MSQMS 2014 Solution Paper

QMA (Mathematics)

1. (c)

$$xy + yz + zx = \frac{pq(p-q) + qr(q-r) + rq(r-p)}{(p-q)(q-r)(r-p)}$$

$$= \frac{-(p-q)(q-r)(r-p)}{(p-q)(q-r)(r-p)} = -1$$

2. (b)

$$u = tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{y}.\frac{1}{1+\frac{x^2}{y^2}} = \frac{y}{y^2+x^2}$$

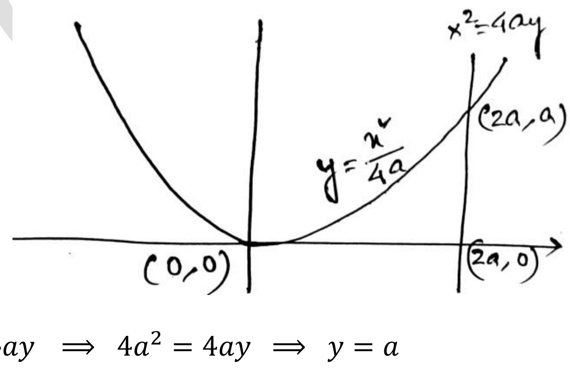
$$\frac{\partial u}{\partial y} = \frac{-x}{y^2+x^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2xy}{(y^2+x^2)^2};\quad \frac{\partial^2 u}{\partial y^2} = \frac{2xy}{(y^2+x^2)^2}$$

$$\frac{\partial^2 u}{\partial z^2} = 0$$

$$So, \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

3. (b)



$x^2 = 4ay \implies 4a^2 = 4ay \implies y = a$

$0 \le x \le 2a, \qquad 0 \le y \le \frac{x^2}{4a}$

$$\int_0^{2a}\int_0^{x^2/4a} xy\,dy\,dx = \int_0^{2a} x\left|\frac{y^2}{z}\right|_0^{x^2/4a} dx = \frac{1}{2}\int_0^{2a} x.\frac{x^4}{16a^2}dx = \frac{1}{32a^2}.\frac{1}{6}.64a^6 = \frac{1}{3}\,a^4$$

4. (d)

$$\int \frac{1}{x + x\log x}dx$$

$$= \int \frac{1}{(1+\log x)}.d(\log x)$$

$$= \log|1+\log x| + c.$$

5. (d)

$$\frac{2}{3!} + \frac{4}{5!} + \cdots$$

$$= \frac{3-1}{3!} + \frac{5-1}{5!} + \cdots$$

$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$$

$$= e^{-1}$$

6. (d)

$$\frac{ds}{dt} = 100e^{-0.25t}$$

$$s_t = -400e^{-0.25t} + c$$

Sales promoted during the fifth year $= s_5 - s_4 = 400(e^{-1} - e^{-1.25}) = 400(e^{-1} - e^{-1.25})$

$$= 32.549 \approx 32.55$$

7. (d)

Total number of ways $= 2_{C_1}.5_{C_4}.9_{C_6} + 2_{C_1}.5_{C_5}.9_{C_5} = 840 + 252 = 1092.$

8. (b)

$$\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} - \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (abc - 1)\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (abc - 1)(a - b)(b - c)(c - a) = 0$$

Since $a \neq b \neq c,\ so\ abc\ = 1$

9. (d) Highest power of 5 in 100! is

$$\left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] + \left[\frac{100}{5^3}\right] = 20 + 4 + 0 = 24$$

10. (a)

$$f(x) = |x| + \sin x + cos^3 x = x\left(\frac{|x|}{x} + \frac{\sin x}{x}\right) + cos^3 x$$

$$RHL\ = \underset{x \to 0^+}{lt}\ f(x) = 0(1 + 1) + 1 = 1$$

$$LHL = \underset{x \to 0^-}{lt}\ f(x) = 0(-1 + 1) + 1 = 1$$

$LHL = RHL = f(0)$; so f(x) is continuous at $x = 0$

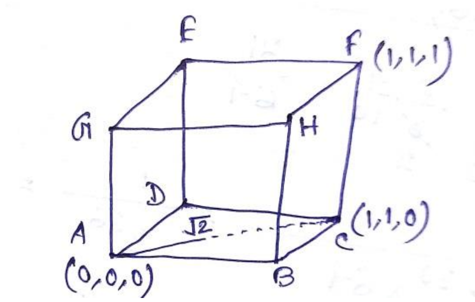
If f'(x) exists, then f'(x) = $\frac{|x|}{x} + \cos x - 3cos^2 x \sin x$

$$LHD = \underset{x \to 0^-}{lt} f'(x)\ = -1 + 1 = 0$$

$$RHD = \underset{x \to 0^+}{lt} f'(x) = 1 + 1 = 2$$

$LHD \neq RHD$, so it's not differentiable at $x = 0$.

11. (e)



Ant should travel from A(0,0,0) to F(1,1,1).

For the shortest distance it should go from AC $= \sqrt{2}$ *and then* $CF = 1$.

So, total distance $= (\sqrt{2} + 1)m$.

12. (d) $x^4 + x^2 = 1$

$$\Rightarrow \left(x^2 + \frac{1}{2}\right)^2 = \frac{5}{4}$$

$$\Rightarrow x^2 + \frac{1}{2} = \pm\frac{\sqrt{5}}{2}$$

$$\Rightarrow x^2 = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

So, x will have one positive, one negative, and two complex roots.

13. (b) Let the books be A and B with selling prices Rs. 30 each. For A, profit is 25%, for B loss is 25%.

Let the cost price for A is x and for B is y.

$$Then \qquad \frac{125}{100} \times x = 30; \qquad \frac{75}{100} y = 30$$

$$\therefore x = 24 \qquad \therefore y = 40$$

Net cost price $= 64$

So, loss $= 64 - 60 = Rs. 4$.

14. (a) $f(x) = x^3 + ax^2 + bx + c$

$$f(i) = 0 = -i - a + bi + c = (c - a) - i(1 - b)$$

$$f(-i) = 0 = -i - a + bi + c$$

So, $a = c, b = 1$.

15. (d)

$$E = \frac{1}{4} \cdot \frac{2}{6} \cdot \frac{3}{8} \cdot \frac{4}{10} \ldots \ldots \frac{31}{64} = \frac{1}{2^{30} \times 64} = \frac{1}{2^{36}} = 8^x$$

$$\therefore 8^x = 8^{-12}; \quad so, \qquad x = -12$$

16. (b) $B = -A^{-1}BA$

$$\rightarrow AB = -BA$$

$$\rightarrow AB + BA = 0$$

$$(A + B)^2 = A^2 + AB + BA + B^2 = A^2 + B^2$$

QMB (Mathematics)

1. (a) Number of subsets of n elements= $2n+1_{C_n}$

Number of subsets of $\overline{n-1}$ elements $= 2n+1_{C_{n-1}}$

⋮

⋮

Total number of subsets $= 2n+1_{C_n} + 2n+1_{C_{n-1}} + \cdots + 2n+1_{C_0}$

$$= \frac{1}{2}(2^{2n+1}) = 2^{2n}$$

(b) One way is by Induction.

Alternate way:

Take matrix A as a function f:

$$f(x).f(y) = \begin{pmatrix} \cos x\cos y - \sin x \sin y & \cos x \sin y + \sin x \cos y \\ -\sin x \cos y - \cos x \sin y & -\sin x \sin y + \cos x \cos y \end{pmatrix}$$

$$= f(x+y)$$

So, $f(\alpha).f(\alpha) = f(2\alpha)$

$f(2\alpha).f(\alpha) = f(3\alpha) = [f(\alpha)]^3$

⋮

⋮

$[f(\alpha)]^n = f(n\alpha)$

$$So, A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$$

2. (a) $|f(x) - f(y)| \leq (x-y)^2$

$\Rightarrow \left|\frac{f(x)-f(y)}{x-y}\right| \leq |x-y|$

$\Rightarrow \underset{x\to y}{lt}\left|\frac{f(x)-f(y)}{x-y}\right| \leq \underset{x\to y}{lt}|x-y| \quad [\because f \text{ is continuous over } \mathbb{R}]$

$\Rightarrow |f'(x)| \leq 0$

$\Rightarrow f'(x) = 0 \Rightarrow f(x) is\ a\ constant\ function.$

(b)

$x^n + n\,ax - b = 0$

$\Rightarrow (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n) = x^n + n\,ax - b$ -------------------(*)

Differentiating (*) w.r.t. x ;

$$\Rightarrow (x-\alpha_2)(x-\alpha_3)\dots(x-\alpha_n) + (x-\alpha_1)(x-\alpha_3)\dots(x-\alpha_n) + (x-\alpha_1)(x-\alpha_2)(x-\alpha_4)\dots(x-\alpha_n) + \cdots + (x-\alpha_1)(x-\alpha_2)\dots.(x-\alpha_{n-1}) = nx^{n-1} + na$$

Put $x = \alpha_1$;

$$(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\dots(\alpha_1-\alpha_n) = n(\alpha_1^{n-1} + a).$$

3. (a) Involves huge amount of calculation. Try yourself.

(b)

$$\begin{vmatrix} x & l & m & 1 \\ \alpha & x & n & 1 \\ \alpha & \beta & x & 1 \\ \alpha & \beta & y & 1 \end{vmatrix} = \begin{vmatrix} x-\alpha & l-\beta & m-y & 0 \\ 0 & x-\beta & n-y & 0 \\ 0 & 0 & x-y & 0 \\ \alpha & \beta & y & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x-\alpha & l-\beta & m-y \\ 0 & x-\beta & n-y \\ 0 & 0 & x-y \end{vmatrix}$$

$$= (x-\alpha)(x-\beta)(x-y)$$

4. (a) $x = r\cos\theta$ $\qquad |J| = r$

$y = r\sin\theta$

$$\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx = \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} \frac{r^2\cos\theta\,dr\,d\theta}{r}$$

$$= \int_{\pi/4}^{\pi/2} \left[\frac{r^2}{2}\right]_0^{\sqrt{2}} \cos\theta\,d\theta = \int_{\pi/4}^{\pi/2} \cos\theta\,d\theta$$

$$Ans \ = 1 - \frac{1}{\sqrt{2}}$$

(b) (By Induction)

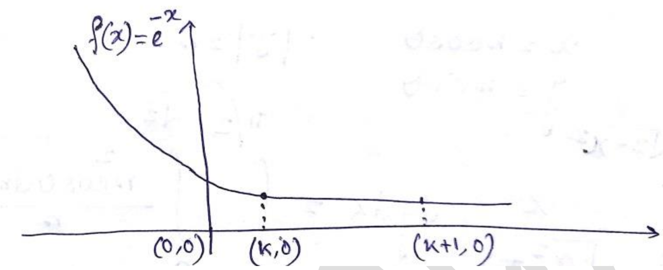
$$For\ n = 2,\ \ A_{2\times2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = Zero\ matrix$$

$$For\ n = m, A^m = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

$$For\ n = m + 1, A^{m+1} = A^m . A = \begin{bmatrix} 0 & 0 \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 \cdots & 0 \end{bmatrix}_{K\times K} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{K_1} \end{bmatrix}_{K\times 1} = \underset{\sim}{0}$$

5. (a)



Let Area $A(k) = \frac{1}{2}\left[e^{-k} + e^{-(k+1)}\right] - \int_k^{k+1} e^{-x}dx \ = \frac{3}{2}\ e^{-(k+1)} - \frac{1}{2}e^{-k}$.

$$A = \frac{3}{2}\sum_{k=0}^{\infty} e^{-(k+1)} - \frac{1}{2}\sum_{k=0}^{\infty} e^{-k}$$

$$= \frac{3}{2}[e^{-1} + e^{-2} + e^{-3} + \cdots] - \frac{1}{2}[1 + e^{-1} + e^{-2} + \cdots]$$

$$= [e^{-1} + e^{-2} + \cdots] - \frac{1}{2}$$

$$= \frac{e^{-1}}{1 - e^{-1}} - \frac{1}{2} = \frac{1}{e-1} - \frac{1}{2}.$$

(b)

$$f(x)=\begin{cases}(\alpha-x)^2.e^{(\beta-x)^2} & ;x>0\\ \alpha^2e^{\beta^2} & ;x=0\\ (\alpha+x)^2.e^{(\beta+x)^2} & ;x<0\end{cases}$$

$$\lim_{x\to0^+}f(x)=\lim_{x\to0^+}(\alpha-x)^2.e^{(\beta-x)^2}=\alpha^2e^{\beta^2}$$

$$\lim_{x\to0^-}f(x)=\lim_{x\to0^-}(\alpha+x)^2.e^{(\beta+x)^2}=\alpha^2e^{\beta^2}$$

So, f(x) is continuous $\forall\ (\alpha,\beta)\in\mathbb{R}$

$$\lim_{h\to0^+}\frac{f(h)-f(0)}{h}=\lim_{h\to0^+}\frac{(\alpha-h)^2e^{(\beta-h)^2}-\alpha^2e^{\beta^2}}{h}$$

$$f'(0^+)=e^{-\frac{2}{\alpha}-2\beta};\quad f'(0^-)=e^{\frac{2}{\alpha}+2\beta}$$

$$So,\qquad -\frac{2}{\alpha}-2\beta=\frac{2}{\alpha}+2\beta.$$

$$\therefore\frac{4}{\alpha}+4\beta=0\quad\Rightarrow\alpha\beta=-1$$

$\therefore f(x)$ is differentiable $\alpha\beta=-1$.