MSQMS 2015 Solution Paper

QMA (Mathematics)

1. (c)

$$xy + yz + zx = \frac{pq(p-q) + qr(q-r) + rq(r-p)}{(p-q)(q-r)(r-p)}$$

$$= \frac{-(p-q)(q-r)(r-p)}{(p-q)(q-r)(r-p)} = -1$$

2. (b)

$$u = tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{y}.\frac{1}{1+\frac{x^2}{y^2}} = \frac{y}{y^2+x^2}$$

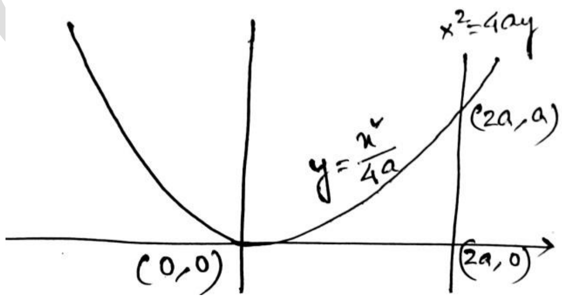
$$\frac{\partial u}{\partial y} = \frac{-x}{y^2+x^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2xy}{(y^2+x^2)^2};\quad \frac{\partial^2 u}{\partial y^2} = \frac{2xy}{(y^2+x^2)^2}$$

$$\frac{\partial^2 u}{\partial z^2} = 0$$

$$So, \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

3. (b)



$x^2 = 4ay \implies 4a^2 = 4ay \implies y = a$

$0 \le x \le 2a, \qquad 0 \le y \le \frac{x^2}{4a}$

$$\int_0^{2a}\int_0^{x^2/4a} xy\,dy\,dx = \int_0^{2a} x \left|\frac{y^2}{z}\right|_0^{x^2/4a} dx = \frac{1}{2}\int_0^{2a} x.\frac{x^4}{16a^2}dx = \frac{1}{32a^2}.\frac{1}{6}.64a^6 = \frac{1}{3}\,a^4$$

4. (d)

$$\int \frac{1}{x + x\log x}dx$$

$$= \int \frac{1}{(1+\log x)}.d(\log x)$$

$$= \log|1 + \log x| + c.$$

5. (d)

$$\frac{2}{3!} + \frac{4}{5!} + \cdots$$

$$= \frac{3-1}{3!} + \frac{5-1}{5!} + \cdots$$

$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$$

$$= e^{-1}$$

6. (d)

$$\frac{ds}{dt} = 100e^{-0.25t}$$

$$s_t = -400e^{-0.25t} + c$$

Sales promoted during the fifth year $= s_5 - s_4 = 400(e^{-1} - e^{-1.25}) = 400(e^{-1} - e^{-1.25})$

$$= 32.549 \approx 32.55$$

7. (d)

Total number of ways $= 2_{C_1}.5_{C_4}.9_{C_6} + 2_{C_1}.5_{C_5}.9_{C_5} = 840 + 252 = 1092.$

8. (b)

$$\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} - \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (abc-1)\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (abc-1)(a-b)(b-c)(c-a) = 0$$

Since $a \neq b \neq c,\ so\ abc\ = 1$

9. (d) Highest power of 5 in 100! is

$$\left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] + \left[\frac{100}{5^3}\right] = 20 + 4 + 0 = 24$$

10. (a)

$$f(x) = |x| + \sin x + cos^3 x = x\left(\frac{|x|}{x} + \frac{\sin x}{x}\right) + cos^3 x$$

$$RHL\ = \underset{x \to 0^+}{lt}\ f(x) = 0(1+1) + 1 = 1$$

$$LHL = \underset{x \to 0^-}{lt}\ f(x) = 0(-1+1) + 1 = 1$$

$LHL = RHL = f(0)$; so f(x) is continuous at $x = 0$

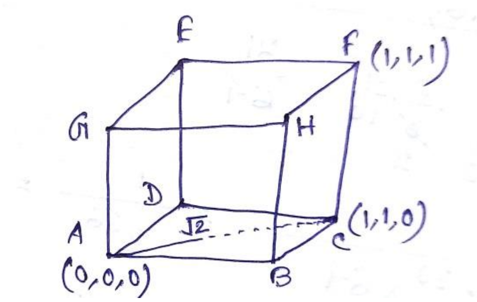
If f'(x) exists, then f'(x) = $\frac{|x|}{x} + \cos x - 3cos^2 x \sin x$

$$LHD = \underset{x \to 0^-}{lt} f'(x)\ = -1 + 1 = 0$$

$$RHD = \underset{x \to 0^+}{lt} f'(x) = 1 + 1 = 2$$

$LHD \neq RHD$, so it's not differentiable at $x = 0$.

11. (e)



Ant should travel from A(0,0,0) to F(1,1,1).

For the shortest distance it should go from AC $= \sqrt{2}$ $and\ then\ CF = 1$.

So, total distance $= (\sqrt{2}+1)m$.

12. (d) $x^4 + x^2 = 1$

$$\Rightarrow \left(x^2 + \frac{1}{2}\right)^2 = \frac{5}{4}$$

$$\Rightarrow x^2 + \frac{1}{2} = \pm\frac{\sqrt{5}}{2}$$

$$\Rightarrow x^2 = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

So, x will have one positive, one negative, and two complex roots.

13. (b) Let the books be A and B with selling prices Rs. 30 each. For A, profit is 25%, for B loss is 25%.

Let the cost price for A is x and for B is y.

$$Then \qquad \frac{125}{100} \times x = 30; \qquad \frac{75}{100} y = 30$$

$$\therefore x = 24 \qquad \therefore y = 40$$

Net cost price $= 64$

So, loss $= 64 - 60 = Rs. 4$.

14. (a) $f(x) = x^3 + ax^2 + bx + c$

$$f(i) = 0 = -i - a + bi + c = (c - a) - i(1 - b)$$

$$f(-i) = 0 = -i - a + bi + c$$

So, $a = c, b = 1$.

15. (d)

$$E = \frac{1}{4}.\frac{2}{6}.\frac{3}{8}.\frac{4}{10}\dots..\frac{31}{64} = \frac{1}{2^{30} \times 64} = \frac{1}{2^{36}} = 8^x$$

$$\therefore 8^x = 8^{-12}; \quad so, \qquad x = -12$$

16. (b) $B = -A^{-1}BA$

$$\rightarrow AB = -BA$$

$$\rightarrow AB + BA = 0$$

$$(A + B)^2 = A^2 + AB + BA + B^2 = A^2 + B^2$$

17. (a) For the candidate to be successful has to pass at least 5 papers.

So, total number of ways in which he can be successful $= 9_{C_5} + 9_{C_6} + 9_{C_7} + 9_{C_8} + 9_{C_9} = 256$.

18. (b) $e^{e^x} = a_0 + a_1 x + a_2 x^2 + \cdots$

$$Put\ x = 0,$$

$$e = a_0$$

19. (b)

$$\underset{x \to 0}{lt} \frac{\sin(\pi cos^2 x)}{x^2}$$

$$= \underset{x \to 0}{lt} \frac{\sin(\pi cos^2 x)}{\pi cos^2 x} \times \frac{cos^2}{x^2} = \pi$$

Alternate way: Applying L'Hospital Rule:

$$\lim_{x \to 0} -\frac{\cos(\pi cos^2 x)\pi \sin 2x}{2x} = \lim_{x \to 0} \frac{-\pi \cos(\pi cos^2 x)}{1} \times \frac{(\sin 2x)}{(2x)}$$

$$= 1 \times \pi \times 1 = \pi$$

20. (d) P(there is a girl child in a family)

$$= \frac{3C_1}{2^3} = \frac{3}{8}$$

QMB (Mathematics)

1. (a) Number of subsets of n elements= $2n+1_{C_n}$

Number of subsets of $\overline{n-1}$ elements $= 2n+1_{C_{n-1}}$

$\vdots$

$\vdots$

Total number of subsets $= 2n+1_{C_n} + 2n+1_{C_{n-1}} + \cdots + 2n+1_{C_0}$

$$= \frac{1}{2}(2^{2n+1}) = 2^{2n}$$

(b) One way is by Induction.

Alternate way:

Take matrix A as a function f:

$$f(x).f(y) = \begin{pmatrix} \cos x\cos y - \sin x \sin y & \cos x \sin y + \sin x \cos y \\ -\sin x \cos y - \cos x \sin y & -\sin x \sin y + \cos x \cos y \end{pmatrix}$$

$$= f(x+y)$$

So, $f(\alpha).f(\alpha) = f(2\alpha)$

$f(2\alpha).f(\alpha) = f(3\alpha) = [f(\alpha)]^3$

$\vdots$

$\vdots$

$[f(\alpha)]^n = f(n\alpha)$

$$So, A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$$

2. (a) $|f(x) - f(y)| \le (x-y)^2$

$\Rightarrow \left|\frac{f(x)-f(y)}{x-y}\right| \le |x-y|$

$\Rightarrow \underset{x\to y}{lt}\left|\frac{f(x)-f(y)}{x-y}\right| \le \underset{x\to y}{lt} |x-y| \quad [\because f \text{ is continuous over } \mathbb{R}]$

$\Rightarrow |f'(x)| \leq 0$

$\Rightarrow f'(x) = 0 \Rightarrow f(x) is\ a\ constant\ function.$

(b)

$x^n + n\,ax - b = 0$

$\Rightarrow (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n) = x^n + n\,ax - b$ -------------------(*)

Differentiating (*) w.r.t. x ;

$$\Rightarrow (x-\alpha_2)(x-\alpha_3)\dots(x-\alpha_n) + (x-\alpha_1)(x-\alpha_3)\dots(x-\alpha_n) + (x-\alpha_1)(x-\alpha_2)(x-\alpha_4)\dots(x-\alpha_n) + \cdots + (x-\alpha_1)(x-\alpha_2)\dots.(x-\alpha_{n-1}) = nx^{n-1} + na$$

Put $x = \alpha_1$;

$$(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\dots(\alpha_1-\alpha_n) = n(\alpha_1^{n-1} + a).$$

3. (a) Involves huge amount of calculation. Try yourself.

(b)

$$\begin{vmatrix} x & l & m & 1 \\ \alpha & x & n & 1 \\ \alpha & \beta & x & 1 \\ \alpha & \beta & y & 1 \end{vmatrix} = \begin{vmatrix} x-\alpha & l-\beta & m-y & 0 \\ 0 & x-\beta & n-y & 0 \\ 0 & 0 & x-y & 0 \\ \alpha & \beta & y & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x-\alpha & l-\beta & m-y \\ 0 & x-\beta & n-y \\ 0 & 0 & x-y \end{vmatrix}$$

$$= (x-\alpha)(x-\beta)(x-y)$$

4. (a) $x = r\cos\theta$ $\quad |J| = r$

$y = r\sin\theta$

$$\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx = \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} \frac{r^2\cos\theta\,dr\,d\theta}{r}$$

$$= \int_{\pi/4}^{\pi/2} \left[\frac{r^2}{2}\right]_0^{\sqrt{2}} \cos\theta \, d\theta = \int_{\pi/4}^{\pi/2} \cos\theta \, d\theta$$

$$Ans \ = 1 - \frac{1}{\sqrt{2}}$$

(b) (By Induction)

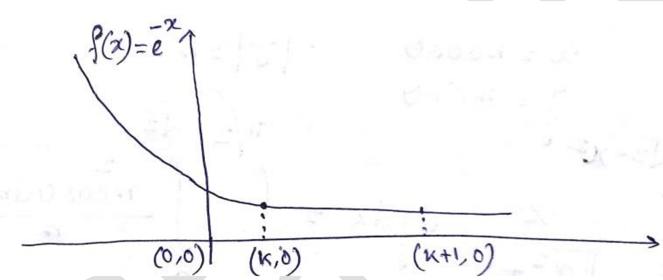
$$For\ n = 2,\ \ A_{2\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = Zero\ matrix$$

$$For\ n = m, A^m = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

$$For\ n = m+1, A^{m+1} = A^m . A = \begin{bmatrix} 0 & 0 \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 \cdots & 0 \end{bmatrix}_{K\times K} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{K_1} \end{bmatrix}_{K\times 1} = \underset{\sim}{0}$$

5. (a)



Let Area $A(k) = \frac{1}{2}\left[e^{-k} + e^{-(k+1)}\right] - \int_k^{k+1} e^{-x} dx \ = \frac{3}{2}\ e^{-(k+1)} - \frac{1}{2} e^{-k}$.

$$A = \frac{3}{2} \sum_{k=0}^{\infty} e^{-(k+1)} - \frac{1}{2} \sum_{k=0}^{\infty} e^{-k}$$

$$= \frac{3}{2}\left[e^{-1} + e^{-2} + e^{-3} + \cdots\right] - \frac{1}{2}\left[1 + e^{-1} + e^{-2} + \cdots\right]$$

$$= \left[e^{-1} + e^{-2} + \cdots\right] - \frac{1}{2}$$

$$= \frac{e^{-1}}{1 - e^{-1}} - \frac{1}{2} = \frac{1}{e-1} - \frac{1}{2}.$$

(b)

$$f(x) = \begin{cases} (\alpha - x)^2 . e^{(\beta - x)^2} & ; x > 0 \\ \alpha^2 e^{\beta^2} & ; x = 0 \\ (\alpha + x)^2 . e^{(\beta + x)^2} & ; x < 0 \end{cases}$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (\alpha - x)^2 . e^{(\beta - x)^2} = \alpha^2 e^{\beta^2}$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (\alpha + x)^2 . e^{(\beta + x)^2} = \alpha^2 e^{\beta^2}$$

So, f(x) is continuous $\forall \ (\alpha, \beta) \in \mathbb{R}$

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{(\alpha - h)^2 e^{(\beta - h)^2} - \alpha^2 e^{\beta^2}}{h}$$

$$f'(0^+) = e^{-\frac{2}{\alpha} - 2\beta} ; \quad f'(0^-) = e^{\frac{2}{\alpha} + 2\beta}$$

$$So, \qquad -\frac{2}{\alpha} - 2\beta = \frac{2}{\alpha} + 2\beta.$$

$$\therefore \frac{4}{\alpha} + 4\beta = 0 \quad \Rightarrow \alpha\beta = -1$$

$\therefore f(x)$ is differentiable $\alpha\beta = -1$.

6. (a)

$$\int_0^\infty \frac{\beta}{\eta} \left(\frac{x - \mu}{\eta} \right)^{\beta - 1} \exp\left[-\left(\frac{x - \mu}{\eta} \right)^\beta \right] dx \ ; \ \beta > 0, \eta > 0.$$

$$= \frac{\beta}{\eta^\beta} \int_0^\infty (x - \mu)^{\beta - 1} e^{-\left(\frac{x - \mu}{\eta} \right)^\beta} dx \ = \int_{\left(\frac{-\mu}{\eta} \right)^\beta}^\infty e^{-t} dt$$

$$\left[Take \ \left(\frac{x - \mu}{\eta} \right)^\beta = t, \qquad so \qquad \frac{\beta}{\eta^\beta} (x - \mu)^{\beta - 1} dx \ = dt \right]$$

$$Ans. = \left(\frac{1}{e} \right)^{\left(\frac{-\mu}{\eta} \right)^\beta}$$

(b) $g(x) = x^6 - x^5 + x^2 - x + 3$; $-\infty < x < \infty$

$g(x) = x(x-1)(x^4+1) + 3;$

If x > 1, then g(x) > 0.

Now when x < 0, then g(x) > 0

So, when $x \in [0, 1]$; we have to examine g(x) > 0 or not.

Even at $x = 0, g(x) = 3,$

$at\ x = 1, g(x) = 3$

So, checking should be done when $x \in (0, 1)$

In (0,1); we have $-x(x-1) \leq \frac{1}{4}$

$1 + x^4 \leq 2.$

$So, x(1-x)(1+x^4) < \frac{1}{2} < 3.$

$So, g(x) = 3 - [x(1-x)(1+x^4)]\ when\ x \in (0,1)$

$\therefore g(x) > 0$

So, g(x) > 0 for all x.

7. (a)

$$S = \frac{1}{1.2} + \frac{1.3}{1.2.3.4} + \frac{1.3.5}{1.2.3.4.5.6.} + \cdots$$

$$= \frac{1}{2} + \frac{1}{2.4} + \frac{1}{2.4.6.} + \cdots$$

$$= \frac{1}{2}.\frac{1}{1!} + \frac{1}{2^2}.\frac{1}{2!} + \frac{1}{2^3.3!} + \cdots$$

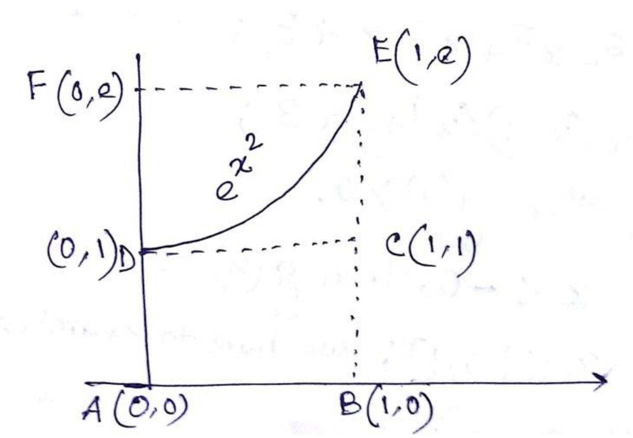
$$= \sum_{r=1}^{\infty} \frac{\left(\frac{1}{2}\right)^r}{r!}$$

$$Now,\ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty$$

$$e^{1/2} = 1 + \frac{1}{2} + \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^3}{3!} + \cdots \infty$$

$$So, \qquad S = e^{1/2} - 1$$

(b)



$$Area(ABCD) < \int_0^1 e^{x^2} dx < Ar(ABEF)$$

$$\Rightarrow 1 \times 1 \int_0^1 e^{x^2} dx < 1 \times e$$

$$\Rightarrow 1 < \int_0^1 e^{x^2} dx < e$$

8. (a) $x + y = k$

We have to find maximum value of $x^3y^2 = z$ (say)

$$z = x^3y^2 = x^3(k-x)^2 = x^5 - 2kx^4 + x^3k^2 = f(x)$$

To find maximum, $f'(x) = 5x^4 - 8kx^3 + 3k^2x^2$

Putting $f'(x) = 0 \quad \Rightarrow 5x^2 - 8kx + 3k^2 = 0$

$\Rightarrow (x-k)(5x-3k)=0$

$So, x=\frac{3k}{5}, \quad y=\frac{2k}{5}\ will\ give\ (x^3y^2)\ max$

(b) Radius $= r$, height $= h$

$\pi r^2 h = 25\frac{1}{7} = \frac{176}{7}$

Amount of sheet used should be minimum.

So, we are to find $\min\{\pi(r^2+2rh)\}$

$\pi(r^2+2rh) = \pi\left(r^2+2r.\frac{176}{7\pi r^2}\right) = \pi\left(r^2+\frac{2\times 176}{7\pi}\right) = f(r)$

$f'(r) = 2r - \frac{2\times 176}{7\pi r^2} = 0$

$\Rightarrow r^3 = \frac{176}{7\pi} = 8.$

$\Rightarrow r = 2.\ \ so, h = \frac{176}{22\times 4} = 2.$

9. (a)

$$a = \underbrace{1\,1\,1\,1\ldots 1}_{63\ digits} = \frac{10^{63}-1}{9}$$

$$b = 1+10+10^2+\cdots+10^6 = \frac{10^7-1}{9}$$

$$c = 1+10^7+10^{14}+\cdots+10^{56} = \frac{10^{63}-1}{10^7-1}$$

$$bc = \frac{10^{63}-1}{9} = a\ \ (QED)$$

(b)

$$a(b-c)x^2 + b(c-a)x + c(a-b) = 0$$

We take $x = 1, then$

$$a(b-c) + b(c-a) + c(a-b) = 0$$

So, two roots are $x = 1, 1$

So, sum of roots

$$= \frac{b(c-a)}{a(c-b)} = 2$$

$$\Rightarrow \frac{1}{a} - \frac{1}{c} = \frac{2}{b} - \frac{2}{c}$$

$$\Rightarrow \frac{2}{b} = \frac{1}{a} + \frac{1}{c}$$

$$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c} are\ in\ A.P.$$

$$\Rightarrow a, b, c\ are\ in\ H.P.$$

10. (a) Five digit number > 40000

Taking 1st digit as 4, total such numbers with 0, 1, 2, 3, 5 with no repetitions $= 5! = 120$ ways

Taking 1st digit as 5, total such numbers with 0,1, 2, 3, 4 with no repetitions $= 5! = 120\ ways$

So, total number of ways $= 240\ ways$.

(b) $g(x) = (3+x)^n = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$.

Putting $x = 1$, $4^n = a_0 + a_1 + \cdots + a_n = 4096 = 4^6$

$\therefore n = 6 \dots \dots (i)$

$(3+x)^6 = 3^6 + 6.3^5.x + 15.3^4.x^2 + 20.3^3.x^3 + 15.3^2.x^4 + 6.3.x^5 + x^6$

$\qquad = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6$

So, largest $a_j = a_1 = 1458$.