $\frac{X}{Y}$ has a standard Cauchy distribution. What would be the distn of $\frac{X}{|Y|}$?

Soln:- Here, $f_{X,Y}(x, y) = \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}}$, (x, y) $\in \mathbb{R}^2$

Let, U=$\frac{X}{Y}$ and V = y.

$\therefore$ u = $\frac{x}{y}$, v= y [$-\infty < u < \infty$, $-\infty < v < \infty$]

$\Rightarrow$ x= uv , y=v

$\therefore$ J = $\begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix}$= v = $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

Clearly, (u,v) $\in \mathbb{R}^2$

The PDF (U, V) is ___

$f_{U,V}(u, vy) = \frac{1}{2\pi} e^{-(1+u^2)\frac{v^2}{2}} |v|$, (u,v) $\in \mathbb{R}^2$

The PDF of U is

$f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u, v)\, dv =$
$\int_{-\infty}^{\infty} \frac{1}{2\pi}\, e^{-(1+u^2)\frac{v^2}{2}} |v| dv$

Or, $=[2 \int_0^{\infty} \frac{v}{2\pi} e^{-\frac{1}{2}v^2(1+u^2)} dv]$ =
$\frac{2}{\pi} \int_0^{\infty} e^{-(1+u^2)\frac{v^2}{2}} v dv$

$[\int_0^{\infty} \frac{1}{\pi} \frac{1}{(1+u^2)} . e^{-Z} dZ \frac{1}{2} v^{2(1+u^2)=Z]}$
$= \frac{1}{\pi} \int_0^{\infty} e^{-(1+u^2)Z}\, dZ$

[where, $Z=\frac{v^2}{2}$, ⇒dZ = vdv]

$[\frac{1}{\pi(1+u^2)} [-e^{-Z}]]$
$= \frac{1}{\pi} . \frac{\Gamma(1)}{(1+u^2)}$, u ∈ $\mathbb{R}$

$[\frac{1}{\pi(1+u^2)}]$
$= \frac{1}{\pi(1+u^2)}$, u ∈ $\mathbb{R}$

Hence, $U=\frac{X}{Y}$ ~ C(0, 1) distn

Let, w = $\frac{x}{|Y|}$, The PDF of w is $f_W(w)$= P [W≤ w/Y < 0]P [Y < 0]+ P [W≤ w/Y > 0]P [Y > 0]

$= \frac{1}{2}$ {P [$\frac{X}{-Y} \le w$] + P [$\frac{X}{Y} \le w$]}

$= \frac{1}{2}$ {P[-U≤ w] + P [U ≤ w]}

$= \frac{1}{2}$ .2 .P [U≤ w] [∵ U~ C (0, 1)is symmetrical about '0']

[⇒$f_U(-u) = f_U(u)$

⇒ u and –U have identical distribution]

∴ $f_W(w) = f_U(w)$ ∀ w

⇒ W = $\frac{X}{|Y|}$~ C (0, 1).

Q44. If X, Y ~ N(0, 1). Find the distns of U = $\sqrt{X^2+Y^2}$ and V = $\frac{X}{Y}$

Solution:- $f_{x,Y}(x, y) = \frac{1}{2\pi} . e^{-\frac{1}{2}(x^2+y^2)}$, (x, y)∈ $\mathbb{R}^2$

Note that, u= $\sqrt{x^2+y^2}$, v = $\frac{x}{y}$

⇒ u = |y|. $\sqrt{1+v^2}$, x= vy

⇒ x = ± $\frac{uv}{\sqrt{1+v^2}}$, y = ± $\frac{u}{\sqrt{1+v^2}}$

Let, $x_1 = \frac{uv}{\sqrt{1+v^2}}$, $y_1 = \frac{u}{\sqrt{1+v^2}}$

Then for a pair (U, V), there are two points of (x, y):

$(x_1, y_1), (-x_1, -y_1)$

The transformation is not one – to one,

Clearly, 0 < u < ∞ , v ∈ $\mathbb{R}$

Now, $J_1 = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial y_1}{\partial u} & \frac{\partial y_1}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{v}{\sqrt{v^2+1}} & \frac{u}{(\sqrt{v^2+1})^{3/2}} \\ \frac{1}{\sqrt{v^2+1}} & \frac{uv}{(\sqrt{v^2+1})^{3/2}} \end{vmatrix}$

$= -\frac{u}{1+v^2} = J_2$

Hence, The PDF of (U, V)is

$f_{U,V}(u, v) = \begin{cases} f_{x,Y}(x_1, y_1)|J_1| + f_{x,Y}(-x_1, -y_1)|J_2|, if\ 0 < u < \infty, -\infty < \\ 0 \quad\quad , ow \end{cases}$

$= \begin{cases} \frac{2}{2\pi}\ e^{-\frac{u^2}{2}} . \left|-\frac{u}{1+v^2}\right|, if 0<u< \infty, and -\infty<v< \infty \\ 0 \qquad , ow \end{cases}$

=

$\begin{cases} \left(ue^{-\frac{u^2}{2}}\right) . \frac{1}{\pi(1+v^2)}, if\ 0 < u < \ \infty\ and\ v \in \mathbb{R} \\ 0 \qquad , \qquad ow \end{cases}$

Hence, U = $\sqrt{X^2+Y^2}$ has the PDF

$f_U(u) = \begin{cases} u\ e^{-\frac{u^2}{2}}, 0 < u < \ \infty \\ 0 \quad , ow \end{cases}$

and V ~ Cauchy (0, 1), independently.

Q45. If X, Y ~ N(0, 1), Find the distn of U= $\frac{XY}{\sqrt{X^2+Y^2}}$, and V = $\frac{X^2-Y^2}{\sqrt{X^2+Y^2}}$.

Solution:- $f_{x,Y}(x,y) = \frac{1}{2\pi}\ e^{-\frac{X^2+Y^2}{2}}$, (x, y)∈ $\mathbb{R}^2$

Let, x= rcosθ, y = rsinθ,

Here, 0< r<∞, 0< θ< 2π,

∴ J = r,

The PDF of (r, θ)is ______

g (r, θ)= $\begin{cases} re^{-\frac{r^2}{2}} . \frac{1}{2\pi}, 0<r< \infty\ and\ 0< \theta<2\pi \\ 0 \quad , \qquad ow \end{cases}$

Here, u= rsinθcosθ = $\frac{r}{2} sin^2\theta$

And v= rcos2θ

Clearly, (U, V) ∈ $\mathbb{R}^2$

$J_1 = \frac{\partial(\mathrm{r},\theta)}{\partial(\mathrm{u},\mathrm{v})} = \frac{1}{\frac{\partial(\mathrm{u},\mathrm{v})}{\partial(\mathrm{r},\theta)}} = \frac{1}{\begin{vmatrix} \frac{1}{2}sin^2\theta & \mathrm{r}\cos 2\theta \\ \cos 2\theta & -2sin\theta \end{vmatrix}} = \frac{1}{r} = J_2$

Here, $(2u)^2 + v^2 = r^2$ [a pair(u,v)is a obtained, for two pairs: (r, θ), (r, θ+2π). The transformation is not one-to-one]

$\Rightarrow$ r = $\sqrt{4u^2+v^2}$

The PDF of (U, V)is

$f_{UV}(u,v) = \frac{2 .e^{-\frac{4u^2+v^2}{2}}}{2\pi}$.($\sqrt{4u^2+v^2}$)|$\frac{1}{\sqrt{4u^2+v^2}}$|, (u, v) ∈ $\mathbb{R}^2$

$= \frac{1}{\frac{1}{2}\sqrt{2\pi}}\ .\ e^{-\frac{u^2}{2\frac{1}{4}}} . \frac{1}{\sqrt{2\pi}}\ .\ e^{-\frac{v^2}{2}}$; (u, v) ∈ $\mathbb{R}^2$

= $f_U(u).f_V(u)$, u, v ∈ $\mathbb{R}$

Hence, U ~ N (0, $\frac{1}{4}$) and V ~ N (0, 1), independently.

Q46. Let X_1, X_2 ~ r (0, 1). Find out CDF and hence the PDF of $X_1 + X_2$. How should the above result be modified in case $X_1, and\ X_2$ ~ R (a, b)?

Solution:- $f_U(u)$ = P [U≤ u]

= P [$X_1 + X_2$ ≤ u]

=$\iint_{X_1+X_2}(x_1,x_2)dx_1, dx_2$

Here, U = $X_1 + X_2$ takes values between 0 and 2.

Note that for 0< u <1,

P [U≤ u]= P [$X_1 + X_2 \leq u$]

= $\frac{Area\ of\ the\ region\ A}{Area\ of\ the\ sample\ space\ (\Omega)}$

[Using the concept of geometric probability,

As ($X_1,\ X_2$) is uniformly distributed

Over Ω, here,

Ω = {($X_1,\ X_2$) : 0< $X_1,\ X_2$< 1}

And A ={$(X_1,\ X_2)$: $X_1 +\ X_2 \leq u\}$≤ Ω]

∴ P [U≤ u] = $\frac{\frac{1}{2}u^2}{1^2}$ = $\frac{1}{2}u^2$, for 0< u< 1.

For, 1 ≤ u< 2 , ___

P [U ≤ u]= P [$X_1 +\ X_2 \leq u$]

= $\frac{Area\ of\ the\ region\ A}{Area\ of\ the\ sample\ space\ (\Omega)}$

= $\frac{1^2-\frac{1}{2}(2-u)^2}{1^2}$

= 1 -$\frac{1}{2}(2-u)^2$

Hence the CDF of U is___ $F_U(u)$=

$$\begin{cases} 0, u \leq 0 \\ \frac{1}{2}u^2, 0 < u < 1 \\ 1 - \frac{1}{2}(2-u)^2, 1 \leq u < 2 \\ 1, u \geq 2 \end{cases}$$

And the PDF of U is _____

$$F_U(u)= \begin{cases} u, 0 < u < 1 \\ 2-u, 1 \leq u < 2 \\ 0\ , ow \end{cases}$$

Modification:- X_i~ R (a, b) , i=1, 2

⇒ U_i = $\frac{X_i-a}{b-a}$ ~ R (0, 1), i=1, 2.

Q47. Let X_1, X_2 ~ R (0, 1). Find out CDF and PDF of

i> $|X_1 -\ X_2|$, ii>$X_1\ X_2$

Solution:-

i> Let U=$|X_1 -\ X_2|$

Note that U takes values between 0 and 1.

For, 0< u< 1,

P [U ≤ u]= P [$|X_1 -\ X_2|$ ≤ u]

= P [-u≤ $X_1 -\ X_2$ ≤ u]

=$\frac{Area\ of\ the\ shaded\ region}{Area\ of\ the\ sample\ space\ (\Omega)}$

[Using the concept of Geometric Probability as (X_1, X_2) is Uniformly distributed over Ω]

P [U ≤ u]= $\frac{1^2-\frac{1}{2}(1-u)^2}{1^2}$

= 1- $(1-u)^2$

Hence, the CDF of u is

$$f_U(u)= \begin{cases} 0\ \ , u \leq 0 \\ 1 -\ (1-u)^2, 0 < u < 1 \\ 1\ \ ,\ u \geq 1 \end{cases}$$

And the PDF of U is

$$f_U(u)= \begin{cases} 2\ (1-u), 0 < u < 1 \\ 0\ ,\ \ ow \end{cases}$$

ii> Let U= $X_1\ X_2$

Then U takes value between 0 and 1.

For, 0< u< 1:

P [U ≤ u]= P [$X_1\ X_2 \leq$ u]

= $\frac{Area\ of\ the\ shaded\ region}{Area\ of\ the\ sample\ space\ (\Omega)}$

=$\frac{uX_1+\int_u^1 x_2 dx_1}{1^2}$

=u +$\int_u^1 \frac{u}{x_1}\ dx_1$

= u+ u [1nx_1]1,u =u(1-|nu)

Q48. X and Y ~ R (0, 1) X & Y are independent,

i> X+Y ~ ?

ii> **X- Y ~?**

iii> **XY ~?**

iv> $\boldsymbol{\frac{X}{Y}}$ **~?**

v> $\boldsymbol{|X-Y|}$**~?**

ANs:-i) Z= X+Y

0< X, Y<1

⇒ 0< Z < 2.

Distribution function of Z is

$F_Z(z)$= P [z≤ z]

= P[Y≤z-X]

$$= \begin{cases} 0\,, z \leq 0 \\ \frac{1}{2}z^2\,, if\ 0 < z < 1 \\ 1-\frac{1}{2}(2-z)^2, if\ 1 < z < 2 \\ 1 \quad\quad ,\ if\ z \geq 2 \end{cases}$$

PDF of Z is, ______

$$F_Z(z)= \begin{cases} z\,, if\ 0 < z < 1 \\ 2-z\,, if\ 1 < z < 2 \\ 0 \quad , \quad\quad ow \end{cases}$$

ii) Z = X- Y

0< X, Y< 1

⇒ -1< z< 1

$F_Z(z)$ = P [X-Y ≤z]

=P [Y ≥ $X-z$]

$$= \begin{cases} 0\,, if\ z \leq\ -1 \\ \frac{1}{2}(z+1)^2, if -1 < z < 0 \\ 1-\frac{1}{2}(1-z)^2, if\ 0 < z < 1 \\ 1 \quad ,\ z \geq 1 \end{cases}$$

PDF of Z is, ________

$$F_Z(z)= \begin{cases} z+1\,, if -1 < z < 0 \\ 1-z\,, if\ 0 < z < 1 \\ 0 \quad , \quad ow \end{cases}$$

iii) Z=XY

0< X, Y< 1 ⇒ 0< XY <1.

⇒ 0<x < 1

$F_Z(z)$= P [XY ≤ z]

=

$$\begin{cases} 0\ \ , if\ z\ \leq 0 \\ \int_0^1 \int_0^z dxdy + \int_z^1 \int_0^{z/x} dxdy\ \ , if\ 0 < z < 1 \\ 1\,, \quad if\ z \geq 1 \end{cases}$$

= z+$\int_z^1 \frac{z}{x} dx$, if 0< z< 1

= z + z [|n|-|nz]

= z- z|nz = z (1-|nz). , if 0< z<1

∴ CDF of z is, ______

$$F_Z(z)= \begin{cases} 0\ \ ,\ if\ z \leq 0 \\ z\,(1-|nz), if\ 0 < z < 1 \\ 1 \quad , if\ z \geq 1 \end{cases}$$

PDF of z is, ___

$$F_Z(z)= \begin{cases} -1nz\,, if\ 0 < z < 1 \\ 0 \quad\quad , ow \end{cases}$$

iv) Z= $\frac{X}{Y}$

0< X, Y< 1

⇒ 0<$\frac{X}{Y}$ < ∞

∴ 0< Z <∞

P [Z ≤ z]

=P [$\frac{X}{Y}$ ≤ z]

= P [Y ≥ ½ X]

$$= \begin{cases} 0\,, if\ z \le 0 \\ \frac{1}{2}z\,, if\ 0 < z < 1 \\ 1 - \frac{1}{2z}, if\ z \ge 1 \end{cases}$$

$$\therefore F_Z(z) = \begin{cases} \frac{1}{2}, if\ z \le 0 \\ \frac{1}{2z^2},\ if\ z\ \ge 1 \\ 0 \quad,\ ow \end{cases}$$

v) Z = $|X - Y|$

0< X, Y< 1

⇒ 0<$|X - Y|$ < 1

P [Z ≤z]

= P [X –Z ≤Y ≤ X+Z]

$$= \begin{cases} 0\,, if\ z \le 0 \\ 1 - (1-z)^{2\,, if\ 0<z<1} \\ 1, if\ z\ \ge 1 \end{cases}$$

$$\therefore f_Z(z) = \begin{cases} 2(1-z)\,, if\ 0 < z < 1 \\ 0 \quad, \quad ow \end{cases}$$

Q49. X ~ R(0, a)

Y ~ R(0, b)

X and Y are independent & a> b

i> X+Y ~ ?

ii> X- Y ~?

iii> XY ~?

iv> $\frac{X}{Y}$ ~?

v> $|X - Y|$~?

ANS :- i) X + Y= Z,

0< x < a, 0<y < b

⇒ 0< x+y< a+b

∴ P [Z ≤z]= P [Y≤z-X]

$$= \begin{cases} 0\,, z \le o \\ \frac{1}{ab} \times \frac{1}{2}z^2\,, if\ 0 < z\ \le b \\ \frac{1}{ab} \times \frac{b}{2} \times (2z - b), b < z\ \le 0 \\ \frac{ab - (\frac{a+b-z)^2}{2}}{ab}\ a < z < a + b \\ 1, if\ z\ \ge a + b \end{cases}$$

$$= \begin{cases} \frac{z}{ab}\,, a < z\ \le b \\ \frac{1}{a}, b < z \le a \\ \frac{a+b-z}{ab}\,, a < z < a + b \\ o\ ,\ ow \end{cases}$$

ii) X- Y = z

P [Z ≤ z]

= P [X-Y ≤ z]

= P [Y ≥ X- z]

$$= \begin{cases} 0\,, z\ \le\ -b \\ \frac{1}{2ab}(b+z)^2\,, -b < z\ \le 0 \\ \frac{1}{2ab}\ (b + 2z)b, 0 < z\ \le a - b \\ 1 - \frac{1}{2ab}(a-z)^2\,, a - b < z < a \\ 1\ ,\ \ z \ge a \end{cases}$$

$$\therefore f_z(z) = \begin{cases} \frac{1}{ab}(b+z), -b < z\ \le 0 \\ \frac{1}{a}\,, 0 < z\ \le a - b \\ \frac{a-z}{ab}\,, a - b < z < a \\ 0\ , ow \end{cases}$$

iii) Z= XY , 0< z < ab

P [Z ≤ z]

$= P\left[Y \leq \frac{z}{X}\right]$

$$= \begin{cases} 0 \ , z \leq 0 \\ \frac{1}{ab}\left[z + \int_{\frac{z}{b}}^{a} \frac{Z}{X} dx\right], 0 < z < ab \\ 1 \ , \ z \geq ab \end{cases}$$

∴

$f_z(z)$=

$$\begin{cases} [\, 1 - b + 1nba - 1nz]\frac{1}{ab}, 0 < z < ab \\ 0 \ , \ ow \end{cases}$$

iv) Z = $\frac{X}{Y}$, 0< z

P $\left[\frac{X}{Y} \leq z\right]$

P$\left[Y \geq \frac{X}{z}\right]$

$$= \begin{cases} 0 \ , \ z \leq 0 \\ \frac{1}{2ab} \times bz \times b \, , 0 < z < 1 \\ 1 - a.\frac{a}{z}.\frac{1}{2ab} \ , 1 \leq z \end{cases}$$

$$\therefore f_z(z) = \begin{cases} \frac{b}{2a} \ , 0 < z < 1 \\ \frac{a}{2b}\left(\frac{1}{z^2}\right), z \geq 1 \\ 0 \ , \ ow \end{cases}$$

v) Z = |X- Y| , 0< z < a

P [X-z ≤Y ≤ X + z]

$$= \begin{cases} 0 \ , z \leq 0 \\ 1 - \frac{1}{2}(b-z)^2 - \frac{1}{2}(a-z)^2 \, , 0 < z < a \\ 1 \ , \ z \geq a \end{cases}$$

$$\therefore f_z(z) = \begin{cases} a + b - 2z \, , 0 < z < a \\ 0 \ , \ ow \end{cases}$$

Q50. let X_1, X_2, X_3 be iid RV's with PDF

F(x) = $\begin{cases} \boldsymbol{e^{-x}} \, , \boldsymbol{x} > 0 \\ \boldsymbol{0} \, , \boldsymbol{ow} \end{cases}$

Show that ⟶ $Y_1 = X_1 + X_2 + X_3$

$\boldsymbol{Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3}}$

$\boldsymbol{Y_3 = \frac{X_1}{X_1 + X_2}}$ **, are independently distributed.**

Identify their distribution.

Solution :- $f_{X_1, X_2, X_3}(x_1, x_2, x_3)$=

$$\begin{cases} e^{-(x_1+x_2+x_3)} \, , if \ x_i > 0 \ \forall \ i = 1, 2, 3 \\ 0 \ , \qquad ow \end{cases}$$

Here, $x_1 + x_2 + x_3 = y_1$

$x_1 + x_2 = y_1 y_2$

$[\because x_1 + x_2 < x_1 + x_2 + x_3 \Rightarrow \frac{x_1+x_2}{y_1} < 1, \frac{x_1}{x_1+x_2} < 1 \ ]$

$x_3 = y_1 \, (1 - y_2)$

Clearly, 0< y_1 < ∞ and 0< y_2, y_3< 1

The Jacobian is J= $\begin{matrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{matrix}$

$$= \begin{matrix} y_2 \, y_3 & y_1 y_3 & y_1 \, y_2 \\ y_2(1 - y_3) & y_{1(1-y_3)} & -y_1 \, y_2 \\ 1 - y_2 & -y_1 & 0 \end{matrix} = \begin{matrix} 1 & 0 & 0 \\ y_2(1 - y_3) & y_1(1 - y_3) & -y_1 \, y_2 \\ 1 - y_2 & -y_1 & 0 \end{matrix}$$

= - $y_1{}^2 y_2$

The PDF of (Y_1, Y_2, Y_3)is

$f_{Y_1, Y_2, Y_3}(y_1, y_2, y_3)$=

$$\begin{cases} e^{-y_1} |\, y_1{}^2 y_2 | \ ,if \ 0 < y_1 < \infty \ and \ 0 < y_2 \, , y_3 < 1 \\ 0 \ , \qquad ow \end{cases}$$

$$= \begin{cases} \frac{e^{-y_1} . {y_1}^{3-1}}{\Gamma(3)} . 2y_2 .1 , if\ 0 < y_1 < \infty , 0 < y_{2,} \ y_3 < 1 \\ 0 \quad , \quad ow \end{cases}$$

Where, $f_{Y_1}(y_1) = \begin{cases} \frac{e^{-y_1} . {y_1}^{3-1}}{\Gamma(3)} , 0 < y_1 < \infty \quad \therefore Y_1 \sim Gamma(3) \\ 0 \quad , \quad o\, w \end{cases}$

$$f_{Y_2}(y_2) = \begin{cases} \frac{{y_{2,}}^{2-1}(1- y_{2,})^{1-1}}{\beta(2,1)} , 0 < y_{2<1} \therefore Y_2 \sim \beta(2,1) \\ 0 \quad , \quad ow \end{cases}$$

And $f_{Y_3}(y_3) = \begin{cases} 1 \ , 0 < y_3 < 1 \quad , \therefore \ Y_3 \ \sim U(0,1) \\ 0 \quad , \quad ow \end{cases}$

[Due to independence]

Q51. let $f_{X,Y,Z}(x,y,z)$= $\begin{cases} \frac{6}{(1+x+y+x)^4} , if\ x,y,z > 0 \\ 0 \ , \ ow \end{cases}$

Be the PDF of (X, Y, Z). find the distn of U = X+ Y+ Z.

Solution :- U = X+ Y+ Z

V= $\frac{X+Y}{X+Y+Z}$ $here, 0 < U < \infty \ and\ 0 < V , w < 1$

W= $\frac{X}{X+Y}$

J= - $u^2 v$

$f_{U,V,W}(u,v,w)$= $\begin{cases} \frac{6}{(1+u)^4} . |- u^2 v|, 0 < u < \infty \ and\ 0 < v,w < 1 \\ 0 \quad , \quad ow \end{cases}$

The PDF of U is , ____________

$f_U(u)$= $\begin{cases} \int_0^1 \left(\int_0^1 \frac{6u^2 v}{(1+u)^4} dv \right) dw \ , \ if\ 0 < u < \infty \\ 0 \quad , \quad ow \end{cases}$

$= \begin{cases} \frac{1}{\beta(3,1)} . \frac{u^{3-1}}{(1+u)^{3+1}} , 0 < u < \infty \\ 0 \ , \ ow \end{cases}$

Hence U ~ second kind Beta (3, 1)

Q. If X_1, X_2, X_3 ~ N(0, 1). Find the distn s of

$Y_1 = \frac{X_1 + X_2 + X_3}{\sqrt{3}}$

$Y_2 = \frac{X_1 - X_2}{\sqrt{2}}$

$Y_3 = \frac{X_1 + X_2 - 2X_3}{\sqrt{6}}$

Solution:-$f_{X_1,X_2,X_3}(x_1,x_2,x_3)$=$(\frac{1}{2\pi})^{3/2} . e^{-\frac{1}{2}({x_1}^2 + {x_2}^2 + {x_3}^2)}$; xi ∈ ℝ

Note that, y = $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

= a x , wher A is orthogonal, i. e. $AA^T = I_3$.

∴ x= $A^{-1}y = A^T y$ and

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$\Rightarrow x_1 = \frac{y_1}{\sqrt{3}} + \frac{y_2}{\sqrt{2}} + \frac{y_3}{\sqrt{6}}$

$x_2 = \frac{y_1}{\sqrt{3}} - \frac{y_2}{\sqrt{2}} + \frac{y_3}{\sqrt{6}}$

$x_3 = \frac{y_1}{\sqrt{3}} + 0.y - \frac{2y_3}{\sqrt{6}}$

Jacobian = $|\frac{\partial(old\ variable)}{\partial(old\ variable)}| = |\frac{\partial x}{\partial y}|$ or $|\frac{\partial(x_1,x_2,x_3)}{\partial(y_1,y_2,y_3)}|$

$$= \begin{matrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{matrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix} =$$

$|A^T| = \pm 1$.

Note that, y′y = x′ A′Ax = x′x $\Rightarrow \sum_{i=1}^{3} {y_i}^2 = \sum_{i=1}^{3} {x_i}^2$

Clearly, $y_i \in \mathbb{R}$, i=1, 2, 3

The PDF of Y_1, Y_2, Y_3 is

$f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3) = (\frac{1}{2\pi})^{3/2} . e^{-\frac{1}{2}\sum_{i=1}^{3} {y_i}^2} .| \pm 1|$, $y_i \in \mathbb{R}$

$= \prod_{i=1}^{3} \{\frac{1}{\sqrt{2\pi}} . e^{-\frac{1}{2}{y_i}^2}\} = \prod_{i=1}^{3} f_{Y_i}(y_i)$

Hence, $Y_i \sim N(0, 1)$, i= 1, 2, 3.

Q52. Let $X_1, X_2, \dots, X_n$ be a r. s. from N (μ, μ), $\mu > 0$.

(a) Find a consistent estimator of μ^2. Is it unbiased?

(b) Find out an UE which is consistent?

Solution :- (a) $\bar{x} \sim N(\mu, \frac{\mu}{n})$

$\Rightarrow E(\bar{x}) = \mu$

$V(\bar{x}) = \frac{\mu}{n} \longrightarrow 0$ as $n \longrightarrow \infty$

Hence $\bar{x}$ is consistent for μ.

By invariance property, $\bar{x}^2$ is consistent for μ^2.

But, $E(\bar{x}^2) = v(\bar{x}) + E^2(\bar{x})$

$= \frac{\mu}{n} + \mu^2 \neq \mu^2$ [$\because X_i \sim N(\mu, \mu)$]

i.e. $\bar{x}^2$ is biased for μ^2.

(b) In a normal sample, $\bar{x}$ and S^2 are independently distributed.

Also, $E(\bar{x}) = \mu$ and $E(S^2) = \mu$

Hence, $E(\bar{x}. S^2) = E(\bar{x}) .E(S^2)$, due to independence.

$= \mu^2$

And $var(\bar{x}.S^2) = E(E^2)^2 - E^2(\bar{x}.S^2)$

$= E(\bar{x}^2.S^4) - \mu^4$

$= E(\bar{x}^2). E(.S^4) - \mu^4$

$= \{v(\bar{x}) + E^2(\bar{x})\} \quad \{v(S^2) + E^2(S^2)\} - \mu^4$

$= \{\frac{\mu}{n} + \mu^2\} \{\frac{2\mu^2}{n-1} + \mu^2\} - \mu^4$

$\longrightarrow 0$ as $n \longrightarrow \infty$

Hence, $\bar{x}.S^2$ is consistent as well as unbiased for μ^2.

Q52. give an example of an estimator which is

(i) Consistent but not unbiased,

(ii) Unbiased but not consistent,

(iii) Consistent as well as unbiased.

ANs :- (i) Let $T_1 = \bar{x} + \frac{1}{n}$

Clearly, $T_1 = \bar{x} + \frac{1}{n}$ is consistent but E $(T_1)= \mu + \frac{1}{n} \neq \mu$

So, it is not unbiased.

[if $\{T_n\}$ is consistent for θ, the $\{T_n + a_n\}$ is consistent for θ if $\lim_{n\to\infty} a_n = 0$.]

(ii) note that, T = $\frac{X_1+X_n}{2}$ is an unbiased estimator of μ.

T ~ n(μ, $\sigma^2/2$)

Now, P [|T- μ|< ∈] = P [|$\frac{T-\mu}{\frac{\sigma}{\sqrt{2}}}$| < $\frac{\epsilon\sqrt{2}}{\sigma}$]

= 2 I[$\frac{\epsilon\sqrt{2}}{\sigma}$]-1

$\nrightarrow$1 as n $\longrightarrow\infty$

Hence, T is unbiased but not consistent for μ.

(iii) Let $X_1, X_2, \ldots, X_n$ be a r. s. from N (μ, σ^2)

Then $\bar{x}$ ~ n (μ, σ^2/n)

E ($\bar{x}$)= μ , v($\bar{x}$) = $\sigma^2/n \longrightarrow 0$ as n$\longrightarrow \infty$

$\Rightarrow \bar{x}$ is consistent as well as unbiased.

Q53. Show that for a r. s. from Cauchy distribution with location parameter μ, i.e. C(μ, 1), the sample mean is not consistent for μ but the sample median is consistent for μ.

ANS: - let $X_1, X_2, \ldots, X_n$ be a r. s. from C (μ, 1)

Then $\bar{x}$ ~ c (μ, 1)

Now, P [|$\bar{x}$ -μ|< ∈]= P [μ- ∈ <$\bar{x}$ < μ+ ∈]

$=\int_{\mu-\epsilon}^{\mu+\epsilon} \frac{d\bar{x}}{\pi\{1+(\bar{x}-\mu)^2\}}$

$= [\frac{1}{\pi} \tan^{-1}(\bar{x}-\mu)]_{\mu-\epsilon}^{\mu+\epsilon}$

$=\frac{2}{\pi} \tan^{-1} \epsilon \nrightarrow 1$ as n $\longrightarrow\infty$

Hence $\bar{x}$ is not consistent for μ.

It can be shown that for large samples,

$\xi_p \sim N\left(\xi_p, \frac{P(1-P)}{n.f^2(\xi_p)}\right)$,

Where, f(.)is the PDF of the distribution.

For, C (μ, 1) distribution, $\xi_p \sim N\left(\xi_p, \frac{1}{4nf^2(\mu)}\right)$

$\Rightarrow \bar{x} \sim N\left(\mu, \frac{\pi^2}{4n}\right)$ [$\because f(\mu)=\frac{1}{\pi}$]

Hence, for large n, E ($\bar{x}$)=μ

V ($\bar{x}$)= $\frac{\pi^2}{4n} \longrightarrow 0$ as n $\longrightarrow\infty$

$\Rightarrow \bar{x}(\xi_{\frac{1}{2}})$ is consistent for μ.

Remarks: - By khinchinte's WLLN: $\bar{x} \xrightarrow{P} \mu$, provided E($X_1$)=$\mu$, the population mean exists. In Cauchy population, the population mean does not exist and μ is not the populations mean but it is the population median. Hence for μ, $\bar{X}$ is not consistent, but $\bar{x}$ is consistent.

Q54. let $X_1, X_2, \ldots, X_n$ be a r. s. from the population with PDF

f (x; θ) = $\begin{cases} e^{-(x-\theta)}, \text{if } x>\theta \\ 0 \ , ow \end{cases}$

Show that $X_{(1)}$ is consistent for θ.

ANS :- $f_{X_{(1)}(x)}$= n[1 − $\int_0^x e^{-(x-\theta)}dx]^{n-1}.e^{-(x-\theta)}$; x> θ

= n $[1+e^{-(x-\theta)} - 1]^{n-1}.e^{-(x-\theta)}$

= n$e^{-n(x-\theta)}$; x> θ

P [$|X_{(1)} - \theta| < \epsilon$]= P [$\theta < X_{(1)} < \theta + \epsilon$] = n$\int_\theta^{\theta+\epsilon} e^{-n(x-\theta)}dx$

= n$e^{n\theta}$ $[\frac{e^{-nx}}{-n}]_\theta^{\theta+\epsilon}$

= 1- $e^{-n\epsilon}$

⟶1 as n ⟶∞

∴ $X_{(1)}$ is consistent for θ.

Q55. If $X_1, X_2, \ldots, X_n$ be a r. s. from f(x) = $\frac{1}{2}$ (1+θx);

-1< x<1, -1<θ <1. Find a consistent estimator of θ.

Solution :- f(x)= $\frac{1}{2}$ (1+θx)I -1< x< 1

∴ E(X) = $\frac{1}{2}\int_{-1}^{1}(1+\theta x)x dx = \frac{\theta}{3}$

Now, E ($\bar{X}$) = $\frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{\theta}{3}$

⇒ E (3$\bar{X}$) = θ

Now, E (X^2) = $\frac{1}{2}\int_{-1}^{1} x^2(1+\theta x)dx = \frac{1}{2}\int_{-1}^{1}(x^2+\theta x^3)dx = \frac{1}{3}$

∴ V(X) = E (X^2) -$E^2(X)$

⇒ V(X) = $\frac{1}{3} - \frac{\theta^2}{9}$

V ($\bar{X}$) = $\frac{1}{n^2}.n(\frac{1}{3} - \frac{\theta^2}{9}) = \frac{1}{n}(\frac{1}{3} - \frac{\theta^2}{9})$

∴ $\underset{n\to\infty}{\text{lt}} V(3\bar{X}) = 9 \underset{n\to\infty}{\text{lt}} V(\bar{X}) = 9 \lim_{n\to\infty}\frac{1}{n}(\frac{1}{3} - \frac{\theta^2}{9}) = 0$

∴ 3$\bar{X}$ is a consistent estimator of θ.

Q56. Examine whether the WLLN holds for the following sequences $\{X_n\}$ of independent R.Vs:

I> P [$X_n = -2^n$] = 2^{-2n-1} = P[$X_n = 2^n$]

P [$X_n = 0$]= 1- 2^{-2n}

II> P [$X_n = -\frac{1}{n}$]= ½ =P [$X_n = \frac{1}{n}$]

Solution :- i> μk= E(X_k)= (-2^k). 2^{-2k-1} + (2^k). $2^{-2k-1} + 0.(1-2^{-2k})$

= 0

And var(X_k)=$\boldsymbol{\sigma}k^2 = E(\text{X}k^2)$

= $(-2^k)^2 . 2^{-2k-1} + (2^k)^2 . 2^{-2k-1} + 0$

= 1 , k ∈ℕ

Now, $\frac{1}{n^2}\sum_{k=1}^{n}\sigma k^2 = \frac{1}{n^2}\sum_{k=1}^{n} 1 = \frac{1}{n}$ ⟶0 as n ⟶∞

Hence, $\{X_n\}$ obeys WLLN , by chebyshev's WLLN.

II) Here μk = 0 and $\boldsymbol{\sigma}k^2 = V(X_k) = E(X^2k) = \frac{1}{k^2}$, n ∈ℕ

Now, $\frac{1}{n^2}\sum_{k=1}^{n}\sigma k^2 = \frac{1}{n^2}\sum_{k=1}^{n}\frac{1}{k^2} < \frac{C}{n^2}$

[$\sum_{k=1}^{n}\frac{1}{k^2}$ is a convergent p-series,

$\Rightarrow \sum_{k=1}^{n}\frac{1}{k^2} < \sum_{k=1}^{\infty}\frac{1}{k^2} = c$, a finite quantity]

Hence, $\{X_n\}$obeys WLLN, by chebyshev's WLLN.

Q57. Let P [$X_n = -n^P$]= $\frac{1}{2}$ = $P[X_n = n^P]$

Show that WLLN holds for the sequence $\{X_n\}$ of independent R.V.'s if P < $\frac{1}{2}$

Solution :- here, $\mu_k = E\ (X_k) = 0$

σk^2= V $(X_k) = E(X_k)^2 = (-k^P)^2.\frac{1}{2} + (k^P)^2.\frac{1}{2}$

= k^{2P}, k $\in\mathbb{N}$

Now, $\frac{1}{n^2}\sum_{k=1}^{n}\sigma k^2 = \frac{1}{n^2}\sum_{k=1}^{n}k^{2P} < \frac{1}{n^2}\int_1^n x^{2P}dx$

= $\frac{n^{2p+1}-1}{n^2(2P+1)}$

Now, $0 \le \frac{1}{n^2}\sum_{k=1}^{n}\sigma k^2 < \frac{n^{2p+1}-1}{n^2(2P+1)} < \frac{n^{2p-1}}{(2P+1)} \longrightarrow 0$ as n $\longrightarrow\infty$ [if 2P-1< 0, if p < $\frac{1}{2}$]

$\Rightarrow$ if p < $\frac{1}{2}$, $\frac{1}{n^2}\sum_{k=1}^{n}\sigma k^2 \longrightarrow 0$ as n $\longrightarrow\infty$

Hence, $\{X_n\}$ obeys WLLN if p < $\frac{1}{2}$.

Q58. Decide whether WLLN holds for the sequence,{ X_n} o0f independent R.V.'s :

P [$X_n = \pm 2^{-n}$]= $\frac{1}{2}$

$\Leftrightarrow$ P [$X_n = -2^{-n}$]= $\frac{1}{2}$= P[$X_n = 2^{-n}$]

Solution :- Here, μk=0,

And $\sigma k^2 = V\ (X_k) = E\left({X_k}^2\right) = 2^{-2k}, k \in \mathbb{N}$

Now, $\frac{1}{n^2}\sum_{k=1}^{n}\sigma k^2 =$
$\frac{1}{n^2}\sum_{k=1}^{n}2^{-2k} = \frac{1}{n^2}.\frac{\frac{1}{4}\{1-(\frac{1}{4})^n\}}{1-\frac{1}{4}}$

= $\frac{1}{3}.\frac{1}{n^2}\{1-\left(\frac{1}{4}\right)^n\} < \frac{1}{3n^2} \longrightarrow 0$ as n $\longrightarrow\infty$

Hence, $\lim_{n\longrightarrow k}\frac{1}{n^2}\sum_{k=1}^{n}\sigma k^2 = 0$

$\Rightarrow X_n\}$ obeys WLLN by chebyshev's WLLN.

II) μk=0

And $\sigma k^2 = E\left({X_k}^2\right) = (-k)^2.\frac{1}{2\sqrt{k}} + (k)^2.\frac{1}{2\sqrt{k}} + 0$

= $k^{3/2}$

Now, $\frac{1}{n^2}\sum_{k=1}^{n}\sigma k^2 = \frac{1}{n^2}\sum_{k=1}^{n}k^{3/2}$

For, large n, $\frac{1}{n}\sum_{k=1}^{n}(\frac{k}{n})^{3/2} \simeq \int_0^1 x^{3/2}dx = \frac{2}{5}$

$\Rightarrow \sum k^{3/2} \simeq \frac{2n^{5/2}}{5} = \frac{2}{5}\sqrt{n} \nrightarrow 0 \longrightarrow \infty$ as n $\longrightarrow \infty$

As $\frac{1}{n^2}\sum_{k=1}^{n}\sigma k^2 \nrightarrow 0$ as n $\longrightarrow\infty$

We cannot draw any conclusion by chebyshev's WLLN, whether WLLN holds or not.

Q59. Let (X_1, X_2, X_3) be a r.s. from Bin(1, p). Is T= $X_1 + 2X_2 + X_3$ sufficient for p ? is $X_1 + X_2 + X_3$ is sufficient for p?

ANS :- (i) Here T takes the value 0, 1, 2, 3, 4.

P $[X_1 = 1, X_2 = 0, X_3 = 1 | T = 2]$

$= \frac{P[X_1=1,X_2=0,X_3=1\ ;T=2]}{P[T=2]}$

$= \frac{P[X_1=1,X_2=0,X_3=1\ ]}{P[X_1=1,X_2=0,X_3=1]+P[P[X_1=0,X_2=1,X_3=0]}$

$= \frac{p^2(1-p)}{p^2(1-p)+\ p(1-p)^2} = \frac{p}{p+1-p}$ = p, which depends on p.

Hence T is not sufficient for p

(ii) Here, $X_1 +\ X_2 +\ X_3 = T$

Let us consider a specific case, $X_1 = 1, X_2 = 1, X_3 = 0$ and T =1

Here, $X_1 +\ X_2 +\ X_3 = 1$ for,

$\{(X_1 = 1, X_2 = 1, X_3 = 0),(X_1 = 1, X_2 = 0, X_3 = 1),(X_1 = 0, X_2 = 1, X_3 = 1), (X_1 = 0, X_2 = 0, X_3 = 1)\}$

$\therefore$ P $[X_1 = 1, X_2 = 1, X_3 = 0 | T = 1]$

$= \begin{cases} \frac{\text{P }[X_1=1,X_2=1,X_3=0]}{P[T=1]} & \text{, if T} = 1 \\ 0 & ,\quad ow \end{cases}$

$= \begin{cases} \frac{p^2(1-p)}{3p^2(1-p)+\ (1-p)^2p} & , if\ T = 1 \\ 0 & ,ow \end{cases}$

$= \begin{cases} \frac{P}{2P+1} & , if\ T = 1 \\ 0 & ,\ ow \end{cases}$

E. T is not sufficient for p.

Q60. Let $X_1, X_2, \dots, X_n$ be a r. s. from the following PDF & find the non trival sufficient statistic in each case :

(i) $\mathbf{f(x\ ;\ \theta)} = \begin{cases} \boldsymbol{\theta x^{\theta-1}} & ; \mathbf{0} < x < 1 \\ \mathbf{0} & ,\ \boldsymbol{ow} \end{cases}$

(ii) $\mathbf{f(x\ ;\ \mu)} = \frac{\mathbf{1}}{|\boldsymbol{\mu}|\sqrt{\mathbf{2\pi}}} \cdot \boldsymbol{e}^{-\frac{(x-\mu)}{2\mu^2}}$ **; x** $\in\mathbb{R}$

(III) $\mathbf{f\ (x\ ;\ \alpha,\beta)} = \begin{cases} \frac{\boldsymbol{x^{\alpha-1}(1-x)^{\beta-1}}}{\boldsymbol{\beta(\alpha,\beta)}} & ,\mathbf{0} < x < 1 \\ \mathbf{0} & ,\ \boldsymbol{ow} \end{cases}$

(iv) $\mathbf{f(x\ ;\ \mu,\ \lambda)} = \begin{cases} \frac{\mathbf{1}}{\boldsymbol{\lambda}}\ \boldsymbol{e}^{-\frac{(x-\mu)}{\lambda}} & ,\boldsymbol{if\ x>\mu} \\ \mathbf{0} & ,\ \boldsymbol{ow} \end{cases}$

(v) $\mathbf{f(x;\ \mu,\ \sigma)} =$
$\begin{cases} \frac{\mathbf{1}}{\boldsymbol{x\sigma\sqrt{2\pi}}}\ \boldsymbol{e}^{-\frac{1}{2\sigma^2}(\boldsymbol{\ln x-\mu})^2} & ,\ \boldsymbol{if\ x>0} \\ \mathbf{0} & ,\ \boldsymbol{ow} \end{cases}$

(vi) $\mathbf{f(x;\alpha,\theta)} = \begin{cases} \frac{\boldsymbol{\theta\ \alpha^\theta}}{\boldsymbol{x^{\theta+1}}} & \boldsymbol{if x} > \alpha \\ \mathbf{0} & ,\ \boldsymbol{ow} \end{cases}$

(vii) $\mathbf{f(x;\ \theta)} = \begin{cases} \frac{\mathbf{2(\theta-x)}}{\boldsymbol{\theta^2}} & ;\mathbf{0} < x < \theta \\ \mathbf{0} & ,\ \boldsymbol{ow} \end{cases}$

Ans:- (I) The joint PDF of $X_1, X_2, \dots, X_n$ is

f(x) = $\theta^n(\prod_{i=1}^n x_i)^{\theta-1}$

= $g_{\theta\ \{\prod_{i=1}^n x_i\}}$. h(x). where h(x)=1

And T (x)= $\prod_{i=1}^n x_i$

$\therefore$ By Neyman – Fisher factorization criterion,

T = $\prod_{i=1}^n x_i$ is sufficient for θ.

(ii) f(x; μ, σ)= $\frac{1}{|\mu|\sqrt{2\sigma}} \cdot e^{-\frac{(x-\mu)}{2\sigma^2}}$

So, X ~ N (μ, μ^2), where $\mu \neq 0$.

By Ex.(3). T (x)= $(\sum_{i=1}^n X_i, \sum_{i=1}^n {X_i}^2)$ is sufficient for μ.

Note : - If in the range of X_i , there is the parameter of the distribution present then we have to use the concept of indicator function($X_{(1)}$ or $X_{(n)}$) or $\min_i\{\chi_i\}$.

(iii> $f_\theta(x) = \frac{1}{B(\alpha,\beta)}\ x\,.(1-x)$, if 0< x< 1 , α, β> 0

$\therefore$ Joint PDF of $X_1, \ldots, X_n$ is

f(x̰)= $\left[\frac{1}{B(\alpha,\beta)}\right]^n (\prod_{i=1}^n x_i)^{\alpha-1}(\prod_{i=1}^n 1-x_i)^{\beta-1}$

= g(T((x̰) ; α, β)h(x̰), where, h(x̰)= 1 and T (x̰) =($\prod_{i=1}^n x_i$, $\prod_{i=1}^n 1-x_i$) is jointly sufficient for (α, β)

(iv> f(x̰)= $\frac{1}{\theta^n}\ .e^{-\sum_{i=1}^n \frac{(x_i-\mu)}{\sigma}}$ if $x_i > \mu$

= $\frac{1}{\sigma^n}$. exp{$\frac{-\sum_{i=1}^n x_i - n\mu}{\sigma}$}. I ($x_{(1)\ ,\mu}$) where I (a, b)= 1 if a $\geq$ b

= 0 ow

= g($\sum_{i=1}^n X_i, x_{(1)\ ;\,\sigma,\mu}$) . h(x̰), where h(x̰)= 1.

Thus, $X_{(1)}$ and $\sum_{i=1}^n X_i$, are jointly sufficient statistic for μ and $\boldsymbol{\sigma}$.

<v> f(x; μ, $\boldsymbol{\sigma}$)= $\frac{1}{x\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2\sigma^2}(|nx-\mu)^2\ ,\ if\ x>0}$

The joint PDF of x̰ is

F(x̰) = $\frac{1}{(\prod_{i=1}^n x_i)\sigma^n(\sqrt{2\pi})^n}$. exp{$-\frac{1}{2\sigma^2}\ \sum_{i=1}^n (|nx_i - \mu)^2$,} if $x_i > 0$

= $\frac{1}{\sigma^n(\sqrt{2\pi})^n}\ .e^{-(\frac{\sum(1nx_i)^2}{2\sigma^2} - \frac{\mu\sum 1nx_i}{\sigma^2} + \frac{n\mu^2}{\sigma^2})}.\frac{1}{(\prod_{l=1}^n x_i)}$

= T ($\sum_{i=1}^n |nx_i, \sum_{i=1}^n (|nx_i)^2\ ;\ \mu, \sigma$). h(x̰) ; where,

h(x̰) = $\frac{1}{\prod_{i=1}^n x_i}$; T (x̰)= $\sum_{i=1}^n |nx_i, \sum_{i=1}^n (|nx_i)^2$

is sufficient for μ and $\boldsymbol{\sigma}$.

(vi) f (x̰)= $\theta^n \frac{(\alpha^\theta)^n}{\prod_{i=1}^n (x_i{}^{\theta+1})}$ if $x_i > \alpha$

= $(\theta\alpha^\theta)^n . \frac{1}{\prod_{i=1}^n \{x_i\}^{\theta+1}}\ I(x_{(1),\alpha})$ if $x_{(1)>\alpha}$

where I (a, b)=1 if a > b

= 0 ow

= g ($\prod_{i=1}^n x_i$, $x_{(1);\ \alpha,\ \alpha}$). h (x̰); where , h (x̰)= 1 and hence

T =$\prod_{i=1}^n x_i$, $x_{(1)}$ is sufficient for θ and α.

(vii> f (x̰) $= \frac{2^n}{\theta^{2n}}\prod_{i=1}^n (\theta - x_i)$; $0 < x_i < \theta$

$= \left(\frac{2^n}{\theta^{2n}}\right)^n .(\theta - x_1)(\theta - x_2)\ldots.(\theta - x_n)$; $0 < x_i < \theta$

These cannot be expressed in the form of factorization criterion.

So ($X_1, X_2, \ldots, X_n$) or($X_{(1)}, X_{(2)}, \ldots, X_{(n)}$) are trivally sufficient for here ,̰.there is no non- trival sufficient statistic.

Q61. Let $X_1, \ldots, X_n$ be a r.s. from gamma distn with pdf

$$f_\theta(x) = \frac{\alpha^P}{\Gamma(P)}\ \exp[-\alpha x]x^{P-1}\quad \text{if } 0< x< \infty$$

, where α> 0, P > 0

Show that $\sum_i X_{i\ and} \prod_i X_i$ are jointly sufficient for (α, P)

Solution:- f(x̰)=
$\{\frac{\alpha^P}{\Gamma(P)}\}^n \cdot \exp[-\alpha \sum_i X_i] \cdot (\prod_i X_i)^{P-1}$

= g (T(x̰); , P). h (x̰); where h(x̰)=1

∴ T (x̰)= ($\sum_{i=1}^{n} X_i, \prod_{i=1}^{n} X_i$) is jointly sufficient for (α, P).

Q62. If f(x)= $\frac{1}{\theta} e^{-x/\theta}; 0 < x < \theta. Find\ a\ sufficient\ estimator\ for\ \theta.$

Solution:- f(x̰)= $\frac{1}{\theta^n} \cdot exp\,\{-\frac{1}{\theta}\sum_{i=1}^{n} x_i\}$

= g $\{\sum_{i=1}^{n} x_i, \theta\}$. h (x̰); where h (x̰)= 1.

∴T = $\sum_{i=1}^{n} x_i$ is sufficient statistic for θ.

Q63. If $f_\theta(x)$= ½ ; θ -1< x < θ -1 , then show that $X_{(1)}$ *and* $X_{(n)}$ are jointly sufficient for θ. ($X_i \sim U(\theta - 1, \theta + 1)$)

Solution:- f(x̰)=$(\frac{1}{2})^n$

= $\frac{1}{2^n} \cdot I\left(\theta - 1, x_{(1)}\right) I\left(x_{(n)}, \theta + 1\right); \theta - 1 < x_{(1)} < x_{(n)} < \theta + 1$

Where I (a, b)= $\begin{cases} 1 & if\ a < b \\ 0 & if\ a \geq b \end{cases}$

= g (T (x̰); θ)h(x̰); where h(x̰)= $\frac{1}{2^n}$.

∴ T (x̰) = ($X_{(1)}, X_{(n)}$)is jointly sufficient for θ.

Q64. let $X_1, X_2, \ldots, X_n$ be a R.S. from c (θ, 1), where θ is the location parameter, S. T. there is no sufficient statistic other than the trivial statistic ($X_1, X_2, \ldots, X_n$) or ($X_{(1)}, X_{(2)}, \ldots, X_{(n)}$)

If a random sample of size n ≥ 2 from a Cauchy distn with p.d.f. $f_\theta(x)$= $\frac{1}{\pi[1+(x-\theta)^2]}$, where -∞ < θ <∞, is considered.

Then can you have a single sufficient statistic for θ?

Solution:- The PDF of ($X_1, \ldots, X_n$) is

$$\prod_{i=1}^{n} f\,(X_i, \theta) = \frac{1}{\pi^n\{\prod_{i=1}^{n}[1 + ((X_i - \theta)^2]\}}$$

Note that, $\prod_{i=1}^{n}\{1 + ((X_i - \theta)^2\}$

= $\{1 + ((x_1 - \theta)^2\}\{1 + ((x_2 - \theta)^2\}\ldots.\{1 + ((x_n - \theta)^2\}$

= 1+ term involving one X_i+ term involving two $X_i's$ + ……. + term involving all $X_i's$.

= 1+ $\sum_i (x_i - \theta)^2 + \sum_i \sum_{\neq j} (x_i - \theta)^2 \left((x_j - \theta\right)^2 + \ldots. + \prod_{i=1}^{n}((x_i - \theta)^2$

Clearly, $\prod_{i=1}^{n} f(x_i, \theta)$cannot be written as g (T (x̰), θ). h(x̰)

For a statistic other than the trivial choices

($X_1, \ldots, X_n$) or ($X_{(1)}, \ldots, X_{(n)}$).

Hence there is no non-trivial sufficient statistic

Therefore, in this case, <u>no reduction in the space is possible</u>.

⇒ The whole set ($X_1, \ldots, X_n$) is jointly sufficient for θ.

Q65. Let X_1 *and* X_2 be iid RVS having the discrete uniform distribution on {1, 2, ……, N}, where n is unknown. Obtain the conditional distribution of X_1 , X_2 given (T = max (X_1 , X_2))

Hence, show that T is sufficient for n but $X_1 + X_2$ is not.

ANS :- (i) p (T= t)= P [max $(X_1, X_2) = t$]

= P [$X_1 < t,\ X_2 = t$]+ P [$X_1 = t,\ X_2 < t$]+ P [$X_1 = t,\ X_2 = t$]

= P [$X_1 < t$] P [$X_2 = t$]+ P [$X_1 = t$] P [$X_2 < t$] + P [$X_1 = t$] P [$X_2 = t$]

Now, P [$X_1 < t$]= P [$X_2 = 1$]+ P [$X_2 = 2$]+ …. + P [$X_2 = t-1$]

$= \underbrace{\frac{1}{N} + \frac{1}{N} + \ \dots . + \frac{1}{N}}$

(t-1) times

$= \frac{t-1}{N}$ & P [$X_1 = t$]= P [$X_2 = t$]= $\frac{1}{N}$

∴ P [T = t]= $\frac{1}{N} \cdot \frac{t-1}{N} + \frac{t-1}{N} \cdot \frac{1}{N} + \frac{1}{N} \cdot \frac{1}{N}$

$= \frac{2\,(t-1)+1}{N^2}$

∴ P [$X_1 = x_1, X_2 = x_2 \mid T = t$]=
$\begin{cases} \frac{P[X_1 = x_1, X_2 = x_2]}{P\,[T=t]}, if\ Max(x_1, x_2) = t \\ 0\ ,\ ow \end{cases}$

$= \frac{\frac{1}{N} \cdot \frac{1}{N}}{\frac{2\,(t-1)+1}{N^2}} = \frac{1}{2\,(t-1)+1}$, which is independent of N.

(ii) T = $X_1 + X_2$ then,

For $2 \le t \le N+1$;P [T=t]= P [$X_1 = 2, X_2 = t-1$]+ P [$X_1 = 2, X_2 = t-2$] + …+P [$X_1 = 2, X_1 = t-1, X_2 = 1$]

$= \frac{(t-1)}{N^2}$

For N +2$\le t \le 2N$; P [T=t]= P[$X_1 = t-N, X_2 = N$] + P [$X_1 = t-N+1, X_2 = N-1$]+ …..+P [$X_1 = N, X_2 = t-N$]

$= \frac{2N-t+1}{N^2}$

∴ P [$X_1 = x_1 ;\ X_2 = x_2 \mid T = t$]= $\frac{P\,[X_1 = x_1\ ;\ X_2 = x_2\ ]}{P\,[X_1 + X_2 = t]}$

$= \begin{cases} \frac{\frac{1}{N^2}}{\frac{(t-1)}{N^2}} = \frac{1}{t-1}\ \ if\ X_1 + X_2 = t \\ \frac{\frac{1}{N^2}}{\frac{2N-t+1}{N^2}} = \frac{1}{2N-t+1}\ \ if\ X_1 + X_2 = t \end{cases}$

Which depends on N, so for the 2nd case $(X_1 + X_2)$is not sufficient.

Q66. Let $X_1, X_2, \dots, X_n$ be a R.S. from one of the following two PDFs

If θ = 0, f(x/θ)= $\begin{cases} \mathbf{1}\ , \mathbf{0} < x < 1 \\ \mathbf{0}\ ,\ \ \boldsymbol{ow} \end{cases}$

If θ=1, , f(x/θ)= $\begin{cases} \frac{\mathbf{1}}{\mathbf{2}\sqrt{x}}\ , \mathbf{0} < x < 1 \\ \mathbf{0}\ ,\ \ \boldsymbol{ow} \end{cases}$

Find the MLE of θ.

Solution: - The Likelihood function is

L (θ/ $\underset{\sim}{x}$)= $\prod_{i=1}^{n} f\ \left(\frac{x_i}{\theta}\right), \theta \in \Omega = (0, 1)$

When θ = 0, L (θ/ $\underset{\sim}{x}$) =
$\begin{cases} 1\ \ if\ \ 0 < x_i < 1\ \forall\ i = 1(1)n \\ \ \ 0\ \ \ \ \ ,\ \ \ \ \ \ 0w \end{cases}$

When θ= 1, L (θ/ $\underset{\sim}{x}$)
$= \begin{cases} \frac{1}{2^n \sqrt{\prod_{i=1}^{n} x_i}}\ , 0 < x_i < 1\ , i = 1(1)n \\ \ \ \ 0\ \ ,\ \ \ ow \end{cases}$

Now, $\frac{L\,(\theta = 1/\ \underset{\sim}{x})}{L\,(\theta = \frac{0}{\underset{\sim}{x}})} \gtreqless 1$

iff $\frac{1}{\sqrt{4^n G^n}} \gtreqless 1$, where G = $\left(\prod_{i=1}^{n} x_i\right)^{1/n}$

iff 4 G $\lesseqgtr$ 1 iff G$\lesseqgtr \frac{1}{4}$

Hence MLE of θ is θ' = $\begin{cases} 1 \; if \; G < \frac{1}{4} \\ 0 \; if \; G > \frac{1}{4} \\ 0,1 \; if \; G = \frac{1}{4} \end{cases}$

Q67. Let $X_1, \ldots, X_n$ be a R.S. from N (μ, σ^2), $\mu \epsilon$R, σ > 0

Find the MLE of (μ, σ^2).

Solution: - Likelihood function:

L (μ, $\sigma^2/\underset{\sim}{x}$) = $\frac{1}{(2\pi\sigma^2)^{n/2}} \cdot e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$; $x_i \, \epsilon\mathbb{R}$ where μ ϵ R, σ > 0

⇒ |nL (μ, $\sigma^2/\underset{\sim}{x}$)= constant ($-\frac{n}{2}|n\sigma^2 - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2$)

$0 = \frac{\partial |nL}{\partial \mu} = -\frac{1}{2\sigma^2}\sum 2(x_i-\mu)(-1) = \frac{\sum x_i}{\sigma^2} - \frac{n\mu}{\sigma^2}$

$0 = \frac{\partial |nL}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{\sum(x_i-\mu)^2}{2\sigma^4}$

⇒ $\begin{cases} \acute{\mu} \doteq \bar{x} \\ \sigma^2 = \frac{1}{n}\sum(x_i-\bar{x})^2, \end{cases}$ the likelihood function has a unique solution.

Note that, the matrix of second order partial derivatives at (μ, σ^2) is

$\begin{pmatrix} \frac{\delta^2|nL}{\partial\mu^2} & \frac{\delta^2|nL}{\delta\mu\delta\sigma^2} \\ \frac{\delta^2|nL}{\delta\sigma^2\delta\mu} & \frac{\delta^2|nL}{\delta(\sigma^2)^2} \end{pmatrix}$ (μ, σ^2)= (μ, σ^2)

$= \begin{pmatrix} -\frac{n}{\sigma^2} & 0 \\ 0 & -\frac{n}{2\sigma^4} \end{pmatrix}$ is negative definite (n. d.)

Hence, L (μ, $\sigma^2/\underset{\sim}{x}$) is maximum at ($\mu$, σ^2)= (μ, σ^2)

Therefore, the MLE of (μ, σ^2) is

(μ, σ^2) = ($\bar{x}$, S^2) where nS^2= $\sum_{i=1}^{n}(X_i-\bar{x})^2$.

Q68. Let $X_1, \ldots, X_n$ be a R.S. from f (x; μ,)= $\frac{1}{2\sigma}$ $e^{-\frac{|x-\mu|}{\sigma}}$; $x \, \epsilon \, R$,where $\mu \, \epsilon$R, σ > 0. Find the MLE of μ and σ.

Solution:- The log- likelihood function is

L (μ, $\sigma^2/\underset{\sim}{x}$) = -n|n2- n|n$\sigma$- $\frac{1}{\sigma}\sum|x_i-\mu|$; $\mu \, \epsilon R, \sigma > 0$

[As$\sum|x_i-\mu|$is not differentiable w.r.t. μ, hence the derivative technique is not applicable for maximizing |nL w.r.t.μ]

We adopt two stage maximization:-

First fix σ, and then maximize |nL for variation in μ.

For fixed σ, lnL is maximum,

iff, $\sum|x_i-\mu|$is minimum

iff, μ=x̄= the sample median

= μ, say.

Now, we maximize |nL (μ, $\sigma^2/\underset{\sim}{x}$)=-n|n2- n|n$\sigma$- $\frac{1}{\sigma}\sum|x_i-\mu|, w.r.t. \sigma$

Note that $\frac{\delta}{\delta\sigma}$|nL ($\mu$, $\sigma^2/\underset{\sim}{x}$)

= $-\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum|x_i-\mu|$

= $-\frac{n}{\sigma^2}\{\sigma - \frac{1}{n}\sum|x_i-\mu|\}$

$$\begin{cases} > 0, \sigma < \frac{1}{n}\sum |x_i - \mu| \\ < 0, \sigma > \frac{1}{n}\sum |x_i - \mu| \end{cases}$$

By, 1st derivative test, |nL (μ, $\sigma^2/\underset{\sim}{x}$) is maximum at $\sigma = \frac{1}{n}\sum_{i=1}^{n}|x_i - \mu|$

Hence, the MLE of μ and σ are $\mu = \bar{x}$, $\sigma = \frac{1}{n}\sum |x_i - \bar{x}|$.

Q69. let $X_1, X_2, \ldots, X_n$ be a R.S. from

F(x; μ,)= $\begin{cases} \frac{1}{\sigma} e^{-(x-\mu)/\sigma}, if\ x > \mu \\ 0 \quad , ow \end{cases}$

Where $\mu \in$ R, $\sigma > 0$. Find the MLE of (i) μ and σ

(ii)μ when $\sigma = \mu$ (>0)

Solution:- (i) The likelihood function is

L (μ, $\sigma/\bar{x}$)= $\begin{cases} \frac{1}{\sigma^n} \cdot e^{-\frac{\sum(x_i-\mu)}{\sigma}}; if\ x_{(1)} \geq \mu \\ 0, \ ow \end{cases}$

$\mu \in$ R, $\sigma > 0$

We adopt two stage maximization.

First fix σ, then maximize L (μ, $\sigma/\bar{x}$) w.r.t.μ

For fixed σ, L (μ, $\sigma/\bar{x}$) is maximum

iff $\sum(x_i - \mu)$ is minimum subject to $\mu \leq x_{(1)}$

iff μ is as large as possible subject to the restriction $\leq x_{(1)}$.

iff $\mu = x_{(1)} = \mu$ (say)

Now we shall maximize L (μ, $\sigma/\bar{x}$) w.r.t.σ

Now, in L(μ, $\sigma/\bar{x}$)= -nlnσ-$\frac{\sum(x_i-\mu)}{\sigma}$

Note that, $\frac{\delta}{\delta\sigma}$ lnL(μ, $\frac{\sigma}{\bar{x}}$)= $-\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum(x_i - \mu)$

$= \frac{n}{\sigma^2}\{\sigma - (\bar{x} - x_{(1)})\}$

$$\begin{cases} > 0\ if\ \sigma < \bar{x} - x_{(1)} \\ < 0\ if\ \sigma > \bar{x} - x_{(1)} \end{cases}$$

Hence, L(μ, $\sigma/\bar{x}$) is maximum at $\sigma = \bar{x} - x_{(1)} = \sigma$

Therefore, The MLE of μ and σ are $\mu = x_{(1)}$ and $\sigma = \bar{x} - x_{(1)}$

(ii) When $\sigma = \mu > 0$

L ($\mu/\underset{\sim}{x}$) = $\begin{cases} \frac{1}{\mu^n} e^{-\frac{\sum(x_i-\mu)}{\mu}}; x_{(1)} \geq \mu \\ 0 \quad , ow \end{cases}$

L ($\mu/\underset{\sim}{x}$) is maximum iff

For $\mu \leq x_{(1)}$

$$\frac{\delta}{\delta u} ln\text{L} = \frac{\delta}{\delta u}\{-nln\mu - \frac{1}{\mu}\sum(x_i - \mu)\}$$

$= -\frac{n}{\mu^2}(\mu - \bar{x})$

$$\begin{cases} > 0\ if\ \mu < \bar{x} \\ < 0\ if\ \mu > \bar{x} \end{cases}$$

$\Rightarrow$ L ($\mu/\underset{\sim}{x}$) is maximum at $\mu = \bar{x}$

From the graph for $\mu \leq x_{(1)}$, L ($\mu/\underset{\sim}{x}$) is maximum at $\mu = x_{(1)}$,

Therefore, $\mu = x_{(1)}$, is the MLE of μ.

Q70. Let X be a single observation from the PDF

F (x; θ) = {$\frac{1}{\pi\{1+(x-\theta)^2\}}$, $x \in \mathbb{R}$

Show that the test

$$\Phi\ (x)=\begin{cases} 1, \frac{f(x,1)}{f(x,0)} > k \\ 0 \ \ , ow \end{cases}$$

Is an MP test of H_0: $\theta = 0$ aginst H_1: $\theta = 1$ of its size.

Solution:- For a particular value of k, the test

$$\Phi\ (x)=\begin{cases} 1, \frac{f(x,1)}{f(x,0)} > k \\ 0 \ \ , ow \end{cases}$$

Is an MP test of H_0: $\theta = 0$ aginst H_1: $\theta = 1$ of its size, by NP lemma,

Now, $\frac{f(x,1)}{f(x,0)} > k \Rightarrow \frac{1+x^2}{1+(x-1)^2} > k$

$\Rightarrow x^2(k-1) - 2kx + (2k-1) < 0$

[If (k-1)> 0, $x^2 - \frac{2k}{(k-1)}x + \frac{2k-1}{k-1} < 0$

$\Rightarrow$ (x- α)(x-β)< 0

Where, α+β= $\frac{2k}{(k-1)}$, and $\alpha\beta$= $\frac{2k-1}{k-1}$

$\Rightarrow$ α< x< β

In the given MP test α=1, β=3

Hence, 1+3 =$\frac{2k}{(k-1)}$ $\Rightarrow$ k=2]

Set, k=2, $\frac{f(x,1)}{f(x,0)}$ > 2

$\Rightarrow$ 1< x < 3

For k=2, the test Φ (x)=$\begin{cases} 1 \ , 1 < x < 3 \\ 0 \ , ow \end{cases}$

Is an MP test of H_0 against H_1 of its size

= E [Φ (x)/ H_0]= P [1< x<3/θ=0]

= $\int_1^3 \frac{1}{\pi(1+x^2)}dx$= $\frac{1}{\pi}$ [$\tan^{-1} x$]3,1

=$\frac{1}{\pi}$ [$\tan^{-1} 3 - \tan^{-1} 1$]

= $\frac{1}{\pi}\tan^{-1}\frac{3-1}{1+3.1}$

= $\frac{1}{\pi}\tan^{-1}\frac{1}{2}$.

Q71. Find an MP test of testing H_0 such that H_0 : $X \sim f_0(x)$ against H_1 : $X \sim f_1(x)$ of its size, where

$f_0(x)$= $\frac{1}{\sqrt{2\pi}}$ $e^{-x^2/2}$, $x \in R$

$f_1(x)$= $\frac{1}{2}$ $e^{-|x|}$, $x \in R$

S.T. the power of the test is greater than its size.

Solution:- By N-P lemma, for a particular value of k, the test

$$\Phi(x)=\begin{cases} 1 \ , \frac{f_1(x)}{f_0(x)} > k \\ 0 \ \ , ow \end{cases}$$

Is an MP test of H_0 against H_1 of its size.

Now, $\frac{f_1(x)}{f_0(x)} > k$

$\Rightarrow e^{\frac{1}{2}\{x^2-2|x|\}} > k_1$

$\Rightarrow e^{\frac{1}{2}}\{(|x|-1)^2 - 1\} > k_1$

$\Rightarrow (|x|-1)^2 > {k_2}^2$, $k_2 > 0$

$\Rightarrow$ |x|-1 < -k_2 or |x|-1>k_2

$\Rightarrow$ |x|< C_1 or |x|> C_2

[Alternative: - note that $f_1(x)$ has more probability in its tails and near 0 than $f_0(x)$ has. If either a very large or very small value of x is observed, we suspect that H_1 is true rather thanH_0. For some C_1 andC_2, we shall reject H_0 iff $\frac{f_1(x)}{f_0(x)} > k$

To |x|< C_1 or |x|<C_2.]

Hence, for some C_1 andC_2, the test

Φ(x) = $\begin{cases} 1 \text{ , } |x| < C_1 \text{ or } |x| < C_2 \\ \quad 0 \text{ , } ow \end{cases}$

Is an MP test of H_0 against H_1 of its size

Note, that, $\beta_{\Phi(}(f_1) = P[1 \times 1 < C_1 or\ 1 \times 1 < C_2\ ]$

= $\int_w^{f_1} f_1(x)dx\ , w = \{x: |x| < C_1\ or\ |x| > C_2\ \}$

>$\int_w\ f_0\ (x)dx$, as $f_1(x) >\ f_0(x) \forall x \epsilon w$

= $P_{f_0}[1 \times 1 < C_1\ or\ 1 \times 1 < C_2\ ]$

= $\beta_\Phi(f_0)$. (Proved).

Q72. Find an MP test of H_0 : X~N(0, ½) against H_1 : X~c (0, 1) of its size .

Solution :- For a given K , the test Φ (x) = $\begin{cases} 1, \frac{f_1(x)}{f_0(x)} > k \\ \quad 0, ow \end{cases}$

Is an MP test of H_0 against H_1 of its size,

By N-P lemma,

Note that , $\frac{f_1(x)}{f_0(x)} > k$

$\Rightarrow \frac{e^{x^2}}{1+x^2} > k_1$, say

Let u (x) = $\frac{e^{x^2}}{1+x^2}$

Now, u′ (x)= $\frac{(1+x^2)e^{x^2}.2x - e^{x^2}.2x}{(1+x^2)^2}$

= $\frac{2x^3.e^{x^2}}{(1+x^2)^2}$

[u′ (0)=0 $\Rightarrow 2x^3.e^{x^2} = 0 \Rightarrow x = 0\ or e^{x^2} = 0 \Rightarrow x^2 = \infty$]

= $\begin{cases} < 0, if\ x < 0 \\ > 0, if\ x > 0 \end{cases}$

From the graph ,u(x) >k_1

$\Leftrightarrow$ |x|>c_1

Hence, for a particular value of c_1, the test

Φ(x) = $\begin{cases} 1 \text{ , } |x| > C_1 \\ \quad 0 \text{ , } ow \end{cases}$

Is an MP test H_0 against H_1 of its size.

Q73. Find on MP test at level α= 0.05 for testing H_0: X ~ N (0, 1)against H_1: X ~C(0, 1).

Solution:- for a given k, the test Φ(x) = $\begin{cases} 1 \text{ , } \frac{f_1(x)}{f_0(x)} > k \\ \quad 0 \text{ , } ow \end{cases}$

Is an MP test of H_0 against H_1 of its size, by NP lemma. Note that. R (x)= $\frac{f_1(x)}{f_0(x)} > k$

$\Rightarrow \frac{e^{x^2/2}}{1+x^2} > k_1$, say.

Let u (x) =$\frac{e^{x^2/2}}{1+x^2}$

Note that, u′ (x)= $\begin{cases} < 0\,, x < -1 \\ > 0\,, -1 < x < 0 \\ < 0\,, 0 < x < 1 \\ > 0\,, x > 1 \end{cases}$

[for k> 0.7979, then the critical region:

$|x| > c_2$ with size< 0.1118.

For 0.6524≤ k≤ 0.7979,

Then critical region:

$|x|>c_1$ or $|x| > c_2$ with size ϵ (0.1118, 0.3913)

For, k < 0.6524, the critical region: x ϵR with size=1]

For α= 0.05, a small quantity, then u(x)> k_1, where k_1 is such that P [u(x)> k_1/H_0]=0.05 and from the graph u(x)> k_1⇔ |x|> c_2.

Hence, $\Phi(x) = \begin{cases} 1\,, |x| > C_2 \\ 0\,, ow \end{cases}$

Is an MP test of H_0 against H_1 at level α= 0.05, where

0.05= P [$|x| > c_2/H_0$]

=P [[$|x| > c_2$/X ~ N (0, 1)]

= 2 [1- ϕ (c_2)]

⇒ 1- ϕ(c_2) = 0.025 = 1-ϕ (0.025)

⇒ c_2= 0.025

=1.96

Hence, $\Phi(x) = \begin{cases} 1\,, |x| > 1.96 \\ 0\,, ow \end{cases}$

Is an MP test for testing H_0: X ~ N (0, 1) against H_1: X ~c (0, 1) at level α= 0.05.

Q74. Let $X_1, \dots, X_n$ be a R.S. from f(x; θ) = $\begin{cases} \theta e^{-\theta x}, if\ x > \theta \\ 0\,, ow \end{cases}$

Find the size α LRT of (i) H_0:θ= θ_0 $against$ H_1: $\theta \neq \theta_0$

(ii) H_0:θ= θ_0 $against$ H_1: $\theta > \theta_0$

($iii >$ H_0:$\theta \geq \theta_0$ $against$ H_1: $\theta < \theta_0$

Solution:- The likelihood function is

$$L\ (\underset{\sim}{x};\ \theta) = \begin{cases} \theta^n e^{-\theta \sum_{i=1}^{n} x_i}, if\ x_i > \theta \\ 0\ , ow \end{cases}$$

Where , θ> 0

(i) To test H_0:θ= θ_0 $against$ H_1: $\theta \neq \theta_0$:~

Here $\Omega_0 = \{\theta_0\}$ and Ω= {0, θ> 0}

[∵$e^{-\theta_0 n\bar{x} + \frac{1}{\bar{x}} n\bar{x}}$

= $e^{-\theta_0 n\bar{x} + n}$]

The likelihood ratio is

$\lambda = \frac{\theta^{sup}{\epsilon\Omega_0} L\ (\underset{\sim}{x};\ \theta)}{\theta^{sup}{\epsilon\Omega} L\ (\underset{\sim}{x};\ \theta)} = \frac{L\ (\underset{\sim}{x};\ \theta_0)}{(\theta\hat{})^n . e^{-\theta \sum_{i=1}^{n} x_i}}$

Where, $\theta\hat{} = \frac{1}{x}$ is the MLE of θ under Ω.

Here, λ= $(\theta_0\bar{x})^n . e^{-n(\theta_0\bar{x}-1)}$

= $y^n . e^{-n(y-1)}$, where y = $\theta_0\bar{x}$

Now, $\frac{d\lambda}{dy}$= $y^n . e^{-n(y-1)}$ (-n)+$ny^{n-1} . e^{-n(y-1)}$

$= ny^{n-1}.e^{-n(y-1)}\ \{1-y\}$

$= \begin{cases} > 0\ if\ y < 1 \\ < 0\ \ if\ y > 1 \end{cases}$

From graph, λ< c

$\Rightarrow$ y < k_1 *or* y > k_2

$\Rightarrow 2\ \theta_0 \sum_{i=1}^{n} x_i < a\ or\ 2\ \theta \sum_{i=1}^{n} x_i > b$

Where, $2nk_1 = a, 2nk_2 = b$

Here, the size αLRT is given by:

Reject H_0 iff λ > c iff $2\ \theta_0 \sum_{i=1}^{n} x_i\ \notin [a,b]$

Where 'a', 'b' are such that

α = PH_0 $[2\ \theta_0 \sum_{i=1}^{n} x_i\ \notin [a,b]]$

= 1- PH_0[a $\leq\ 2\ \theta_0 \sum x_i\ \leq b]$

= 1- P [a $\leq\ {\chi^2}_{2n}\ \leq b]$

= 1- F${\chi^2}_{2n}(b)$+ ${\chi^2}_{2n}(a)$ and (k_1)= $\lambda(k_2)\Rightarrow$ ${k_1}^n.e^{-n(k_1-1)}$

$={k_2}^n.e^{-n(k_2-1)}$

(II) To test H_0:$\geq \theta_0$ *against* H_1: θ> θ_0:-

Here, $\Omega_0 = \{\theta_0\}$ and Ω= { $\geq\ \theta_0$}

The likelihood ratio is

$\lambda= \frac{\theta^{sup}{\in \Omega_0}\text{L}\ (\underset{\sim}{x};\theta)}{\theta^{sup}{\in \Omega}\text{L}\ (\underset{\sim}{x};\theta)} = \frac{{\theta_0}^n.e^{-\theta_0 \sum_{i=1}^{n} x_i}}{\theta^{sup}{\geq \theta_0}\{\text{L}\ (\underset{\sim}{x};\theta)\}}$

For θ > 0, L $(\underset{\sim}{x};\ \theta)$ is maximum at θ= $\frac{1}{\bar{x}}$ = $\theta\hat{}$

$\therefore$ Sup L $(\underset{\sim}{x};\ \theta)$ $\ \theta \geq \theta_0$

$= \begin{cases} (\theta\hat{})^n e^{-\theta\hat{}2x_i},if\theta_0<\theta\hat{} \\ {\theta_0}^n.e^{-\theta_0 \sum x_i},if\theta_0>\theta\hat{} \end{cases}$

Now, $\lambda= \begin{cases} (\theta_0\bar{x})^n.e^{-n(\theta_0\bar{x}-1)},if\ \theta_0\bar{x}<1 \\ 1\ \ ,if\ \theta_0\bar{x} \geq 1 \end{cases}$

From graph, λ< c(<1)

$\Rightarrow$ y< k

$\Rightarrow 2\theta_0 \sum_{i=1}^{n} x_i$ < a , say

The size α LRT is given

by: reject H_0 iff λ < c, iff $2\ \theta_0 \sum_{i=1}^{n} x_i < a$

Where 'a' is such that α= PH_0 [2 $\theta_0 \sum_{i=1}^{n} x_i < a]$

$\therefore \alpha$ = P $[{\chi^2}_{2n} < a]$

$\Rightarrow$ a=${\chi^2}_{1-\alpha;2n}$

Therefore the size α LRT is given by:

Reject H_0 iff $\sum_{i=1}^{n} x_i < \frac{{\chi^2}_{1-\alpha;2n}}{2\theta_0}$.

(III) Hint:- $\lambda = \frac{\theta^{sup}{\geq \theta_0}\text{ L }(\underset{\sim}{x};\theta)}{\theta^{sup}{\in R}\text{ L }(\underset{\sim}{x};\theta)}$

The size α LRT is given by: Reject H_0 iff $\sum_{i=1}^{n} x_i > \frac{{\chi^2}_{\alpha;2n}}{2\theta_0}$.

Q75. let $X_1, X_2, \ldots, X_n$ be a R.S. from N (θ, σ^2), σ known. Derive size α LRT for testing

(i) H_0:θ= θ_0 *against* H_1: $\theta \neq \theta_0$

(ii) H_0:θ= θ_0 *against* H_1: θ> θ_0

Show that the LRT's obtained are unbiased.

Solution:- the likelihood function is

L $(\underset{\sim}{x};\ \theta) =$
$(\frac{1}{\sigma\sqrt{2\pi}})^n . e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2}$; $where\ \theta\ \epsilon\mathbb{R}$

(i) To test H_0:θ= θ_0 $against$ H_1: $\theta \neq \theta_0$:-

Here $\Omega_0 = \{\theta_0\}$ and Ω=$\{\theta_0: \theta\epsilon\mathbb{R}\}$

The likelihood ratio is $\lambda = \frac{\theta^{sup}\epsilon\Omega_0 \text{L } (\underset{\sim}{x}; \theta)}{\theta^{sup\epsilon}\Omega \text{L } (\underset{\sim}{x}; \theta)}$

$= \frac{\text{L } (\underset{\sim}{x},\theta_0\ )}{\theta^{sup\epsilon}\text{RL } (\underset{\sim}{x}; \theta)}$

$= \frac{(\frac{1}{\sigma\sqrt{2\pi}})^n . e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta_0)^2}}{(\frac{1}{\sigma\sqrt{2\pi}})^n . e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2}}$

$= e^{-\frac{1}{2\sigma^2}\{\sum(x_i-\theta_0)^2 - \sum(x_i-\bar{x})^2\}}$

$= e^{-\frac{1}{2\sigma^2}.n\ (\bar{x}-\theta_0)^2}$

Note that λ< c

$\Rightarrow e^{-\frac{n}{2\sigma^2}\ (\bar{x}-\theta_0)^2 < c}$

$\Rightarrow \frac{n(\bar{x}-\theta_0)^2}{\sigma^2} > c_1$

$\Rightarrow |\frac{\sqrt{n}\ (\bar{x}-\theta_0)}{\sigma}|$> k, say.

The size α LRT is given by:

Reject H_0 iff λ> c iff $|\frac{\sqrt{n}\ (\bar{x}-\theta_0)}{\sigma}|$> k, where k is such that

α = P_{H_0} $\left[\left|\frac{\sqrt{n}\ (\bar{x}-\theta_0)}{\sigma}\right| > \ k\right] = P\ [|z| > k]$, z ~N (0, 1)

$\Rightarrow$ k = $\alpha/2$

(II) to test H_0:θ= θ_0 $against$ H_1: θ> θ_0:-

Here $\Omega_0 = \{\theta_0\}$ and Ω= $\{\theta_0: \theta \geq\ \theta_0\}$

The likelihood ratio is λ= $\frac{\theta^{sup}\epsilon\Omega_0 \text{L } (\underset{\sim}{x}; \theta)}{\theta^{sup\epsilon}\Omega \text{L } (\underset{\sim}{x}; \theta)}$

$= \frac{\text{L } (\underset{\sim}{x}; \theta)}{\theta^{\text{sup } \geq} \theta_0 \text{L } (\underset{\sim}{x}; \theta)}$

Here L $(\underset{\sim}{x};\ \theta)$ is maximum at θ= $\bar{x}$= $\hat{\theta}$

Now, $\theta^{\text{sup } \geq} \theta_0$L $(\underset{\sim}{x};\ \theta) =$
$\begin{cases} \text{L } (\underset{\sim}{x};\ \hat{\theta}), \text{if } \bar{x} > \theta_0 \\ \text{L } (\underset{\sim}{x};\ \theta_0)\ , \text{if } \theta_0 \geq \bar{x} \end{cases}$

Here, λ= $\begin{cases} e^{-\frac{n}{2\sigma^2}\ (\bar{x}-\theta_0)^2, if\,\theta_0<\bar{x}} \\ \ \ 1\ \ \ , if\ \theta_0 \geq \bar{x} \end{cases}$

Note that, λ <c (< 1)

$\Rightarrow\ e^{-\frac{n}{2\sigma^2}\ (\bar{x}-\theta_0)^2 < c}$, where $\theta_0 < \bar{x}$

$\Rightarrow \frac{n\ (\bar{x}-\theta_0)}{\sigma} > c_1$, where $\bar{x}$>θ_0

$\Rightarrow \frac{\sqrt{n}\ (\bar{x}-\theta_0)}{\sigma}$ > k , as $(\bar{x} - \theta_0)$> 0

The size α LRT is given by: Reject H_0 iff λ < c

iff $\frac{\sqrt{n}\ (\bar{x}-\theta_0)}{\sigma}$ > k , where k is such that

α = P_{H_0} $\left[\frac{\sqrt{n}\ (\bar{x}-\theta_0)}{\sigma} > \ k\right]$

$= P\ [z > k]$, z ~N (0, 1)

The size LRT is given by: Reject H_0 iff $\bar{x} > \theta_0 + \frac{\sigma}{\sqrt{n}}\ \alpha$,

Which is the UMP test for testing H_0: θ = θ_0 $against$ H_1: θ > θ_0 and is unbiased.

H_1: θ ≠ θ_0 is $\beta(\theta)$= P_{H_0} $\left[\left|\frac{\sqrt{n}\ (\bar{x}-\theta_0)}{\sigma}\right| > \frac{\alpha}{2}\right]$

= 1- $\Phi\ (\frac{\sqrt{n}\ (\bar{x}-\theta_0)}{\sigma} + \frac{\alpha}{2})$+ $\Phi(\frac{\sqrt{n}\ (\theta_0-\theta)}{\sigma} - \frac{\alpha}{2})$

Note that, $\beta'(\theta)$= $\Phi(\frac{\sqrt{n}\,(\theta_0-\theta)}{\sigma}+\frac{\alpha}{2})(\frac{\sqrt{n}}{\sigma})$- $\Phi(\frac{\sqrt{n}\,(\theta_0-\theta)}{\sigma}-\frac{\alpha}{2})\times(\frac{\sqrt{n}}{\sigma})$; if $\theta > \theta_0$

Now, $\beta'(\theta)=\begin{cases} > 0 \; if \; \theta > \theta_0 \\ < 0 \; if \theta < \theta_0 \end{cases}$

Clearly, β (θ)> (θ_0) $\forall$ $\theta \neq \theta_0$

⇒ power > size

i.e the LRT is unbiased

Q76. Let $X_1, \dots, X_n$ be a R.S. from B (1, p) population. Derive a LRT of its size of $H_0: p = \{po\} against H_1: p \neq po$

Solution: - Here $\Omega_0 = \{p_0\}$ and $\Omega =$ $\{p\,; 0 < p < 1\}$

The likelihood function is

L $(\underset{\sim}{x}$; p)= $\begin{cases} p^{\sum x_i}(1-p)^{n-\sum x_i}, if \; x_i = 0, 1 \\ \quad 0 \quad , ow \end{cases}$

The LR is

λ= $\frac{p^{sup}\epsilon\Omega_0 \text{L}\ (\underset{\sim}{x};\text{p})}{p^{sup\epsilon}\Omega\text{L}\ (\underset{\sim}{x};\ \text{p})}$ = $\frac{\text{L}\ (\underset{\sim}{x};p_0)}{\text{L}\ (\underset{\sim}{x};\ \hat{\text{p}})}$, where

λ= $\frac{{p_0}^t(1-p_0)^{n-t}\hat{\text{p}}}{\hat{\text{p}}^t(1-\hat{\text{p}})^{n-t}}$, $t = \sum x_i$

= $(\frac{np_0}{t})^t(\frac{n\,(1-p_0)}{n-t})^{n-t}$

Now, ln λ= tln$(\frac{np_0}{t})$+ (n-t)ln $\{\frac{n\,(1-p_0)}{n-t}\}$

And $\frac{d}{dt}\,ln\lambda$ = ln$(\frac{np_0}{t})$- ln $\{\frac{n\,(1-p_0)}{n-t}\}$

= | n$(\frac{np_0}{t}.\frac{n-t}{n\,(1-p_0)t})$

= $\begin{cases} > 0\,, if \; t < np_0 \\ < 0\,, if \; t > np_0 \end{cases}$

From graph, λ < c ⇒ t > k_2 or $t < k_1$

The LRT of its size is given by;

Reject H_0 iff λ < c iff $\sum_{i=1}^{n} x_i < k_1$ or $\sum_{i=1}^{n} x_i > k_2 \ni \lambda\,(k_1)$= $\lambda\,(k_2)$

Q77. let X be a discrete random variable with P [X= -1]= P and P [x= k]= $(1-p)^2P^k$, where P ϵ (0, 1)is unknown. Show that U(X)is an unbiased estimator of θ iff U (k)= ak , k = -1, 0, 1, 2, …. for some a.

Solution:- E [U(x)] = $\sum_{k=-1}^{\infty} a_k . P\ [X = k]$

= -ap + $\sum_{k=0}^{\infty} k(1-p)^2P^k$

= -ap + a$(1-p)^2$[p+ $2p^2 + 3p^3 +$ … ∞]

=-ap +a$(1-p)^2$p$(1-p)^2$

= -ap+ ap

= 0.

Now, $\sum_{x=-1}^{\infty} u\ (x)f(x) = 0$

⇨ u (-1) P [X= -1]+ $\sum_{x=0}^{\infty} u\ (x)f(x) = 0$

⇨ u (-1) p + $\sum_{x=0}^{\infty} u\ (x)(1-p)^2p^x$= 0

⇨ p (u (-1)) + $(1-p)^2\sum_{x=0}^{\infty} u\ (x)\ p^x$=0

⇨ $\sum_{x=0}^{\infty} u\ (x)\ p^x$= $\frac{-pu(-1)}{q^2}$

⇨ $\sum_{x=0}^{\infty} ux\,p^x$= -u (-1) p $(1-p)^2$

= -u (-1) p (1+ 2p +3$p^2 + \cdots$)

= -u (-1) [p+2$p^2 + 3p^3 + \cdots$]

= -u (-1) $\sum_{x=0}^{\infty} x\,p^x$ [comparing power series from both side]

⇨ u (x) = -u (-1)x

⇨ u(x) = ax.

Q78. $\mathbf{f(x)} = \begin{cases} e^{-(x-\theta)} \ if\ x \geq \theta \\ 0 \quad ow \end{cases}$

Find (a) MLE of θ (b) 95% C. I. for θ

Solution:- (a) let $X_1, \dots, X_n$ be a R.S. from f(x). then,

$f_\theta(x) = \exp[-\sum_{i=1}^{n}(x_i - \theta)].I\ X_{(1)} \geq \theta;$

$= L(\theta|\underset{\sim}{x})$

The likelihood function will be maximum when $\sum_i (x_i - \theta)$is minimum

i.e. , when θ is maximum.

i.e. $\hat{\theta}_{MLE} = X_{(1)}$

(b) CI will based on sufficient statistic

$f_\theta(\underset{\sim}{x}) = \begin{cases} e^{-\sum_i x_i + n\theta} \ ; X_{(1)} \geq \theta \\ 0 \quad ; ow \end{cases}$

$= e^{-\sum_i x_i + n\theta}\ \Phi\ (\theta, X_{(1)})$; where

$\Phi\ (\theta, X_{(1)}) = \begin{cases} 1\ if\ \theta \leq i \\ 0\ \ if\ ow \end{cases}$

$\therefore f_\theta(\underset{\sim}{x}) = g_\theta\ (t). h\ (\underset{\sim}{x})$

Where, $g_\theta\ (t) = e^{n\theta}. \Phi\ (\theta, t)$ and $h\ (\underset{\sim}{x}) = e^{-\sum x_i}$

$\therefore$ by NF Factorization theorem $X_{(1)}$ is sufficient.

Now, the PDF of $X_{(1)}$ is given by $f_{X_{(1)}}(y) = ne^{-n\,(y-\theta)} I\ \ y \geq \theta$

Note that, it is a shifted exponential distribution,

$\therefore$ 2n $(X_{(1)} - \theta) \sim {\chi^2}_2$

$\therefore p_\theta\ [{\chi^2}_{2;1-\frac{\alpha}{2}} \leq 2n(X_{(1)} - \theta) \leq {\chi^2}_{2;\frac{\alpha}{2n}}] = 1 - \alpha$

$\Rightarrow P_\theta \left[\frac{{\chi^2}_{2;1-\frac{\alpha}{2}}}{2n} \leq X_{(1)} - \theta \leq X_{(1)}\right] = 1 - \alpha$

$\Rightarrow P_\theta \left[X_{(1)} - \frac{{\chi^2}_{2;1-\frac{\alpha}{2}}}{2n} \geq \theta \geq X_{(1)} - {\chi^2}_2; \frac{\alpha}{2n}\right] = 1 - \alpha$

$\therefore$ Confidence interval for θ is

$(X_{(1)} - {\chi^2}_2; \frac{\alpha}{2n}, X_{(1)} - \frac{{\chi^2}_{2;1-\frac{\alpha}{2}}}{2n})$

Q79. suppose $X_1, X_2, \dots, X_n$ are i.i.d. N (θ, 1), $\theta_0 \leq \theta \leq \theta_1$, where $\theta_0 < \theta_1$ are two specified numbers. Find the MLE of θ and show that it is better than the sample mean x̄ in the sense of having smaller mean squared error.

Solution:- L = $\prod_{i=1}^{n} f\ (x_i, \theta) = \frac{1}{(\sqrt{2\pi})^n} . e^{-\frac{1}{2}\sum(x_i - \theta)^2}\ ; x_i \epsilon\ \mathbb{R}$

Here we wish to maximize L w.r.t. θ subject to the condition$\theta_0 \leq \theta \leq \theta_1$; L will be maximum iff $e^{\frac{1}{2}\sum(x_i - \theta)^2}$ is minimum. Iff $\sum(x_i - \theta)^2$ is minimum at $\hat{\theta}_{MLE} = \theta_0$

For x̄,

lnL = c-$\frac{1}{2}\sum(x_i - \theta)^2$

$$\frac{\partial |nL}{\partial \theta} = \sum(x_i - \theta) = 0$$

$\Rightarrow \sum x_i = n\theta$

$\Rightarrow \hat{\theta}_{MLE} = \bar{x}$; where $\theta_0 \leq \bar{x} \leq \theta_1$ as $\theta_0 \leq \theta \leq \theta_1$

E $(X - \theta_0)^2 = E\ [X - \bar{x}) + (X - \theta_0)]^2$

= E $(X - \bar{x})^2$+ E $(X - \theta_0)^2$

$\Rightarrow$ E $(X - \theta_0)^2 \leq$ E $(X - \bar{x})^2$.

Q80. if T: $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear transformation defined by

T (x, y, z) = (x+y+z, y+z, z) then find T^n(x, y, z).

Solution: - T^2(x, y, z) = T (x + y+ z, y+z, z)

= (x+ 2y + 3z, y+ 2z, z)

T^3(x, y, z) = T (x + 2y + 3z, y + 2z, z)

= (x+ 3y + 6z, y+ 3z, z)

⋮

T^n(x, y, z) = (x+ ny +$\frac{n\,(n+1)}{2}.z, y+nz, z$).

Q81. Let $X_1, X_2, \dots, X_n$ be i.i.d random variables with family f(θ ; x ; x ϵ $\mathbb{R}$, θ ϵ (0, 1) be the unknown parameter. Suppose that there exit an unbiased estimator T of θ based on sample size one, i.e.

E [T (X_1)] = θ. Assume that V (T (X_1)) < ∞.

(I) find the estimator V_n for θ, E (X_n) ∍ V_n is constant.

(II) Let s_n be the MVUE of θ based on $X_1, X_2, \dots, X_n$ s.t. $\underset{n\to\infty}{\text{lt}}\ V(s_n) \longrightarrow 0$

Solution:- (I) $X_i \sim f_\theta$(x)

Y = T (X_1) ~f_θ(t)

Now, $\bar{\text{Y}} \longrightarrow$ E (Y_1)

i.e. $V_n = \frac{1}{n}\sum_{i=1}^{n}$ T (X_i) ⟶ E [T(X_1) = θ]

(II) E ($\bar{\text{Y}}$) = 0

$$V\,(s_n) \le V\,(V_n) - \frac{V\,(\mathrm{T}(X_1))}{n^2}$$

$= \frac{V\,(\mathrm{T}(X_1))}{n}$

⇨ $\underset{n\to\infty}{\text{lt}}\ V\,(s_n) = 0$.

Q82. Let f (0)=0 then show that $\lim\limits_{h\to 0} \frac{f\,(h)+f\,(-h)}{h^2} = f''(0)$

Solution:- f (h)= f(0)+hf′(0)+$\frac{h^2}{2!}f''(0)+$ $\frac{h^3}{3!}f'''(c)$ [∵ f(0)= 0]

Similarly, f(-h)= - hf′(0)+ $\frac{h^2}{2!}f''(0) -$ $\frac{h^3}{3!}f'''(c)$

---

f (h)+ f (-h)= $h^2 f''(0)$

$\therefore f''(0) = \underset{h\to 0}{\text{lt}} \frac{f\,(h)+f\,(-h)}{h^2}$.

Q83. Suppose x has a normal distribution with mean 0 and variance 25.

Let Y be an independent RV taking values -1 and +1 with equal probability ½ .define

S= X y + $\frac{X}{y}$, T = xy -$\frac{x}{y}$.

(I) Find the probability distribution of S.

(II) Find the probability distribution of ($\frac{S+T}{10}$)^2

Solution:- (i) $F_S(s) = P\,[S \le s]$

= P [S$\leq s \mid Y = -1]P\ [Y = -1] + \ P\ [S \leq s|Y = 1]\ P\ [Y = 1]$

$= \frac{1}{2}\ P\ [-2X\ \leq s] + \frac{1}{2}\ P\ [-2x \leq s]$

=P [x $\leq \frac{s}{2}$], since 'X' is symmetrically distributed about '0',

= P [$\frac{x-0}{S}\ \leq\ \frac{s-0}{10}$]

= $\Phi(\frac{s}{10})$

∴ S ~ N (0, 10^2).

(Ii) s + T =2 xy

$(S+T)^2 = 4\,x^2y^2 = 4\,x^2$, since P [$Y^2 =$ 1]=1

⇨ $(S+T)^2 = (\frac{X}{5})^2 \sim\ \chi^2$.

Q84. using an appropriate probability distribution or otherwise find the value of $\lim\limits_{n\to\infty} \frac{1}{2^{n/2}\,\Gamma(\frac{n}{2})}\cdot\int_{n+\sqrt{2}\,n}^{\infty} e^{-t/2}\ t^{n/2-1}\ dt$

Solution:- Let $\{x_n\}$ be a sequence of i.i.d. Random variables following Gamma $(\frac{1}{2}, \frac{1}{2})$

Here , f (x)= $\begin{cases} \frac{e^{-x/2}.x^{1/2-1}}{2^{1/2}\cdot\Gamma(\frac{1}{2})} & if\ x>0 \\ 0 & ow \end{cases}$

Then $\sum_{k=1}^{n} X_k = S_n \sim Gamma(\frac{1}{2}, \frac{n}{2})$

E (S_n)= n, V(S_n)= 2n. by lindeberg- lavy central Limit theorem

$$\lim_{n\to\infty} P\left[\frac{S_n - E\ (S_n)}{\sqrt{V(S_n)}} \leq x\right] = \ \Phi\ (x)$$

⇨ $\lim\limits_{n\to\infty} P\left[\frac{S_n-n}{\sqrt{2n}} \leq 1\right] = \ \Phi(1)$

⇨ $\lim\limits_{n\to\infty} P\left[S_n\ \leq n+\sqrt{2n}\,\right] = \Phi(1)$

⇨ $\lim\limits_{n\to\infty} P\left[S_n > n+\sqrt{2n}\,\right] = 1-\Phi(1)$

⇨ $\lim\limits_{n\to\infty}\left\{\frac{1}{2^{n/2}\,\Gamma(\frac{1}{2})}\int_{n+\sqrt{2}\,n}^{\infty} e^{-t/2}\ t^{n/2-1}\ dt\right\} = \Phi(-1)$

Q85. letY_1, Y_2, Y_3, Y_4 has four uncorrelated r.v.s with E (Y_i) = iθ,

V (Y_i) = $i^2\sigma^2$, i= $1(1)^4$, where θ, σ> 0 are unknown parameter

Find the values of c_1, c_2, c_3, c_4 for which $\sum_{i=1}^{4} c_iY_i$ is unbiased for and has least variance.

Solution:- E $(\sum_{i=1}^{4} c_iY_i)$= $(\sum_{i=1}^{4} i_\theta c_i)\theta$, $\sum c_i = 1$

Again, V $(\sum_i c_iY_i)$= $\sum {c_i}^2\ i^2\sigma^2$ =$\sigma^2 = \sum i^2{c_i}^2$

Now, $\sum c_iY_i$is unbiased.

So, 1 = $(\sum_i ic_i)^2\ \leq (\sum_i i^2{c_i}^2)(\sum_i 1)$, by C-S inequality.

'=' holds when ic_i = k= $\frac{1}{4}$ ⇨c_i=$\frac{1}{4i}$.

Alternative way:-

$$X_i \sim (\theta, \sigma^2)$$

$Y_i \sim\ (i_\theta, i^2\sigma^2\ )$

$\frac{Y_i}{i} \sim (\theta, \sigma^2)$

$\bar{\text{Y}} \xrightarrow{BLUE} \theta$

$\frac{1}{4}\sum\frac{Y_i}{i}$ is BWE for θ

$\sum c_i Y_i$ is BWE for θ, where $c_i = \frac{1}{4i}$

Q86**. Let $X_1, X_2, \dots, X_n$ be independently distributed random variables with densities**

f $(x_i;\ \theta)$= $\begin{cases} e^{i\theta - x_i} & ,if\ x_i \geq i\theta \\ 0 & ,ow \end{cases}$ **[Here $x_i' sare\ not\ random\ samples$]**

Find a one –dimensional sufficient statistic forθ.

Solution:- the joint PDF of $X_1, X_2, \dots, X_n$ is

$$\prod_{i=1}^{n} f\ (x_i;\ \theta)$$

$$= \begin{cases} e^{\theta\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i}\ ; if x_i \geq i_\theta, \forall\ i = 1(1)n \\ 0 \qquad , \qquad ow \end{cases}$$

$$= \begin{cases} e^{\frac{n(n+1)\theta}{2} - \sum_{i=1}^{n} x_i}\ ; if\ \frac{x_i}{i} \geq \theta \forall\ i = 1(1)n \\ 0 \qquad ; \quad ow \end{cases}$$

$$= \begin{cases} e^{\frac{n(n+1)\theta}{2} - \sum_{i=1}^{n} x_i}; if\ \min\limits_i\{\frac{x_i}{i}\} \geq \theta \\ 0 \qquad ; ow \end{cases}$$

$=e^{\frac{n(n+1)\theta}{2} - \sum_{i=1}^{n} x_i}. I\ (\theta, \min\{\frac{x_i}{i}\})$; where

I(a,b)= $\begin{cases} 1 & if\ a \leq b \\ 0 & if\ a > b \end{cases}$

$= e^{\frac{n(n+1)\theta}{2}}. I\ (\theta, \min\{\frac{x_i}{i}\}).\ e^{-\sum_{i=1}^{n} x_i}$

= g (T ($\underset{\sim}{x}$) ;θ). h($\underset{\sim}{x}$), where h($\underset{\sim}{x}$)= $e^{-\sum x_i}$

so, T ($\underset{\sim}{x}$) = $\min\limits_i\{\frac{x_i}{i}\}$ is sufficient for θ, by NFFT.

Q87. If f(x) = $\frac{\theta a^\theta}{x^{\theta+1}}$ $I\ a < x < \infty, \theta > 0, a >$ 0;

(a) Find UMVUE of θ, when a is known,

(b) Find UMVUE of a, when θ is known

Solution: - (a) from OPEF, the complete sufficient statistic is $\sum_{i=1}^{n} |nX_i$ and hence$\sum_{i=1}^{n} |n\frac{X_i}{a}$ will also be complete sufficient statistic.

Now, $|n\frac{X_i}{a} \sim$ Exp (θ)

$\therefore 2\theta\sum_{i=1}^{n} |n\frac{X_i}{a} \sim {\chi_{2n}}^2$

⇨ E $[\frac{1}{2\theta\ \sum_{i=1}^{n} |n\frac{X_i}{a}}] = \frac{1}{(2n-2)}$ [If X $\sim {\chi_n}^2$ E $(\frac{1}{X}) = \frac{1}{n-2}$]

⇨ E $[\frac{n-1}{\sum_i |n\frac{X_i}{a}}] = \theta$

⇓

Function of complete sufficient statistic

$\therefore \frac{n-1}{\sum_i |n\frac{X_i}{a}}$ is the required UMVUE.

(b) $f_a(\underset{\sim}{x}) = \frac{\theta^n a^{n\theta}}{(\prod_{i=1}^{n} x_i)^{\theta+1}}\ .\ \Phi(a,x)$; where

$\Phi(a,x) = \begin{cases} 1\ , if\ a < x \\ 0,\ \ ow \end{cases}$

$\therefore g_a(t) = a^{n\theta}. \Phi(a,t)$ With T = $X_{(1)}$

$\therefore X_{(1)}$ is the sufficient statistic for θ.

Now, $f_{X_{(1)}}(t) = P\ [X_{(1)} \leq\ t]$= 1-P $[X_{(1)} > t]$= 1- $(P[X > t])^n$

Where, P [X > t]= $\int_t^\infty \theta a^\theta x^{-\theta-1}\, dx = \frac{a^\theta}{t^\theta}$

$\therefore f_{X_{(1)}}(t) = -n\theta\left(\frac{a}{t}\right)^{n\theta-1}.(-\frac{a}{t^2})$

= n . $\frac{a^{n\theta}}{t^{n\theta+1}}\ I_a > 0$

Now, if (t) is any arbitrary function of t, then E [ψ(t)]=0

⇨ $\int_a^\infty n\theta\frac{a^{n\theta}}{t^{n\theta+1}}\psi(t)dt = 0$

⇨ $\int_a^\infty \frac{\psi(t)}{t^{n\theta+1}}dt = 0$

g (t), say

⇨ G (∞)-G (a)=0

⇨ g (∞)-0 - g (a)-1=0

⇨ g (a)=0 ⇨ ψ(a)=0

$\therefore X_{(1)}$ is complete sufficient for θ

Let $X_{(1)} = T$

Now, E (T)= $\int_a^\infty t.n\theta\frac{a^{n\theta}}{t^{n\theta+1}}dt$

=$\frac{n\theta-a}{n\theta+1}$

$\therefore$ E [t. $\frac{n\theta-1}{n\theta}$]= a

$\therefore$ UMVUE of a is$\frac{n\theta-1}{n\theta} = (1-\frac{1}{n\theta})$.

Q88. let $X_1, X_2, \ldots, X_n$~ Rec (0, θ) with an unknown θ (1, ∞)

Suppose we only observe $z_i = \begin{cases} X_i\ \ if X_i \geq 1 \\ 1\ \ if\ X_i < 1 \end{cases}$

Derive UMVUE of θ.

Solution:- let T ($X_{(n)}$) be an unbiased estimator of 0.

$\therefore$ E [T ($X_{(n)}$)]=0

⇨ $\int_0^\infty t(x_{(n)}).\frac{nx(n)^{n-1}}{\theta^n}\ dx_{(n)} = 0$

⇨ $\int_0^\theta t(x_{(n)}).\ x(n)^{n-1}dx_{(n)} = 0$⇨ G ($\theta$)-G(0)=0

⇨ g (θ)= 0 ⇨ u ($x_{(n)}$). $x(n)^{n-1}$=0 ⇨ u ($x_{(n)}$)=0

$\therefore X_{(n)}$ is complete sufficient for 0.

To find UE of θ based on$X_{(n)}$, let us consider the function,

h (^$X_{(n)}$)= $\begin{cases} a\ \ if\ X_{(n)} < 1 \\ bX_{(n)}\ \ if\ X_{(n)} \geq 1 \end{cases}$

$\therefore$ E [h ($X_{(n)}$)] = θ [∵ h ($X_{(n)}$) is UE of θ]

⇨ a. P [$X_{(n)} \leq 1$]+ b $x_{(n).}\frac{nx(n)^{n-1}}{\theta^n}dx_{(n)}$=$\theta$

⇨ a $\int_0^1 \frac{nx(n)^{n-1}}{\theta^n}\ dx_{(n)} +$
$b\ \int_1^\theta n.\frac{x_{(n)}}{\theta^n}\ dx_{(n)} = \theta$

⇨ a. $\frac{n}{\theta^n}.\frac{1}{n} + \frac{b.n}{(n+1)\theta^n} = \theta$

⇨ a+ $\frac{bn}{n+1}(\theta^{n+1}-1) = \theta^{n+1}$

⇨ a+ $\frac{bn}{n+1}\theta^{n+1}$ -$\frac{bn}{n+1}$= θ^{n+1}

$\therefore \frac{bn}{n+1} = 1, a = 1, \frac{bn}{n+1} = 1$. [Equating coefficients of θ]

$\therefore$ h (X) = $\begin{cases} 1\ \ if X_{(n)} < 1 \\ \frac{n+1}{n}X_{(n)} \geq 1 \end{cases}$ is UMVUE of θ.

Q89. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution having pdf

f(x; x_0, α)= $\begin{cases} \frac{\alpha x_0}{x^{\alpha+1}} \; for\; x > \; x_0, \\ 0 \qquad\qquad ow \end{cases}$

Where, x_0, > 0, α> 0. Find the maximum likelihood estimator of α if x_0 is known.

Solution: - Likelihood function, L (x, α) is given by,

L (x, α) = $\frac{\alpha^n {x_0}^{\alpha n}}{{x_1}^{(\alpha+1)}{x_2}^{(\alpha+1)}\dots {x_n}^{(\alpha+1)}}$

⇨ $\log L = n\log\alpha + \alpha n \log x_0 - (\alpha + 1)\sum x_i$

⇨ $\frac{1}{L}.\frac{\partial L}{\partial\alpha} = \frac{n}{\alpha} + \; n\log x_0 - \sum x_i \frac{\partial L}{\partial\alpha} = 0$

⇨ $\frac{n}{\alpha}$+n|n$x_0 = \sum x_i$

⇨ $\alpha^{-1} = \frac{\sum x_i}{n} - |\text{n}x_0$

⇨ $\alpha = \frac{1}{\frac{\sum x_i}{n} - |\text{n}x_0}$

Thus MLE α is given by $\alpha = \frac{1}{\frac{\sum x_i}{n} - |\text{n}x_0}$

Q90. A fair coin is flipped 2n times. Find the probability that it comes up heads more often than it comes up tail.

Solution: - P (No. of Heads > No. of Tails)+ P (No. of Heads= No. of Tails)+ P (No. of Heads < No. of Heads < No. of Tails)=1.

Assuming you are tossing a fair coin, by symmetry, we also have that

P (No. of Heads > No. of Tails)

= P (No. of Heads < No. of Tails)

If we want to get k heads in 2n tosses, where the probability of getting a head is P then the probability is.

$$\binom{2n}{k} P^k (1-P)^{2n-k}$$

In our case, if we want the number of heads to be the same as number of tails then k=n and if we are tossing a fair coin then P =$^1/_2$. Hence, we get

P (No. of Heads=No. of Tails)

= $\binom{2n}{n}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^n = \frac{1}{2^{2n}}\binom{2n}{n}$

Hence, we get that P (No. of Heads > No. of Tails)= P (No. of Heads < No. of Tails)

= $\frac{1-\frac{\binom{2n}{n}}{2^{2n}}}{2} = \frac{1}{2} - \frac{\binom{2n}{n}}{2^{2n+1}}$.

Q91. Suppose $a_n \geq 0$ *and* $\sum a_n$ is convergent. Show that

$$\sum \frac{1}{n^2 a_n} \; is\; divergent.$$

Solution: Using CS Identity

$$\left[\left(\sqrt{a_1}\right)^2 + \left(\sqrt{a_2}\right)^2 + \cdots + \left(\sqrt{a_n}\right)^2\right]\left[\left(\frac{1}{1.\sqrt{a_1}}\right)^2 + \left(\frac{1}{2\sqrt{a_2}}\right)^2 + \cdots + \left(\frac{1}{n\sqrt{a_n}}\right)^2\right] \geq \left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right]^2$$

RHS is divergent and $\sum a_n$ is convergent. Hence

$$\sum \frac{1}{n^2 a_n} \; is\; divergent$$

Q92. Let F : $\mathbb{R}^n \to \mathbb{R}$ be defined by

$$F(x_1, x_2, \dots, x_n) = \max\{|x_1|, |x_2|, \dots, |x_n|\}$$

Show that F is a uniformly continuous function.

Solution: Take

$$X_1 = (X_{11}, X_{12}, X_{13}, \dots, X_n), X_2 = (X_{21}, X_{22}, \dots, X_{2n})$$

$$|F(X_{11}, X_{12}, \dots, X_n) - F(X_{21}, X_{22}, \dots, X_{2n})| = \max\{|X_{11}|, |X_{12}|, \dots, |X_{1n}|\} - \max\left\{\begin{matrix}|X_{21}|, |X_{22}|, \dots, \\ |X_{2n}|\end{matrix}\right\}$$

$$\leq |\max\{|X_{11} - X_{21}|, |X_{12} - X_{22}|, \dots, |X_{1n} - X_{2n}|\}|$$

$$\leq \sqrt{(X_{11} - X_{21})^2 + (X_{12} - X_{22})^2 + \cdots + (X_{1n} - X_{2n})^2}$$

$$\leq |X_1 - X_2|$$

So, for any given $\in > 0, choose\ \delta = \in$ $, then\ |F(X_1) - F(X_2)| < \in\ \forall\ |X_1 - X_2| < \delta$

Q93. Let f be a continuous function on [0, 1]. Evaluate

$$\lim_{n\to\infty} \int_0^1 x^n f(x)\, dx$$

Solution: Applying Mean Value theorem for integrals

$$f : [0, 1] \to \mathbb{R} \quad and \quad g : [0, 1] \to \mathbb{R}$$

be two integrable functions and g(X) has the same sign on [0, 1]

$$Then \quad \int_a^b f(x)\, g(x) dx = \mu \int_a^b g(x) dx$$

Take $g(x) = x^n\ ; x^n\ does\ not\ change\ sign\ on\ [0, 1]$

$$Hence \quad \int_0^1 x^n f(x) dx = \mu \int_0^1 x^n dx$$

$$\Rightarrow \int_0^1 x^n f(x)\, dx = \frac{\mu}{n+1}$$

$$\Rightarrow \lim_{n\to\infty} \int_0^1 x^n f(x) dx = \lim_{n\to\infty} \frac{\mu}{n+1} = 0$$

Q94. (a) Suppose the series

$$\sum_{n=1}^{\infty} \boldsymbol{a_n}$$

is convergent $(\boldsymbol{a_n} \geq \boldsymbol{0})$

Is it true that

$$\sum_{n=1}^{\infty} \sqrt{\boldsymbol{a_n a_n - 1}}\ \boldsymbol{also\ convergent}?$$

(b) Is the converse of the statement (a) true?

Solution: (a) AM ≥ GM gives

$$\sqrt{a_n a_{n-1}} \leq \frac{a_n + a_{n-1}}{2}$$

If

$$\sum_{n=1}^{\infty} a_n$$

is convergent then

$$\sum_{n=1}^{\infty} \frac{a_n + a_{n-1}}{2}$$

is also convergent.

Therefore $\sqrt{a_n a_{n-1}}$ is convergent.

(b) Take $a_n = n^2$ (for odd n); $a_n = n^{-102}$ (for even n). Then

$$\sum_{n=1}^{\infty} a_n$$

is divergent but

$$\sum_{n=1}^{\infty} \sqrt{a_n a_{n-1}}\ is\ convergent$$

(By p−series)

Q95. Evaluate the value of

$$\lim_{n\to\infty} \int_0^a e^{nx} dx$$

Solution:

$$I = \underset{n\to\infty}{lt} \int_0^a e^{nx}\,dx = \underset{n\to\infty}{lt} \left[\frac{e^{nx}}{n}\right]_0^a$$

$$= \underset{n\to\infty}{lt} \frac{1}{n}[e^{na} - 1]$$

$$= \underset{n\to\infty}{lt}\ ae^{na}$$

$$I = \begin{cases} \infty & ,a > 0 \\ 0 & ,a \le 0 \end{cases}$$

Q96. Let $u, v : [a, b] \to \mathbb{R}$ be continuous. Define $f: [a, b] \to \mathbb{R}$ by

$$f(x) = \begin{cases} u(x)\ \ if\ x\ is\ rational \\ v(x)\ if\ x\ is\ irrational \end{cases}$$

Show that f is Riemann integrable on [a, b] if and only if $u(x) = v(x)\ for\ all\ x \in [a, b]$

Solution: Now pick any partition 'P'.

Construct two Riemann sums

$$S(P, f, \xi)\ and\ S'(P, f, \xi')$$

$S \to$ choose rationals as intermediate points.

$S' \to$ choose irrationals as intermediate points.

We know that

$$\lim_{\|P\|} S \to B\ and\ \lim_{\|P\|} S' \to B$$

As 'f' is Riemann integrable

$$\Rightarrow \lim_{\|P\|\to 0} S - S' \to 0$$

$$\Rightarrow \lim_{\|P\|\to 0} u(x)(b-a) - v(x)(b-a) \to 0$$

$$\Rightarrow \lim_{\|P\|\to 0} u(x) - v(x) \to 0$$

$$\Rightarrow u(x) = v(x)$$

- If $u(x) = v(x)\ for\ x \in [a, b]$ and given u is continuous function, f is integrable (To show).

If 'f' is Riemann integrable then it satisfies the following property that for every partition 'P' and choice of intermediate points ξ,

$\lim_{\|P\|\to 0} S(P, f) = B$ ↜ $a\ real\ number, when\ S(P, f, \xi)$ is the Riemann sum.

Q97. Let f be a bounded twice differentiable real valued function on $\mathbb{R} \ni f''(x) \ge 0$ for all x. Show that f is a constant.

Solution: $f'' \ge 0 \Rightarrow f$ is convex

$$f(x) \ge f(x_0) + f'(x_0)(x - x_0)$$

Case I: $f'(x_0) > 0$

$$\Longrightarrow f(x) \to \infty\ as\ x \to \infty$$

$$\Longrightarrow f(x) is\ constant\ \ [\because f\ is\ bounded]$$

Case II : $f'(x_0) < 0$

$$\Rightarrow f(x) \to \infty\ as\ x \to -\infty$$

$$\Rightarrow f(x)\ is\ constant\ as\ f\ is\ bounded$$

Case III: $f'(x_0) = 0$

$$f'(x) = 0\ \forall\ x \in \mathbb{R}$$

$$\Rightarrow f\ is\ a\ constant$$

Q98. Find conditions on a, b, c ∈ ℝ to ensure that the following system is consistent, and in that case, find the general solutions:

$$\boldsymbol{x + 3y - 2z = a}$$

$$\boldsymbol{-x - 5y + 3z = b}$$

$$\boldsymbol{2x - 8y + 3z = c}$$

Solution:

$$A_{3\times3}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 3 & -2 \\ -1 & -5 & 3 \\ 2 & -8 & 3 \end{pmatrix} \underbrace{R'_2 \leftrightharpoons}R_1$$

$$+R_2\begin{pmatrix} 1 & 3 & -2 \\ 0 & -2 & 1 \\ 2 & -8 & 3 \end{pmatrix}$$

$$\underbrace{R'_3 = R_3 - 2R_1}\begin{pmatrix} 1 & 3 & -2 \\ 0 & -2 & 1 \\ 0 & -14 & 7 \end{pmatrix}$$

$$\underbrace{R'_3 = R_3 - 7R_2}\begin{pmatrix} 1 & 3 & -2 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

So, Rank (A) = 2

$$\bar{A} = \left(\begin{array}{ccc|c} 1 & 3 & -2 & a \\ -1 & -5 & 3 & b \\ 2 & -8 & 3 & c \end{array}\right)\underbrace{some\ operations}$$

$$\begin{pmatrix} 1 & 3 & & -2 & a \\ 0 & -2 & & 1 & a+b \\ 0 & 0 & 0 & & c-9a-7b \end{pmatrix}$$

For rank(A) $= rank\ (\bar{A})\ \Rightarrow c - 9a - 7b = 0 \Rightarrow$ $9a + 7b = c$ is the required condition.

For general solution:

$$\left.\begin{matrix} x + 3y - 2z = a \\ -2y + z = a + b \end{matrix}\right\}$$

Take $z = t$,

$$y = -\left(\frac{a+b-t}{2}\right);\ x = \frac{5a+3b+t}{2}$$

Q99. Prove or disprove:

(a) ∃ a linear map $\boldsymbol{T} :\ \mathbb{R}^2 \to \mathbb{R}^4$ such that

Range (T) = $\{(\boldsymbol{x_1, x_2, x_3, x_4}) \in \mathbb{R}^4 :\ \boldsymbol{x_1 + x_2 + x_3 + x_4 = 0}\}$

(b) ∃ a linear map $\boldsymbol{T} :\ \mathbb{R}^2 \to \mathbb{R}^3\ \ni$

Range (T) = $\{\ (\boldsymbol{x_1, x_2, x_3})\ \in \mathbb{R}^3 :\ \boldsymbol{x_1 + x_2 + x_3 = 0}\}$

Solution: (a) NO

$$T :\ \mathbb{R}^2 \to \mathbb{R}^4$$

$$\dim(R^2) = \dim(Range\ (T) + \ker(T))$$

$$\Rightarrow 2 = 3 + x$$

$$\Rightarrow \dim\big(Range\ (T)\big) = 3$$

(b)$\dim\big(Range\ (T)\big) = 3$

$$\dim(R^2) = 2 + \dim(\ker T)$$

$$\Rightarrow \dim(\ker T)\ = 0,$$

Yes, linear transformation is possible.

$$T :\ \mathbb{R}^2 \to \mathbb{R}^3$$

$$T(x_1\ , x_2) \to (x_1\ , x_2, -x_1 - x_2)$$

Q100. Consider a circle which is tangent to the y–axis at 0. Show that the slope at (x, y) satisfies

$$\boldsymbol{\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}}$$

Solution: General equation of such circle would be, $(x - r)^2 + y^2 = r^2$ ……….(*)

$$\Rightarrow x^2 + y^2 - 2xr = 0 \ldots\ldots\ldots (1)$$

Differentiating (*), $2(x - r) + 2yy' = 0$

$$y' = \frac{r - X}{2y} \ldots \ldots (2)$$

Eliminating 'r' from (1) & (2) gives

$$y' = \frac{y^2 - x^2}{2xy}$$

MODEL TEST PAPERS WITH SOLUTIONS

MODEL TEST PAPER 1

Q1. **Find positive numbers n and $a_1, a_2, \ldots, a_n$ such that $a_1 + a_2 + \cdots + a_n = 10^3$ *and* $a_1 a_2 \ldots a_n$ is as large as possible.**

Sol.:- We are led to discover that in a max product (i) no a_i will >4 (ii) no $a_i$will = 1, (iii) all $a_i's$ can be taken to be 2 or 3 (As 4 = 2 × 2 and 4 = 2+ 2), (iv) at most two $a_i's$ will equal 2 (As $2 \times 2 \times 2 < 3 \times 3$ and 2+ 2+ 2 = 3+ 3).

Each of this is easy to establish. Thus, when the parameter is 1000 as in the problem at hand, the maximum product must be $3^{332} \times 2^2$.

Q2. **Each of the numbers $a_1, a_2 \ldots, a_n$ is 1 on -1. We have S = $a_1a_2a_3a_4 + a_2a_3a_4a_5 + \cdots + a_na_1a_2a_3$.Then show that 4/n.**

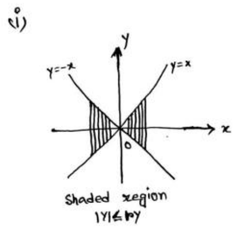
Sol. This is a number theoretic problem, but it can also be solved by invariance. If we no place any a_i by - a_i, Then s. does not change mod 4 since four cyclically adjacout terms change their sing. Indeed, if two of those terms are positive and two negative, nothing changes. It one or three have the same sing, S changes by ± 4.

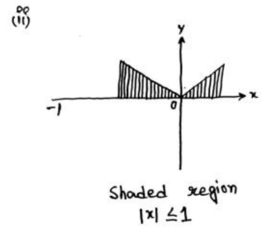
Finally, if all four are of the same sing, then S changes by ± 8.

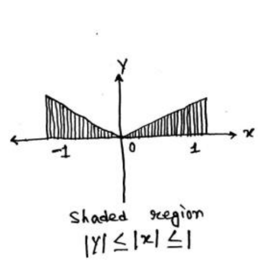
Initially we have S = 0 which implies S = 0 mods. Now, step by step, we change each negative sing into a positive sing. This does not change S mod = 4. At the end, we still have S= 0 mod 4, but also S = $\frac{n_i . s^4}{n}$.

Q3. Draw the graph $|y| \leq |x| \leq 1$.

Sol. The graph of $|y| \leq |x| \leq 1$. will be







Q4. **In a triangle ABC the bisectors AD, BE, CF meet at the point I. Show that $\frac{1}{4} < \frac{IA}{AD} . \frac{IB}{BE} . \frac{IC}{CF} = \frac{8}{2}$.**

Sol. A bisector of a triangle divides the opposite side in the ratio of the other two sides.

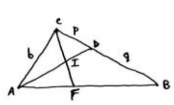
Hence, p = CD =$\frac{(ab)}{(b+c)}$, q = DB = $\frac{(ac)}{(b+c)}$.

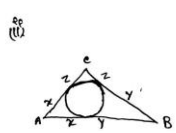
Thus we have

$$\frac{AI}{ID} = b; p\frac{b+c}{a}, \frac{AI}{AD} = \frac{AI}{AI+ID} = \frac{b+c}{a+b+c}$$

Similarly,

$$\frac{BI}{BE} = \frac{a+c}{a+b+c}, \frac{CI}{CF} = \frac{a+c}{a+b+c}$$





Applying the GM- AM inequality to the numerator we get

f(a, b, c) = $\frac{AI}{AD} . \frac{BI}{BE} . \frac{CI}{CF} = \frac{(a+b)(b+c)(c+a)}{(a+b+c)^3} \leq \frac{8}{(a+b+c)^3}\left(\frac{a+b+c}{3}\right)^3$ $which\ is\ \frac{8}{27}$.

This is the right side of the inequality chain. To prove the left side, we use the triangle inequality

(a+ b –c) (a +c –b) (b+ c –a) >0 (2)

For a more economical evaluation, we introduce the elementary symmetric functions.

= a+ b +c, v = ab + bc + ca, w = abc (3)

Putting (3) into (2), we get $-\mu^3 + 4\mu v - 8w > 0$ (4)

On the other band $\frac{1}{4} < f(a, b, c)$ (5)

Give $-\mu^3 + 4\mu v - 4w > 0$ (6)

Now (4) is obviously correct. Hence (6) is also correct. Here we profitably used the elementary symmetric functions. They are useful in cares when we are dealing with functions which are symmetric in their variables.

Here is the simplest proof of (5) : set a = y+ z, b = z+ x, c = x+ y. with r = $\frac{x}{x+y+z}, s = \frac{y}{(x+y+z)}, t = \frac{z}{(x+y+z)}$,

We get $\frac{AI}{ID} = \frac{1}{2}(1+r)$,

$$\frac{BI}{BE} = \frac{1}{2}(1+s), \frac{CI}{CF} = \frac{1}{2}(1+t), r+s+t = 1,$$

$$f(a, b, c) = \frac{1}{8}(1+r)(1+s)(1+t) = \frac{1}{8}$$

$$(1 + 1 + rs + st + tr + rst) > \frac{1}{4}.$$

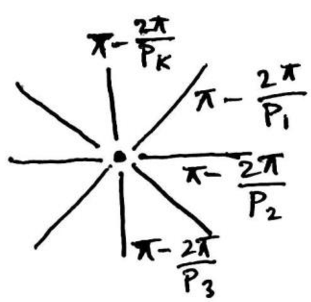
Q5. Suppose regular polygon of number of sides $p_1, p_2, \ldots, p_k$ meet at a common verstex

so that no gaps is left, what is the relation between $\boldsymbol{p_1, p_2, \dots, p_k}$?

Sol.: The sum of the exterior angles of a convex polygon of any number of sides is 2π. For a regular polygon of p sides, each extarias angle would be $\frac{2\pi}{p}$ and each interior angle would be $\pi - \frac{2\pi}{p}$.

If k polygons of $p_1, p_2, \dots, p_k$ sides meet at a point and there is no gap, then since the sum of the angles at a point is 2II, it follows that $\pi - \frac{2\pi}{p_1} + \left(\pi - \frac{2\pi}{p_2}\right) + \cdots + \left(\pi - \frac{2\pi}{p_k}\right) = 2\pi$

$$or, \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} = \frac{k}{2} - 1$$



Q6. Find the number of solutions in positive integers $\boldsymbol{x_1 \le 5}$, $\boldsymbol{x_1 + x_2 \le 5}$, $\boldsymbol{x_1 + x_2 + x_3 \le 5}$,

$\boldsymbol{x_1 + x_2 + x_3 + x_4 \le 5, x_1 + x_2 + x_3 + x_4 + x_5 \le 5.}$

Make a conjecture about the number of solution on positive integers of $\boldsymbol{x_1 + x_2 + \cdots + x_k \le x, x \ge k.}$

Sol. (i) $x_1 \le 5 \implies x_1 == 5\ or\ x_1 = 4\ or\ x_1 = 3\ or\ x_1 = 2\ or\ x_1 = 1$ number of solution in positive integers is

$$4c_0 + 4c_0 + 2c_0 + 1c_0 + 0c_0 = 5 = 5c_1$$

(ii)$x_1 + x_2 \le 5 \implies x_1 + x_2 = 5, 4, 3, 2.$ Number of solutions in positive integers is

$$4c_1 + 3c_1 + 2c_1 + 1c_1 += 10 = 5c_2$$

(iii) $x_1 + x_2 + x_3 \le 5 \implies x_1 + x_2 + x_3 = 5, 4, 3.$ Number of solutions in positive integers is

$$4c_2 + 3c_2 + 2c_2 = 10 = 5c_3$$

(i) $x_1 + x_2 + x_3 + x_4 \le 5 \implies x_1 + x_2 + x_3 + x_4 = 5, 4.$ Number of solutions in positive integers is $4c_3 + 3c_3 = 5 = 5c_4$.

(ii) $x_1 + x_2 + x_3 + x_4 + x_5 \le 5 \implies x_1 + x_2 + x_3 + x_4 + x_5 = 5.$ Number of solutions in positive integers is $4c_4 = 5c_5$.

Total number of solutions in positive integer is

$$5c_1 + 5c_2 + 5c_3 + 5c_4 = 2^5 - 1 = 31$$

The conjecture is that number of solutions in positive integers of

$x_1 + x_2 + \cdots + x_k \le x\ is\ nc_k.$

Q7. **Let f and g be real valued functions defined for all real values of x and y and satisfying the equation**

f(x+ y) + f(x –y)= 2 f(x) .g(y) ∀ x, y. Prove that if f(x) is not identically zero and if 1 f(x) │≤│∀x, then │g(y)│ ≤ 1∀y.

Sol. Since │f (x) │ is bounded, it has a least upper bound M and since f is not identically 0. M > 0.

MW, suppose that the inequality │g(y)│ ≤1 does not hold fordly. Then these is a point y_0 such that │g(y_0)│ >1.

Using the given equation and the triangle inequality. We get 2│f(x)│ │g(y_0)│= │f(x+ y_0)+ f(x -y)│≤ │1 f(x+ y_0)│+│f(x -y_0)│≤ 2M.

Hence │f(x)│≤ $\frac{M}{|g(y_0)|} < M$, contradicting the fact that M is least upper bound of │f(x)│. We conclude that │g(y)│≤ for all y.

Q.8. **Show that for any odd prime p, there exists a positive integers n such that n, n^n, n^{n^n}……….all leave the same remainder upon division by p where n does not leave a remainder of 0 or 1 upon division by p.**

Sol. We claim that n = 2p -1 will satisfy the given conditions. First 1n = 2p -1 is odd and n = 2p -1 = -1 (mod p). So for any power of n, we have $n^{n^n} = (-1)^{n^n} = -1 (mod\ p)$.

∴ All the numbers $n, n^n, n^{n^n}, \dots \dots$ leave a remainder of p -1 when divided by p. Finally, since p is an odd prime, 2p -1 cannot be 0 or 1.

Q9. **If a circle intersects the hyperbola $y = \frac{1}{x}$ at four distinct points (x_i, y_i), i = 1, 2, 3, 4, then show that $x_1x_2 = y_3y_4$.**

Sol. Equation of circle $x^2 + y^2 = r^2$

$$Now, y = \frac{1}{x}, \Longrightarrow x^2 + \frac{1}{x} = r^2,$$
$$\Longrightarrow x^4 - r^2x^2 + 1 = 0,$$

∴ Product of roots, i.e. $x_1x_2x_3x_4 = 1$,

$$= x_1x_2 = \frac{1}{x_3}.\frac{1}{x_4}. = x_1x_2 = y_3y_4$$

Q10. **Prove the identity $sin^{2p}x + sin^{2p}2x + sin^{2p}3x + \cdots + sin^{2p}nn = \frac{1}{2} + n\left(\frac{1.3.5\dots(2p-1)}{2.4.6\dots 2p}\right)$; if x = $\frac{\pi}{2n}$ $and\ p < 2n$ (p is a positive integer).**

Sol. S = $sin^{2p}x + sin^{2p}2x + \cdots + sin^{2p}nx$

$$= \sum_{l=1}^{n} sin^{2p} \times sin^{2p} l\alpha = \frac{1}{2p-1}(-1)^p.$$

$$= \sum_{k=0}^{p-1}(-1)^k 2p_{C_k}\, cos2(p-k)l\alpha + \frac{1}{i^{2p}}.2p_{C_p}.$$

∴ S = $\frac{(-1)^p}{2p-1}.\sum_{k=0}^{p-1}(-1)$: $2p_{C_k}\, \sum^n cos2(p-k)l\alpha + \frac{n}{p}.2p_{C_p}$.

Put 2 (p –k)= λ, then $\sum_{l=\lambda}^{n} 2cos2(p-k)l\alpha = cos\lambda + \cdots + cosn\lambda = \frac{sinn\lambda \cos\frac{n+1}{2}}{\sin\frac{\lambda}{2}}$ ……………. (i)

Case A: Equation. (i) = 0 (if k is of the same point y as p) (k ≡p) and

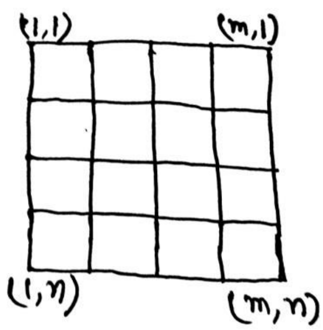
Case B: Equation. (i) = -1(if k and p are of different point) k = (p +1).

S=$\frac{(-1)^p}{2p-1}.\sum_{k=0}^{p-1}(-1)^k 2p_{C_k} + n.\frac{1}{2^{2p}}.2p_{C_p} = \frac{1}{2^{2p}-1} \times \sum_{k=0}^{p-1} 2p_{C_k} + \frac{n}{2^{2p}}.2p_{C_p} = \frac{1}{2} + n\left(\frac{1.3.5\dots.(2p-1)}{2.4.6\dots.2p}\right)$.

MODEL TEST PAPER - 2

Q1. In a city, there are m – roads from North to South and x–roads going from East to West, what is the length of the shortest path from North – East corner to the South-West corner and how many such shortest paths are there? What will be the lengths of other possible paths, if no part of a road is allowed to be travelled twice?

Sol.



Let F(m, n) be the number of paths for reaching from (1, 1) to (m, n) then are has to reach either (m -1, n) or (m, n -1) before reaching the vertex (m, n). This gives the functional equations.

F(m, n) = F(m- n) + F(n, m- 1). Knowing that F(1, 2) = 1, F(2, 1) = 1, we can deduce successively. The values of F(2, 1) , F(3, 1); …….. F(m, n). Alternatively we know that we have to cover m-1 step towards the west and n -1 steps towards the South out of a total of m+ n- 2 steps so that the number of different paths is

$\binom{m+n-2}{m-1} = \frac{(m+n-2)!}{(m-1)!(n-1)!}$

The shortest path is of length m+ n- 2 and the longest path has the length mn if mn are both even and is mn−1 if one or both are odd and the other lengths will be in AP with the first term as m+ n- 2 and common difference 2 and the last term as given above.

Q2. **Show that when** $\boldsymbol{p} > 1, \frac{\mathbf{1}}{\mathbf{1^p}} + \frac{\mathbf{1}}{\mathbf{2^p}} + \cdots + \frac{\mathbf{1}}{\boldsymbol{n^p}} > \int^{x+1} \frac{1}{x^p} \boldsymbol{dx} > \frac{\mathbf{1}}{\mathbf{2^p}} + \frac{\mathbf{1}}{\mathbf{3^p}} \ldots + \frac{\mathbf{1}}{\boldsymbol{n^p}}$

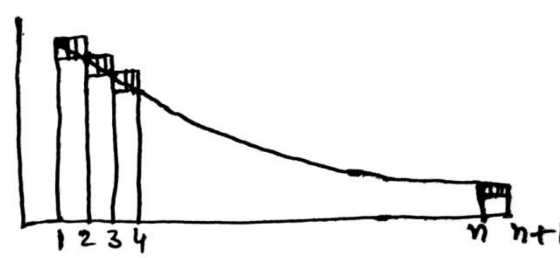
$$\boldsymbol{or\ S_1 > \frac{1}{p+1}\left\{1 - \frac{1}{(n+1)p+1}\right\} > S_p + (n+1)^p - 1}$$

$$\boldsymbol{or \frac{1}{p+1}\left\{1 - \frac{1}{(n+1)p+1}\right\} < S_p < \frac{1}{p+1}\left(1 - \frac{1}{(n+1)^{p+1}}\right) + 1 - \frac{1}{(n+1)^p}}$$

show also that as n → ∞ **and p > 1.**

$\boldsymbol{\frac{1}{p+1} \leq \sum_1^\infty \frac{1}{n^p} \leq 1 + \frac{1}{p+1}}$

Sol.



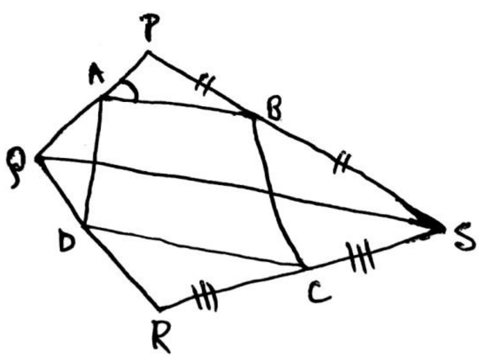
This follows from the fact that the area under the curve 4 = $\frac{1}{x^p}$ between x= 1 and x = n+ 1 lies between the area of the horizontally shaded rectangles and the sum of the horizontally and vertically shaded rectangles.

Q3. **Given a parallelogram ABCD, can you construct a quadrilateral PQRS of which the middle points of the sides are the four vertices**

A, B, C, D of the given parallelogram? How many such quadrilaterals

Can be constructed? What are the maximum and minimum areas of the these quadrilaterals? What can you say about their perimeter?

Sol. Given any quadrilateral PQRS we can easily show that if we join the middle points of its sides. The quadrilateral obtained will always be a parallelogram. Here we are concerned with the inverse problem. We are given the parallelogram and we want to reconstruct the quadrilateral from which the given parallelogram can be obtained by joining the middle points of the sides.



Let ABCD be the parallelogram. Through A, we draw any line PAQ inclined at an arbitrary angle O to the side AB such that AP = AQ = l where l is any arbitrary length.

Now join P to B and produce it to S, that PB = BS. Now join SC and QD and produce then to meet at R.

In the triangle PQS, A and B are the middle points of the two sides (by our construction) and therefore AB is l/2 and is parallel to QS in the triangle RSQ, CD is parallel to QS and is equal to ½ of QS. Since CD is parallel and = AB, therefore C and D must be the middle points of QR and RS. As such PQRS is the required quadrilateral. However in this quadrilateral angle Q and length l are arbitrary and therefore in general we can obtain a double infinity set of quadrilaterals with their middle points of the sides at the vertices of the given parallelogram.

In the given figure the area of the triangle PAB = $\frac{1}{4}$.

The area of the triangle PQS and the area of the triangle RQC is $\frac{1}{4}$ the area of the triangle RQS, therefore the seem of the areas of the triangle APB and RCD is $\frac{1}{4}$ the area of the quadrilateral PQRS. Similarly seem of the areas of triangles QAD and SBC = ¼ of the quadrilateral PQRS, therefore area of the quadrilateral outside the parallelogram= the area of parallelogram and therefore the area of the quadrilateral is double the area of parallelogram.

But the area of parallelogram is fixed, therefore the area of quadrilateral is also fixed. Thus although we have an infinity of quadrilaterals for the same parallelogram and these will have different sides and angles, the areas of these quadrilaterals will be the same. The quadrilateral with the minimum perimeter will arise when the quadrilateral is a square.

Q4. n! is defined only for n = 0 and for positive integral values of n. Define a function for all positive real values of x so that it reduces to n! when x is non-negative integer n. How many such functions can be defined? Can you define such a function which is both continuous and differentiable for all non-negative real values of x?

Sol. We know that f(0) = 1, f(1) = 1, f(2) = 2, f(3) = 6, f(4) = 24, f(50 = 120, ………..

Now join (0, 1) to (1, 1) : (1, 1) to (2, 2) and (2, 2) to (3, 6) and show by straight line segments and find equations of all the straight segments.

Thus we find f(x) = 1, 0 ≤x ≤ 1

= x, 1 ≤x ≤ 2.

= 2(2x -3), 2 ≤ x ≤ 3.

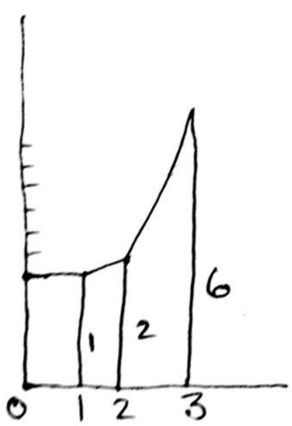
= 6(3x -8), 3 ≤ x ≤ 4.

= …………

= n! [1+ n(x –n)], n ≤ x ≤ n+1.

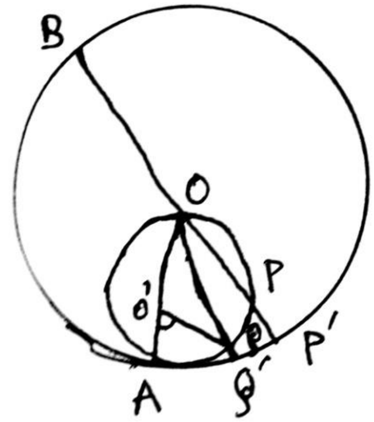
Thus the above function is defined for all non-negative values of x and it reduce to n! when x is a non-negative integer n. The function is continuous at all positive integral values of n.

In the above discussion, we have fitted straight line segments. If we fit parabolic segments of the second degree, we can get a function which is both continuous and differentiable everywhere, but its second order derivative will not exist for positive integral values of x. If we want there to exist, we would have to fit parabolic segments of the third degree.



Q5. A circular disc rolls inside a circle of double its radius. A hole is made in the disc near the circumstance. A pencil is passed through the hole. What will be the curve traced by this pencil as the disc rolls.

Sol. Let the rolling disc. Be initially at the lowest position and let P the point market on the disc. Let Q be other point on the disc, then as the disc rolls every point on the circumference of the disc between A and Q cares in contact in succession with points on the ring between A and Q arc AQ = arc AQ so that $\angle AOQ \; = \theta =$ arc AQ' /a = arc.



AQ/a and as the disc continuous, to roll further the point P will come into contact with point P' where AOP = d AO'P' = 2d and OP and P' lie on a straight line. Let P" be the new position of P then O'P" P' is a straight line. This shows that locus of P is the line OP'. As the disc rolls the marked point moves from P to P' along the diameter through O and P and this point moves from P' to P, then from P to O, then from O to B and finally from B to the initial position of the marked point. Thus the locus of any point on the rim of the disc will be the diameter of the vig passing through the initial position of the marked point.

Q6. a, b, c are any three digits from 0 to 9. Show that

1. **abcabc is divisible by 7, 11, 13.**
2. **If abc ≤ 500 then abc (twice abc) abc is divisible by $(7 \times 11 \times 13)^2$**
3. **If abc ≤ 333 then abc (three times abc) (three times abc) abc is divisible by $(7 \times 11 \times 13)^3$**

What is the final quotient in each care?

Sol. $7 \times 11 \times 13 = 1001 \implies (abc) \times (1001) = abcabc$

$\implies (abc) \times (1001)^2 = abc\ (twice\ abc)abc$

$\implies (abc) \times (1001)^3 = abc\ (three\ abc)$

$(three\ abc)abc.$

Q7. **Let αbe a fixed real number such that 0 < α < π and F(θ) = $\frac{\sin\theta+\sin(\theta+\alpha)}{\cos\theta-\cos(\theta+\alpha)}$ $where\ 0 \le \theta \le \pi - \alpha$. Show that F is a constant.**

Sol. Suppose that F is a const. then F(θ) + F(O) ∀ θ, $0 \le \theta \le \pi - \alpha$. That is

$\frac{\sin\theta+\sin(\theta+\alpha)}{\cos\theta-\cos(\theta+\alpha)} = \frac{sin\alpha}{1-cos\alpha}$ .... (i),

$[\sin\theta + \sin(\theta+\alpha)][1 - cos\alpha] = sin\alpha$

$\cos\theta - \cos(\theta+\alpha)$ .... (ii)

$\sin\theta + \sin(\theta+\alpha) - sin\theta cos\alpha - \sin(\theta + \alpha)\cos\alpha$

$= sin\alpha cos\theta - sin\alpha\cos(\theta+\alpha)$ .... (iii),

$\sin\theta + \sin(\theta+\alpha) - [sin\theta cos\alpha + sin\alpha cos\theta] - [\sin(\theta+\alpha)\cos\alpha - sin\alpha cos(\theta + \alpha)] = \theta$ .... (iv)

$\sin\theta + \sin(\theta+\alpha) - \sin(\theta+\alpha) - \sin(\theta + \alpha - \alpha) = 0$............. (v)

The last equation is an identity. For the proof, we must reverse these steps. The only questionable step is from (e) to 1: the proof is valid only if we do not divide by zero in going from (2) to (1). Then do yourself.

Q8. **If F_1 denote the term in the Fibonacci sequence, then show that $F_{n+1}{}^2 + F_n{}^2 = F_{2n+1}$.**

Sol. The result holds for n= 1. So suppose the result holds for integer k, then

$F_{k+2}{}^2 + F_{k+1}{}^2 = (F_{k+1} + F_k)^2 + F_{k+1}{}^2 = F_{k+1}{}^2 + 2F_{k+1}F_k + F_k{}^2 + F_{k+1}{}^2 = (F_{k+1}{}^2 + F_k{}^2) + (2F_{k+1}F_k + F_{k+1}{}^2) = F_{2k+1} + (2F_{k+1}F_k + F_{k+1}{}^2)$, the last step by the inductive assumption.

We should be done it we could show $2F_{k+1}F_k + F_{k+1}{}^2 = F_{2k+1}$, for we could then continue the previous argument and then you do yourself.

Q9. Show that $\int_0^1 \log x \log(1-x)dx = 2 - \frac{\pi^2}{6}$.

Sol. $1 = \int_0^1 \log x \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) dx$

$$= \int_0^1 \log x \sum_{k=1}^{\infty} -x^k + dx$$

$$= -\sum_{k=1}^{\infty} \frac{1}{k} \int_0^1 x^k\ logxdx$$

$$Now \int_0^1 x^k\ dx = \left(\frac{1}{k+1}\right)$$

Differentiating centre the sign of integration with respect to k, we get $\int_0^1 x^2\ logxdx = -\frac{1}{(k+1)^2}$

$$I = \sum_{k=1}^{\infty} \frac{1}{k(k+1)^2} = \sum_{k=1}^{\infty} \frac{k+1-k}{k(k+1)^2}$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{k(k+1)} - \frac{1}{(k+1)^2}\right)$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) - \sum_{k=1}^{\infty} \frac{1}{(k+1)^2}$$

$$= 1 - \left(\frac{\pi^2}{6} - 1\right) = 2 - \frac{\pi^2}{6}.$$

Q10. Let r and s be nonzero integers. Prove that $(r^2 - s^2)x^2 - 4rsxy - (r^2 - s^2)y^2$ has no solution in integers x, y.

Sol. The L.H.s. can be factorized so that$\{(r-s)x - (r+s)y\}\,\{(r+s)x - (r-s)y\} = 1.$ Both factors are integers. As such the only possibility is that (r –s)x – (r +s)y- k = 0. (r+ s)x + (r –s)y –k = 0,

Where k is either 1 or -1, so that $\frac{x}{kr} = \frac{y}{-ks} = \frac{1}{r^2+s^2}$.

$$x = \frac{kr}{r^2+s^2}, y = \frac{ks}{r^2+s^2}, so\ that\ x^2 + y^2 = \frac{k^2}{r^2+s^2},$$

$$or\ (x^2+y^2)(r^2+s^2) = k^2 = 1, = 1\ r^2 + s^2 = 1, \Longrightarrow r = 1, s = 0, or\ r = 0, s = 1.$$

But r and s are non-zero integers.

MODEL TEST PAPER – 3

Q1. In a round robin tournament with n-players $p_1, p_2, \ldots, p_n$ where n > 1, each player plays are game with each of other players and rules are such that no ties can occur. If $W_n\ and\ L_n$ be the number of games won and last respectively by player p_n,Show that $\sum_{r=1}^{n} {W_r}^2 = \sum_{r=1}^{n} {L_r}^2$.

Sol:

Let $\sum_{r=1}^{n} {W_r}^2 = \sum_{r=1}^{n} {L_r}^2\ then\ \sum_{r=1}^{n}({W_r}^2 - {L_r}^2) = 0.$

$\sum_{r=1}^{n}(W_r - L_r)(W_r + L_r) = 0\ but\ W_r + L_r = n-1$for each. r, so $(n-1)\sum_{r=1}^{n}(W_r - L_r) = 0$,

$$\Longrightarrow \sum_{r=1}^{n}(W_r - L_r) = 0, \Longrightarrow \sum_{r=1}^{n} W_r = \sum_{r=1}^{n} L_r.$$

This last equation is true, since the total number of games won by the players has to equal the total number of games lost.

Q2. If x, y, z, are positive integers, show that (x, y) (x, z) (y, z) $[x, y, z]^2 = [x, y][x, z][y, z](x, y, z)^2$, where (a, …, g) and [a, …, g] denote gcd (a, …, g) and lcm (a, …, g) respectively.

Sol. Because of unique factorization, if suffices to show that for each prime p, the power of p on the left side (in its prime factorization) is equal to the power of n on the right side. So let x = $p^a r, y = p^b s\ and\ z = p^c t$, for integers r, s, t each relatively prime to p. We may assume (as of symmetry and by relabeling if necessary that) a ≤ b≤ c. Then the power of p in the unique factorization of $[x, y, z]^2$ is 2c; the powers of p in (x, y), (x, z) and (y, z) are a, a and b respectively. Hence the power of p on the left side is 2a + b + 2c. In the same manner, the power of p on the R. S. is b + c + c + 2a = 2a+ b + 2c.

Q3. **Suppose f(x) = a, $\sin x + a_2 \sin 2x + \cdots + a_n \sin nx$ $where$ $a_1, a_2, \ldots, a_n$ are real numbers and n is a positive integer. If $(x)1 \le \sin x$ $\forall$ x, show that $1a, +2a_2 + \cdots + na_n 1 \le 1$.**

Sol. Let we try inducting on the number of terms in f(x). When n= 1, f(x) = $a_1 \sin x$ $and\ since\ 1\ f(x)1 \le| \sin x$ | $,it\ follows\ that$ | a_1 |=| $a_1 \sin\left(\frac{\pi}{2}\right)$ |= | f $\left(\frac{\pi}{2}\right)$ |≤ | $\sin\left(\frac{\pi}{2}\right)$ |= $1.$

Suppose the result holds for k and consider the functions f(x) = $a_1 \sin x + a_2 \sin 2x + \cdots + a_k \sin k_2 + a_k + 1^{sin}(k+1)x$ for some choice of real numbers $a_1, a_2, \ldots, a_{k+1}$ and suppose that if| $f(x)$ |≤| $\sin x$ | $\forall$ x. Since $\sin(k+1)\,x = \sin kx\ cosx + \sin x \cos x\ kx,$ we can coride $f(x) = (a_1 + a_k + 1\alpha \sin kx)$. We have now recoritten $f(a)$ as a sum of k terms, more or less of type from which we can apply the induction assumption. Then you do yourself.

Q4. **Find all functions which are everywhere differentiable and satisfy f(x) + f(y) = f $\left(\frac{x+y}{1-xy}\right)$ $\forall$ x, y for which xy ≠ 1.**

Sol. Differentiating partially with respect to x, y

f'(x) = $f'\left(\frac{x+y}{1-xy}\right)\frac{1+y^2}{(1-xy)^2}$

$f'(y) = f'^{\left(\frac{x+y}{1-xy}\right)}\frac{1-y^2}{(1-xy)^2}$

Dividing$(1+x^2)f'(x) = (1+y^2)f'(y)$.

Since L.H.S. is a function of x only and R.H.S. is a function of y only, each must be a constant, so that

$$f(x) = \frac{C}{1+x^2}, f(x) = C\tan^{-1}x + D.$$

$$Also\ f(x) + f(0) = f(x) \Rightarrow f(0) = 0 \Rightarrow D = 0.$$

$$f(x) = c\tan^{-1}x.$$

Q5. **Find two non-congruent similar triangles with sides of integral length having the length of two sides of one triangle equal to two sides of the other triangle.**

Sol. Let a, b, c ; b, c, d, be the lengths of the sides of the two triangles and let a < b. Since the triangles are similar.

$\frac{a}{b} = \frac{b}{c} = \frac{c}{d}, so\ that\ c = \frac{b^2}{c}, d = \frac{c^2}{b}$ but c < a+ b

$\Rightarrow \frac{b^2}{a} < b + a \Rightarrow \frac{b^2}{a^2} - \frac{b}{a} - 1 < 0.$

$\Rightarrow \left(\frac{b}{a} - r\right)\left(\frac{b}{a} + \frac{1}{r}\right) < 0\ where\ r = \frac{Us+1}{2}.$

$= 1.618 \ldots.. \Rightarrow \frac{b}{a} < r.$

As such we can take $\frac{b}{a}$ any rational number between 1 and 1.6. The suppose we take $\frac{b}{a} = \frac{4}{3}.$

$\therefore a = 3k, b = 4k, c = \frac{16k}{3}, d = \frac{64k}{9}$

If we choose k = 9

a = 27, b = 36, c = 48, d = 64.

Length of all the sides are integers and

$\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$ is satisfied.

Q6. **Let a, b, c be three real numbers such that a < b < c f(x) is continuous in [a, c] and differentiable in (a, c). Also f'(x) is strictly increasing in (a, c). Prove that (c - b) f(a) + (b - a) f(c) > (c - a) f (b).**

Sol. : By mean value theorem $\frac{f(b) - f(a)}{b-a} = f'(u), a < u < b$

$\frac{f(c)-f(b)}{c-b} = f'(v), b < v < c$

Since f'(x) is strictly increasing f'(u) < f'(v)

$\frac{f(b)-f(a)}{b-a} < \frac{f(c)-f(b)}{c-b}$

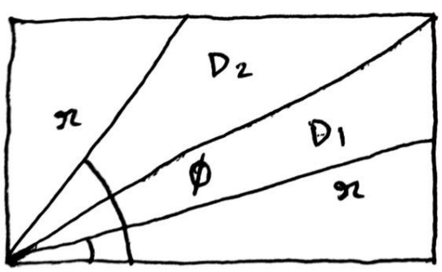
$or\ f(b)(c-b+b-a) - f(a)(c-b) - f(c)(b-a) < 0.$

$or, (c-b)f(a) + (a-c)f(b) + (b-a)f(c) > 0.$

$or, (b-c)f(a) + (c-a)f(b) + (a-b)f(c) < 0.$

Q7. Evaluate bin $\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{1}{j^2+k^2}$.

Sol.



I= $\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{\frac{j}{n}}{\frac{j^2}{n^2}+\frac{k^2}{n^2}}\cdot\frac{1}{n^2} = \int\int\frac{x\,dx\,dy}{x^2+y^2} = \int^{D_1+D_2}\int_{D_1}\frac{x\,dx\,dy}{x^2+y^2} + \int\ \int_{D_2}\frac{x\,dx\,dy}{x^2+y^2}$

$$= \int_0^{\frac{\pi}{4}}\cos\phi\ d\phi\int^{sec\phi}dr$$

$$+ \int^{\frac{\pi}{4}}\cos\phi\ d\phi\int^{cosec\phi}dr$$

$$= \frac{\pi}{4} + \int_0^{\frac{\pi}{2}}\cot\phi\ d\phi = \frac{\pi}{4} + \left[log^{\pi/4}sin\phi\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4} - \log\frac{1}{\sqrt{2}} = \frac{\pi}{4} + \frac{1}{2}\log 2.$$

Q8. **Find the number of points in the cartesian plane with integral co- ordinate satisfying the inequalities |x| ≤ k, |y| ≤ k, |x - y| ≤ k.**

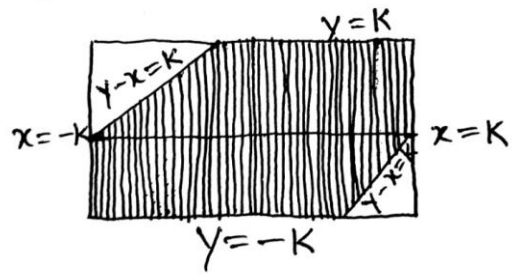
Sol. : We must –k ≤ x ≤ k, -k ≤ y ≤ k, -k ≤ y – x ≤ k.

The points are bounded by the straight lines

x = -k , x = k

y = -k , y = k

y = x – k, y = x + k



When

$$\left.\begin{array}{r} x = -\text{k, we get } y = -\text{k}, -(\text{k}-1), \dots .0\ \text{k}+1 \\ x = -(\text{x}-1)y = -\text{k}, \dots\dots 1, \text{k}+2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots .. \end{array}\right\}$$

$$\left.\begin{array}{r} x = -1,\ \ y = -k, \dots k-1, k+k \\ x = 0,\ \ y = -k, \dots k, k+k+1 \\ x = 1,\ \ y = -(k+1), \dots ., k, k+k \\ x = 2,\ \ y = -(k-2), \dots ., k, k+k-1 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots .. \\ x = k \qquad y = 0 \dots\dots\dots\dots k, k+1 \end{array}\right\}$$

Total number of points

= 2 [(k +1) + …+(k +k)] + 2k +1

= k (3k +1)+ 2k + 1

= $3k^2 + 3k + 1$

= $(k+1)^3 - k^3$.

Q9. **Let $ax^2 + bxy + cy^2$ and $Ax^2 + Bxy + Cy^2$ be two positive definite forms which are not proportional. Prove the from**

$(aB - bA)x^2 + 2(aC - cA)xy + (bC - cB)y^2$ is indefinite.

Sol. : Since the two forms are positive definite.

a > 0 : c > 0 $\quad b^2 - 4ac < 0$

A > 0 : C > 0 $\quad B^2 - 4ac < 0$

To show that the new form is indefinite, we have to show that its discriminant D > 0. Now

D = $4(aC - cA)^2 - 4(aB - bA)(bC - cB)$

$a^2D = \{2a\,(aC - cA) - b(aB - bA)\}^2 - (b^2 - 4ac)(aB - bA)^2$

And D would be > 0 unless

$aB - bA = 0, aC - cA = 0$

i.e. unless $\frac{a}{A} = \frac{b}{B} = \frac{c}{C}$

Which is given to be not true. As such D > 0 and the given form is indefinite.

Q10. **Prove that there is no equilateral triangle all of where vertices have integral co-ordinates.**

Let $(x_1, y_1), (x_2, y_2)$ be the vertices of the triangle and let a be the length of the side of the equilateral triangle.

The area of the triangle = $\frac{1}{2} \left| \begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix} \right| = an\ integer$

By another method, area = $\frac{1}{2} a^2 \sin 60° = \frac{\sqrt{3}}{4} a^2 = \frac{\sqrt{3}}{4} [(x_1 - x_2)^2 + (y_1 - y_2)] = \frac{\sqrt{3}}{4} \times an\ integer$

And this is not an integer, since $\sqrt{3}$ is irrational.

The contradiction in the result obtained by the two methods proves the required result.

MODEL TEST PAPER - 4

Q1. **Solve $x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right)$, x_0 $a\ given\ complex\ number$**

$$and\ find\ \lim_{n\to\infty} x_n.$$

Sol. : Substituting $x_n = \frac{1+z_n}{1-z_n}, z_n = \frac{x_n - 1}{x_n + 1}$

We get, $\frac{1+ z_n+1}{1-z_n+1} = \frac{1}{2}\left(\frac{1-z_n}{1-z_n} + \frac{1-z_n}{1+z_n}\right) = \frac{1+z_n{}^2}{1-z_n{}^2}$

$z_{n+1} = z_n{}^2 = z_{n-1}{}^4 = \cdots = z_0{}^{2n+1}$ | $z_n + 1$ | = | z_0 | 2^{n+1}

$$or, \left(\frac{x_{n-1}}{x_{n+1}}\right) = \left(\frac{x_{0+1}}{x_{0+1}}\right)^{2^4}$$

$$Also\ |\ z_n\ | = |\ z_0\ |^{2^4}$$

$$As\ n \to \infty, if\ |\ z_0\ | > 1, |\ z_n\ | \to \infty$$

$if\ |\ z_0\ | < 1, |\ z_n\ | \to 0.$

$If\ |\ z_0\ | = 1, z_0 =$ $\qquad Then\ z_{n\to 1}$
if k is of the form

$\frac{a}{2^b}$, a, b are integers and z_n diverges if k is not of this form Thus if Re $x_0 > 0, then\ x_n \to 1$

Re $x_0 < 0, then\ x_n \to -1$

Re $x_0 = 0, then\ x_n diverges.$

Q2. **If $C_0, C_1, C_2, \ldots\ldots, C_n$are real numbers satisfying $C_0 + C_1 + \cdots + C_n = 0$, Show that $C_0 + 2c, x + \cdots + (n+1)C_n x^n = 0$ has at least one real roof.**

Sol. : If f(x)= $C_0 + 2c, x + \cdots + (n+1)C_n x^n$

$$\int_0^1 f(x)dx = C_0 + C_1 + \ldots\ldots + C_n = 0$$

By the mean value theorem of integral calculus, there exist on (0 <f < 1) such that f(f) = 0.

So that f(x) = 0 has at least one real roof.

Q3**. Prove that the product of four consecutive positive integers cannot be a perfect square on a perfect cube.**

Sol. :Let the form integers be n – 1, n, n + 1, n + 2 so that P = $(n-1)n(n+1)(n+2) = (n^2+n-2)(n^2+n) = (n^2+n-1)^2 - 1$. If P is a perfect square, then P and (n^2+n-1) would be two consecutive integers., both of which one square, which is impossible. As such P cannot be a perfect square.

Now when n = 1, p = 0 and when n = 2, p = 24 which are not perfect cubes. So we take n > 2.

Again out of two integers n and n+1, only one can be odd cubes, case (i) Let n be odd, then n-1, n +1, are even and so n is relatively prime to both n- 1 and n +1 .

Again it is also relatively prime to n +2. Since 2m + 1 and 2m +3 cannot have a common factor because if they had it would divide their difference viz. 2. As such it is relatively prime to (n- 1) (n +1) (n +2).

Now, P = [n] [(n- 1) (n+ 1) (n+ 2)] since the two factors in squares brackets are relatively prime.

P is a perfect cube if (n- 1) (n+ 1) (n+ 2)= $n^3 + 2x - x - 2$ is a perfect cube, but if n > 2.

$n^3 < n^3 + 2n(n-2) + 3n - 2$

$(n+1)^3 > n^3 + 2n^2 - n - 2$, so that we have a cube lying between two consecutive cubes which is not possible. As such when n is odd. P cannot be a perfect cube.

Care (ii) let n be even so that (n+ 1) and (n- 1) are odd and n+ 2 is a consecutive even term so that if p is a perfect cube then so will be n (n -1) (n+ 2)= $n^3 + n^2$ and if n > 2 , $n^3 < n^3 + n^2 + 2n < (n+1)^3$

So that if p is a perfect cube., we shall again have a cube numbers lying between two consecutive cubes which is again impossible.

Thus p cannot be a perfect cube whether n is even on odd.

Q4. **Prove that if Z_1, Z_2 represent two vertices of an equilateral triangle in the Argand plane, then the third vertex is given by $-wz_1 - w^2 z_2$ where w is a cube root of unity.**

Sol. : If z_1, z_2, z_3 are the vertices of the equilateral triangle, then since the sides are of equal length.

$$|\ z_3 - z_1\ | = |\ z_2 - z_1\ | = |\ z_3 - z_2\ |\ so\ that\ if\ z_2 - z_1 = ae^{i\theta}$$

$z_3 - z_1 = ae^{i\theta \pm \frac{\pi}{3}}$.

So that $\frac{z_3 - z_1}{z_3 - z_1} = e^{\pm i\frac{\pi}{3}} = k$. Where k is a root of the equation $k^2 - k^1\left(2\cos\frac{\pi}{3}\right) + 1 = 0\ or, k^2 - k + 1 = 0$

So that k = -w where w is a imaginary cube root of unity and $1+\omega+\omega^2 = 0$

$$z_3 = kz_2 + z_1(1-k) = \omega z_2 + z_1(1+\omega) = -\omega z_2 - \omega^2 z_1$$

Since w, w^2 are two imaginary cube roots of unity, the third vertex is either$-\omega z_2 - \omega^2 z, or - \omega^2 z_2 - \omega z_1$.

Q5. **Find all pairs (m, n) of integer larger then > 1 such that | $p^m - q^n$ |= 1, where p and q are primes.**

Sol. : If is immediate that not both p and q are odd, for this would imply that $p^m - q^n$ is even. So let that q =2. We will show, by using only algebraic identities of this section, that the only solution is that formal. Suppose m and n are larger than 1 and that $|p^m - 2^n| = 1$. It cannot be the case that m and n are both even for if m = 2r and n = 2s, then

$1 = |p^m - 2^n| = |p^{2r} - 2^{2s}| = |p^r - 2^s||p^r + 2^s|$ and this is impossible ($\because p^r + 2^s > 1$). Then you do yourself.

Q6. **Evaluate $\sum_{n=1}^{\infty} 3^{n-1}\ sin^3\left(\frac{x}{3^n}\right)$**

Sol. : Using the moivre's theorem,

$\sin 3\theta = I_m(e^{3i\theta}) = I_m\left(\left(e^{i\theta}\right)^3\right) = I_m[cos\theta + isin\theta]^3 = I_m[cos^3\theta + 3cos^2\theta isin\theta + 3cos\theta i^2 sin^2\theta + i^3 sin^3\theta]$

$= 3cos^2\theta sin\theta - sin^3\theta = 3[(1 - sin^2\theta)sin\theta] - sin^3\theta$

$= 3sin\theta - 4sin^3\theta$. It follows that $sin^3\theta = \frac{3}{4}sin\theta - \frac{1}{4}sin3\theta$. Thus, $S_k =$

$\sum_{n=1}^{k} 3^{n-1}.sin^3\left(\frac{x}{3^n}\right) = \sum_{n=1}^{k} 3^{n-1}\left[\frac{3}{4}\sin\left(\frac{x}{3^n}\right) - \frac{1}{4}\sin\left(\frac{x}{3^{n-1}}\right)\right]$. Then do yourself.

Q7. **Find all positive continuous functions f(x) defined in [0, 1] for which $\int_0^1 f(x)de = 1$, $\int_0^1 x\,f(x)dx = a$, $\int_0^1 x^2 f(x)dx = a^2$.**

Sol. : Multiply these equations by $a^2, -2a$ and 1 respectively and odd we get .

$\int_0^a (a-x)^2 f(x)\,dx = 0$ and this cannot be satisfied if f(x) is positive and continuous in [0, 1].

As such there is no positive function satisfying the conditions of the problem.

Q8. **Find all twice differentiable real valued functions with domain the set of all real numbers and satisfying the functional equation $f^2(x) - f^2(y) = f(x+y)f(x-y)$.**

Sol. : Putting x = 0, y = 0 we get f(0) = 0.

Differentiating the given equation first partially with respect to x and then with respect to y, we get

$2f(x)f'(x) = f'(x+y)f(x-y) + f(x+y)f'(x-y)$

$0 = f''(x+y)f(x-y) - f'(x+y)f'(x-y) + f'(x+y)f'(x-y) - f(x-y)f''(x-y)$

$= f''(x+y)f(x-y) - f''(x-y)f(x+y)$.

$If\ x+y = \mu, x-y = v : this\ given$

$f''(\mu)f(v) - f''(v)f(\mu) = 0$

$or, f''(\mu)/f(\mu) = \frac{f''(\mu)}{f(v)}\ so\ that\ \frac{f''(\mu)}{f(\mu)} = c.$

Which gives a using f(0) = 0

f(μ) = A sin mu, C = m^2

= A sin mu, C= $-m^2$

= Au C = 0

There include the solution f(u) ≡ 0 when A ≡ 0.

Q9. **Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}-1=0$. Interprete your result.**

Sol. : Differentiating we get

$$\frac{x^2}{a^2+\lambda}+\frac{yp}{b^2+\lambda}=0; p=\frac{dy}{dx}.$$

$$So\ that\ \frac{a^2+\lambda}{\frac{1}{py}}+\frac{b^2+\lambda}{-\frac{1}{x}}=-\frac{1}{xy(xp-y)}$$
$$=\frac{a^2-b^2}{\frac{1}{py}+\frac{1}{x}}$$

So that the differentiable equation the family is

$$(xp-y)(x+py)=(a^2-b^2)py$$

To get the orthogonal trajectories, we replace p by $-\frac{1}{p}$, so that the differential equation of the family of orthogonal trajectories is

$$\left(-\frac{x}{p}-y\right)\left(x-\frac{y}{p}\right)=(a^2-b^2)\left(-\frac{y}{p}\right)$$

$$or,\quad (x+py)(xp-y)=(a^2-b^2)py$$

Which is the same as the original family, so that the family is self-orthogonal.

Let $a^2>b^2$, then so long as $\lambda>-b^2$, the family represents ellipses. When λ lies between $-b^2\ and\ -a^2$, it represents hyperbolas with the same face and when $\lambda>-a^2$. Those become imaginary ellipse. Every real ellipse intersects every hyperbola at right angles at every point of intersection.

Q10. Let p be a point in the interior of ΔABC and r_1, r_2, r_3 denote the distance from p to the sides a_1, a_2, a_3 of the triangle respectively. If R denote the circumradius of ΔABC. Show that

$$\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3}\le\frac{1}{\sqrt{2R}}({a_1}^2+{a_2}^2+{a_3}^2)^{\frac{1}{2}}$$

with equality if and only if Δ ABC is equilateral and p is the in centre.

Sol. : By the Cauchy-schwarz-inequality.

$\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3}=\sqrt{a_1r_1}\sqrt{\frac{1}{a_1}}+\sqrt{a_2r_2}\sqrt{\frac{1}{a_2}}+\sqrt{a_3r_3}\sqrt{\frac{1}{a_3}}\le(a_1r_1+a_2r_2+a_3r_3)^{\frac{1}{2}}\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right)^{\frac{1}{2}}$ with equality if and only if $\frac{\sqrt{a_1r_1}}{\sqrt{\frac{1}{a_1}}}=\frac{\sqrt{a_2r_2}}{\sqrt{\frac{1}{a_2}}}=\frac{\sqrt{a_3r_3}}{\sqrt{\frac{1}{a_3}}}$, or equivalently, if and only if ${a_1}^2r_1={a_2}^2r_2={a_3}^2r_3$.

$$\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3}$$
$$\le\left(\frac{a_1\,a_2\,a_3}{2R}\right)^{\frac{1}{2}}\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right)^{\frac{1}{2}}$$
$$=\left(\frac{a_1\,a_2\,a_3}{2R}\right)^{\frac{1}{2}}\times\left(\frac{a_2a_3+a_3a_1+a_1a_2}{a_1a_2a_3}\right)^{\frac{1}{2}}$$
$$=\frac{1}{\sqrt{2}R}$$

$$(a_2a_3+a_3a_1+a_1a_2)^{\frac{1}{2}}$$

Now, again by the Cauchy-Schwarz inequality,

$$a_2a_3 + a_3a_1 + a_1a_2 \le (a_2{}^2 + a_3{}^2 + a_1{}^2)^{\frac{1}{2}} (a_3{}^2 + a_1{}^2 + a_2{}^2)^{\frac{1}{2}}$$

$= (a_1{}^2 + a_2{}^2 + a_3{}^2)^{\frac{1}{2}}$ with equality if and only of $\frac{a_2}{a_3} = \frac{a_3}{a_1} = \frac{a_1}{a_2} = \frac{a_2+a_3+a_1}{(a_3+a_1+a_2)} = 1$ or equivalently if and only if $a_1 = a_2 = a_3$.

Thus, we have $\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3} \le \frac{1}{\sqrt{2R}}(a_1{}^2 + a_2{}^2 + a_3{}^2)^{\frac{1}{2}}$ with equality if and only if $a_1{}^2r_1 = a_2{}^2r_2 = a_3{}^2r_3$ and $a_1 = a_2 = a_3$, i.e. $a_1 = a_2 = a_3$ and $r_1 = r_2 = r_3$.

MODEL TEST PAPER - 5

Q1. Show that $x^n + y^n = z^n$ where n is an nteger >2 has no solution in integers x, y, z with 0 ≤ x ≤ n; 0 ≤ y ≤ n.

Sol. : We assume that without loss of generality x ≤ y, then z is an integer > y so that

$$z^n > (y+1)^n = y^n + ny^{n-1} + nc_2y^{n-2} + \cdots + \cdots > y^n + ny^{n-1} > y^n + y^n \; when\; n > y.$$

$> x^n + y^n. since\; y > x.$

So that $z^n > x^n + y^n\; or, z^n \ne x^n + y^n$.

Q2. Suppose x_1 *and* x_2 be the roots of the equation $\boldsymbol{x^2 - (a+d)x + (ad-bc) = 0}$. *Show that* $x_1{}^3$ *and* $x_2{}^3$ are the roots of

$$\boldsymbol{y^2 - \left(a^2 + d^2 + 3abc + 3bcd\right)y + (ad - bc)^3 = 0.}$$

Sol. : We know that $x_1 + x_2 = a + d, x_1x_2 = ad - bc$

$\because (x_1+x_2)^3 = x_1{}^3 + 3x_1{}^2x_2 + 3x_1x_2{}^2 + x_2{}^3, we\; have$

$$x_1{}^3 + x_2{}^3 = (x_1+x_2)^3 - 3x_1{}^2x_2 - 3x_1x_2{}^2 = (a+d)^3 - 3x_1x_2(a_1 + x_2)$$

$$= (a+d)^3 - 3(ad-bc)(a+d) = (a+d)(a^2 + 2ad + d^2 - 3ad + 3bc)$$

$$= (a+d)(a^2 - ad + d^2 + 3bc) = a^3 + d^3 + 3abc + 3bcd.$$

Furthermore, $x_1{}^3 + x_2{}^3 = (ad-bc)^3$, and that the proof is complete.

Q3. **Let f is a function on the positive integers which satisfies f(2k) = 2f(k)- 1, f(k +1) = 2f(k) + 1 and also an arbitrary positive integer whose binary representation is**

$\boldsymbol{a = a_na_{n-1} \ldots a_2a_1a_0 = a_n2^n + a_{n-1}2^{n-1} + \cdots + a_12 + a_0}$. ***Show that*** $\boldsymbol{f(a) = b_n2^n + b_{n-1}2^{n-1} + \cdots b_12 + b_0}$, ***where*** $\boldsymbol{b_1 = 1, a_i = 1; = -1, a_i = 0}$.

Sol. : We will induct on the number of digits in the binary representation of a, The result is true for a = 1, so suppose it holds whenever a has lower that k +1 digits. Now consider an integer a with k+ 1 digits (in base 2). Say

$a = a_ka_k - 1 \ldots. a_2a_1a_0. If\; a_0 = 0, then$

$a = 2(a_ka_k - 1 \ldots. a_1). f(a) = 2f(a_k \ldots. a_1) - 1 = 2[b_k2^{k-1} + \cdots + b_22 + b_1] - 1 = b_k2^k + \cdots + b_22^2 + b_1{}^2 + b_0$

And the result holds. If $a_0 = 1, then$

$a = 2(a_ka_k - 1 \ldots. a_1) + 1. f(a) = 2f(a_k \ldots. a_1) + 1$

$2\left(p_k 2^{k-1} + \cdots + b_1\right) + 1 = b_k 2^k + \cdots + {b_1}^2 + b_0$ and again the result holds.

Q4. **Let 0 <** $\boldsymbol{x_i} < \pi, i = 1, \ldots, n$ *and set* $x = \frac{\mathbf{1}}{\boldsymbol{n}}(\boldsymbol{x_1} + \boldsymbol{x_2} + \cdots + \boldsymbol{x_n})$.

Prove that $\frac{\boldsymbol{n}}{\boldsymbol{\Pi}}\left(\frac{\sin x_i}{x_i}\right) \leq \left(\frac{\sin x}{x}\right)^{\boldsymbol{n}}$

Sol. : The prob. Is equivalent to proving that

$$\sum_{i=1}^{n} \log \frac{\sin x_i}{x_i} \leq n \log \frac{\sin x}{x}.$$

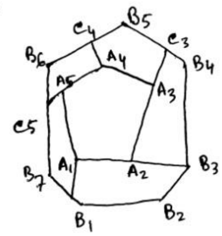
Consider the function f(t) = $\log \frac{S\,int}{t}$. It is straight forward matter to show that f is concave $(f''(t) < 0)$ *on the interval* $(0, \pi)$. So,

$f\left(\frac{x_1+x_2}{2}\right) \geq \{f(k_1) + f(k_2)\}$. In a manner completely analogous it follows that $f\left(\frac{x_1+\cdots x_n}{n}\right) \geq \frac{f(x_1)+\cdots f(x_n)}{n}$. Direct substitution into this inequality completes the proof.

$$\log \frac{\sin x}{x} \geq \frac{1}{n}\left(\log \frac{\sin x_1}{x_1} + \cdots + \log \frac{\sin x_n}{x_n}\right).$$

Q5. **An m- sided polygon is inside an n-side polygons (n > m). Prove that perimeter of the m- sided polygon is loss than the perimeter of the n-sided polygon.**

Sol. : Consider the fig. where the pentagon is inside a heptagon.



$A_1C_1 + C_1B_2 + B_2C_2 > A_1A_2 + A_2C_2$

$A_2C_2 + C_2B_3 + B_2B_4 + B_4C_2 > A_2A_2 + A_2C_3$

$A_3C_2 + C_2B_5 + B_5C_4 > A_2A_4 + A_4C_4$

$A_4C_4 + C_4B_6 + B_6B_7 + B_7C_5 > A_4A_5 + A_5C_5$

$A_5C_5 + C_5B_1 + B_1C_5 > A_5A_1 + A_1C_1$

Adding, we find that the perimeter of the outer polygon > perimeter of the interior polygon. The Proof is quite general and holds in the general care.

Q6. **Let** $\boldsymbol{A_i}(\boldsymbol{x_i}, \boldsymbol{y_i})(\boldsymbol{i} = \mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{k})$ **be the k points in a plane and log P(x, y) and point in the plane satisfying** $\boldsymbol{PA_1} + \boldsymbol{PA_2} + \cdots + \boldsymbol{PA_k} = \boldsymbol{k}$. **Show that the equation of the locus of P is f(x, y) = k where f(x, y) is a convex function of x and y. Show also that if** $\boldsymbol{Q_1}, \boldsymbol{Q_2}, \ldots, \boldsymbol{Q_k}$, **one the angles, which**

$\boldsymbol{PA_1}, \boldsymbol{PA_2}, \ldots, \boldsymbol{PA_k}$ **make with the normal to this locus at a point P, then** $\sum_{i=1}^{k} \sin \boldsymbol{Q_i} = \mathbf{0}$.

Discuss the special cases when k = 1, 2,.

Sol. : The equation of the locus is given by f(x, y)

$$= \sum_{i=1}^{k} \{(x - x_i)^2 + (y - y_i)^2\}$$

$$So\ that\ \frac{df}{dx} = \sum_{i=1}^{k} (x - x_i)[(x - x_i)^2 + (y - y_i)^2] - y_2$$

$$\frac{d^2f}{dx^2} = \sum_{i=1}^{k}[(x-x_i)^2 + (y-y_i)^2]^{-\frac{1}{2}} - (x-x_i)^2\,[(x-x_i)^2 + (y-y_i)^2]^{\frac{3}{2}}$$

$$= \sum_{i=1}^{k}[(x-x_i)^2 + (y-y_i)^2]^{-\frac{3}{2}}\,[(y-y_i)^2]$$

$$= \sum_{i=1}^{k}\{A_i(y-y_i)\}^2$$

$$and\,\frac{d^2f}{dxdy} = -\sum_{i=1}^{k}[(x-x_i)^2 + (y-y_i)^2]^{\frac{3}{2}}\,(x-x_i)(y-y_i)$$

$$= -\sum_{i=1}^{k}A_i(x-x_i)A_i\,(y-y_i).$$

$$where\; A_i = \{(x-x_i)^2 + (y-y_i)^2\}^{\frac{3}{4}}$$

By using Cauchy-Schwarz is equality we find

$$\frac{d^2f}{dx^2} > 0, \frac{d^2f}{dy^2} > 0, \frac{d^2f}{dx^2} - \frac{d^2f}{dy^2} - \left(\frac{d^2f}{dxdy}\right)^2 > 0.$$

So that f(x, y) is a convex function and f(x, y) = k is closed convex curve.

When k = 1, it reduces to the ordinary circle.

When k = 2, it reduces to the ordinary ellipse which is defined as the locus of a point, the sum of whose distances from two fixed points in a plane (called foci is constant).

When k is a positive integer > 2, its locus will be called a k-ellipse.

Since $r_1 + r_2 + \cdots + r_k = k$

$$\frac{dr_1}{ds} + \frac{dr_2}{ds} + \cdots + \frac{dr_k}{ds} = 0.$$

$$\cos\phi_1 + \cos\phi_2 + \cdots + \cos\phi_k = 0$$

Where $\phi_1, \phi_2, \ldots, \phi_k$ are the angles which there radius vectors make with the tangent to the k-ellipse at P. It $\theta_1, \theta_2, \ldots, \theta_k$ are the angle which there radius vectors make with the normal at P, then we get,

$$\sin\theta_1 + \sin\theta_2 + \cdots + \sin\theta_k = 0.$$

If k = 1, it gives $\sin\theta = 0$ i.e. the normal to a circle at every point coincides with the radius.

If k = 2, it gives $\sin\theta_1 + \sin\theta_2 = 0$ i.e. normal at any point of an ellipse makes equal angles with the lies the point to the foci.

Q7. **Find the point with in a triangle, the sum of which distances from the three vertices is minimum. Find also the minimum distance in terms of the lengths of the sides of the triangle.**

Sol. : (Draw the figure yourself) Suppose we put three small pulleys O at A, B, C and a smooth ring at P. A string passes through the ring and over pulleys and equal weights are attached at the ends of the strings hanging over pulleys at A. B. C. in the position of equilibrium, the potential energy is minimum so that sum of the lengths of the three strings within the triangle is minimum.

However if the string is smooth, the tensions in the three string are equal and by lamis the orem, sines of the three angles APB, BPC, CPA are equal, so that each of three angles is 120°.

Thus the point P which is chosen such that PA + PB + PC is minimum is such that AB, BC, CA subtend equal angles of 120° at point P. The point P will be called steiner's point of the triangle.

There are other methods of solving this problem, but we have given this method to illustrate Horos physical principles can also help us in solving purely mathematical problems.

If PA = x, PB = y, PC = z, then from the figure.

$$a^2 = y^2 + z^2 + y^2 \| \ b^2 = z^2 + x^2 + xz \| \ c^2 = x^2 + y^2 + xy$$

$$= \left(y + \frac{z}{2}\right)^2 + \frac{3z^2}{4} | = \left(z + \frac{x}{2}\right)^2 + \frac{3x^2}{4} | = \left(x + \frac{y}{2}\right)^2 + \frac{3y^2}{4}$$

Also if area of the triangle is $\mathit{\Delta}$, then

$$\mathit{\Delta} = \frac{1}{2} yz \sin 120° + \frac{1}{2} zx \sin 120° + \frac{1}{2} xy \sin 120°$$
$$= (xy + yz + zx) \times \frac{\sqrt{3}}{4}$$

$$a^2 + b^2 + c^2 = 2(x^2 + y^2 + z^2) + (xy + yz + zx)$$

$$= 2\,\{(x + y + z)^2 - 2(xy + yz + zx)\} + (xy + yz + zx)$$

$$= 2(x + y + z)^2 - 3(xy + yz + zx)$$

$$= 2(x + y + z)^2 - 3 \times \frac{4\mathit{\Delta}}{\sqrt{3}}$$

$$or, 2(x + y + z)^2 = a^2 + b^2 + c^2 + 4\sqrt{3}\mathit{\Delta}$$

$$L = x + y + z =$$

$$\left[\frac{\left(a^2 + b^2 + c^2 + 4\sqrt{3}\mathit{\Delta}\ \right)\sqrt{s(s-a)(s-b)(s-c)}}{2}\right]^{\frac{1}{2}}$$

$$where\ s = \frac{a + b + c}{2}$$

Which gives the minimum sum of distance from A, B, C.

Q8. I**f B(m, n) is a beta function with parameters m, n then Prove that**

$$\boldsymbol{\log B(m, n) \leq (m-1) \frac{d}{dm} in\ B(m, n) + (n-1) \frac{d}{dn} in\ B(m, n) and\ if\ m, n\ are\ integers}.$$

$$\boldsymbol{\log \frac{(m-1)!\,(n-1)!}{(m+n-1)!} \leq -(m-1)}$$

$$\boldsymbol{\left[\frac{1}{m+n-1} + \frac{1}{m+n-2} + \ \dots 1m\right]}$$

$$\boldsymbol{-(n-1)\left[\frac{1}{m+n-1} + \frac{1}{m+n-2} + \cdots \frac{1}{n}\right]}$$

Sol. : The entropy of any probability distribution with density function f(x) over [0, 1] is given by

$$-\int_0^1 f(x) \log f(x)\, dx$$

The entropy is maximum for the uniform distribution. So that for any probability density function f(x). We have

$$-\int_0^1 f(x) \log f(x)\, dx \leq -\int_0^1 1. \log 1\, dx$$

$$or, \int_0^1 f(x) \log f(x)\, dx \geq 0$$

We can get any number of inequalities by taking different density functions over the interval [0, 1). In particular if we take.

$$f(x) = \frac{1}{B(m, n)} x^{m-1} (1-x)^{n-1}$$

$$we\ get, \int_0^1 f(x) \left[\log \frac{1}{B(m, n)} + (m-1) \log x + (n-1) \log(1-x)\right]$$

$$or, -\log B(m,n) + \frac{m-1}{B(m,n)}\int_0^1 x^{m-1}(1-x)^{n-1}\log x\,dx + \frac{n-1}{B(m,n)}\int_0^1 x^{m-1}(1-x)^{n-1}\log(1-x)\,dx \geq 0.$$

$$Now, B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$$

Differentiating partially with respect to m, n we get.

$$\frac{d}{dm}B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\log x\,dx$$

$$\frac{d}{dn}B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\log(1-x)\,dx$$

So that the above inequality becomes

$$-\log B(m,n) + (m-1)\frac{d}{dm}\log B(m,n) + (n-1)\frac{d}{dn}\log B\ (m,n) \geq$$

Which was the inequality to be proved, so that

$$log\frac{T(m)T(n)}{T(m+n)} - (m-1)\left\{\frac{T'(m)}{T(m)} - \frac{T'(m+n)}{T(m+n)}\right\} - (n-1)\left[\frac{T'(n)}{T(n)} - \frac{T'(m+n)}{T(m+n)}\right] \leq 0$$

$$Now\ T(N) = (N-1)T(n-1)$$

$$so\ that\ \frac{T'(n)}{T(n)} = \frac{1}{N-1} + \frac{T'(N-1)}{T(n-1)}$$

$Thus,$

$$\frac{\Gamma'(m+n)}{\Gamma(m+n)} - \frac{\Gamma(m)}{\Gamma(m)} = \frac{1}{m+m-1} + \frac{1}{n+m-2} + \cdots + \frac{1}{m}$$

$$\frac{\Gamma'(m+n)}{\Gamma(m+n)} - \frac{\Gamma(n)}{\Gamma(n)} = \frac{1}{m+m-1} + \frac{1}{n+m-2} + \cdots + \frac{1}{n}$$

Substituting there values, we get the second inequality to be proved.

Q9. **Does $y = x^{x^{x^x}}$ represent a function of x? Can you find its derivative and area under this curve?**

Sol. : $y = x^x$ is a function of x for all positive values of x. However y $=x^{x^x}$ has two interpretations viz. $=x^{(x)^{(x)}}$ or $(x^2)^x$ and these are different. Similarly if x is raised to power which consists of n x's, we can have $\frac{(2n-2)!}{(n-1)}$ we can restore uniqueness by defining the function recursively by $u_0(x) = 1, u_1(x) = x^{u_0(x)}, u_2(x) = x^{u_1(x)}, \ldots . u_n(x) = x^{u_{n-1}(x)}$ … ..

$So\ that \log u_n(x) = u_{n-1}(x)\log x\,; \log u_{n-1}(x) = u_{n-1}(x)\log x.$

$$or\ \log\frac{u_n(x)}{u_{n-1}(x)} = \{u_{n-1}(x) - u_{n-2}(x)\}\log x$$

$If\ x > 1, u_1(x) > x, u_2(x) > u, u_3(x) > u_2(x) \ldots u_n(x) > u_{n-1}(x)$ … and the sequence can be a divengent sequence. If $x < 1, u_0(x) = 1, u_1(x) = x < 1,$

$$\log\frac{u_2(x)}{u_1(x)} = \{u_1(2) - u_0(x)\}\log x$$
$$= (x-1)\log x > 0$$

$So\ that\ u_2(x) >\ u_1(x).$

$$Again \log\frac{u_3(x)}{u_2(x)} = \{u_2(x) - u_1(x)\}\log x < 0.$$

$So\ that\ u_3(x) <\ u_2(x).$

In general $u_n(x) \gtrless u_{n-1}(x)$ according as $u_{n-1}(x) \gtrless\ u_{n-1}(x)$.

Here we have made use of the fact that

$$\log\frac{u_n(x)}{u_{n-2}(x)} = \{u_{n-1}(x) - u_{n-2}(x)\}\log x$$

So that $u_n\ \gtrless\ u_{n-2}(x)$ according as $u_{n-1}(x) \gtrless u_{n-1}(x)$

Thus the sequence $\{u_{2n}(x)\}$ is monotonic decreasing and the sequence $\{u_{2n} - (k)\}$ is monotonic increasing.

Also $\frac{u_n(x)}{u_{n-2}(x)} =\ x^{u_{n-1}(x)} - u_{n-3}(x)$

If the sequence $\{u_n(x)\}$ convenges to y, then

$$y =\ x^x \Rightarrow \log y = y\log x \Rightarrow \frac{1}{y}\frac{dy}{dx}$$
$$= \log x\ \frac{dy}{dx} + \frac{y}{x}$$

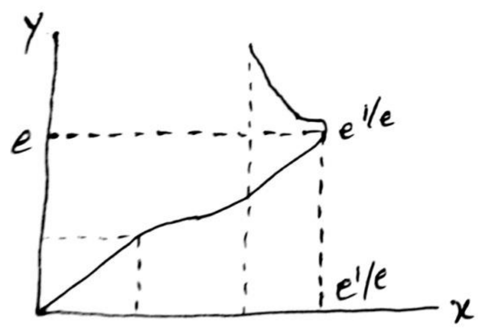
$$\Rightarrow \frac{dy}{dx} = \frac{\frac{y}{x}}{\frac{1}{y} - \log x} = \frac{\frac{y^2}{x}}{1 - y\log x} = \frac{\frac{y^2}{2}}{1-\log y}$$

$$\operatorname*{Lt}_{y\to 0}\frac{\log y}{y} = -\infty,\ \operatorname*{Lt}_{y\to 0} y^{\frac{1}{4}} = 0.$$

$$\operatorname*{Lt}_{y\to\infty}\frac{\log y}{y} =\ \operatorname*{Lt}_{y\to\infty}\frac{1}{y} = 0 \Rightarrow Lt\ \ y^{\frac{1}{4}} =\ e^0 = 1$$

Thus as y goes from 0 to 1 x goes from 0 to 1 and as y goes from 1 to∞ , x goes from 1 to $c^{\frac{1}{e}}$ and then from $c^{\frac{1}{e}}\ to\ 1$.

For every value of y, there is a unique value of x, but for every value of x > 1, but <$c^{\frac{1}{e}}$, there are two values of y at one of which the derivative is positive and at the other the derivative is negative. When x = <$c^{\frac{1}{e}}$ there is only one value of y i.e. c and at this point the tangent is parallel to the axis of y.



However for every value of x, between 0 and 1, there is only one value of y lying between 0 and 1 and the derivative at all points is positive and we are concerned with only the values of y when x lies between 0 and 1.

Now area under the curve up to 'x' < 1 is

$$\int_0^x y\ dx = xy - \int_0^y x\ dy = xy - \int_0^y y\ dy^{\frac{1}{y}}$$
$$=\ y^{\frac{1}{4}+1} - \int_0^y y^{1/y}\,dy.$$

Which can be numerically evaluated.

Q10. **Let E be the ellipse with centre at again O whose major and minor axis are 2a and 2b respectively. If θ be the acute angle of which E**

is cut by a circle with centre at the origin (i.e. θ is the acute angle of intersection). Prove that the maximum possible value of θ is

$$\tan\left(\frac{a^2-b^2}{2ab}\right).$$

Sol : $tangent \equiv xx_1 + yy_1 = r^2$

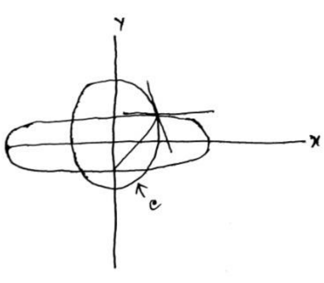
$$slope \equiv x_1 + y_1\frac{dy}{dx} = 0, \Rightarrow \frac{dy}{dx} = -\frac{x_1}{y_1} = m_1$$

$$tangent\ E \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1, \Rightarrow \frac{x_1}{a_1} + \frac{b_1}{b^2}.\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\frac{x_1}{a^2}}{\frac{y_1}{b^2}} = -\frac{x_1}{y_1}.\frac{b^2}{a_1} = m_{2.}$$

Now, $\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right| = \left|\frac{-\frac{x_1}{y_1}+\frac{x_1b^2}{y_1a^2}}{1+\frac{\left(\frac{x_1}{y_1}\right)b^2}{a}}\right|$

$$= \left|\frac{\frac{a^2-b^2}{a^2+\left(\frac{xy}{y_1}\right)^2 b^2}}{-\left(\frac{a_1}{y_1}\right)}\right|$$



Now, for max angle $\frac{x_1}{y_1} = \frac{a}{b}$.

$$\therefore \tan\theta = \left|\frac{\frac{a^2-b^2}{a^2+\frac{a^2}{b^2}.b^2}}{-\frac{a}{b}}\right| = \left|\frac{a^2-b^2}{-ab-ab}\right| = \frac{a^2-b^2}{2ab}$$

$$\therefore \theta = \tan^{-1}\left(\frac{a^2-b^2}{2ab}\right).$$

MODEL TEST PAPER – 6

Q1. Find all solutions in integers of the equations : $x^8 + y^8 = 1984$; $\mu^3 + v^6 + \omega^8 = 1985$.

Hence on otherwise find solution of $a^3 + b^3 = c^3 + d^3$ with a, b, c, d as different positive integers.

Sol. : $1^8 = 1$, but 1983 is not a perfect cube.

$2^8 = 256\ and\ 1984 - 256 = 1728 = 12^3$ is a perfect cube.

∴Only solution in integers of the first equation is x = 12, y = 2.

Next we try v= 1; ω= 1; v = 1; ω = 2; v = 1; ω= 3; v = 2; ω = 2; v = 1; ω = 2; v = 2; ω = 3.

We find that $12^3 + 2^8 + 1^6 = 1985\ and\ 10^3 + 2^8 + 3^6 = 1985$

$\therefore 12^3 + 2^8 + 1^6 = 10^3 + 3^6 + 2^8 = 1985$ give the only solutions.

From the above $12^3 + 1^6 = 10^3 + 3^6$

$or, 12^3 + 1^3 = 10^3 + 9^3 = 1729$

Q2. **For what real values of a does the $eq^x.ax^2t\ \ x + a - 1 = 0$ have two distinct real root α and β. Satisfying the inequality $\left|\frac{1}{\alpha} - \frac{1}{\beta}\right| > 1$ and 'a' belongs the domain of definition of the function**

$\boldsymbol{f(a) = \frac{1}{\left[a+\frac{1}{2}\right]} + \sqrt{a - \frac{11}{10}}}$**, |.| being the greatest integer.**

Sol. : $D > 0; 1 - 4a\,(a-1) > 0, \Rightarrow 1 - 4a^2 + 4a > 0.$

$\Rightarrow 4a^2 - 4a - 1 < 0, \frac{1-\sqrt{2}}{2} < \alpha < \frac{1+\sqrt{2}}{2}$

$| \left|\frac{1}{\alpha} - \frac{1}{\beta}\right| \, | > 1. \Rightarrow | \beta - \alpha | > | \alpha\beta |, \Rightarrow (\alpha+\beta)^2 - 4\alpha\beta > \alpha^2\beta^2$

∴ 0 < a < 615 for f(a) to be defined 0 ≤ a+ ½ ≮1.

∴ reqd. value of a, a $\left[\frac{11}{10}, \frac{6}{5}\right]$.

Q3. In a club of 80 numbers, 10 members play name of Tennis, Badminton and Cricket, 30 members play exactly one of these three games and 30 members play exactly two of these games. 45 members play at least one of the games among Tennis and Badminton, whereas 18 members play both Tennis and Badminton. Determine the number of Cricket playing members.

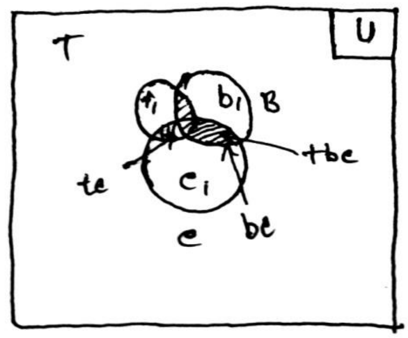
Sol. : Members playing either Tennis (T) or Cricket (C) or Badminton (B). Winner , no. of members playing exactly one of these three games = $t_1 + b_1 + c_1 = 30$.No. of members playing exactly two of there three games = tb +bc +ct = 30 and no. of members playing three games = tbc = 70 – 30- 30 = 10.

∵ 45 members are playing at least Tennis and Badminton.

$C_1 = 70 - 45 = 25 \; and \; bc + tc = 30 - tb = 30 - 18 = 12$

$[\because tb = 18(winner)]$

∴ No. of member playing Cricket = $C_1 + tc + tbc + b = 25 + 12 + 10 = 47$.



Q4. **Let $\boldsymbol{P_n(x)}$ be the polynomial$\boldsymbol{P_n(x) = 1 + 2x + 3x^2 + \cdots + (x+1)x^n}$. Show that $\boldsymbol{P_n(x)}$, has no real zero if n is even and exactly one real zero if m is odd and this zero lies between -1 and 0.**

Sol. : When x > 0, $P_n(x) > 0 \; and \; so \; P_n(x) = 0$ can have no positive real root.

Now, $P_n(x) = 1 + 2x + 3x^2 + \cdots + (x+1)x^n$

$xP_n(x) = x + 2x^2 + 3x^3 + \cdots + nx^n + (n+2)x^{n+1}$

$(1-x)P_n(x) = 1 + x + x^2 + \cdots + x^n - (n+1)x^{n+1}$

$$P_n(x) = \frac{1 - (n+2)x^{n+1} + (n+1)x^{n+2}}{(1-x)^2}$$

For negative values of x, $P_n(x)$ will vanish whenever

$f(x) \equiv 1 - (n+2)x^{n+1} + (n+1)x^{n+2} = 0.$

$f(x) \equiv 1 - (n-2)(-1)^{n+1}x^{n+1} + (n+1)(-1)^{n+2}x^{n+2}$

If n is even, there is no changes of sing in this expression and so there is no negative real root also. If n is odd , there is one change of sign there can be one negative real root.

In this care f(-1) = 1 – (n +2) - (n+ 1) = -2n – 2 <0, f(0) = 1 >0. As such when n is odd, the real root lies between 0 and -1.

Q5. **Show that the set S=** $\{.\mathbf{2}, \mathbf{2^3}, \mathbf{2^5}, \dots\dots\}$ **is such that sum of any subset of element cannot be a perfect square.**

Sol. : Let N = $2^{S_1} + 2^{S_2} + \cdots + 2^{S_k} = 2^{S_1}(1 + 2^{S_1 - S_2} + \cdots + 2^{S_k - S_1})$

Without loss of generality, we can assume $S_1 < S_2 < S_3 \dots < S_k$, so that the expression within breaketer is an odd number which may or may not be a perfect square.

Q6. Decide whether the following statements are true or false. There exist number $\boldsymbol{a_1, a_2, \dots, a_n}$ such that $\boldsymbol{f(x) \equiv a, \cos x + a_2 \cos 2x + \dots + a_n \cos nx} > 0$ for all x.

Sol. : Suppose the statement is true, then

$$\int_0^{2\Pi} (a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx) dx > 0.$$

$$or, \quad \left[a_1 \sin x + a_2 \frac{\sin 2x}{2} + \cdots + a_n \frac{\sin nx}{n}\right]_0^{2\Pi} > 0.$$

But L.H.S. = 0, so that there is a contradiction and the given statement is false.

Q7. Let f (x, y) be a function satisfying the functional equation f(x, y) = f(2x+ 2y; 2- 2x).For all real numbers x, y. Define g(x) by g(x) = f($\boldsymbol{2^x, 0}$). Prove that g(X) is a periodic function with period 12, i.e. show that g(x +12) = g(x).

Sol. : Using the functional equation again and again

$g(x) = f(2^x, 0) = f(2^{x+1}, -2^{x+1}) = f(0, -2^{x+3})$

$= f(-2^{x+4}, -2^{x+4}) = f(-2^{x+4}, 0) = f(-2^{x+7}, 2^{x+7}) = f(0, 2^{x+9}) = f(2^{x+10}, 2^{x+10})$

$= f(2^{x+12}, 0) = g(x + 12)$. So that g(x) is a periodic function with period 12.

Q8. **If a(x), b(x), c(x) and d(x) are polynomials is x, show that**

$$\boldsymbol{\int_1^x a(x)\ c(x) dx \quad \int_1^x b(x)\ d(x) dx - \int_1^x a(x) d(x)\, dx \int_1^x b(x)\, c(x) dx\ is\ divisible\ by\ (x - 1)^4}.$$

Sol. : Denote the expression in question by F(x). Notice that F(x) is a polynomial in x. Also notice that F(1) = 0 and therefore (x- 1) is a factor of F(x).

Because F is a polynomial, we know that $(x - 1)^4$ is a root of F(x) = 0 if and only if F(t)= 0. We can compute fundamental theorem.

$$F'(x) = ac \int_1^x bd + bd \int_1^x ac - ad \int_1^x bc - bc \int_1^x ad.$$

(Note that F'(x) = 0 and hence $(x - 1)^2$ is a root of F(x) = 0) the derivative F'' and F''' are done in a similar manner; it turns out that

F''' (1) = $[(ac)^1 bd + (bd)^1 ac - (ad)^1 bc - (bc)^1 ad]_{x=1} = 0$

This completes the proof.

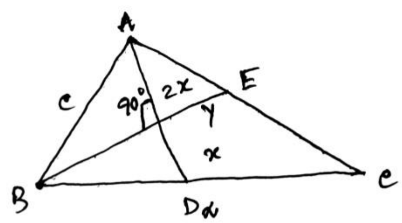
Q9. **The medians of two sides of a triangle meet at right angle. The two sides have lengths a and b units. Find condition on a and b for this to be possible and express that length of the third side is terms of a and b.**

Sol. : Let G be the centroid of the triangle since it trisects the medians.

AG = 2x, GD = x

BG = 2y, GE = y

Then using Pythagorean theorem, we get from the figure.



$4(x^2 + y^2) = c^2$

$4x^2 + y^2 = \frac{b^2}{4}, 4y^2 + x^2 = \frac{a^2}{4}$

Eliminating x^2, y^2 we get,

$$c^2 = 4\frac{a^2 + b^2}{20} = \frac{a^2 + b^2}{5}.$$

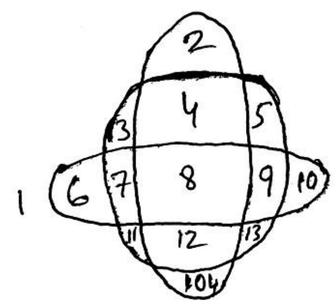
It is also clear that $c^2 < b^2, c^2 < a^2$ so that c is the smallest side and $\frac{a^2+b^2}{5} < a^2, \frac{a^2+b^2}{5} < b^2$, $or,\ b^2 < 4a^2, a^2 < 4b^2$,

$$\frac{b}{a} < 2, \frac{a}{b} < 2, \frac{1}{2} < \frac{b}{a} < 2.$$

Q10. **A number of elliptic curves are drawn in the planes any two of them intersecting in 4 points and no three of the curves are concurrent. Into how many regions would the plane be divided, if n elliptic curves are drawn?**

Sol. : One curve divides the plane into two regions. Two curves divide the plane into 6 region and three curves into 14 region, so that if (R (n) is the number of regions, then R(1) = 2, R(2) = 6, R(3) = 14.

Now, suppose n curves have already drawn and (x +1) curves is now drawn. It cuts each of the previous n curves in 4 distinct points and there 4n points divide its perimeter into 4n across.



Each of these cuts across one of the earthier R_n region resulting from the previous n curves dividing into two pieces gives rise to 4_n new regions so that.

$$R_{n+1} = R_n + 4_n$$

$$or, R_n = R_{n-1} + 4(n-1) = R_{n-2} + 4(n - 2) + 4(n-1) = \cdots$$

$$= R_2 + 4(2 + 3 + \cdots + \overline{n-1}) = R_1 + 4(1 + 2 + \cdots + n - 1)$$

$$= 2 + 4\left(\frac{(n-1)n}{2}\right) = 2n^2 - 2n + 2.$$

<u>**MODEL TEST PAPER - 7**</u>

Q1. The real number μ, ν, ω satisfy $0 < \mu < 1$; $0 < \nu < 1$; $0 < \omega < 1$ prove that at least one of the

3 numbers $\boldsymbol{\mu(1-\nu), \nu(1-\omega), \omega(1-\mu)}$ *is* $\leq \frac{1}{4}$.

Sol. : Let $P = \mu(1-\nu), Q = \nu(1-\omega), R = \omega(1-\mu)$

$Then\ PQR = \mu(1-\mu)\nu(1-\nu)\omega(1-\omega)$

$but \mu(1-\mu) \leq \frac{1}{4}, \nu(1-\nu) \leq \frac{1}{4}, \omega(1-\omega) \leq \frac{1}{4}$

$So\ that\ PQR\ \leq \left(\frac{1}{4}\right)^3.$

And that one of P, Q, R ≤ $\frac{1}{4}$.

Q2. **Find a six digits number which is multiplied by the factor 6, if the final 3 digit are removed and placed (without changing their order) at the beginning.**

So. : Let N be the number, A be the number consisting of the first 3 digits and B be the number consisting of last 3 digits.

Then N = 1000A+ B and 6N = 1000B + A

So that 1000B + A = 6(1000A+ B)

Or, 994B = 5999A

Or, 142B = 857A

We can chose B = 857, A = 142, since there are co-prime integers N = 142, 857.

Q3. **The numerical sequence $\boldsymbol{x_1, x_2}$ satisfies $\boldsymbol{x_1 = \frac{1}{2}, x_{k+1} = {x_k}^2 + x_k}$ for all natural number k. Find the integer part of the sum $\boldsymbol{\frac{1}{x_1+1} + \frac{1}{x_2+1} + \cdots + \frac{1}{x_{1000}+1}}$**

Sol. : $\frac{1}{x_{k+1}} = \frac{1}{x_k(x_{k+1})} = \frac{1}{x_k} - \frac{1}{x_{k+1}}$

$$\frac{1}{x_k+1} = \frac{1}{x_k} - \frac{1}{x_{k+1}}$$

$$\sum_{k=1}^{1000} \frac{1}{x_{k+1}} = \sum_{k=1}^{1000} \left(\frac{1}{x_k} - \frac{1}{x_{k+1}}\right) = \frac{1}{x_1} - \frac{1}{x_{1001}}$$

Now $x_{k+1}. > x_k > x_k(x_k)$ is a positive monotonic increasing sequence and $\left\{\frac{1}{x_k}\right\}$ is a positive monotonic decreasing sequence.

$$x_1 = \frac{1}{2}, x_2 = \frac{3}{4}, x_3 = \frac{21}{16}, x_k > 1, when\ k > 2$$

$\therefore 0 < \frac{1}{x_{1001}} < 1, \frac{1}{x_1} = 2.$

Integral part of $\frac{1}{x_1} = \frac{1}{x_{1001}}$ is unity.

In fact the integral part of $\frac{1}{x_1} = \frac{1}{x_k}$ is unity for all k > 2.

Q4. **Prove that in an acute angled trianglecos $\boldsymbol{A \cos B \cos C \leq \frac{1}{8}}$. Deduce that**

$$\boldsymbol{\frac{1 + \cos A + \cos B + \cos C}{2 \cos A \cos B \cos C} \geq 10}.$$

Sol. : Let y = $\log cos\ x\ , o < x < \frac{\Pi}{2}$

$$\frac{dy}{dx} = -\tan x\, , \frac{d^2y}{dx^2} = -sec^2 x < 0$$

⟹ y is a concave function.

$\log \cos A \cos B \cos C$ is also a concvave function and its maximum value subject to A + B + C = π arises when $A = B = C = \frac{\pi}{3}$.

$\cos A \cos B \cos C$ has its maximum value when $A = B = C = \frac{\pi}{3}$.

$\therefore \cos A \cos B \cos C \leq \frac{1}{8}$.

And the equality sign holds only for the case of the equilateral triangle.

$$\frac{1+\cos A+\cos B+\cos C}{\cos A \cos B \cos C}$$
$$=\frac{1}{\cos A \cos B \cos C}+\frac{\sec B \sec C+\sec C}{\sec A\ +\sec A \sec B}$$

Using the above inequality and the arithmetic geometric mean inequality.

$$\frac{1+\cos A+\cos B+\cos C}{\cos A \cos B \cos C}$$
$$\geq 8+3\sqrt[3]{sec^2B\ sec^2C\ sec^2A}$$

$$=8+3\frac{1}{(\cos A \cos B \cos C)^{\frac{2}{3}}} \geq 8+3.(8)^{\frac{2}{3}}$$
$$=20.$$

$\frac{1+\cos A+\cos B+\cos C}{2\cos A \cos B \cos C} \geq 0$

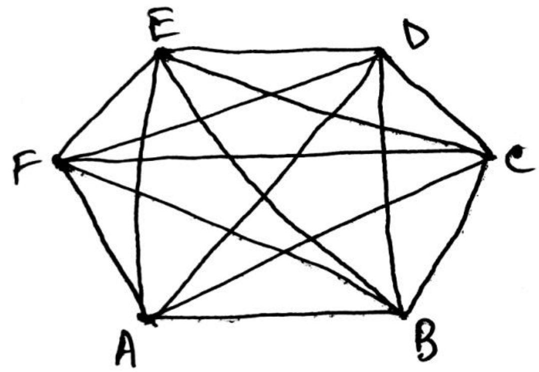
And the equality sign holds only for the equilateral triangle.

Q5. **Prove that if ABCDEF is a hexagon (not necessarily convex) such that AB ‖DE, BC ‖ EF and CD‖ FA, then area of triangle ACE = area of triangle BDF, where signed areas are used.**

Sol. : BC ‖ EF⟹ area of ΔCER = area of ΔBRF

CD ‖ FA ⟹ area of Δ CQA = area of ΔFQD

AB ‖ DE ⟹ area of Δ ADE = area of Δ DPB



$\Delta ADE = area\ of\ \Delta DPB$

If we add there three equalities and at ΔPQR to both sides, we get ΔACE = ΔBDF.

Q6. **Suppose f(x) be differentiable on [0, 1] with f(0)= 0 and f(1)= 1. For each positive integer n, so that there exist distinct points**

$x_1, x_2, \ldots, x_n$ in [0, 1] such that

$$\sum_{i=1}^{n} \frac{1}{f'(x_1)} = n.$$

Sol. : To help generate ideas, consider the case n = 1, we wish to find x_1 in [0, 1] such that

$\frac{1}{f'(x_1)} = 1.$

This is possible by the mean value theorem since on the interval [0, 1] there is a point x_1 such that $f'(x_1)$.

Consider the case n = 2. Consider the subintervals [0, x] and [x, 1] where x is save number between 0 and 1 yet to be determine.

By the mean value theorem, there is on x_1 in (0, x) and $x_2\ in\ (x, 1)$ such that

$$f'(x_1) = \frac{f(x) - f(0)}{x - 0} \; and \; f'(x_2) = \frac{f(1) - f(x)}{1 - x}$$

$$Thus \frac{1}{f'(x_1)} + \frac{1}{f'(x_2)} = 2$$

$$if \; and \; only \; if \; \frac{x}{f(x)} + \frac{1 - x}{1 - f(x)} = 2,$$

$x\big(1 - f(x)\big) + (1 - x)f(x) = 2f(x) - 2\big(f(x)\big)^2,$

$x - xf(x) + f(x) - xf(x) - 2f(x) + 2\big(f(x)\big)^2 = 0$

$x - 2x\,f(x) - f(x) + 2\big(f(x)\big)^2 = 0$

$x\big(1 - 2f(x)\big) - f(x)\big(1 - 2f(x)\big) = 0$

$[x - f(x)][1 - 2f(x)] = 0.$

Now had we chosen x in $(0, 1)$ so that $f(x) = \frac{1}{2}$ (this could be done by intermediate value theorem), the proof would be complete upon reversing the previous steps. With this background we can consider the care to an arbitrary positive integer x, let c_i be the smallest number in $[0, 1]$ such that f(c_i) $= \frac{i}{n}$ (the existence of this number is a consequence of the intermediate value theorem together with the assumption of continuity). Then

$0 < c_1 < c_2 < \cdots < c_{n-1} < 1. Define \; c_0 = 0 \; and \; c_n = 1$ and for each internal $(c_1 - 1, c_i) i = 1, 2, \ldots, n$. chose $x_i$such that $f'(x_1) = \frac{f(c_i) - f(c_i - 1)}{c_1 - c_{i-1}}$

(This can be done by the I mean-value theorem). Then

$f'(x_i) = \frac{\frac{1}{n} - \frac{i-1}{n}}{c_i - c_{i-1}} = \frac{1}{n(c_i - c_{i-1})}$ $so \; that \; \sum_{i=1}^{n} \frac{1}{f'(x_i)} = \sum_{i=1}^{n} n(c_i - c_{i-1}) = n.$

Q7. **Let f(x) be a polynomial with integer co-efficients it is known that f(b) – f(a) = 1 (whose a and b are integers.) Prove that a and b differ by unity.**

Sol. : $Let \; f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$

$Then \; f(b) - f(a) = c_1(b - a) + c_2(b^2 - a^2) + \cdots + c_n(b^n - a^n)$

$= (b - a)[c_1 + c_2(b + a) + \cdots + c_n(b^{n-1} + b^{n-2}a + \cdots + a^n)]$

$= (b - a)I$

Where I is an integer.

Since (b - a)I = 1, b - a can be ≠ 1. So that a and b differ by unity.

Q8. **Prove that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the distance between the centre and any normal does not exceed | a- b|. Find a point on the ellipse normal at which is at a distance of |a -b| from center.**

Sol. : Let $P(a\cos\theta, b\sin\theta)$ represents any pt. at on the given ellipse and the slope of the tangent at this pt is $\frac{dy}{dx} = -\frac{b^2 x}{ay} = -\frac{b\cot\theta}{a}$ at P. So, the slope of the normal at the pt. P is $\frac{a}{b}\tan\theta$, so that the equation of the normal is

$$y - b\sin\theta = \frac{a\tan\theta}{b}(n. a\cos\theta),$$

$$\Rightarrow \left(-\frac{a\tan\theta}{b}\right)$$

$x + y + (-b\sin\theta + a^2\sin\theta) = 0$ ……………….(i)
The distance from (0, 0) to the line (i) is

$$D = \left| \left| \frac{b\sin\theta - \frac{a\sin\theta}{b}}{\sqrt{\frac{a^2 + a^2\theta}{b^2} + 1}} \right| \right| = \left| \left| \frac{(b^2 - a)\sin\theta}{\sqrt{b^2 + a^2 tan^2\,\theta}} \right| \right|$$

$$= \left| \left| \frac{b^2 - a^2}{\sqrt{b^2\,cosec^2\theta + a\,n\,sec^2\theta}} \right| \right|$$

For fixed a and b, D is max. when S = $b^2 cosec^2\theta + a^2\,sec^2\theta$ is min. This happens when $\frac{ds}{d\theta} = -2b^2 \times cosec\,\theta(cosec\theta\,cot\theta) + 2a^2\sec\theta\,(sec\theta\,tan\theta)$

$$= -\frac{2b^2 cos\theta}{sin^3\theta} + \frac{2a^2\sin\theta}{cos^3\theta} = -\frac{2(b^2 cos^4\theta - a^2\,sin^4\theta)}{sin^3\theta\,cos^3\theta} = 0,$$

i.e., when $tan^2\theta = \frac{b}{a}$.

$$\therefore \frac{d^2s}{d\theta^2} = \frac{2b^2\sin\theta}{sin^3\theta} + \frac{6b^2 cos^2\theta}{sin^4\theta} + \frac{2a^2 cos\theta}{cos^3\theta} + \frac{ba^2 sin^2\theta}{cos^4\theta}$$

$$= \frac{2b^2}{sin^2\theta} + \frac{6b^2 cos^2\theta}{sin^4\theta} + \frac{2a^2}{cos^2\theta} + \frac{6a^2 sin^2\theta}{cos^4\theta} > 0.\forall\,\theta.$$

$\therefore tan^2\theta = \frac{b}{a}$ corresponds to the men of S given by

S min = $\| \ b^2(1 + cos^2\theta) + a^2\ (1 + a^2\theta)| | \ + a^2\theta = \frac{b}{a}$

$$= b^2\left(1 + \frac{a}{b}\right) + a^2\left(1 + \frac{b}{a}\right) = (a + b)^2$$

Hence the max. value of the reqd. distance is

$$D_{max} = \left| \frac{b^2 - a^2}{a + b} \right| = |\ b - a|\ \ Proving\ that\ D \leq |\ b - a|$$

Equality occurs when $+a^2\theta = \frac{b}{a}\ \ one\ such\ \theta = \tan^{-1}\frac{\sqrt{b}}{\sqrt{a}}, \Longrightarrow \sin\theta = \frac{\sqrt{b}}{a+b}.\cos\theta = \frac{\sqrt{a}}{a+b},$

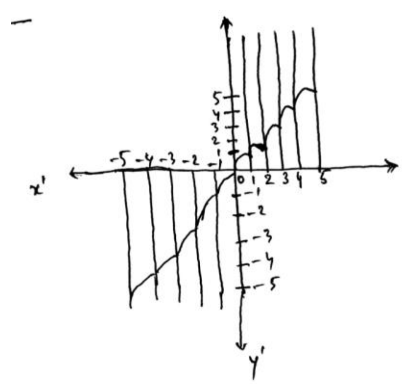
$$\Longrightarrow P \equiv \left(a\ \frac{\sqrt{a}}{a + b}, b\ \frac{\sqrt{b}}{a + b}\right).$$

Q9. **Let y = f(x) = [x] + $\sqrt{x - [x]}$ be defined for all real members x where [x] denote the greatest integer of x. Sketch f(x) is the range -5 ≤ x ≤ 5. Also show that given any real number y_0 there is a real number x_0 such that $y_0 = f(x_0)$.**

Sol. :$y = f(x) = [x] + \sqrt{n - [x]} = [x] + \sqrt{\{x\}}$ where {x} = fractional part of x(i) we know, 0 ≤ {x} < 1,

$\Longrightarrow \sqrt{\{x\}} \geq \{x\}, \Longrightarrow [x] + \sqrt{\{x\}} \geq [x] + \{x\}$

$\Longrightarrow f(x) \geq$
$x\ equality\ holds\ when\ x\ taxes\ integral\ value.$



∴ The graph of f(x) will be: (ii) again,

y = f(x) = [x] + $\sqrt{\{x\}}$.

As $0 \leq \{x\} < 1$, hence $\sqrt{\{x\}}$ is always real,

$\Longrightarrow$ f(x) is always real, $\Longrightarrow$ there is $a\ x_0, \forall\ y_0 \in R$ $and\ x_0\ also\ \in R, such\ that\ y_0 = f(x_0)$.

Q10. **On [0, 1], let f have a continuous derivative satisfying 0 < f[1] (t) ≤ 1 also f(0) = 0, then show that**

$$\left[\int_0^t \boldsymbol{f(t)\,dt}\right]^2 \leq \int_0^t [\boldsymbol{f(t)}]^3\,\boldsymbol{dt}.$$

Sol. : For $0 \leq x \leq 1$,

Let F(x) $\equiv \left[\int_0^x f(t)\,dt\right]^2 - \int_0^x \left(f(t)\right)^3 dt$.

$then\ F(0) = 0\ and\ F'(x) = 2$

$$\left[\int_0^x f(t)\,dt\right] f(x) - [f(x)]^3$$

$$= f(x)\left[2\int_0^x f(t)\,dt - [f(x)]^2\right]$$

We do know that f(x) ≥ 0 for $0 < x < 1$ (since we are given f(0) $= 0$ and f(x) > 0) however, it is not clear that the second factor in the last expression for F' is nonnegative therefore, let

$G(x) = 2\int_0^x f(t)dt - [f(x)]^2, 0 \leq x \leq 1$

$then\ G(0) = 0, and\ G'(x) = 2f(x) - 2f(x)f'(x)$

$= 2f(x)[1 - f'(x)] \geq 0.$

(The last inequality holds because f(x) ≥ 0 and by hypothesis 1- f'(x) ≥ 0). It follows from their arguments that F(x) ≥ 0 for all x, $0 \leq x \leq 1$; in particular, F(1) ≥ 0 and the proof is complete).

MODEL TEST PAPER – 8

Q1. Let $a_1, a_2, \dots, a_n$ are real (n > 1) and $A + \sum_{i=1}^n a_i^2 < \frac{1}{n-1}(\sum_{i=1}^n a_i)^2, show\ that\ A < 2a_i\,a_j\ for\ 1 \leq i \leq j \leq u$

Sol. : By the Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^n a_i\right)^2 = [(a_1 + a_2) + \cdots + a_n]^2$$

$$< (1 + \cdots + 1)[(a_1 + a_1)^2 + \cdots + a_n^2]$$

$$= (n-1)\left[\sum_{i=1}^n a_i^2 + 2a_1a_2\right]$$

This together with the given inequality, implies that

$$A < -\left(\sum_{i=1}^n a_i^2\right) - \frac{1}{n-1}\left(\sum_{i=1}^n a_i\right)^2$$

$$< -\left(\sum_{i=1}^n a_i^2\right) - \frac{1}{n-1}\left[(n-1)\left[\sum_{i=1}^n a_i^2 + 2a_1a_2\right]\right]$$

$= 2a_1a_2.$

In a similar manner, $A < 2a_i\,a_j\ for\ 1 \leq i \leq j \leq n$.

Q2. **Let m be an integer >1 and define the numbers m_1, m_2 … by postulating $m_1 = m, m_{i+1} = m_i^2 - m_{i+1}, i = 1, 2, 3, \dots$. Show that none of the numbers m_2, m_3, m_4 is divisible by m.**

Sol. : $m_2 = m^2 - m + 1 = m(m-1) + 1 = m_2 + 1$ where q_2 is an integer. As such m_1 is not divisible by m.

Now we assume $m_k = mq_k + 1, when\ q_k$ is an integer and prove that $m_k + 1 = mq_{k+1} + 1$, when q_{k+1} is also an integer.

$m_k + 1 = m_k(m_{k-1}) + 1 = (mq_k + 1)mq_k + 1 = mq_{k+1} + 1$

$whose\ q_{k+1} = q_k(mq_{k+1})$.

Q3. **Consider the Fibonacci sequence 0, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ….. in which every terms beginning from the third awards is the sum of two proceeding terms. Show that there exists a number terminating with 4 zeroes in the first 100, 000, 001 terms.**

Sol. : Suppose we divide by 7, the sequence of remainders is 0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 0, 1, 1……similarly if we divide by any positive integer n, we shall get a sequence of remainders.

Since, the original sequence is determined by two consecutive terms, this sequence of remainders is also determined by two consecutive remainders. Since the sequence starts with 0 and since there are only n possible different remainder and n^2 possible pairs of remainder i.e. pair (0, 1) will occur in every block of n^2 terms. If we choose n = 10^4, there will be a remainder 0 in every block of 10^8 terms and the corresponding member in the original sequence will be divisible by 10^4 and so will terminate in 4 zeroes.

Q4. n letters $a_1, a_2, \dots, a_n$ are written and have to be placed in corresponding envelopes marked $A_1, A_2, \dots, A_n$. The letters are placed at random in there envelopes. Find the number of ways in which every latter goes in a wrong envelopes.

Sol. : Let u_n be the required number of ways consider two letters a_1, a_2 and corresponding envelops A_1, A_2 then for all the letters to be in wrong envelops, two mutually exclusive possibilities arise.

(i) a_i goes into $A_2\ and\ a_2$ goes into A_1 and the remaining (n -2)

letters are placed wrongly into the other (n-2) envelops. The number of ways for this is u_{n+2}.

(ii) a_i goes into A_2 but a_2 does not go into A_1. In this case (n −1) letters go wrongly into (n−1) enevlops. This can be done in u_{n-1} ways.

Thus the number of ways in which a_1 goes into $A_2\ and$ all the letters are in the wrong envelops is $u_{n-1} + u_{n-2}$. We get the same number of ways in which a_1 goes into $A_3, A_4, \dots, A_n$ so that the total number of ways is $(n-1)(u_{n-1} + u_{n-2})$, but those must be same as u_n, so that $u_n = (n-1)(u_{n-1} + u_{n-2})$.

This is a difference equation of the second order. To solve this, we write it as

$$u_n - nu_{n-1} = -(u_{n-1} - \overline{n-1}\, u_{n-2})$$

$$u_n - \overline{n-1}\, u_{n-1} = -(u_{n-2} - \overline{n-2}\, u_{n-2})$$

……………………………………………………………………….

$$u_3 - 3u_2 = -(u_2 - 2u_1).$$

Multiplying there equations, we get,

$$u_n - nu_{n-1} = (-1)^{n-2}\,(u_2 - 2u_1).$$

Now $u_1 = 0$, since if there is only one letter and one envelope there is no way of the latter going into the wrong envelope. Also $u_2 = 1$, since if there are only two letters and two envelopes, there is one way of putting letters in the wrong envelope and one way of putting there in the right envelope, so that

$$u_n + nu_{n-1} = (-1)^{n-2}$$

Dividing this equation by n we get,

$$\frac{u_n}{n!} - \frac{u_{n-1}}{(n-1)!} = \frac{(-1)^{n-2}}{n!}$$

$$so\ that\ \frac{u_{n-1}}{(n-1)} - \frac{u_{n-2}}{(n-2)!} = \frac{(-1)^{n-2}}{(n-1)!}$$

$$\frac{u_2}{2!} - \frac{u_1}{1!} = \frac{(-1)}{2!}$$

Adding there together we get

$$\frac{u_n}{n!} - \frac{u_1}{1!} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \dots . + \frac{(-1)^{n-2}}{n!}$$

$$u_n = n!\left[\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right]$$

Q5. Find the number of real roots of the equation $\cos x = \frac{x}{50}$.

Sol. : Let $f(x) = \cos x = \frac{x}{50}$.

We have to find the points of intersection $y = \cos x\ and\ y = \frac{x}{50}$ since $-1 \le \cos x \le 1,$ the roots lie between -50 and 50. We also find that $f(14\pi) > 0, f\left(\frac{29\pi}{2}\right) < 0, f(15\pi) < 0$

$$f\left(-\frac{29\pi}{2}\right) > 0, f(-15\pi) < 0, f\left(-\frac{31\pi}{2}\right) > 0, f(16\pi) > 0.$$

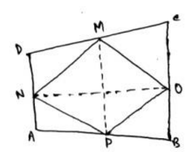
Using similar results, we find that one root lies between $0\ and \frac{\pi}{2}$ and two roots each lie between

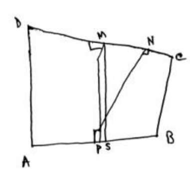
$$\frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}, \frac{13\pi}{2}, \frac{15\pi}{2}, \frac{17\pi}{2}, \frac{19\pi}{2},$$

$$\frac{21\pi}{2}, \frac{23\pi}{2}, \frac{25\pi}{2}, \frac{27\pi}{2}, \frac{29\pi}{2}, -\frac{31\pi}{2}, total\ 16\ roots.$$

Thus total number of real roots is 31.

Q6. ABCD is a cyclic quadrilateral and M, N, P, Q are the mid point of the sides CD, DA, AB, BC respectively. MS, NT, PN, QV are perpendicular to AB, BC, CD, DA respectively. Prove that they MN and PQ are each parallel to AC and each equal to $\frac{1}{2}$ AC so that MNPQ is a parallelogram.





Middle point of MP coincides with the middle point of NQ. Let 0 be the centre of the circumcircle of ABCD and let PV and intersect at X.

MP ⊥AB, MS ⊥ AB

∴ OP ∥ MS

Similarly since OM ⊥ CD, PV ⊥ CD

∴ ON ∥ PX

OPVXM is a parallelogram.

$\therefore$ They middle point of OX coincides with the middle point of MP. Similarly y be the point of intersecting of perpendiculars drawn from N and Q of opposite sides, then by the same argument the middle point of NQ coincides with the middle point of OY.

However, the middle points MP and NQ coincide. Therefore, the middle point of OX and OY coincide, therefore X and Y coincide.

As such perpendiculars from P, Q, M, N on opposite sides are concurrent.

Q7. Let P be a point outside a square ABCD. Find the focus of P if the shortest angle between two rays starting from P and including the whole square between them is θ.

Show that the locus is a closed curve consisting of eight circular arcs. Find the perimeter of this curve and also the area enclosed by it.

Sol. : The symmetries of the square will also be the symmetries of the desired locus, so that the desired locus will be symmetrical about the two right bisectors of the two pairs of opposite parallel sides as well as about the two diagonals. It will also by unchanged if the square is rotated about the centre through a right angle on two right angles or three rt. Angles.

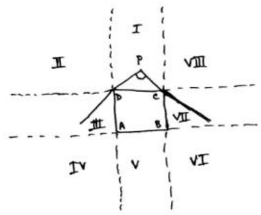
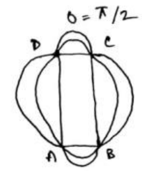


Fig (1)



Fig(2)

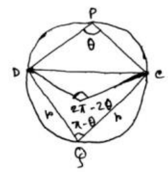


Fig (3)

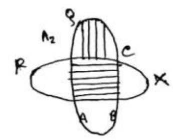


Fig (4)

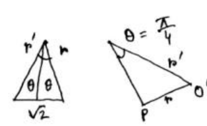


Fig (5)

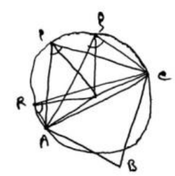


Fig (6)

Case A. P is a in region 1: The square is obviously contained between the rays PC and PD and so $\angle CPD = \theta$ is the smallest angle between the rays for which the rays can include the square. Since $\angle CPD$ is const., the locus of P is in region I is obviously the one of segment of a circle (fig. I).

Now three cases arise:

(i) $\theta = \frac{\pi}{2}$. In this case the locus of P is obviously the semicircle with CD as diameter for convenience we shall take the side of the square as of unit length. The locus in this case is given by four semi-circle areas (fig. 2). The total areas enclosed = $1 + 4.\frac{\pi}{2}\left(\frac{1}{2}\right)^2 = 1 + \frac{\pi}{2}$ and the perimeter of the fig (2) locus is $1 + 4 \times \pi \frac{1}{2} = 2\pi$.

(ii) $\theta > \frac{\pi}{2}$.In this case we shall get fig (3) and the centre of the circular are will be written the square. If O is the centre of the circle ∠COD = 2π - 2θ. If r is the radius of the circle $2r\sin(\pi - \theta) = 1 \; or \; r = \frac{1}{2}\sin\theta$. Area enclosed by arc CPD and the st. line CD = area of sec OCPD − area of Δ COD $=\frac{1}{2}r^2(2\pi - 2\theta) - \frac{1}{2}r^2\sin(2\pi - 2\theta) = r^2$

$$\left(\pi - \theta - \frac{1}{2}\sin 2\theta\right) = \frac{1}{85ir^2\theta}$$

$(2\pi - 2\theta - \sin 2\theta)$. Thus total area enclosed by the locus when $\theta > \frac{\pi}{2}$ is given by $A(\theta) = \frac{1}{25in^2\theta}(2\pi - 2\theta - \sin 2\theta) + 1$. Now Lt $A(\theta) = 1 + \frac{\pi}{2}$ as expected. Also $\underset{0\to\pi}{\text{Lt}} \frac{1}{2sin^2\theta}(2\pi - 2\theta - \sin 2\theta)$

$= \underset{0\to\pi}{\text{Lt}} \frac{-2-2\cos 2\theta}{4\sin\theta\,\cos\theta} =$

$\underset{0\to\pi}{\text{Lt}} \frac{4\sin 2\theta}{4(cos^{2\theta} - sin^2\theta)} = 0,$

So that $\underset{0\to\pi}{\text{Lt}} [A(\theta)] = 1.$

This is also obvious since in this case the locus of P coincides with the four sides of the square itself. Again if P(θ) is the length of the perimeter $P(\theta) = 4r\,(2\pi - 2\theta) = \frac{4(\pi-\theta)}{\sin\theta}$.

This approaches 2π as $\theta \to \frac{\pi}{2}$ as expected. If also approaches $\underset{0\to\pi}{\text{Lt}} \frac{4(\pi-\theta)}{\sin\theta} = \underset{0\to\pi}{\text{Lt}} \frac{-4}{\cos\theta} = 4$, again as expected. Thus we get the results $A\theta = 1 + \frac{2}{2sin^2\theta}(2\pi - 2\theta - \sin 2\theta); P(\theta) = \frac{4(\pi-\theta)}{\sin\theta};\; \theta \geq \frac{\pi}{2}$.

(iii) When $\theta \leq \frac{1}{2}$. In this case the locus in region I is an arc of circle fig. (4) and the area enclosed under it in region 1 = area of segment + area of rectangle $= \frac{1}{2}r^2 2\theta + \frac{1}{2}1.x = r^2\theta + \frac{1}{2}\cot\theta = \frac{4}{4sin^2\theta} + \frac{1}{2}.\frac{\cos\theta}{\sin\theta}$

Which approaches π as $\theta \to \frac{\pi}{2}$ and $\to\infty \; as \; \theta \to 0$. Also length of the circular arc = r.2.θ = $\frac{2\pi}{2\sin\theta} = \frac{\theta}{\sin\theta}$.

Case B. P. lies in region II : The locus in this region on is still the arc QR of a circle whose centre will be within the square if $\theta > \frac{\pi}{4}$ and fig (5) and outside the square if $\theta > \frac{\pi}{4}$. The square is contained bet can be rays PA and PC so that ∠ APC= θ, ∠AQC = θ, ∠ARC = θ.

∴∠ QCD $= \frac{\pi}{2} - \theta, \angle AO'C = 2\theta, \angle QO'R = 2\angle QCR = 2\angle QCD = \pi - 2\theta$

∴ ∠QO'C +∠RO'A

= 2π - 2θ- (π- 20) = π but by symmetry there angles are equal, so that

∠QO'C = ∠RO'A = $\frac{\pi}{2}$. Area of RPQDR = Area of sec.

QO'R - 2 areas 4QDO' $= \frac{1}{2}r'2$

$$(\pi - 2\theta) - 2.\frac{1}{2}r'xsin\left(\theta - \frac{\pi}{4}\right)$$

$$= \frac{1}{2}.\frac{1}{2sin^2\theta}(\pi - 2\theta) - \frac{1}{\sqrt{2}\sin\theta}\cot\theta\left(\sin\theta\frac{1}{\sqrt{2}} - \cos\theta\frac{1}{\sqrt{2}}\right) = \frac{1}{4sin^2\theta}$$

$$(\pi - 2\theta) - \frac{1}{2}\frac{\cos\theta}{sin^2\theta}(\sin\theta - \cos\theta)$$

It can be shown that the same results holds when $\theta < \frac{\pi}{2}$, the locus is given in fig (4) and A(θ) = 4× $area\ A_1$ + 4 area $A_2 + A_3 i\ A(\theta) = \frac{4}{sin^2\theta} + \frac{2\cos\theta}{sin\theta} + \frac{\pi - 2\theta}{sin^2\theta} - \frac{2\cos\theta}{sin^2\theta}(\sin\theta - \cos\theta) +$ $10 \leq \frac{\pi}{2}$, then you do yourself.

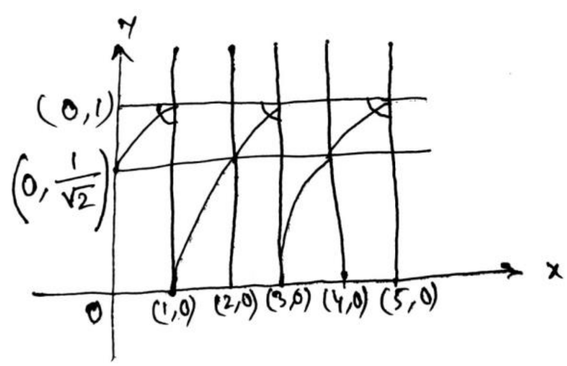
Q8. Consider the function f(x) = $\sin\{\frac{\pi}{4}(x - [x])\}$ if [x] is odd, x ≥0 $= \cos\{\frac{\pi}{4}(1 - x + [x]\}$ $if\ [x]$ is even x ≥ 0, where [x] denote the greatest integer ≤ x, sketch the graph of the function f(x) determine the points of discontinuities of f(x) and the points where f(x) is not differentiable.

Sol. : Let $2x > x > 2m - 1$, we can write, x = (2m- 1)+ 8, where 1 > 8 ≥ 0

∴ [x] = (2m- 1) which is odd.

∴ f(x) = $\sin\left(\frac{\pi}{4}(x - [x])\right) = \sin\left[\frac{\pi}{4}\{(2m - 1) + \delta - (2m - 1)\}\right] = \sin\frac{\pi}{4}\delta, where\ 1 > \delta \geq 0.$

Let, 2m +1 > x ≥ 2m.



∴ x = 2m + δ'

Where 1 > δ' ≥ 0.

[x] = 2m, which is even.

$\therefore f(x) = \cos\left[\frac{\pi}{4}\{1 - x + [x]\}\right] = \cos\left[\frac{\pi}{4}\{1 - (2m + \delta_1) + 2m\}\right] = \cos\left\{\frac{\pi}{4}(1 - \delta_1)\right\}$

Let us examine the case when m = 1, [x] = odd.

2 >x ≥ 1.

∴ x = 1+ δ, where 0 ≤ δ <1. δ = 0, x = 4, δ → 1, x → 2. f(a) = $\sin\frac{\pi}{4}\ \delta[x] = even.$

2 ≤ x < 3, x = 2 + δ' where 0 ≤ δ' < 1, δ' = 0, x = 2.

δ' → 1, x → 3, f(x) = $\cos\left\{\frac{\pi}{4}(1 - \delta_1)\right\}$. From the curve we can safely conclude that f(x) is discontinuous for every odd integral value of x, hence it is also not differentiable.

Q9. Let $\{C_n\}$ be an infinite sequence of circles lying in the positive quadrant of the xy-plane, with strictly decreasing radii and satisfying the following conditions. Each C_n touches both the X-axis and Y-axis. Further for all n ≥ 1, the circle C_{n+1} touches the circle C_n externally. If C_1 has radius 10 cm, then show that sum of the areas of all these circle is $\frac{25\pi}{3\sqrt{2}-4}cm^2$.

Sol. : $OO_1 = R_1\sqrt{2}$.

$\therefore OP = R_1\sqrt{2} - R_1$

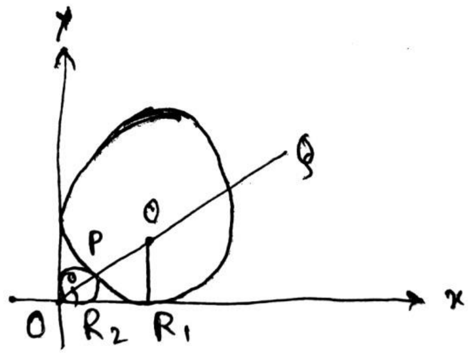
$\therefore OQ = R_1\sqrt{2} + R_1 = R_1(\sqrt{2} + 1)$.

$\therefore R_1 = \frac{OQ}{\sqrt{2}+1}$.

$$Now, OP = R_2(\sqrt{2} + 1), R_2 = \frac{OP}{\sqrt{2} + 1}$$

$$= R_1\frac{\sqrt{2} - 1}{\sqrt{2} + 1}.$$

$$\therefore R_3 = R_2 . \frac{\sqrt{2}-1}{\sqrt{2}+1} = R_1 \left(\frac{\sqrt{2}-1}{\sqrt{2}+1} . \right)$$



$$\therefore Area = \pi\left({R_1}^2 + {R_2}^2 + \cdots + \infty\right)$$

$$= \pi \left\{ {R_1}^2 + {R_2}^2 \left(\frac{\sqrt{2}-1}{\sqrt{2}+1} \right) + {R_1}^2 \left(\frac{\sqrt{2}-1}{\sqrt{2}+1} \right)^4 + \cdots + \infty \right\}$$

$$= \pi {R_1}^2 \left\{ 1 + \left(\frac{\sqrt{2}-1}{\sqrt{2}+1} \right)^2 + \left(\frac{\sqrt{2}-1}{\sqrt{2}+1} \right)^4 + \cdots \infty \right\}$$

$$= \pi {R_1}^2 \left\{ \frac{1}{1 - \left(\frac{\sqrt{2}-1}{\sqrt{2}+1} \right)^2} \right\} = \pi {R_1}^2 \frac{3+2\sqrt{2}}{4\sqrt{2}}$$

$$= \pi {R_1}^2 \frac{3\sqrt{2}+4}{8} = \frac{\pi}{8} . {R_1}^2 \frac{18-16}{3\sqrt{2}-4}$$

$$= \frac{\pi}{4} . 100 . \frac{1}{3\sqrt{2}-9}$$

$$(\because R_1 = 10\ cm) = \frac{25\pi}{(3\sqrt{2}-4)} sq.cm.$$

Q10. Prove that for $0 \le a < b < \frac{\pi}{2}, \frac{b-a}{cos^2 a} <$ $\tan b - \tan a < \frac{b-a}{cos^2 b}$.

Sol. : Consider the function f(x) is tan x on [a, b]. According to the mean value theorem there is a point c on (a, b) such that

$\frac{f(b) - f(a)}{b-a} = f'(c)$

In this case, this means that

$\frac{\tan b - \tan a}{b-a} = sec^2 c$ - for same c in (a, b).

$$sec^2 a < sec^2 c < sec^2 e\ for \ldots 0 < a < b < \frac{\pi}{2}$$

Let $f''\ R \ \rightarrow R\ such\ that\ f''(x) \ge 0$, then

$$f\ \frac{x+y}{2} \le \frac{f(x)+f(y)}{2}$$

And if $f''(x) \le 0, then\ f\ \left(\frac{x+y}{2}\right) > \frac{f(x)+f(y)}{2}$

For example, for real number a and b,

$$f\left(\frac{x+y}{2}\right)^2 < \frac{x^2+y^2}{2}.$$

Because f(x) = x^2 is a convex function. As another example if 0 < x, y < π

$$\sin\left(\frac{x+y}{2}\right) \ge \frac{\sin x + \sin y}{2}$$

Because f(x) = sin x is a concave function on $(0, \pi)$.

MODEL TEST PAPER 9

Q1. If n arithmetic means $A_1, A_2, \ldots, A_n$ and n geometric means $G_1, G_2, \ldots, G_n$ we inscribed between two positive numbers a and b, show that $\frac{1}{x}(A_1 + A_2 + \cdots A_n) \ge \sqrt[n]{G_1, G_2, \ldots G_n}$

Sol. : If d is the common difference of the A.P and r is the common ratio of G.P., then

$a + (n+1)d = b, ar^{n+1} = b,$

$$so\ that\ Ai + a + id = a + i\frac{b-a}{n+1}$$
$$= \frac{n+1-i}{n+1}a + \frac{i}{n+1}b$$

$$and\ Gi = ar^i = a\left(\frac{a}{b}\right)\frac{i}{n+1}$$
$$= a\ \frac{n+1-i}{n+1}.\frac{i}{b^n+1}.$$

So that Ai and Gi are the weighted arithmetic and geometric mean between the two numbers a and b, when the weights one $\frac{(n+1-i)}{(n+1)}$ and $\frac{i}{(n+1)}$ but the weighted arithmetic mean of two positive numbers ≥ weighted geometric means. If the weights are the same and the equality sign holds only when the two numbers are equal so that.

$Ai \geq Gi\ and\ Ai = Gi\ , if\ a = b, i = 1, 2, \ldots n$ then each arithmetic mean ≥ corresponding geometric mean. Again since, we get $\frac{A_1+A_2+\cdots+A_n}{n} \geq \sqrt[n]{A_1, A_2, \ldots A_n} \geq \sqrt[n]{G_1, G_2, \ldots G_n}$ and the equality sign holds if a = b.

Q2. Let [m] denote the largest integer ≤ m, such that the equation $[x] + [2x] + [4x] + [8x] + [16x] + [32x]$= M has no real so has no real solution unless M is of the form 63k or 63k +1 or 63k + 7 or 63k + 15 or 63k+ 31 where, k is any positive integer, negative or zero.

Sol. : Let $f(x) = [x] + [2x] + [4x] + [8x] + [16x] + [32x]$

Then f(x) = 63k when $k \leq x < k + \frac{1}{32}$.

$$= 63k + 1\ when\ k + \frac{1}{32} \leq x < k + \frac{1}{16}$$

$$= 63k + 3\ when\ k + \frac{1}{16} \leq x < k + \frac{1}{8}$$

$$= 63k + 7\ when\ k + \frac{1}{8} \leq x < k + \frac{1}{4}$$

$$= 63k + 15\ when\ k + \frac{1}{4} \leq x < k + \frac{1}{2}$$

$$= 63k + 31\ when\ k + \frac{1}{2} \leq x < k + 1$$

So that f(x) can take only when six sets of values and f(x) = M has no real solution when M is different from these value.

Q3. If a, b, c are the roots of $x^3 - x^2 - x - 1 = 0$, so that (i) a, b, c are distinct; (ii) $\frac{a^{1992}-b^{1992}}{a-b} + \frac{b^{1992}-c^{1992}}{b-c} + \frac{c^{1992}-a^{1992}}{c-a}$ is an integer.

Sol. : The polynomial f(x) =$x^3 - x^2 - x - 1$ has only are change of sign but f(-x) =$-x^3 - x^2 + x - 1$ has two change of sign. As such by Descartes rules of signs f(x) = 0 can have at the positive real root and at most two negative real roots and all the roots can be real. Now $f(-\infty) = -\infty < 0, f(0) = -1, f(1) = -2 < 0, f(2) = 1 > 0, f(\infty = \infty > 0)$. Thus as x goes from $-\infty$ to 0, f(x) goes $-\infty\ to - 1$ and as x goes from 0 to ∞, f(x) goes $-1\ to\ \infty$.

Again, f'(x)= $3x^2 - 2x - 1$

$$f'(x) = 0 \Rightarrow x = 1\ or, x = -\frac{1}{3}$$

$$and\ f\left(\frac{1}{3}\right) = -\frac{1}{27} - \frac{1}{9} + \frac{1}{3} - 1 = \frac{32}{27} < 0$$

$$f''(1) = 4 > 0. f''\left(-\frac{1}{2}\right) = -9 < 0.$$

Thus f(x) has a local maximum at x = $-\frac{1}{3}$ when the local maximum value is $-\frac{22}{27}$ and has a local minimum at 2 = 1

Where the local maximum value is -2.

Then f(x) = 0 has three roots one real and positive lying between 1 and 2 and the other two are complex in the negative real part. All the three roots are distinct.

For the second part,

Let $x_n = \frac{a^n-b^n}{a-b}, y_n \frac{b^n-c^n}{b-c}, z_n \frac{c^n-a^n}{c-a},$

$$so\ that\ x_{n+3} = \frac{a^{n+3}-b^{n+3}}{a-b}$$
$$= \frac{a^n(a^2+a+1)-b^n(b^2+b+1)}{a-b}$$

$$= \frac{a^{n+2}-\ b^{n+2}}{a-b} + \frac{a^{n+1}-b^{n+1}}{a-b} + \frac{a^n-b^n}{a-b}$$

$= x_{n+2} + x_{n+1} + x_n.$

$similarly, y_{n+3} = \ y_{n+2} + y_{n+1} + y_n$

$z_{n+3} = \ z_{n+2} + z_{n+1} + z_n.$

So that $(x_{n+3} + y_{n+3} + z_{n+3}) = (x_{n+2} + y_{n+2} +\ z_{n+2}) +\ (x_{n+1} + y_{n+1} + z_{n+1}) + (x_n +\ y_n +\ z_n)$

$let\ u_n = \ x_n + y_n +\ z_n.$

$so\ that\ u_{n+3} = \ u_{n+2} + u_{n+1} + u_n$

So that if u_n, u_{n+1}, u_{n+2} one integers, then u_{n+1} *has also be can integer*.

Now $u_1 = 3, u_2 = 2(a+b+c) = 2, u_3 = 2(a^2+b^2+c^2) + ab + bc + ca = 2[1+2] - 1 = 5$.

So that u_1, u_2, u_3 are integers, then applying the result proved earlier in succession are we get u_4, u_5, u_6 are all integers. In fact we get the sequence 3, 2, 5, 10, 17, 32, 59, 108, 199, 366, ….

Q4. Given any 13 real numbers, show that there are two of those a_i, a_j such that $0 \leq \frac{a_i-a_j}{1+a_i\,a_j} \leq \frac{\sqrt{3}-1}{\sqrt{3}+1}$.

Sol. : Consider the function y = tan x, then as x goes from $-\frac{\pi}{2}\ to \frac{\pi}{2}$ y goes from $-\infty\ to\ \infty$. Divide the interval $-\frac{\pi}{2}\ to \frac{\pi}{2}$ into 12 equal parts each of length $\frac{\pi}{12}$. If $C_1, C_2, \ldots . C_{12}$ are given real numbers then $\tan^{-1} C_1 \tan^{-1} C_2 \ldots . \tan^{-1} C_{12}$ lie between $-\frac{\pi}{2}, \frac{\pi}{2}$ but by the pigon hole principle all the thirteen cannot be in 12 distinct intervals unless at least two of there are in the same interval. Let $\tan^{-1} e_i$ and $\tan^{-1} e_j$ be in the same interval, then

$0 \leq c_i - c_j \leq 15°$

$0 \leq\ \tan^{-1}(c_i - c_j) \leq\ \tan^{-1} 15°$

$$0 \leq \frac{\tan^{-1} c_i - \tan^{-1} c_j}{1 + \tan^{-1} c_i \tan^{-1} c_j} \leq \frac{\sqrt{3}-1}{\sqrt{3}+1}.$$

Q5. If x_1, x_2 are positive and $x_{n+1} = \frac{1}{2}(x_n + x_{n-1})$. Show that the sequences, x_1, x_3, x_5 …. and x_2, x_4, x_6 … .. are the one increasing and the other decreasing and both approach to a common limit $\frac{1}{2}(x_1 + 2x_2)$.

Sol. : $x_{n+1} = \frac{1}{2}(x_n + x_{n-1})$

$$x_{n-1} - x_n = \frac{1}{2}(x_{n-1} - x_n) = \ -\frac{1}{2}(x_n - x_{n-1})$$

As such $x_n - x_{n-1}$ and $x_{n+1} - x_n$ have opposite signs and if $x_n > x_{n-1}\ than\ x_{n+1} < x_n$.

Also $x_n = \frac{1}{2}(x_{n-1} + x_{n-2}), x_{n-1} = \frac{1}{2}(x_{n-2} + x_{n-3})$

So that $x_{n+1} = \frac{1}{2}\left[\frac{1}{2}x_{n+1} + \frac{1}{2}x_{n-2} + x_{n-1}\right] = \frac{1}{2}\left[\frac{-3}{2}x_{n-1} + \frac{1}{2}(2x_{n-1} - x_{n-2})\right]$

$$= \frac{1}{4}[5x_{n-1} - x_{n-3}].$$

$$\Longrightarrow x_{n+1} - x_{n-1} = \frac{1}{4}[x_{n-1} - x_{n-3}]$$

So that $x_{n-1} - x_{n-1}$ has the same sign as $x_{n-1} - x_{n-3}$

So that $x_{n+1} - x_{n-1} = \left(\frac{1}{4}\right)^2 (x_{n-3} - x_{n-5}) = \left(\frac{1}{4}\right)^3 (x_{n-5} - x_{n-7})$

$$x_{2n+1} - x_{2n+1} = \left(\frac{1}{2}\right)^{2n-2} (x_3 - x_1)$$
$$= \left(\frac{1}{2}\right)^{2n-1} (x_2 - x_1)$$

In the same way,

$$x_{2n} - x_{2n-2} = \left(\frac{1}{2}\right)^{2n-4} (x_1 - x_2)$$
$$= \left(\frac{1}{2}\right)^{2n-4} \frac{1}{2}(x_3 - x_2)$$
$$= \left(\frac{1}{2}\right)^{2n-2} (x_1 - x_1)$$

Then if $x_1 > x_2, x_2 < x_4 < x_6 < x_8 \ldots < x_n <$ and the sequence $\{x_{2n}\}$ is a monotonic decreasing sequence. Also, $x_1 > x_3 > x_5 > x_7 \ldots > x_{2n-1} > \cdots$ and the sequence $\{x_{2n+1}\}$ is monotonic decreasing sequence.

Similarly, if $x_1 < x_2$, the sequence $\{x_{2n}\}$ is monotonic decreasing and the sequence $\{x_{2n+1}\}$ is monotonic increasing.

In either case, out of the sequences $\{x_{2n-1}\} and \{x_{2n}\}$, one is increasing and the other is decreasing.

Also $\underset{n\to\infty}{\mathrm{Lt}} (x_{2^n} + 2 - x_{2^n}) = 0$, $\underset{n\to 0}{\mathrm{Lt}} (x_{2^n+1} + 2 - x_{2^n-1}) = 0$.

So that both the odd and even numbered sequences are convergent. If $x_{2^n} \to a, x_{2^n-1} \to b, then\ b = \frac{1}{2}(a+b)a, b = a$.

Thus both sequences approach a common limit.

Q6. Let f; R→ R satisfies $f\left(\frac{x+y}{2}\right) < \frac{f(x)+f(y)}{2}$ $\forall x, y$ in an interval (a, b) x ≠ y, show that $f\left\{\frac{1}{n}(x_1 + x_2 + \cdots + x_n)\right\} < \frac{1}{n}$

$\{f(x_1) + f(x_2) + \cdots + f(x_n)\}$ whenever the $x_i's$ are in (a, b) with $x_i \neq x_j$ for at least are pair (i, j).

Sol. : Assume the result holds for n = m_i we will show if holds also for n = 2m. We have

$$f\left(\frac{x_1 + \ \ldots. + x_{2m}}{2m}\right)$$
$$= f\left\{\frac{1}{2}\left(\frac{x_1 + \cdots + x_m}{m} + \frac{x_{m+1} + \cdots + x_{2m}}{2m}\right)\right\}$$
$$\leq \frac{1}{2}\left[f\left(\frac{x_1 + \cdots + x_m}{m}\right) + f\left(\frac{x_{m+1} + \cdots + x_{2m}}{m}\right)\right]$$
$$< \frac{1}{2}\left\{f\ \frac{(x_1) + \cdots + f(x_m)}{m} + f\left(\frac{x_{m+1} + \cdots + x_{2m}}{m}\right)\right\}$$
$$= \frac{f(x_1) + f(x_2) + \cdots + f(x_{2m})}{2m}$$

Thus, by induction, the result holds for 4 all powers of 2. Now suppose that n >2 and n is not a power of 2; i.e. let

$2^{m-1} < n < 2^m$ for some integer m. Let k = $2^m - n$ and set $y_i = \frac{(x_1 + \cdots + x_n)}{n}$ for i = 1, 2, …, k. Then $x_1, x_2, \ldots., k_n, y_1, \ldots, y_k$ are 2^m

numbers in the interval (a, b) and so our preceding argument implies that

$$f\ \frac{(x_1\ +\cdots+x_n+y_1\ +\cdots+y_k)}{2^m} < \frac{f(x_1\ )+\cdots+f(y_k)}{2_m}.$$

But note that $f\ \frac{(x_1+\cdots+x_n+y_1+\cdots+y_k)}{2^m}$

$$= f\frac{x_1\ +\cdots+x_n+k(x_1\ +\cdots+x_n)}{2^m}$$

$$= f\left(\frac{n(x_1\ +\cdots+x_n)+(2^m-n)(x_1\ +\cdots+x_n)}{n\times 2^m}\right)$$

$$= f\,\frac{(x_1\ +\cdots+x_n)}{n}.$$

Making this substitution into the last inequality,

$$f\ \frac{(x_1\ +\cdots+x_n)}{n} < \frac{f(x_1\ )+\cdots+f(x_n)+f(y_1\ )+\cdots+f(y_n)}{2^m}$$

$$= \frac{f(x_1\ )+\cdots+f(x_{1n})+kf(x_1\ +\cdots x_n)/n}{2^m}$$

Multiplying each side by 2^m yields $2^m f\left(\frac{x_1+\cdots+x_n}{n}\right) <\ f(a_1\ )+\cdots+f(x_n\ )+ (2^m-n)f\left(\frac{x_1+\cdots+x_n}{n}\right) < \frac{f(x_1\ )+\cdots+f(x_n\ )}{n}.$

Q7. Among all triangular having a fixed LA and an inscribed circle of fixed radius r, determine the triangle has the least perimeter.

Sol. : Perimeter 2S = a + b+ c

$$= r\left[\cot\frac{B}{2}+\cot\frac{C}{2}\right]+r\left[\cot\frac{C}{2}+\cot\frac{A}{2}\right] + r\left[\cot\frac{A}{2}+\cot\frac{B}{2}\right]$$

$$= 2r\left[\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right]$$

Since A and r are fixed, the perimeter is minimum, when

$\cot\frac{B}{2}+\cot\frac{C}{2} = \frac{\sin\frac{B+C}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}$ is minimum when $\sin\frac{B}{2}\sin\frac{C}{2}$

$= \frac{1}{2}\left[\cos\frac{B-C}{2}-\cos\frac{B+2}{2}\right] = \frac{1}{2}\left[\cos\frac{B-C}{2}-\sin\frac{A}{2}\right]$ is maximum i.e. when $\cos\frac{B-C}{2}$ is maximum i.e. when B = C i.e. when the triangle is isosceles.

Q8. Suppose f(x) is a real valued differentiable function defined a [1, ∞) with f(1) = 1. Also f(x) satisfies

$$\boldsymbol{f'(x) = \frac{1}{x^2+f^2(x)}}.$$

Prove that $\boldsymbol{\lim_{x\to\infty} f(x)}$ ***exists and is less than*** $\boldsymbol{1+\frac{1}{4}\pi}$.

Sol. : By the fundamental theorem of calculus

$$f(x)-\ f(1) =\ \int_1^x f'(x)\,dx$$

Observe that f(x) is increasing, moreover, f(x) > 1 for all x > 1.

Since f(1) = 1 and f'(x)>0. Therefore

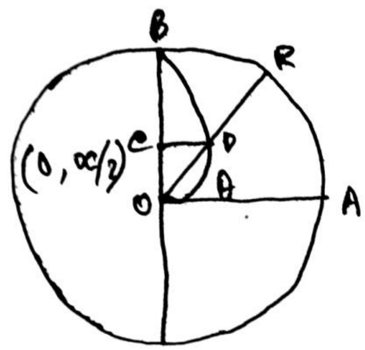
$$f(x)-f(1) = \int_1^x \frac{dx}{x^2+f^2(x)} < \int^x \frac{dx}{1+x^2}$$

$$= [\tan^{-1}x]_1^x =\ \tan^{-1}x-\frac{\pi}{4} < \frac{\pi}{2}-\frac{\pi}{4} < \frac{\pi}{4}.$$

Thus f(x) is increasing and bounded above by $1+\frac{1}{4}\pi$, and consequently, $\lim_{x\to\infty} f(x)$ exists and is less than $1+\frac{1}{4}\pi$.

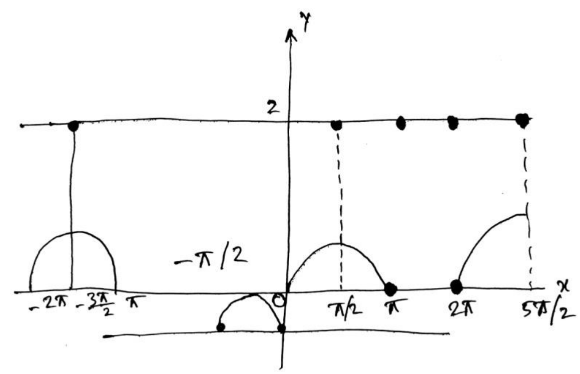
Q9. A rabbit runs round a circular path with a certain uniform speed v. A dog follows it from the origin in such a way that the origin, dog, rabbit are always on the same straight line and the dog runs with the same speed v. Show that the dog will catch the rabbit when it has run round a quarter of the circle.

Sol. : Consider the semi-circle with centre at $\left(0, \frac{a}{2}\right)$ *and radius* $\frac{a}{2}$ and let D be the position of the dog on the semicircle and R be the position of the rabbit on the circle, so that ODR is a straight line .If $\angle$AOD $= \theta$, $\angle$ OCD$=$ 2θ and arc OD $=$ arc AR, since the distance moved by the dog $=$ distance moved by the rabbit. The semicircle and the circle meet at B and the dog catches the rabbit here.



Q10. Draw the graph of y = [sin x]+ sin x, where [x] denotes the greatest integer function.

Sol. : -1 ≤ sin x ≤ 1. ∴ 0 ≤ |sin x| ≤ 0 and [sin x] = -1, 0, 1, then -1 ≤ [sin x] + |sin x|≤ 2.



<u>MODEL TEST PAPER - 10</u>

Q1. Let x_i >0 for i = 1, 2, ..., n. For each non-negative integer k, Prove that $\frac{1}{n}\left(x_1{}^2 + \cdots + x_n{}^2\right) \le \frac{x_1{}^{k+1} + \cdots + x_n{}^{k+1}}{x_1 + \cdots + x_n}$

Sol. : Let $x_1 + \ \ldots + x_n = 1$, for if not, replace x_i by $x_i = \frac{x_i}{(x_1 + \cdots + x_n)}$.

The results holds when k = 0. Let the result holds for all non-negative integer < k. By Cauchy Schwarz. Inequality.

$$\frac{\sum_{i=1}^{n} x_i{}^k}{n} = \sum_{i=1}^{n} x_i{}^{(k+1)/2} \frac{x_i{}^{(k-1)/2}}{n}$$
$$\le \left(\sum_{i=1}^{n} x_i{}^{k+1}\right)^{1/2} \left(\sum_{i=1}^{n} \frac{x_i{}^{k-1}}{n}\right)^{1/2}$$

By the inductive assumption, $\sum_{i=1}^{n} \frac{x_i{}^{k-1}}{n} \le \sum_{i=1}^{n} x_i{}^k$ and so continuing from last in equation, we have

$$\left(\sum_{i=1}^{n} x_i{}^{k+1}\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \frac{x_i{}^{k-1}}{n^2}\right)^{\frac{1}{2}}$$
$$\le \left(\sum_{i=1}^{n} x_i{}^{k+1}\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \frac{x_i{}^{k}}{n}\right)^{\frac{1}{2}}$$

$$thus \sum_{i=1}^{n} \frac{x_i{}^k}{n} \leq \left(\sum_{i=1}^{n} x_i{}^{k+1}\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \frac{x_i{}^k}{n}\right)^{\frac{1}{2}},$$

$$\Rightarrow \left(\sum_{i=1}^{n} \frac{x_i{}^k}{n}\right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{n} x_i{}^{k+1}\right)^{\frac{1}{2}},$$

Then you do yourself.

Q2. Find all positive integer x, y, z, ω which satisfy ω! = x!+ y! + z!.

Sol. : Without less of generality. We can assume x ≤ y ≤ z.

It is obvious that ω ≥ z +1, so that

(z+ 1)! ω! = x! + y! +z! ≤ 3z!

So that z +1 ≤3 a, z ≤ 2 and x ≤ 2, y ≤ 2 calculations show that x = y = z = 2 and ω = 3 give the only solution.

Q3. n elements $a_1, a_2, \dots, a_n$ are permuted among themselves so that the total number of permutations is n! Let f_n denotes the number of permutations in which no element remain fixed i.e. every element changes its position and let $g_n$denote the number of permutations where only one element remains fixed and the (n−1) remaining elements change their position. Show that $f_n{}^2 g_n = 1$.

Sol. : By a transformation let a_i go into a_j. Now there are two posibilities either a_j goes into a, or a_j does not go into a_i.

In the first case n -2 elements change their positions and there are f_{n-2} permutations of this type corresponding to each j = 1, 2, 3, …, n so that this give (n -1) f_{n-2} permutations.

In the second case there are f_{n-1} permutations corresponding to each f so that there are (n- 1) f_{n-1} permutations of this type. Thus the total number of permutations in which no elements changes is (n -1) $(f_{n-2} + f_{n-1})$ and this would be sane as f_n so that

$f_n = (n-1)(f_{n-1} + f_{n-2})$ Also be definition $g_n = nf_{n-1}. so\ that\ f_{n-1} = nf_n + nf_{n-1}, g_{n-1} = (n+1)f_n$

$\therefore f_{n+1} - g_{n+1} = nf_{n-1} - f_n = g_n - f_n = -(f_n - g_n)$

$= (-1)^2(f_{n-1} - g_{n-1}) = (-1)^3(f_{n-2} - g_{n-2})$

$= (-1)^n(f_1 - g_1) = (-1)^{n+1}, since\ f_1 = 0, g_1 = 1$

$\therefore |f_n - g_n| = 1$ or, $f_n - g_n = 1$.

Q4. Let $x_1, x_2, \dots, x_n$ be n real numbers between o and 1. Find the greatest and smallest values of

$\boldsymbol{\delta = \sum_{1 \leq ij \leq n} |x_i - x_j|}$

Sol. : Since we can choose $x_1, x_2, \dots, x_n$ all equal, the minimum value of S is zero. For finding the largest value, we assume without loss of generality that

$0 \leq x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_{n-1} \leq x_n \leq 1$

Then S = $(x_2 - x_1) + [(x_3 - x_1) + (x_3 - x_2)] + [(x_4 - x_1) + (x_4 - x_2) + (x_4 - x_3)] + [(x_n - x_1) + \cdots + (x_n - x_{n-1})]$

On the R.H.S., x_1 occurs (n -1) lines with a negative sign giving a total of – (n- 1) x. Also x_2 occurs (n-2) lines with a negative sign and once with a positive sign giving a total of – (n-3) x_2.

Proceeding in this way we get x_k occurs (n - k) lines with a negative sign and k -1 line with a positive sign.

So that S = $\sum_{k=1}^{n}[(k-1)-(n-k)]x_k = \sum_{k=1}^{n}(2k-n-1)x_k$

Case (i) n is even say n = 2m, then S = $\sum_{k=1}^{2m}(2k-2m-1)x_k$

If k = 1, 2, ..., m the coefficients are negative and since we want to maximize S, we choose $x_1 = x_2 = \cdots = x_{2m} = 1$

So that S = 1+ 3+ ...+ (2_{m-1})= $m^2 = \frac{n^2}{4}$

Case (ii) n is odd say n = 2m +1, then

S= $\sum_{k=1}^{2m+1}(2k-2m-2)\, x_k$

If k = 1, 2, ..., m, the coefficient are negative and in order to maximize S, we choose

$$x_1 = x_2 \ldots = x_m = 0,$$

If k = m +1, co-efficient of $x_m + 1$is zero and we can give any value to $x_m + 1$.

If k=m +2, ..., $2_m + 1$ the coefficients are positive and in order to maximize S, we choose

$$x_{m+2} = x_{m+3} = \cdots x_{2m+1} = 1$$

So that S = (2+ 4+ ...+ 2m)= m(m+ 1) = $\frac{\boldsymbol{n-1}}{\boldsymbol{2}}\,\frac{\boldsymbol{n+1}}{\boldsymbol{2}} = \frac{\boldsymbol{n^2-1}}{4}$. Thus when n is even S= $\frac{n^2}{4}$ and when n is odd S = $\frac{(n^2-1)}{4}$ so that we can say that S= $\left[\frac{n^2}{4}\right]$.

Q5. Given that the polynomial f(x) = $\boldsymbol{x^n} + \boldsymbol{a_1x^{n-1}} + \cdots + \boldsymbol{a_n}$ with integral coefficients is equal to 5 for 4 distinct integers a, b, c, d. Show that there is no integer k such that f(k) = 7.

Sol: f(x) ≡ $(x-a)(x-b)(x-c)(x-d)\phi(x) + 5$ *where* ϕ(x) is a polynomial with integral coefficients of (n−4)th degree. If f(k) =7, then

(k -a) (k -b) (k -c) (k -d) ϕ(k) = 2

Now, four distinct integral factors of 2 are 1, -1, 2, -2 since (k -a), (k -b), (k -c), (k -d) are distinct, their product cannot be 2.

Q6. Let $\boldsymbol{x_1 = \tan^{-1} 2 > x_2 > x_3 > \cdots}$ are positive real numbers satisfying $\boldsymbol{\sin(x_{n-1} - x_n) + 2^{-n+1}\sin x_{n+1} = 0}$ for n ≥ 1. Find $\boldsymbol{\cot x_n}$and also show that $\boldsymbol{\lim\limits_{x\to\infty}\frac{\pi}{4}}$.

Sol. : By the problem $\sin x_{n+1}\cos x_n - \cos x_{n+1}\sin x_n + 2^{-(n+1)}\sin x_n \sin x_{n+1} = 0, \Rightarrow \sin x_{n+1}$

$$\{\cos x_{n+2} - (n-1)\times \sin x_n\} = \cos x_{n+1}\sin x,$$

$$\Rightarrow \cot x_{n+1} = \cot x_{n+2} - (n+1)(\textit{dividing}\sin x_{n+1}\sin x_n)$$

$$\cot x_n = \cot x_{n-1} + 2^{-n} = \cot x_{n-2} + 2^{-(n+1)} + 2^{-x}$$

$= \cot x_{n-3} + 2^{-(n-2)} + 2^{-(n-1)} + 2^{-n} = \cdots = \cot x_1 + 2^{-2} + 2^{-3} + \cdots + 2^n = 2^{-1} + 2 + 2 + \cdots + 2^{-n} (\because \tan^{-1} 2 = x_1)$

$$= \frac{1}{2}\left[\frac{1-\left(\frac{1}{2}\right)^n}{1-\left(\frac{1}{2}\right)}\right] = 1-\left(\frac{1}{2}\right)^n \therefore \cot x_n = 1-\left(\frac{1}{2}\right)^n.$$

$$\textit{Now}, \operatorname*{Lt}_{n\to x}\cot x_n = \operatorname*{Lt}_{n\to x} 1-\left(\frac{1}{2}\right)^n = 1, \Rightarrow \cot\left(\operatorname*{Lt}_{n\to x} x_n\right) = 1,$$

$\Rightarrow \underset{n\to x}{\mathrm{Lt}}\, x_n$
$= \frac{\pi}{4}\left[\cot x \ \ is\ a\ continuous\ function\ of \frac{\pi}{4}\right].$

Q7. A hexagon is inscribed in a circle. Show that its shortest side cannot be less than the radius. What is the length of the largest side?

Sol. : If n is the radius of the circle, the length of the circumference is 2πr. This is divided into six area by the six vertices of the hexagon. All there cannot be greater than $\frac{2\pi r}{6}$. As such the length of shortest one has to $\leq \frac{\pi r}{3}$ and angle θ subtended by it at the centre $\leq \frac{\pi}{3}$ radius. The length of the shortest side $\leq 2r\sin\frac{\pi}{6} = r \leq$ radius of the circle and it will actually by equal to r if the hexagon is regular.

The length of the largest side can of course equal to 2r.

Q8. In a rectangle ABCD, AB = 240 a, BC = a. The side AB is divided into 240 parts by 239 points $\boldsymbol{O_1}, \boldsymbol{O_2}$ …. $\boldsymbol{O_{239}}$. The point C is joined to $\boldsymbol{O_1}, \boldsymbol{O_{235}}$ and $\boldsymbol{O_{239}}$ to give angles $\boldsymbol{\alpha, \beta, \gamma}$. Show that $\boldsymbol{4\beta - \alpha = \gamma}$.

Sol. : $\tan\alpha = \frac{a}{239a} = \frac{1}{239}\tan\beta = \frac{a}{5a} = \frac{1}{5}, \tan\gamma = \frac{a}{a} = 1$

$$\tan 2\beta = \frac{2\tan\beta}{1 - tan^2\beta} = \frac{\frac{2}{5}}{1-\frac{1}{25}} = \frac{5}{12}\tan 4\beta$$
$$= \frac{2\tan 2\beta}{1 - tan^2 2\beta}$$

$$= \frac{\frac{5}{6}}{1 - \frac{25}{144}} = \frac{120}{119}$$

$$\tan(4\beta - \alpha) = \frac{\tan 4\beta - \tan\alpha}{1 + \tan 4\beta\tan\alpha}$$
$$= \frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119}\times\frac{1}{239}}$$

$$= \frac{120\times 239 - 119}{119\times 239 + 120} = \frac{119\times 239 + 120}{119\times 239 + 120} = 1$$
$$= \tan\infty$$

$\therefore 4\beta - \alpha = \gamma$

Q9. If $\boldsymbol{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n}$ are the roots of the equation $\boldsymbol{x^n - nax - b = 0\ and (\alpha_1 - \alpha_2).(\alpha_1 - \alpha_3)\dots(\alpha_1 - \alpha_n) = A}$, then find the value of $\boldsymbol{A - n\alpha_1{}^{n-1}}$.

Sol. : $x^n - nax - b = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\dots(x-\alpha_n) \Rightarrow \frac{x^n - nax - b}{(x-\alpha_1)} = (x-\alpha_2)(x-\alpha_3)\dots(x-\alpha_n)$,

$$\underset{x\to a}{\mathrm{Lt}}\frac{x^n - nax - b}{(x-\alpha_1)} = \underset{x\to\alpha_1}{\mathrm{Lt}}(x-\alpha_2)(x-\alpha_3)\dots(x-\alpha_n),$$

$$\Rightarrow \underset{x\to\alpha_1}{\mathrm{Lt}}\frac{nx^{n-1} - na}{1} = \underset{x\to\alpha_1}{\mathrm{Lt}}(x-\alpha_2)(x-\alpha_3)\dots(x-\alpha_n),$$

$$\Rightarrow x\alpha_1{}^{n-1} - na = (\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)\dots(\alpha_1-\alpha_n),$$

$\Rightarrow n\alpha_1{}^{n-1} - na = A. \therefore A - n\alpha_1{}^{n-1} = -na.$

Q10. Draw the graph of $\boldsymbol{f(x) = x^2 + x + \frac{1}{x} + \frac{1}{x^2}}$ for x ≠0. Show that the function f(x) defined for positive real numbers attains a unique

minimum. What is the minimum value of the function? What is the value of x of which the minimum is attained?

Sol. : $y = x^2 + x + \frac{1}{x} + \frac{1}{x^2}; x \neq 0$

$$\therefore \frac{dy}{dx} = 2x + 1 - \frac{1}{x^2} - \frac{2}{x^3} = 0, \Longrightarrow x \neq 1$$

$$\frac{d^2y}{dx^2} = 2 + \frac{2}{x^3} + \frac{6}{x^4}.$$

$$\therefore \left[\frac{dy}{dx^2}\right]_{x=\neq 1} = 2 \neq 2 + 6$$

$$= y \neq 2 > 0(\min pt).$$

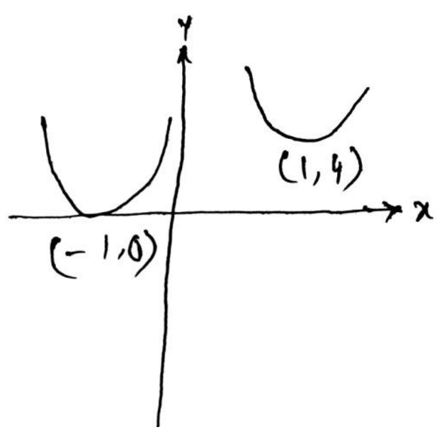
$x = 1, y = 4$

$x = -1, y = 0$

$x \to +\infty, y \to +\infty$

$x \to -\infty, y \to +\infty$

$but\ x > 0, so\ \ min. pt. \to (1, 4)(unique)$



MODEL TEST PAPER - 11

Q1. Given that m and n are relatively prime positive integers greater than one, show that $\frac{log_{10^m}}{log_{10^n}}$ is not a rational number.

Sol. : If possible, let $\frac{log_{10^m}}{log_{10^n}} = \frac{a}{b}$ where a, b are positives integers with no common factor so that $a\ log_{10^n} = a\ log_{10^m}\ or \log {n_{10}}^a = \log {m_{10}}^b$, $or\ n^a = m^b$, but this cannot be true since a and b are integers and n and m are relatively prime i.e. here no common factor accept 1.

Q2. Prove that if the coefficients of the quadratic equation $ax^2 + bx + c = 0$ are odd integers, then the roots of the equations cannot be rational numbers.

Sol. : Let $a = 2m + 1, b = 2n + 1, c = 2k + 1$, then the roots will be rational if $(2n + 1)^2 - 4(2m + 1)(2k + 1)$ is a perfect square.

Since the number is odd, if can be only the square of an odd number, so that$(2n + 1)^2 - 4(2m + 1)(2k + 1) = (21 + 1)^2$,

$or\ (2n + 1)^2 - (21 + 1)^2 = 4(2m + 1)(2k + 1)$,

$or\ (2n + 21 + 2)(2n - 21) = 4(2m + 1)(2k + 1) or\ (n + l + 1)(n - l)$

$= (2m + 1)(2k + 1)$

Now n and l cannot be both even or odd, since then L.H.S will be even and the R. H. S. would be odd. If one of them is even and the other is odd, the first factor is again even and this again gives a contradiction.

Q3. Suppose $n(n+1)a_{n+1} = n(n-1)a_n - (n-2)a_{n-1}$ for every positive n ≥1. Given that $a_0 = 1, a_1 = 2$. Find $\frac{a_0}{a_1} + \frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{100}}{a_{101}}$.

Sol. : If $a_0 = 1, a_1 = 2$, the given equation gives

$$2a_2 = a_0 = 1, \Longrightarrow a_2 = \frac{1}{2} = \frac{1}{21}$$

$$6a_3 = 2a_2 - 0a_1 = 1, \Longrightarrow a_3 = \frac{1}{6} = \frac{1}{31}$$

a	b	c	d
d	c	b	a
b	a	d	c
c	d	a	b

$$12a_4 = 6a_3 - 2 = 1 - \frac{1}{2}, \Longrightarrow a_4 = \frac{1}{24} = \frac{1}{41}.$$

The suggests $a_k = \frac{1}{k_1}$. Suppose this is true for $a_1, a_2, \ldots, a_k$, then

$$k(k+1)a_{k+1} = k(k-1)a_k - (k-2)a_{k-1}$$

Aa	Bb	Cc	Dd
Bd	Ac	Db	Ca
Cb	Da	Ad	Bc
Dc	Cd	Ba	Ab

$$= \frac{k(k-1)}{k!} - \frac{k-2}{(k-1)!}$$

$$= \frac{0}{(k-2)!} - \frac{1}{(k-2)!} + \frac{1}{(k-1)!}$$

$$\therefore a_{k+1} = \frac{1}{(k+1)!}$$

This shows by mathematical induction that $a_k = \frac{1}{k_i}$ for all positive values of k.

Now $\frac{a_0}{a_1} + \frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{110}}{a_{101}} = \frac{1}{2} + \frac{2}{\frac{1}{2}!} + \frac{\frac{1}{2}!}{\frac{1}{3}!} + \cdots + \frac{\frac{1}{100}!}{\frac{1}{101}!} = \frac{1}{2} + 4 + 3 + 4 + 5 + \cdots + 101$

$$= \frac{3}{2} + 1 + 2 + 3 + \cdots + 101 = \frac{3}{2} + \frac{101 - 102}{2}$$

$$= 5151 + 1\,\frac{1}{2} = 5152\,\frac{1}{2}.$$

Q4. Show that it is possible to put pair wise distinct positive integers less than 100 in the cells of a 4× 4 table so that the product of all the numbers in every row and every column are row equal to each other.

Sol. : This can be achieved by using orthogonal latin squares.

A	B	C	D
B	A	D	C
C	D	A	B
D	C	B	A

(1)

(2)

(3)

Fig (1) gives a Latin square since in it each of the letters A, B, C, D occurs once in every row and once in every column. Similarly the (2) fig. represents another latin square in which each of the letters a, b, c, d occurs once in every row and once in every column. The 3rd fig. is obtained by superposing there so that each of the four letters A, B, C, d are each of the four letter a, b, c, d occurs in every row and column and each of the sixteen pairs of these elements occurs once only. If A, B, C, D, a, b, c, d represents numbers and Aa stands for A× a etc. then it is easily seen that the product of elements is every row and every column is ABCD abcd and so is the same for every row and column. We choose there as prime numbers i.e. we take A = 1, B = 2, C = 3, D = 5, a = 7, b = 11, c = 13, d = 17 to get the result for fig (3).

7	22	39	85
34	13	15	21
33	35	17	26
65	51	14	11

The const. is $1 \times 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 = 510510$

Q5. A sequence $[a_n]$ is determine by the rule $a_0 = 9$ and for any positive r, $a_{n+1} = 3a^4n + 4a^3n$. Prove that $a_n = -1\left(mod\ 10^{2^n}\right)$.

Sol. : $a_0 = -1(mod\ 10)$. Suppose the result is true for some n, then $a_n = b_n\ 10^{2n-1}, then$

$$a_n + 1 = 3\left(b_n 10^{3^n} - 1\right)^4 + 4\left(b_n 10^{2^n} - 1\right)^3$$
$$= \left(10^{2^n} b_n - 1\right)^3 \left(3 \times 10^{2^n} b_n + 1\right)$$
$$= \left(10^3 \times\ 2^n {b_n}^3 - 3 \times 10^{2.2^n} {b_n}^2 + 3.10^{2^n} b_n - 1\right)\left(3.10^{2^n} b_n + 1\right)$$
$$= \left(3.10^{2^n} b_n - 1\right)\left(3 \times 10^{2^n} - 1\right)(mod\ 10^{2n+1})$$

$= -1(mod\ 10^{2n+1})$, so that by induction our result follows.

Q6. Prove that $(\boldsymbol{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n})^2 \le \left(\boldsymbol{{a_1}^2 + {a_2}^2 + \cdots + {a_n}^2}\right)\left(\boldsymbol{{b_1}^2 + {b_2}^2 + \cdots + {b_n}^2}\right)$ and the equality sign holds, only when $\boldsymbol{\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}}$.

Deduce that the root mean square ≥ arithmetic mean.

Sol. : Consider the quadratic equation.

$(a_1 x - b_1)^2 + (a_2 x - b_2)^2 + \cdots + (a_n x - b_n)^2 = 0$. This can have real roots only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n} = \frac{a}{b}$ (say) and in this case the two real roots councicle and each is equal to $\frac{a}{b}$.

In all other cases, the roots are complex and the discriminate of the quadratic equation is negative. The quadratic equation is

$x^2({a_1}^2 + {a_2}^2 + \cdots + {a_n}^2) - 2x$

$(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n) - \left({b_1}^2 + {b_2}^2 + \cdots + {b_n}^2\right) = 0$

Since its discriminant ≤ 0, we get

$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq ({a_1}^2 + {a_2}^2 + \cdots + {a_n}^2)({b_1}^2 + {b_2}^2 + \cdots + {b_n}^2)$ and the equality sign holds only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$.

Putting $b_1 = b_2 = \cdots b_n = 1$, we get

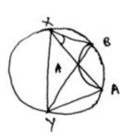
$(a_1 + a_2 + \cdots + a_n)^2 \leq n({a_1}^2 + {a_2}^2 + \cdots + {a_n}^2)$.

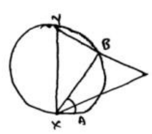
$$\Rightarrow \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^2 \leq \frac{{a_1}^2 + {a_2}^2 + \cdots + {a_n}^2}{n}$$

$or,\ A.M. \leq R.M.S.$

Q7. If A and B are fixed points on a given circle not collinear with centre O of the circle and if XY is a variable diameter, find the locus of P (the intersection of the line through A and X and the line through B and Y).

Sol. :





∠XBY =90°(angle in a semi-circle) ∠ A× $B = \alpha$(const. angle subtended by given are AB at any point on the circle)

∠XPB = 90° $- \alpha$ and ∠APB = 90° $+ \alpha$ = const. so that the arc of AB subtends a const.

angle 90° $+ \alpha$ at P.

∴ Locus of P is the arc of a circle passing through A and B and subtending a const. angle 90° $+ \alpha$ at all points on this arc in fig.(2), the locus is again the arc of a circle passing through A and B with angle 90°X.

Q8. Find the max. and min. values of sin x sin 2x and draw its graph.

Sol. : Let $z = \sin x \sin 2x = \frac{1}{2}[\cos x - \cos 3x] = \frac{1}{2}$

$$[\cos x - 4cos^3 x + 3\cos x] = 2\cos x\,(1 - cos^2 x) = 2(y - y^3),$$

$$y = \cos x\,, so\ that \frac{dz}{dy} = 2(1 - 3ye), \frac{d^2z}{dy^2} - 12y, \frac{dz}{dy} = 0, \Rightarrow y =\neq \frac{1}{\sqrt{3}}, \frac{d^2y}{dz^2} < 0, when$$

$y = \frac{1}{\sqrt{3}}$ and it is > 0 when $y = -\frac{1}{\sqrt{3}}$ there is a max. value when $\cos x = \frac{1}{\sqrt{3}}$ and a min. value when $\cos x = -\frac{1}{\sqrt{3}}$. The max. value is 2 $\frac{1}{\sqrt{3}}\left(1 - \frac{1}{3}\right) = \frac{4}{3\sqrt{3}} = .7698, x = .3041\pi > \frac{\pi}{4}$.

The main value is $-2.\frac{1}{\sqrt{3}}\left(1 - \frac{1}{3}\right) = -\frac{4}{3+3} = -7698, x = .6959\pi < \frac{3\pi}{4}$.

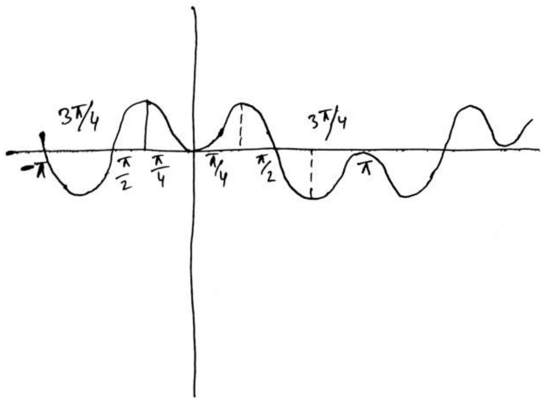
Now $\frac{dz}{dx} = \frac{dz}{dy}.\frac{dy}{dx} = -\frac{dz}{dy}\sin x\,, so\ that \frac{dz}{dx}$ also vanishes when $\sin x = 0. \cos x\ =\neq 1, z = 0$

$Let \cos\alpha = \frac{1}{\sqrt{3}}, 0 < \alpha < \frac{\pi}{2}$, then the given function has max value of .7698 at ≠ α, ≠ α≠ 2π, ≠α≠ 4π, …, ≠

$(\pi - \alpha) \neq 2k\pi, k = 0, 1, 2, 3, \ldots ..$

It has also min. and min. value at x = xπ, when n is an integer, positive, negative on zero.

The graph is, therefore, as shown below.



Q9. Let f be a real function with a continuous third derivative such that $\boldsymbol{f(x), f^n(x), f^m(x)}$ are positive for all real x suppose that $\boldsymbol{f^m(x) \leq f(x) \forall x}$.

Show that $\boldsymbol{f'(x) < 2f(x) \forall\, x}$.

Sol. : If f is a differentiable function on all of R $\lim_{n\to\infty} f(x) \geq 0 \; and \; f'(x) > 0 \forall\, x \in R$, then

$f(x) > 0 \forall x \in R$. If f(y) <0 for some x, then f(x) < f(y) ∀ x< y.

∵ f' > 0but then $\operatorname{lt}_{n\to\infty} f(x) \leq f(y) < 0$ from the inequality $f'''(x) \leq f(x)$. We obtain

$$f''f''(x) \leq f''(x).f(x)$$
$$< f''(\alpha).f(x) + f'2(x).$$

∴ $f'(x)$ is positive.

∴ $\frac{1}{2}(f''(x))^2 < f(x).f'(x)$ ………… (1)

∵f(x) and f'''(x) are both positive for all x we have $2f'(x).f''(x) < 2f(x).f'''(x)$ ……………..(2)

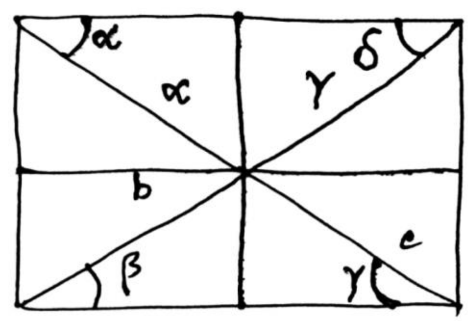
From (1) and (2), $\frac{1}{2}\left\{\frac{f'(x)^2}{2f(x)}\right\} < \frac{1}{2}\left(f''(x)\right)^2 < f(x).f'(x), \Rightarrow \left(f'(x)\right)^3 < \left(f(x)\right)^3$.

∴ $f'(x) < \; 2f(x)$.

Q10. M is an interior pt. of a rectangle ABCD and S is its area prove that S ≤AM. CM + BN .DM.

Sol. : Let G, F, GH be the middle points of the AB, BC, CD, DA.

S_1= area of rectangle AHME + area of rectangle BFME + area of rectangle CGMF + area of rectangle GDHM = $bd\cos\beta\sin\gamma\, s + ac\cos\alpha\sin\gamma + \; bd\cos\delta\sin\beta + \; ac\cos\gamma\sin\alpha = bd\;(\cos\beta\sin\delta + \cos\delta\sin\beta) + \; ac(\cos\alpha\sin\gamma + \sin\alpha\sin\gamma) = bd\cos(\beta+\delta) + ac\sin(\alpha+\gamma) \leq bd + ac = BM.MD + AM.AC$.



MODEL TEST PAPER - 12

Q1. Show that the equation $\boldsymbol{x^n - x^{n-1} - x \ldots.1 = 0}$ has only one positive real root which lines between 1 and 2 and this root aproaches 2 as n→∞.

Sol. : Let $f(x) = \; x^n - x^{n-1} - x^{n-2} \ldots. -1$.

Since there is only one change of sign in f(x) by Descartes rule of signs, these cannot be more than one positive real root of f(x) = 0. Also f(0) = -1, f(1) = -(x- 1)<0, f(2) = $2^n \; \frac{2^n-1}{2-1} = 1 > 0$.

So that f(x) changes sign as x goes from 1 to 2. As such f(x) = 0 has exactly one root lying between 1 and 2.

Let this root by 2- y, then $(2-y)^{n+1} - 2(2-y)^n + 1 = 0$,

$$\Rightarrow\ 2^{n+1}\left(1 - n + 1\frac{y}{2} + \frac{(x+1)n}{2}.\frac{y^2}{4}\ldots..\right)$$
$$-\,2^{n+1}\left(1 - n.\frac{y}{2} + \frac{x(n-1)}{2}.\frac{y^2}{4}\ldots..\right) + 1 = 0,$$

$\Rightarrow 1 - 2^n y + 2^{n-1}y^2 n + \cdots =$
$0\ so\ that\ as\ h \to \infty$, y → 0 and the root of the equation approaches 2.

Q2. Let n be a number consisting of 1991 one's i.e. $n = \underbrace{\mathbf{111\ldots.1\ldots1}}_{\mathbf{1991}}$ show that n is not a prime number.

Sol. : Since 1991 = 11× 181; we can write

$n = \underbrace{\underbrace{11\ldots1}_{11}\ \underbrace{111\ldots1}_{11}\ldots\underbrace{111\ldots.1}_{11}}_{181}$ so that $n =$
$a(10^{1980} + 10^{1959} + 10^{1958} + \cdots + 10^{11} + 1)$

$a = \underbrace{11\ldots1}_{n}$ so that n is not a prime number.

Q3. Find $\sum_{r=1}^{6N-1}\mathbf{min}\left(\left\{\frac{\mathbf{r}}{\mathbf{3N}}\right\},\left\{\frac{\mathbf{r}}{\mathbf{6N}}\right\}\right)$, $\boldsymbol{where}\ \{\boldsymbol{a}\} = \mathbf{min}(\boldsymbol{a} - [\boldsymbol{a}], [\boldsymbol{a}] - \boldsymbol{a} + \mathbf{1})$ the distance to the nearest integer ([a] represents G.I.F).

Sol. : Middle term of the sequence is $\left\{\frac{3N}{3N}\right\} = 0$ and $\left\{\frac{6N-r}{3N}\right\} = \left\{\frac{r}{3N}\right\}, \left\{\frac{6N-r}{6N}\right\} = \left\{\frac{r}{6N}\right\}$

Reqd. sum = $2\sum_{r=1}^{3N-1}\min\left(\left\{\frac{r}{3N}\right\},\left\{\frac{r}{6N}\right\}\right) =$
$2\sum_{r=1}^{2N}\left\{\frac{r}{6w}\right\} + 2\sum_{2N+1}^{3N-1}\left\{\frac{r}{3N}\right\}$

$$= 2\sum_{r=1}^{2N}\frac{r}{6N} + 2\sum_{r=1}^{N-1}\frac{(N-1)N}{2} = N.$$

Q4. Show that if there are $\boldsymbol{n_k} + \mathbf{1}$ pigeons and only n holds for them, then one of the holes has to have k+1 on more pigeons.

Sol. : This is obvious since if all the holes have k or fewer pigeons. Thus the total number of pigeons would be less than or equal to n_k and cannot be $n_k + 1$. This principle is known as the pigeon-hole principle.

Q5. Prove that if $\boldsymbol{n_1}, \boldsymbol{n_2}, \ldots, \boldsymbol{n_k}$ are any integers and a is any number, they $\left[\frac{\boldsymbol{n_1}+\boldsymbol{n_2}+\cdots+\boldsymbol{n_K}}{\boldsymbol{a}}\right] \ge \left[\frac{\boldsymbol{n_1}}{\boldsymbol{a}}\right] + \left[\frac{\boldsymbol{n_2}}{\boldsymbol{a}}\right] + \cdots + \left[\frac{\boldsymbol{n_k}}{\boldsymbol{a}}\right]$, where [x] denotes the integral part of x. Deduce that the high e = t power of a prime p contained in n! is $\left[\frac{\boldsymbol{n}}{\boldsymbol{p}}\right] + \left[\frac{\boldsymbol{n}}{\boldsymbol{p^2}}\right] + \left[\frac{\boldsymbol{n}}{\boldsymbol{p^3}}\right] + \cdots$

Sol. : Let $n_1 = aq_1 + r_1, n_2 = aq_2 + r_2, \ldots n_k = aq_k + r_k\ so\ that\ \frac{n_1+n_2+\cdots+n_k}{a} = \frac{r_1+r_2+\cdots+r_k}{a}$,

$$\Rightarrow \left[\frac{n_1 + n_2 + \cdots + n_k}{a}\right] \ge\ q_1 + q_2 + \cdots + q_k$$

$$=\ \left[\frac{n_1}{a}\right] + \left[\frac{n_2}{a}\right] + \cdots + \left[\frac{n_k}{a}\right].$$

Now out of the numbers 1, 2, …., n. $\left[\frac{n}{p}\right]$ are divisible by r, $\left[\frac{n}{p^2}\right]$ are divisible by p^2 $\left[\frac{n}{p^3}\right]$ are divisible by p^3 and so on, so that the highest power of the prime r contained in n! is $\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots$

Q6. For a cyclic quadrilateral ABCD inscribed in a circle, we have BC = CD. Prove that area of the quadrilateral $\frac{1}{2}(AC)^2 \sin A$.

Sol. : In the quadrilateral ∠DAC = ∠DBC (angles in the same segment), ∠DBC = ∠BDC (∵ BD = CD),

∠BCD = ∠CAB (angles in the same segment), ∠DCA = ∠DBA (angles in the same segment) .

∴Δ's ABE and ACD are similar, so that

$\frac{BE}{AC} = \frac{AE}{AD} = \frac{AB}{AC}$. Similarly since Δ's CED and CAD are similar, we get

$\frac{CE}{CD} = \frac{DE}{DA} = \frac{CD}{AC}$.

Now area of ABCD = area of ΔABD + Area of Δ BCD = $\frac{1}{2} AB \times AD\ Sin\ A + \frac{1}{2} BC \times CD \sin(\pi - A) = \frac{1}{2}\sin A\,[AB \times AD + BC \times BD] = \frac{1}{2}\sin A\,[AC \times AE + AC \times CE]$

$= \frac{1}{2}\sin A\,.\,AC(AE + EC) = \frac{1}{2}\sin A\ \ (AC)^2$.

Q7. Is it possible to divide the plane into polygons so that each polygon is transformed into itself under same rotation by $\frac{2\pi}{7}$ about same pt.? All sides of there polygon must be greater than 1 cm. (A polygon is the part of a plane bounded by one non-self intersecting closed broken line, not necessary convex)?

Sol. : Try yourself.

Q8. Compute the area of the shaded fig. knowing that all areas come from circles with radius R and knowing that A and B are mutually perpendicular axes of symmetry.

Sol. : Since OD = R, OE =R, ED = R, ΔOED is an equilateral triangle and ∠DOE= 60°, ∠DOF = 30°.

Area of shaded strip = $\frac{AR^2}{6} - \frac{1}{2}R.R\cos 30° = R^2$

$\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right).\, Area\ of\ half\ this\ strip$

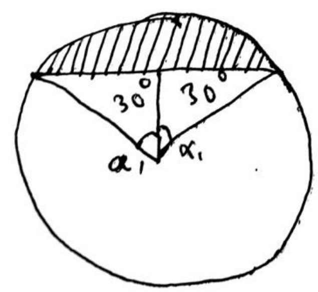
$$= \frac{R^2}{2}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = area\ I.$$

Area of II = $R^2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$.

Area of sector ODF = $\frac{\pi R^2}{12}$.

$\frac{1}{8} of\ reqd.\,area = \frac{\pi R^2}{12} - \frac{3R^2}{2}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$.

∴ Reqd. area =$8R^2\left(\frac{\pi}{12} - \frac{3\pi}{12} + \frac{3\sqrt{3}}{4}\right) - R^2\left(6\sqrt{3} - \frac{4\pi}{3}\right)$.



Q9. Find all solutions $(x_1, x_2, x_3, x_4, x_5)$ of the system of inequalities

$(x_1{}^2 - x_3x_5)(x_2{}^2 - x_3x_5) \le 0$;

$(x_2{}^2 - x_4x_1)(x_3{}^2 - x_4x_1) \le 0$;

$(x_3{}^2 - x_5x_2)(x_4{}^2 - x_5x_2) \le 0$;

$(x_4{}^2 - x_1x_3)(x_5{}^2 - x_1x_3) \le 0\ and$ $(x_5{}^2 - x_2x_4)(x_1{}^1 - x_2x_4) \le 0, where\ x_1, x_2, x_3, x_4, x_5$ **are positive real numbers.**

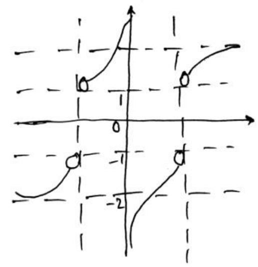
Sol. : Each inequality is of the form $(x_i{}^2 - x_{i+2}x_i + 4)(x_{i+1}{}^2 - x_{i+2}x_i + 4) \le 0$, where the idices are read modulo 5, i.e. $x_j + 5 = x_j$. If we multiply out each expression on the left and then odd all the inequalities, we find that all (5) = 10 terms of the from $x_i{}^2 x_j{}^2 (i \neq j)$ appear the sum, as well as ten cross terms, five of the form $-x_1{}^2 x_1 + 1x_1 + 3$ and five of the form $-x_1{}^2 x_1 + 2x_1 + 4$. This suggests a sum of squares of the form $\frac{1}{2}(y_1{}^2 + y_2{}^2 + \cdots + y_{10}{}^2)$, where to each cross terms, we associate ay, for example, to the cross term $x_2{}^2 - x_3 x_5$, we associate $y_1 = x_2 x_3 - x_2 x_5$, knowing that the terms $x_2{}^2 x_3{}^2, x_2{}^2 x_5{}^2$ appearing in $y_1{}^2$ also appear in our sum. Thus we arrive at the following representation of the sum of the given inequalities.

$$0 \geq \sum_{i=1}^{5} (x_i{}^2 - x_{i+2} x_i + 4)(x_{i+1}{}^2 - x_{i+2} x_i x_{i+4}) = \frac{1}{2} \sum_{i=1}^{5} \{(x_i x_{i+1} - x_i x_{i+3})^2 + (x_{i-1} x_{i+1} - x_{i-1} x_{i+3})^2\}$$

Since this sum of squares cannot be negative. We conclude that it is zero, which means that each term vanishes. This implies that $x_1 = x_2 = x_3 = x_4 = x_5$. Every set of five equal positive numbers is a solution of the given system of inequalities.

Q10. Draw the graph of y = $\frac{(x-1)}{|\ |x|| - 1} + \frac{|\ |x+1||}{(x+1)} + \frac{1}{x}$.

Sol. :



It is clear y is not defined at x = 0, ≠ 1.

MODEL TEST PAPER – 13

Q1. If p(x), Q(x), R(x), S(x) are polynomials so that

$\boldsymbol{P(x^5) + xQ(x^5) + x^2K(x^5)(x^4 + x^3 + x^2 + x + 1)}$**S(x) …… (i), then show that x- 1 is a factor of P(x).**

Sol. : λ Let $\omega = e^{2\pi i/5}$, so that $\omega^5 = 1$. We see for x in (1), $\omega, \omega^2, \omega^3, \omega^4$ successively and get the following equations.

$P(1) + \omega Q(1) + \omega^2 . R(1) = 0.$

$P(1) + \omega^2 Q(1) + \omega^4 R(1) = 0.$

$= 2\int_{f(a)}^{f(b)} x \left[b - \int^{1} (x) \right] dx,$

$P(1) + \omega^4 Q(1) + \omega^3 R(1) = 0,$

$-\omega\ P(1) - \omega^2 Q(1) - \omega^3 R(1) = 0,$

$-\omega^2 P(1) - \omega^4 Q(1) - \omega R(1) = 0,$

$-\omega^3 P(1) - \omega Q(1) - \omega^4 R(1) = 0,$

$-\omega^4 P(1) - \omega^3 Q(1) - \omega^2 R(1) = 0.$

Using $1+\omega + \omega^2 + \omega^3 + \omega^4 = 0$, we get the see m 5P(1)= 0, i.e. $\frac{(x-1)}{P(x)}$.

Q2. Find x in 88 ….. 8x 999….9 (there are 50 eights and 50 mines) so that the number is divisible by 7.

Sol. : If the number is divisible by 7, then so is 11….1x 22……2.

No 111111 is divisible by 7, so that we can remove 48 one's from the left hand side and 48 two's from the R.H.s. if the number without affecting divisibility, so that the given number is divisible by 7 if 11x 22 is divisible by 7.

The requires $2 + 3 \times 2 + 2 \times x + 6 \times 1 + 4 \times 1 \equiv 0 (mod\ 7)$,

$\Rightarrow 2x + 4 \equiv 0\ (mod\ 7), \Rightarrow x = 5$.

Q3. Prove that the equation $\boldsymbol{x^3 + 11^3 = y^3}$ has no solution for positive integers x and y.

Sol. : $11^3 = y^3 - x^3 = (y - x)(y^2 + x^2 + xy)$

$\because y - x\ divides\ 11^3$, we have the following possibilities.

$y - x = 1, \qquad y^2 + x^2 = xy = 11^3$

$y - x = 11, \qquad y^2 + xy + x^2 = 11^2$

$y - x = 11^2, \qquad y^2 + xy + x^2 = 11$

$y - x = 11^3, \qquad y^2 + xy + x^2 = 1$

In the last three cases y >11 and so $y^2 + xy + x^2 > 11^2$ and so no solution is possible. In the first case, we get

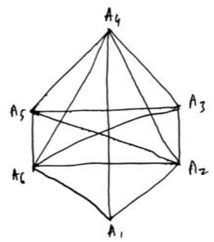
$(x + 1)^2 + x^2 + x(x + 1) = 1331, \Rightarrow 3x^2 + 3x - 1330 = 0$, but 1330 is not divisible by 3, so no solution in integers is possible in this case also.

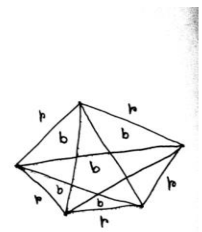
Q4. Show that if in a party there are 6 persons then there must be of least 3 persons who are either mutual acquaintances or mutual strangers.

Sol. : We represents the 6 persons by the 6 vertices $A_1, A_2, A_3, A_4, A_5, A_6$ of a hexagon so that no 3 of those points are collinear and we can draw $6c_2 = 15$ lines joining pair of those 6 points. This gives us 6 edges of a hexagon $A_1A_2A_3A_4A_5A_6$ and 9 diagonals of this hexagon. We draw is of these 15 lines either in red or blue. We draw a red or blue line joining pts. A_i, A_j according as A_i, A_j are acquaintances or strangers. Some of the 15 lines will be red and some will be blue. If we get a triangle all of whose sides are red, we get 3 mutual acquaintances and if we get a triangle all of whose sides are blue, we shall have 3 persons who are mutual strangers. Thus we have to show that in the complete graph of 6 vertices and 15 lines, these must be either one blue triangle or one red, i.e. there must be a monochromatic triangle, whatever be the way in which we colour 15 lines as red or blue.

From the vertex, A_i, 5 edges emerge, some of which are red and some of which blue, by pigeon hole principle at least 3 of these must be of the same colour. For definiteness, let us take there as red. Then the 3 other vertices of their 3 edges will from a triangle with 3 edges of its own. If one of their 3 edges is red, we get a red triangle with 3 red edges and out result is proved.

However if none of their 3 edges is red, we get a blue triangle will all its sides blue. In either case, we get either a red triangle or a blue triangle and our result is proved. The result does not hold if there are only 5 persons in the party since then we shall have a complete graph of 5 edges and 5 diagonals of a pentagon and we can have a graph of the type shown in the following fig 1 where all the sides have been coloured blue.





Q5. Prove that A_1, A_2, A_3, A_4 are the angles of a convex quadrilateral, then $\sin\frac{A_1}{2} + \sin\frac{A_2}{2} + \sin\frac{A_3}{2} + \sin\frac{A_4}{2} \leq 4\sin\frac{\pi}{4}$.

Sol. : Let $A_1, A_2, \dots, A_n$ be the angles of n-sided convex polygon, then $A_1 + A_2 + \cdots A_n = (n-2)\pi, \frac{A_1}{n-2} + \frac{A_2}{n-2} + \cdots + \frac{A_n}{n-2} = \pi$, so that each of the angles $\frac{A_1}{n-2}, \frac{A_2}{n-2}, \dots, \frac{A_n}{n-2}$ is less than π. Now if f(x) = sin x, f'(x) = $\cos x, f''(x) = -\sin x$,

So that if x is between 0 and π, f(a) is a concave function.

Thus $\sin\frac{A_1}{n-2}, \sin\frac{A_2}{n-2}, \dots, \sin\frac{A_n}{n-2}$ are all concave functions and therefore their sum is also a concave function of max. value subjected to $\frac{A_1}{n-2} + \frac{A_2}{n-2} + \cdots + \frac{A_n}{n-2} = \pi$ arise when A : = $\frac{\pi(n-2)}{n}$ for each i, so that the max. value of $\sin\frac{A_1}{n-2} + \sin\frac{A_2}{n-2} + \cdots + \sin\frac{A_n}{n-2} = n.\sin\frac{\pi}{n}, so\ that \sin\frac{A_1}{n-2} + \sin\frac{A_2}{n-2} + \cdots + \sin\frac{A_n}{n-2} \leq n\sin\frac{\pi}{n}$.

Q6. Find the condition that in a triangle of lengths of sides a, b, c you can draw one or more st. lines each of which simultaneously bisects the perimeter and the area of the triangle.

Sol. : If possible let PQ be such a line, then since it bisects the perimeter $x + y = \frac{1}{2}(a+b+c)$ and since the area $\frac{1}{2}xy\sin A = \frac{1}{2}\left(\frac{1}{2}bc.\sin A\right)$,

$$\Rightarrow xy = \frac{1}{2}bc, so\ that\ (x-y)^2$$

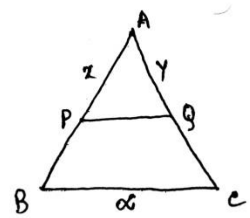
$$= \left(\frac{a+b+c}{2}\right)^2 - 2bc.$$

$$x, y = \frac{1}{2}\left[\frac{a+b+c}{2} \neq \sqrt{\left(\frac{a+b+c}{2}\right)^2 - 2bc}\right].$$

For the line PQ with the reqd. properties to exist, if is necessary that

(i) $(a+b+c)^2 \geq 8bc$,

(ii) $x \leq c, y = b\ or\ x \leq b, y \leq c$.



Similarly, the condition that a line exists intersecting AB and BC and also bisecting perimeter and area simultaneously are (i) $(a + b + c)^2 \geq 8bc$,

(ii)It is possible to cut off lengths $\frac{1}{2}\left[\frac{a+b+c}{2} \pm \sqrt{\left(\frac{a+b+c}{2}\right)^2 - 2bc}\right]$ from the sides AB and BC.

A similar set of conditions can be written down for a line to exist intersecting AC and BC and simultaneously bisecting the perimeter and area of the triangles.

Q7. Check whether the function defined by $f(x+\lambda) = \lambda + \sqrt{2f(x) - f^2(x)}\ \forall x \in R$, is periodic on not, if periodic, then find its period.

Sol. : The given function is true if $2f(x) - f^2(x) \geq 0$,

$\Rightarrow f(x)[f(x) - 2] \leq 0, \Rightarrow 0 \leq f(x) \leq 2 \ldots .(i)$.

Also from the given function, it is clear that $f(x+\lambda) \geq 1, \Rightarrow f(x) \geq 1 \ldots \ldots .(ii)$

From (i) and (ii), we conclude that $1 \leq f(x) \leq 2$.

Again, we have $\{f(x+\lambda) - 1\}^2 = 2f(x) - f^2(a)1$

$\Rightarrow \{f(x+\lambda) - 1\}^2 = -\{f(x) - 1\}^2 + 1 \ldots \ldots \ldots (iii)$

Replacing x by x + λ in above equation, we get

$\{f(x+\lambda) - 1\}^2 \ -\{f(x) - 1\}^2 + 1 \ldots \ldots \ldots (iv)$

From (iv) –(iii), we get

$\{f(x+2\lambda) - 1\}^2 = \{f(x+\lambda) - 1\}^2, \Rightarrow f(x+2\lambda) = f(x), \Rightarrow f$ is a periodic function with period 2λ.

Q8. f : R→ [0, ∞) be a function satisfying $f(x+y) - f(x-y) = f(x)[f(y) - f(-y)]$, $f'(0) = \log a, f(0) = 1$ (for all values of a except 1), then solve the differential $\frac{dy}{dx} = \frac{\{\log a(f(x)f(y)\}^2}{(\log a\, f(x)+2)(\log a\, f(y)-2)}$.

Sol. : Given that $f(x+y) - f(x-y) = f(x)[f(y) - f(-y)] \ \therefore f'(0) = \underset{n\to 0}{\text{Lt}} \frac{f(n)-f(0)}{n}$

$= \underset{n\to 0}{\text{Lt}} \frac{f(n)-1}{n} = \log a \ \ldots . .(i) and$

$f'(x) = \underset{n\to 0}{\text{Lt}} \frac{f(x+n) - f(x-n)}{2n}$

$= \underset{n\to 0}{\text{Lt}} \frac{f(x)[f(n) - f(-n)]}{2n}$

$\Rightarrow f'(x) = \frac{f(x)}{2} \underset{n\to 0}{\text{Lt}} \left[\frac{f(n)-1}{n} + \frac{f(-n)-1}{-n}\right]$

$= \frac{f(x)}{2} 2\log a, \Rightarrow \frac{f'(x)}{f(x)} = \log a, \Rightarrow \log f(a)$

$= x\log a + \log c \ \ (where\ c\ is\ const.), \Rightarrow \log f(x)$

$= \log(a^x c), \Rightarrow f(x) = a^x c \ldots \ldots . .(ii)$

Putting x = 0 in (ii), we get f(0) = c, ⟹ c = 1.

∴ f(x) = $a^x \ldots .(iii)$

∴ Given differential equation $\frac{dy}{dx} = \frac{\{\log a(f(x)f(y)\}^2}{(\log a\, f(x)+2)(\log a\, f(y)-2)}\ becomes \frac{dy}{dx} = \frac{(x+y)^2}{(a+2)(y-2)}$ $(\because f(x) = a^x, \Rightarrow \log a\ \ f(x) = x)$,

$\Rightarrow \frac{dy}{dx} = \frac{(x+2+y-2)^2}{(x+2)(y-2)}, \Rightarrow \frac{dy}{dx} = \frac{(x+y)^2}{xy}$

(Putting x + y = x, y – 2 = y) …….. (iv)

Putting y = +x, so that $\frac{dy}{dx} = t + x\frac{dt}{dx}$

∴ Equation (iv) becomes $\Rightarrow t + x\frac{dy}{dx} = \frac{(1+t)^2}{t} \Rightarrow t + x\frac{dt}{dx} = \frac{1}{t} + t + 2, \Rightarrow x\frac{dt}{dx} = \frac{1+2t}{t}, \Rightarrow \frac{tdt}{1+2t} = \frac{dx}{x}$,

$\Rightarrow \frac{1}{2}\int \frac{(2t+1-1)}{(2t+1)} = \int \frac{dx}{x}$,

$\Rightarrow t - \frac{1}{2}\log|| \ (1+2t)| \ |$

$= 2\log x + c, \Rightarrow \left(\frac{y-2}{x+2}\right)$

$-\frac{1}{2}\log| \ 1 + 2\left(\frac{y-2}{x+2}\right)|$

$= 2\log(x+2) + c.$

Q9. If f(x) is monotonic and differentiable to real valued f function and a, b are two real numbers, show that

$$\int_a^b [f(x) + f(a)]\,[f(x) - f(a)]dx = 2\int_{f(a)}^{f(b)} x\left[b - \int^{1}(x)\right]dx.$$

Sol. : To prove that

$\int_a^b x[f(x) + f(a)]\,[f(x) - f(a)]dx \;=\; 2\int_{f(a)}^{f(b)} x\,[b - f^{-1}(x)]\,dx$….. (i)

We know that a differentiable function is also continuous, so if f is differentiable if will be continuous, f is also monotonous (given). So f is bijective and its inverse also exists. Now, let $f^{-1}(x) = y; \Longrightarrow x = f(x), \Longrightarrow dx = f(y)f'(y)dy,$

$$2\int_{f(a)}^{f(b)} x\,[b - f^{-1}(x)]\,dx$$

$$= \int_a^b 2f(y)(b - y)\,f'(y)dy \quad \big(L.H.S.of\ (i)\big)$$

$$\int_a^b 2xf(y)\,(b-y)dy$$

$$= [(b-y)f^2(y)]_a^b + \int_a^b f^2(y)\,dy$$

$$= -(b-a)f^2(a) + \int_a^b f^2(y)\,dy$$

$$= \int_a^b [f^2(y) - f^2(a)]\,dy$$

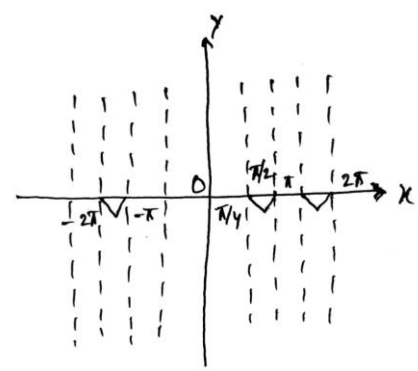
$$= \int_a^b [f^2(x) - f^2(a)]\,dx$$

$$= \int_a^b [f(x) + f(a)]\ [f(a) - f(a)]dx$$

$= R.H.S.of\ (i)proved.$

Q10. Draw the graph of y = $\frac{1}{2}(\tan x + \cot x) + \left\| \frac{1}{2}(\tan x - \cot x)\right\|$.

Sol. : If $\tan x \geq \cot x\,, then\ y = \tan x\ \ and\ if\ \tan x \leq \cot x\,, then\ y = \cot x.$



<u>MODEL TEST PAPER - 14</u>

Q1. Show that A = 10101…… 101 is not a prime number, unless A = 101.

Sol. : $A = \ 10^{2n} + \ 10^{2n+2} + \cdots + \ 10^2 + 1$

$\Longrightarrow 100A = \ 10^{2n+2} + 10^{2n} + 10^6 + 10^4 + 10^2$

$\therefore\!\rightarrow 9A = \ 10^{2n+2} - 1 = (10^{n+1} + 1)(10^{n+1} - 1)$

When n >1, $10^{n+1} - 1 = 9999$ or 999999, or …. And the 2nd factor is divisible by 99. Except when n = 1.

The quotient is x1 and A has two proper factors. If n is even $10^{n+1}+1=1001, 100001, \ldots$ and all there are divisible by 11. Thus A is nonprime except when it is 101.

Q2. Let a, b, c, d be any four positive integers. Let a_1, b_1, c_1, d_1 be the differences $\| a - b\|, || b-c| |, || c-d||, \| d-a||$. In the same way define $a_{n+1} = | |a_n - b_n| |, || b_n + 1||, \| b_n - c_n\|, c_n + 1 = |c_n - d_n|, d_n + 1 = || d_n - a_n||$, show that whatever be the four numbers we start with ultimately four zerox must be obtained.

Sol. : We easily note the following properties:

(i) max $(a_n, b_n, c_n, d_n] \leq \max(a_{n-1}, b_{n-1}c_{n-1.}\, d_{n-1})$ for all $n \geq 1$.

(ii) If ultimately zeros are obtained when we start with kn, kb, kc, kd we shall also obtain zeros when we start with a, b, c, d where k is any positive integers.

(iii) After at most 4 stages, 4 even number are obtained. If we denote by A_n, B_n, C_n, D_n the four numbers after n stages of the process and after cancelling out any common factors we

(iv) $\max(A_n+4, B_n+4, C_n+4, D_n+4) \leq \max(A_n, B_n, C_n, D_n)$

(v) $\max(A_{4k}, B_{4k}, C_{4k}, D_{4k}) \leq \frac{1}{2_k}\max(A_0, B_0, C_0, D_0)$

Hence we must have $\max(A_{4k}, B_{4k}, C_{4k}, D_{4k}) = 0$ for sufficiently large k and the destined result is established.

Q3. The real numbers $a_0, a_1, \ldots, a_n, \ldots$ satisfy the condition: 1 = $a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$ the numbers $b_1, b_2, \ldots, b_n, \ldots$. are defined by

$$b_n = \sum_{k=1}^{n}\left(1 - \frac{a_k - 1}{a_k}\right)\frac{1}{\sqrt{a_k}}.$$

(a) Prove that $0 \leq b_n < 2 \;\forall\, n$.

(b) Given C with $0 \leq c < 2$, prove that there exist numbers $a_0, a_1, \ldots$ with the above properties such that $b_n > c$ for large enough n.

Sol. :

(a) We note that $\frac{a_{k-1}}{a_k} \leq 1, so\ that\ b_n \geq 0$ for all n denote $\sqrt{a_k}\ by\ a_k$. Then K th term of the sum b_n is

$$\left(1 - \frac{{a_k}^2 - 1}{{a_k}^2}\right)\frac{1}{a_k}$$
$$= \frac{{a_k}^2 - 1}{a_k}\left(\frac{1}{{\alpha_{k-1}}^2} - \frac{1}{{a_k}^2}\right.$$
$$= \left.\frac{{a_k}^2 - 1}{\alpha_k}\right)$$
$$\times\left(\frac{1}{\alpha_{k-1}} + \frac{1}{\alpha_k}\right)\left(\frac{1}{\alpha_{k-1}} - \frac{1}{\alpha_k}\right)$$
$$= \frac{\alpha_{k-1}}{\alpha_k}\left(1 + \frac{\alpha_{k-1}}{\alpha_k}\right)$$
$$\times\left(\frac{1}{\alpha_{k-1}} - \frac{1}{\alpha_k}\right)$$
$$\leq 2\left(\frac{1}{\alpha_{k-1}} - \frac{1}{\alpha_k}\right).$$

Adding there inequalities for k = 1, 2, …, n. We observe that the right side from a telescoping sum, and we get $0 \leq b_n < 2\left(\frac{1}{\alpha_0} - \frac{1}{\alpha_n}\right) = 2\left(\frac{1}{\sqrt{\alpha_0}} - \frac{1}{\sqrt{\alpha_n}}\right) = 2\left(1 - \frac{1}{\sqrt{\alpha_n}}\right) < 2\ for\ all\ n.$

(b) Winners, $0 \leq c < 2$, we shall establish the existence of the appropriate a_i by constructing them as terms of a geometric series.

Set $\frac{1}{\sqrt{a_k}} = d^k, then\ the\ k-th\ term\ of\ the\ sum\ b_n\ is\ \left(1-\frac{d^{-2(k-1)}}{d^{-2k}}\right)d^k = (1-d^2)d^k$.

Hence $b_n = \sum_{k=1}^{n}(1-d^2)d^k = (1-d^2)\sum_{k=1}^{n} d^k = (1-d^2)\times\frac{d-d^{n+1}}{1-d} = d(1+d)(1-d^n)$ we must pick d between 0 and 1 so that $b_n = d(1+d)(1-d^n) > c$ for large enough n. We containly need to have d(1 +d) >c.

This can be achieved for any c <2. Since d (1+ d) approaches 2 as d tends to 1.

(In fact the reader can easily verify that d(1+ d) >c if d = $\frac{2}{\sqrt{\frac{c}{2}}}$). Now since d <1, we see that $1-d^n$ is as close to 1 as we please for all sufficiently larger. In particular, $1-d^h > \frac{c}{d}(1+d), i.e. d(1+d)(1-d^h) > c$ for all sufficiently large n. we suggest that the reader find a number N depending on the given C such that $b_n > c$ for all n > N.)

Q4. Let G be the centroid of the Δ ABC. Under rotation by $120°$ about the point G, the pt. B is taken to the pt. P and under n rotation by $240°$ about G, the pt. C is taken to the pt. Q. Prove that either APQ is an equilattienal triangle on the points A, P, Q. Coincide.

Sol. :

Let G be the original and let A, B, C. be represented by complex number a, b, c then after rotation B goes to P is represented by $\frac{be^{2\pi i}}{3}$ and C goes to Q represented by $\frac{Ce^{4\pi i}}{3}$. Also since G is the origin a + b + c = 0, so that points A, P, Q are –(b + c), $\frac{be^{2\pi i}}{3}, \frac{Ce^{4\pi i}}{3}, or -(b+c), bw^2, Cw^4$, when w = $\frac{e^{2\pi i}}{6}$, so that $w^2 - w + 1 = 0, w^3 = -1$.

Then A, P, Q are –(b + c), bw^2, $-cw$. If A and P coincide the $n - b - c - bw^2 = 0\ on - c - bw = 0, or\ c + bw = 0, or\ bw^2 = -cw = cw^4$. So that P and Q also coincide. If A and P do not coincide, then $P - a = bn^2 - a = (b+a)\omega^2 - a(1+\omega^2) = -c\omega^2 - a\omega = c\omega^5 - a\omega = q\omega - a\omega = (q-a)\omega = (q-a)e^{i\pi/6}$

∴ ∠PAQ = 60°. Also $p - q = \frac{be^{2\pi i}}{3} - \frac{Ce^{4\pi i}}{3} = b\omega^2 - c\omega^4 = \omega^2 - (-a-b)\omega^4$

$$= b\omega^2 + (a+b)\omega^4 = b\omega^2 - (a+b)\omega$$
$$= -b - a\omega = b\omega^3 - a\omega$$
$$= (b\omega^2 - a)\omega = (b-a)\omega.$$

∴ ∠QPA = 60°, so that the ΔAPQ is equilateral.

Q5. Let f(x) be a continuous function is [-1, 1] and satisfies $f(2x^2-1) = 2xf(x) \forall\ x \in [-1,1]$.Prove that f(x) is identically zero for all x ∈ [-1, 1].

Sol. : We have $f(2x^2-1) = 2xf(x)\forall x \in [-1,1]$ ……. (i)

Replacing x → -x, we get $f(2x^2-1) = -2xf(x)$ ……. (ii)

From equation (i) and (ii), we get 2xf(x) = -2xf(-x), on f(x) = -f (–x) ∀ $x \in [-1,1]$ ……. (iii)

Hence f(x) is an odd function. Now putting x = 0 in equation (iii), we have f(0) = -f(0), ⟹f(0) = 0 …..(iv).

Putting $x = \cos\theta$ in equation (i), we get
$f(\cos 2\theta) = 2\cos\theta . f(\cos 2\theta) =$
$2\cos\theta . f\left(2cos^2\frac{\theta}{2} - 1\right)$

$$= 2\cos\theta . 2\cos\frac{\theta}{2} f\left(\cos\frac{\theta}{2}\right)$$
$$= 2^{n+1}. \cos\theta . \cos\frac{\theta}{2}$$

$$\cos\frac{\theta}{2^n - 1} . \cos\frac{\theta}{2^n}\ f\left(\cos\frac{\theta}{2^n}\right.$$
$$= \sin 2\theta . f\left(\cos\frac{\theta}{2^n}\right)\Big) / \sin\frac{\theta}{2^n}$$

Taking limit on the both sides as n → ∞, we get

$$\underset{n\to\infty}{\mathrm{Lt}} f(\cos 2\theta) = \sin 2\theta \underset{n\to\infty}{\mathrm{Lt}} \frac{f\left\{\cos\left(\frac{\theta}{2^n}\right)\right\}}{\sin\left(\frac{\theta}{2^n}\right)},$$

$$\Rightarrow f(\cos 2\theta)$$
$$= \sin 2\theta , \underset{n\to\infty}{\mathrm{lt}} \frac{f\left(2cos^2\frac{\theta}{2^{n+1}} - 1\right)}{2\sin\frac{\theta}{2^{n+1}}\cos\frac{\theta}{2^{n+1}}} = \sin 2\theta.$$

$$\underset{n\to\infty}{\mathrm{lt}} \frac{f\left(2sin^2\frac{\theta}{2^{n+1}} - 1\right)}{2\sin\frac{\theta}{2^{n+1}}\cos\frac{\theta}{2^{n+1}}} (\because f(x) is\ an\ odd\ function)$$

$$= -\sin 2\theta \left\{\underset{n\to\infty}{\mathrm{lt}} \frac{2\sin\frac{\theta}{2^n+1} f\left(\sin\frac{\theta}{2^n+1}\right)}{2\sin\frac{\theta}{2^n+1}\cos\frac{\theta}{2^n+1}}\right\}$$
$$= 0$$

$$= \sin 2\theta . \left\{\frac{f(0)}{1}\right\} = 0, \Rightarrow f(\cos 2\theta) = 0\ \forall\ 0$$
$$\in R, \Rightarrow f(x) = 0 \forall\ x \in [-1, 1].$$

Q6. Let $f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right) = \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}$ where $x_i \epsilon R\ and\ n \epsilon N$. If f(x) is differentiable and f'(0)=a, f(0) $= b$. Evaluate $\frac{d^k f(x)}{d(x_k)}(k \geq 4)$.

Sol: Given

$$f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

Taking n = 2, we have $f\left(\frac{x_1+x_2}{2}\right) = \frac{1}{2}\{f(x_1) + f(x_2)\}$

(Where $x_1\ and\ x_2$ are independent variables) …. (iii). Now differentiating (ii) w. r. t. x_1, we get $f'\left(\frac{x_1+x_2}{2}\right)\frac{1}{2}\left(1 + \frac{dx_2}{dx_1}\right) = \frac{1}{2}\left\{f'(x_1) + f'(x_2)\frac{dx_2}{dx_1}\right\}$….. (iii) . Since x_2 is independent fx_1 so $\frac{dx_2}{dx_1} = 0$

∴ (iii) $\Rightarrow \frac{1}{2}f'\left(\frac{x_1+x_2}{2}\right) = \frac{1}{2}f'(x_1)$.

Putting $x_1 = 0\ and\ x_2 = x'$, we get

$$\frac{1}{2}f'\left(\frac{0 + x'}{2}\right) = \frac{1}{2}f'(0), \Rightarrow f'\left(\frac{x'}{2}\right) = a,$$
$$\Rightarrow f'(x) = a$$

(Putting x'/2 = x). Integrating above equation. W. r. t. x, we get f(x) = ax + c.

Putting x = 0, we get f(0) = 0+ c, ⟹ c = b

∴ f(x) = ax + b. ∴ $f'(x) = a, f''(x) = 0, f'''(x) = 0, \frac{d^k f(x)}{dx^k} = 0.$

Q7. If f(x) is an increasing function from R →R such that $f''(x) > 0, f(x) \neq 0\ and\ f^{-1}$ exists, then show that $\frac{d^2\{f^{-1}(x)\}}{dx^2} < 0.$

Sol. : Let f be an increasing function $\Rightarrow f'(a) > 0' and\ f''(x) > 0$ …..(i) (given).

Let $g(x) = f^{-1}(x)$ …… (ii), then f{g(x)} = x, $\Rightarrow f'\{g(x)\}. g'(x) \equiv 1, \Rightarrow g'(x) = \frac{1}{f'\{g(x)\}}$ …… (iii).

Again differentiating both sides w. r. t. x, we get $g''(x) = -1\frac{1}{\{f(g(x))\}} f''\{g(x)\}g'(x)$ …. (iv).

Let g(x) = $f^{-1}(x) = y, \Rightarrow x = f(y), \Rightarrow 1$

$$= f'(y)\frac{dy}{dx}, \Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)} > 0, \Rightarrow g'(x) > 0$$

∴ From equation (iv), we have $\frac{d^2}{dx^2}\{g(x)\} < 0$

(∵ f′(g(x))>0), $\Rightarrow \frac{d^2\{f^{-1}(x)\}}{dx^2} < 0$

$[\because g(x) = f^{-1}(x)]$.

Q8. Let a_1, a_2 … be a non-decreasing sequence of positive integers. For m ≥1, define $b_m = \min\{n: a_n \geq m\}$, $i.e.\ b_m$ is the minimum value of n such that $a_1 \geq m$. If $a_{19} = 85$, determine the maximum value of $a_1 + a_2 + \cdots + a_{19} + b_1 + b_2 + \cdots + b_{85}$.

Sol. : We will show that if $a_q = p$, then $S_{pq} = a_1 + a_2 + \cdots + a_q + b_1 + b_2 + b_p = p(q+1)$. In particular, for the case q = 19, p = 85, we have the sum $p(q+1) = 1700$.

If $a_i = p\ for\ all\ 1 \leq i \leq q, then\ b_j = 1\ for\ all\ 1 \leq j \leq p.$

Hence, $s_{pq} = pq + p$ as reqd. If not, let t be the largest index such that $a_i < p. Let\ a_i = u. If\ a_i$ is increased by 1, then all the b_j remain unchanged except for b_{n+1} which decreases by 1. Hence the value of the desired sum is unchanged. By repeating this increment process (in decreasing order of the subscript so as to maintain a non-decreasing sequence) as long as necessary, we will eventually arrive at the const. sequence which gives the desired result.

Q9. Find out the area bounded by the curve y =

$$\int_{1/8}^{sin^2 x} (\sin^{-1}\sqrt{t})\, dt + \int_{1/8}^{cos^2 x} (\cos^{-1}\sqrt{t})\, dt\ (0 \leq x \leq \pi\ 12)$$

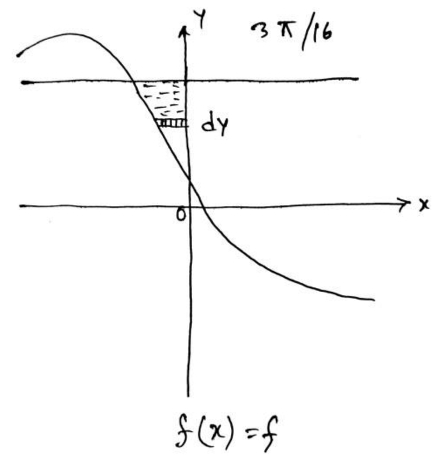
And the curve satisfying the differential equation $y(x + y^3)dx = x(y^3 - x)dy$ passing through $(4, -12)$.

Sol. : Given differential equation is $x(ydx + xdy) = y^3(xdy - ydx). \Rightarrow xd(xy)y^3x^2\left(\frac{xdy-ydx}{x^2}\right)$.

$$\Rightarrow xd(xy) = x^2y^3 d\left(\frac{y}{x}\right), \Rightarrow \frac{d(xy)}{(xy)^2} = \frac{y}{x}.d\left(\frac{y}{x}\right).$$

On integrating,

$-\frac{1}{xy} = \frac{1}{2}\left(\frac{y}{x}\right)^2 + c$…….. (i)



Curve (i) passes through

$(4, -2), \Rightarrow \frac{1}{8} = \frac{1}{8} + c \Rightarrow c = 0.$

Hence curve (i) becomes

$y^3 + 2x = 0, \implies y = (-2x)^{1/3}, \implies f(x) = (-2x)^{1/3}$

The second equation is

$y = \int_{1/8}^{sin^2 x} \sin^{-1}\sqrt{t}\, dt + \int_{1/8}^{cos^2 x} \cos^{-1}\sqrt{t}\; dt$ .... (ii)

$\implies y' = x.2\sin x\cos x + x.2\cos x\,(-\sin x) = 0$

$\implies y = c_1(const.). Now\ putting \sin x = \cos x = \frac{1}{\sqrt{2}}$ is equation (ii), we get y = $\int_{1/8}^{1/2}(\sin^{-1}\sqrt{t} + \cos^{-1}\sqrt{t})\, dt = \int_{1/8}^{1/2}\left(\frac{\pi}{2}\right) dt = \frac{3\pi}{16}$........... (iii)

Now reqd. area

$$= \int_0^{3\pi/16} xdy = \left|\left(\int_0^{\frac{3\pi}{16}}\left(-\frac{y^3}{2}\right) dy\right)\right|$$
$$= \frac{1}{8}\left(\frac{3\pi}{16}\right)^4.$$

Q10. If $\Psi$is the difference of eccentric angles of two points on an ellipse, the tangents of which are at right angles. Prove that $ab\sin\Psi = d_1 d_2$, where d_1, d_2, are the semi-diameters parallel to the tangents at the points and a, b, are semi-axes of the ellipse.

Sol. : Let the given ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...... (i). Let P(α) and Q(β) be two pts. On (i) such that $\Psi = \alpha - \beta$...... (iii). Given that tangents at P and Q are at right angles.

$\therefore \left(-\frac{b}{a}\cot\alpha\right)\left(-\frac{b}{a}\cot\beta\right) = -1, \implies a^2\sin\alpha\sin\beta + b^2$

$\cos\alpha\cos\beta = 0$ ...... (iii). But the diameter parallel to the tangent at P(x) will be conjugate to the diameter CP then its extremities will be. $(-a\ sin\ \alpha, b\cos\alpha)$.

$\therefore {d_1}^2 = a^2\ sin^2\alpha + b^2 cos^2\alpha.$

$Similarly, {d_2}^2 = a^2\ sin^2\beta + b^2 cos^2\beta, \implies {d_1}^2 {d_2}^2$

$= (a^2\ sin^2\alpha + b^2 cos^2\alpha)(a^2\ sin^2\beta + b^2 cos^2\beta)$

$= (a^2\ sin\alpha\sin\beta + b^2\cos\alpha\cos\beta)^2 + a^2b^2(sin\alpha\ cos\beta - cos\alpha\ sin\beta)^2 = 0 + a^2b^2 sin^2(\alpha - \beta)$

$[from\ (\beta)] = a^2b^2 sin^2\Psi\ [from\ (2)]. \therefore \sin\Psi = d_1 d_2.$

MODEL TEST PAPER - 15

Q1. Let $G_n = x^n\sin nA + y^n\sin nB + z^n\sin nC$, where x, y, z, A, B, C are real and A + B+ C is an integral multiple of π. Prove that if $G_1 = G_2 = 0, then\ G_n = 0$ for all positive integral n.

Sol. : A standard trick is to recognize that G_n is the imaginary part of the expression $H_n = x^n e^{inA} + y^n e^{inB} + z^n e^{inC}$.

Suppose that H_n is real for n = 0, 1 , ... , k and consider H_{k+1}. We have

$H_1 H_k = H_{k+1} + H, where\ H = xe^{iA}y^k e^{ikB} + xe^{iA}\, z^k\, e^{ikC} + ye^{iB}\, x^k\, e^{ikA} + ye^{ikB} = e^{ikC} + ze^{iC}\, x^k\, e^{ikA} + z\, e^{iC}\, y^k\, e^{ikB} = xye^{i(A+B)}\left[y^{k-1}e^{i(k-1)B} + x^{k-1}e^{i(k-1)A}\right] + xze^{i(A+C)}\left[z^{k-1}e^{i(k-1)C} + x^{k-1}e^{i(k-1)A}\right] + yze^{i(B+C)}\left[y^{k-1}e^{i(k-1)B} + z^{k-1}e^{i(k-1)C}\right] = xy\, e^{i(A+B)} \times \left[H_{k-1} - z^{k-1}e^{i(k-1)C}\right] + xze^{i(A+C)}\left[H_{k-1} - y^{k-1}e^{i(k-1)B}\right] + yze^{i(B+C)}\left[H_{k-1} - x^{k-1}e^{i(k-1)A}\right] =$

$H_{k-1}\left[xye^{i(A+B)}+xze^{i(A+C)}+yze^{i(B+C)}\right]-xyze^{i(A+B+C)}H_{k-2}$

$=H_{k-1}k-xyze^{i(A+B+C)}H_{k-2}, where\ k = xye^{i(A+B)}+xze^{i(A+C)}+yze^{i(B+C)}$

Observe that $H_2 = {H_1}^2 + 2k$ and since $H_1\ and\ H_2$ are real, by hypothesis, if must be the case that k is real also, by the inductive assumption, $H_{k-1}\ and\ H_{k-2}$ are real.

Because A + B + C is a multiple of π, $e^i(A+B+C)$ is real. Putting there facts together, the formula of the last paragraph show that it is real. Now since H_k is real, by the inductive assumption and since $H_{k+1} = H_1H_{k-1}$, it follows that H_{k+1} is real. Thus, the result of the prob. follows by mathematical induction.

Q2. Let $\{x_n\} and\ \{y_n\}$ denote two sequences of integers defined as follows:

$\boldsymbol{x_0 = 1, x_1 = 1, x_{n-1} = x_n + 2x_{n-1}(n = 1, 2, 3, \ldots.)}$

$\boldsymbol{y_0 = 1, y_1 = 7, y_{n+1} = 2y_n + 3y_{n-1}(n = 1, 2, 3, \ldots.)}$

Thus the first few terms of the sequences are:

X: 1, 1, 3, 5, 11, 21, ….

Y: 1, 7, 17, 55, 161, 487 …….

Prove that, except for the '1', there is no term which occurs both sequences.

Sol. : Mod 8, the first few terms of the two sequences are X: 1, 2, 3, 5, 3, 5 and Y: 1, 7, 1, 7, 1, 7, …… An easy induction shows that this alternate periodic behavior persists. Thus 1 is the only common term of the two sequences. If a, b, x_0, x_1 are given numbers and $x_2, x_3, \ldots$ are determined recursively by means of $x_{n+1} = ax_n + bx_{n-1}, n = 1, 2, 3, \ldots$ and if $a^2 + 4^b \neq 0$, then x_n can be expressed in terms of a, b, x_0, x_1 by the formula

$x_n = \frac{(x_1-k_1x_1){k_2}^n-(x_1-k_2x_1){k_1}^n}{k_2-k_1}$ where k_1, k_2 are the roots of $k^2 - ak - b = 0(if\ a^2 + 4b = 0, k_1 = k_2\ and\ the\ formula\ for\ x_n\ is\ different)$.

1st case: We have $x_0 = 1, x_1 = 1, a = 1, b = 2$ and find $k_2 = 2, k_1 = -1$(or vice-ressa), we get $x_n = \frac{1}{3}[2^{n+1} + (-1)^n]$ for the 2nd sequence, we find $y_n = 2.3^n - (-1)^n. To\ get\ x_n = y_m$, we must have $3^{m+1} - 2^n = \frac{1}{2}[3 - (-1)^m + (-1)^n]$. If n = 0 or 1, we see that m = 0 is the only solution. Hence forth, take n ≥ 2. If m and n are both even or odd, the right member of take this equation is even, but the left member is odd. If m and n are of opposite parity, the equation is invalid mod 4.

Q3. An integer n will be called good if we can write $\boldsymbol{n = a_1 + a_2 + \cdots + a_k}$, where $\boldsymbol{a_1, a_2, a_k}$ are positive integers (not necessarily distinct) satisfying $\boldsymbol{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} = 1}$. Given the information that the integers 33 through 73 are good, prove that every integer ≥ 33 is good.

Sol. : From n good integer n, we produce the two larger good integers 2n + 8 and 2n + 9 as follows: let $(a_1, a_2, \ldots, a_k)$ be a partition of n which is good, then $\frac{1}{2a_1} + \frac{1}{2a_2} + \cdots + \frac{1}{2a_k} = \frac{1}{2}. Since\ \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \frac{1}{3} + \frac{1}{6}$, it follows that the two partitions

$(4, 4, 2a_1, 2a_2, \dots, 2a_k) and\ (3, 6, 2a_1, 2a_2, \dots, 2a_k)$ also have the prop that the sum of the reciprocals is 1.

There are partitions of the integers 2n + 8 and 2n + 9 respectively. So (i) if n is good, so also 2n +8 and 2n +9. So "33 is good" implies that 74, 75 are good. We use the hypothesis to fill the gate between n = 33 and 2n +8 = 74: let s_n denote the statement "all the integers n, n+ 1, …., 2n +7 are good". We begin an induction with the given information that S_{33} is valid. By (i) we conclude that $S_n \to S_{n-1}$. Hence by induction, S_n is valid for all n ≥ 33, giving the desired result.

Q4. Show that $\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} \dots + (-\mathbf{1})^{n+1}.\frac{1}{n}\binom{n}{n} = \mathbf{1} + \frac{1}{2} + \cdots + \frac{1}{n}$.

Sol. : The left side of the identity looks like the definite integral of a binomial series and this provides the idea for the following argument.

$$(1-x)^n = \binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 \dots$$

$$1-(1-x)^n \binom{n}{1}x - \binom{n}{2}x^2 + \binom{n}{3}x^3 \dots,$$

$$\frac{1-(1-x)^n}{x} = \binom{n}{1} - \binom{n}{2}x + \binom{n}{3}x^2 \dots$$

We are now set up to integrate each side from 0 to 1 and we get;

$$\int_0^1 \frac{1-(1-x)^n}{x}dx = \binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} \dots$$

To finish the prob., we must show the integral on the left is equal to $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Let y = 1 –x , then

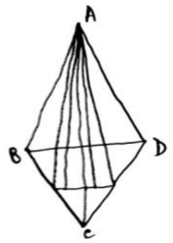
$\int_0^1 \frac{1-(1-x)^n}{x}dx = \int_0^1 \frac{1-y^n}{1-y}dy = \int_0^1 (1 + y + y^2 + \cdots + y^{n-1})\,dy = \left[y + \frac{1}{2}y + \cdots + \frac{1}{n}y^n\right]_0^1 = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

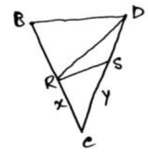
Q5. 9 Mathematicians meet at an international conference and discover that among any 3 of them, at least 2 speak a common language. If each of the mathematicians can speak at most 3 languages. Prove that there are at least 3 of mathematicians who can speak the same language.

Sol. : We assume that at most 2 mathematicians speak a common language. Each mathematician can speak to at most 3 others, one for each language he or she knows. Suppose mathematician M, can only speak with M_2, M_3, M_4. Now mathematicians M_5 can speak with at most three of M_2, M_3, M_4 or at most 3 of M_6, M_7, M_8, M_9. This leaves one of the last 4 who cannot speak with $M_1\ or\ M_5$ giving the desired contradiction.

Q6. Two pts. P and Q lie in the interior of a regular tetrahedron ABCD. Prove that ∠APQ = ∠60°.

Sol. : We can assume without loss of generally that each edge of ABCD= 1, that P an Q lie in the interior of ΔBCD and that line PQ intersects BC in R and CD in S as in fig. Then ∠PAQ ∠RAS. We now show that RS is the shortest side of ΔARS and this implies that ∠RDS ∠60° . Thus RD > RS. Since AR = RD (from congruent triangles BDR and BAR), AR >RS. Similarly, AS > RS. Hence RS is the shortest side of ΔARS.





Q7. Let A_0 denotes the area bounded by $f_n(x) = \left|\frac{\sin 8nx + \cos 8nx}{x}\right|, x - axis, y - axis\ and\ line\ x = \frac{\pi}{8}$. Then prove that $A_n > \frac{2\sqrt{2}}{\pi}\left[1 + \frac{1}{2} + \cdots + \frac{1}{n}\right]$ $(n \in N)$.

Sol. : $f_n(x) \geq 0$, so reqd. area,

$$A_n = \int_0^{\frac{\pi}{8}} \left|\frac{\sin 8nx + \cos 8nx}{x}\right| dx. Put\ 8nx = t,$$

$$\Rightarrow dx = \frac{dt}{8n}.$$

$$\Rightarrow A_n = \int_0^{n\pi} \left|\frac{\sin t + \cos t}{t}\right| dt$$

$$= \int_{(n-1)\pi}^{\pi} \left|\frac{\sin t + \cos t}{t}\right| dt$$

Now, $t \in (0, \pi)\ \int_0^{\pi} \left|\frac{\sin t + \cos t}{\pi}\right| < \int_0^{\pi} \left|\frac{\sin t + \cos t}{\pi}\right|$

$$t \in (\pi, 2\pi) \int_0^{2\pi} \left|\frac{\sin t + \cos t}{t}\right|$$

$$< \int_{\pi}^{2\pi} \left|\frac{\sin t + \cos t}{2\pi}\right|$$

$$t \in \{(n - 1)\pi, n\pi\} \int_{(n-1)\pi}^{n\pi} \left|\frac{\sin t + \cos t}{t}\right| \int_{(n-1)\pi}^{n\pi} \left|\frac{\sin t + \cos t}{n\pi}\right|$$

So, $A_n => \frac{1}{\pi}\left[\int_0^{\pi} |\sin t + \cos t|\, dt + \frac{1}{2}\int_{\pi}^{2\pi} |\sin t + \cos t|\, dt \ldots\right]$

$\because |\sin t + \cos t|$ is periodic with period π.

$\Rightarrow \int_0^{\pi} |\sin t + \cos t|\, dt = \int_{\pi}^{2\pi} |\sin t + \cos t|\, dt \ldots$

$$= \int_{(n-1)\pi}^{n\pi} |\sin t + \cos t|\, dt$$

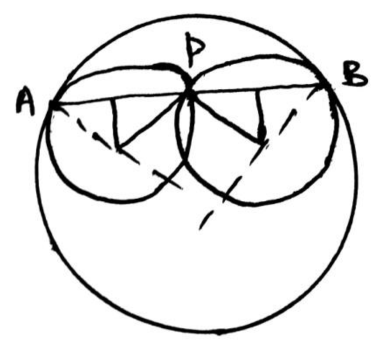
Now, $\int_0^{\pi} \mid \sin t + \cos t \mid dt = \int_{\pi}^{2\pi/4} \mid \sin t + \cos t \mid dt - \int_{3\pi/4}^{3\pi/4} \mid \sin t + \cos t \mid dt = 0$

$\therefore A_n > \frac{1}{n}\left[2\sqrt{2} + \frac{2\sqrt{2}}{2} + \frac{2\sqrt{2}}{3} + \cdots + \frac{2\sqrt{2}}{n}\right]$

$$\therefore A_n > \frac{2\sqrt{2}}{\pi}\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right].$$

Q8. Two given circles intersect in two pts. P and Q. Show how to construct a segment AB passing through P and terminating on the two circles such that AP.PB is a maximum.

Sol. : Since AP = $2\sin\alpha\ and\ BP = 2\sin\beta$, we want to maximize $\sin\alpha \sin\beta$. We note that since $\angle O_1PO_2$ is fixed, so also is the sum $\alpha + \beta$. Now $2\sin\alpha\sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$, and since cosine is a decreasing function, the max. occurs when $\alpha = \beta$. This implies that $AO_1 \parallel PO_2\ and\ BO_2 \parallel PO_2$ and the rest follows as before.



Q9. Find the area enclosed by the curve $y = 2^x\ and\ \max\{|x|, |y|\} = 1$.

Sol. : Given max{ $|x|$, $|y|$ } $= 1$ .... (i) This curve represents four lines.

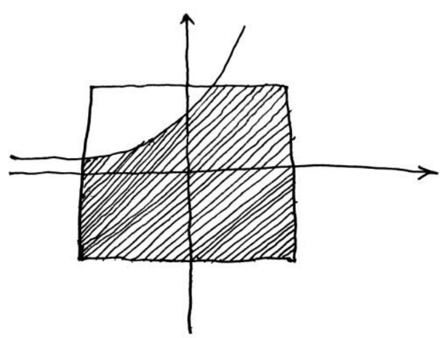
Case I: If $|x| > |y|, \Rightarrow |x| = 1 \therefore x = \neq 1$.

Case II: If $|x| < |y|, \Rightarrow |y| = 1, \therefore y = \neq 1$.

Hence the curve (i) represents a sequence. The shaded area represents the reqd. area enclosed by the given curves.

∴ Reqd. area A= 3 × area of (OTMP + OPQS)

$$= 3 \times (1 \times 1) + \int_{-1}^{0} 2^x \, dx = 3 + \frac{1}{2}\log_e 2.$$



Q10. Sol. : $\frac{dy}{dx} = \sqrt{\frac{1}{2} + \int_{cos^4 z}^{sin^4 z} \frac{\sqrt{f(m)}}{\sqrt{f(\cos 2z - m)} + \sqrt{f(m)}} dm}$,

$$where\ z = x + y\ and\ \frac{\pi}{2} < (x + y) < \frac{3\pi}{4}.$$

Sol.: Given that

$$\frac{dy}{dx} = \sqrt{\frac{1}{2} + \int_{cos^4 z}^{sin^4 z} \frac{\sqrt{f(m)}}{\sqrt{f(\cos 2z - m)} + \sqrt{f(m)}} dm} \ldots .. (i)$$

Let I = $\int_{cos^4 z}^{sin^4 z} \frac{\sqrt{f(m)}}{\sqrt{f(\cos 2z - m)} + \sqrt{f(m)}} dm$ ...... (ii)

⟹I $= \int_{cos^4 z}^{sin^4 z} \frac{\sqrt{f(\cos 2z - m)}}{\sqrt{f(m)} + \sqrt{f(\cos 2z - m)}} dm\ (using\ prop.)$ .... (iii)

Adding (ii) and (iii), we get

2I = $\int_{cos^4 z}^{sin^4 z} \frac{\sqrt{f(m)} + \sqrt{f(\cos 2z - m)}}{\sqrt{f(m)} + \sqrt{f(\cos 2z - m)}} dm = \int_{cos^4 z}^{sin^4 z} dm = -[sin^4 z + cos^4 z]$,

$$\therefore 1 = -\frac{1}{2}[sin^4 z + cos^4 z] = -\frac{1}{2} + \frac{1}{4} sin^2 2z$$

Hence, differential equation (i) becomes

$$\frac{dy}{dx} = \sqrt{\frac{1}{2} - \frac{1}{2} + \frac{1}{4}\ sin^2 2z}$$
$$= \frac{1}{2}|\sin 2z|, \left|\frac{1}{2} sin^2(x + y)\right|$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2} sin^2(x + y) \ldots .. (iv) \left[\frac{\pi}{2} < x + y < \frac{3\pi}{4}\right]$$

Given that $x + y = z, \Rightarrow 1 + \frac{dy}{dx} = \frac{dz}{dx}$

So, equation (iv) becomes $\frac{dz}{dx} = 1 - \frac{1}{2}\sin 2z$,

$$\Rightarrow \frac{dz}{1 - \sin z \cos z} = dx, \Rightarrow \int \frac{dz}{1 - \sin z \cos z} = \int dx$$

$$\Rightarrow \int \frac{\sec^2 z dt}{\sec^2 z - \tan z} = \int dx,\ \Rightarrow \int \frac{dr}{1 + r^2 - r} = \int dx$$

(Putting tan z = r, so that$\sec^2 z dt = dr$),

$\Rightarrow \int \frac{dr}{\left(r - \frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)^2} = \int dx, \Rightarrow \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2r-1}{\sqrt{3}}\right) = c + x$

$$\Rightarrow \frac{2r-1}{\sqrt{3}} = \tan x\left(\frac{\sqrt{3}}{2}(x+c)\right),$$

$$\Rightarrow \frac{2\tan(x+y)-1}{\sqrt{3}} = \tan\left\{\frac{\sqrt{3}}{2}(x+c)\right\}$$

$$\therefore 2\tan(x+y) = \sqrt{3}\tan\left\{\frac{\sqrt{3}}{2}(x+c)\right\}+1.$$

PROBLEMS WITH SOLUTIONS FOR I.S.I. MSQE. MSQMS, MTECH ENTRANCE TESTS

1. **Prove that for all natural numbers n ≥ 3 there exist odd natural numbers x_n, y_n such that $7{x_n}^2 + {y_n}^2 = 2^n$.**

Sol: For n = 3, we have $x_3 = y_3 = 1$. Now suppose that for a given natural number n we have odd natural numbers x_n, y_n such that $7x_n^2 + y_n^2 = 2^n$ we shall exhibit a pair (X, Y) such that $7X^2 + Y^2 = 2^n$ we shall exhibit a pair (X, Y) such that $7{x_n}^2 + {y_n}^2 = 2^{n+1}$. In fact,

$$7\left(\frac{x_n \pm y_n}{2}\right)^2 + \left(\frac{7x_n \pm y_n}{2}\right)^2 = 2(7{x_n}^2 + {y_n}^2) = 2^{n+1}$$

One of $\frac{(x_n + y_n)}{2}$ *and* $\frac{|x_n - y_n|}{2}$ is odd (as their sum is the larger of x_n *and* y_n which is odd), giving the desired pair.

2. **Let a, b, c be real numbers and let M be the maximum of the function $y = |4x^3 + ax^2 + bx + c|$ in the interval |-1, 1|. Show that M ≥ 1 and find all cases where equality occurs.**

Sol.: a = 0, b = -3, c = 0, where M = 1, with the maximum achieved at -1, -1/2, ½ , 1. On the other hand, if M < 1 for some choice of a, b, c, then

$$(4x^3 + ax^2 + bx + c) - (4x^3 + 3x)$$

Must be positive at -1, negative -1/2, positive at ½, and negative at 1, which is impossible for a quadratic function. Thus M ≥ 1, and the same argument shows that equality only occurs for (a, b, c) = (0, -3, 0). (Note: this is a particular case of the minimum deviation property of Chebyshev polynomials).

3. **The real numbers $a_1, a_2, \ldots, a_n (n \geq 3)$ from an arithmetic progression. There exists a permutation $a_{i1}, a_{i2}, \ldots, a_{in}$ *of* $a_1, a_2, \ldots, a_n$ which is a geometric progression. Find the numbers $a_1, a_2, \ldots a_n$ if they are all different and the largest of them as equal to 1996.**

Sol.: Let $a_1 < a_2 < - < a_n = 1996$ and let q be the ratio of the geometric progression $a_{i1} \ldots \ldots \ldots a_{in}$; clearly q$\neq 0 \pm 1$. By reversing the geometric progression if needed, we may assume |q| > 1, and so $|a_{i1}|$< $|a_{i2}|$ < −$|a_{in}|$. Note that either all of the terms are positive, or they alternate in sign; in the latter case, the terms of either sign form a geometric progression by themselves.

There cannot be three positive terms, or else we would have a three term geometric progression a, b, c which is also an arithmetic progression, violating the AM –GM inequality.

Similarly, there cannot be three negative terms, so there are at most two terms of each sign and n ≤4.

If n = 4, we have $a_1 < a_2 < 0 < a_3 < a_4 and\ 2a_2 = a_2 + a_3, 2a_3 = a_2 + a_4$. In this case, q < -1 and the geometric progression is either a_3, a_2, a_4, a_1 *or* a_2, a_3, a_1, a_4. Suppose the former occurs (the argument in similar in the latter case): then

$2a_3q = a_3q^3 + a_3\ and\ 2a_3 + a_3q^3 + a_3q^2$, giving q =1, a contradiction.

We deduce n = 3 and consider two possibilities. If $a_1 < a_2 < 0 < a_3 = 1996, then\ 2a_2 = a_2q^2 + a_2q$, so $q^2 + q - 2 = 0\ and\ q = -2$, yielding $(a_1, a_2, a_3) = (-3992, -998, 1996)$. If $a_1 < 0 < a_2 < a_3 = 1996$, then

$2a_2 = a_2q + a_2q^2$, so again q = -2, yielding $(a_1, a_2, a_3) = (-998, 499, 1996)$.

4. Find all prime numbers p, q for which pq divides $(5^p - 2^p)(5^q - 2^q)$.

Sol.: If p $\mid 5^p - 2^p, then p \mid$ 5 -2 by Fermat's theorem,

So p = 3, suppose p, q ≠3; then p $\mid 5^q - 2^q$ and q $\mid 5^p - 2^p$. Without lose of generality assume p >q, so that (p, q -1) = 1. Then if a is an integer such that 2a ≡5 (mod q), then the order of a mod q divides p as well as q -1, a contradiction.

Hence one of p, q is equal to 3. If q ≠ 3, then q $\mid 5^3 - 2^3 = 9.13$. so q = 13, and similarly p ∊(3, 13).

Thus the solutions are (p, q) = (3, 3), (3, 13), (13, 3).

5. If α, β, γ are the roots of $x^3 - x - 1 = 0$, compute $\frac{1-\alpha}{1+\alpha} + \frac{1-\beta}{1+\beta} + \frac{1-\gamma}{1+\gamma}$.

Sol.: The given quantity equals

$$2\left(\frac{1}{\alpha+1} + \frac{1}{\beta+1} + \frac{1}{\gamma+1}\right) - 3.$$

Since $P(x) = x^3 - x - 1$ has roots α, β, γ, the polynomial $P(x-1) = x^3 - 3x^2 + 2x - 1$ has roots α+1, β+1, γ+1.

By a standard formula, the sum of the reciprocals of the roots of $x^3 + c_2x^2 + c_1x + c_0\ is - c_1/c_0$, so the given expression equals 2(2)-3= 1.

6. Find all real solution to the following system of equations:

$$\frac{4x^2}{1+4x^2} = y$$
$$\frac{4y^2}{1+4y^2} = z$$
$$\frac{4z^2}{1+4z^2} = x.$$

Sol.: Define $f(x) = \frac{4x^2}{(1+4x^2)}$; the range of f is [0, 1), so x, y, z must lie in that interval. If one of x, y, z is zero, then all three are, so assume they are nonzero. Then $\frac{f(x)}{x} = \frac{4x}{(1+4x^2)}$ is at least 1 but the AM –GM inequality, with equality for x = ½ . Therefore x ≤y ≤ z ≤ x, and so equality holds everywhere, implying x = y = z = ½ . Thus the solutions are (x, y, z) = (0, 0, 0), (½ , ½ , ½).

7. Let f(n) be the number of permutations $a_1, \dots, a_n$ of the integers 1, …, n such that

(i) $a_1 = 1$;

(ii) $|a_i - a_{i+1}| \le 2, i = 1, \dots, n-1.$

Determine whether f(1996) is divisible by 3.

Sol.: Let g(n) be the number of permutations of the desired form with $a_n = n$. Then either $a_{n-1} = n-1$ or $a_{n-1} = n-2$; in the latter case we must have $a_{n-2} = n-1$ and $a_{n-3} = n-3$. Hence g(n) = g(n-1) + g(n -3) for n ≥4. In particular, the values of g(n) modulo 3 are g(1) = 1, 1, 1, 2, 0, 1, 0, 0….. repeating with period 8.

Now let h(n) = f(n) – g(n); h(n) counts permutations of the desired from where n occurs in the middle, sandwiched between n-1 and n -2. Removing n leaves an acceptable permutation, and any acceptable permutation on n -1 symbols can be so produced except those ending in n -4, n -2, n -3, n -1. Hence h(n) = h(n -1)+ g(n -1) –g(n -4) = h(n -1)+ g(n -2); one checks that h(n) modulo 3 repeats with period 24.

Since 1996 ≡ 4 (mod 24), we have f(1996) ≡ f(4) = 4(mod 3), so f(1996) is not divisible by 3.

8. Let ΔABC be an isosceles triangles with AB = AC. Suppose that the angle bisector of ∠B meets AC at D and that BC = BD + AD. Determine ∠A.

Sol.: Let $\alpha =\angle A$, $\beta=\frac{(\pi-\alpha)}{4}$ and assume $AB = 1$.

Then by the Law of Sines,

$$BC = \frac{\sin\alpha}{\sin 2\beta}, BD = \frac{\sin\alpha}{\sin 3\beta}, AD = \frac{\sin\beta}{\sin 3\beta}.$$

Thus we are seeking a solution to the equation $\sin(\pi-4\beta)\sin 3\beta = (\sin(\pi-4\beta) + \sin\beta)\sin 2\beta$.

Using the sum-to-product formula, we rewrite this as

$$\cos\beta - \cos 7\beta = \cos 2\beta - \cos 6\beta + \cos\beta - \cos 3\beta.$$

Cancelling $\cos\beta$, we have $\cos 3\beta - \cos 7\beta = \cos 2\beta - \cos 6\beta$, which implies

$$\sin 2\beta \sin 5\beta = \sin 2\beta \sin 4\beta.$$

Now $\sin 5\beta = \sin 4\beta$, so $9\beta = \pi$ and $\beta = \frac{\pi}{9}$.

9. Let $r_1, r_2, \dots, r_m$ be a given set of positive rational numbers whose sum is 1. Define the function f by $f(n) = n - \sum_{k=1}^{m} \lfloor\lfloor r_k n\rfloor\rfloor$ for each positive integer n. Determine the minimum and maximum values of f(n).

Sol.: Of course $\lfloor\lfloor r_k n\rfloor\rfloor \leq r_k n, so\ f(n) \geq 0$, with equality for n = 0, so 0 is the minimum value. On the other hand, we have $r_k n - \lfloor\lfloor r_k n\rfloor\rfloor < 1, so\ f(n) \leq m-1$.

Here equality holds for n = t- 1 if t is the least common denominator of the r_k.

10. Find the smallest positive integer K such that every K-element subset of (1, 2, …, 50) contains two distinct elements a, b such that a+b divides ab.

Sol.: The minimal value is k = 39. Suppose a, b∈S are such that a + b divides ab. Let c = gcd (a, b) and put a = $ca_1, b = cb_1$, so that a_1 and b_1 are relatively prime. Then c$(a_1 + b_1) divides\ c^2 a_1 b_1, so\ a_1 + b_1\ divides\ ca_1 b_1$.

Since a_1 and b_1 have no common factor, neither do a_1 and $a_1 + b_1$, or b_1 and $a_1 + b_1$.In short, $a_1 + b_1$ divides c.

Since S ⊆ $\{1, \dots, 50\}$, we have a +b ≤99, so c$(a_1 + b_1) \leq 99$, which implies $a_1 + b_1 \leq 9$, on the other hand, of course $a_1 + b_1 \geq 3$. An exhaustive search produces 23 pairs, a, b satisfying the conditions.

$a_1 + b_1 = 3$ (6, 3), (12, 6), (18, 9), (24, 12), (30, 15), (36, 18), (42, 21), (48, 24)

$a_1 + b_1 = 4$ (12, 4), (24, 8), (36, 12), (48, 16)

$a_1 + b_1 = 5$ (20, 5), (40, 10), (15, 10), (30, 20), (45, 30)

$a_1 + b_1 = 6$ (30, 6)

$a_1 + b_1 = 7$ (42, 7), (35, 14), (28, 21)

$a_1 + b_1 = 8$ (40, 24)

$a_1 + b_1 = 9$ (45, 36)

Let M = {6, 12, 15, 18, 20, 21, 24, 35, 40, 42, 45, 48} and T = {1, …, 50} –M. Since each pair listed above contains an element of M, T does not have the desire property. Hence we must take k $\geq |\mathrm{T}| + 1 = 39$. On the other hand, from the 23 pairs mentioned above we can select 12 pairs which are mutually disjoint:

$(6, 3), (12, 4), (20, 5), (42, 7), (24, 8), (18, 9), (40, 10), (35, 14), (30, 15), (48, 16), (28, 21), (45, 36)$.

Any 39-element subset must contain both elements of one of these pairs. We conclude the desired minimal number is k = 39.

11. **Eight singers participate in an art festival where m songs are performed. Each song is performed by 4 singers, and each pair of singers performs together in the same number of songs. Find the smallest m for which this is possible.**

Sol.: Let r be the number of songs each pair of singers performs together, so that

$$m\binom{4}{2} = r\binom{8}{2}$$

And so m = $\frac{14r}{3}$; in particular, m ≥14. However, m = 14 is indeed possible, using the arrangement

{1, 2, 3, 4} {5, 6, 7, 8} {1, 2, 5, 6} {3, 4, 7, 8}

{3, 4, 5, 6} {1, 3, 5, 7} {2, 4, 6, 8} {1, 3, 6, 8}

{2, 4, 5, 7} {1, 4, 5, 8} {2, 3, 6, 7} {1, 4, 6, 7}

{1, 2, 7, 8} {2, 3, 5, 8}

12. **In triangle ABC, $\angle C = 90°, \angle A = 30°$ and $BC = 1$. Find the minimum of the length of the longest side of a triangle inscribed in ABC (that is, one such that each side of ABC contains a different vertex of the triangle).**

Sol.: We first find the minimum side length of an equilateral triangle inscribed in ABC. Let D be a point on BC and put x = BD. Then take points E, F on CA, AB, respectively, such that CE = $\frac{\sqrt{3}x}{2}$ and $BF = 1 - \frac{x}{2}$. A calculation using the Law of Cosines shows that

$$DF^2 = DE^2 = EF^2 = \frac{7}{4}x^2 - 2x + 1$$
$$= \frac{7}{4}\left(x - \frac{4}{7}\right)^2 + \frac{3}{7}$$

Hence the triangle DEF is equilateral, and its minimum possible side length is $\sqrt{\frac{3}{7}}$.

We know argue that the minimum possible longest side must occur for some equilateral triangle. Starting with an arbitrary triangle, first suppose it is not isosceles. Then we can side one of the endpoints of the longest side so as to decrease its length; we do so until there are two longest sides, say DE and EF. We now fix D, move E so as to decrease DE and move F at the same time so as to decrease EF; we do so until all three sides become equal in length. (It is fine if the vertices move onto the extensions of the sides, since the bound above applies in that case as well.)

Hence the minimum is indeed $\sqrt{\frac{3}{7}}$, as desired.

13. Prove that if a sequence $\{G(n)\}_{n=0}^{\infty}$ of integers satisfies

G(0) = 0,

G(n) = $n - G\{G(n)\}$ (n= 1, 2, 3, ….)

then

(a) $G(k) \geq G(k-1)$ for any positive integer k;

(b) No integer k exists such that G(k - 1) = G(k) = G(k +1).

Sol.:

(a) We show by induction that $G(n) - G(n-1) \in \{0, 1\}$ for all n. If this holds up to n, then

$$G(n+1) - G(n) = 1 + G\big(G(n-1)\big) - G\big(G(n)\big).$$

$If\ G(n-1) = G(n), then\ G(n+1) - G(n) = 1$; otherwise, $G(n-1) and\ G(n)$ are consecutive integers not greater than n, so $G\big(G(n)\big) - G\big(G(n-1)\big) \in \{0, 1\}$, again completing the induction.

(b) Suppose that G(k -1)= G(k) = G(k+1)+A for some k, A. Then

A= G(k+ 1)= k +1 –G(g(k))= k+1-G(A)

And similarly A = k –G(A) (replacing k +1 with k above), a contradiction.

Note: It can be shown that G(n) = $\lfloor n\omega \rfloor$ for $\omega = \frac{(\sqrt{5}-1)}{2}$.

14. For which integers k does there exist a function f : N →Z such that

(a) f(1995) = 1996, and

(b) f(xy) = f(x) + f(y) + kf(gcd(x, y))for all x, y ∈ N?

Sol.: Such f exists for k = 0 and k = -1. First take x = y in (b) to get $f(x^2) = (k+2)f(x)$. Applying this twice, we get

$$f(x^4) = (k+2)f(x^2) = (k+2)^2 f(x).$$

On the other hand,

$$f(x^4) = f(x) + f(x^3) + kf(x) = (k+1)f(x) + f(x^3)$$

$$= (k+1)f(x) + f(x) + f(x^2) + kf(x)$$

$$= (2k+2)f(x) + f(x^2) = (3k+4)f(x).$$

Setting x = 1995 so that f(x) ≠ 0, we deduce $(k+2)^2 = 3k+4$, which has roots k = 0, -1. For k = 0, an example is given by

$$f({p_1}^{e_1} \dots {p_n}^{e_n}) = e_1 g(p_1) + \cdots + e_n g(p_n).$$

Where g(5) = 1996 and g(p) = 0 for all primes p ≠5 for k = 1, as example is given by

$$f({p_1}^{e_1} \dots {p_n}^{e_n}) = g(p_1) + \cdots + g(p_n)$$

15. A triangle ABC and points K, L, M on the sides AB, BC, CA respectively, are given such that

$$\frac{AK}{AB} = \frac{BL}{BC} = \frac{CM}{CA} = \frac{1}{3}$$

Show that if the circumcircles of the triangles of the triangles AKM, BLK, CML are congruent, then so are the in circles of these triangles.

Sol.: We will show that ABC is equilateral, so that AKM, BLK, CML are congruent and hence have the same in radius.

Let R be the common circumradius; then

KL = 2R sin A, LM = 2R sin B, MK = 2R sin C,

So the triangles KLM and ABC are similar. Now we compare areas:

[AKM] = [BLK] = [CLM] = $\frac{2}{9}$[ABC],

So, [KLM] = $\frac{1}{3}$[ABC] and the coefficient of similarity between KLM and ABC must be $\sqrt{\frac{1}{3}}$.

By the law of cosines applied to ABC and AKM.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\frac{1}{3}a^2 = \left(\frac{2p}{3}\right)^2 + \left(\frac{c}{3}\right)^2 - 2\frac{2b}{3}\frac{c}{3}\cos A.$$

From these we deduce $a^2 = 2b^2 - c^2$, and similarly $b^2 = 2c^2 - a^2, a^2 = 2a^2 - b^2$. Combining these gives $a^2 = b^2 = c^2$, so ABC is equilateral, as desired.

16. **Let ABC be a triangle and construct squares ABED, BCGF, ACHI externally on the sides of ABC. Show that the points D, E, F, G, H, I are concyclic if and only if ABC is equilateral or isosceles right.**

Sol.: Suppose D, E, F, G, H, I are concyclic; the perpendicular bisectors of DE, FG, HI coincide with those of AB, BC, CA respectively, so the center of the circle must be the circumcenter O of ABC. By equating the distances OD and OF, we find

$$(\cos B + 2\sin B)^2 + sin^2 B$$
$$= (\cos C + 2\sin C)^2 = sin^2 C$$

Expanding this end cancelling like terms, we determine $sin^2 B + \sin B\cos B = sin^2 C + \sin C\cos C$.

Now note that

$$2(sin^2\theta + \sin\theta\cos\theta) = 1 - \cos 2\theta + \sin\theta$$

$$= 1 + \sqrt{2}\sin\left(2\theta - \frac{\pi}{4}\right).$$

Thus we either have B = C or $2B - \frac{\pi}{4} + 2C - \frac{\pi}{4}$

$$= \pi, or\ B + C = \frac{3\pi}{4}.$$

In particular, two of the angles must be equal, say A and B, and we either have A = B = C, so the triangle is equilateral, or $B + (\pi - 2B) = \frac{3\pi}{4}$, in which case A = B = $\frac{\pi}{4}$ and the triangle is isosceles right.

17. **Let a, b be positive integers with a odd. Define the sequence $\{\boldsymbol{u_n}\}$ as follows: $\boldsymbol{u_0 = b}$ & n ∈ ℕ.**

$$\boldsymbol{u_{n+1}} = \begin{cases} \frac{1}{2}\boldsymbol{u_n} & \boldsymbol{if\ u_n\ is\ even} \\ \boldsymbol{u_n + a} & \boldsymbol{otherwise} \end{cases}$$

(a) Show that $\boldsymbol{u_n \le a}$ for some n ∈ℕ.
(b) Show that the sequence $\{\boldsymbol{u_n}\}$ is periodic from some point onwards.

Sol:

(a) Suppose $u_n > a$, if u_n is even, $u_{n+1} = \frac{u_n}{2} < u_n$; if $u_n$is odd, $u_{n+2} = \frac{(u_n+a)}{2} < u_n$. Hence for each term greater than a, there is a smaller subsequent term. These form a decreasing subsequence which must eventually terminate, which only occurs once $u_n \le a$.

(b) If $u_m \le a$, then for all n ≥ m, either $u_n \le a, or, u_n$ is even and $u_n \le 2a$, by induction on n. In particular, $u_n \le 2a\ for\ all\ m \ge n$, and so some value of u_n eventually repeats, leading to a periodic sequence.

18. (a) Find the minimum value of x^x for x a positive real number.

(b) If x and y are positive real numbers, show that $x^x + y^x > 1$.

Sol.:

(a) Since $x^x = e^{x\log x}$ and e^x is an increasing function of x, it suffices to determine the minimum of x log x. This is easily done by setting its derivative 1+ log x to zero, yielding $x = \frac{1}{e}$. The second derivative $\frac{1}{x}$ is positive for x > 0, so the function is everywhere convex, and the unique extremum is needed a global minimum. Hence x^x has minimum value $e^{-1/e}$.

(b) If x ≥ 1, then $x^y \geq 1$ for y > 0, so we may assume 0< x, y<1. Without loss of generality, assume x ≤y; now note that the function $f(x) = x^x + y^x$ has derivative $f'(x) = x^x \log x + y^{x-1}$. Since $y^x \geq x^x \geq x^y$ for $x \leq y$ and $\frac{1}{x} \geq -\log x$, we see that $f'(x) > 0$ for $0 \leq x \leq y$ and so the minimum of f occurs with x = 0, in which case f(x) = 1; since x > 0, we have strict inequality.

19. Starting at (1, 1), a stone is moved in the coordinate plane according to the following rules:

(i) From any point (a, b), the stone can move to (2a, b) or (a, 2b).

(ii) From any point (a, b), the stone can move to (a –b, b) if a > b, or to (a, b –a) if a < b.

For which positive integers x, y can the stone be moved to (x, y)?

Sol.: It is necessary and sufficient that gcd(x, y) = 2^x for some nonnegative integer s. We show necessity by nothing that gcd(p, q) = gcd(p, q –p), so an odd common divisor can never be introduced, and nothing that initially gcd(1, 1)= 1.

As for sufficiency, suppose gcd(x, y) = 2^x. Of those pairs (p, q) from which (x, y) can be reached, choose one to minimize p +q. Neither p and q can be even, else one of $\left(\frac{p}{2}, q\right)$ or $\left(p, \frac{q}{2}\right)$ is an admissible pair. If p > q, then (p, q) is reachable from $\left(\frac{(p+q)}{2}, q\right)$, a contradiction; similarly p < q is impossible. Hence p = q, but gcd(p, q) is a power of 2 and neither p nor q is even. We conclude p =q = 1, and so (x, y) is indeed reachable.

20. Prove that every integer k > 1 has a multiple less than k^4 whose decimal expansion has at most four distinct digits.

Sol.: Let n be the integer such that $2^{n-1} \leq k \leq 2^n$. For n ≤6 the result is immediate, so assume n > 6.

Let S be the set of nonnegative integers less than 10^n whose decimal digits are all 0s or 1s. Since $|S| = 2^n > k$, we can find two elements a < b of S which are congruent modulo, k, and b − a only has the digits 8, 9, 0, 1 in its decimal representation. On the other hand,

$$b - a \leq b \leq 1 + 10 + \cdots + 10^{n-1} < 10^n < 16^{n-1} \leq k^4,$$

Hence b – a is the desired multiple.

21. Given 81 natural numbers whose prime divisors belong to the set {2, 3, 5}, prove there exist 4 numbers whose product is the fourth power of an integer.

Sol.: It suffices to take 25 such numbers. To each number, associate the triple (x_2, x_3, x_5) recording the parity of the exponents of 2, 3 and 5 in its prime factorization. Two numbers have the same triple if and only if their product is a perfect square. As long as there are 9 numbers left, we can select two whose product is a square, in so doing, we obtain 9 such pairs. Repeating the process with the square roots of the products of the pairs, we obtain four numbers whose product is a fourth power.

22. Prove the following inequality for positive real numbers x, y, z:

$$\boldsymbol{(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\geq\frac{9}{4}.}$$

Sol.: After clearing denominators, the given inequality becomes

$$\sum_{sym} 4x^5y - x^4y^2 - 3x^3y^3 + x^4yz - 2x^3y^2z + x^2y^2z^2 \geq 0$$

Where the symmetric sum runs over all six permutations of x, y, z. (In particular, this means the coefficient of x^3y^3 in the final expression is -6, and that $x^2y^2z^2$ is 6.)

Recall the inequality:

$$x(x-y)(x-z)+y(y-z)(y-x)+z(z-x)(z-y)\geq 0$$

Multiplying by 2xyz and collecting symmetric terms, we get

$$\sum_{sym} x^4yz - 2x^3y^2z + x^2y^2z^2 \geq 0$$

On the other hand,

$$\sum_{sym}(x^5y-x^4y^2)+3(x^5-x^3y^3)\geq 0$$

By two applications of AM-GM; combining the last two displayed inequalities gives the desired result.

23. Prove that for every pair m, k of natural numbers, m has a unique representation in the from

$$\boldsymbol{m=\binom{a_k}{k}+\binom{a_{k-1}}{k-1}+\cdots+\binom{a_t}{t}}$$

$$\textbf{where } \boldsymbol{a_k > a_{k-1} > \cdots > a_t \geq t \geq 1}.$$

Sol.: We first show uniqueness. Suppose m is represented by two sequences $a_k, \ldots, a_t$ and $b_k, \ldots, b_t$. Find the first position in which they differ, without loss of generally, assume this position is k and that $a_k > b_k$. Then

$m \leq \binom{b_k}{k}+\binom{b_{k-1}}{k-1}+\cdots+\binom{b_k-k+1}{1} < \binom{b_k+1}{1} \leq m$, a contradiction.

To show existence, apply the greedy algorithm: find the largest a_k such that $\binom{a_k}{m} \leq m$, and apply the same algorithm with m and k replaced by $m-\binom{a_k}{k}$ *and* $k-1$.

We need only make sure that the sequence obtained is indeed decreasing, but this follows because by assumption, $m < \binom{a_k+1}{m}$, *and so* $m-\binom{a_k}{k} < \binom{a_k}{k-1}$.

24. The top and bottom edges of a chessboard are identified together, as are the left and right edges, yielding a torus. Find the maximum number of knights which can be placed so that no two attack each other.

Sol.: The maximum is 32 knights; if the chessboard is alternately colored black and white in the usual fashion, an optimal arrangement puts a knight on each black square. To see that this cannot be improved, suppose that k knights are placed. Each knight attack 8 squares, but no unoccupied square can be attacked by more than 8 knights. Therefore 8k ≤ 8(64 –k), where k ≤ 32.

25. Let P(x) be a polynomial with rational coefficients such that $P^{-1}(Q) \subseteq Q$. Show that P is linear.

Sol: By a suitable variable substitution and constant factor, we may assume P(x) is monic and has integer coefficients; let P(0)= c_0. If p is a sufficiently large prime, the equation P(x)= p +c_0 has a single real root, which by assumption is rational and which we may also assume is positive (since P has positive leading coefficient). However, by the rational root theorem, the only rational roots of P(x) – p - c_0 can be ± 1 *and* $\pm p$. Since the root must be positive and cannot be 1 for large p, we have P(p) –p -$c_0 = 0$ for infinitely many p, so P(x) = x +c_0 is linear.

26. For each positive integer n, find the greatest common divisor of n! +1 and (n+1)!.

Sol: If n + 1 is composite, then each prime divisor of (n+ 1)! is a prime less than n, which also divides n! and so does not divide n! +1. Hence f(n) = 1. If n +1 is prime, the same argument shows that f(n) is a power of n +1, and in fact n +1 | n! +1 by Wilson's theorem. However, $(n+1)^2$ does not divide (n +1)!, and thus f(n) = n +1.

27. For each positive integer n, let S(n) be the sum of the digits in the decimal expansion of n. Prove that for all n, $S(2n) \leq 2S(n) \leq 10S(2n)$ & show that there exists n such that S(n) = 1996S(3n).

Solution: It is clear that S(a +b) ≤ S(a) + S(b), with equality if and only if there are no carries in the addition of a and b. Therefore S(2n) ≤ 2S(n). Similarly S(2n) ≤ 5S(10n) = 5S(n). An example with S(n) = 1996S(3n) is 133 … 35 (with 5968 threes).

28. Let F_n denote the Fibonacci sequence, so that $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for n ≥0. Prove that

(i) The statement "$F_{n+k} - F_n$ is divisible by 10 for all positive integers n" is true if k = 60 and false or any positive integer k < 60;

(ii) The statement "$F_{n+t} - F_n$ is divisible by 100 for all positive integers n" is true if t = 300 and false or any positive integer t<300.

Solution: A direct computation shows that the Fibonacci sequence has period 3 modulo 2 and 20 modulo 5(compute terms until the initial terms 0, 1 repeat, at which time the entire sequence repeats), yielding (a). As for (b), one computes that the period mod 4 is 6. The period mod 25 turns out to be 100, which is awfully many terms to compute by hand, but knowing that the period must be a multiple of 20 helps, and verifying the recurrence $F_{n+8} = tF_{n+4} + F_n$, where t is an integer congruent to 2 modulo 5, shows that the period divides 100, finally, an explicit computation shows that the period is not 20.

29. Prove that for all positive integers n, $\mathbf{2^{1/2}.4^{1/4} \ldots .(2^n)^{1/2^n}} < 4$.

Solution: It is sufficient to show

$$\sum_{n=1}^{x} \frac{n}{2^n} = 2;$$

$$\sum_{n=1}^{x} \frac{n}{2^n} = \sum_{n=1}^{x}\sum_{n=1}^{x} \frac{1}{2^k} = \sum_{n=1}^{x} \frac{1}{2^{n-1}} = 2.$$

30. Let p be a prime number and a, n positive integers.
Prove that if $\mathbf{2^p + 3^p = a^n}$, then n = 1.

Solution: If p = 2, we have $2^2 + 3^2 = 13$ and n = 1. If p > 2, then p is odd, so 5 divides $2^p + 3^p$ and so 5 divides a. Now if n > 1, then 25 divides a^n and 5 divides $\frac{2^p+3^p}{2+3} = 2^{p-1} - 2^{p-2}.3 + \cdots + 3^{p-1} \equiv p2^{p-1}$ (mod 5), a contradiction if p ≠ 5. Finally, if p = 5, then $2^5 + 3^5 = 753$ is not a perfect power, so n = 1 again.

31. Prove that the equation $\mathbf{a^2 + b^2 = c^2 + 3}$ has infinitely many integer solutions (a, b, c).

Sol.: let a be any odd number, let b = $\frac{(a^2-5)}{2}$ *and* $c = \frac{(a^2-1)}{2}$. Then

$$c^2 - b^2 = (c+b)(c-b) = a^2 - 3.$$

32. Let A and B be opposite vertices of a cube of edge length 1. Find the radius of the sphere with center interior to the cube, tangent to the three faces meeting at A and tangent to the three edges meeting at B.

Solution: Introduce coordinates so that A = (0, 0, 0), B = (1, 1, 1) and the edges are parallel to the coordinate axes. If r is the radius of the sphere, then (r, r, r) is its center, and (r, 1, 1) is the point of tangency of one of the edges at B. Therefore $r^2 = 2(1-r)^2, giving\ r^2 - 4r + 2 = 0$ and so r = $2 - \sqrt{2}$ (the other root puts the center outside of the cube).

33. Given an alphabet with three letters a, b, c find the number of words of n letters which contain an even number of a's.

Solution: If there are 2k occurrences of a, these can occur in $\binom{n}{2k}$ places, and the remaining positions can be filled in 2^{n-2k} ways. So the answer is

$$\sum_k \binom{n}{2k} 2^{n-2k}.$$

To compute this, note that

$$(1+x)^n + (1-x)^n = 2\sum_k \binom{n}{2k} x^{2k}.$$

So the answer is

$$\frac{1}{2}2^n\left[\left(1+\frac{1}{2}\right)^n + \left(1-\frac{1}{2}\right)^n\right] = \frac{1}{2}(3^n+1).$$

34. What is the minimum number of squares that one needs to draw on a white sheet in order to obtain a complete grid with n squares on a side?

Solution: It suffices to draw 2n -1 squares: in terms of coordinates, we draw a square with opposite corners (0, 0) and (i, i) for 1 ≤ i ≤ n and a square with opposite corners (i, i) and (n, n) for 1 ≤ i≤ n -1.

To show this many squares are necessary, note that the segments from (0, i) to (1, i) and from (n -1, i) to (n, i) for 0 < i < n all must lie on different squares, so surely 2n -2 squares are needed. If it were possible to obtain the complete grid with 2n -2 squares, each of these segments would lie on one of the squares, and the same would hold for the segments from (i, 0) to (i, 1) and from (i, n-1) to (i, n) for 0 < I < n. Each of the aforementioned horizontal segments shares a square with only two of the vertical segments, so the only possible arrangements are the one we gave above without the square with corners (0, 0) and (n, n), and the 90° rotation of this arrangement, both of which are insufficient. Hence 2n -1 squares are necessary.

35. Consider a triangulation of the plane, i.e. a covering of the plane with triangles such that no two triangles have overlapping interiors, and no vertex lies in the interior of an edge of another triangle. Let A, B, C be three vertices of the triangulation and let θ be the smallest angle of the triangle ΔABC. Suppose no vertices of the triangulation lie inside the circumcircle of ΔABC. Prove there is a triangle σ in the triangulation such that $\sigma \cap \Delta ABC \neq \theta$ and every angle of σ is greater than θ.

Sol.: We may assume $\theta = \angle$A. The case where ABC belongs to the triangulation is easy, so assume this is not the case. If BC is an edge of the triangulation, one of the two triangles bounded by BC has common interior points with ABC, and this triangle satisfies the desired condition. Otherwise, there is a triangle BEF in the triangulation whose interior intersects BC. Since EF crosses BC at an interior point, ∠BEF < ∠BAF < ∠BAC, so triangle BEF satisfies the desired condition.

36. Let m and n be positive integers with gcd(m, n) = 1. Compute gcd($\mathbf{5^m} +$ $\mathbf{7^m}$, $\mathbf{5^n} + \mathbf{7^n}$).

Sol.: Let $s_n = 5^n + 7^n. If\ n \geq 2m$, note that

$$s_n = s_m s_{n-m} - 5^m 7^m s_{n-2m},$$

So gcd(s_m, s_n) = gcd(s_m, s_{n-2m}) … similarly, if m < n < 2m, we have gcd(s_m, s_n)= gcd(s_m, s_{n-2m}). Thus by the Euclidean algorithm, we conclude that if m + n is even, then gcd(s_m, s_n) = gcd(s_1, s_2) = 12, and if m + n is odd, then gcd(s_m, s_n) = gcd(s_0, s_1) = 2.

37. Let x > 1 be a real number which is not an integer. For n = 1, 2, 3, …., let $\boldsymbol{a_n} =$ $\lfloor \lfloor \boldsymbol{x^{n+1}} \rfloor \rfloor - \boldsymbol{x} \lfloor \lfloor \boldsymbol{x^n} \rfloor$. Prove that the sequence $\{\boldsymbol{a_n}\}$ is not periodic.

Solution: Assume, on the contrary, that there exist p > 0 such that $a_{p+n} = a_n$ for every n. Since $\lfloor x_n \rfloor \longrightarrow \infty\ as\ n\ \rightarrow \infty$,we have $\lfloor \lfloor x^{n+p} \rfloor \rfloor - \lfloor \lfloor x^n \rfloor \rfloor > 0$ for some n; then setting $a_{n+p} = a_n$ and solving for x, we get

$$x = \frac{\lfloor x^{n+p+1} \rfloor - \lfloor x^{n+1} \rfloor}{\lfloor x^{n+p} \rfloor - \lfloor x^n \rfloor}$$

And so x is rational.

Put y = x^p and

$$b_m = \sum_{k=0}^{p-1} x^{p-k-1} a_{mp+k}$$
$$= \lfloor \lfloor x^{m+p} \rfloor - x^p \lfloor \lfloor x^m r \rfloor \rfloor$$
$$= \lfloor \lfloor y^{m+1} \rfloor - y \lfloor \lfloor y^m \rfloor \rfloor .$$

Since $a_{p+n} = a_p$, we have $b_{m+1} = b_m$, and y is also rational number which is not an integer. Now put $c_m = \lfloor \lfloor y^{m+1} - y^m \rfloor \rfloor$; *then* $c_{m+1} = yc_m = y^m c_1$. This means c_m cannot be an integer for large m, a contradiction.

38. Let θ be the maximum of the six angles between the edges of a regular tetrahedron and a given plane. Find the minimum value of θ over all positions of the plane.

Sol.: Assume the edges of the tetrahedron Γ = ABCD have length l. If we place the tetrahedron so that AC and BC are parallel to the horizontal plane H. We obtain $\theta = 45°$, and we shall show this is the minimum angle. Let a, b, c, d be the projections of A, B, C, D to the horizontal plane H, and $l_1, \dots, l_6$ the projection of the edges $L_1, \dots, L_6$. Since the angle between L_1 and H has cosine l, it suffices to consider the shortest l_i.

If a, b, c, d from a convex quadrilateral with largest angle at a, then one of ab or ad is at most $\frac{1}{\sqrt{2}}$ since bd $\leq$ 1. Otherwise, it is easily shown that one of the l_1 originating from the vertex inside the convex hull has length at most $\frac{1}{\sqrt{3}}$.

39. Find all pairs (n, r), with n a positive integer and r a real number, for which the polynomial $(\boldsymbol{x+1})^{\boldsymbol{n}} - \boldsymbol{r}$ is divisible by $\boldsymbol{2x^2 + 2x + 1}$.

Sol.: Let t = $\frac{(-1+i)}{2}$ be one of the roots of $2x^2 + 2x + 1$; then $(x+1)^n - r$ is divisible by $2x^2 + 2x + 1$ for r real if and only if $(t+1)^n = r$. Since the argument of t + 1 is $\frac{\pi}{4}$, this is possible if and only if n = 4m, in which case $(t+1)^4 m = (-4)^4$. Hence $(4m, (-4)^m)$ are the only solutions.

40. For a natural number k, let p(k) denote the smallest prime number which does not divide k. If p(k) > 2, define q(k) to be the product of all primes less than p(k), otherwise let q(k)= 1. Consider the sequence. $\boldsymbol{x_0 = 1}$, $\boldsymbol{x_{n+1} = \frac{x_n p(x_n)}{q(x_n)}}$; $\boldsymbol{n = 0, 1, 2}, \dots$

Determine all natural numbers n such that $\boldsymbol{x_n = 111111}$.

Sol.: An easy induction shows that if $p_0, p_1, \dots$, are the primes in increasing order an n has base 2 representations $c_0 + 2c_1 + 4c_2 + \cdots$, then $x_n = p_0{}^{c_0} p_1{}^{c_1} \dots$ in particular, 111111 = 3.7.11.13.37 = $p_1 p_3 p_4 p_5 p_{10}$, *so* $x_n = 111111$ if and only if n $= 2^{10} + 2^5 + 2^4 + 2^3 + 2^1 = 1082$.

41. Find the greatest positive integer n for which there exist n nonnegative integers $\boldsymbol{x_1, x_2, \dots, x_n}$, not all zero, such that for any sequence $\boldsymbol{\epsilon_1, \epsilon_2, \dots, \epsilon_n}$, of elements of {-1, 0, 1}, not all zero, $\boldsymbol{n^3}$ does not divide $\boldsymbol{\epsilon_1 x_1 + \epsilon_2 x_2 + \cdots + \epsilon_n x_n}$.

Solution: The statement holds for n = 9 by choosing $1, 2, 2^2, \dots, 2^8$, since in that case

$$\left| \epsilon_1 + \cdots + \epsilon g 2^8 \right| \leq 1 + 2 + \cdots + 2^8 < 9^3.$$

However, if n = 10, then $2^{10} > 10^3$, so by the pigeonhole principle, there are two subsets A and B of $\{x_1, \dots, x_{10}\}$ whose sums are congruent modulo 10^3. Let $\epsilon_1 = 1$ if x_i occurs in A but not in B, -1 if x_i occurs in B but not in A, and 0 otherwise; then $\sum \epsilon_i x_i$ is divisible by n^3.

42. Let x, y be real numbers. Show that if the set

$$\{\cos(n\pi x) + \cos(n\pi y) \mid n \in N\}$$

Is finite, then x, y ∈ Q.

Sol.: Let $a_n = \cos n\pi x \ and \ b_n = \sin n\pi x$. Then

$$(a_n + b_n)^2 + (a_n - b_n)^2 = 2\left({a_n}^2 + {b_n}^2\right) = 2 + (a_{2n} + b_{2n}).$$

If $\{a_n + b_n\}$ is finite, it follows that $\{a_n - b_n\}$ is also a finite set, and hence that $\{a_n\}$ is finite, since

$$a_n = \frac{1}{2}[(a_n + b_n)(a_n - b_n)].$$

And similarly $\{b_n\}$ is finite. In particular, $a_m = a_n$ for some m < n, and so (n –m)πx is an integral multiple of π. We conclude x and y are both rational.

43. Let n ≥ 3 be an integer and $x_1, x_2, \dots, x_{n-1}$ nonnegative integers such that

$$x_1 + x_2 + \cdots + x_{n-1} = n$$
$$x_1 + 2x_2 + \cdots + (n-1)x_{n-1} = 2n - 2.$$

Find the minimum of the sum

$$F(x_1, \dots, x_{n-1}) = \sum_{k=1}^{n-1} kx_k\,(2n - k).$$

Sol.: The desired sum can be written as

$$2n(2n-2) - \sum_{k=1}^{n-1} k^2 x_k.$$

Now note

$$\sum_{k=1}^{n-1} k^2 x_k = \sum_{k=1}^{n-1} x_k + (k-1)(k+1)x_k \le n + n$$

$$\sum_{k=1} n - 1\,(k-1)x_k = n + n(2n - 2 - n) = n^2 - n.$$

Hence the quantity in question is at most

$2n(2n-2) - (n^2 - n) = 3n^2 - 3n$, with equality for $x_1 = n - 1, x_2 = \cdots = x_{n-2} = 0, x_{n-1} = 1$.

44. Find all prime numbers p, q for which the congruence $\alpha^{3pq} \equiv \alpha(mod\ 3pq)$ holds for all integers α.

Sol.: Without loss of generality assume p ≤ q; the unique solution will be (11, 17), for which one many check the congruence using the Chinese Remainder Theorem.

We first have $2^{3pq} \equiv 2(mod\ 3)$, which means p and q are odd. In addition, if α is a primitive root mod p, then $\alpha^{3pq-1} \equiv 1\ (mod\ p)$ implies that p -1 divides 3pq -1 as well as 3pq -1-3q(p -1) = 3q -1, and conversely that q -1 divides 3p -1. If p = q, we now deduce p = q = 3, but $4^{27} \equiv 1$(mod 27), so this fails. Hence p < q.

Since p and q are odd primes, q ≥ p +2, so $\frac{(3p-1)}{(q-1)} < 3$. Since this quantity is an integer, and it is clearly greater than 1, it must be 2. That is, 2q = 3p +1. On the other hand, p -1 divides 3q -1= $\frac{(9p+1)}{2}$ as well as (9p +1) –(9p -9) = 10. Hence p = 11, q = 17.

45. Let n ≥ 3 be an integer and p ≥ 2n -3 a prime. Let M be a set of n points in the plane, no three collinear, and let f: M ⟶{0, 1, ..., p -1} be a function such that:

(i) Only one point of M maps to 0, and

(ii) If A, B, C are distinct points in M and k is the circumcircle of the triangle ABC, then

$$\sum_{P\in M\cap k} \boldsymbol{f(P) \equiv 0(mod\ P)}.$$

Show that all of the points of M lie on a circle.

Solution: Let X be the point mapping to 0. We first show that if every circle through X and two points of M contains a third point of M, then all of the points of M lie on a circle. Indeed, consider an inversion with center at X. Then the image of M – {X} has the property that the line through any two of its points contains a third point; it is a standard result that this means the points are collinear. (Otherwise, find a triangle ABC minimizing the length of the altitude AH; there is another point N on BC, but then either ABN OR CAN has a shorter altitude than AH, contradiction).

Now suppose the points of M do not lie on a circle. By the above, there exists a circle passing through M and only two points A, B of M. Let f(A) = i, so that by the hypothesis, f(B) = p –i. Let a be the number so circles passing through X, A and at least one other point of M, let b be the number of circles passing through X, B and at least one other point of M, and let S be the sum of f(P) over all P in M. By adding the relations obtained from the circles through X and A, we get S + (a -1) i ≡ 0 (mod p), and similarly, S + (b -1) (p –i) ≡ 0 (mod p). Therefore a + b -2 ≡ 0 (mod p) ; since a + b ≤2n +4 < p, we have a + b = 2 and so a = b = 1, contradicting the assumption that the points do not all lie on a circle.

46. Let x, y, z be real numbers. Prove that the following conditions are equivalent.

(i) x, y, z > 0 and $\boldsymbol{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\le 1}$.

(ii) For every quadrilateral with sides a, b, c, d, $\boldsymbol{a^2x+b^2y+c^2z>d^2}$.

Sol.: To show (i) implies (ii), note that

$$a^2x+b^2y+c^2z \ge (a^2x+b^2y+c^2z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \ge (a+b+c)^2 > d^2.$$

Using Cauchy-Schwarz after the first inequality.

To show (i) implies (ii), first note that if x ≤ 0, we may take a quadrilateral of sides a = n, b = 1, c = 1, d = n and get y + z >$n^2(1-x)$, a contradiction for large n. Thus x > 0 and similarly y > 0, z > 0. Now use a quadrilateral of sides $\frac{1}{x},\frac{1}{y},\frac{1}{z}$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{1}{n}$, where n is large. We then get $\frac{x}{x^2}+\frac{y}{y^2}+\frac{z}{z^2}>\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{1}{n}\right)^2$

Since this holds for all n, we may take the limit as n⟶∞ and get

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\ge\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{1}{n}\right)^2$$

And hence $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\le 1$.

47. Let n be a positive integer and D a set of n concentric circles in the plane. Prove that if the function f : D ⟶D satisfies

d(f(A), f(B)) ≥ d(A, B) for all A, B, ϵ, D, then d(f(A), f(B)) = d(A, B) for every A, B, ϵD.

Sol.: Label the circles $D_1, \dots, D_n$ in increasing order of radius, and let r_1 denote the radius D_i. Clearly the maximum of d(A, B) occurs when A and B are antipodal points on D. Let ABCD be the vertices of a square inscribed in D_n; then f(A) and f(C) are antipodal, as are f(B) and f(D). In addition, each of the minor arcs f(A) f(B) and f(B) f(C) must be at least a quarter arc, thus f(B) bisects one of the semicircles bounded by f(A) and f(C), and f(D) bisects the other. Now if P is any point on the minor arc AB, then the arcs f(P) f(A) and f(P) f(B), which are at least as long as the arc PA and PB, and up to the quarter arc f(P) f(B). We conclude f is isometric on D_n.

Since f is clearly injective and is now bijective on $D_n f$ maps $D_1 U \dots . U D_{n-1}$ into itself. Thus we many repeat the argument to show that f is isometric on each D_i. To conclude, it suffices to show that distances between adjacent circles, say D_1 and D_2, are preserved. This is easy; choose a square ABCD on D_1 and A', B', C', D' be the points on D_2 closet to A, B, C, D, respectively. Then A'B'C'D' also from a square, and the distance from A to C' is the maximum between any point on D_1 and any point on D_3. Hence the eight points maintain their relative position under f, which suffices to prove isometry.

48. Let n ≥ 3 be an integer and X ⊆ {1, 2, …, $\boldsymbol{n^3}$} a set of $\boldsymbol{3n^2}$ elements. Prove that one can find the distinct numbers $\boldsymbol{a_1, \dots, a_9}$ in X such that the system

$$\boldsymbol{a_1 x + a_2 y + a_3 z = 0}$$
$$\boldsymbol{a_4 x + a_5 y + a_6 z = 0}$$
$$\boldsymbol{a_7 x + a_8 y + a_9 z = 0}$$

Has a solution $\boldsymbol{(x_0, y_0, z_0)}$ in nonzero integers.

Sol.: Label the elements of X in increasing order $x_1 < \cdots < a_{3n^2}$, and put

$$X_1 = \{x_1, \dots, x_{n^2}\}, X_2 = \{x_{n^2+1}, \dots ., x_{2n^2}\}, X_3 = \{x_{n^2+1}, \dots, x_{3n^2}\},$$

Define the function f: $X_1 \times X_2 \times X_3 \to X \times X$ as follows: f(a, b, c) = (b –a, c –b).

The domain of f contains n^6 elements. The range of f, on the other hand, is contained in the subset of X × X of pairs whose sum is at most n^3, a set of cardinality.

$$\sum_{k=1}^{n^3-1} k = \frac{n^3(n^3-1)}{2} < \frac{n^6}{2}.$$

By the pigeonhole principle, some three triples (a_i, b_i, c_i) (i = 1, 2, 3) map to the same pair, in which case x =$b_1 - c_1, y = c_1 - a_1, z = a_1 - b_1$ is a solution in nonzero integers. Note that a_i, cannot equal b_j since X_1 and X_2 and so on, and that $a_1 = a_2$ implies that the triple (a_1, b_1, c_1) and (a_2, b_2, c_2) are identical, a contradiction. Hence the nine numbers chosen are indeed distinct.

49. Which are there more of among the natural numbers from 1 to 1000000, inclusive: numbers that can be represented as the sum of a perfect square and a (positive) perfect cube, or numbers that cannot be?

Sol.: There are more numbers that not of this form. Let n = $k^2 + m^3$, where k, m, n ∈ N and n ≤ 1000000. Clearly k ≤ 1000 and m ≤ 100.

Therefore there cannot be more numbers in the desired from than the 1000000 pairs (k, m).

50. Let x, y, p, n, k be natural numbers such that

$$x^n + y^n = p^k.$$

Prove that if n > 1 is odd, and p is an odd prime, then n is a power of p.

Sol.: Let m = gcd(x, y). Then x = mx_1, $y = my_1$ and by virtue of the given equation, $m^n({x_1}^n + {y_1}^n) = p^k$, and so m = p^α for some nonnegative integer α. It follows that ${x_1}^n + {y_1}^n = p^{k-n^\alpha}$. (1)

Since n is odd,

$$\frac{{x_1}^n + {y_1}^n}{x_1 + y_1} = {x_1}^{n-1} - {x_1}^{n-2}y_1 + {x_1}^{n-3}{y_1}^2 - \cdots - x_1{y_1}^{n-2} + {y_1}^{n-1},$$

Let A denote the right side of the equation. By the condition p > 2, it follows that at least one of x_1, y_1 is greater than 1, so since n > 1. A > 1.

From (1) it follows that A$(x_1 + y_1) = p^{k-n^\alpha}$, so since $x_1 + y_1 > 1$, A >1, both of these numbers are divisible by p, moreover, $x_1 + y_1 = p^\beta$ for some natural number β.

Thus

$$A = {x_1}^{n-1} - {x_1}^{n-2}(p^\beta - x_1) + \cdots - x_1(p^\beta - x_1)^{n-2} + (p^\beta - x_1)^{n-1} = n{x_1}^{n-1} + Bp.$$

Since A is divisible by p and x_1 is relatively prime to p, it follows that n is divisible by p.

Let n = pq. Then $x^{pq} + y^{pq} = p^k$ *or* $(x^p)^q + (y^p)^q = p^k$. If q > 1, then by the same argument, p divides q. If q =1, then n = p. Repeating this argument, we deduce that n = p^l for some natural number l.

51. In the Duma there are 1600 delegates, who have formed 16000 committees of 80 persons each. Prove that one can find two committees having no fewer than four common members.

Solution: Suppose any two committees have at most three common members. Have two deputies count the possible ways to choose a chairman for each of three sessions of the Duma. The first deputy assumes that any deputy can chair any session, and so gets 1600^3 possible choices. The second deputy makes the additional restriction that all of the chairmen belong to a single committee. Each of the 16000 committees yields 80^3 choices, but this is an over count; each of the 16000 $\frac{(16000-1)}{2}$ pairs of committees give at most 3^3 overlapping choices. Since the first deputy counts no fewer possibilities than the second, we have the inequality

$$1600^3 \geq 16000.80^3 - \frac{16000.15999}{2}3^3.$$

However,

$$1600.80^3 - \frac{16000.15999}{2}3^3 > 16000.80^3 - \frac{16000.15999}{2}\frac{4^2}{2}$$

$$= \frac{16000.4^3}{4} + 2^{13}.10^6 - 2^{12}.10^6 > 2^{12}.10^6 = 1600^3.$$

We have a contradiction.

52. Show that in the arithmetic progression with first term 1 and ratio 729, there are infinitely many powers of 10.

Sol.: We will show that for all natural numbers n, $10^{81n} - 1$ is divisible by 729. In fact, $10^{81n} - 1 = (10^{81})^n - 1^n = (10^{81} - 1).A, and$

$10^{81n} - 1 = \underbrace{9 \dots 9}_{81}$

$= \underbrace{9 \dots 9}_{9} \dots \underbrace{10 \dots 01}_{8}\underbrace{10 \dots 01}_{8} \dots \underbrace{10 \dots 01}_{8}$

$= 9\underbrace{1 \dots 9}_{9} \dots \underbrace{10 \dots 01}_{8}\underbrace{10 \dots 01}_{8} \dots \underbrace{10 \dots 01}_{8}$

The second and third factors are composed of 9 units, so the sum of their digits is divisible by 9, that is, each is a multiple of 9. Hence $10^{81n} - 1$ is divisible by $9^3 = 729$, as is $10^{81n} - 1$ for any n.

53. Two piles of coins lie on a table. It is known that the sum of the weights of the coins in the two piles are equal, and for any natural number k, not exceeding the number of coins in either pile, the sum of the weights of the k heaviest coins in the first pile is not more than that of the second pile. Show that for any natural number x, if each coin (in either pile) of weight not less than x is replaced by a coin of weight x, the first pile will not be lighter than the second.

Sol.: Let the first pile have n coins of weights

$x_1 \geq x_2 \geq \cdots \geq x_n$, and let the second pile have m coins of weights $y_1 \geq y_2 \geq \cdots \geq y_m, where\ x_1 \geq \cdots \geq x_s \geq x \geq x_{s+1}$ k and $y_1 \geq \cdots \geq y_t \geq x \geq y_{t+1} \geq \cdots \geq y_m$. (If there are no coins of weight greater than x, the result is clear). We need to show that $xs + x_{s+1} + \cdots + x_n \geq xt + y_{t+1} + \cdots + y_m$. Since $x_1 + \cdots + x_n = y_1 + \cdots + y_m = A$, this inequality can be equivalently written $xs + (A - x_1 - \cdots - x_m) \geq xt + (A - y_1 - \cdots - y_t)$, which in turn can be rewritten

$x_1 + \cdots + x_s + x(t - s) \leq y_1 + \cdots + y_t,$

This is what we will prove,

If t ≥ s, then

$$\begin{aligned} x_1 + \cdots + x_s + x(t-s) &= (x_1 + \cdots + x_s) + \underbrace{(x + \cdots + x)}_{t-s} \\ &\leq (y_1 + \cdots + y_s) + (y_{s+1} + \cdots + y_t), \end{aligned}$$

Since $x_1 + \cdots + x_s \leq y_1 + \cdots + y_s$ (from the given condition) and $y_{s+1} \geq \cdots \geq y_t \geq x$.

If t < s, then $x_1 + \dots + x_s + x(t - s) \leq y_1 + \cdots + y_t$ is equivalent to

$x_1 + \cdots + x_s \leq y_1 + \cdots + y_t + \underbrace{(x + \cdots + x)}_{t-s}$

The latter inequality follows from the fact that

$$\begin{aligned} x_1 + \cdots + x_s \leq y_1 + \cdots + y_s &= (y_1 + \cdots + y_t) \\ &+ (y_{t+1} + \cdots + y_s) and\ y_s \leq \cdots \\ &\leq y_{t+1} \leq x. \end{aligned}$$

54. Can a 5×7 checkerboard be covered by L's (figures formed from a 2×2 square by removing one of its four 1×1 corners),

not crossing its borders, in several layers so that each square of the board is covered by the same number of L's?

Solution: Color the cells of the checkerboard alternately black and white, so that the corners are all black. In each black square we write the number -2, and in each white square 1. Note that the sum of the numbers in the cells covered by each L is non-negative, and consequently if we are given a covering of the board in k layers, the sum over each L of the numbers covered by that L is nonnegative. But if this number is S and s is the sum of the numbers on the board, then

S = ks = k (-2.12+23.1)= -k <0. We have a contradiction.

55. Find all natural numbers n, such that there exist relatively prime integers x and y and an integer k >1 satisfying the equation $\boldsymbol{3^n = x^k + y^k}$.

Sol.: The only solution is n = 2.

Let $3^n = x^k + y^k$, where x, y are relatively prime integers with x > y, k > 1, and n a natural number. Clearly neither x nor y is a multiple of 3. Therefore, if k is even$x^k and\ y^k$ are congruent to 1 mod 3, so their sum is congruent to 2 mod 3, and so is not a power of 3.

If k is odd and k > 1, then $3^n = (x + y)(x^{k-1} - \cdots + y^{k-1})$. Thus x + y = 3^m for some m ≥ 1. We will show that n ≥ 2m. Since $\frac{3}{k}$ (see the solution to Russia 3), by putting $x_1 = x^{k/3}\ and\ y_1 = y^{k/3}$ we may assume k = 3.

Then $x^3 + y^3 = 3^m$ and x + y = 3^n. To prove the inequality n ≥ 2m, it suffices to show that $x^3 + y^3 \geq (x+y)^2$, or $x^2 - xy + y^2 \geq x + y$. Since x ≥ y +1, $x^2 - x = x(x-1) \geq xy, and\ (x^2 - x + xy) + (y^2 - y) \geq y(y-1) \geq 0$, and the inequality n ≥ 2m follows.

From the identity $(x+y)^3 - (x^3 + y^3) = 3xy(x+y)$ it follows that $3^{2m-1} - 3^{n-m-1} = xy$.

But 2m -1 ≥ 1, and n –m -1 ≥n -2m ≥0. If strict inequality occurs in either place in the last inequality, then $3^{2m-1} - 3^{n-m-1}$ is divisible by 3 while xy is not. Hence n –m -1 = n -2m = 0, and so m = 1, n = 2 and $3^2 = 2^3 + 1^3$.

Note: The inequality $x^2 - xy + y^2 \geq x + y$ can alternatively be shown by nothing that

$x^2 - xy + y^2 - x - y = (x-y)^2 + (x-1)(y-1) - 1 \geq 0$,

Since $(x-y)^2 \geq 1$.

56. Show that if the integers $\boldsymbol{a_1, \ldots, a_m}$ are nonzero and for each k = 0, 1, ..., m(n < m -1),

$$\boldsymbol{a_1 + a_2 2^k + a_3 3^k + \cdots + a_m m^k = 0},$$

Then the sequences $\boldsymbol{a_1, \ldots, a_m}$ contains at least n + 1 pairs of consecutive terms having opposite sings.

Solution: We many assume $a_m > 0$, since otherwise we may multiply each of the numbers by -1. Consider the sequence $b_1, \ldots, b_m$, where $b_i = \sum_{j=0}^{n} c_j i^j$ for an arbitrary sequence of real numbers $c_0, \ldots, c_n$. From the given condition,

$$\sum_{i=1}^{m} a_i b_i = \sum_{i=1}^{m} a_i \sum_{j=0}^{n} c_j i^j = \sum_{j=0}^{n} c_i \sum_{j=1}^{n} a_i i^j = 0.$$

Suppose now that the sequence $a_1, \dots, a_m$ has k pairs of neighbors that differ in sign, where k < n +1, and let $i_1, \dots, i_k$ be the indices of the first members of these pairs.

Let $b_i = f(i) = (i - x_1)(i - x_2) \dots (i - x_k)$, where $x_k = i_k + \frac{1}{2}(i = 1, 2, \dots, k)$. The function f changes sign only at the points $x_1, \dots, x_k$, and so b_1 *and* b_{i+1} have different signs if and only one of the x_l falls between them, which means i = i_l. We deduce that the sequences $a_1, \dots, a_m$ *and* $b_1, \dots, b_m$ have the same pairs of neighbors of opposite sign. Since a_m *and* b_m are positive, we have that a_i *and* b_i have the same sign for i = 1, …., m, so $\sum_{i=1}^{m} a_i b_i$>0, a contradiction.

57. At the vertices of a cube are written eight pair wise district natural numbers, and on each of its edges is written the greatest common divisor of the numbers at the end points of the edge. Can the sum of the numbers written at the vertices be the same as the sum of the numbers written at the edges?

Sol.: This is not possible. Note that if a and b are natural numbers with a > b, then gcd(a, b) ≤ b and gcd(a, b) ≤$\frac{a}{2}$.It follows that if a ≠ b, then gcd(a, b)≤$\frac{(a+b)}{3}$. Adding 12 such inequalities, corresponding to the 12 edges, we find that the desired condition is only possible if gcd(a, b) = $\frac{(a+b)}{3}$ in each case. But in this case the larger of a and b is twice the smaller; suppose a = 2b. Consider the numbers c and d assigned to the vertices of the other end points of the other two edges coming out of the vertex labeled a. Each of these is either half of or twice a. If at least one is less a, it equals b; otherwise, both are equal. Either option contradicts the assumption that the numbers are distinct.

58. Can the number obtained by written the numbers from 1 to n in order (n >1) be the same when read left-to-right and right-to left?

Sol.: This is not possible. Suppose N = 123 …321 is an m digit symmetric number, formed by writing the numbers from 1 to n in succession. Clearly m >18. Also let A and B be the numbers formed from the first and last k digits, respectively, of N, where k = $\lfloor m/2 \rfloor$ ∟. Then if 10^p is the largest power of 10 dividing A, then n> 2. 10^{p+1}, that is, n has at most p +2 digits. Moreover, A and B must contain the fragments

$$\underbrace{99 \dots 9}_{p}\ \underbrace{100 \dots 01}_{p} \quad and \quad \underbrace{100 \dots 0}_{p}\ \underbrace{199 \dots 9}_{p}$$

Respectively, which is impossible.

59. Do there exist three natural numbers greater than 1, such that the square of each, minus one, is divisible by each of the others?

Sol.: Such integers do not exist. Suppose a ≥ b ≥ c satisfy the desired condition. Since $a^2 -$ 1is divisible by b, the numbers a and b are relatively prime. Hence the number $c^2 - 1$, which is divisible by a and b, must be a multiple of ab, so in particular $c^2 - 1 \geq ab. But\ a \geq c\ and\ b \geq c, so\ ab \geq\ c^2$, a contradiction.

60. Does there exist a finite set M of nonzero real numbers, such that for any natural number n a polynomial of degree no less than n with coefficients in M, all of whose roots are real and belong M?

Solution: Such a set does not exist. Suppose on the contrary that M = $\{a_1, a_1, \dots, a_n\}$ satisfies the desired property. Let m = min $\{|a_1|, \dots, |a_n|\}$ and M = max $\{|a_1|, \dots, |a_n|\}$; the condition implies M ≥ m > 0.

Consider the polynomial P(x) = $b_k x^k + \cdots + b_1 x + b_0$ all of whose coefficients $b_0, \dots, b_k$ are roots $x_1, \dots, x_k$ lie in M. By Vieta's theorem.

$$-\frac{b_{k-1}}{b_k} = x_1 + \cdots x_k$$

$$x_1x_2 + x_1x_3 + \cdots + x_{k-1}x_k = \frac{b_{k-2}}{b_k}$$

And so

$$x_1{}^2 + \cdots + x_k{}^2 = \frac{b_{k-1}{}^2}{b_k{}^2} - 2\frac{b_{k-2}}{b_k}.$$

It follows that

$$km^2 \le x_1{}^2 + \cdots + x_k{}^2 = \frac{b_{k-1}{}^2}{b_k{}^2} - 2\frac{b_{k-2}}{b_k} \le \frac{M^2}{m^2} + 2\frac{M}{m}.$$

Hence $k \le \frac{M^2}{m^4} + \frac{2M}{m^3}$, contradiction the fact that P may have arbitrarily large degree.

61. The natural numbers a and b are such that $\frac{a+1}{b} + \frac{b+1}{a}$

is an integer. Show that the greatest common divisor of a and b is not greater than $\sqrt{a+b}$.

Solution: Let d = gcd(a, b) and put a = md and b = nd. Then we have $\frac{(md+1)}{nd} + \frac{(nd+1)}{md} = \frac{(m^2d+m+n^2d+n)}{mnd}$ is an integer, so that in particular, d divides $m^2d + m + n^2d + n$ and also m + n. However, this means d ≤ m + n, and so d ≤ $\sqrt{d(m+n)} = \sqrt{a+b}$.

62. Let G be the centroid of the triangle ABC. Prove that if AB + GC = AC + GB, then ABC is isosceles.

Sol.: Let a, b, c, be the lengths of sides BC, CA, AB, respectively. By Stewart's theorem and the fact that G trisects each median (on the side further from the vertex), we deduce

$$9GB^2 = 2a^2 + 2c^2 - b^2, 9GC^2 = 2a^2 + 2b^2 - c^2.$$

Now assume b > c. Assuming AB + GC = AC + GB, we have

$$3(b-c) = \sqrt{2a^2 + 2b^2 - c^2} - \sqrt{2a^2 + 2c^2 - b^2}$$

$$= \frac{3(b^2 - c^2)}{\sqrt{2a^2 + 2b^2 - c^2} + \sqrt{2a^2 + 2c^2 - b^2}}$$

$$< \frac{3(b^2 - c^2)}{\sqrt{2\,(b-c)^2 + 2b^2 - c^2} + \sqrt{2\,(b-c)^2 + 2c^2 - b^2}}$$

Since $a^2 > (b-c)^2$ by the triangle inequality. However,

$2\,(b-c)^2 + 2b^2 - c^2 = (2b-c)^2$, so we have

$$3(b-c) < \frac{3(b^2-c^2)}{2b-c+|2c-b|}.$$

If b ≤2c then the two sides are equal, a contradiction. If b ≤ 2c we get $9(b-c)^2 <$

$3(b^2 - c^2)$; upon dividing off 3(b –c) and rearranging, we get 2b < 4c, again a contradiction. Thus we cannot have b > c or similarly b < c, so b = c.

63. Find all real solutions of the equation

$$\sqrt{x^2 - p} + 2\sqrt{x^2 - 1} = x$$

For each real value of p.

Sol.: Squaring both sides, we get

$$x^2 = 5x^2 - 4 - p + 4\sqrt{(x^2 - p)(x^2 - 1}$$

Isolating the radical and squaring again, we get

$$16(x^2 - p)(x^2 - 1) = (4x^2 - p - 4)^2,$$

Which reduces to $(16 - 8p)x^2 = p^2 - 8p + 16$. Since x ≥ 0(it is the sum of two square roots), we have x = $\frac{|p-4|}{\sqrt{16-8p}}$

If a solution exists. We need only determine when this value actually satisfies. Certainly we need p ≤ 2. In that case plugging in our claimed value of x and multiplying through by $\sqrt{16 - 8p}$ gives │3p -4│+2│p│ = 4 –p.

If p ≥ $\frac{4}{3}$ this becomes 6p = 8, or p = $\frac{4}{3}$; if 0 ≤ p ≤ $\frac{4}{3}$ this holds identically; if p ≤ 0 this becomes 4p = 0, or p = 0. We conclude there exists a solution if and only if $0 \leq p \leq 4/3$, in which case it is the solution given above.

64. Let $\prod_{n=1}^{1996}(1 + nx^{3^n}) = 1 + a_1x^{k_1} + a_2x^{k_2} + \cdots + a_mx^{k_m}$,

where $a_1, a_2, \ldots, a_m$ are nonzero and $k_1 < k_2 < \cdots < k_m$. Find a_{1996}.

Sol.: Note that k_i is the number obtained by writing i in base 2 and reading the result as a number in base 3, and a_i is the sum of the exponents of the powers of 3 used. In particular, $1996 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^3 + 2^2, so\ a_{1996} = 10 + 9 + 8 + 6 + 6 + 3 + 2 = 45$.

65. In a parallelogram ABCD with ∠A <90°, the circle with diameter AC meets the lines CB and CD again at E and F, respectively, and the tangent to this circle at A meets BD at P. Show that P, F, E are collinear.

Sol.: Without loss of generality, suppose B, D, P occur in that order along BD. Let G and H be the second intersection of AD and AB with the circle. By Menelaus' theorem, it suffices to show that

$$\frac{CE.BP.DF}{EB.PD.FC} = 1$$

Find note that

$$\frac{BP}{AB}\frac{AD}{DP} = \frac{\sin\angle BAP \sin\angle APD}{\sin\angle APB \sin\angle DAP} = \frac{\sin\angle BAP}{\sin\angle DAP}$$

Since AP is tangent to the circle, ∠BAP = ∠HAP

= π - ∠HCA = π -∠FCA; similarly, ∠DAP = ∠GCA = ∠EAC. We conclude

$$\frac{BP}{AB}\frac{AD}{DP} = \frac{\sin\angle FAC}{\sin\angle EAC} = \frac{FC}{EC}$$

Finally we note that $\frac{DF}{BE} = \frac{DA}{AB}$ because the right triangles AFD and AED have the same angles at B and D and are thus similar. This prove the claim.

66. Given real numbers $0 = x_1 < x_2 < \cdots < x_{2n} < x_{2n+1} = 1$ with $x_{i+1} - x_i \leq h$ for 1

≤ i ≤ 2n, show that $\frac{1-h}{2}$ <
$\sum_{i=1}^{n} x_{2i}\,(x_{2i+1} - x_{2i-1}) < \frac{1+h}{2}$.

Sol.: The different between the middle quantity and $\frac{1}{2}$ is the difference between the sum of the areas of the rectangles bounded by the lines x = $x_{2i-1}, x = \ x_{2i+1}, y = 0, y = \ x_{2i}$ and the triangle bounded by the lines y = 0, x = 1, x = y. The area contained in the rectangles but not the triangle is a union of triangles of total base less than 1 and height at most h, as is the area contained in the triangle but not the rectangles. Hence the sum differs from $\frac{1}{2}$ but at most $\frac{h}{2}$, as desired.

67. Find the maximum number of pair wise disjoint sets of the from $S_{a,b} = \{n^2 + an + b: n \in Z\}$ $with\ a, b\ \in Z$.

Solution: Only two such sets are possible, for example, with (a, b) = (0, 0) and (0, 2) (since 2 is not a difference of squares). There is no loss of generality in assuming a ∈ {0, 1} by a suitable shift of n, and the sets generated by (0, a) and (1, b) have the common value $(a-b)^2 + a = \ (a-b)^2 + (a-b) + b$.Thus we have a = 0 or a = 1 universally.

First suppose a = 0. If b –c ≠ 2 (mod 4), then (0, b) and (0, c) gives a common value because b –c is a difference of squares, clearly this precludes having three disjoint sets. Now suppose a = 1. If b –c is even, we can find x, y such that b –c = (x +y +1) (x –y), and so $x^2 + x + b = y^2 + y + c$, again, this precludes having three disjoint sets.

68. For which ordered pairs of positive real numbers (a, b) is the limit of every sequence $\{x_n\}$ satisfying the condition.

$$\lim_{n\to\alpha}(ax_{n+1} - bx_n) = 0$$

Sol.: The holds if and only if b < a, if b > a, the sequence $x_n = \left(\frac{b}{a}\right)^n$ satisfies the condition but does not go to zero, if b = a, the sequence $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ does likewise. Now suppose b < a. If L and M are the limit inferior and limit superior of the given sequence, the condition implies M ≤ $\left(\frac{b}{a}\right) L$; since L ≤ M, we have M ≤ $\left(\frac{b}{a}\right) M$, and so L, M ≥ 0. Similarly, the condition implies L ≥ $\left(\frac{b}{a}\right) M$, and since M ≥ L, we have L ≥ $\left(\frac{b}{a}\right) L$, so L, M ≤ 0; therefore L = M = 0 and the sequence converges to 0.

69. Consider the pair of four –digit positive integers (M, N) = (3600, 2500). Notice that M and N are both perfect squares, with equal digits in two places, and differing digits in the remaining two places. Moreover, when the digits differ, the digit in M is exactly one greater than the corresponding digit in N. Find all pairs of four –digit positive integers (M, N) with these properties.

Sol.: If M = $m^2\ and\ N = \ n^2$, then (m+ n) (m –n) ∈{11, 101, 110, 1001, 1010, 1100}.

Since M and N are four-digit numbers, we must have 32 ≤ n <m ≤ 99, and so 65 ≤ m +n ≤ 197. Moreover, m +n and m –n are both odd or both even, so 11, 110 and 1010 lead to no solutions. From this we get exactly five acceptable factorizations.

101 = (m+ n) (m –n) = 101 × 1

1001 = (m +n) (m –n) = 143 × 7

1001 = (m +n) (m –n) = 91 × 11

1001 = (m+ n) (m –n) = 77 × 13

1100 = (m+ n) (m –n) = 110 × 10

Giving the solutions (M, N) = (2601, 2500), (5625, 4624), (2601, 1600), (2025, 1024), (3600, 2500).

70. A function f defined on the positive integers satisfies f(1) = 1996 and
$$\boldsymbol{f(1) + f(2) + \cdots + f(n) = n^2 f(n)} (\boldsymbol{n} > 1).$$

Sol.: An easy induction will show that

$$f(n) = \frac{2 \times 1996}{n(n+1)}$$

Namely,

$$f(n) = \frac{1}{n^2-1}\left(\frac{3992}{1.2} + \cdots + \frac{3992}{(n-1)n}\right)$$

$$= \frac{3992}{n^2-1}\left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n}\right)$$

$$= \frac{3992}{(n+1)(n-1)}\left(1 - \frac{1}{n}\right)$$

$$= \frac{3992}{(n+1)(n-1)} \frac{n-1}{n} = \frac{3992}{n(n+1)}$$

In particular, f(1996) = $\frac{2}{1997}$

71. Define $\boldsymbol{q(n) = \left\lfloor \frac{n}{\lfloor\sqrt{n}\rfloor} \right\rfloor} \ (\boldsymbol{n = 1, 2, \ldots})$.

Determine all positive integers n for which q(n) > q(n +1).

Sol.: We have q(n) > q(n+1) if and only if n +1 is a perfect square. Indeed, if n +1 = m^2, then

$$q(n) = \left\lfloor \frac{m^2-1}{m-1} \right\rfloor = m + 1, q(n+1) = \left\lfloor \frac{m^2}{m} \right\rfloor = m$$

On the other hand, for n = $m^2 + d$ with $0 \le d \le 2m$, $q(n) = \left\lfloor \frac{m^2+d}{m} \right\rfloor = m + \left\lfloor \frac{d}{m} \right\rfloor$

Which is non-decreasing.

72. Let a, b, c be positive real numbers.

(a) Prove that $\boldsymbol{4(a^3 + b^3) \ge (a + b)^3}$

(b) Prove that $\boldsymbol{9(a^3 + b^3 + c^3) \ge (a + b + c)^3}$

Sol.: Both parts follow from the Power Mean inequality: for r > 1 and $x_1, \ldots, x_n$ positive,

$$\left(\frac{{x_1}^r + \cdots + {x_n}^r}{n}\right)^{1/r} \ge \frac{x_1 + \cdots + x_n}{n},$$

Which in turn follows from Jensen's inequality applied to convex function x^r.

73. Find all Solutions in non-negative integers x, y, z of the equation.

$$\boldsymbol{2^x + 3^y = z^2}$$

Sol.: If y = 0, then $2^x = z^2 - 1 = (z+1)(z-1)$, so z +1 and z -1 are powers of 2. The only powers of 2 which differ by 2 are 4 and 2, so (x, y, z) = (3, 0, 3).

If y > 0, then 2^x is a quadratic residue modulo 3, hence x is even. Now we have $3^y = z^2 - 2^x = \left(z + 2^{\frac{x}{2}}\right)\left(z - 2^{\frac{x}{2}}\right)$. The factors are powers of 3, say z+$2^{x/2} = 3^m$ and z $-2^{x/2} = 3^n$, but then $3^m - 3^n = 2^{\frac{x}{2}+1}$. Since the right side is not divisible by 3, we must have n = 0 and $3^m - 1 = 2^{\frac{x}{2}+1}$.

If x = 0, we have m = 1, yielding (x, y, z) = (0, 1, 2). Otherwise, $3^m - 1$ is divisible by 4, so m is even and $2^{\frac{x}{2}+1} = \left(3^{\frac{m}{2}} + 1\right)\left(3^{\frac{m}{2}} - 1\right)$. The two factors on the right are powers of 2 differing by 2, so they are 2 and 4, giving x = 4 and (x, y, z) = (4, 2, 5).

74. The sides a, b, c and u, v, w of two triangles ABC and UVW are related by the equations.

$$\boldsymbol{u(v+w-u) = a^2,}$$

$$\boldsymbol{v(w+u-v) = b^2,}$$

$$\boldsymbol{w(u+v-w) = c^2.}$$

Prove that ABC is acute, and express the angles U, V, W in terms of A, B, C.

Sol.: Note that $a^2+b^2-c^2 = w^2-u^2-v^2+2uv = (w+u-v)(w-u+v) > 0$ by the triangle inequality, so $\cos C > 0$. By this reasoning, all of the angles of triangle ABC are acute. Moreover,

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

$$= \sqrt{\frac{(w+u-v)(w-u+v)}{4uv}}$$

$$= \sqrt{\frac{w^2-u^2-v^2+2uv}{4uv}} = \frac{1}{\sqrt{2}}\sqrt{1-\cos U}$$

From which we deduce U = $1-2cos^2A = \cos(\pi-2A)$.

Therefore U = π -2A, and similarly V = π- 2B, W = π- 2C.

75. Find all solutions in positive real numbers a, b, c, d to the following system of equations:

a + b+ c + d = 12

abcd = 27 +ab +ac +ad +bc +bd +cd.

Sol.: The first equation implies abcd = ≤ 81 by the arithmetic geometric mean inequality, with equality holding for a = b = c = d = 3. Again by AM-GM,

abcd ≥ 27 +6 $(abcd)^{1/2}$

However, $x^2-6x-27 \geq 0$ for $x \leq -3$ or $x \geq 9$, so $(abcd)^{1/2} \geq 9$, hence abcd ≥ 81. We conclude abcd = 81, and hence a = b= c= d= 3.

76. Prove that the average of numbers n $\boldsymbol{\sin n^\circ\ (n = 2, 4, 6, \ldots, 180)}$ *is* $\boldsymbol{\cot 1^\circ}$.

Solution: All arguments of trigonometric functions will be in degrees. We need to prove

$$2\sin 2 + 4\sin 4 + \cdots + 178\sin 178 = 90\cot 1 \qquad (2)$$

Which is equivalent to

$$2\sin 2\sin 1 + 2(2\sin 4\,.\sin 1) + \ \ldots + 89$$

$$(2\sin 178\,.\sin 1) = 90\cos 1. \qquad (3)$$

Using the identity $2\sin a\,.\sin b = \cos(a-b) - \cos(a+b)$, we find

$2\sin 2\,.\sin 1 + 2(2\sin 4\,.\sin 1) + \cdots + 89$

$(2\sin 178\,.\sin 1)$

$= (\cos 1 - \cos 3) + 2(\cos 3 - \cos 5) + \cdots + 89(\cos 177 - \cos 179)$

$= \cos 1 + \cos 3 + \cos 5 + \cdots + \cos 175$

$\cos 177 - 89\cos 179$

$$= \cos 1 + (\cos 3 + \cos 177) + \cdots + (\cos 89 + \cos 91) - 89\cos 179$$

$= \cos 1 + 89\cos 1 = 90\cos 1,$

So (1) is true.

Note: An alternate solution involves complex numbers. One expresses sin n as

$\frac{\left(e^{\frac{\pi in}{180}}-e^{\frac{-\pi in}{180}}\right)}{(2i)}$ *and uses the fact that*

$x + 2x^2 + \cdots + nx^n = (x + \cdots + x^n) + (x^2 + \cdots + x^n) + \cdots + x^n$

$= \frac{1}{x-1}[(x^{n+1} - x) + (x^{n+1} - x^2) + \cdots + (x^{n-1} - x^n)]$

$= \frac{nx^{n+1}}{x-1} - \frac{x^{n+1}-x}{(x-1)^2}$.

77. For any nonempty set S of real numbers, let σ(S) denote the sum of the elements of S. Given a set A of n positive integers, consider the collection of all distinct sums σ(S) as S ranges over the nonempty subsets of A. Prove that this collection of sums can be partitioned into n classes so that in each classes, the ratio of the largest sum of the smallest sum does not exceed 2.

Sol.: Let A = $\{a_1, a_2, \dots, a_n\}$ where $a_1 < a_2 < \cdots < a_n$. For i = 1, 2, …, n let $s_i = a_1 + a_2 + \cdots + a_i$ *and take* $s_0 = 0$. All the sums is question are less than or equal to s_n, and if σ is one of them, we have

$$s_{i-1} < \sigma < s_i \qquad (1)$$

For an appropriate i. Divide the sums into n classes by letting C_i denote the class of sums satisfying (1). We claim that these classes have the desired property. To establish this, it suffices to show that (1) implies.

$$\frac{1}{2}s_i < \sigma < s_i \qquad (2)$$

Suppose (1) holds. The inequality $a_1 + a_2 + \cdots + a_{i-1} = s_{i-1} < \sigma$ shows that the sum σ contains at least one addend a_k *with* $k \geq i$. Then since then $a_k \geq a_i$, we have

$s_i - \sigma < s_i - s_{i-1} = a_i \leq a_k \leq \sigma$, which together with $\sigma \leq s_i$ implies (2).

Note: The result does not hold if 2 is replaced by any smaller constant c. To see this, choose n such that $c < 2 - 2^{-(n-1)}$ and consider the set $\{1, \dots, 2^{n-1}\}$. If this set is divided into n subsets, two of 1,……, 2^{n-1}, 1 + …….+ 2^{n-1} must lie in the subset, and their ratio is at least (1+…..+ $2^{n-1})/(2^{n-1}) = 2 - 2^{(n-1)}$ > c.

78. An n –term sequence $\{\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n}\}$ in which each term is either 0 or 1 is called a binary sequence of length n. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let $\boldsymbol{b_n}$ be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $\boldsymbol{b_{n+1} = 2a_n}$ for all positive integers n.

Sol.: We refer to the binary sequences counted by (a_n) and (b_n) as "type A" and "type B", respectively. For each binary sequence $(x_1, x_2, \dots, x_n)$ there is a corresponding binary sequence $(y_0, y_1, \dots, y_n)$ obtained by setting $y_0 = 0$ *and* $y_1 = x_1 + x_2 + \cdots + x_i$ mod 2, i = 1, 2, …, n. (2)

(Addition mod 2 is defined as follows: 0 +0 = 1 +1 = 0 and 0 + 1 = 1 +0 = 1.) Then

$$x_i = y_i + y_{i-1} \, mod \, 2, i = 1, 2, \dots, n,$$

And it is easily seen that (1) provides a one-to – one correspondence between the set of all binary sequences of length n and the set of binary sequences of length n +1 in which the first term is 0. Moreover, the binary sequence $(x_1, x_2, \ldots, x_n)$ has three consecutive terms equal 0, 1, 0 in that order if and only if the corresponding sequence $(y_0, y_1, \ldots, y_n)$ has four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order, so the first is of type A if and only if the second is of type B. The set of Type B sequences of length n +1 in which the first term is 0 is exactly half the total number of such sequences, as can be seen by means of the mapping in which 0's and 1's are interchanged.

79. Solve the system of equations:

$$\sqrt{3x}\left(1+\frac{1}{x+y}\right) = 2$$
$$\sqrt{7y}\left(1-\frac{1}{x+y}\right) = 4\sqrt{2}$$

Sol.: Let u =$\sqrt{x}, y = \sqrt{y}$, so the system becomes

$$u+\frac{u}{u^2+v^2}=\frac{2}{\sqrt{3}}$$

$$v-\frac{v}{u^2+v^2}=\frac{4\sqrt{2}}{\sqrt{7}}.$$

Now let z = u +vi; the system then reduces to the single equation

$$z+\frac{1}{z}=2\left(\frac{1}{\sqrt{3}}+\frac{2\sqrt{2}}{\sqrt{7}}i\right).$$

Let t denote the quantity inside the parentheses; then

$$z = t \pm \sqrt{t^2-1}$$
$$=\frac{1}{\sqrt{3}}+\frac{2\sqrt{2}}{\sqrt{7}}i \pm \left(\frac{2}{\sqrt{21}}+\sqrt{21}\right)$$

From which we deduce

$$u = \left(\frac{1}{\sqrt{3}} \pm \frac{2}{\sqrt{21}}\right)^2, v = \left(\frac{2\sqrt{2}}{\sqrt{7}} \pm \sqrt{2}\right)^2.$$

80. Determine, as a function of n, the number of permutations of the set {1, 2, …, n} such that no three of 1, 2, 3, 4 appear consecutively.

Sol.: There are n! permutations in all. Of those, we exclude (n -2)! Permutations for each arrangement of 1, 2, 3, 4 into an ordered triple and one remaining element, or 24(n -2)! in all. However, we have twice excluded each of the 24(n-3)! Permutations in which all four of 1, 2, 3, 4 occur in a block. Thus the number of permutations of the desired from is n! -24 (n -2)! + 24(n -3)!

81. Determine all function f: ℕ ⟶ℕ satisfying (for all n ∈ℕ)

$$\boldsymbol{f(n)+f(n+1) = f(n+2)f(n+3) - 1996}.$$

Sol.: From the given equation, we deduce

$$f(n)-f(n+2) = f(n+3)[f(n+2) - f(n+4)]$$

If f(1) > f(3), then by induction, f(2m -1) > f(2m +1) for all m > 0, giving an infinite decreasing sequence f(1), f(3), …. Of positive integers, a contradiction. Hence f(1) ≤ f(3), and similarly f(n) ≤ f(n +2) for all n.

Now note that

$$0 = 1996 + f(n) + f(n+1) - f(n+2)f(n+3) \leq 1996 + f(n+2) + f(n+3) - f(n+2)f(n+3)$$

$$= 1997 - [f(n+2) - 1][f(n+3) - 1].$$

In particular, either f(n +2) = 1 or f(n +3) ≤ 1997, and vice versa. The numbers f(2m+1) – f(2m-1) are either all zero or all positive, and similarly for the numbers f(2m+ 2) –f(2m). If they are both positive, eventually f(n +2) and f(n+3) both exceed 1997, a contradiction.

We now split into three cases. If f(2m) and f(2m +1) are both constant, we have [f(2m) - 1][f(2m+1)-1] = 1997 and so either f(2m) = 1 and f(2m +1) = 1997 or vice versa. If f(2m +1) is constant but f(2m) is not, then $f(2m+1) = 1$ for all m and $f(2m+2) = f(2m) + 1997, so\ f(2m) = 1997(m-1) + f(2)$.

Similarly, if f(2m) is not constant, then f(2m)= 1 and f(2m +1) = 1997m + f(1).

82. Let a, b, c, d be four nonnegative real numbers satisfying the condition

$$\boldsymbol{2(ab + ac + ad + bc + bd + cd) + abc + abd + acd + bcd = 16}$$

Prove that

$$\boldsymbol{a + b + c + d \geq \frac{2}{3}(ab + ac + ad + bc + bd + cd)}$$

And determine when equality occurs.

Sol.: For i = 1, 2, 3, define s_i as the average of the products of the i-element subsets of {a, b, c, d}. Then we must show

$$3s_2 + s_3 = 4 \Rightarrow s_1 \geq s_2.$$

It suffices to prove the (unconstrained) homogeneous inequality

$$3{s_2}^2{s_1}^2 + s_3{s_1}^3 \geq 4{s_2}^3,$$

As then $3s_2 + s_3 = 4$ will imply

$$(s_1 - s_2)^3 + 3({s_1}^3 - {s_2}^3) \geq 0.$$

We now recall two basic inequalities about symmetric means of nonnegative real numbers. The first is Schur's inequality:

$$3{s_1}^3 + s_3 \geq 4s_1s_2.$$

While the second,

$${s_1}^2 \geq s_2$$

Is a case of Maclaurin's inequality ${s_i}^{i+1} \geq {s_{i+1}}^i$.These combine to prove the claim:

$$3{s_2}^2{s_1}^2 + s_3{s_1}^3 \geq 3{s_2}^2{s_1}^2 + \frac{{s_2}^2 s_3}{s_1} \geq 4{s_2}^3.$$

Finally, for those who have only seen Schur's inequality in three variables, note that in general any inequality involving $s_1, \dots, s_k$ which holds for n ≥ k variables also holds for n +1 variables, by replacing the variables $x_1, \dots, x_{n+1}$ by the roots of the derivative of the polynomial $(x - x_1) \dots (x - x_{n-1})$.

83. Let m and n be positive integers such that n ≤ m. Prove that

$$\boldsymbol{2^n n! \leq \frac{(m+n)!}{(m-n)!} \leq (m^2 + m)^n}.$$

Sol.: The quantity in the middle is $(m+n)(m+n-1) \dots (m-n+1)$. If we pair off terms of the form $(m+x) and\ (m+1-x)$, we get products which do not exceed $m(m+1)$, since the function $f(x) = (m+x)(m+1-x)$ is a concave parabola with maximum at $x = \frac{1}{2}$. From this the right inequality follows. For the left, we need only show $(m+x)(m+1-x) \geq 2x$ for x ≤n; this rearranges to $(m-x)(m+1+x) \geq 0$, which holds because $m \geq n \geq x$.

84. Let a, b and c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}$$

And determine when equality occurs.

Sol.: By the triangle inequality, $b+c-a \; and \; c+a-b \; abc$ positive. For any positive x, y, we have

$$2(x+y) \geq x+y+2\sqrt{xy} = \left(\sqrt{x}+\sqrt{y}\right)^2$$

By the AM-GM inequality, with equality for x = y. Substituting $x = a+b-c, y = b+c-a$ we get

$$\sqrt{a+b-c} + \sqrt{b+c-a} \leq 2\sqrt{a},$$

Which added to the two analogous inequalities yields the desired result. Inequality holds for $a+b-c = b+c-a = c+a-b, i.e. a = b = c$.

85. Let k ≥1 be an integer. Show that there are exactly 3^{k-1} positive integers n with the following properties:

(a) The decimal representation of n consists of exactly k digits.
(b) All digits of k are odd.
(c) The number n is divisible by 5.
(d) The number m = $\frac{n}{5}$ has k odd (decimal) digits.

Sol.: The multiplication in each place must produce an even number of carries, since these will be added to 5 in the next place and an odd digit must result. Hence all of the digits of m must be 1, 5 or 9 and the first digit must be1, since m and n have the same number of decimal digits. Hence there are 3^{k-1} choices for m and hence for n.

86. The polynomials $P_n(x)$ are defined by $P_0(x) = 0, P_1(x) = x$ and $P_n(x) = xP_{n-1}(x) + (1-x)P_{n-2}(x) \;\; n \geq 2$.
For every natural number n ≥ 1, find all real numbers x satisfying the equation $P_n(x) = 0$.

Sol.: One shows by induction that

$$P_n(x) = \frac{x}{x-2}[(x-1)^n - 1]$$

Hence $P_n(x) = 0$ if and only if x = 0 or $x = 1 + e^{2\pi ik/n}$ for some k ∈ {1, …, n -1}.

87. The real numbers x, y, z, t satisfy the equalities $x+y+z+t = 0 \; and \; x^2 + y^2 + z^2 + t^2 = 1$. Prove that

$$-1 \leq xy + yz + zt + tx \leq 0.$$

Sol.: The inner expression is $(x+z)(y+t) = -(x+z)^2$, so the second inequality is obvious. As for the

first, note that

$$1 = (x^2+z^2) + (y^2+t^2) \geq \frac{1}{2}[(x+z)^2 + (y+t)^2] \geq [(x+z)(y+t)]$$

By two applications of the power mean inequality.

88. Natural numbers k, n are given such that 1 < k < n. Solve the system of n equations.
$\boldsymbol{x_i^3(x_i^2 + \cdots + x_{i+k-1}^2) = x_{i-1}^2\ 1 \le i \le n}$
in n real unknowns $x_1, \dots, x_n$. (Note: $x_0 = x_n, x_1 = x_{n+1}, etc.$)

Sol.: The only solution is $x_1 = \cdots x_n = k^{-1/3}$. Let L and M be the smallest and largest of the x_i, respectively. If M = x_i, then

$$kM^3L^2 \le x_i^3(x_i^2 + \cdots + x_{i+k-1}^2) = x_{i-1}^2 \le M^2$$

And so M ≤ $\frac{1}{(kL^2)}$. Similarly, if L = x_j, then

$$kL^3M^3 \ge x_1^3(x_i^2 + \cdots + x_{i-k+1}^2) = x_{i-1}^2 \ge L^2$$

and so $L \ge \frac{1}{(kM^2)}$. Putting this together, we get

$L \ge \frac{1}{kM^2} \ge kL^4$

And so $L \ge k^{-1/3}; similarly, M \ge k^{-1/3}$. Obviously L ≤ M, so we have L = M = $k^{-1/3}\ and\ x_1 = \cdots = x_n = k^{-1/3}$.

89. Shows that there do not exist nonnegative integers k and m such that k! +48 = 48 $(k+1)^m$.

Sol.: Suppose such k, m exist. We must have 48 | k!, so k ≥ 6; one checks that k = 6 does not yield a solution, so k ≥ 7. In that case k! is divisible by 32 and by 9, so that $\frac{(k!+48)}{48}$ is relatively prime to 6, as then is k +1.

If k +1 is not prime, it has a prime divisor greater than 3, but this prime divides k! and not k! +48. Hence k +1 is prime, and by Wilson's theorem k! +1 is a multiple of k +1. Since k! +48 is as well, we find k +1 = 47, and we need only check that $\frac{46!}{48+1}$ is not a power of 47. We check that $\frac{46!}{48+1} = 29(mod\ 53)$ (by cancelling as many terms as possible in 46! Before multiplying), but that 47 has order 13 modulo 53 and that none of its powers is congruent to 29 modulo 53.

90. Let O and G be the circumcenter and centroid, respectively, of triangle ABC, If R is the circumradius and r the inradius of ABC, show that

$$\boldsymbol{OG \le \sqrt{R(R-2r)}}.$$

Sol.: Using vectors with original at O, we note that $OG^2 = \frac{1}{9}(A+B+C)^2 = \frac{1}{3}R^2 + \frac{2}{9}R^2(\cos 2A + \cos 2B + \cos 2C)$.

Hence $R^2 - OG^2 = \frac{(a^2+b^2+c^2)}{9}$. On the other hand, by the standard area formula K = $rs = \frac{abc}{4R}$, we have $2rR = \frac{abc}{(a+b+c)}$. We now note that

$$(a^2+b^2+c^2)(a+b+c) \ge 9abc$$

By two applications of the AM-GM inequality, so $2rR \le R^2 - OG^2$, proving the claim.

91. Show that there exists a subset A of the set {1, 2, …, 1996} having the following properties:

(a) 1, $2^{1996} - 1 \in A$;

(b) **Every element of A, except 1, is the sum of two (not necessarily distinct) elements of A;**

(c) **A contains at most 2012 elements.**

Sol.: We state the problem a bit differently: we want to write down at most 2012 numbers, starting with 1 and ending with $2^{1996}-1$, such that every number written is the sum of two numbers previously written. If 2^n-1 has been written, then $2^n(2^{n-1})$ can be obtained by n doublings, and 2^n-1 can be obtained in one more step.

Hence we can obtain $2^2-1, 2^4-1, \dots, 2^{256}-1\ in\ (1+1)+(2+1)+\cdots+(128+1)=263$ steps. In 243 steps, we turn $2^{256}-1\ into\ 2^{499}-2^{243}$. Now notice that the numbers $2^{243}-2^{115}, 2^{115}-2^{51}, 2^{51}-2^{19}, 2^{19}-2^3, 2^3-2^1, 2^1-1$ have all be written down; in 6 steps, we now obtain $2^{499}-1$. We make this into $2^{998}-1$ in 500 steps, and make $2^{1996}-1$ in 999 steps. Adding 1 for the initial 1, we count

$$1+263+243+6+500+999=2012$$

Numbers written down, as desired.

92. Let $\mathbb{Z}^+$ denote the set of nonzero integers. Show that an integer p > 3 is prime if an only if for any a, b ∈ $\mathbb{Z}^+$, exactly one of the numbers

$$\boldsymbol{N_1=a+b-6ab+\frac{p-1}{6}},$$
$$\boldsymbol{N_2=a+b+6ab+\frac{p+1}{6}}$$

belongs to $\mathbb{Z}^+$.

Sol.: If $N_1=0$, then $p=(6a-1)(6b-1)$ is composite; similarly, $N_2=0\ implies\ p=-(6a+1)(6b+1)$ is composite. Conversely, suppose that p is composite. If p ≡ 0, 2, 3 or 4 (mod 6), then $N_1\ and\ N_2$ are not integers.

Otherwise, all divisors of p are congruent to ± 1 (mod 6). So there exist natural numbers c, d such that

$$p=(6c+1)(6d+1) or (6c-1)(6d-1) or (6c+1)(6d-1).$$

In the first case, N_2 is not an integer and $N_1=0$ for $a=-c, b=-d$.

In the second case, N_2 is not an integer and $N_1=0$ for $a=c, b=d$.

In the third case, N_1 is not an integer and N_2= 0 for $a=c, b=-d$.

93. Let M be a nonempty set and *a binary operation on M. That is, to each pair (a, b) ∈$\boldsymbol{M\times M}$ one assigns an element a * b. Suppose further that for any a, b ∈M, (a* b)*b = a and a*(a* b) = b.

(a) Show that a *b = b *a for all a, b ∈M.

(b) For which finite sets M does such a binary operation exist?

Sol.:

(a) First note that [a* (a* b)] *(a* b) = a by the first rule. By the second rule, we may rewrite the left side as b* (a *b), so b* (a *b) = a and so b *a = b*[b* (a* b)]. b* a = b*[b*(a* b)]. By the second rule this equals a *b, so a* b = b* a.

(b) Such sets exist for all finite sets M. Identify M with {1, …, n} and define a* b = c ⟺a + b+ c = 0 (mod n). It is immediate that the axioms are satisfied.

94. Determine whether there exist a function f: ℤ⟶ℤ such that for each k = 0, 1, …, 1996

and for each m $\in\mathbb{Z}$ the equation $\boldsymbol{f(x)+bx=m}$ has at least one solution x $\in\mathbb{Z}$.

Sol.: Each integer y can written uniquely as 1997m +k with m $\in\mathbb{Z}$ and k ∈ {0, …, 1996}. Define the function f by $f(y) = m - ky$; then $f(x) + kx = m$ has the solution $x = 1997m + k$, so the condition satisfied.

95. Two sets of intervals A, B on a line are given. The set A contains $\boldsymbol{2m-1}$ intervals, every two of which have a common interior point. Moreover, each interval in A contains at least disjoint intervals of B. Show that there exists an interval in B which belongs to at least m intervals from A.

Sol.: Let $\alpha_1 = [a_i, b_i](i = 1, \dots, 2m-1)$ be the intervals, indexed so that $a_1 \leq a_2 \leq \cdots \leq a_{2m-1}$. Choose k $\in\{m, \dots, 2m-1\}$ to minimize b_k. By assumption, the interval α_k contains two disjoint intervals from B, say $\beta_1 = [c_1, d_1]\ and\ \beta_2 = [c_2, d_2]$. Without loss of generality, assume

$$a_k \leq c_1 < d_1 < c_2 < d_2 \leq b_k.$$

If $d_1 \leq b_i$ for i = 1, 2, …, m, then $\beta_1 \subset \alpha_1$ for i = 1, 2, …, m, so β_1 satisfies the desired property. Otherwise, $d_1 > b_x$ for some s ∈ {1, 2, …, m}. By assumption, $c_2 > d_1 > b_8$. Since no two of the α are disjoint, we have $b_8 \geq a_i$ for all i, so $c_2 > a_i$. On the other hand, by the choice of $k, b_k < b_1$ for i = m, …, $2m_1$. Therefore $a_i < c_2 < d_2 \leq b_k \leq b_i$ for each $i \in \{m, m+1, \dots, 2m-1\}, and\ so\ \beta_2$ has the desired property.

96. Let n be a natural number. A cube of side length n can be divided into 1996 cubes whose side lengths are also natural numbers. Determine the smallest possible value of n.

Sol.: Since 1996 >12^3, we must have n ≥ 13, and we now show n = 13 suffices, Inside a cube of edge 13, we place one cube of edge 5, one cube length 4, and 2 of length 2, and fill the remainder with cubes of edge 1. The number of cubes used is

$13^3 - (5^3 - 1) - (4^3 - 1) - 2(2^3 - 1) = 2197 - 124 - 63 - 2(7) = 1996$, as desired.

97. Find all positive integers n such that $\boldsymbol{3^{n-1} + 5^{n-1}\ divides\ 3^n + 5^n}$.

Sol.: This only occurs for n = 1. Let $s_n = 3^n + 5^n$ and note that

$$s_n = (3+5)s_{n-1} - 3.5.s_{n-2}$$

So s_{n-1} must also divide $3.5.s_{n-2}$. If n >1, then s_{n-1} is coprime to 3 and 5, so s_{n-1} must divide s_{n-2}, which is impossible since $s_{n-1} > s_{n-2}$.

98. Prove that for every polynomial $\boldsymbol{x^2+px+q}$ with integer coefficients, there exists a polynomial $\boldsymbol{2x^2+rx+s}$ with integer coefficients such that the sets of values of the two polynomials on the integers are disjoint.

Sol.: If p is odd, then $x^2 + px + q$ has the same parity as q for all integers x, and it suffices to choose r even and s of the opposite parity as q. If p = 2m is even, then $x^2 + px + q = (x+m)^2 + (q-m)^2$ which is congruent to $q - m^2\ or\ q - m^2 + 1$ modulo 4. Now it suffices to choose r even and s congruent to $q - m^2 + 2$ modulo 4.

99. Sergey found 11 different solutions to the equation $f\left(19x - \frac{96}{x}\right) = 0$. Prove that if he had tried harder, he could have found at least one more solution.

Sol.: The equations $19x - \frac{96}{x} = t$ can be rewritten $19x^2 - tx - 96 = 0; since\ t^2 + 19.96 > 0$, it always has two real roots. Therefore the number of zeroes of f (if finite) is an even integer, so Sergey can find at least one more zero.

100. There are 2000 towns in a country, each pair of which is linked by a road. The Ministry of Reconstruction proposed all of the possible assignments of one way traffic to each road. The ministry of Transportation rejected each assignment that did not allow travel from any town to any other town. Prove that more of half of the assignments remained.

Sol.: We will prove the same statement for n ≥ 6 towns. First suppose n = 6. In this case there are 2^{15} assignments, and an assignment is rejected only if either one town has road to all of the others in the same direction, or if there are two sets of three towns, such that within each town the roads point in a circle, but all of the roads from one set to the other point in the same direction. There are 5.2^{11} had assignments of the first kind and 20.8 of the second kind, so the fraction of good assignments is at least $\frac{5}{8}$.

For n ≥ 6, we claim that the fraction of good assignments is at least

$$\frac{5}{8}\prod_{i=6}^{n-i}\left(1 - \frac{1}{2^{i-1}}\right).$$

We show this by induction on a good assignment or r h -1 vertices can be extended to a good assignment on vertices simply by avoiding having all edges from the last vertex pointing in the same direction, which occurs in 2 cases out of 2^{n-1}.

Now it suffices to show that the above expression is more than $\frac{1}{2}$.

In fact,

$$\prod_{i=5}^{\infty}\left(1 - \frac{1}{2}\right)^{-1} \le 1 + \sum_{i=5}^{\infty}\frac{i-4}{2^i}$$

$$= 1 + \frac{1}{2^5}\sum_{i=0}^{\infty}\frac{i+1}{2^i}$$

$$= 1 + \frac{1}{2^5}\sum_{i=0}^{\infty}\sum_{k=i}^{\infty}\frac{1}{2^i}$$

$$= 1 + \frac{1}{2^5}\sum_{i=0}^{\infty}\frac{1}{2^i - 1}$$

$$= 1 + \frac{4}{2^5} = \frac{9}{8}$$

Thus the fraction of good assignments is at least $\left(\frac{5}{8}\right)\left(\frac{8}{9}\right) = \frac{5}{9} > \frac{1}{2}$.